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Kolmogorov problem in $W^r H^\omega[0,1]$ and extremal Zolotarev ω-splines

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Abstract

The main result of the paper, based on the Borsuk Antipodality Theorem, describes extremal functions of the Kolmogorov–Landau problem

$$(*) \qquad f^{(m)}(\xi) \to \sup, \quad f \in W^rH^\omega[\xi, b], \ \|f\|_{\mathbb{C}[a,b]} \le B,$$

for all $0 < m \le r$, $\xi \le a$ or $\xi = (a+b)//2$, all $B > 0$ and concave moduli of continuity ω on $\mathbb{R}_+$. It is shown that any extremal function $\mathcal{Z} = \mathcal{Z}_{B,r,m,\omega,\xi}$ of the problem $(*)$ enjoys the following two characteristic properties. First, the function $\mathcal{Z}^{(r)}(\cdot) - \mathcal{Z}^{(r)}(\xi)$ is extremal for the problem

$$(**) \qquad \int_\xi^b h(t)\psi(t)\,dt \to \sup, \quad h \in H^\omega[\xi, b], \ h(\xi) = 0,$$

for an appropriate choice of the kernel ψ with a finite number of sign changes on $[\xi, b]$. Second, the function $\mathcal{Z}$ equioscillates $n = n(B, r, m, \omega, \xi) \ge r+1$ times on the interval $[a, b]$ between $-B$ and B. This analogy with the properties of extremal functions in the linear case $\omega(t) = t$ of the problem $(*)$ makes it natural to call these functions $\mathcal{Z}$ the Zolotarev and Chebyshev ω-splines.

As in the linear case $\omega(t) = t$, the solution of the problem $(*)$ leads to the qualitative description of extremal functions and sharp additive inequalities for intermediate derivatives in the celebrated Kolmogorov problems on the infinite intervals $I = \mathbb{R}$ or $\mathbb{R}_+$:

$$\|f^{(m)}\|_{\mathbb{L}_\infty(I)} \to \sup, \quad f \in W^rH^\omega(I), \ \|f\|_{\mathbb{L}_\infty(I)} \le B, \ 0 < m \le r.$$

0. Introduction

0.1. Inequalities for derivatives of polynomials. A. A. Markov [45], V. A. Markov [46] and S. N. Bernstein [12] were among the first who investigated the properties of the algebraic polynomials P_n of degree n which yield the maximum modulus for the derivatives at a fixed point ξ of a finite interval $[a, b]$. However, the inequalities

$$\max_{x\in[-1,1]} |P_n^{(m)}(x)| \le \frac{n^2(n^2-1)\dots(n^2-(m-1)^2)}{1\cdot 3\dots(2m-1)} \|P_n\|_{\mathbb{C}[-1,1]}, \quad 0 < m < n, \tag{1.1}$$

of the Markov brothers (see also S. N. Bernstein [11] or R. J. Duffin and A. C. Schaeffer [53]) or Bernstein's refinement

$$|P_n^{(m)}(x)| \le \left(\frac{m}{1-x^2}\right)^{m/2} \frac{n!}{(n-m)!} \|P_n\|_{\mathbb{C}[-1,1]}, \quad 0 < m < n, \tag{1.2}$$

neither gave the exact constant $C_{n,m}(x) = \sup_{P_n} |P_n^{(m)}(x)| \cdot \|P_n\|^{-1}_{\mathbb{C}[-1,1]}$ nor described the extremal functions for the sharp inequality

$$|P_n^{(m)}(x)| \le C\|P_n\|_{\mathbb{C}[-1,1]}, \quad x \in [-1, 1]. \tag{1.3}$$

P. L. Chebyshev and E. I. Zolotarev [62] set and successfully solved the problem of finding a polynomial of degree n with *one* or *two* fixed leading coefficients, which deviates least from zero on $[0, 1]$ (see also N. I. Akhiezer [1]). Finally, E. V. Voronovskaya [60] and V. A. Gusev [24] took advantage of Chebyshev and Zolotarev polynomials in carrying the problem of extremal functions for sharp inequalities (1.3) to a complete solution. The reader is referred to Voronovskaya's monograph [61] for a detailed discussion of various extremal problems for polynomials.

