TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART I
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

GRADED MODULES AND THEIR COMPLETIONS

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Suppose R is a G-graded Cohen–Macaulay $k[x_1, \ldots, x_t]$-algebra where G is a group and k a field. We study the completion functor from graded R-modules to $\hat{R}$-modules where $\hat{R}$ is the completion of R, with special emphasis on Cohen–Macaulay modules. We prove that if the orders of elements of finite order in G are invertible in k and the orders of finite subgroups of G are bounded, then almost split sequences of graded Cohen–Macaulay modules go to direct sums of almost split sequences of Cohen–Macaulay modules over $\hat{R}$. Moreover, we show that R is of finite graded Cohen–Macaulay type if and only if $\hat{R}$ is of finite Cohen–Macaulay type.

Introduction

Before stating the main results in this paper, we need some notation and definitions concerning graded rings and modules for arbitrary groups.

Let G be a group which we write additively even though it may not be commutative. By a *G-graded ring* R we mean a ring (not necessarily commutative) together with a direct sum decomposition $R = \coprod_{g \in G} R_g$ such that $R_h R_g \subset R_{h+g}$ for all h and g in G. A *G-graded R-module* C is an R-module C together with a direct sum decomposition $C = \coprod_{g \in G} C_g$ such that $R_h C_g \subset C_{h+g}$ for all h and g in G. An R-module morphism $\alpha\colon B \to C$ of

This paper is in final form and no version of it will be submitted for publication elsewhere.

G-graded R-modules B and C is said to be of *degree* 0 if $\alpha(B_g) \subset C_g$ for all g in G. We denote by $\mathrm{Mod}(\mathrm{gr} R)_0$ the category of G-graded R-modules together with degree 0 morphisms.

Throughout this paper T denotes a polynomial ring $k[X_1, \ldots, X_d]$ with k a field and the X_i indeterminates. We always assume that a G-grading on T has the following properties: (a) $T_0 = k$; (b) each X_i is homogeneous; (c) each T_g is of finite dimension over $k = T_0$. By a *G-graded T-algebra R* we mean throughout this paper that R is a T-algebra by means of the ring morphism f: $T \to R$ and that T and R are G-graded rings such that R is a finitely generated G-graded T-module and f: $T \to R$ is a degree 0 morphism of G-graded T-modules. Hence R is left and right noetherian. Also for each finitely generated R-module C we denote by $\hat{C}$ the completion of C in the $\mathfrak{m}$-adic topology where $\mathfrak{m} = (X_1, \ldots, X_d)$, the ideal in T generated by the indeterminates $X_1, \ldots, X_d$.

Suppose R is a G-graded T-algebra. Then a G-graded R-module C is said to be *Cohen–Macaulay* if it is free when viewed as a T-module. We say that R is a *Cohen–Macaulay G-graded T-algebra* if R is a G-graded Cohen–Macaulay R-module. We say that R *has isolated singularities* if $\mathrm{gl.dim}\, R_{\mathfrak{p}} = \dim T_{\mathfrak{p}}$ for all nonmaximal prime ideals $\mathfrak{p}$ of T. Throughout the rest of this introduction we assume that R is a Cohen–Macaulay G-graded T-algebra with isolated singularities.

Our main results are the following theorems connecting the subcategories $\mathrm{CM}(\mathrm{gr} R)_0$ and $\mathrm{CM}(\hat{R})$ of Cohen–Macaulay modules in the categories $\mathrm{mod}(\mathrm{gr} R)_0$, finitely generated G-graded R-modules with degree zero morphisms, and $\mathrm{mod}\,\hat{R}$, finitely generated $\hat{R}$-modules, respectively.

THEOREM A. (a) *The category* $\mathrm{CM}(\mathrm{gr} R)_0$ *has almost split sequences.*

(b) *If* $0 \to A \to B \to C \to 0$ *is an almost split sequence in* $\mathrm{CM}(\mathrm{gr} R)_0$, *then* $0 \to \hat{A} \to \hat{B} \to \hat{C} \to 0$ *is a direct sum of almost split sequences in* $\mathrm{CM}(\hat{R})$ *provided the orders of elements of finite order in* G *are invertible in* k.

This result about almost split sequences is used to prove the following result about when R is of *finite graded Cohen–Macaulay type*, i.e. when there are only a finite number of isomorphism classes of indecomposable modules in $\mathrm{CM}(\mathrm{gr} R)_0$ up to shift.

THEOREM B. *Suppose G has the following properties*: (a) *the orders of elements of finite order in G are invertible in k, and* (b) *the orders of finite subgroups of G are bounded. Then R is of finite graded Cohen–Macaulay type if and only if $\hat{R}$ is of finite Cohen–Macaulay type.*

These results generalize our results in [4]–[6]. These generalizations were inspired by conversations with H. Lenzing, who pointed out interesting examples from his work with Geigle on two-dimensional graded rings where indecomposability is not preserved under completion. Lenzing has given a different (independent) proof for the first theorem (when G is abelian), using the result for $G = Z$. We take the opportunity to thank Lenzing for his hospitality during our stay in Paderborn. We would also like to thank M. Van den Bergh for helpful conversations about graded rings.

