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THE CANONICAL ALGEBRAS

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with an Appendix by William Crawley-Boevey

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For k an algebraically closed field, the canonical k-algebras have been introduced and investigated in [R2]. A different approach for obtaining the structure of the module category of a canonical algebra over an algebraically closed field has been displayed by Geigle and Lenzing [GL] and it seems that there may be further interest in this class of algebras. An extension of these results to the case of an arbitrary base field had been announced in [R2]: "it should be easy to outline the necessary changes both in the formulations and the proofs". Actually, the author became aware only recently of a reasonable definition of canonical algebras in general, and the purpose of this note is to give an outline of the theory. The general approach seems to be of interest also in the special case when the base field is algebraically closed. The paper was stimulated by penetrating questions of William Crawley-Boevey and the author is strongly endebted to him for his remarks and encouragements. In particular, he has contributed an appendix presenting a straightforward definition of the canonical algebras which may be read without preknowledge on the representation theory of tame bimodules.

The methods which will be used are tilting theory [HR] and tubular extensions [ER]. The structure theory for the category of representations of a tame bimodule [DR1], [R1] will be presupposed, the additional factorization property for knowing to deal with a separating tubular family will be derived in Section 6. We also show that any connected hereditary algebra with semidefinite quadratic form has a preprojective tilting module whose endomorphism ring is a canonical algebra. As a consequence, we obtain a new proof for the classification of the indecomposable modules of a tame hereditary algebra as presented in [DR1]. For unexplained notation, we refer to [R2].

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. The definition

Let k be a (commutative) field. The algebras which we will consider will be finite-dimensional k-algebras, we will assume that k operates centrally on any given bimodule, and all modules will be assumed to be finite-dimensional over k.

Let $F(a)$, $F(b)$ be division algebras, and $M = {}_aM_b$ an $F(a)$-$F(b)$-bimodule such that $\dim_{F(a)}M \cdot \dim M_{F(b)} = 4$. The representation theory of these bimodules (they may be called *tame* bimodules) is known (see [DR1], [R1]), we recall the essential features. A representation of M is of the form $X = (X_a, X_b, \varphi_X)$, where X_a is an $F(a)$-space, X_b an $F(b)$-space, and φ_X: $X_a \otimes_{F(a)} {}_aM_b \to X_b$ is $F(b)$-linear; given two representations X, Y of M, a map f: $X \to Y$ is of the form $f = (f_a, f_b)$ where f_a: $X_a \to Y_a$ is $F(a)$-linear, f_b: $X_b \to Y_b$ is $F(b)$-linear and $\varphi_Y(f_a \otimes 1) = f_b\varphi_X$. Sometimes, it will be convenient to consider a representation of M just as an $F(b)$-linear map $X_a \otimes_{F(a)} M \to X_b$.

If X is a representation of M, the pair $\underline{\dim}\, X = (\dim(X_a)_{F(a)}, \dim(X_b)_{F(b)})$ in $\mathbf{N}_0^2$ is called the *dimension vector* of X. The linear map ∂: $\mathbf{Z}^2 \to \mathbf{Z}$, defined by $\partial(x_a, x_b) = 2x_a - mx_b$, where $m = \dim_{F(a)}M$, will be called the *defect* (it is a positive multiple of the one used in [DR1]). A representation X of M is said to be *simple regular* provided its endomorphism ring is a division ring and $\partial(\underline{\dim}\, X) = 0$. We denote by $\Omega = \Omega(M)$ the set of isomorphism classes of simple regular representations of M.

The set Ω will be used as the basic index set for our considerations. An element of Ω (or also a fixed representative of an element of Ω) will be denoted by a small Greek letter, like ϱ. (Note that for k algebraically closed, there is only one possible choice for M, namely $M = k^2$, and in this case $\Omega = \Omega(k^2) = \mathbf{P}_1 k$.) Let $\mathbf{T}$: $\Omega \to \mathbf{N}_1$ be a function with $\mathbf{T}(\varrho) = 1$ for almost all $\varrho \in \Omega$. We are going to define the canonical algebra of type $\mathbf{T}$ (so it depends on the division algebras $F(a)$, $F(b)$, the $F(a)$-$F(b)$-bimodule M and the function $\mathbf{T}$).

Let $\varrho_1, \ldots, \varrho_t$ be the pairwise different elements of Ω with $n_i := \mathbf{T}(\varrho_i) > 1$. We consider ϱ_i as a representation of M, say as an $F(b)$-linear map ϱ_i: $U_i \otimes_{F(a)} M \to V_i$. Let D_i be the endomorphism ring of the representation ϱ_i. Then U_i is a D_i-$F(a)$-bimodule, V_i is a D_i-$F(b)$-bimodule, and ϱ_i: $U_i \otimes_{F(a)} M \to V_i$ is D_i-$F(b)$-linear. Let ϱ_i^+: $V_i^+ \to U_i \otimes_{F(a)} M$ be the kernel of this map ϱ_i. Then V_i^+ is again a D_i-$F(b)$-bimodule, and ϱ_i^+ is D_i-$F(b)$-linear. Since the map ϱ_i is surjective, ϱ_i^+ determines uniquely (the isomorphism class) ϱ_i. Since U_i is a D_i-$F(b)$-bimodule, $U_i^* := \mathrm{Hom}_{F(a)}(U_i, F(a)_{F(a)})$ is an $F(a)$-D_i-bimodule, and we may consider the adjoint map $\tilde{\varrho}_i$: $U_i^* \otimes_{D_i} V_i^+ \to M$ of ϱ_i^+. The species $\mathscr{S} = (F(i), {}_iM_j)_{i,j}$ which we construct has underlying quiver as shown in Fig. 1, the division algebras are for $i = a$ and $i = b$ the given ones $F(a)$ and $F(b)$, whereas $F(i, j) = D_i$, and the bimodules are

$$ {}_aM_b = M, \quad {}_aM_{(i,1)} = U_i^*, \quad {}_{(i,n_i-1)}M_b = V_i^+, $$

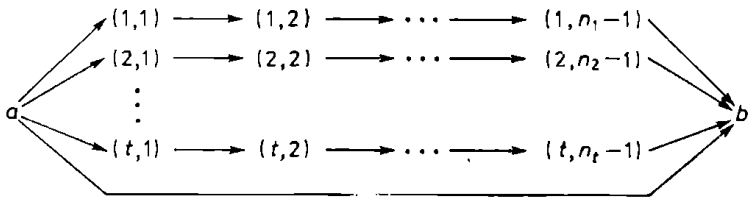


Fig. 1

and finally, ${}_{(i,j)}M_{(i,j+1)}$, for $1 \leqslant j \leqslant n_i - 2$, is the canonical D_i-D_i-bimodule D_i. Let $\mathscr{T}$ be the tensor algebra of $\mathscr{S}$. Let $\mathscr{R}$ be the ideal of $\mathscr{T}$ which is generated by the elements of the form

$$u \otimes 1 \otimes 1 \otimes \ldots \otimes 1 \otimes v - \tilde{\varrho}_i(u \otimes v) \quad \text{with } u \in U_i^*,\ v \in V_i^+$$

(this is an element of

$$({}_aM_{(i,1)} \otimes_{(i,1)} M_{(i,2)} \otimes \ldots \otimes_{(i,n_i-2)} M_{(i,n_i-1)} \otimes_{(i,n_i-1)} M_b) \oplus {}_aM_b).$$

The *canonical algebra of type* $\mathbf{T}$ (over M) is, by definition, $C = C(\mathbf{T}) = \mathscr{T}/\mathscr{R}$.

It will be convenient to simplify the presentation of C when this is possible. Assume that for some s, the map $\tilde{\varrho}_s$: $U_s^* \otimes_{D_s} V_s^+ \to M$ is surjective. In this case, we may omit the arrow $a \to b$ from the underlying quiver of $\mathscr{S}$ and change the ideal to be factored out accordingly. (The new ideal will be generated by the elements of the form

$$u \otimes 1 \otimes \ldots \otimes 1 \otimes v - \sum_r u_r \otimes 1 \otimes \ldots \otimes 1 \otimes v_r$$

with $u \in U_i^*$, $v \in V_i^+$, all $u_r \in U_s^*$, $v_r \in V_s^+$, such that $\tilde{\varrho}(u \otimes v) = \sum_r \tilde{\varrho}_s(u_r \otimes v_r)$. In particular, it will contain the elements $\sum_r u_r \otimes 1 \otimes \ldots \otimes 1 \otimes v_r$ with $\sum_r u_r \otimes v_r$ in the kernel of $\tilde{\varrho}_s$.) More generally, assume that the image of $\tilde{\varrho}_s$ is a direct summand of the $F(a)$-$F(b)$-bimodule M. Let $\bar{M}$ be a direct complement for the image of $\tilde{\varrho}_s$ in M. Then we may replace M by $\bar{M}$ and change $\mathscr{R}$ correspondingly. In case $\bar{M}$ is nonzero, the underlying quiver of $\mathscr{S}$ is still as displayed above. Finally, it may happen that there are s, s' such that M is the direct sum of the images of $\tilde{\varrho}_s$ and $\tilde{\varrho}_{s'}$. In this case, we may again omit the arrow $a \to b$ and convey the appropriate changes to $\mathscr{R}$.

PROPOSITION. *The opposite of a canonical algebra is a canonical algebra.*

Proof. First, consider the case $t = 1$, $\mathbf{T}(\varrho_1) = 2$. Thus, we deal with the representation $\varrho = \varrho_1$: $U \otimes_{F(a)} M \to V$ of A, with endomorphism ring D.

We will need various dual modules. Let ${}^*M = \mathrm{Hom}_{F(a)}(M, {}_{F(a)}F(a))$; this is an $F(b)$-$F(a)$-bimodule. The kernel ϱ^+: $V^+ \to U \otimes_{F(a)} M$ of ϱ has adjoint $\bar{\varrho}$: $V^+ \otimes_{F(b)} {}^*M \to U$, and we denote its kernel by $\Phi\varrho$: $U^+ \to V^+ \otimes_{F(b)} {}^*M$. Observe that $\Phi\varrho$ is a D-$F(a)$-bimodule map. We apply $\mathrm{Hom}_{F(a)}(-, F(a)_{F(a)})$ to the exact sequence

$$0 \to U^+ \xrightarrow{\Phi\varrho} V^+ \otimes_{F(b)} {}^*M \xrightarrow{\bar{\varrho}} U \to 0,$$

and use the fact that

$$\begin{aligned}\mathrm{Hom}_{F(a)}(V^+ \otimes_{F(b)} {}^*M,\ F(a)_{F(a)}) &\cong \mathrm{Hom}_{F(b)}(V^+,\ \mathrm{Hom}_{F(a)}({}^*M,\ F(a)_{F(a)}))\\ &\cong \mathrm{Hom}_{F(b)}(V^+,\ M),\end{aligned}$$

since $\mathrm{Hom}_{F(a)}({}^*M, F(a)_{F(a)})$ is the $F(a)$-bidual of M. We obtain the exact sequence

$$0 \to U^* \xrightarrow{\bar{\varrho}^*} \mathrm{Hom}_{F(b)}(V^+, M) \xrightarrow{(\Phi\varrho)^*} U^{+*} \to 0$$

of $F(a)$-D-bimodules. The map $\bar{\varrho}^*\colon U^* \to \mathrm{Hom}_{F(b)}(V^+, M)$ has as adjoint map just the map $\tilde{\varrho}\colon U^* \otimes V^+ \to M$.

Given a ring R, we denote by R^{op} its opposite ring. If we consider M as an $F(b)^{\mathrm{op}}$-$F(a)^{\mathrm{op}}$-bimodule, we denote it by M^{op}. Let A be the tensor algebra of ${}_{F(a)}M_{F(b)}$. Then A^{op} is the tensor algebra of M^{op}.