0.2. General problem of sharp inequalities for intermediate derivatives. Let I be either the entire line $\mathbb{R}$ or the half-line $\mathbb{R}_+$. Let also $p, s, q \in [1, \infty]$, and $r, m \in \mathbb{N}$ with $m < r$.

Definition 0.2.1. A function f belongs to the class $W^r_{p,s}(I)$ if $f^{(r-1)}$ is absolutely continuous on any interval $[\sigma, \xi] \subset I$, and both norms $\|f\|_{\mathbb{L}_p(I)}$ and $\|f^{(r)}\|_{\mathbb{L}_s(I)}$ are finite.

The first results concerning inequalities for derivatives of functions $f \in W^r_{p,s}(I)$ in the multiplicative form

$$\|f^{(m)}\|_{\mathbb{L}_q(I)} \le K\|f\|^{\alpha}_{\mathbb{L}_p(I)} \|f^{(r)}\|^{\beta}_{\mathbb{L}_s(I)}, \tag{2.1}$$

are due to E. Landau [40] and J. Hadamard [25], who constructed extremal functions for the sharp inequalities (2.1) in the case $m = 1$, $r = 2$, $p = q = s = \infty$ for $I = \mathbb{R}_+$ and $I = \mathbb{R}$, respectively.

V. N. Gabushin [22] described the exponents α and β in the inequalities (2.1): if $(r-m)/p + m/s \geq r/q$, then α and β can be uniquely determined:

$$\alpha = \frac{r-m-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}}, \quad \beta = 1-\alpha.$$

In the late 1930's G. E. Shilov [14] found sharp inequalities (2.1) in the case $p=q=s=\infty$, $I=\mathbb{R}$, $2\leq r\leq 5$, and formulated the hypothesis that proved to be true: *the set of extremal functions for the inequality* (2.1) *for* $p=q=s=\infty$ *and* $I=\mathbb{R}$ *coincides with the functions of the form* $f(t)=\gamma\phi_{\lambda,r}(t+\rho)$ *for* $\gamma,\rho\in\mathbb{R}$ *and* $\lambda\in\mathbb{R}_+$, *where* $\phi_{\lambda,r}$ *is a* $2\pi/\lambda$*-periodic function such that* $\phi^{(r)}_{\lambda,r}(t)=\operatorname{sign}\sin(\lambda t)$. These functions had occurred in the works of J. Favard [20], N. I. Akhiezer and M. G. Krein [2] and even earlier in Euler's investigations; sometimes they are referred to as the *Euler splines*.

In 1939 A. N. Kolmogorov [33] confirmed Shilov's hypothesis and characterized the sharp constants in the inequality (2.1) in the case $p=q=s=\infty$, $I=\mathbb{R}$:

$$\|f^{(m)}\|_{\mathbb{L}_\infty(\mathbb{R})} \leq c_{rm}\|f\|^{1-m/r}_{\mathbb{L}_\infty(\mathbb{R})}\|f^{(r)}\|^{m/r}_{\mathbb{L}_\infty(\mathbb{R})}, \tag{2.2}$$

where $c_{rm} := K_{r-m}/K_m^{1-m/r}$, and

$$K_l := \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(l+1)}}{(2\nu+1)^{l+1}}, \quad l\in\mathbb{Z}_+,$$

are known as the *Favard constants*. A. S. Cavaretta [16], [18] suggested an elementary proof and refinements of the Kolmogorov inequalities (2.2).

