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REAL-VALUED CONTINUOUS FUNCTIONS AND THE SPAN OF CONTINUA

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Suppose that X is a metric space and that f is a function defined on X. In this paper we study the structure of the set

$$\Delta_f = \{\mathrm{dist}(x, y)\colon f(x) = f(y)\},$$

and prove that, under some circumstances, Δ_f contains an interval $[0, a]$, where $a = a(X) > 0$ is a number which depends only on X. The notion of the span, introduced by the first author [3], is essential to our approach.

Let X be a non-empty continuum, i.e. a connected compact metric space. We define the *span* $\sigma(X)$ of X to be the least upper bound of the set of real numbers a which satisfy the following condition: there exists a continuum $C \subset X \times X$ such that $p_1(C) = p_2(C)$ and $a \leqslant \mathrm{dist}(x, y)$ for each point $(x, y) \in C$, where p_1 and p_2 denote the projections of $X \times X$ onto X, i.e. $p_1(x, y) = x$ and $p_2(x, y) = y$ for $x, y \in X$ (see [3], p. 209).

THEOREM. *If X is a unicoherent locally connected continuum and $f\colon X \to R$ is a real-valued continuous function, then*

$$[0, \sigma(X)] \subset \Delta_f.$$

Proof. The inclusion is trivial for $\sigma(X) = 0$. Let us then assume $0 < \beta < \sigma(X)$, where β is an arbitrary number. By the definition of $\sigma(X)$, there exist a real number a and a continuum $C \subset X \times X$ such that

$$p_1(C) = p_2(C) = X_0, \tag{1}$$

$$\beta < a \leqslant \mathrm{Min}\{\mathrm{dist}(x, y)\colon (x, y) \in C\}, \tag{2}$$

and, by the compactness of X_0, there exists a point $x_0 \in X_0$ such that

$$f(x_0) = \mathrm{Min}\{f(x)\colon x \in X_0\}. \tag{3}$$

The closed set $B = \{(x, y)\colon \operatorname{dist}(x, y) = \beta\}$ separates the product $X \times X$ into the two sets

$$A_0 = \{(x, y)\colon \operatorname{dist}(x, y) < \beta\}, \quad A_1 = \{(x, y)\colon \operatorname{dist}(x, y) > \beta\},$$

and we have $a_0 = (x_0, x_0) \in A_0$ as well as $C \subset A_1$, according to (2). Let us take a point $a_1 \in C$ and notice that B separates the two points a_0 and a_1 of the continuum $X \times X$ which is unicoherent and locally connected (see [2], p. 434 and 438). Consequently, there is a continuum $K \subset B$ separating a_0 and a_1 in $X \times X$ (ibidem). It follows from (1) that both continua $\{x_0\} \times X_0$ and $X_0 \times \{x_0\}$ meet the continuum C. Hence the sets

$$C' = C \cup (\{x_0\} \times X_0), \quad C'' = C \cup (X_0 \times \{x_0\})$$

are continua joining a_0 and a_1 in $X \times X$. But since

$$C \subset A_1 \subset (X \times X) \setminus B \subset (X \times X) \setminus K,$$

the continuum K must meet both $\{x_0\} \times X_0$ and $X_0 \times \{x_0\}$; otherwise, one of the continua C', C'' would connect a_0 and a_1 in $(X \times X) \setminus K$ which is not the case. Thus there are points $x_1, x_2 \in X_0$ such that $(x_0, x_1) \in K$ and $(x_2, x_0) \in K$. By (3), we get $f(x_0) \leqslant f(x_1)$ and $f(x_2) \geqslant f(x_0)$. The function f being continuous, the connectedness of K implies the existence of a point $(x^*, y^*) \in K$ such that $f(x^*) = f(y^*)$, whence

$$\beta = \operatorname{dist}(x^*, y^*) \in \Delta_f,$$

as $K \subset B$. We conclude that each interior point of the interval $[0, \sigma(X)]$ belongs to Δ_f. However, since X is compact and f is continuous, the set Δ_f is closed in R; so it must contain the whole interval $[0, \sigma(X)]$ and the proof is completed.

By a *simple triod* we mean any metric space T which is the union $T = A_0 \cup A_1 \cup A_2$ of three arcs A_i having a common end-point v such that $A_i \cap A_{i+1} = \{v\}$ for $i = 0, 1, 2$, and the subscripts of A_j taken mod 3. We use the notation

$$w(T) = \operatorname{Min}\{\operatorname{Max}\{\varrho(x, A_{i+1} \cup A_{i+2})\colon x \in A_i\}\colon i = 0, 1, 2\},$$

where the distance $\varrho(x, A)$ between x and A is defined by the formula

$$\varrho(x, A) = \operatorname{Inf}\{\operatorname{dist}(x, a)\colon a \in A\}.$$

LEMMA. *If T is a simple triod, then $w(T) \leqslant \sigma(T)$.*

Proof. For $i = 0, 1, 2$, let $x_i \in A_i$ be a point such that

$$\varrho(x_i, A_{i+1} \cup A_{i+2}) = \operatorname{Max}\{\varrho(x, A_{i+1} \cup A_{i+2})\colon x \in A_i\},$$

whence $w(T) \leqslant \varrho(x_i, A_{i+1} \cup A_{i+2})$. Let

$$C_i = \{x_i\} \times (A_{i+1} \cup A_{i+2}), \quad C_i' = (A_{i+1} \cup A_{i+2}) \times \{x_i\},$$

and $C = C_0 \cup C_1' \cup C_2 \cup C_0'$. Thus $w(T) \leqslant \text{dist}(x, y)$ for each point $(x, y) \in C$, and all the sets C_i, C_i' are continua. On the other hand, each of the three intersections $C_0 \cap C_1'$, $C_1' \cap C_2$ and $C_2 \cap C_0'$ is non-empty since they contain the points (x_0, x_1), (x_2, x_1) and (x_2, x_0), respectively. It follows that C is a continuum, and $p_1(C) = p_2(C) = T$. Consequently, we have $w(T) \leqslant \sigma(T)$, by the definition of the span.

COROLLARY. *If X is a metric space, $T \subset X$ is a simple triod and $f: X \to R$ is a real-valued continuous function, then*

$$[0, w(T)] \subset \Delta_f.$$

Remarks. As is suggested by a construction due to Ingram [1], given any number $\varepsilon > 0$, there exists a simple triod T such that $\sigma(T) > 1$ and $w(T') < \varepsilon$ for each simple triod $T' \subset T$. We see that the inequality $w(T) \leqslant \sigma(T)$ from the above Lemma cannot be replaced by the equality. Also, our Corollary (or Theorem) provides an affirmative answer to a question asked in [4] by the first author whose forthcoming paper will discuss, among other things, the relationship between the span and the numbers $w(T)$ for some classes of continua.

The work represented here began with a proof of the above Corollary; the authors wish to thank Richard Bourgin and Bob Lee for a number of very helpful conversations in connection with the proof of this result.

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