We want to rewrite $(\Phi\varrho)^*$ as a representation of M^{op}. Let $V^{+*} = \mathrm{Hom}_{F(b)}(V^+, F(b)_{F(b)})$; it is an $F(b)$-D-bimodule, and we have $\mathrm{Hom}_{F(b)}(V^+, M) \cong M \otimes_{F(b)} V^{+*}$ as $F(a)$-D-bimodules. Thus, we may consider $(\Phi\varrho)^*$ as a map

$$(\Phi\varrho)^*\colon\ M \otimes_{F(b)} V^{+*} \to U^{+*}$$

of left $F(a)$-modules, thus as a representation of M^{op}. This map $(\Phi\varrho)^*$ is an $F(a)$-D-bimodule map; it follows that D is the endomorphism ring of $(\Phi\varrho)^*$ considered as a left A-module (first, we only know that D is a subring of $D' = \mathrm{End}_A((\Phi\varrho)^*)$; however, we can reverse the construction and embed D' into the endomorphism ring of ϱ). If we consider $(\Phi\varrho)^*$ as a right A^{op}-module, then its endomorphism ring (now acting on the left) is D^{op}. A straightforward calculation shows that together with ϱ also $(\Phi\varrho)^*$ has defect zero, thus $(\Phi\varrho)^*$ is a simple regular representation of M^{op}.

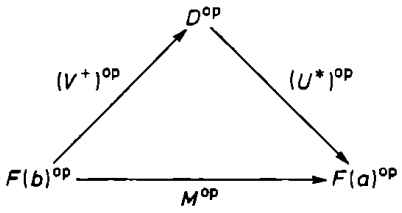
We are going to construct the canonical algebra $C(\mathbf{T}')$ over M^{op}, where $\mathbf{T}'$: $\Omega(M^{\mathrm{op}}) \to \mathbf{N}_1$ is defined by $\mathbf{T}'((\Phi\varrho)^*) = 2$, and $\mathbf{T}'(\sigma) = 1$ for the remaining σ. Note that the kernel of $(\Phi\varrho)^*$ is

$$(\Phi\varrho)^{*+} = \bar{\varrho}^*\colon\ U^* \to M \otimes_{F(b)} V^{+*}.$$

In order to construct $((\Phi\varrho)^*)^{\sim}$ we have to form the $F(b)$-dual of V^{+*}; but this is the $F(b)$-bidual of V^+, which may be identified with V^+. Thus

$$((\Phi\varrho)^*)^{\sim} = \tilde{\varrho}\colon\ U^* \otimes V^+ \to M.$$

It follows that $C(\mathbf{T}')$ is obtained from the tensor algebra of the species



by factoring out the ideal corresponding to the map

$$\tilde{\varrho}\colon (V^+)^{\mathrm{op}} \otimes_{D^{\mathrm{op}}} (U^*)^{\mathrm{op}} \to M^{\mathrm{op}},$$

thus $C(\mathbf{T}') = C(\mathbf{T})^{\mathrm{op}}$.

This calculation obviously extends to the case of a general $\mathbf{T}$. In case $t = 1$ and $\mathbf{T}(\varrho_1) \geqslant 3$, we have to insert the trivial bimodules ${}_{D^{\mathrm{op}}}(D^{\mathrm{op}})_{D^{\mathrm{op}}}$. In case $t \geqslant 2$, the t paths can be treated separately, as above. We only have to observe that nonisomorphic simple regular representations ϱ, ϱ' of M lead to nonisomorphic representations $(\Phi\varrho)^*$ and $(\Phi\varrho')^*$. This completes the proof.

Remark. We may identify the $F(b)$-$F(a)$-bimodule $^*M = \mathrm{Hom}_{F(a)}(M, {}_{F(a)}F(a))$ with $\mathrm{Hom}_{F(b)}(M, F(b)_{F(b)})$, in this way dealing with dual bimodules in the sense of [DR1] (for a proof, see [D], Lemma 0.2). But note that such an identification is quite arbitrary. Having made such an identification, $\Phi\varrho$ may be considered as a representation $\Phi\varrho\colon U^+ \otimes M \to V^+$ of M. This functor Φ is the Coxeter functor as considered in [DR1]. Also, we may identify U^{+*} with $\mathrm{Hom}_k(U^+, k)$ and V^{+*} with $\mathrm{Hom}_k(V^+, k)$; in this way, $(\Phi\varrho)^*$ becomes the usual k-dual of the A-module $\Phi\varrho$.

2. Examples

2.1. Let $F(a) = F(b) = k$, thus $M = k^2$, and let $\varrho_1, \ldots, \varrho_t$ be pairwise different indecomposable representations of M with dimension vector $(1, 1)$. Choose numbers $n_i > 1$, for $1 \leqslant i \leqslant t$, and let $\mathbf{T}\colon \Omega \to \mathbf{N}_1$ be defined by $\mathbf{T}(\varrho_i) = n_i$, and $\mathbf{T}(\varrho) = 1$ for the remaining $\varrho \in \Omega$. Note that we may identify U_i^* and V_i^+ with the canonical bimodule k, and $\tilde{\varrho}_1\colon U_i^* \otimes V_i^+ \to M$ is the embedding of a one-dimensional subspace of k^2, and the subspaces are pairwise different, for the various i.

If $t = 0$, we deal with the Kronecker quiver itself. If $t = 1$, we may replace M by $\bar{M} = M/\mathrm{Image}(\tilde{\varrho}_1)$, and we may identify $\bar{M}$ with the canonical bimodule k. The canonical algebra which we obtain is the path algebra of the quiver of Fig. 2. For $t \geqslant 2$, we observe that $M = \mathrm{Image}(\tilde{\varrho}_1) \oplus \mathrm{Image}(\tilde{\varrho}_2)$, so we may

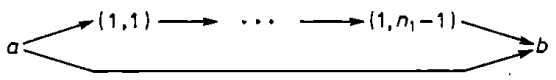


Fig. 2

delete the arrow $a \to b$. We obtain the path algebra of the quiver of Fig. 3 modulo a $(t-2)$-dimensional ideal which is a generic subspace of the vector space of paths from a to b (in the sense of [R2], p. 161).

If k is algebraically closed, this case 2.1 is the only possible one, so the notion of a canonical k-algebra coincides with the one considered in [R2].

2.2. Assume the (tame) bimodule $M = {}_{F(a)}M_{F(b)}$ is not simple, say there

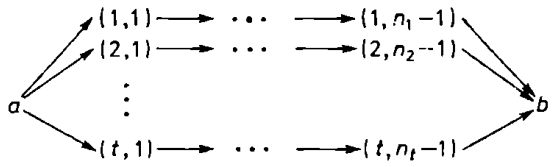


Fig. 3

is given a proper nonzero $F(a)$-$F(b)$-submodule M'. It follows immediately that $\dim {}_{F(a)}M = \dim M_{F(b)} = 2$, and $\dim {}_{F(a)}M' = \dim M'_{F(b)} = 1$. The canonical projection $\pi_{M'}$: $M \to M/M'$ may be considered as a representation $\pi_{M'} = (F(a), M/M', \pi_{M'})$ of M, and its endomorphism ring is $F(a)$. Note that the corresponding map $\tilde{\pi}_{M'}$ is the inclusion map $F(a) \otimes_{F(a)} M' \subsetneq M$. Let $\mathbf{T}$: $\Omega \to \mathbf{N}_1$ be the function with $\mathbf{T}(\varrho) = 1$ for all $\varrho \neq \pi_{M'}$, and $\mathbf{T}(\pi_{M'}) \geqslant 2$. The canonical algebra $C(\mathbf{T})$ is obtained from the tensor algebra of the species of Fig. 4 by

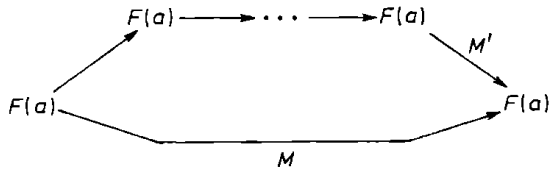


Fig. 4

factoring out the ideal given by the elements of the form $1 \otimes 1 \otimes \ldots \otimes 1 \otimes x - x$ with $x \in M'$. In case an arrow $c \to c'$ of the graph of a species is equipped with $F(c) = F(c') = D$ and the canonical bimodule ${}_DD_D$, we will usually omit the bimodule and simply write $D \to D$.

This algebra and its representations have been studied in [DR2]. If M is indecomposable as a bimodule, the algebra $C = C(\mathbf{T})$ has the remarkable property that there exists a subalgebra C' of C with $C' \oplus \operatorname{rad} C = C$, whereas $\operatorname{rad}^2 C$ is not a direct summand of $\operatorname{rad} C$ as a C'-C'-bimodule (thus k cannot be perfect in this case). It should be remarked that choosing a nonzero element $m \in M'$ yields an isomorphism ι: $F(a) \to F(b)$ via $xm = m\iota(x)$. If we identify in this way $F(a)$ and $F(b)$, say $F(a) = F(b) = F$, then M' may be identified with ${}_FF_F$.

2.3. Before we proceed, let us insert the following (easy) result.

LEMMA. *Let* ϱ: $U \otimes_{F(a)} M \to V$ *be a simple regular representation of* M *and assume the image of* $\tilde{\varrho}$: $U^* \otimes V^+ \to M$ *is a proper submodule* M' *of* M. *Then* $\dim {}_{F(a)}M = \dim M_{F(b)} = 2$, *and* ϱ *is isomorphic to the representation* $\pi_{M'} = (F(a), M/M', \pi_{M'})$.

It follows that in dealing with a canonical algebra $C = C(\mathbf{T})$, we may almost always delete the arrow $a \to b$ from the presentation of C; the only exception is the case when $\mathbf{T}(\varrho) > 1$ for at most one ϱ, and this ϱ is of the form

$(F(a), M/M', \pi)$, where M' is a proper $F(a)$-$F(b)$-submodule of M and π is the canonical projection.

Also, if ϱ is a simple regular representation with endomorphism ring D, say $\varrho\colon U\otimes_{F(a)} M \to V$, and $\dim_k(U^*\otimes_D V^+) = \dim_k M$, then $\tilde{\varrho}\colon U^*\otimes_D V^+ \to M$ is bijective. Thus, if $C = C(\mathbf{T})$ is the canonical algebra of type $\mathbf{T}$ and $\mathbf{T}(\varrho) > 1$, then we may delete the arrow $a \to b$, and rewrite the relations. Actually, in this case, the generating set for the ideal to be factored out becomes simpler.

2.4. We return to the case $F(a) = F(b) = k$, thus $M = k^2$. But we assume now that k is not algebraically closed, so that there exist simple regular representations of M with dimension vector (d, d), where $d > 1$. Always, a simple regular representation ϱ different from $k \underset{1}{\overset{0}{\rightrightarrows}} k$ is of the form $D \underset{x}{\overset{1}{\rightrightarrows}} D$, where $k \subseteq D$ is a finite field extension with a primitive element x (and we denote the multiplication by x on D again by x). In particular, D is commutative, and is the endomorphism ring of ϱ. So suppose $d = \dim_k D > 1$, thus ϱ is not isomorphic to a representation of the form $\pi_{M'}$. The canonical algebras $C(\mathbf{T})$ with $\mathbf{T}(\varrho) \geqslant 2$ and $\mathbf{T}(\varrho') = 1$ for the remaining ϱ' are given by the species

$$k \xrightarrow{{}_kD_D} D \to \ldots \to D \xrightarrow{{}_DD_k} k$$

and the ideal of the tensor algebra to be factored out is $(d-2)$-dimensional over k.

2.5. Certain extensions $G \subseteq F$ of division algebras of index 2 give rise to bimodules ${}_{F(a)}M_{F(b)}$ with $\dim {}_{F(a)}M = 2 = \dim M_{F(b)}$.

First, let G_1, G_2 be division subalgebras of the division algebra F, with $\dim_{G_1} F = 2 = \dim_{G_2} F$, and let ${}_{F(a)}M_{F(b)} = {}_{G_1}F_{G_2}$. Let $\mu\colon F_{G_1}\otimes {}_{G_1}F_{G_2} \to F_{G_2}$ be the multiplication map. Then $\mu = (F_{G_1}, F_{G_2}, \mu)$ is a representation of M with endomorphism ring F and $\partial(\underline{\dim}\,\mu) = 0$, thus μ is simple regular. The canonical algebra $C(\mathbf{T})$ with $\mathbf{T}(\mu) = n \geqslant 2$ and $\mathbf{T}(\varrho) = 1$ for $\varrho \neq \mu$ is the tensor algebra of the species

$$G_1 \xrightarrow{{}_{G_1}F_F} F \to F \to \ldots \to F \xrightarrow{{}_FF_{G_2}} G_2$$

with $n+1$ vertices.