Kolmogorov's result led to the development of the theory of sharp inequalities for intermediate derivatives on $\mathbb{R}$ and $\mathbb{R}_+$. Since 1939, the complete solution of the *problem of sharp constants*

$$K = K^{r,m}_{p,q,s} = \sup_{f\in W^r_{p,s}(I),\ f\neq 0} \|f^{(m)}\|_{\mathbb{L}_q(I)}\cdot\|f\|^{-\alpha}_{\mathbb{L}_p(I)}\cdot\|f^{(r)}\|^{-\beta}_{\mathbb{L}_s(I)}$$

in (2.1) for all m,r with $0<m<r$, and some fixed constants p,q,s, has been obtained in three cases for the entire line $\mathbb{R}$: G. H. Hardy, J. Littlewood, G. Pólya [26], $p=q=s=2$; E. M. Stein [56], $p=q=s=1$; L. V. Taĭkov [58], $q=\infty$, $p=s=2$; and in two cases for the half-line $\mathbb{R}_+$: N. P. Kuptsov [39], $p=q=s=2$; V. N. Gabushin [23], $q=\infty$, $p=s=2$. Moreover, a series of results by B. Sz.-Nagy [57], H. Cartan [15], V. V. Arestov [3], V. N. Gabushin [21] and G. G. Magaril-Il'yaev [42] deal with the problem of exact constants in the inequality (2.1) in special partial cases (i.e. for some fixed m,r and p,q,s). A comprehensive survey of the Kolmogorov inequalities for various choices of p,q,s and r,m in (2.1) can be found in the commentary by V. M. Tikhomirov and G. G. Magaril-Il'yaev in [34].

0.3. Kolmogorov–Landau problems in Sobolev classes $W^n_\infty(I)$. If a calculation of the exact constant K in (2.1) does not yield an explicit expression (as in the Kolmogorov case), the solution of the problem is understood in the sense of the qualitative characterization of extremal functions for the inequality (2.1). Due to the homogeneity of the classes $W^r_{p,s}(I)$, it is sufficient to consider classes of functions f with norms $\|f^{(r)}\|_{\mathbb{L}_s(I)}$ bounded by 1.

DEFINITION 0.3.1. Let $I = [a, b]$, $-\infty \le a < b \le \infty$. The *Sobolev class* $W_\infty^n(I)$ is defined as follows:

$$W_\infty^n(I) := \{f \in W_{\infty,\infty}^n(I) \mid \|f^{(n)}\|_{\mathbb{L}_\infty(I)} \le 1\}, \quad n \in \mathbb{N}. \tag{3.1}$$

In the present section we discuss the *Kolmogorov problem*

$$\|f^{(m)}\|_{\mathbb{L}_\infty(I)} \to \sup, \quad f \in W_\infty^n(I),\ \|f\|_{\mathbb{L}_\infty(I)} \le B,\ 0 < m < n,\ I = \mathbb{R} \text{ or } \mathbb{R}_+, \tag{3.2}$$

and the *Kolmogorov–Landau problem* on the finite interval $[a, b]$:

$$f^{(m)}(\xi) \to \sup, \quad f \in W_\infty^n[a, b],\ \|f\|_{\mathbb{L}_\infty[a,b]} \le B,\ 0 < m < n,\ \xi \in \mathbb{R}. \tag{3.3}$$

Note that the inequalities of (3.3) in the Sobolev classes $W_\infty^n[a, b]$ can be viewed as extensions of the corresponding Markov–Bernstein inequalities for polynomials of degree n with leading coefficient $1/n!$. Moreover, the familiar Chebyshev and Zolotarev polynomials (cf. [1]) serve as extremal functions of problems (3.3) for all sufficiently large B. Furthermore, the maximizing functions of (3.2) and (3.3) inherit extremal structural properties of the aforementioned polynomials. In the remaining part of the section we briefly mention some results relevant to the content of this paper.

The problem (3.2) for $r = 2$, $I = [0, 1]$ was solved simultaneously by M. Sato [52] and A. Zvyagintsev and A. Lepin [63]. The characterization of extremal functions and sharp constants in the corresponding inequalities of (3.2) for $r = 2$ and $I = \mathbb{R}_+$ is due to A. P. Matorin [47] and S. B. Stechkin [55].

In 1970, A. S. Cavaretta and I. J. Schoenberg [17] described the solution of the problem (3.2) for $I = \mathbb{R}_+$. The extremal function for (3.2) is the *Chebyshev perfect spline* $T(x)$ uniquely characterized by the property of infinite equioscillation between $-B$ and B. We remark that both the Euler and the Chebyshev extremal perfect splines of (3.2) for $I = \mathbb{R}$ and $\mathbb{R}_+$ could be obtained as the limiting functions of Chebyshev perfect splines with a fixed number of knots constructed by V. M. Tikhomirov [59] (see also S. Karlin [31]).

