GRAPHS AND THEIR SPECTRA

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§ 1. Introduction

Can one hear the shape of a drum? By this suggestive title M. Kac [17] posed the question whether in certain problems of partial differential equations eigenfrequencies determine boundary conditions. A discrete version amounts to the question whether a graph is determined by its spectrum, i.e., by the eigenvalues of its adjacency matrix and their multiplicities. In certain cases the answer is positive, and these are all the more interesting if they characterize the group of the graph as well. This applies to the automorphism group $O^-(6, F_2)$ of the Schläfli graph, which describes the 27 lines on a cubic surface and has spectrum $(16^1, 4^6, (-2)^{20})$. This also applies to McLaughlin's group, which is characterized as the group of the graph on 275 vertices having spec $= (162^1, 27^{22}, (-3)^{252})$. But in general the answer to the question is negative. Pairs of nonisomorphic cospectral graphs exist already for order 8. For higher orders things can really go out of hand: there exist at least 16448 nonisomorphic cospectral graphs on $n = 36$ vertices having spec $= (15^1, 3^{15}, (-3)^{20})$ (cf. [4]). Yet, the spectrum of a graph can tell us something about the structural properties of a graph. We illustrate this in a number of examples, and refer to the literature for more complete expositions (cf. [1], [6], [8], [12], [13]).

We shall in particular deal with the largest, the second largest, and the smallest eigenvalue of the adjacency matrix of a finite graph. In Section 2 we show that $\alpha_{max} = 2$ characterizes the Coxeter-Dynkin graphs, as a consequence of certain lemmas on cliques and claws in a graph. The Hoffman-Shearer theory about $\alpha_{max} > 2$ is the subject of Section 3. We briefly indicate in Section

4 the recent developments about expander graphs and the second largest eigenvalue. Finally, in Section 5 we illustrate the application of root systems in the characterization of graphs having smallest eigenvalue -2, and make some further remarks about $\alpha_{min} < -2$.

Throughout the paper we use the following notions and notations. A graph is described by its adjacency matrix A (entries 1 for adjacent vertices and 0 otherwise). The eigenvalues of the graph (of the matrix A) are denoted by

$$\alpha_{max} = \alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n = \alpha_{min},$$

and the automorphisms by permutation matrices P such that $A = PAP'$. A graph is *regular* with *valency* k whenever $AJ = kJ$ (J is the all-one matrix) and *strongly regular* whenever
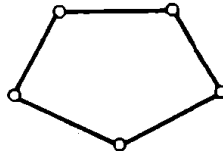
$$A^2 = kI + \lambda A + \mu(J - I - A).$$

The factorization

$$(A - rI)(A - sI) = \mu J$$

leads to the spectrum (k^1, r^f, s^g) of a strongly regular graph, with the multiplicities $1, f, g$ of the eigenvalues k, r, s.
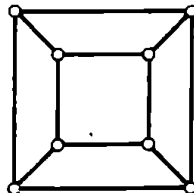
EXAMPLE. The pentagon graph has the spectrum $2, \tau^{-1}, \tau^{-1}, -\tau, -\tau$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ denotes the golden ratio, with

$$\tau^2 = \tau + 1, \quad 2\cos 72^\circ = \tau^{-1}, \quad 2\cos 144^\circ = -\tau.$$

EXAMPLE. The cubic graph has

$$A = \begin{bmatrix} 0 & J - I \\ J - I & 0 \end{bmatrix}, \quad \text{spec} = (3, -3, 1^3, (-1)^3),$$

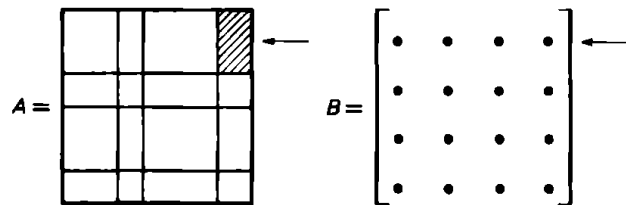and is a bipartite graph with two complementary cocliques of size 4.

Finally, we recall two important theorems involving spectral properties of graphs (cf. [1], [8], [13]).

For a connected graph Γ the *Perron–Frobenius theorem* says that

1. α_{max} is simple, and has a positive eigenvector;
2. each $\alpha_i \geqslant -\alpha_{max}$, and $\alpha_{min} = -\alpha_{max}$ if and only if Γ is bipartite;
3. $\alpha_{max} \leqslant$ max valency, with equality if and only if Γ is regular;
4. α_{max}(subgraph) $< \alpha_{max}(\Gamma)$, a strict inequality for any proper subgraph of Γ.

From the *interlacing theorem* for eigenvalues we only need the following:

1. Between any two eigenvalues of Γ there is one eigenvalue of the graph $\Gamma\setminus\{x\}$, obtained from Γ by deleting any vertex x.