Assume now, in addition, that $G_1 = G_2 = G$. The bimodule ${}_GF_G$ has the G-G-submodule ${}_GG_G$, thus we also may consider the simple regular representation $\pi_G = (G_G, (F/G)_G, \pi_G)$ with $\tilde{\pi}_G\colon G\otimes_G G \to F$ the inclusion map. The canonical algebra $C(\mathbf{T})$ with $\mathbf{T}(\mu) \geqslant 2$, $\mathbf{T}(\pi_G) \geqslant 2$, and $\mathbf{T}(\varrho) = 1$ for the remaining $\varrho \in \Omega$ is obtained from the tensor algebra of a species of the form shown in Fig. 5 by factoring out the ideal given by the elements of the form $1\otimes 1\otimes\ldots\otimes 1\otimes g - 1\otimes 1\otimes\ldots\otimes 1\otimes g$ with $g \in G$ (the first summand being an element of the tensor product along the upper path, the second of the tensor product along the lower path).

Next, let G be a division algebra which is embedded into two

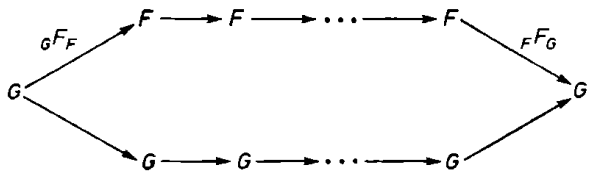


Fig. 5

division algebras F_1, F_2 such that $\dim_G F_1 = 2 = \dim_G F_2$, and let ${}_{F_1}M_{F_2} = {}_{F_1}(F_1)_G \otimes {}_G(F_2)_{F_2}$. The projection $\varepsilon\colon M \to M/G\otimes_G F_2$ gives a simple regular representation $\varepsilon = ((F_1)_{F_1}, (M/G\otimes_G F_2)_{F_2}, \varepsilon)$. Note that the endomorphism ring of ε is G and that

$$\tilde{\varepsilon}\colon {}_{F_1}(F_1)_G \otimes {}_G(G\otimes_G F_2)_{F_2} \to M$$

is the identity map. The canonical algebra $C(\mathbf{T})$ with $\mathbf{T}(\varepsilon) = n \geqslant 2$ and $\mathbf{T}(\varrho) = 1$ for $\varrho \neq \varepsilon$ is the tensor algebra of the species

$$F_1 \xrightarrow{F_1(F_1)G} G \to G \to \ldots \to G \xrightarrow{G(F_2)F_2} F_2$$

with $n+1$ vertices.

If we assume now, in addition, that $F_1 = F_2 = F$, then the bimodule ${}_FM_F = {}_FF_G \otimes {}_GF_F$ has a proper nonzero F-F-submodule M', namely the kernel of the multiplication map $F\otimes_G F \to F$. The projection $\pi_{M'}\colon F\otimes_G F = M \to M/M' = F$ is just the multiplication map and we obtain a simple regular representation $\pi_{M'} = (F_F, F_F, \pi_{M'})$, with endomorphism ring F. The canonical algebra $C(\mathbf{T})$ with $\mathbf{T}(\varrho) = 1$ if and only if $\varrho \notin \{\varepsilon, \pi_{M'}\}$ is obtained from the path algebra of a species of the form shown in Fig. 6 by factoring out the ideal given

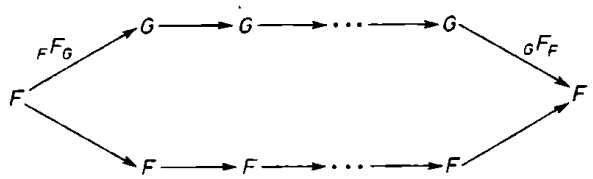


Fig. 6

by the elements of the form $f\otimes 1\otimes\ldots\otimes 1\otimes 1 - 1\otimes 1\otimes\ldots\otimes 1\otimes f - f\otimes 1\otimes\ldots$ $\ldots\otimes 1\otimes 1$, with $f \in F$, where the first two summands belong to the tensor product along the upper path, the third to the tensor product along the lower path.

2.6. Consider now a bimodule ${}_{F(a)}M_{F(b)}$ with $\dim {}_{F(a)}M = 4$, $\dim M_{F(b)} = 1$. Thus, we may consider $F(a)$ as a division subalgebra of $F(b)$, and identify M with the canonical bimodule ${}_{F(a)}F(b)_{F(b)}$. Assume that there exists an intermediate division algebra $F(a) \subset D \subset F(b)$ with $\dim {}_{F(a)}D = 2$, thus also $\dim_D F(b) = 2$. The multiplication map $\mu\colon D_{F(a)} \otimes {}_{F(a)}F(b) \to F(b)$ gives a simple regular representation $\mu = (D_{F(a)}, F(b)_{F(b)}, \mu)$ of M with endomorphism ring D. The canonical algebras $C(\mathbf{T})$ with $\mathbf{T}(\mu) = n \geqslant 2$ and $\mathbf{T}(\varrho) = 1$ for $\varrho \neq \mu$ are just

the tensor algebras of the species

$$F(a) \xrightarrow{F(a)D_D} D \to \ldots \to D \xrightarrow{D^{F(b)}F(b)} F(b)$$

with $n+1$ vertices.

2.7. Let R be a hereditary k-algebra. Then

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle = \dim_k \operatorname{Hom}(X, Y) - \dim_k \operatorname{Ext}^1(X, Y)$$

defines a bilinear form on the Grothendieck group $K_0(R)$. We denote by χ_R the corresponding quadratic form, thus $\chi_R(x) = \langle x, x\rangle$.

PROPOSITION. *Let R be a connected hereditary k-algebra with χ_R semidefinite (but not definite). Then there exists a preprojectve tilting module T_R such that* $\operatorname{End}(T_R)$ *is a canonical algebra.*

The proof uses the classification of these algebras (see [DR1] and [DR2]). In case R is of type $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, or $\widetilde{\mathbf{BC}}$, the basic algebra for R is already a canonical algebra, as we have seen above. Note that this includes the exceptional case treated in [DR2]. In case R is of type $\tilde{\mathbf{D}}_n$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$, or $\tilde{\mathbf{E}}_8$, a preprojective tilting module is constructed as in the special case of k being algebraically closed (see [R2], 4.3.4).

For the remaining cases, we indicate in Fig. 7 the position of the indecomposable direct summands in the preprojective component of R.

First we deal with the four cases which involve an extension $G \subset F$ of division algebras of index 2. In the case $\widetilde{\mathbf{BD}}_n$, we obtain the canonical algebra $C(\mathbf{T})$ over ${}_FF_G \otimes {}_GF_F$ with $\mathbf{T}(\varepsilon) = n-1$, $\mathbf{T}(\pi_{M'}) = 2$ (and $\mathbf{T}(\varrho) = 1$ otherwise).

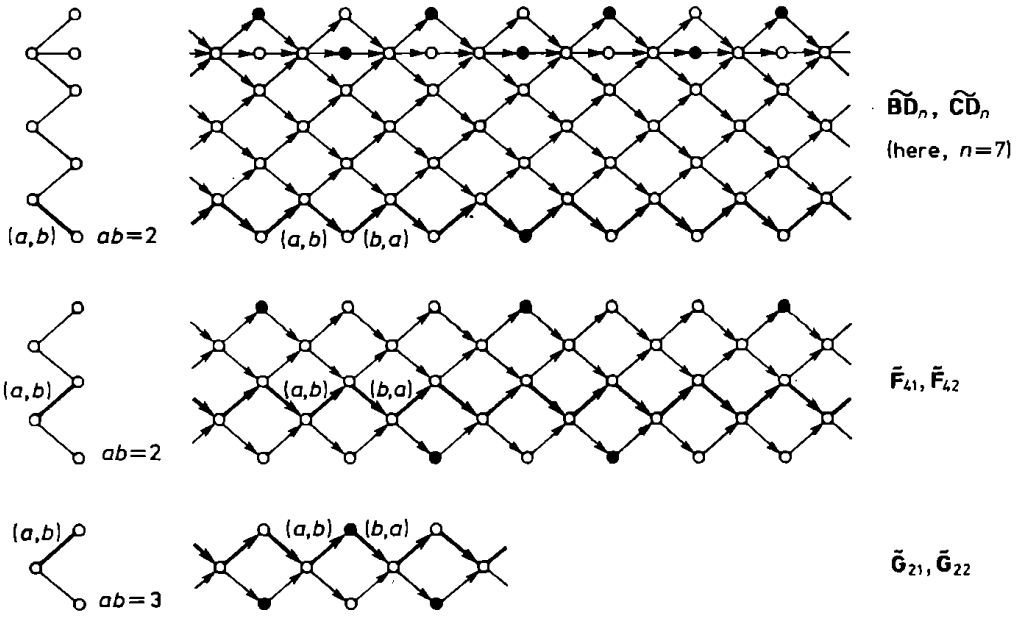


Fig. 7

For $\tilde{\mathbf{CD}}_n$, we obtain the canonical algebra $C(\mathbf{T})$ over ${}_GF_G$ with $\mathbf{T}(\mu) = n-1$, $\mathbf{T}(\pi_G) = 2$. For $\tilde{\mathbf{F}}_{41}$, we obtain $C(\mathbf{T})$ over ${}_GF_G$ with $\mathbf{T}(\mu) = 2$, $\mathbf{T}(\pi_G) = 3$. Finally, for $\tilde{\mathbf{F}}_{42}$, we obtain $C(\mathbf{T})$ over ${}_FF_G \otimes {}_GF_F$ with $\mathbf{T}(\varepsilon) = 2$, $\mathbf{T}(\pi_{M'}) = 3$.

Second, assume $G \subset F$ is an extension of division algebras of index 3 which is involved in $\tilde{\mathbf{G}}_{21}$ or $\tilde{\mathbf{G}}_{22}$. These cases are slightly more technical since one has to use dual bimodules (see [DR1], § 2), so we only indicate the result. Always, we obtain a canonical algebra $C(\mathbf{T})$ with $\mathbf{T}(\varrho) \neq 1$ for precisely one ϱ (and $= 2$ for this ϱ). Both the bimodule M and the representation ϱ to be used may be read off from the corresponding tables in [DR1].

3. The module category of a canonical algebra

Let ${}_{F(a)}M_{F(b)}$ be a tame bimodule, $\Omega = \Omega(M)$ the set of isomorphism classes of simple regular representations of M, and $C(\mathbf{T})$ the canonical algebra of type $\mathbf{T}$: $\Omega \to \mathbf{N}_1$ over M.

The tensor algebra of ${}_{F(a)}M_{F(b)}$ will be denoted by A, and j^*: $\operatorname{mod} C \to \operatorname{mod} A$ denotes the restriction functor. We denote by $e(a)$ and $e(b)$ the idempotents of C corresponding to the vertices a and b, respectively, and $e = e(a)+e(b)$. Note that $e(a)Ce(b)$ is canonically isomorphic to ${}_{F(a)}M_{F(b)}$ (due to the choice of the ideal $\mathscr{R}$), thus we can identify A with eCe, and $j^* = -\otimes_C Ce = \operatorname{Hom}_C(eC, -)$. The defect ∂ on $K_0(A)$ gives rise via j^* to a corresponding function $K_0(C) \to \mathbf{Z}$ which again will be denoted by ∂ and called the defect; thus, for a C-module X_C we have $\partial(\underline{\dim}\, X) = \partial(\underline{\dim} j^* X)$ $= 2\dim X_a - m \cdot \dim X_b$ (where $X_i = Xe(i)$ and $m = \dim {}_{F(a)}M$).