In 1975, the Kolmogorov–Landau extrapolation problem (3.3) for all $\xi \in \mathbb{R} \setminus (a, b)$ was treated by S. Karlin [29], [30], who constructed the family of *Zolotarev perfect splines* $\{\mathcal{Z}_B\}_{B>0}$. For each $B > 0$, the function $\mathcal{Z}_B$ of norm B has $n = n(B) \ge 0$ knots and oscillates $n + r + 1$ times between $B = \|\mathcal{Z}_B\|_{\mathbb{C}[0,1]}$ and $-B$. In particular, the classical Zolotarev polynomials are extremal for the problem (3.3) for all ξ outside (a, b) and all B exceeding the norm of the corresponding Chebyshev polynomial. Finally, A. Pinkus [50] offered the solution of the pointwise problem (3.3) for $\xi \in (a, b)$ that generalizes the Voronovskaya–Gusev results [61].

By definition, functions $f \in W_\infty^{r+1}(I)$ obey the restriction $\|f^{(r+1)}\|_{\mathbb{L}_\infty(I)} \le 1$. This inequality is equivalent to the constraint $\omega(f^{(r)}; t) \le t$, where $\omega(g; t) := \inf_{|x-y|\le t} |g(x) - g(y)|$ stands for the *modulus of continuity* of the continuous function g.

In our generalizations, we consider constraints of the form $\omega(f^{(r)}; t) \le \omega(t)$ for some fixed concave modulus of continuity ω. This discussion leads us to the following setting of new problems and formulation of results in the theory of function classes defined by a common majorizing modulus of continuity.

0.4. Function classes $W^r H^\omega(I)$. In the Jackson inequalities the errors of approximation of an individual function $f \in \mathbb{C}^r[a, b]$ by a specified finite-dimensional subspace are expressed in terms of the modulus of continuity of the rth derivative of f (consult, e.g., [27] and [35]–[37]). Instead, S. M. Nikol'skiĭ suggested considering classes of functions with a common bounding concave modulus of continuity ω.

DEFINITION 0.4.1. A function $\omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *concave modulus of continuity* if the following conditions are satisfied:

(4.1) (i) $\omega(0) = 0$;

(ii) $\omega(t_1) \le \omega(t_2)$ for $t_2 > t_1 \ge 0$;

(iii) $\omega(\alpha t_1 + (1-\alpha)t_2) \ge \alpha\omega(t_1) + (1-\alpha)\,\omega(t_2)$

for $\alpha \in (0,1),\ t_1, t_2 \in \mathbb{R}_+,\ t_1 \ne t_2$.

If *strict* inequality holds in (4.1)(iii), then ω is a *strictly concave modulus of continuity.*

DEFINITION 0.4.2. Let ω be a concave modulus of continuity. The function class $W^rH^\omega[a,b]$ is defined by

$$W^rH^\omega[a,b] := \{x \in \mathbb{AC}^r[a,b] \mid \omega(x^{(r)};t) \le \omega(t) \text{ for } t \in [0, b-a]\}, \tag{4.2}$$

where $\mathbb{AC}^r[a,b]$ is the subset of functions in $\mathbb{C}^r[a,b]$ with absolutely continuous derivative $f^{(r)}$. For $r = 0$ we use the notation

$$H^\omega[a,b] := W^0H^\omega[a,b].$$

The standard Sobolev class $W_\infty^{r+1}[a,b]$ can be viewed as a particular case of $W^rH^{\widetilde{\omega}}[a,b]$ for $\widetilde{\omega}(t) = t$. Another example is provided by the *Hölder moduli of continuity* $\omega_\alpha(t) = t^\alpha,\ 0 < \alpha \le 1$. In this case, we use the notation

$$W^rH^\alpha[a,b] := W^rH^{\omega_\alpha}[a,b].$$

We also consider the subset of functions vanishing at a point $\tau \in [a,b]$:

$$H_\tau^\omega[a,b] := \{f \in H^\omega[a,b] \mid f(\tau) = 0\}. \tag{4.3}$$

The classes $W^rH^\omega[a,b]$ were introduced in 1946 by S. M. Nikol'skiĭ [49] in connection with approximation of functions by their Fourier sums. Naturally, a number of problems arose concerning best characteristics of approximation of these classes by algebraic and trigonometric polynomials and other finite-dimensional subspaces.