1. Preliminaries

In this section we first give some preliminary results about graded T-algebras. We end the section by giving an existence theorem for almost split sequences.

Denote by $\operatorname{rad} R$ the ordinary radical of R and by $\operatorname{gr-rad} R$ the graded radical. For general background on graded theory we refer to [11]. For the first 3 lemmas we assume that if t is the order of an element of finite order in G, then t is invertible in k.

LEMMA 1.1. *Assume that R is a finite-dimensional k-algebra.*

(a) *If R is a graded division ring, then $H = \{g \in G;\ R_g \neq 0\}$ is a finite subgroup of G. Also R is semisimple and is the direct sum of at most $|H|$ indecomposable R-modules, where $|H|$ denotes the order of H. If H is trivial, then R is a division ring.*

(b) $\operatorname{gr-rad} R = \operatorname{rad} R$.

Proof. (a) It is well known and easy to see that H is a finite subgroup of G. Since by assumption $|H|$ is invertible in k, $\operatorname{rad} R = (0)$ by [8]. Further $R_0 = D$ is a division ring, and as a left D-module, $R_g \simeq D$ for each $g \in H$ [11]. Hence the last claim follows.

(b) Since R is artin, $\operatorname{gr-rad} R$ is nilpotent, so that $\operatorname{gr-rad} R \subseteq \operatorname{rad} R$. By a graded version of the Wedderburn theorem

$$R/\operatorname{gr-rad} R \simeq M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t),$$

where each D_i is a graded division ring and $M_{n_i}(D_i)$ denotes the ring of $n_i \times n_i$ matrices over D_i (see [11]). By (a) each D_i is semisimple, so that $\operatorname{rad}(R/\operatorname{gr-rad} R) = (0)$, and hence $\operatorname{gr-rad} R = \operatorname{rad} R$.

LEMMA 1.2. $(\operatorname{gr-rad} R)_{\mathfrak{m}} = \operatorname{rad} R_{\mathfrak{m}}$ *and hence* $(\operatorname{gr-rad} R)^{\wedge} = \operatorname{rad} \hat{R}$.

Proof. By the graded version of Nakayama's lemma we have $\mathfrak{m}R \subset \operatorname{gr-rad} R$ [11], and we also have $\mathfrak{m}R_{\mathfrak{m}} \subset \operatorname{rad} R_{\mathfrak{m}}$. By Lemma 1.1 we have for $\Lambda = R/\mathfrak{m}R \simeq R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ that $\operatorname{gr-rad} \Lambda = \operatorname{rad} \Lambda$, since Λ is a finite-dimensional k-algebra. Taking preimages we get $(\operatorname{gr-rad} R)_{\mathfrak{m}} = \operatorname{rad} R_{\mathfrak{m}}$. We have $\operatorname{rad} \hat{R}_{\mathfrak{m}} \subseteq \operatorname{rad} \hat{R}$, and since $(R_{\mathfrak{m}}/\operatorname{rad} R_{\mathfrak{m}})^{\wedge} = R_{\mathfrak{m}}/\operatorname{rad} R_{\mathfrak{m}}$ is semisimple, we have $\operatorname{rad} \hat{R}_{\mathfrak{m}} = \operatorname{rad} \hat{R}$, and consequently $(\operatorname{gr-rad} R)^{\wedge} = (\operatorname{rad} R_{\mathfrak{m}})^{\wedge} = \operatorname{rad} \hat{R}$.

LEMMA 1.3. *Let A be indecomposable in* $\operatorname{mod}(\operatorname{gr} R)_0$.

(a) *If A is not projective, then $P(\hat{A}, \hat{A}) \subset \operatorname{rad} \operatorname{End}(\hat{A})$, where $P(\hat{A}, \hat{A})$ denotes the maps $f\colon \hat{A} \to \hat{A}$ which factor through a projective module.*

(b) *If A is not projective, then $\hat{A}$ has no nonzero projective summands.*

(c) *$\hat{A}$ decomposes into a direct sum of at most n indecomposable summands, where n is the order of the group $H = \{g \in G;\ \Lambda_g \neq (0)\}$, where*

$$\Lambda = \mathrm{End}_R(A)/\text{gr-rad}\,\mathrm{End}_R(A).$$

(d) *If G has no nonzero elements of finite order, then*

$$\mathrm{End}_R(A)/\text{gr-rad}\,\mathrm{End}_R(A) \quad \textit{and} \quad \mathrm{End}_{R_\mathfrak{m}}(A_\mathfrak{m})/\mathrm{rad}\,\mathrm{End}_{R_\mathfrak{m}}(A_\mathfrak{m})$$

are division rings and $\hat{A}$ is indecomposable in $\mathrm{mod}\,\hat{R}$.