We denote by $\mathscr{P}$, $\mathscr{T}$, and $\mathscr{Q}$ the module classes given by the indecomposable C-modules X_C satisfying $\partial(\underline{\dim} X_C) < 0$, $= 0$, or > 0, respectively.

THEOREM 1. *The category $\mathscr{T}$ is abelian (with exact inclusion functor), it is a stable tubular Ω-family of type $\mathbf{T}$, and $\mathscr{T}$ separates $\mathscr{P}$ from $\mathscr{Q}$.*

That $\mathscr{T}$ separates $\mathscr{P}$ from $\mathscr{Q}$ means the following: first of all, $\operatorname{Hom}(\mathscr{Q}, \mathscr{P}) = \operatorname{Hom}(\mathscr{Q}, \mathscr{T}) = \operatorname{Hom}(\mathscr{T}, \mathscr{P}) = 0$, and, second, given a map from $\mathscr{P}$ to $\mathscr{Q}$ and any $\varrho \in \Omega$, this map can be factored through the tube in $\mathscr{T}$ with index ϱ.

COROLLARY. *Let X be an indecomposable C-module. Then*

(a) *X belongs to $\mathscr{P}$ if and only if j^*X is a nonzero preprojective A-module.*
(b) *X belongs to $\mathscr{T}$ if and only if j^*X is a regular A-module.*
(c) *X belongs to $\mathscr{Q}$ if and only if j^*X is a nonzero preinjective A-module.*

Before we show how to derive the corollary, let us introduce some notation. Let $\mathscr{Z}$ be the class of all C-modules X with $j^*X = 0$. Of course, $\mathscr{Z} \subseteq \mathscr{T}$ and obviously there are only finitely many isomorphism classes of indecomposable modules in $\mathscr{Z}$ (any such module lives on a unique path from

a to b, and there are $\begin{bmatrix} n_i - 1 \\ 2 \end{bmatrix}$ isomorphism classes of indecomposable C-modules which live on the path with first index i; they form a wing, see the shaded region in the tube displayed in Fig. 8).

The left adjoint of j^* will be denoted by $j_!$, the right adjoint by j_*.

Proof of the corollary. Let X be in $\mathscr{P}$, and Y an indecomposable A-module which is preprojective or regular. Since $j^* j_! = \mathrm{id}$, the C-module $j_! Y$ is indecomposable and belongs to $\mathscr{P} \vee \mathscr{T}$. Thus $0 = \mathrm{Hom}_C(j_! Y, X) \cong \mathrm{Hom}_A(Y, j^* X)$. This shows that $j^* X$ is preprojective. If $j^* X$ were zero, X would belong to $\mathscr{X} \subseteq \mathscr{T}$, a contradiction. This finishes the proof of (a). Similar arguments, using also the right adjoint j_* of j^* (and $j^* j_* = \mathrm{id}$), give the remaining two assertions.

There is a natural transformation $\eta: j_! \to j_*$ with $j^*(\eta_Y)$ the identity map $Y = j^* j_! Y \to j^* j_* Y = Y$, where Y is an A-module. Given $\varrho \in \Omega$ (considered as an A-module), we denote by $E(\varrho)$ the image of the map η_ϱ.

THEOREM 2. *The simple objects in $\mathscr{T}$ are, first, the simple C-modules which belong to $\mathscr{X}$, and, second, the C-modules of the form $E(\varrho)$ with $\varrho \in \Omega$. Any tube in $\mathscr{T}$ contains precisely one module of the form $E(\varrho)$, and we denote this tube by $\mathscr{T}(\varrho)$. The rank of $\mathscr{T}(\varrho)$ is $\mathbf{T}(\varrho)$. Thus, if $\mathbf{T}(\varrho) = 1$, then $E(\varrho)$ is the only module on the mouth of $\mathscr{T}(\varrho)$. For $\mathbf{T}(\varrho_i) = n_i \geqslant 2$, the mouth of the tube consists of the module $E(\varrho_i)$ and the $E(i, j)$ with $1 \leqslant j \leqslant n_i - 1$, and we have $\tau E(i, j) = E(i, j+1)$ for $0 \leqslant j \leqslant n_i - 1$, where by definition $E(i, 0) = E(\varrho) = E(i, n_i)$.*

It follows that the tube $\mathscr{T}(\varrho_i)$ looks as shown in Fig. 8. The shaded part consists of the modules in $\mathscr{T}(\varrho_i)$ which belong to $\mathscr{X}$.

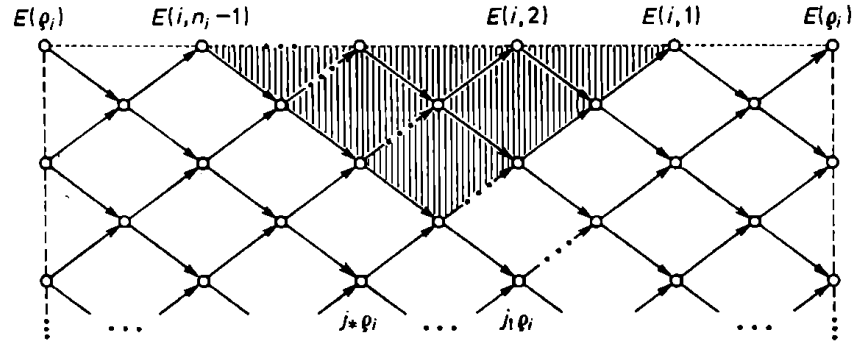


Fig. 8

4. The squid corresponding to a canonical algebra

As before, A denotes the tensor algebra of the bimodule ${}_{F(a)}M_{F(b)}$, and $\varrho_1, \ldots, \varrho_t$ are the pairwise different elements of Ω with $n_i = \mathbf{T}(\varrho_i) \geqslant 2$. We consider ϱ_i as an A-module and denote by D_i the endomorphism ring of ϱ_i.

Consider the star $\mathbf{T}_{n_1,\ldots,n_t}$ with the orientation so that the center z is the unique sink (see Fig. 9). We endow the vertices (i, j) with the division algebra D_i, the vertex z with A; the arrow $(i, n_i-1)\to z$ with the bimodule ${}_{D_i}(\varrho_i)_A$, and the arrows of the form $(i, j)\to(i, j+1)$ (where $1 \leqslant j \leqslant n_i-2$) with the canonical

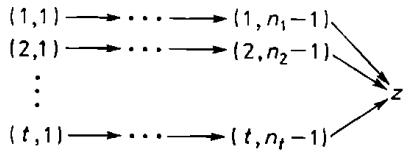


Fig. 9

bimodule ${}_{D_i}(D_i)_{D_i}$. The tensor algebra obtained from these data will be denoted by $B = B(\mathbf{T})$.

We can apply the theory of tubular extensions as presented in [ER] (see also the account in [R2] for the special case of k being algebraically closed). The algebra B is a tubular extension of A, thus the structure of mod B is known: The preprojective A-modules form a component $\mathscr{P}(B)$ of mod B. There is a tubular Ω-family $\mathscr{T}(B)$ obtained from the tubular Ω-family in mod A by inserting into the tube with index ϱ_i precisely n_i-1 rays, for $1 \leqslant i \leqslant t$. Any of these inserted rays starts with an indecomposable projective B-module. An indecomposable B-module belongs to $\mathscr{T}(B)$ if and only if its restriction to A has zero defect and is nonzero (and even indecomposable). We denote by $\mathscr{Q}(B)$ the module class given by the indecomposable B-modules whose restriction to A is preinjective and nonzero. And $\mathscr{Z}(B)$ denotes the class of all B-modules with zero restriction to A. There are only finitely many indecomposable B-modules which belong to $\mathscr{Z}(B)$ (and they form wings). The indecomposable injective B-module $Q_B(i, j)$, $1 \leqslant j \leqslant n_i-1$, corresponding to the vertex (i, j) belongs to $\mathscr{Z}(B)$, the remaining ones, $Q_B(a)$ and $Q_B(b)$, belong to $\mathscr{Q}(B)$. The module class $\mathscr{T}(B)$ separates $\mathscr{P}(B)$ from $\mathscr{Q}(B) \vee \mathscr{Z}(B)$. Also, $\mathscr{Z}(B)$ is the torsion class, and $\mathscr{P}(B) \vee \mathscr{T}(B) \vee \mathscr{Q}(B)$ the torsion-free class of a split torsion pair.

We are going to construct a cotilting B-module with endomorphism ring $C = C(\mathbf{T})$. We denote by $Q_B(i, j)$, $Q_B(a)$, $Q_B(b)$ the indecomposable injective B-modules corresponding to the vertex (i, j), a, and b, respectively. Let B_0 be obtained from B by deleting the vertex b. Since b is a sink, the injective B_0-modules are injective as B-modules. Observe that B_0 is a hereditary algebra, thus it has a preinjective component and this component contains all $Q_B(i, j)$ and $Q_B(a)$. The slice inside this component with $Q_B(a)$ as only sink contains besides $Q_B(a)$ the modules $\tau^{n_i-j}Q_B(i, j)$, with $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant n_i-1$. The direct sum of these modules is a cotilting module for B_0, thus a partial cotilting module for B. Observe that $\tau^{n_i-j-1}Q_B(i, j)$ belongs to $\mathscr{Z}(B)$, thus $\operatorname{Hom}(\tau^{n_i-j-1}Q_B(i, j), Q_B(b)) = 0$, therefore $\operatorname{Ext}^1(Q_B(b), \tau^{n_i-j}Q_B(i, j)) = 0$. It

follows that

$$S_B = Q_B(a) \oplus \Big(\bigoplus_{i,j} \tau^{n_i - j} Q_B(i, j)\Big) \oplus Q_B(b)$$

is a cotilting module for B.

LEMMA 1. $\mathrm{End}(S_B) = C(\mathbf{T})$.

Proof. We consider first the case $t = 1$, $\mathbf{T}(\varrho_1) = 2$. Thus, let $\varrho = \varrho_1$: $U \otimes_{F(a)} M \to V$, with endomorphism ring D. In this case $S_B = Q_B(a) \oplus \tau Q_B(1, 1) \oplus Q_B(b)$. The homomorphism sets between the indecomposable projective B-modules are

$$\mathrm{Hom}\big(Q(b), Q(a)\big) = M, \quad \mathrm{Hom}\big(Q(a), Q(1, 1)\big) = U,$$

$$\mathrm{Hom}\big(Q(b), Q(1, 1)\big) = V$$

(where we have dropped the reference to B), and the composition of maps

$$\mathrm{Hom}\big(Q(a), Q(1, 1)\big) \otimes \mathrm{Hom}\big(Q(b), Q(a)\big) \to \mathrm{Hom}\big(Q(b), Q(1, 1)\big)$$

is just the map ϱ. The sink map for $Q(1, 1)$ is the evaluation map $U \otimes_{F(a)} Q(a) \to Q(1, 1)$, thus $\tau Q(1, 1)$ is defined by the following exact sequence:

$$0 \to \tau Q(1, 1) \to U \otimes_{F(a)} Q(a) \to Q(1, 1) \to 0.$$

This is a sequence of D-B-bimodules, in particular $\mathrm{End}\big(\tau Q(1, 1)\big) = D$. Applying $\mathrm{Hom}(-, Q(a))$ to this sequence, we obtain the exact sequence

$$0 \to \mathrm{Hom}\big(Q(1, 1), Q(a)\big) \to \mathrm{Hom}\big(U \otimes_{F(a)} Q(a), Q(a)\big) \to \mathrm{Hom}\big(\tau Q(1, 1), Q(a)\big) \to 0;$$

here, $\mathrm{Hom}\big(Q(1, 1), Q(a)\big) = 0$, and $\mathrm{Hom}\big(U \otimes_{F(a)} Q(a), Q(a)\big) = U^*$. Also, applying $\mathrm{Hom}\big(Q(b), -\big)$ to the sequence, we obtain

$$0 \to \mathrm{Hom}\big(Q(b), \tau Q(1, 1)\big) \xrightarrow{\alpha} \mathrm{Hom}\big(Q(b), U \otimes Q(a)\big) \xrightarrow{\beta} \mathrm{Hom}\big(Q(b), Q(1, 1)\big) \to 0;$$

this is an exact sequence, since we have started with an Auslander–Reiten sequence and $Q(b)$ is not isomorphic to $Q(1, 1)$. Now, β is just the map ϱ: $U \otimes M \to V$, thus we can identify α with ϱ^+: $V^+ \to U \otimes M$. In particular, $\mathrm{Hom}\big(Q(b), \tau Q(1, 1)\big) = V^+$. It follows easily that the composition

$$\mathrm{Hom}\big(\tau Q(1, 1), Q(a)\big) \otimes_D \mathrm{Hom}\big(Q(b), \tau Q(1, 1)\big) \to \mathrm{Hom}\big(Q(b), Q(a)\big)$$

is just $\tilde{\varrho}$: $U^* \otimes V^+ \to M$, thus $\mathrm{End}(S_B) = C(\mathbf{T})$, in this case. It is straightforward that the same calculation can also be used in the general case: In case $t = 1$, and $\mathbf{T}(\varrho_1) = n \geqslant 3$, we arrange the indecomposable direct summands of S_B as follows:

$$Q(a), \ \tau^{n-1} Q(1, 1), \ldots, \tau^1 Q(1, n-1), Q(b),$$

and obtain the presentation of $C(\mathbf{T})$ introduced in Section 1. In case $t \geqslant 2$, we

observe that the t branches of B give t paths between a and b which can be handled separately. This finishes the proof of the lemma.