The problem of *n-widths* of the periodic classes $\widetilde{W^rH^\omega} = \{f \in W^rH^\omega(\mathbb{R}) \mid f(t+2\pi) = f(t) \text{ for } t \in \mathbb{R}\}$ in the uniform and integral norm was treated by N. P. Korneĭchuk [35]–[37], V. I. Ruban [51] and V. P. Motornyĭ and V. I. Ruban [48].

An excellent summary of other results in the theory of W^rH^ω classes up to 1985 and a comprehensive bibliography are given by N. P. Korneĭchuk in his survey article [38] dedicated to the contribution of S. M. Nikol'skiĭ to Approximation Theory.

0.5. The Kolmogorov–Landau problem in $W^rH^\omega(I)$. The main objective of this paper is to characterize extremal functions of the problems

$$f^{(m)}(\xi) \to \sup, \quad f \in W^rH^\omega(I),\ \|f\|_{\mathbb{L}_\infty(I)} \le B, \tag{5.1}$$

for all concave moduli of continuity ω, $0 < m \le r$, an arbitrary finite interval $I = [a,b]$, all positive B, and for $\xi \in \mathbb{R} \setminus (a,b)$ and $\xi = \frac{1}{2}(a+b)$.

This work concludes the following cycle of investigations in the area of sharp Kolmogorov–Landau inequalities in $W^rH^\omega(I)$. In [6] we described solutions $\mathcal{Z}_B$ of the problem (5.1) on $I = [0,1]$ for $\xi = 0$, $r \in \mathbb{N}$, and all *sufficiently large* B. They inherit the corresponding properties of classical Zolotarev polynomials: they have $r+1$ points of alternance and the *full modulus of continuity* on $[0,1]$: $\omega(\mathcal{Z}_B^{(r)};t) = \omega(t)$ for $t \in [0,1]$. This analogy with the case of the linear modulus of continuity $\omega(t) = t$ makes it natural to call such functions the *Zolotarev ω-polynomials*.

In the process of solving (5.1) for $0 < m < r$ and all positive B, we ran into the necessity of characterizing extremal functions of the problem

$$\int_a^b h(t)\psi(t)\,dt \to \sup, \quad h \in H_a^\omega[a,b], \tag{5.2}$$

for kernels ψ with zero mean on $[a,b]$ and a finite number n–1 of sign changes in $[a,b]$. The detailed description of structural features and limiting properties of extremal functions of (5.2) can be found in Chapter 1 of our paper [4]. In Chapter 1 of the present paper we list the main results of [4] employed in our constructions.

The solution of all variations of problems (5.1) for $m = r$ and $\xi \in \mathbb{R} \setminus (a,b)$ requires the description of the form and properties of extremal functions of the problem

$$\int_a^b h(t)\psi(t)\,dt \to \sup, \quad h \in H_a^\omega[a,b], \tag{5.3}$$

for kernels ψ with a nonzero mean on $[a,b]$ and a finite number of sign changes in $[a,b]$. The reader is referred to [5] for the solution of (5.3), while the corresponding formulas for extremal functions of (5.3) are given in Theorem 1.2.1 of Chapter 1.

Furthermore, in [5] we also characterize the solution of the problem

$$\int_a^0 h(t)\psi_1(t)\,dt + \int_0^b h(t)\psi_2(t)\,dt \to \sup, \quad h \in H_0^\omega[a,b],\ a < 0 < b, \tag{5.4}$$

where the kernels ψ_1 and ψ_2 have monotonically ordered countable sets of sign changes in the respective intervals $[a,0]$ and $[0,b]$. Extremal functions of (5.4) for appropriate choices of ψ_1 and ψ_2 feature as the rth derivatives of solutions of the pointwise Kolmogorov–Landau problem (5.1) for $m = r$ and $\xi \in (a,b)$.