2. Let the adjacency matrix A of Γ be partitioned into parts having constant line sums, and let B denote the matrix consisting of these line sums. Then each eigenvalue of B is an eigenvalue of A.

$$A = \begin{bmatrix} & & & \diagdown \\ & & & \\ & & & \\ & & & \end{bmatrix} \longleftarrow \qquad B = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \longleftarrow$$

§ 2. Coxeter graphs

· We shall start with two lemmas on cliques and claws in graphs, in the spirit of Hoffman. A *coclique* is a subgraph none of whose vertices are adjacent. A *claw* of size $c + 1$ consists of a coclique of size c and a vertex which is adjacent with all vertices of the coclique.

LEMMA. *For a k-regular graph Γ the size c of a coclique satisfies*

$$c = |\text{coclique}| \leqslant \frac{-n\alpha_n}{k - \alpha_n}.$$

Proof. Label the vertices of Γ so that

$$A = \begin{bmatrix} 0 & P \\ P' & Q \end{bmatrix}, \qquad z = \begin{bmatrix} (c-n)j \\ cj \end{bmatrix}_{n-c}^{c}.$$

Then an easy computation involving the $n \times 1$ vector z yields

$$\alpha_n \leqslant \frac{z'Az}{z'z} = \frac{-nc^2k}{c(n-c)n} = \frac{-ck}{n-c}. \qquad \blacksquare$$

Remark. For arbitrary graphs the inequality reads

$$c \leqslant \frac{-n\alpha_1\alpha_n}{v_{min}^2 - \alpha_1\alpha_n},$$

where v_{min} is the minimal valency (cf. [13]). The colouring number χ of the graph satisfies

$$\chi c \geqslant n, \quad \chi \geqslant 1 - \alpha_1/\alpha_n.$$

LEMMA. *For a claw of size* $c+1$ *in a graph we have*

$$c \leqslant \alpha_{max}^2.$$

Proof. Write $\alpha := \alpha_{max}$. The size n matrix $\alpha I - A$ is positive-semidefinite, hence so is its size $(c+1)$ submatrix:

$$\alpha I - A = \begin{bmatrix} \alpha & - & - & - & \\ - & \alpha & & & \\ - & & \cdot & & P \\ \vdots & & & \cdot & \\ - & & & & \alpha \\ \hline & & P' & & Q \end{bmatrix}, \quad \begin{bmatrix} \alpha & -j' \\ -j & \alpha I \end{bmatrix} \approx \begin{bmatrix} \alpha - c/\alpha & 0' \\ 0 & \alpha I \end{bmatrix}.$$

It follows that $\alpha^2 \geqslant c$. Furthermore, if $\alpha^2 = c$ then P and Q must vanish. ∎

We are now in a position to define the notion of a Coxeter graph, to give the examples $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, and to prove that these are all Coxeter graphs.

DEFINITION. A *Coxeter graph* is a connected graph having $\alpha_{max} = 2$.

For a Coxeter graph (V, E) with adjacency matrix A and eigenvector $x = (x_1, \ldots, x_n)$ corresponding to $\alpha_{max} = 2$ we have

$$Ax = 2x, \quad 2x_p = \sum_{(i,p)\in E} x_i.$$

It is easy to prove that the following graphs are Coxeter graphs, by guessing the (positive) components $x_1, \ldots, x_n$.
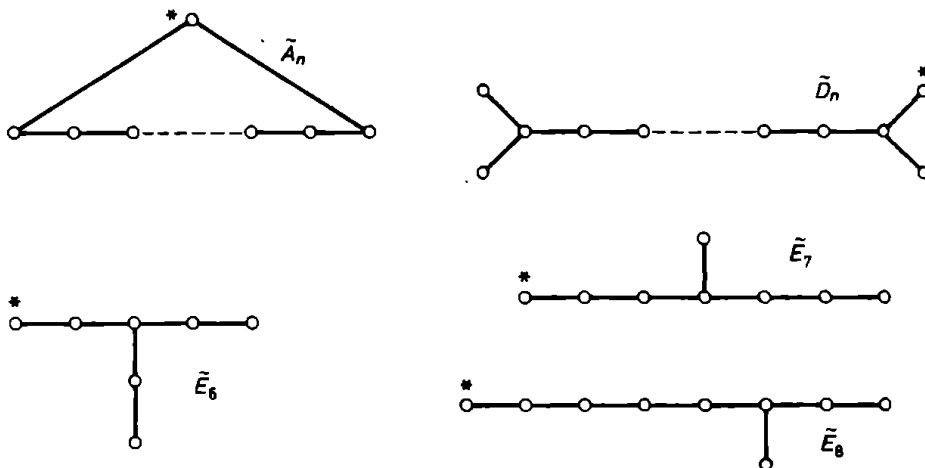
THEOREM. *The graphs* $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ *are the only Coxeter graphs.*