Proof. (a) Since the endomorphism rings of indecomposable modules in $\mathrm{mod}(\mathrm{gr}\,R)_0$ are local, $\mathrm{mod}(\mathrm{gr}\,R)_0$ has projective covers. Let $P \to A$ be a projective cover for A in $\mathrm{mod}(\mathrm{gr}\,R)_0$. We then have an exact sequence $\mathrm{Hom}_R(A, P) \to \mathrm{Hom}_R(A, A) \to \underline{\mathrm{Hom}}_R(A, A) \to 0$, which under completion goes to an exact sequence

$$\mathrm{Hom}_{\hat{R}}(\hat{A}, \hat{P}) \to \mathrm{Hom}_{\hat{R}}(\hat{A}, \hat{A}) \to \underline{\mathrm{Hom}}_{\hat{R}}(\hat{A}, \hat{A}) = (\underline{\mathrm{Hom}}_R(A, A))^\wedge \to 0.$$

It follows that $(P(A, A))^\wedge = P(\hat{A}, \hat{A})$. Clearly $P(A, A) \subset \text{gr-rad}\,\mathrm{End}_R(A)$, so $P(\hat{A}, \hat{A}) \subset \mathrm{rad}\,\mathrm{End}_R(\hat{A})$ then follows from Lemma 1.2.

(b) This is a direct consequence of (a).

(c) Λ decomposes into at most n indecomposable summands by Lemma 1.1 (b). By Lemma 1.2, $\Lambda = \mathrm{End}_{\hat{R}}(\hat{A})/\mathrm{rad}\,\mathrm{End}_R(\hat{A})$, so that also $\hat{A}$ decomposes into at most n indecomposable summands.

(d) This is a direct consequence of Lemma 1.1 (a), Lemma 1.2 and part(c).

LEMMA 1.4. *Let A and B be indecomposable in* $\mathrm{mod}(\mathrm{gr}\,R)_0$.

(a) $\mathrm{End}_R(A)$ *is a graded local ring.*

(b) *If $\hat{A} \simeq \hat{B}$, then there is some $g \in G$ such that $A \simeq B(g)$, where $B(g)_h = B_{h+g}$ for all $h \in G$.*

Proof. (a) Let u and v be elements in Γ_g, where $\Gamma = \mathrm{End}_R(A)$, which do not have left inverses. It is then sufficient to show that $u+v$ does not have a left inverse. Assume to the contrary that there is some $w \in \Gamma_{-g}$ such that $w(u+v) = 1 = wu+wv$. Since Γ_0 is a local ring, wu or wv must have a left inverse, so we have a contradiction.

(b) This is proved as in [6, Prop. 9].

LEMMA 1.5. *If P is indecomposable projective in* $\mathrm{mod}(\mathrm{gr}\,R)_0$, *then* gr-rad P *is the unique graded maximal ideal in P.*

Proof. Since the endomorphism rings of indecomposable objects in $\mathrm{mod}(\mathrm{gr}\,R)_0$ are local, the category $\mathrm{mod}(\mathrm{gr}\,R)_0$ has projective covers. Hence P is a projective cover for each nonzero factor module, in particular for $P/M_1 \cap M_2 = P/M_1 \amalg P/M_2$ if M_1 and M_2 are distinct graded maximal submodules of P. Since P is also a projective cover for P/M_1, this easily leads to a contradiction.

To state the next theorem we need to observe that the subgroup H generated by $\{g \in G;\ T_g \neq 0\}$ is commutative since T is a commutative domain.

THEOREM 1.6. *Let R be a Cohen–Macaulay G-graded T-algebra.*

(a) *If C is an indecomposable nonprojective module in $\mathrm{CM}(\mathrm{gr}\,R)_0$, there is an almost split sequence $0 \to A \to B \to C \to 0$ in $\mathrm{CM}(\mathrm{gr}\,R)_0$ if and only if $C_\mathfrak{p}$ is a projective $T_\mathfrak{p}$-module for each nonmaximal prime ideal $\mathfrak{p}$ in T.*

(b) *If A is an indecomposable noninjective object in $\mathrm{CM}(\mathrm{gr}\,R)_0$, there is an almost split sequence $0 \to A \to B \to C \to 0$ in $\mathrm{CM}(\mathrm{gr}\,R)_0$ if and only if $A_\mathfrak{p}$ is an injective object in $\mathrm{CM}(R_\mathfrak{p})$ for each nonmaximal prime ideal $\mathfrak{p}$ in T.*

(c) *If there is an almost split sequence $0 \to A \to B \to C \to 0$ in $\mathrm{CM}(\mathrm{gr}\,R)_0$, then $A = D\mathrm{Tr}_L(-\tilde{d})$, where $\tilde{d} = \deg X_1 + \ldots + \deg X_d$.*

Proof. This is a generalization of [4, Th. 1.1] (and [5]), and the proof is similar to the proof in [4]. We only make a few extra comments. We first recall that D denotes the duality $\mathrm{Hom}_T(\ ,T)$ between $\mathrm{CM}(\mathrm{gr}\,R)_0$ and $\mathrm{CM}(\mathrm{gr}\,R^{\mathrm{op}})_0$, Tr denotes the transpose and $\mathrm{Tr}_L = \Omega^d\mathrm{Tr}$, where Ω^d denotes the dth syzygy.