The description of the extremal functions of (5.2)–(5.4), called *perfect ω-splines*, was followed by the solution of the problem of sharp Kolmogorov–Landau inequalities associated with the problem (5.1). It turns out that if $I = [0,1]$ and even $I = \mathbb{R}$ or $\mathbb{R}_+$, then sharp Kolmogorov–Landau inequalities assume the additive form: for all $f \in W^rH^\omega(I)$ with $\|f\|_{\mathbb{L}_\infty(I)} \le B$,

$$|f^{(m)}(\xi)| \le C_B B + D_B(\omega'), \tag{5.5}$$

for some constant C_B and integral functional $\omega' \mapsto D_B(\omega')$. We find expressions for $C_B(r,m,B,\omega,\xi)$ and $D_B(\omega';r,m,B,\omega,\xi)$ in Chapter 3 for the endpoints $\xi = a$ or b, in Chapter 5 for $\xi \in \mathbb{R} \setminus [a,b]$, and in Chapter 7 for $\xi = \frac{1}{2}(a+b)$.

As already mentioned, the construction of the Chebyshev perfect polynomial splines in the case of $\omega(t) = t$ is due to V. M. Tikhomirov [59]. In [10] we constructed families of Chebyshev functions $\{\mathcal{Z}_n(t) = \mathcal{Z}_{n,r,m,\omega}(t)\}_{n\geq r}$ in $W^rH^\omega[0,1]$ extremal for various problems of Approximation Theory. Like the corresponding polynomial Chebyshev spline, the function $\mathcal{Z}_n$ has a complete set of $n+2$ alternance points and $n-r$ knots on $[0,1]$. An application of the limiting procedure to the rescaled and normalized Chebyshev functions $\mathcal{Z}_n$ then enabled us to construct the set of extremal functions of the Kolmogorov problem

$$\|f^{(m)}\|_{\mathbb{L}_\infty(I)} \to \sup, \quad f \in W^rH^\omega(I),\ \|f\|_{\mathbb{L}_\infty(I)} \leq B, \tag{5.6}$$

in the Hölder classes $W^rH^\alpha(\mathbb{R}_+)$ or $W^rH^\alpha(\mathbb{R})$, i.e. for $I = \mathbb{R}_+$ or $\mathbb{R}$ and $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$. This result generalizes the Cavaretta–Schoenberg [17] and Kolmogorov [33] solution of the problem (5.6) for $\omega(t) = t$. Finally, the problem (5.6) for $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and arbitrary concave moduli of continuity ω on $\mathbb{R}_+$ was solved by different methods in [7] or [8], and [9].

0.6. Labeling of internal references. The paper divides into chapters, sections and subsections. Definitions or notations, theorems, propositions or lemmas are labeled as follows. For example, Theorem K.L.M is the Mth theorem in Chapter K, Section K.L. We also assign labels to the formula tags in the following manner. The two-entry tag (X.Y) is used to identify the Yth formula in Section C.X of the current Chapter C. However, whenever reference is made outside the current Chapter C, we use three-entry tags (B.X.Y) to refer to the Yth formula in Section B.X of Chapter B.

0.7. Organization of the paper. In general, one can judge the content of each chapter and section by its title. Chapter 1 starts with the introduction of the notion of a simple kernel Ψ and the rearrangement $\Re(\Psi;\cdot)$ of the simple kernel. The important Korneĭchuk lemma describes the extremal functions and the numerical value of the maximum for the problem (5.2), if ψ is the derivative of a simple kernel Ψ (i.e. ψ is an integrable function with zero mean and one sign change on $[a,b]$). Then we present the major fact lying in the foundation of the theory of extremal problems in W^rH^ω: Theorem 1.2.1 on the structural and limiting properties of extremal functions of problems (5.2) and (5.3), and the quantitative solution of these problems.