Proof. By use of the second lemma above we observe that in a Coxeter graph claws of size $c+1$ must have $c \leqslant 4$, and that for $c = 4$ the only Coxeter graph is $\tilde{D}_4$. As a consequence of part 4 of the theorem by Perron and Frobenius, a Coxeter graph cannot have three claws of size $3+1$:
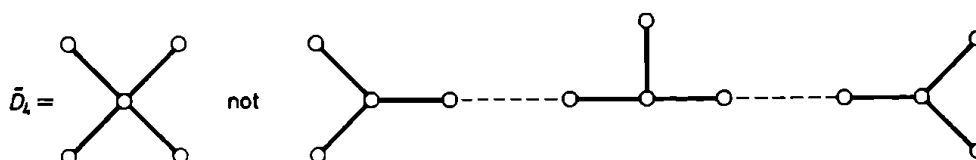
If the number of claws of size $3+1$ equals two or one or zero, then the Coxeter graph must be $\tilde{D}_n$ or $\tilde{E}_{6,7,8}$ or $\tilde{A}_n$, respectively. ∎

The present theorem represents Coxeter's reflections and goes back to Smith [21]. It implies the characterization of connected graphs having $\alpha_{max} < 2$. These are the graphs A_n, D_n, E_n, obtained from $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$ by deleting the starred vertex. For the value of α_{max}, also called the *spectral radius* of the graph, the following is well known:

$$\|A_n\| = 2\cos\frac{\pi}{n+1}, \qquad \|D_n\| = 2\cos\frac{\pi}{2n-2},$$

$$\|E_6\| = 2\cos\frac{\pi}{12}, \qquad \|E_7\| = 2\cos\frac{\pi}{18}, \qquad \|E_8\| = 2\cos\frac{\pi}{30}.$$

§ 3. Largest eigenvalue $\alpha_{max} > 2$

As first examples the following two infinite families of graphs are considered:
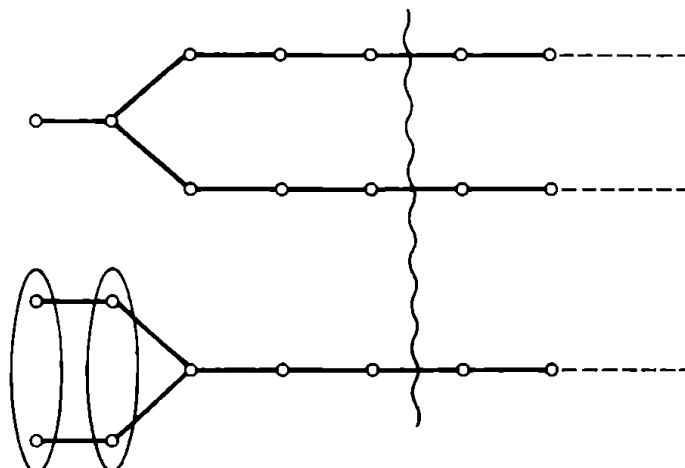
We first recognize the graphs $\tilde{E}_7$ and $\tilde{E}_6$ with $\alpha_{max} = 2$. Extending these graphs to the right we obtain graphs having

$$2 < \alpha_{max} =: 2\cosh\alpha = e^{\alpha} + e^{-\alpha}, \quad \alpha > 0.$$

Our interest is the limit of α_{max} as $n \to \infty$. To that end we consider the maximum eigenvalue of the matrix of the average row sums of the indicated blocks (of size $2, 2, 1, 1, 1, \ldots$) of the second graph Γ_n:

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 1 & 0
\end{bmatrix},
\quad
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & & \text{etc.} & & & \\
\hline
1 & 0 & 1 & 0 & 0 & & & & & \\
\hline
0 & 2 & 0 & 1 & 0 & & & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & & & x & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & & & -1 & x & -1 \\
& & \text{etc.} & & & & & 0 & -1 & x
\end{bmatrix}.
$$

The characteristic polynomial $P_n(x)$ of this matrix satisfies

$$P_{n+2}(x) = xP_{n+1}(x) - P_n(x).$$

Putting $x := e^{\xi} + e^{-\xi}$ we obtain

$$P_{n+2}(x) - e^{-\xi}P_{n+1}(x) = e^{\xi}\big(P_{n+1}(x) - e^{-\xi}P_n(x)\big)$$

$$= \ldots = e^{(n-1)\xi}\big(P_3(x) - e^{-\xi}P_2(x)\big)$$

$$= e^{(n-1)\xi}\left[\begin{vmatrix} x & -1 & 0 \\ -1 & x & -1 \\ 0 & -2 & x \end{vmatrix} - e^{-\xi}\begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix}\right]$$

$$= e^{(n-2)\xi}(e^{4\xi} - e^{2\xi} - 1).$$

This expression has largest root $e^{2\xi} = \tau$, the golden ratio. Hence by application of a theorem by Hurwitz we find

$$\lim_{n \to \infty} \alpha_{max} = e^{\xi} + e^{-\xi} = \tau^{1/2} + \tau^{-1/2} = \tau^{3/2}.$$