Since we prefer tilting modules to cotilting modules, we deal with the dual situation: There is given a tubular coextension B' of the tensor algebra A' of a tame bimodule, and the category mod B' is partitioned as follows: $\mathscr{Z}(B')$ is the class of modules with restriction to A' zero, $\mathscr{P}(B')$ is the module class given by the indecomposable B'-modules with restriction to A' nonzero preprojective, and $\mathscr{R}(B')$ is the module class given by the indecomposable B'-modules with restriction to A' nonzero regular or nonzero preinjective. The projective B'-modules belong to $\mathscr{P}(B')$, the injective B'-modules to $\mathscr{R}(B')$. In fact, the restriction of $Q_{B'}(i, j)$ to A' is simple regular. There are two torsion pairs: in one case, $\mathscr{P}(B') \vee \mathscr{R}(B')$ are the torsion modules, and $\mathscr{Z}(B')$ the torsion-free ones, in the other case, $\mathscr{R}(B')$ are the torsion modules, and $\mathscr{Z}(B') \vee \mathscr{P}(B')$ the torsion-free ones. The vertices of B' are labelled as shown in Fig. 10 (the arrows

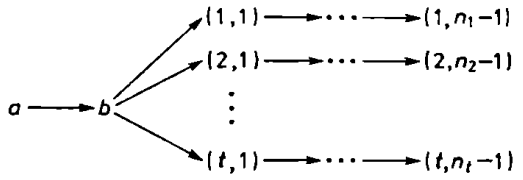


Fig. 10

being endowed with appropriate bimodules: $a \to b$ with a tame bimodule, $(i, j) \to (i, j+1)$ with a bimodule of the form ${}_DD_D$, and so on).

The tilting module which we consider is

$$T_{B'} = P_{B'}(a) \oplus \Big(\bigoplus_{i,j} \tau^{-n_i+j} P_{B'}(i, j)\Big) \oplus P_{B'}(b),$$

and, as we know, its endomorphism ring is the opposite of a canonical algebra, thus a canonical algebra. Now, any canonical algebra arises in this way, thus we may assume that $\operatorname{End}(T_{B'}) = C(\mathbf{T}) = C$. Note that we can identify A' with $A = \operatorname{End}\big(P_{B'}(a) \oplus P_{B'}(b)\big)$. For mod C we will use the notation introduced in Section 3.

Let $\Sigma = \operatorname{Hom}_{B'}({}_CT_{B'}, -)$, and $\Sigma' = \operatorname{Ext}^1_{B'}({}_CT_{B'}, -)$. Observe that $j^*\Sigma$ is just the restriction functor mod $B' \to$ mod A. On the other hand, $j^*\Sigma' = 0$, since $\operatorname{Ext}^1_{B'}(P_{B'}(a), -) = \operatorname{Ext}^1_{B'}(P_{B'}(b), -) = 0$, thus the image of Σ' is contained in $\mathscr{Z}$. Actually, the image of Σ' is all of $\mathscr{Z}$; namely, the construction of $T_{B'}$ shows that $\mathscr{F}(T_{B'}) = \mathscr{Z}(B')$, and the number of indecomposables in $\mathscr{Z}(B')$ and in $\mathscr{Z}$ is the same. This shows that $\mathscr{X}(T_{B'}) = \mathscr{Z}$.

LEMMA 2. *If Z is indecomposable in $\mathscr{Z}$, the A-module $j^*\tau Z$ is simple regular or zero.*

Proof. The Auslander–Reiten sequences in $\mathscr{Z}(B')$ are carried under Σ' to

Auslander–Reiten sequences in $\mathscr{X}(B)$, and the connecting lemma asserts that $\tau\Sigma' P_{B'}(i,j) = \Sigma Q_{B'}(i,j)$. If Z is indecomposable in $\mathscr{X}$ and not isomorphic to any $\Sigma' P_{B'}(i,j)$, then $\tau Z \in \mathscr{X}$, thus $j^*\tau Z = 0$, whereas $j^*\tau\Sigma' P_{B'}(i,j) = j^*\Sigma Q_{B'}(i,j)$ is simple regular.

By duality, we also have the dual assertion:

LEMMA 2*. *If Z is indecomposable in $\mathscr{X}$, the A-module $j^*\tau^- Z$ is simple regular or zero.*

LEMMA 3. $\mathrm{Hom}(\mathscr{T} \vee \mathscr{Q}, \mathscr{P}) = 0$, *and $\mathscr{P}$ is closed under τ and τ^-.*

Proof. We have $\mathscr{G}(T_{B'}) = \mathscr{P}(B') \vee \mathscr{R}(B')$. Let $\mathscr{P}'$ be the image of $\mathscr{P}(B')$ under Σ, and $\mathscr{R}'$ the image of $\mathscr{R}(B')$ under Σ. Since Σ: $\mathscr{G}(T_{B'}) \to \mathscr{Y}(T_{B'})$ is an equivalence, we have $\mathscr{Y}(T_{B'}) = \mathscr{P}' \vee \mathscr{R}'$, and $\mathrm{Hom}(\mathscr{R}', \mathscr{P}') = 0$. We claim that $\mathrm{Ext}^1(\mathscr{P}', \mathscr{X}(B)) = 0$. For, given Z in $\mathscr{X}$, there is an exact sequence $0 \to X \to \tau^- Z \to Y \to 0$, with $X \in \mathscr{X}(T_{B'}) = \mathscr{X}$, and $Y \in \mathscr{Y}(T_{B'})$. Application of the exact functor j^* shows that $j^*\tau^- Z \cong j^* Y$. According to Lemma 2*, the A-module $j^* Y$ is regular. On the other hand, $Y = Y_1 \oplus Y_2$ with $Y_1 \in \mathscr{P}'$ and $Y_2 \in \mathscr{R}'$. If Y_1 is nonzero, then $j^* Y_1$ is a nonzero preprojective A-module. Thus, $j^* Y = j^* Y_1 \oplus j^* Y_2$ shows that $Y_1 = 0$, therefore $Y \in \mathscr{R}'$. Let $P' \in \mathscr{P}'$. Then $\mathrm{Hom}(X, P') = 0$, since $X \in \mathscr{X}(T_{B'})$ and $P' \in \mathscr{Y}(T_{B'})$, and $\mathrm{Hom}(Y, P') = 0$, since $Y \in \mathscr{R}'$ and $P' \in \mathscr{P}'$. It follows that $\mathrm{Hom}(\tau^- Z, P') = 0$, thus $\mathrm{Ext}^1(P', Z) = 0$.

As a consequence, $\mathrm{mod}\, C = \mathscr{X}(T_{B'}) \overline{\rfloor} \mathscr{Y}(T_{B'}) = \mathscr{P}' \vee \mathscr{X} \overline{\rfloor} \mathscr{R}'$. For P' indecomposable in $\mathscr{P}'$, the A-module $j^* P'$ is nonzero preprojective, thus $\partial(\underline{\dim}\, P') < 0$, therefore $\mathscr{P}' \subseteq \mathscr{P}$. The modules in $\mathscr{X} \overline{\rfloor} \mathscr{R}'$ have nonnegative defect, since they are extensions of a module in $\mathscr{X}$ by a module in $\mathscr{R}'$ (the class of such extensions is closed under direct summands) and the modules in $\mathscr{X}$ have zero defect, those in $\mathscr{R}'$ have nonnegative defect. This shows that $\mathscr{X} \overline{\rfloor} \mathscr{R}' \subseteq \mathscr{T} \vee \mathscr{Q}$. Since $\mathrm{mod}\, C = \mathscr{P}' \vee \mathscr{X} \overline{\rfloor} \mathscr{R}'$, it follows that $\mathscr{P}' = \mathscr{P}$ and $\mathscr{X} \overline{\rfloor} \mathscr{R}' = \mathscr{T} \vee \mathscr{Q}$. Since $\mathrm{Hom}(\mathscr{X}, \mathscr{P}') = 0$, $\mathrm{Hom}(\mathscr{R}', \mathscr{P}') = 0$, we see that $\mathrm{Hom}(\mathscr{T} \vee \mathscr{Q}, \mathscr{P}) = \mathrm{Hom}(\mathscr{X} \overline{\rfloor} \mathscr{R}', \mathscr{P}') = 0$.

It remains to show that $\mathscr{P}$ is closed under τ and τ^-. Let P be indecomposable in $\mathscr{P}$. If X is an indecomposable C-module with $\mathrm{Hom}(X, P) \neq 0$, then X belongs to $\mathscr{P}$, since $\mathrm{Hom}(\mathscr{T} \vee \mathscr{Q}, \mathscr{P}) = 0$. Thus τP is in $\mathscr{P}$. Consider now $\tau^- P$. There is an exact sequence $0 \to X \to \tau^- P \to Y \to 0$ with $X \in \mathscr{X}(T_{B'}) = \mathscr{X}$, and $Y \in \mathscr{Y}(T_{B'})$. Note that P is not relative injective in $\mathscr{Y}(T_{B'})$, since the indecomposable relative injective modules in $\mathscr{Y}(T_{B'})$ are the images of the indecomposable injective B'-modules under Σ, and therefore belong to $\mathscr{R}'$. According to Hoshino [H], we know that Y is indecomposable and is the relative τ^--translate of P in $\mathscr{Y}(T_{B'})$. Since $\mathscr{P}(B')$ is closed under τ^- in $\mathrm{mod}\, B'$, it follows that Y belongs to $\mathscr{P}'$. Now $\mathrm{Ext}^1(\mathscr{P}', \mathscr{X}) = 0$ shows that $X = 0$, thus $\tau^- P = Y \in \mathscr{P}'$. This completes the proof.

LEMMA 3*. $\mathrm{Hom}(\mathscr{Q}, \mathscr{P} \vee \mathscr{T}) = 0$, *and $\mathscr{Q}$ is closed under τ and τ^-.*

COROLLARY 1. *$\mathscr{T}$ is closed under τ and τ^-.*

Proof. Let T be indecomposable in $\mathscr{T}$. Then τT belongs to $\mathscr{P} \vee \mathscr{T}$, since $\operatorname{Hom}(\mathscr{Q}, \mathscr{T}) = 0$. But $\mathscr{P}$ is closed under τ^-, thus τT belongs to $\mathscr{T}$. Similarly, $\tau^- T$ belongs to $\mathscr{T}$.

COROLLARY 2. *If X belongs to $\mathscr{P} \vee \mathscr{T}$, then* $\operatorname{proj.dim} X \leqslant 1$. *If X belongs to $\mathscr{T} \vee \mathscr{Q}$, then* $\operatorname{inj.dim} X \leqslant 1$.