In Chapter 2 we list fundamental results such as the Borsuk theorem, the Chebyshev theorem, and a series of auxiliary technical facts employed in our proofs.

Chapter 3 begins with a review of classical Chebyshev and Zolotarev perfect splines in $W^{r+1}_\infty[0,1]$. Then we obtain numerical differentiation formulae and derive sufficient conditions of extremality of $f \in W^rH^\omega[0,1]$ for the problem (5.1) for $\xi = 0$.

Chapter 4 is a core of the paper. We prove one of the main results of the paper, Theorem 3.3.1, which describes the family of Chebyshev ω-splines $\{\mathcal{Z}_B\}_{B>0}$ of the Kolmogorov–Landau problem for the endpoints of the basic interval $[0,1]$.

In Chapter 5 we describe the solution of the extrapolation problem (5.1) for all ξ outside $[a,b]$ and $0 \leq m \leq r$. As we have remarked, the solution of (5.1) for $\xi \in (a,b)$ and $m = r$ requires the characterization of extremal functions for the problem

$$\int_a^b h(t)\psi(t)\,dt \to \sup, \quad h \in H^\omega_\xi[a,b], \tag{7.1}$$

for kernels ψ with a nonzero mean. Chapter 6 offers a detailed analysis of such functions.

In Chapter 7 we solve the extremal pointwise Landau problem (5.1) in the case of the midpoint $\xi = \frac{1}{2}(a+b)$. Finally, in Appendix A we reconstruct the Chebyshev polynomial splines, and in Appendix B we give a new topological construction of Zolotarev polynomial splines.

0.8. References to theorems. We conclude the introductory chapter by giving the list of references to all results used in the paper.

Proposition 1.1 is proved in [35], Lemma 1.1.2 is verified in [10]. The full proof of Korneĭchuk's Lemma 1.1.3 is given in [35]; see also [36], [37]. The fundamental Theorem 1.2.1 on the structure of extremal functions of the problem (5.2) for kernels ψ with zero mean is established in [4], while the solution of (5.3) for kernels ψ with a nonzero mean is obtained in [5]. The reader is referred to [4], [5] for the proof of all corollaries to Theorem 1.2.1 mentioned in Chapter 1.

Borsuk's Antipodality Theorem 2.1.1 originates in [13], a modern proof is offered in [19]. The proof of Chebyshev's Theorem 2.1.2 on the structure of the best polynomial approximator of a continuous function can be found in [35] or [37]. The reader can consult [10] for the proof of Proposition 2.2.1 and Lemmas 2.2.3 and 2.2.4. Proposition 2.2.2 on the existence of perfect splines satisfying the zero boundary conditions is formulated in [44] or [37].

V. M. Tikhomirov initially constructed the Chebyshev perfect splines (Theorem 3.0.1) in [59]; we also recommend the method based on Borsuk's theorem (consult Appendix A or [44]). The monotonicity of the norms of Chebyshev splines in Proposition 3.0.2 is under consideration in [59], [44], [29], [30]. S. Karlin's Theorem 3.0.3 from [29] or [30] (see also our proof in Appendix B) describes the structural properties of Zolotarev splines of a given norm.

All results in Chapter 6 concerning the solution of the problem (5.4) are borrowed from our paper [5]. Finally, Lemma 7.1.1 is stated in [10].

1. Extrema of functionals in $H^\omega[a,b]$ and perfect ω-splines

1.1. Introduction to the theory of the function classes $H^\omega[a,b]$

1.1.1. *Properties of concave moduli of continuity*

Proposition 1.1.1. *Let ω be a concave modulus of continuity on $\mathbb{R}_+$. Then*:

(a) *at any point $x > 0$, ω has one-sided derivatives*

$$\omega'_-(x) = \lim_{h\to 0+} \frac{\omega(x) - \omega(x-h)}{h}, \quad \omega'_+(x) = \lim_{h\to 0+} \frac{\omega(x+h) - \omega(x)}{h};$$

(b) ω'_+ *and* ω'_- *do not increase on* $(0,\infty)$, *and*

$$\omega'_+(x) \le \omega'_-(x), \quad x > 0;$$

(c) ω *is absolutely continuous on* $(0,\infty)$.