The number

$$\tau^{3/2} = \sqrt{2 + \sqrt{5}} \approx 2.058$$

is not the spectral radius of a graph (cf. [11]), since its algebraic conjugate is not a real number. But it is a limit of spectral radii not only for the graphs Γ_n above, but also for the "apple" graphs σ_n on $n + 1$ vertices, consisting of the n-circuit and one edge sticking out. One can prove [14] that

$$\alpha_{max}(\Gamma_n) \nearrow \tau^{3/2}, \quad \alpha_{max}(\sigma_n) \searrow \tau^{3/2}.$$

More generally, for any given number > 2, say $2\cosh\alpha$, $\alpha > 0$, two questions may be asked. Does there exist a graph whose largest eigenvalue equals $2\cosh\alpha$? If not, then does there exist a sequence of graphs Γ_1, $\Gamma_2, \ldots, \Gamma_n, \ldots$ such that the limit of their largest eigenvalues equals $2\cosh\alpha$? The first question has a general answer, as a consequence of Doob's result [9].

THEOREM. *If* $\alpha_{max}(\Gamma) > 2$ *then* Γ *has a proper subgraph* Δ *with* $\alpha_{max}(\Delta) = 2$.

So the required graphs are the extensions of the Coxeter graphs (cf. [7], [9]).

More specifically, one can ask for all graphs Γ having $2 < \alpha_{max}(\Gamma) < \tau^{3/2}$. These graphs are of two types.

Let $T_{i,j,k}$ denote the graph on $i+j+k+1$ vertices consisting of three paths of i, j, k edges, respectively, and having one common end vertex. For instance:

$$E_{10} := T_{1,2,6} \cong$$
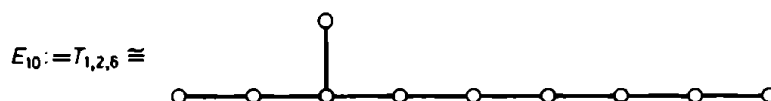
Let $H_{i,j,k}$ denote the graph on $i+j+k+1$ vertices consisting of a path on $u = i+j+k-1$ vertices $x_1, \ldots, x_u$, with two extra edges at x_i and at x_{u+1-k}. For instance:

$$H_{3,3,2} \cong$$
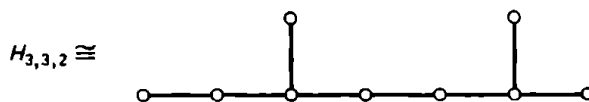
Thanks to the efforts of several authors [14], [21], [7], [12], [2], we now know all graphs having $\alpha_{max} \leqslant \sqrt{2+\sqrt{5}}$. We copy the final result from [2].

THEOREM. *Let* Γ *denote a graph with* $2 < \alpha_{max}(\Gamma) < \tau^{3/2}$. *Then* Γ *is one of the graphs* $T_{i,j,k}$ *or* $H_{i,j,k}$. *The* $T_{i,j,k}$ *occurring have*

$$(i, j, k) = (1, 2, k \geqslant 6); (1, 3, k \geqslant 4); (2, 2, k \geqslant 3);$$

$$(2, 3, 3); (1, j, k) \text{ with } 4 \leqslant j \leqslant k.$$

The $H_{i,j,k}$ *occurring have*

$$(i, j, k) = (i, j \geqslant i+k, k); (3, j \geqslant k+2, k);$$

$$(2, j \geqslant k-1, k); (2, 1, 3); (3, 4, 3);$$

$$(3, 5, 4); (4, 7, 4); (4, 8, 5).$$

Finally, we turn to the second question, posed by Hoffman [14], about limits of spectral radii, i.e., limits of largest eigenvalues of sequences of graphs. The answer is contained in the following theorems, due to Hoffman [14] and Shearer [20], respectively.

THEOREM. *A real number* $2\cosh\alpha$, *with* $2 < 2\cosh\alpha < \tau^{3/2}$, *is a limit of spectral radii iff* $e^{2\alpha}$ *is the largest root of a polynomial*

$$x^q - x^{q-2} - x^{q-3} - \ldots - x - 1.$$

THEOREM. *Any real number* $2\cosh\alpha \geqslant \tau^{3/2}$ *is a limit of spectral radii.*

For both theorems the construction of a sequence of trees suffices. We describe Shearer's construction for the second theorem, for a given number $2\cosh\alpha$, with $e^{2\pi} > \tau$ (for $e^{2\alpha} = \tau$ cf. the construction above).