We note that for each $g \in G$ there is a degree zero isomorphism of graded T-modules $\mathrm{Hom}_R(B, C(g)) \simeq \mathrm{Hom}_R(B, C)(g)$ for all $g \in G$. Also the category $\mathrm{Mod}(\mathrm{gr}\,T)_0$ has enough injectives [11] and injective envelopes. When C is indecomposable in $\mathrm{CM}(\mathrm{gr}\,R)_0$ and $C_\mathfrak{p}$ is $T_\mathfrak{p}$-projective for each nonmaximal prime ideal $\mathfrak{p}$ in R, then $\underline{\mathrm{End}}_R(C)^{\mathrm{op}}$, the endomorphism ring of C modulo projectives, is a finite-dimensional k-algebra, and a crucial point is to show that there is a nonzero degree zero map $f\colon \Gamma \to k$ such that $f(\mathrm{rad}\,\Gamma_0) = 0$. Let $\mathfrak{n} = \text{gr-rad}\,\Gamma$. Then $\mathfrak{n} \cap \Gamma_0 = \mathrm{rad}\,\Gamma_0$, and k is the unique graded simple T-module. Hence we have a nonzero degree zero map $\Gamma/\mathfrak{n} \to k$, and we are done.

We also have the following information on right and left almost split maps.

PROPOSITION 1.7. *Let R be a G-graded T-algebra which is Cohen–Macaulay with isolated singularities.*

(a) *If P is indecomposable projective in $\mathrm{CM}(\mathrm{gr}\,R)_0$, there is a minimal right almost split map $E \to P$.*

(b) *If I is indecomposable injective in $\mathrm{CM}(\mathrm{gr}\,R)_0$, there is a minimal left almost split map $I \to F$.*

Proof. (a) By Lemma 1.5, gr-rad P is a graded maximal submodule of P. By [2] there is a Cohen–Macaulay approximation $g\colon E \to \text{gr-rad}\,P$ in $\mathrm{mod}(\mathrm{gr}\,R)_0$, that is, E is in $\mathrm{CM}(\mathrm{gr}\,R)_0$, and for each map $h\colon C \to \text{gr-rad}\,P$ in $\mathrm{mod}(\mathrm{gr}\,R)_0$, with C in $\mathrm{CM}(\mathrm{gr}\,R)_0$, there is some $t\colon C \to E$ with $gt = h$. Since the endomorphism rings of indecomposable objects in $\operatorname{mod}(\operatorname{gr} R)_0$ are local, it is easy to see that g: $E \to \operatorname{gr-rad} P$ can be chosen to be a minimal map. Hence g: $E \to P$ is a minimal right almost split map.

(b) follows from (a) by duality.

2. Completions of almost split sequences

Let R be a Cohen–Macaulay G-graded T-algebra with isolated singularities. In this section we assume in addition that if G has an element of finite order t, then t is invertible in k. In [6] we proved that if R is commutative positively $\mathbf{Z}$-graded, then almost split sequences in $\operatorname{CM}(\operatorname{gr} R)_0$ stay almost split under completion. This is not true in general, but we prove in this section that an almost split sequence goes to a direct sum of almost split sequences under completion. For this we need the following characterization of a direct sum of almost split sequences.

PROPOSITION 2.1. *Assume that C in $\operatorname{CM}(\hat{R})$ has no nonzero projective summands. Then the following are equivalent for an exact sequence*

$$0 \to D\operatorname{Tr}_L C \xrightarrow{f} B \xrightarrow{g} C \to 0 \quad \text{in } \operatorname{CM}(\hat{R}).$$

(a) *The sequence is a direct sum of almost split sequences.*

(b) *A map $h \in \operatorname{End}_R(C)$ factors through g: $B \to C$ if and only if $h \in \operatorname{rad} \operatorname{End}_R(C)$.*

Proof. (a)⇒(b). Let $C = C_1 \amalg \dots \amalg C_t$ be a decomposition of C into a direct sum of indecomposable modules and denote by

$$0 \to D\operatorname{Tr}_L C_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

the almost split sequences. For h in $\operatorname{End}_R(C)$, denote by h_{ij}: $C_i \to C_j$ the induced maps. Then $h \in \operatorname{rad} \operatorname{End}_R(C)$ if and only if each h_{ij} is not an isomorphism. If $h \in \operatorname{rad} \operatorname{End}_R(C)$, then h_{ij} must factor through g_j: $B_j \to C_j$, so that h factors through g: $B \to C$. If $h \notin \operatorname{rad} \operatorname{End}_R(C)$, then some map h_{ij}: $C_i \to C_j$ is an isomorphism, and hence does not factor through g_j: $B_j \to C_j$. Consequently h does not factor through g: $B \to C$.