Proof. Lext X be in $\mathscr{P} \vee \mathscr{T}$. Any injective module Q belongs to $\mathscr{Q}$, whereas τX belongs to $\mathscr{P} \vee \mathscr{T}$, thus $\operatorname{Hom}(Q, \tau X) = 0$. Similarly, for $Y \in \mathscr{T} \vee \mathscr{Q}$, and P projective, $\operatorname{Hom}(\tau^- Y, P) = 0$.

COROLLARY 3. $\operatorname{Ext}^1(\mathscr{P}, \mathscr{T} \vee \mathscr{Q}) = 0$, $\operatorname{Ext}^1(\mathscr{P} \vee \mathscr{T}, \mathscr{Q}) = 0$.

Proof. Let $P \in \mathscr{P}$, $X \in \mathscr{T} \vee \mathscr{Q}$. Then $\operatorname{Hom}(X, \tau P) = 0$, thus $\operatorname{Ext}^1(P, X) = 0$. Similarly, one shows the second assertion.

COROLLARY 4. *For $P \in \mathscr{P}$, the canonical map $j_! j^* P \to P$ is injective. For $Q \in \mathscr{Q}$, the canonical map $Q \to j_* j^* Q$ is surjective.*

Proof. Let $P \in \mathscr{P}$. The kernel of $\beta: j_! j^* P \to P$ belongs to $\mathscr{X}$, since $j^*(\beta)$ is an isomorphism and j^* is exact. On the other hand, $j_! j^* P$ belongs to $\mathscr{P}$, since $j^* P$ is a preprojective A-module. This shows that the kernel of β is zero. Similarly, one obtains the second assertion.

Remark 1. The name "squid" has been borrowed from Brenner–Butler [BB] where they consider some algebras of the form $B(\mathbf{T})$, or, more precisely, the corresponding opposite algebras.

Remark 2. The reader may have wondered why we have started with the algebras $B(\mathbf{T})$ in order to find a cotilting module with endomorphism ring $C(\mathbf{T})$, and later switched to the dual situation. Whereas the notion of a canonical algebra is fully left-right-symmetric, the presentation which we use is not. The algebra B' which tilts to $C(\mathbf{T})$ may be best described as a tubular coextension of A by the modules $\Phi Q_1, \ldots, \Phi Q_t$, where Φ is a Coxeter transformation for mod A constructed by means of some chosen dual bimodule (see the remark at the end of Section 1).

5. The structure of $\mathscr{T}$

As usual, it follows from the definition of $\mathscr{P}$, $\mathscr{T}$, $\mathscr{Q}$ and from $\operatorname{Hom}(\mathscr{Q}, \mathscr{T} \vee \mathscr{P}) = 0 = \operatorname{Hom}(\mathscr{Q} \vee \mathscr{T}, \mathscr{P})$ that $\mathscr{T}$ is an abelian category:

LEMMA 1. *$\mathscr{T}$ is closed under kernels, images, cokernels and extensions. In particular, it is an abelian subcategory with exact inclusion functor. Thus, it is a length category with no nonzero projective or injective object.*

Proof. If X is a C-module, we write ∂X instead of $\partial(\underline{\dim} X)$. If T belongs to $\mathscr{F}$, and T' is a submodule of T, then T' belongs to $\mathscr{T}$ if and only if $\partial T' = 0$.

For, T' belongs to $\mathscr{P} \vee \mathscr{T}$, thus $\partial T' = 0$ implies that any indecomposable direct summand T'' of T' satisfies $\partial T'' = 0$. Let T_1, T_2 be in $\mathscr{T}$. If $f\colon T_1 \to T_2$ is a map, say with kernel T_1' and image T_2', then $\partial T_1' \leqslant 0$, $\partial T_2' \leqslant 0$, and $0 = \partial T_1 = \partial T_1' + \partial T_2'$ shows that $\partial T_1' = 0$, $\partial T_2' = 0$, thus T_1', $T_2' \in \mathscr{T}$. Similarly, also the cokernel of f belongs to $\mathscr{T}$.

Let $0 \to T_1 \to T' \to T_2 \to 0$ be an extension, and T'' an indecomposable direct summand of T'. If T' is neither isomorphic to a direct summand of T_1 nor to one of T_2, then there is an indecomposable direct summand T_1' of T_1 and an indecomposable direct summand T_2' of T_2 with $\operatorname{Hom}(T_1', T') \neq 0$ and $\operatorname{Hom}(T', T_2') \neq 0$, thus $T' \in \mathscr{T}$. This shows that $\mathscr{T}$ is closed under extensions. Let T be indecomposable in $\mathscr{T}$. Then τT and $\tau^- T$ belong to $\mathscr{T}$. In particular, the Auslander–Reiten sequence ending with T belongs to $\mathscr{T}$, thus T cannot be projective in $\mathscr{T}$, and the Auslander–Reiten sequence starting with T belongs to $\mathscr{T}$, thus T is not injective in $\mathscr{T}$.

LEMMA 2. *The Auslander–Reiten translation τ on $\mathscr{T}$ is a self-equivalence.*

Proof. Let $T \in \mathscr{T}$. Since $\operatorname{proj.dim} T \leqslant 1$, it follows that $\tau T = \operatorname{Hom}_k(\operatorname{Ext}^1_C(T_C, C_C), k)$. Thus the restriction of τ to $\mathscr{T}$ is an equivalence from $\mathscr{T}$ to $\mathscr{T}$.

LEMMA 3. *Let $\mathscr{A}$ be a length category with Auslander–Reiten sequences and without nonzero projective or injective objects. Assume there is a self-equivalence of $\mathscr{A}$ which gives the Auslander–Reiten translation. Then $\mathscr{A}$ is a serial category of global dimension* 1, *its Auslander–Reiten components are quotients of* $\mathbf{ZA}_\infty$, *and an indecomposable object belongs to the boundary of a component if and only if it is simple.*

Proof. Let E be simple in $\mathscr{A}$. Since τ is a self-equivalence, also τE is simple. Let $0 \to \tau E \to X \to E \to 0$ be an Auslander–Reiten sequence. Then X is of length 2, thus indecomposable, thus E and τE belong to the boundary of the corresponding component. If E' is simple with $\operatorname{Ext}^1(E, E') \neq 0$, then clearly $E' \cong \tau E$. Also $\operatorname{Ext}^1(E, \tau E)$ is one-dimensional both as an $\operatorname{End}(E)$-space and as an $\operatorname{End}(\tau E)$-space. This shows that $\mathscr{A}$ is serial. Since there are no nonzero projective objects in $\mathscr{A}$, there exist indecomposable objects of arbitrary length with E as top composition factor, and since there are no nonzero injective objects in $\mathscr{A}$, there exist indecomposable objects of arbitrary length with E being the socle. It follows that the Auslander–Reiten components are quotients of $\mathbf{ZA}_\infty$, with the boundary consisting of simple objects.

LEMMA 4. *The simple objects in $\mathscr{T}$ are, first, the simple C-modules which belong to $\mathscr{Z}$, and, second, the C-modules of the form $E(\varrho)$, with $\varrho \in \Omega$.*

Proof. Let T be simple in $\mathscr{T}$. If $T \in \mathscr{Z}$, then T is a simple C-module, since $\mathscr{Z}$ is closed under subquotients. Thus, assume T does not belong to $\mathscr{Z}$. Consider the adjunction map $\alpha\colon T \to j_* j^* T$. It is a nonzero map, and $j_* j^* T$ is in $\mathscr{T}$, thus α is a monomorphism (since T is simple in $\mathscr{T}$). Let Z be the cokernel of α. Since $j^*\alpha$ is the identity of $j^* T$, and j^* is exact, we see that $Z \in \mathscr{Z}$. In particular,

$\operatorname{Hom}(T, Z) = 0$, since T is simple in $\mathscr{T}$ and not in $\mathscr{Z}$. Let φ be an endomorphism of j^*T. The endomorphism $j_*\varphi$ of j_*j^*T induces an endomorphism φ' of T, with $\alpha\varphi' = (j_*\varphi)\alpha$. If $\varphi' = 0$, then $j_*\varphi$ factorizes through Z, but $\operatorname{Hom}(Z, j_*j^*T) \cong \operatorname{Hom}(j^*Z, j^*T) = 0$, thus $j_*\varphi = 0$. Note that $\operatorname{End}(j_*T) \cong \operatorname{End}(j_*j^*T)$, since j_* is a full embedding. We have shown that this ring embeds into $\operatorname{End}(T)$ via the restriction along α. It follows that together with $\operatorname{End}(T)$, also $\operatorname{End}(j^*T)$ is a division ring, thus j^*T is a simple regular A-module. We denote j^*T by ϱ, and we recall that $\alpha\colon T \to j_*\varrho$ is a monomorphism, with $j^*\alpha$ the identity map.

Similarly, the adjunction map $\beta\colon j_!\varrho = j_!j^*T \to T$ is an epimorphism, with $j^*\beta$ the identity map of ϱ. The composition of α and β is just η_ϱ, therefore $T = E(\varrho)$.

Conversely, we have to show that given $\varrho' \in \Omega$, the C-module $E(\varrho')$ is a simple object of $\mathscr{T}$. Let T be a simple subobject of $E(\varrho')$ in $\mathscr{T}$. Since $\operatorname{Hom}_A(j^*T, \varrho') \cong \operatorname{Hom}_C(T, j_*\varrho') \neq 0$, it follows that T does not belong to $\mathscr{Z}$. By previous considerations, there is $\varrho \in \Omega$ with $T = E(\varrho)$, and $j^*T = \varrho$. But $\operatorname{Hom}_A(\varrho, \varrho') \neq 0$ shows that ϱ and ϱ' are isomorphic, thus $E(\varrho')$ is simple in $\mathscr{T}$.

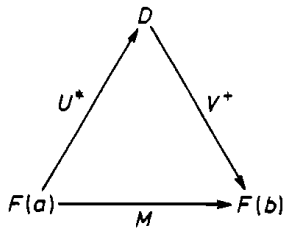
Proof of Theorem 2. According to Lemmas 1 and 2, we can apply Lemma 3 to $\mathscr{T}$. We denote by $\mathscr{T}(\varrho)$ the Auslander–Reiten component of $\mathscr{T}$ (or, what is the same, of mod C) which contains $E(\varrho)$. According to Lemma 3, we know that $\mathscr{T}(\varrho)$ is a quotient of $\mathbf{ZA}_\infty$, and that $E(\varrho)$ belongs to the boundary of $\mathscr{T}(\varrho)$.

The indecomposable object of $\mathscr{T}$ of $\mathscr{T}$-length n and with $\mathscr{T}$-top T will be denoted by $[n]T$; these modules form the coray ending with T. Similarly, in A-mod, we consider a nonsplit exact sequence $0 \to \varrho \to [2]\varrho \to \varrho \to 0$ where $\varrho \in \Omega$. If we apply the right exact functor $j_!$, we obtain an exact sequence $0 \to Z \to j_!\varrho \to j_!([2]\varrho) \to j_!\varrho \to 0$, with $Z \in \mathscr{Z}$, since j^* is exact and $j^*j_!$ is the identity. Together with $[2]\varrho$ also $j_!([2]\varrho)$ is indecomposable. Since $j_!\varrho$ maps onto $E(\varrho)$, we see that both $j_!\varrho$ and $j_!([2]\varrho)$ belong to the coray ending with $E(\varrho)$. If $j_!\varrho = E(\varrho)$, then $j_!\varrho$ is simple and not in $\mathscr{Z}$, thus $Z = 0$, therefore $j_!([2]\varrho)$ is of $\mathscr{T}$-length 2, thus $j_!([2]\varrho) = [2](E(\varrho))$, and its $\mathscr{T}$-socle is $E(\varrho)$. Thus $\tau E(\varrho) = E(\varrho)$, and $\mathscr{T}(\varrho)$ is a stable tube of rank 1.