First determine the infinite sequences a_i, b_i, n_i recursively by $a_0 = b_1 = 1$ and, for $i = 1, 2, \ldots$,

(1) $\qquad 2b_i\cosh\alpha = a_i; \qquad n_i = [(e^\alpha a_i - a_{i-1})/b_i];$

(2) $\qquad a_{i+1} = 2a_i\cosh\alpha - a_{i-1} - n_i a_i/2\cosh\alpha,$ or equivalently,

(2') $\qquad e^\alpha a_i - a_{i-1} = n_i b_i + a_{i+1} - e^{-\alpha}a_i.$

By induction it follows from (2') and (1) that $a_{i+1} > e^{-\alpha}a_i$. Hence the n_i are nonnegative integers and a_i and b_i are positive real numbers. It is possible to interpret $2\cosh\alpha$ and a_i, b_i as eigenvalue and eigenvector of an infinite graph Γ:
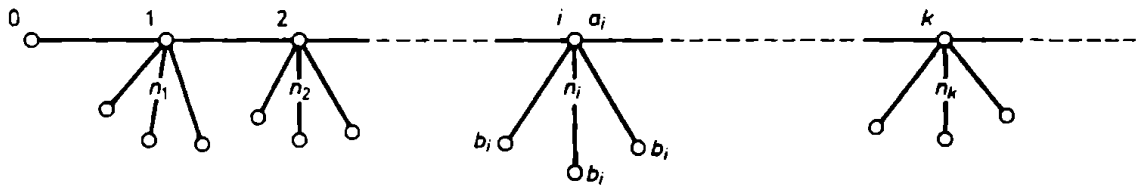
The infinite path $0\text{-}1\text{-}2\text{-}\ldots$ has further edges: n_1 in 1, n_2 in 2, etc. The eigenvector v takes value a_i at vertex i, and b_i at each of the n_i vertices adjacent to i (not on the path). The equations (1) and (2) express that $2\cosh\alpha$ times the value in each vertex equals the sum of the values in the neighbouring vertices:

(3) $\qquad 2\cosh\alpha \cdot v(x) = \sum_{y \sim x} v(y),$

for each vertex x. Now consider the finite subgraph Γ_k on the vertices $0, 1, \ldots, k$ (not $k+1, k+2, \ldots$), and adherents. Denote its maximum eigenvalue by $\lambda(\Gamma_k)$.

LEMMA. $\lambda(\Gamma_1) < \lambda(\Gamma_2) < \ldots < \lambda(\Gamma_k) < 2\cosh\alpha.$

Proof. For fixed n_i, the equations (2) determine a_i as a function of α. Then a_{i+1}/a_i is monotone increasing for $\alpha > 0$, for each $i \geqslant 0$. This follows by induction from $a_1/a_0 = \cosh\alpha$ and from (2) divided by a_i. Now let $\lambda(\Gamma_k) = 2\cosh\alpha'$. Then (2) define the corresponding eigenvector $a_i(\alpha')$, $b_i(\alpha')$. But Γ_k has $a_{k+1}(\alpha') = 0$. Since this is impossible for $\alpha' \geqslant \alpha$, the lemma is proved.

To complete the proof of Shearer's theorem it suffices to show that for

each $\varepsilon > 0$ there is $k = k(\varepsilon)$ such that

$$\lambda(\Gamma_k) > 2\cosh\alpha - \varepsilon.$$

For the vertex set $V_k = V(\Gamma_k)$ it follows from (3) that

$$2\cosh\alpha \cdot \sum_{x \in V_k} v^2(x) = \sum_{x \in V_k}\sum_{y \sim x} v(x)v(y) = \sum_{\substack{x, y \in V_k \\ x \sim y}} v(x)v(y) + a_k a_{k+1}$$

$$\leqslant \lambda(\Gamma_k)\sum v^2(x) + a_k a_{k+1},$$

by a well-known property of the largest eigenvalue. So it remains to show that
for large k we have

$$a_k a_{k+1} \leqslant \varepsilon \sum_{x \in V_k} v^2(x).$$

Now (1) and (2) in addition imply

$$n_i a_i \geqslant (e^\alpha a_i - a_{i-1})(e^\alpha + e^{-\alpha}) - a_i,$$

$$a_{i+1} \leqslant a_i(e^\alpha + e^{-\alpha}) - a_{i-1} - e^\alpha a_i + a_{i+1} + \frac{\alpha_i}{e^\alpha + e^{-\alpha}},$$

$$a_{i+1} \leqslant a_i\left(e^{-\alpha} + \frac{1}{e^\alpha + e^{-\alpha}}\right) \leqslant a_i \tau^{1/2},$$

the last inequality by the assumption. It remains to show that

$$\frac{a_k a_{k+1}}{\sum_{x \in V_k} v^2(x)} \leqslant \tau^{1/2}\frac{a_k^2}{\sum_{i=0}^{k} a_i^2} < \tau^{1/2}\varepsilon.$$

This is true if a_k is bounded for $k \to \infty$, both for convergent and for divergent
$\sum_{i=0}^{\infty} a_i^2$. To prove that a_k is bounded, Shearer distinguishes between the easy
part $2\cosh\alpha \geqslant 2.325$, and the difficult part $2.058 < 2\cosh\alpha < 2.325$. Since we
did not find a shorter reasoning, we now refer back to the original paper [20].

§ 4. The second largest eigenvalue $\alpha_2(\Gamma)$

The second largest eigenvalue α_2 of a graph measures certain properties which
are interesting for computing science. Thus, small α_2 implies large connectivity,
but also large girth and good expansion.

