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ON A CLASS OF SPACES FOR WHICH THE FIXED-POINT PROPERTY IS CHARACTERIZED BY HOMOLOGY GROUPS

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By the Lefschetz fixed-point formula, a polyhedron X, clearly has the fixed-point property (f.p.p.) if it satisfies the following condition:

CONDITION A. X is compact, connected, and the reduced integral homology groups of X are all torsion groups.

This condition is not necessary for a polyhedron X to have the f.p.p. as can be exemplified by the complex projective space $\mathrm{CP}(n)$ for even integers n. We now ask for what spaces the f.p.p. can indeed be characterized by this condition. We call a polyhedron X a *Lefschetz space* provided that X has the f.p.p. if and only if X satisfies Condition A.

In this note*, we shall determine certain Lefschetz spaces and establish a few general properties of Lefschetz spaces. In particular, we shall show that the disk and the real projective plane are the only fixed-point spaces among all the 2-dimensional topological manifolds.

In the following, all manifolds are topological (possibly with non-empty boundary) and all the homology groups are reduced homology groups with integral coefficients. We shall first make two observations.

LEMMA 1. *If a (locally finite) polyhedron X is a fixed-point space, X must be connected and compact.*

Proof. The connectedness part is clear. The compactness of X follows from a theorem of Klee ([3], Theorem 2.7).

LEMMA 2. *Let M_0 be an arbitrary 2-dimensional manifold. If M_1 is a connected sum of M_0 with a torus T, M_1 cannot have the f.p.p.*

Proof. By a *connected sum of M_0 with T* we mean a space obtained by first cutting small open disks D in M_0 and D' in T, then pasting together M_0-D and $T-D'$ by identifying the boundaries of the disks. Let C be

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a longitudinal circle on the torus T which does not touch the disk D'. The circle C is then also a subset of M_1. We contend that C is a retract of M_1.

Let T' be the quotient space obtained from M_1 by collapsing the set $M_1-(T-\mathrm{Cl}(D'))$ into a point, and let q: $M_1 \to T'$ be the quotient map. Clearly, T' is a torus with C as a longitudinal circle. We then let h: $T' \to C$ be a map which collapses the torus T' onto C. Finally, letting $j: C \to M_1$ be the inclusion map, we note that the composition jhq forms a retraction of M_1 onto C.

Clearly, M_1 cannot have the f.p.p., for it has a retract lacking the f.p.p.

THEOREM 1. *All 1-dimensional simplicial complexes are Lefschetz spaces.*

Proof. Since Condition A implies the f.p.p. for polyhedra, we need only to show that if X is a 1-dimensional complex with the f.p.p., then X has to satisfy Condition A. To do this, we note that, by Lemma 1, we need only to show that $H_1(X)$ is a torsion group. In fact, $H_1(X) = 0$.

Suppose the contrary. We can always find a 1-dimensional cycle

$$z = \langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \ldots + \langle v_k, v_0\rangle \quad \text{in } X,$$

where each $\langle v_i, v_j\rangle$ is a 1-simplex of X. Let $|z|$ be the underlying topological space of z. Clearly, $|z|$ is a topological circle.

We define a map f from X onto the standard unit circle S^1 on the Euclidean plane by collapsing every point of X outside the open segment (v_0, v_1) onto a point $p \in S^1$ and carrying the open segment (v_0, v_1) homeomorphically onto $S^1 - \{p\}$. Note that, for any h: $S^1 \to |z|$, the composition hf is a self-map of X, carrying X into the subspace $|z|$. It is not difficult to choose an h such that the map hf is fixed-point free. This is a contradiction. Hence $H_1(X) = 0$.

THEOREM 2. *All 2-dimensional manifolds are Lefschetz spaces.*

Proof. As in the case of Theorem 1, we need only to show that if X is a 2-manifold with the f.p.p., then the homology groups of X are all torsion groups. Let us consider X in its normal form, i. e., X is a 2-sphere with a number of handles, crosscaps and contours. Since X has the f.p.p., by Lemma 2, the number of handles on X has to be zero. Furthermore, the number of crosscaps on X has to be less than 3, for, otherwise, we can always replace two crosscaps by a handle. Hence, X must be a 2-sphere, a projective plane or a Klein bottle, each with a number (possibly zero) of holes punched on the surface. Now, assuming that some homology group of X has a free part, we see that X has to be (1) a 2-sphere, (2) a 2-sphere with two or more holes, (3) a projective plane with r ($r \geqslant 1$) holes, or (4) a Klein bottle with a number (possibly zero) of holes.

Note that, in case (2), X is homeomorphic to a disk with at least one hole. In case (3), X is homeomorphic to a Möbius band with $r-1$ holes. Therefore, it is not difficult to construct a fixed-point free map on X in each of these four cases. Hence, the homology groups of X have to be torsion groups.

COROLLARY. *The disk and the projective plane are the only 2-manifolds with the f.p.p.*

Proof. It is well known that the disk and the projective plane are the only 2-dimensional manifolds which are connected, compact, and whose homology groups are all torsion groups, i. e., they are the only 2-manifolds satisfying Condition A.