Consider now the modules $E(i, j)$, $1 \leqslant j \leqslant n_i - 1$. We fix some i. An easy calculation using a projective presentation of $E(i, j)$ for $1 \leqslant j \leqslant n_i - 2$ shows that for these j, we have $\tau E(i, j) = E(i, j+1)$. Since $E(i, 1)$ is injective in $\mathscr{Z}$, we see that $\tau^- E(i, 1)$ cannot belong to $\mathscr{Z}$, thus $\tau^- E(i, 1) = E(\varrho)$ for some $\varrho \in \Omega$. Similarly, $\tau E(i, n_i - 1) = E(\varrho')$ for some $\varrho' \in \Omega$. Let $j_!\varrho = [n](E(\varrho))$. Since the kernel of the canonical epimorphism $j_!\varrho \to E(\varrho)$ belongs to $\mathscr{Z}$, we see that $n \leqslant n_i$. Now, $j_!([2]\varrho)$ is indecomposable of $\mathscr{T}$-length at most $2n$, and it has the following composition factors in $\mathscr{T}$, starting from the top: first, $E(\varrho)$, then $n-1$ factors in $\mathscr{Z}$, then again $E(\varrho)$, and finally some in $\mathscr{Z}$. But this is possible only in case $n = n_i$ and $E(\varrho') = E(\varrho)$. This shows that $\mathscr{T}(\varrho)$ is a tube of rank n_i containing on the mouth the modules $E(\varrho)$, and $E(i, j)$ with $1 \leqslant j \leqslant n_i - 1$. In order to show that $\varrho = \varrho_i$, we show that $E(i, -1)$ embeds into $j_!\varrho_i$, so that

the indecomposable module $j_!\varrho_i$, and therefore also its $\mathscr{T}$-top $E(\varrho_i)$, belong to $\mathscr{T}(\varrho)$.

We may assume $i = 1$, $n_1 = 2$, and even $\mathbf{T}(\sigma) = 1$ for $\sigma \neq \varrho_1$. Thus, we deal with $\varrho = \varrho_1$: $U \otimes_{F(a)} M \to V$, with endomorphism ring D. The kernel of ϱ is denoted by ϱ^+: $V^+ \to U \otimes_{F(a)} M$, and we have the adjoint map $\tilde{\varrho}$: $U^* \otimes_D V^+ \to M$. We consider representations of the species



say $(X_{F(a)}, Y_D, Z_{F(b)}; X \otimes_{F(a)} U^* \to D,\ Y \otimes_D V^+ \to Z,\ X \otimes_{F(a)} M \to Z)$, which satisfy the commutativity condition

$$\begin{array}{ccc} X \otimes U^* \otimes V^+ & \to & Y \otimes V^+ \\ {\scriptstyle 1 \otimes \tilde{\varrho}} \downarrow & & \downarrow \\ X \otimes M & \to & Z \end{array}$$

Now, $j_!\varrho = (U_{F(a)}, U \otimes_{F(a)} U_D^*, V_{F(a)}; 1_{U \otimes U^*}, \varrho(1 \otimes \tilde{\varrho}), \varrho)$, and we obtain an embedding of $E(1, 1) = (0, D_D, 0; 0, 0, 0)$ into $j_!\varrho$ as follows: Since $U^* = \operatorname{Hom}_{F(a)}(U_{F(a)}, F(a)_{F(a)})$, we can identify $U \otimes_{F(a)} U^*$ with $\operatorname{End}(U_{F(a)})$ via $u \otimes \alpha \mapsto (u' \mapsto u\alpha(u'))$, for $u, u' \in U$, $\alpha \in U^*$, and we denote by $e = \sum u_i \otimes \alpha_i \in U \otimes_{F(a)} U^*$ the element which corresponds to the identity endomorphism. We consider the right D-submodule eD of $U \otimes U^*$ generated by e (which is actually a D-D-submodule of $U \otimes U^*$). We claim that $eD \otimes_D V^+$ is mapped under $\varrho(1 \otimes \tilde{\varrho})$ to zero, thus we obtain in this way a subobject of $j_!\varrho$ isomorphic to $E(1, 1)$. But the composition

$$V^+ \cong eD \otimes_D V^+ \subseteq U \otimes U^* \otimes V^+ \xrightarrow{1 \otimes \tilde{\varrho}} U \otimes M$$

is precisely ϱ^+: $V^+ \to U \otimes M$, and $\varrho\varrho^+ = 0$, by definition of ϱ^+. This completes the proof of Theorem 2.

The component of $\operatorname{mod} A$ which contains ϱ will be denoted by $\mathscr{T}_A(\varrho)$.

PROPOSITION. *If X is a preprojective A-module, both $j_!X$ and j_*X belong to $\mathscr{P}$. If $X \in \mathscr{T}_A(\varrho)$ for some $\varrho \in \Omega$, both $j_!X$ and j_*X belong to $\mathscr{T}(\varrho)$. If X is a preinjective A-module, both $j_!X$ and j_*X belong to $\mathscr{Q}$.*

Proof. We may assume that X is indecomposable, thus also $j_!X$ and j_*X are indecomposable. Since $j^*j_!X \cong X \cong j^*j_*X$, it follows that for X preprojective or preinjective, $j_!X$ and j_*X are both in $\mathscr{P}$, or $\mathscr{Q}$, respectively. So assume $X \in \mathscr{T}_A(\varrho)$; then $j_!X$ and j_*X are in $\mathscr{T}$. There is a nonzero map $X \to \varrho = j^*E(\varrho)$

in mod A, so since $j_!$ is left adjoint to j^*, there is a nonzero map $j_!X \to E(\varrho)$. However, this means that $j_!X$ has $E(\varrho)$ as its $\mathscr{T}$-top. Similarly, there is a nonzero map $j^*E(\varrho) = \varrho \to X$, thus a nonzero map $E(\varrho) \to j_*X$, thus $E(\varrho)$ is the $\mathscr{T}$-socle of j_*X. It follows that $j_!X$ and j_*X belong to $\mathscr{T}(\varrho)$.

6. The factorization property

We want to show that any map $f\colon P \to Q$ with $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ can be factored through a module T which belongs to a prescribed tube $\mathscr{T}(\varrho)$. First, we deal with the case of A itself.

LEMMA 1. *Let P be a preprojective A-module, Q a preinjective A-module, and $\varrho \in \Omega$. Given any map $f\colon P \to Q$, there exists a module T in $\mathscr{T}_A(\varrho)$ such that f can be factored through T.*

Proof. By induction on the length of P, we show that P can be embedded into a module belonging to $\mathscr{T}_A(\varrho)$. Assume $P \neq 0$. There exists a nonzero map $u\colon P \to \varrho$ (namely, some $\tau^{-t}P_0$, with P_0 indecomposable projective, and $t \geqslant 0$, is a direct summand of P, and $0 \neq \operatorname{Hom}(P_0, \tau^t\varrho) \cong \operatorname{Hom}(\tau^{-t}P_0, \varrho)$). Let P' be the kernel of ϱ. By induction, there is $u'\colon P' \to T'$ with $T' \in \mathscr{T}_A(\varrho)$. We form the induced exact sequence

$$\begin{array}{ccccccc} 0 & \to & P' & \to & P & \to & P/P' \to 0 \\ & & {\scriptstyle u'}\downarrow & & \downarrow & & \| \\ 0 & \to & T' & \to & Y & \to & P/P' \to 0. \end{array}$$

If u is surjective, $P/P' = \varrho$ is in $\mathscr{T}_A(\varrho)$, thus Y is in $\mathscr{T}_A(\varrho)$ and $P \to Y$ is the desired embedding. Otherwise, P/P' is a proper submodule of ϱ, thus preprojective, and therefore the induced exact sequence splits. But this means that $Y = T' \oplus P/P'$ is a submodule of $T' \oplus \varrho$, and $T' \oplus \varrho$ belongs to $\mathscr{T}_A(\varrho)$, so we take the embedding $P \to Y \to T' \oplus \varrho$.

So consider now a map $f\colon P \to Q$. Without loss of generality, we can assume that Q is indecomposable. First, assume that Q is injective. Take an embedding $P \to T$ with $T \in \mathscr{T}_A(\varrho)$. Since Q is injective, f can be extended to T, thus we obtain a factorization of f through T. If Q is indecomposable but not injective, $Q = \tau^{-s}Q_0$ for some indecomposable injective module Q_0 and some $s \geqslant 0$. Then $\tau^s f\colon \tau^s P \to Q_0$ can be factored through some $T_0 \in \mathscr{T}_A(\varrho)$, thus f factors through $\tau^{-s}T_0 \cong T_0$. This completes the proof.

We now consider C-modules, where $C = C(\mathbf{T})$ is a canonical algebra. In mod C, we use the notation introduced in Section 3.

LEMMA 2. *Let $P \in \mathscr{P}$. Then P can be embedded into a module $P' \in \mathscr{P}$ such that $P'/P \in \mathscr{T}$ and $\operatorname{Hom}(P, \mathscr{X}) = 0$.*

Proof. Let $X \subseteq P$ be minimal such that $P/X \in \mathscr{X}$. Let $Z = P/X$. Since any

indecomposable object in $\mathscr{T}$ can be embedded into an indecomposable object in $\mathscr{T}$ with $\mathscr{T}$-top of the form $E(\varrho)$, we can embed $Z \subseteq T$ with $T \in \mathscr{T}$, and $\mathrm{Hom}(T, \mathscr{Y}) = 0$. In addition, we can assume that any nonzero submodule of T which belongs to $\mathscr{T}$ intersects Z nontrivially. The epimorphism $\varepsilon\colon P \twoheadrightarrow Z$ induces an epimorphism $\mathrm{Ext}^1(T/Z, P) \twoheadrightarrow \mathrm{Ext}^1(T/Z, Z)$, since $\mathrm{proj.dim}\, T/Z \leqslant 1$. Thus, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & P & \xrightarrow{\mu} & P'' & \xrightarrow{\pi} & T/Z & \to & 0 \\ & & \varepsilon\downarrow & & \varepsilon'\downarrow & & \| & & \\ 0 & \to & Z & \xrightarrow{\mu'} & T & \xrightarrow{\pi'} & T/Z & \to & 0 \end{array}$$

where the lower exact sequence is the canonical one. We show that $\mathrm{Hom}(P'', Z') = 0$ for any $Z' \in \mathscr{Y}$. Thus, let $\alpha\colon P'' \to Z'$ be a map. Since $\mathscr{Y}$ is closed under submodules, and X was minimal with P/X in $\mathscr{Y}$, the map $\alpha\mu$ factors through ε, say $\alpha\mu = \alpha'\varepsilon$. Since the above diagram is a pushout, there is $\beta\colon T \to Z'$ with $\beta\varepsilon' = \alpha$, $\beta\mu' = \alpha'$. But $\mathrm{Hom}(T, \mathscr{Y}) = 0$, thus $\beta = 0$, therefore $\alpha = 0$.

Since P belongs to $\mathscr{P}$ and T/Z belongs to $\mathscr{T}$, we know that P'' is in $\mathscr{P} \vee \mathscr{T}$. Let Y be the maximal submodule of P'' which belongs to $\mathscr{T}$. Then $Y/P \cap Y \cong (P+Y)/P$ embeds into $P''/P = T/Z$, thus $P \cap Y$ is in $\mathscr{T}$ (since $\mathscr{T}$ is closed under kernels). It follows from $P \in \mathscr{P}$ and $P \cap Y \in \mathscr{T}$ that $P \cap Y = 0$. Let $P' = P''/Y$. Then P' belongs to $\mathscr{P}$, we have $\mathrm{Hom}(P', \mathscr{Y}) = 0$, and P embeds into P' with factor module $P''/P+Y$, but $P''/P+Y$ is the cokernel of the inclusion map $Y \cong (P+Y)/P$ into P''/P, thus it belongs to $\mathscr{T}$. This completes the proof.

Proof of the factorization property. Let $\varrho \in \Omega$. Let $P \in \mathscr{P}$, $Q \in \mathscr{Q}$, and $f\colon P \to Q$. According to Lemma 2, we embed P into some $P' \in \mathscr{P}$ with $P'/P \in \mathscr{T}$ and $\mathrm{Hom}(P', \mathscr{Y}) = 0$. Since $\mathrm{Ext}^1(Q, P'/P) = 0$, the map f can be extended to P'. Thus, without loss of generality, we may assume $\mathrm{Hom}(P, \mathscr{Y}) = 0$.