(b)⇒(a). The assumption in (b) means that $\operatorname{Im} \operatorname{Hom}(C, g) = \operatorname{rad} \operatorname{End}(C) \subset \operatorname{End}(C)$. If

$$0 \to D\operatorname{Tr}_L C \xrightarrow{f'} B \xrightarrow{g'} C \to 0$$

is a direct sum of almost split sequences, we have $\operatorname{Im}(C, g') = \operatorname{rad} \operatorname{End}(C)$. By the theory of X-determined morphisms an exact sequence of the form

$$0 \to D\operatorname{Tr}_L C \xrightarrow{f''} E \xrightarrow{g''} C \to 0$$

is uniquely determined up to isomorphism by the $\operatorname{End}(C)^{\mathrm{op}}$-submodule

$\operatorname{Im}(C, g'')$ of $\operatorname{End}(C)^{\mathrm{op}}$ (see [1, Prop. 9.6]). Hence it follows that our two sequences are isomorphic.

We can now prove the main result of this section.

THEOREM 2.2. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an almost split sequence in* $\mathrm{CM}(\operatorname{gr} R)_0$. *Then* $0 \to \hat{A} \xrightarrow{\hat{f}} \hat{B} \xrightarrow{\hat{g}} \hat{C} \to 0$ *is a direct sum of almost split sequences in* $\mathrm{CM}(\hat{R})$.

Proof. C is indecomposable nonprojective in $\mathrm{CM}(\operatorname{gr} R)_0$. By Lemma 1.3, $\hat{C}$ has no nonzero projective summands. Since $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is almost split, we have $\operatorname{Im} \operatorname{Hom}_R(C, g) = \text{gr-rad} \operatorname{End}(C)^{\mathrm{op}}$. Since exactness is preserved by completion, we get $\operatorname{Im} \operatorname{Hom}_{\hat{R}}(\hat{C}, \hat{g}) = (\text{gr-rad} \operatorname{End}_R(C)^{\mathrm{op}})^\wedge$, which by Lemma 1.2 is $\operatorname{rad} \operatorname{End}_{\hat{R}}(\hat{C})^{\mathrm{op}}$. Then we are done by Proposition 2.1.

3. Finite Cohen–Macaulay type

Let R be a G-graded Cohen–Macaulay T-algebra with isolated singularities. In this section we compare the representation type for R and $\hat{R}$. We follow the outline for the corresponding result when R is commutative and $G = \mathbf{Z}$ [6].

In addition to the behavior of almost split sequences given in Theorem 2.2 we need the following result about right and left almost split maps.

PROPOSITION 3.1. (a) *Let P be indecomposable projective in* $\mathrm{CM}(\operatorname{gr} R)_0$ *and* $g\colon E \to P$ *a right almost split map in* $\mathrm{CM}(\operatorname{gr} R)_0$. *Then* $\hat{g}\colon \hat{E} \to \hat{P}$ *is right almost split in* $\mathrm{CM}(\hat{R})$.

(b) *Let I be an indecomposable injective object in* $\mathrm{CM}(\operatorname{gr} R)_0$ *and* $f\colon I \to F$ *a left almost split map in* $\mathrm{CM}(\operatorname{gr} R)_0$. *Then* $\hat{f}\colon \hat{I} \to \hat{F}$ *is left almost split in* $\mathrm{CM}(\hat{R})$.

Proof. (a) We have seen in Section 1 that a right almost split map $g\colon E \to P$ corresponds to a Cohen–Macaulay approximation $g\colon E \to \text{gr-rad}\, P$. From the theory of Cohen–Macaulay approximations [2] it follows that $\hat{g}\colon \hat{E} \to (\text{gr-rad}\, P)^\wedge$ is also a Cohen–Macaulay approximation. From Lemma 1.1 we know that $(\text{gr-rad}\, P)^\wedge = \operatorname{rad} \hat{P}$, and hence we get a right almost split map $\hat{g}\colon \hat{E} \to \hat{P}$.

(b) Similar to (a).

We now get the following main result of this section.

THEOREM 3.2. (a) *If R is of finite graded Cohen–Macaulay type, then $\hat{R}$ is of finite Cohen–Macaulay type, and for every indecomposable module C in* $\mathrm{CM}(\hat{R})$ *there is some indecomposable module B in* $\mathrm{CM}(\operatorname{gr} R)_0$ *such that C is a direct summand of $\hat{B}$.*

(b) *If $\hat{R}$ is of finite Cohen–Macaulay type and there is a bound on the order of the finite subgroups of G, then R is of finite graded Cohen–Macaulay type.*

Proof. (a) Assume that R is of finite graded Cohen–Macaulay type, and let

$B_1, \ldots, B_n$ be a complete set, up to shift, of indecomposable modules in $\mathrm{CM}(\mathrm{gr} R)_0$. Let $\mathscr{L}$ be the finite set of indecomposable modules in $\mathrm{CM}(\hat{R})$ which are summands of some $\hat{B}_i$. Then clearly the indecomposable projective $\hat{R}$-modules are in $\mathscr{L}$, and it is easy to see that $\mathscr{L}$ is closed under irreducible maps, by using Theorem 2.2 and Proposition 3.1 (see [6]). Then it follows from [3] that all indecomposable modules in $\mathrm{CM}(\hat{R})$ are in $\mathscr{L}$.