In 1973 Fiedler (cf. also this volume [10]) defined the *connectivity* of
a graph to be the second smallest eigenvalue of its Laplacian matrix $\Delta - A$.
Here Δ denotes the diagonal matrix of the local valencies of the graph with
adjacency matrix A, and

$$\lambda_2(\Delta - A) := \min_{(z,j) = 0, \|z\| = 1} z'(\Delta - A)z,$$

which equals $k - \alpha_2$ if the graph is regular of valency k. The *girth* of a graph is the length of its smallest circuit. A graph $(V, \sim)$ has good *expansion* if each part S of V has a large neighbourhood

$$\partial S := \{x \in V: \exists_{s \in S}(x \sim s)\}.$$

The relation between expansion and the second eigenvalue is expressed by

$$|S| \leqslant \tfrac{1}{2}|V| \quad \Rightarrow \quad |\partial S| \geqslant \frac{|S| \cdot |V \backslash S|}{|V|} \lambda_2(\Delta - A),$$

for all $S \subset V$ (cf. [11], Lemma 5.7).

The following construction provides an infinite family of graphs having the required properties. Such graphs are called *Ramanujan graphs*. They are regular of valency k, have n vertices, and their $\alpha_2 = 2\sqrt{k-1}$, as small as can be.

Let q and p be primes, both $\equiv 1 \pmod 4$, let $p < q$, and p not a square mod q. Let Q consist of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whose entries are integers mod q, modulo a common factor $\neq 0 \pmod q$, and having determinant $\neq 0$ $\pmod q$. Thus Q constitutes the group $\mathrm{PGL}(2, F_q)$ and has the order $q(q^2 - 1)$.

We represent p as a sum of four squares:

$$p = a_0^2 + a_1^2 + a_2^2 + a_3^2, \quad \text{odd } a_0 \in N; \ a_1, a_2, a_3 \in 2Z.$$

Interpreting $a_0, \ldots, a_3 \bmod q$, we turn this representation into the matrix

$$\begin{bmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{bmatrix}, \quad i^2 \equiv -1 \pmod q.$$

Let P denote the subset of Q consisting of all such representations for p. Then the Cayley graph (Q, P) is defined as follows. The vertices are the elements of Q, and two vertices are adjacent whenever their quotient is in P. This graph has number of vertices and valency

$$n = q(q^2 - 1), \quad k = |P| = p + 1.$$

Indeed, the number of the representations of p as above equals $p + 1 = \sum_{d|p} d$, according to a formula of Jacobi. Much more difficult is the number of the representations

$$p^k = x_0^2 + 4q^2(x_1^2 + x_2^2 + x_3^2), \quad x_i \in Z.$$

Following a conjecture by Ramanujan (1916), proved by Eichler (1954), this number behaves like

$$c \sum_{d|p^k} d + O_\varepsilon(p^{k(1/2 + \varepsilon)}) \quad \text{as } k \to \infty.$$

By use of this formula it is possible to show that our graphs have

$$\text{girth} \geqslant 2\log_p q, \qquad \alpha_2 \leqslant 2\sqrt{p},$$

thus satisfying the required properties; we refer to [19] for the proof, and for the construction in other cases.

For further developments in the spectral properties of the Laplacian we refer to [10] in this volume. This paper also contains references to recent results concerning the multiplicity of the second eigenvalue.

§ 5. Smallest eigenvalue $\alpha_{min} = -2$

It may be convenient to represent the vertex of a graph by a set of vectors in a vector space. The road from graphs to geometry goes via spectra. It uses the main theorem of linear algebra. Recall that every symmetric $n \times n$ matrix M can be diagonalized into $M = SAS'$, where S is orthogonal, and $A = \text{diag}(\Lambda_p^+, \Lambda_q^-, O_r)$ collects the p positive, the q negative, and the r zero eigenvalues of M, $p+q+r = n$. We replace S of size $n \times n$ by T of size $n \times (p+q)$, by deleting the last r columns of S, and by multiplying the other columns j by $|\lambda_j|^{-1/2}$. Then

$$M = T \begin{bmatrix} I_p & O \\ O & -I_q \end{bmatrix} T'.$$

This proves the following theorem, which is phrased in terms of the indefinite vector space $R^{p,q}$ of dimension $p+q$, provided with the inner product

$$(x, y) = x_1 y_1 + \ldots + x_p y_p - x_{p+1} y_{p+1} - \ldots - x_{p+q} y_{p+q}.$$

THEOREM. *Every symmetric matrix of size n is the Gram matrix of n vectors in $R^{p,q}$.*

Now let M denote the (a, b, c)-adjacency matrix of an ordinary graph Γ, defined by its entries

$$m_{ii} = a, \qquad m_{ij} = b \quad \text{for } i \sim j, \qquad m_{ij} = c \quad \text{otherwise}.$$

An (a, b, c)-*representation* of Γ is the set of n vectors in $R^{p,q}$ which corresponds to M according to the theorem. In particular, for the ordinary $(0, 1, 0)$-adjacency matrix A, and for the $(0, -1, 1)$-adjacency matrix C of a graph Γ, we shall frequently use the following representations:

the $(-\alpha_{min}, 1, 0)$-representation: $\quad M = -\alpha_{min} I + A$;
the $(\gamma_{max}, 1, -1)$-representation: $\quad M = \gamma_{max} I - C$.