Consider now the canonical map $\beta_P\colon j_!j^*P \to P$. According to Section 4, this is a monomorphism, and its cokernel Z belongs to $\mathscr{Y}$. Decompose $Z = Z_\varrho \oplus Z'$ with Z_ϱ the maximal direct summand of Z belonging to $\mathscr{T}(\varrho)$. According to Lemma 1, the map $j^*f\colon j^*P \to j^*Q$ can be factored through some $R \in \mathscr{T}_A(\varrho)$, say $j^*f = hg$, with $g\colon j^*P \to R$, $h\colon R \to j^*Q$. We form the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & j_!j^*P & \xrightarrow{\beta_P} & P & \to & Z_\varrho \oplus Z' & \to & 0 \\ & & j_!g\downarrow & & g'\downarrow & & \| & & \\ 0 & \to & j_!R & \to & X & \to & Z_\varrho \oplus Z' & \to & 0 \end{array}$$

Now, $j_!R$ belongs to $\mathscr{T}(\varrho)$, according to Section 5. Since no $\mathscr{T}$-composition factor of Z' belongs to $\mathscr{T}(\varrho)$, we have $\mathrm{Ext}^1(Z', j_!R) = 0$, therefore $X = X_\varrho \oplus Z'$ with $X_\varrho \in \mathscr{T}(\varrho)$. On the other hand, $f\beta_P = \beta_Q j_!j^*f = \beta_Q(j_!h)(j_!g)$, since $\beta\colon j_!j^* \to \mathrm{id}$ is a natural transformation, thus the pushout property of the left

square yields a map $h'\colon X \to Q$ which satisfies $h'g' = f$. Since $X = X_\varrho \oplus Z'$, we can write $f = h_1 g_1 + h_2 g_2$, where $g_1\colon P \to X_\varrho$, $g_2\colon P \to Z'$, $h_1\colon X_\varrho \to Q$, $h_2\colon Z' \to Q$. However, $Z' \in \mathscr{Z}$, thus $\operatorname{Hom}(P, Z') = 0$, therefore $f = h_1 g_1$ is a factorization of f through $X_\varrho \in \mathscr{T}(\varrho)$. This completes the proof.

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Appendix

William Crawley-Boevey

The aim of the appendix is to present an elementary definition of the canonical algebras. In particular, we will not presuppose the structure theory for the category of representations of a tame bimodule.

Let F and G be division rings, and let ${}_FM_G$ be an F-G-bimodule with $(\dim {}_FM)(\dim M_G) = 4$. We denote by χ the number

$$\chi = \sqrt{\frac{\dim {}_FM}{\dim M_G}},$$

thus χ is one of $\frac{1}{2}$, 1, 2.

By an *M-triple* we mean a triple $({}_FN, \varphi, N'_G)$ where ${}_FN$ is a finite-dimensional nonzero left F-module, N'_G a finite-dimensional nonzero right

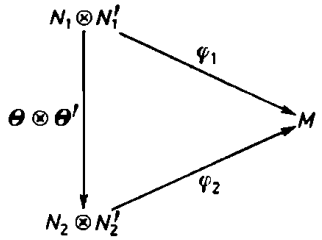
G-module, and φ: ${}_FN \otimes_Z N'_G \to {}_FM_G$ an F-G-homomorphism such that

(1) $$\frac{\dim {}_FN}{\dim N'_G} = \chi,$$

(2) whenever ${}_FX$ and X'_G are nonzero submodules of ${}_FN$ and N'_G, respectively, with $\varphi(X \otimes X') = 0$, then

$$\frac{\dim {}_FX}{\dim {}_FN} + \frac{\dim X'_G}{\dim N'_G} < 1.$$

We call two M-triples (N_1, φ_1, N'_1), (N_2, φ_2, N'_2) *congruent* provided there are isomorphisms Θ: ${}_F(N_1) \to {}_F(N_2)$ and Θ': $(N'_1)_G \to (N'_2)_G$ such that the diagram



commutes.

Define the *middle* D of an M-triple (N, φ, N') to be the set of pairs (d, d') where d is an endomorphism of ${}_FN$ and d' an endomorphism of N'_G such that $\varphi \circ (d \otimes 1) = \varphi \circ (1 \otimes d')$. Clearly, D is a ring under componentwise addition and multiplication, N is an F-D-bimodule, N' a D-G-bimodule, and φ induces a map $N \otimes_D N' \to M$ which again will be denoted by φ.

LEMMA 1. *D is a division ring.*

Proof. Let (d, d') be a nonzero nonunit in D. Since $\varphi(\operatorname{Ker} d \otimes \operatorname{Im} d') = 0$ and the pair $(\operatorname{Ker} d, \operatorname{Im} d') \neq (N, 0)$ or $(0, N')$, it follows that

$$\frac{\dim {}_F\operatorname{Ker} d}{\dim {}_FN} + \frac{\dim \operatorname{Im} d'_G}{\dim N'_G} < 1.$$

Similarly, we also have

$$\frac{\dim {}_F\operatorname{Im} d}{\dim {}_FN} + \frac{\dim \operatorname{Ker} d'_G}{\dim N'_G} < 1.$$

Adding these two inequalities, we obtain

$$\frac{\dim {}_FN}{\dim {}_FN} + \frac{\dim N'_G}{\dim N'_G} < 2,$$

a contradiction.

DEFINITION. A *canonical* ring of type $(n_1, \ldots, n_r)$ where $r \geqslant 0$, $n_i \geqslant 2$, is a ring isomorphic to a matrix ring of the form

$$\begin{array}{c|c|c|c|c|c|}
 & F & N_1 \ldots N_1 & N_2 \ldots N_2 & \ldots & N_r \ldots N_r & M \\
\hline
n_1-1 & 0 & \begin{matrix} D_1 \ldots D_1 \\ \ddots \vdots \\ 0 \quad D_1 \end{matrix} & 0 & \ldots & 0 & \begin{matrix} N_1' \\ \vdots \\ N_1' \end{matrix} \\
\hline
n_2-1 & 0 & 0 & \begin{matrix} D_2 \ldots D_2 \\ \ddots \vdots \\ 0 \quad D_2 \end{matrix} & \ldots & 0 & \begin{matrix} N_2' \\ \vdots \\ N_2' \end{matrix} \\
\hline
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hline
n_r-1 & 0 & 0 & 0 & \ldots & \begin{matrix} D_r \ldots D_r \\ \ddots \vdots \\ 0 \quad D_r \end{matrix} & \begin{matrix} N_r' \\ \vdots \\ N_r' \end{matrix} \\
\hline
 & 0 & 0 & 0 & \ldots & 0 & G \\
\hline
\end{array}$$

where F, G are division rings, ${}_FM_G$ is a bimodule with $(\dim {}_FM)(\dim M_G) = 4$, $(N_1, \varphi_1, N_1'), \ldots, (N_r, \varphi_r, N_r')$ are mutually noncongruent M-triples with middles $D_1, \ldots, D_r$ and the multiplication is given by the action of a division ring on a module, or by the appropriate φ_i.

LEMMA 2. *The centre k of a canonical ring is field.*

Proof. Suppose $x \neq 0$ is in the centre of the matrix ring A. Let e_i be the element of A corresponding to the identity of the ith division ring on the diagonal, and let d_i be an element which is nonzero at precisely the $(1, i)$th position. Then

$$x = \sum_{i,j} e_i x e_j = \sum_{i,j} x e_i e_j = \sum_i x e_i = \sum_i e_i x e_i,$$

so x is diagonal. Also.

$$d_i(e_i x e_i) = d_i x e_i = x d_i = x e_1 d_i = (e_1 x e_1) d_i,$$

so $e_i x e_i$ and $e_1 x e_1$ are zero or nonzero together. Thus x is invertible in A, and hence in k.

LEMMA 3. *The canonical algebras are just the canonical rings which are finite-dimensional over their centres.*

Proof. Let k be a field, let F and G be finite-dimensional k-division rings and ${}_FM_G$ a bimodule centralized by k. In the notation of § 1 it suffices to show that the construction

$$S = (U_F, V_G, \varrho\colon U \otimes {}_FM_G \to V_G) \mapsto T = ({}_FU^*, \tilde{\varrho}\colon {}_FU^* \otimes {}_DV_G^+ \to M, V_G^+)$$

induces a bijection between the isomorphism classes of simple regular representations of ${}_F M_G$ and the congruence classes of M-triples (N, φ, N') with k in their middles and with N and N' centralized by k, and that it takes the endomorphism ring D of S to the middle of T.

Note first that

$$\frac{\dim {}_F U^*}{\dim V_G^+} = \left(\frac{2+\partial(S)}{2\chi} + \dim(\operatorname{Coker}\varrho)_G\right)^{-1}, \tag{3}$$

so if S is simple regular, condition (1) holds for T. Now suppose that $X \leqslant U^*$, $X' \leqslant V^+$ and $\tilde{\varrho}(X \otimes X') = 0$. Set $X^\perp = \{u \in U \mid x(u) = 0 \ \forall x \in X\}$; then

$$S' = (X^\perp, \varrho(X^\perp \otimes M), \varrho|_{X^\perp \otimes M})$$

is a subrepresentation of S. If S is simple regular then $S' = 0$, $S' = S$, or $\partial(S') < 0$. In the first case $X^\perp = 0$, $\tilde{\varrho}(U^* \otimes X') = 0$, so $\varrho^+(X') = 0$ and hence $X' = 0$; if $S' = S$, then $X = 0$; while in the last case, the inequalities $\partial(S') < 0$ and $\dim \varrho(X^\perp \otimes M)_G \leqslant \dim X^\perp \otimes M_G - \dim X'_G$ lead to the inequality of (2). Thus T is an M-triple.

Now suppose that $T = ({}_F N, \varphi, N'_G)$ is an M-triple, and let $\bar{\varphi}$: $N'_G \to N^* \otimes {}_F M_G$ be adjoint to φ; it is injective since $\varphi(N \otimes \operatorname{Ker}\bar{\varphi}) = 0$. Clearly T comes from the representation $S = (N^*, \operatorname{Coker}\bar{\varrho}, \pi)$ where π is the natural projection. By (3), $\partial(S) = 0$. If $S' = (U, V, \varrho)$ is a subrepresentation of S, and $Z = (\bar{\varphi})^{-1}(U \otimes M)$, then $\varphi(U^\perp \otimes Z) = 0$, where $U^\perp = \{n \in N \mid u(n) = 0 \ \forall u \in U\}$. Thus one of three cases occurs:

1. If $U^\perp = 0$, then $U = N^*$ and $S' = S$.

2. If $Z = 0$ and $U^\perp = N$, then $U = 0$ and either $V = 0$ so $S' = 0$, or $\partial(S') = -m \cdot \dim V_G$ is negative.

3. If the inequality in (2) holds, then since $\dim V_G \geqslant \dim U \otimes {}_F M_G - \dim Z_G$, the defect of S' is again negative.

Thus all proper nonzero subrepresentations of S have negative defect, so S is simple regular.

If (f, g) is an isomorphism $(U_1, V_1, \varrho_1) \to (U_2, V_2, \varrho_2)$ of simple regular representations of M, then the corresponding M-triples are congruent via $((f^{-1})^*, h)$, where h is the induced map $V_1^+ \to V_2^+$. Conversely, since the ϱ_i are surjective, any such congruence arises this way. Finally, (f, g) is an endomorphism of (U, V, ϱ) if and only if (f^*, h) is in the middle of $(U^*, \tilde{\varrho}, V^+)$.

Remark. In case the division rings and modules are finite-dimensional over a central field, a triple $({}_F N, \varphi, N'_G)$ satisfying (1) and whose middle is a division ring, also satisfies (2). In general this is not clear, but the M-triples should be regarded as the more natural objects since they correspond to the simple ϱ-torsion modules in the sense of [S, § 1], where ϱ is the rank function obtained by normalizing the defect.

Reference

[S] A. H. Schofield, *Representations of Rings over Skew Fields*, London Math. Soc. Lecture Note Ser. 92, Cambridge Univ. Press, 1985.