The following problems are believed to be open.

PROBLEM 1. Are all 2-dimensional polyhedra Lefschetz spaces? (**P 928**)

Bing raised a similar question in [1] (Question 1 of [1]).

PROBLEM 2. Are all 3-dimensional polyhedra, in particular, all 3-dimensional manifolds, Lefschetz spaces? (**P 929**)

Remark. 4-dimensional manifolds are not necessarily Lefschetz spaces, since CP(2) has the f.p.p. without satisfying Condition A.

We shall now establish a few general properties of Lefschetz spaces. We need first the following lemmas.

LEMMA 3. *If Condition A is satisfied by spaces X and Y, it is also satisfied by $X \times Y$ and $S(X)$ (the suspension of X). Moreover, this condition is also satisfied by $X \cup Y$ when X and Y are subspaces of some topological space and $X \cap Y \neq \emptyset$ is contractible.*

Proof. Since the homology groups of $X \times Y$ and $X \cup Y$ can be evaluated from those of X and Y by the Künneth formula and the Mayer-Vietoris Theorem, respectively, and the homology groups of $S(X)$ are isomorphic to those of X of one lower dimension, it is then easily seen that if X satisfies Condition A, so does $S(X)$, and if both X and Y satisfy Condition A, so do the spaces $X \times Y$ and $X \cup Y$.

LEMMA 4. *Let K be a non-empty, closed subcomplex of a finite simplicial complex X. If $|K|$ is contractible, $|K|$ is a retract of $|X|$.*

Proof. Since $|K|$ is contractible, a map from the i-th skeleton of X into $|K|$ can always be extended into a map from the $(i+1)$-st skeleton of X into $|K|$. It is not difficult to define a retraction from $|X|$ onto $|K|$ inductively on the skeletons of X.

THEOREM 3. *If X and Y are Lefschetz spaces, so are $X \times Y$ and $S(X)$. Furthermore, if X and Y are compact Lefschetz spaces such that $X \cap Y \neq \emptyset$ is a contractible subcomplex of both X and Y, then $X \cup Y$ is also a Lefschetz space.*

Proof. Since Condition A implies the f.p.p. for polyhedra, one needs only to show that if any of the spaces $X \times Y$, $S(X)$ and $X \cup Y$ has the f.p.p., it must also satisfy Condition A. To show this, it suffices, by Lemma 3, to show that if $X \times Y$ or $X \cup Y$ has the f.p.p., then both X and Y satisfy Condition A, and if $S(X)$ has the f.p.p., then X satisfies Condition A. But X and Y are Lefschetz spaces; hence, to show that X or Y satisfies Condition A, one needs only to show that X or Y has the f.p.p.

The case $X \times Y$ is clear, for if f is a fixed-point free map on X or Y, the projection of $X \times Y$, followed by f, forms a fixed-point free map on $X \times Y$. Therefore, if $X \times Y$ has the f.p.p., so do the spaces X and Y. Similarly, suppose f to be a fixed-point free map on X. Let f_s: $S(X) \to S(X)$ be the map induced by f, and let r: $S(X) \to S(X)$ be the reflection of $S(X)$ obtained by interchanging the north and the south poles. Then $f_s r$ is a fixed-point free map on $S(X)$. Hence, the f.p.p. for $S(X)$ also implies the f.p.p. for X.

Finally, consider $X \cup Y$. By the assumption, X and Y are both compact polyhedra such that $X \cap Y$ is a subcomplex of both X and Y. Now, let f be a fixed-point free map, say on X. By Lemma 4, there exists a retraction g: $Y \to X \cap Y$. Then, F: $X \cup Y \to X \cup Y$ defined by $F(x) = f(x)$ if $x \in X$ and by $F(x) = fg(x)$ if $x \in Y$ is a fixed-point free map on $X \cup Y$ (the compactness of X and Y is needed to ensure the continuity of F). Hence, the f.p.p. for $X \cup Y$ also implies the f.p.p. for both X and Y.

Remark. It is known that the f.p.p. for two arbitrary spaces X and Y does not imply the f.p.p. for any of the spaces $X \times X$, $X \times I$, $X \times Y$, $S(X)$, $X \cup_I Y$ and $X \cup_D Y$, where I is the unit interval, $X \cup_I Y$ is the union of two spaces X and Y such that $X \cap Y$ is an arc, and $X \cup_D Y$ is the union of X and Y such that $X \cap Y$ is a disk (see Lopez [5], Theorem 15 of Bing [1], Knill [4], and Theorem 4.9 of Fadell [2]). These implications, however, are true under minor restrictions when X and Y are Lefschetz spaces.

Finally, we observe that Theorem 3 can be used, in particular, to construct spaces with the f.p.p. from some known Lefschetz spaces which have the f.p.p. by forming, successively, the products, suspensions or by pasting together such spaces in a right way.

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