In both cases the graph is represented by a set of vectors of equal length in a positive-definite space. In both cases the number of mutual angles between these vectors equals two. In the case of $M = \gamma I - C$ these angles are

supplementary, and the 1-subspaces ≡ lines which carry the vectors are equiangular.

Line graphs provide examples for such representations. Let a graph with n vertices i and m edges ε be denoted by its $n \times m$ incidence matrix N:

$$N = [n_{i,\varepsilon} := 1 \text{ for } i \in \varepsilon, = 0 \text{ elsewhere}].$$

Then the adjacency matrix A of the graph Γ, and L of its *line graph*, satisfy

$$NN' = \text{Diag} + A, \quad N'N = 2I + L.$$

Since except for the eigenvalue 0, the matrices NN' and $N'N$ have the same eigenvalues and the same multiplicities, we find an easy relation between the spectra of A and L.

EXAMPLE.
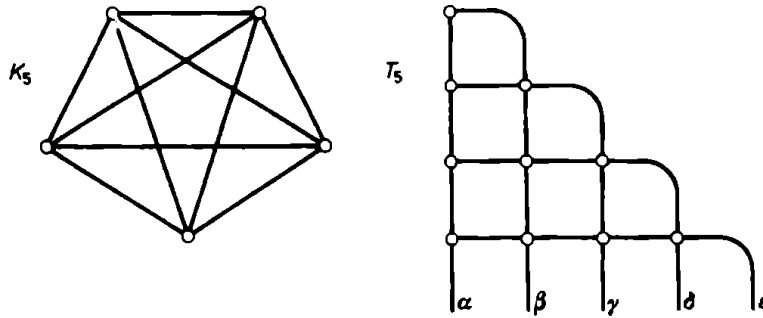
The complete graph K_n has

$$NN' = (n-1)I + J - I,$$

$$\text{spec } NN' = ((2n-2)^1, (n-2)^{n-1}),$$

$$\text{spec } N'N = ((2n-2)^1, (n-2)^{n-1}, 0^{n(n-1)/2-n}),$$

$$\text{spec } T_n = ((2n-4)^1, (n-4)^{n-1}, (-2)^{n(n-3)/2}).$$
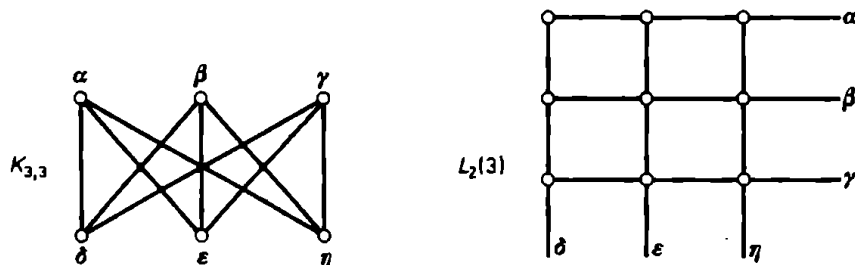
EXAMPLE.

$$\text{spec } L_2(n) = ((n-2)^{2n-2}, (2n-2)^1, (-2)^{n^2-2n+1}).$$

We next consider the problem of determining all graphs having $\alpha_{\min} = -2$. We mention the following examples.

EXAMPLE. Line graphs: $L = N'N - 2I$.

EXAMPLE.

$$\mathrm{spec}\begin{bmatrix} J-I & J-I \\ J-I & J-I \end{bmatrix} = ((2n-2)^1, 0^n, (-2)^{n-1}).$$

EXAMPLE. The graphs of Petersen $(n = 10)$, Clebsch $(n = 16)$, Shrikhande $(n = 16)$, Schläfli $(n = 27)$, Chang $(n = 28)$. The first two graphs are drawn below; for the rest we refer to [6].

In order to determine all graphs with $\alpha_{\min} = -2$ we look at sets of l lines in R^d at $60°$, $90°$. Given such a set, we select $2l$ vectors, of norm 2, two along each line. Their Gram matrix is positive-semidefinite (psd), has entries ± 2, ± 1, 0, and looks as shown below; the upper left corner is psd $(2I - B)$, has $\beta_{\max} \leq 2$, and corresponds to the Coxeter graphs, while the lower right corner is psd $(2I - A)$, has $\alpha_{\min} \geq -2$, and corresponds to the desired graphs.

$$\begin{bmatrix} 2 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ \underline{0/-1 \quad 2} & & & & & & \\ & & & 2 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & 0/1/-1/-2 & & 2 & & & \underline{} \\ & & & & & 2 & \\ & & & & & & \cdot \\ & & & & & & \cdot \\ & & & & & 0/1 & \quad 2 \end{bmatrix}$$

The next theorem determines special classes of sets of lines at $60°$, $90°$ which are irreducible and star-closed, i.e., they contain the third line at $60°$ in the plane containing any two lines at $60°$ of the set.