(b) Let A and B be indecomposable in $\mathrm{CM}(\mathrm{gr} R)_0$. If $\hat{A} \simeq \hat{B}$, there is by Lemma 1.4 some $g \in G$ such that $A \simeq B(g)$. Denote by N the bound on the order of subgroups of G of finite order. Then it follows from Lemma 1.3 that $\hat{A}$ decomposes into a direct sum of at most N indecomposable summands. From this it follows that if $\hat{R}$ is of finite Cohen–Macaulay type, then R is of finite graded Cohen–Macaulay type.

4. Localizations

Throughout this section we assume that R is a graded T-algebra with grading group G. In the previous sections we have been mainly concerned with relations between R and its completion $\hat{R}$, particularly properties of the completion functor from $\mathrm{gr}(R)_0$ to $\mathrm{mod}\,\hat{R}$. In addition to $\hat{R}$, the ring $R_\mathfrak{m}$ is also naturally associated with R. One of the major differences between $\hat{R}$ and $R_\mathfrak{m}$ is that while $\mathrm{mod}\,\hat{R}$ is always a Krull–Schmidt category, $\mathrm{mod}\,R_\mathfrak{m}$ is rarely a Krull–Schmidt category. However, it follows from Lemma 1.2 that if G has no nonzero elements of finite order, then $(\mathrm{mod}\,R_\mathfrak{m})_0$, the full subcategory of $\mathrm{mod}\,R_\mathfrak{m}$ consisting of those $R_\mathfrak{m}$-modules isomorphic to $A_\mathfrak{m}$ for some A in $\mathrm{mod}(\mathrm{gr}\,R)_0$, is a Krull–Schmidt category. For this reason, we assume throughout this section that G has no nonzero elements of finite order.

Our first aim is to study the full subcategory $\mathrm{CM}(R_\mathfrak{m})$ of $\mathrm{mod}\,R_\mathfrak{m}$ when $\mathrm{CM}(\mathrm{gr} R)_0$ is of finite type. To this end we assume now that R is a Cohen–Macaulay T-algebra with isolated singularities.

PROPOSITION 4.1. *Suppose* $\mathrm{CM}(\mathrm{gr} R)_0$ *is of finite type. Then the functor* $\mathrm{mod}(\mathrm{gr}\,R)_0 \to \mathrm{mod}\,R_\mathfrak{m}$ *given by* $A \mapsto A_\mathfrak{m}$ *induces a bijection between the isomorphism classes of indecomposable Cohen–Macaulay modules in* $\mathrm{mod}(\mathrm{gr} R)_0$ *and those in* $\mathrm{CM}(R_\mathfrak{m})$. *In particular,* $\mathrm{CM}(R_\mathfrak{m})$ *is a Krull–Schmidt category of finite type.*

Proof. Suppose A and B are indecomposable modules in $\mathrm{CM}(\mathrm{gr} R)_0$ such that $A_\mathfrak{m} \simeq B_\mathfrak{m}$. Then $\hat{A} \simeq \hat{B}$ and so $A \simeq B$ by [10, Lemma 5.8]. Thus the localization functor $\mathrm{mod}(\mathrm{gr} R)_0 \to \mathrm{mod}\,R_\mathfrak{m}$ induces an injection from the isomorphism classes of Cohen–Macaulay modules in $\mathrm{mod}(\mathrm{gr} R)_0$ to those in $\mathrm{mod}\,R_\mathfrak{m}$.

Next suppose A is an indecomposable Cohen–Macaulay module in $\mathrm{mod}\,R_\mathfrak{m}$. Then $\hat{A}$ is a Cohen–Macaulay module in $\mathrm{mod}\,\hat{R}$. Since $\mathrm{CM}(\mathrm{gr}\,R)_0$ is of finite type, by Theorem 3.2 there is some B in $\mathrm{CM}(\mathrm{gr}\,R)_0$ such that $\hat{A} \simeq \hat{B}$. Since

$\hat{B} \simeq \hat{B}_{\mathfrak{m}}$ we have $A \simeq B_{\mathfrak{m}}$ because modules in $\text{mod}\, R_{\mathfrak{m}}$ with isomorphic completions are isomorphic (slight generalization of [10, Lemma 5.8]). Hence B is an indecomposable module in $\text{CM}(\text{gr}\, R)_0$ such that $B_{\mathfrak{m}} \simeq A$, which finishes the proof of the proposition.

As an immedatie consequence of this result we have the following.

COROLLARY 4.2. *If there is an indecomposable Cohen–Macaulay $R_{\mathfrak{m}}$-module A whose endomorphism ring is not local, then R and $\hat{R}$ are of infinite Cohen–Macaulay type.*

We next want to consider the special case when R is a quasihomogeneous k-algebra, k a field. This means that $R = \coprod_{n \in \mathbf{Z}} R_n$ is a commutative graded ring with grading group $\mathbf{Z}$, the integers, satisfying:

(a) $R_0 = k$.

(b) R_i are finite-dimensional k-vector spaces with $R_i = 0$ for all $i < 0$.

(c) There are a finite number of homogeneous elements $x_1, \ldots, x_n$ such that $R = k[x_1, \ldots x_n]$.

Then R is a graded local ring with graded radical $(x_1, \ldots, x_n)$ which is a maximal ideal in R. Then of course $R_{\mathfrak{m}}$ is a local ring with radical $\mathfrak{m}R_{\mathfrak{m}}$. We now want to investigate the Krull–Schmidt subcategory $(\text{mod}\, R_{\mathfrak{m}})_0$ of $\text{mod}\, R_{\mathfrak{m}}$ in this special case.

PROPOSITION 4.3. *Suppose M is in $(\text{mod}\, R_{\mathfrak{m}})_0$ and $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ is a minimal projective resolution of M. Then $\Omega^i(M) = \text{Im}\, d_i$ is in $(\text{mod}\, R_{\mathfrak{m}})_0$ for all $i \geq 0$.*

Proof. Suppose A in $\text{mod}(\text{gr}\, R)_0$ is such that $A_{\mathfrak{m}} = M$. Let

$$\cdots \to Q_1 \xrightarrow{\bar{d}_1} Q_0 \xrightarrow{\bar{d}_0} A \to 0$$

be a minimal projective resolution of A in $\text{mod}(\text{gr}\, R)_0$. Then $\text{Ker}\, \bar{d}_i \subset \mathfrak{m}Q_i$ for all $i \geq 0$, which implies that

$$\cdots \to (Q_1)_{\mathfrak{m}} \xrightarrow{(\bar{d}_1)_{\mathfrak{m}}} (Q_0)_{\mathfrak{m}} \xrightarrow{(\bar{d}_0)_{\mathfrak{m}}} A_{\mathfrak{m}} \to 0$$

is a minimal projective resolution of $A_{\mathfrak{m}} \simeq M$ over $R_{\mathfrak{m}}$. Since $(\Omega^i_R(A))_{\mathfrak{m}} = \Omega^i(M)$ for all $i \geq 0$, $\Omega^i(M)$ is in $(\text{mod}\, R_{\mathfrak{m}})_0$ for all $i \geq 0$.

Similar arguments show the following.

PROPOSITION 4.4. *Suppose A and B are in $(\text{mod}\, R_{\mathfrak{m}})_0$. Then $\text{Ext}^i_{R_{\mathfrak{m}}}(A, B)$ and $\text{Tor}_i^{R_{\mathfrak{m}}}(A, B)$ are in $(\text{mod}\, R_{\mathfrak{m}})_0$ for all $i \geq 0$.*

Thus we see that the Krull–Schmidt subcategory $(\text{mod}\, R_{\mathfrak{m}})_0$ of $\text{mod}\, R$ is quite large. It would be interesting to have some description of this subcategory.

Now some of the modules in $(\text{mod}\, R_{\mathfrak{m}})_0$ can be described without reference to the graded ring R. For example, let $\mathfrak{r}$ be the radical of $R_{\mathfrak{m}}$. Then $R_{\mathfrak{m}}/\mathfrak{r}^i$ is in $(\operatorname{mod} R_{\mathfrak{m}})_0$ for all $i \geqslant 0$. Therefore each indecomposable summand of $\Omega^j(R_{\mathfrak{m}}/\mathfrak{r}^i)$ has a local endomorphism ring for all j and $i \geqslant 0$. This naturally suggests the following question. If S is a local ring with radical $\mathfrak{r}$, then does every indecomposable summand of $\Omega^j(S/\mathfrak{r}^i)$ have a local endomorphism ring for all j and $i \geqslant 0$? Or can this property be used to test in a fairly effective way whether a local ring is the localization of a quasihomogeneous ring at its graded radical? Clearly the same type of remarks pertain to the class of $R_{\mathfrak{m}}$-modules in $(\operatorname{mod} R_{\mathfrak{m}})_0$ which can be described solely in terms of $\operatorname{mod} R_{\mathfrak{m}}$ without reference to $\operatorname{mod}(\operatorname{gr} R)_0$.

5. Examples

This section is devoted to giving examples illustrating previous results.

A finite-dimensional G-graded k-algebra R is automatically Cohen–Macaulay with isolated singularities, and $\operatorname{CM}(\operatorname{gr} R)_0$ is the same as $\operatorname{mod}(\operatorname{gr} R)_0$. Since further $\hat{R} = R$, we have the following special case of Theorem 2.2.