THEOREM [6]. The irreducible star-closed sets of lines at $60°$, $90°$ are the root systems A_n, D_n, E_6, E_7, E_8.

Now the construction runs as follows. Given a connected graph having adjacency matrix A with $\alpha_{min} = -2$. Then $2I + A$ is the Gram matrix of a set of vectors at $60°$, $90°$. The set of lines at $60°$, $90°$ spanned by these vectors is closed off into a star-closed set of lines at $60°$, $90°$. Hence by the theorem we are left with the known sets A_n, D_n, $E_{6,7,8}$. Using well-known inclusions we arrive at the main theorem.

THEOREM. *Any graph with* $\alpha_{min} = -2$ *is represented by* $2I + A$ *as a subset of the root systems* D_n *or* E_8.

The root systems are defined in terms of an orthonormal basis $e_1, \ldots, e_n$ in R^n.

$$D_n := \{\langle e_i \pm e_j \rangle : i \neq j \in \{1, \ldots, n\}\}, \quad |D_n| = n(n-1);$$

$$A_n := \{\langle e_i - e_j \rangle : i \neq j \in \{1, \ldots, n+1\}\}, \quad |A_n| = \tfrac{1}{2}n(n+1).$$

EXAMPLE. Any $G = (\{e_1, \ldots, e_n\}, E)$ has line graph $L = \{e_i + e_j : \{e_i, e_j\} \in E\}$.

$$L(K_6) = \{e_i + e_j : i \neq j = 1, \ldots, 6\}. \quad L(K_{3,3}) = \{e_i + e_j : i = 1, 2, 3; j = 4, 5, 6\}.$$

$$E_8 := D_8 \cup \{\tfrac{1}{2}\langle \varepsilon_1 e_1 + \ldots + \varepsilon_8 e_8 \rangle : \varepsilon_i = \pm 1, \ \textstyle\prod \varepsilon_i = 1\},$$

having $56 + 64 = 120$ lines in R^8. For

any line l: $\quad E_7 := \{\langle x \rangle : x \in E_8, \ x \perp l\}$, 63 in R^7;

any star s: $\quad E_6 := \{\langle x \rangle : x \in E_8, \ x \perp s\}$, 36 in R^6.

EXAMPLE. The Schläfli graph on $n = 15 + 6 + 6$ vertices is represented in E_8 by

$$\{e_i + e_j : i \neq j = 1, \ldots, 6\} \cup \{\tfrac{1}{2}\textstyle\sum e_k - e_i - e_7\} \cup \{\tfrac{1}{2}\textstyle\sum e_k - e_i - e_8\}.$$

In E_8 graphs $2I + A$ have ≤ 36 vertices and valency ≤ 28. Regular graphs in E_8 have ≤ 28 vertices and valency ≤ 16.

Root systems are very much related to *root lattices*, which are defined as integral lattices spanned by vectors of norm 2. The vectors of norm 2 form a root system, and the analogous definitions for the root lattices are

$$D_n := \{x \in R^n : x_i \in Z, \ \textstyle\sum x_i \in 2Z\},$$

$$E_8 := \langle D_8; \ \tfrac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i : \varepsilon_i = \pm 1 \rangle_Z.$$

The companion theorem reads as follows (cf. [1]):

THEOREM. *Any root lattice is a direct sum of lattices of types* A_n, D_n, E_6, E_7, E_8.

The considerations of the present section essentially determine the graphs having smallest eigenvalue $\alpha_{min} = -2$. The actual classification still is rather

complicated. The same holds for the case $\alpha_{min} > -2$. For these cases we refer to the literature [1], [3], [7], [15].

For $\alpha_{min} < -2$, similar questions may be asked as for $\alpha_{max} > 2$. Interestingly, similar answers are obtained if the constructions involved use bipartite graphs. Indeed, such graphs have $\alpha_{min} + \alpha_{max} = 0$. This applies to the trees constructed in Section 3, hence we have:

THEOREM. *Any real number* $\leqslant -\tau^{3/2}$ *is a limit of smallest eigenvalues of graphs.*

Essentially unknown is the distribution of the minimum eigenvalues of graphs in the interval

$$-\tau^{3/2} < \alpha_{min} < -2.$$

We refer to the literature for scattered results in this region; two such results (cf. [15], [16]) read as follows:

1. $E_{10} \equiv T_{1,2,6}$ is the only graph with $-2.007 < \alpha_{min} < -2$.
2. Graphs with large α_{max} satisfying $-1-\sqrt{2} < \alpha_{min} \leqslant -2$ have $\alpha_{min} = -2$ and are representable in D_n.

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