PROPOSITION 5.1. *Let R be a finite-dimensional G-graded k-algebra, where G is a finite group whose order is invertible in k. If $0 \to A \to B \to C \to 0$ is an almost split sequence in* $\operatorname{mod}(\operatorname{gr} R)_0$, *then it is a direct sum of almost split sequences in* $\operatorname{mod} R$.

This result generalizes the following result about skew group ring constructions from [12].

COROLLARY 5.2. *Let Λ be a finite-dimensional k-algebra and G a group of automorphisms of Λ with the order of G invertible in k. If $0 \to A \to B \to C \to 0$ is an almost split sequence in* $\operatorname{mod} \Lambda$, *then* $0 \to \Lambda G \otimes_\Lambda A \to \Lambda G \otimes_\Lambda B \to \Lambda G \otimes_\Lambda C \to 0$ *is a direct sum of almost split sequences of ΛG-modules.*

Proof. ΛG is a G-graded algebra, where $(\Lambda G)_g = \Lambda_g$ for $g \in G$. The fuctor $\Lambda G \otimes_\Lambda$: $\operatorname{mod} \Lambda \to \operatorname{mod}(\operatorname{gr} \Lambda G)_0$ is an equivalence of categories [11]. The claim is then a direct consequence of Proposition 5.1.

Interesting commutative examples are provided by the graded algebras studied by Geigle and Lenzing [9]. As Lenzing pointed out to us there are examples where indecomposable graded modules decompose under completion, and this inspired us to generalize our results from [6].

For example, let $R = k[x, y, z]/(x^2 + y^3 + z^4)$, where the characteristic of k is not 2, and let G be the abelian group with generators $\vec{x}$, $\vec{y}$, $\vec{z}$ and relations $\vec{x}^2 = \vec{y}^3 = \vec{z}^4$. Then $G \simeq \mathbf{Z} \oplus \mathbf{Z}_2$. We define degree $x = \vec{x}$, degree $y = \vec{y}$, degree $z = \vec{z}$. $\hat{R}$ is known to be of finite Cohen–Macaulay type, and hence R is of finite graded Cohen–Macaulay type. The AR-quiver for R is the translation quiver $\mathbf{Z}\tilde{\mathbf{E}}_7$ [9], and for $\hat{R}$ it is known to be $\mathbf{Z}\mathbf{E}_6/\langle\tau\rangle$, where τ is the translation $D\operatorname{Tr}$.

Using that almost split sequences in $\mathrm{CM}(\mathrm{gr}\,R)_0$ go to a direct sum of almost split sequences in $\mathrm{CM}(\hat{R})$, that the ranks of the indecomposable modules are known in both cases and that rank is preserved by completion, it is easy to see that up to shift there are exactly two indecomposable modules in $\mathrm{CM}(\mathrm{gr}\,R)_0$ which decompose under completion. One module has rank 4 and the completion decomposes into a direct sum of two nonisomorphic indecomposable modules of rank 2, and the other one has rank 2 and the completion decomposes into a direct sum of two nonisomorphic modules of rank 1.

A similarly defined example from [9] is $R = k[x, y, z]/(x^2+y^2+z^2)$, where the characteristic of k is not 2. Then R is graded by the abelian group G with generators $\vec{x}, \vec{y}, \vec{z}$ and relations $\vec{x}^2 = \vec{y}^2 = \vec{z}^2$ in a similar way as before, and $G \simeq \mathbf{Z}\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$. Then R has AR-quiver $\mathbf{Z}\tilde{\mathbf{D}}_4$ [9] and $\hat{R}$ has AR-quiver $\bullet \overset{\rightarrow}{\underset{\leftarrow}{\rightleftarrows}} \bullet$. Here both indecomposable modules in $\mathrm{CM}(\hat{R})$ have rank 1. In $\mathrm{CM}(\mathrm{gr}\,R)_0$ there is only one indecomposable module, up to shift, which has rank greater than 1, and the completion of this module decomposes into a direct sum of two isomorphic modules of rank 1.

As mentioned in the introduction, Lenzing has given a different proof of Theorem 2.2 when G is abelian. He reduces to a skew group construction and the $\mathbf{Z}$-graded case.

If R is a G-graded T-algebra of finite graded Cohen–Macaulay type, it would be interesting to know if there is always a $\mathbf{Z}$-grading on R such that each homogeneous component R_n of R is finite-dimensional over k. It would also be interesting to know if any Cohen–Macaulay $\hat{T}$-algebra of finite Cohen–Macaulay type is of the form $\hat{R}$ for a Cohen–Macaulay $\mathbf{Z}$-graded T-algebra R such that each R_m is finite-dimensional over k, when k is an algebraically closed field of characteristic zero. For finite-dimensional algebras this is known to hold since every k-algebra of finite representation type is standard [7]. (But it is not true if k has characteristic 2.) In the case of two-dimensional tame orders it follows from the work on the classification of such orders of finite representation type [13]. Finally, the only known commutative rings of finite representation type are also completions of graded rings.

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