In this paper we make the following choice from the equivalence class of summable functions, defining the nonincreasing derivative ω' *everywhere* on $\mathbb{R}_+$.

Definition 1.1.1. Let ω be a concave modulus of continuity on $\mathbb{R}_+$. We put

$$\omega'(u) := \tfrac{1}{2}[\omega'_+(u) + \omega'_-(u)], \quad u > 0. \tag{1.1}$$

Lemma 1.1.2. *Let $a < b < c$ and ω be a concave modulus of continuity. Extend $h \in H^\omega[a,b]$ to $[a,c]$ by setting*

$$h(t) = h(b) + \int_b^t \chi\omega'(t-a)\,dt, \quad t \in [b,c],\ \chi \in \{-1,1\}.$$

Then $h \in H^\omega[a,c]$.

1.1.2. *Simple kernels $\Psi(\cdot)$ and their rearrangements $\Re(\Psi;\cdot)$.* The Korneĭchuk lemma describes extremal functions of the functional

$$h \mapsto \int_a^b h(t)\psi(t)\,dt, \quad h \in H^\omega[a,b], \tag{1.2}$$

where ψ is the derivative of a simple kernel on $[a,b]$.

Definition 1.1.2. Let $\psi(\cdot) \in \mathbb{L}_1[a,b]$ be such that $\int_a^b \psi(x)\,dx = 0$ and for some points a', b' with $a < a' \le b' < b$,

(1.3) (i) $\psi(x) < 0$ for a.e. $x \in [a,a']$;

(ii) $\psi(x) = 0$ for a.e. $x \in [a',b']$;

(iii) $\psi(x) > 0$ for a.e. $x \in [b',b]$.

Then $\Psi(x) = \xi\int_a^x \psi(t)\,dt$, $a \le x \le b$, $\xi \in \{1,-1\}$, is called a *simple kernel*.

If Ψ is a simple kernel, the equation $|\Psi(t)| = y$, for $0 < y < \|\Psi\|_{\mathbb{C}[a,b]}$, has precisely two solutions: $\alpha_y \in (a,a')$ and $\beta_y \in (b',b)$ (see Figure 1.1.1). The value of the maximum of the functional (1.2) is expressed in terms of the rearrangement of the simple kernel Ψ.

Definition 1.1.3. Let $\Psi(x)$, $a \le x \le b$, be a simple kernel and $c := \frac{1}{2}(a'+b')$. Let $\rho : [a,c] \to [c,b]$ be derived from the equations

$$\begin{cases} \Psi(t) = \Psi(\rho(t)), & t \in [a,a'],\ \rho(t) \in [b',b], \\ \rho(t) = a' + b' - t, & t \in [a',c]. \end{cases} \tag{1.4}$$

The *rearrangement* $\Re(\Psi;t)$, $0 \le t \le b-a$, of the simple kernel Ψ is defined as follows:

$$\Re(\Psi;t) := \begin{cases} \|\Psi\|_{\mathbb{C}[a,b]}, & t \in [0,b'-a'], \\ |\Psi(y_t)|, & t \in (b'-a',b-a],\ y_t \in [a,a'],\ \rho(y_t) - y_t = t. \end{cases} \tag{1.5}$$

Figure 1.1.1 also illustrates the graph of the rearrangement of a simple kernel Ψ. A systematic exposition of the properties of rearrangements is given in the monograph [26] by G. G. Hardy, G. E. Littlewood and G. Pólya and in A. Zygmund's book [64].

1.1.3. *The Korneĭchuk lemma*

Lemma 1.1.3. *Let $\Psi(t) := \int_a^t \psi(y)\,dy$, $a \le t \le b$, be a simple kernel whose derivative ψ satisfies* (1.3). *Let ω be a concave modulus of continuity. Then*

$$\sup_{f\in H^\omega[a,b]} \int_a^b f(t)\psi(t)\,dt = \int_0^{b-a} \Re(\Psi;t)\omega'(t)\,dt, \tag{1.6}$$

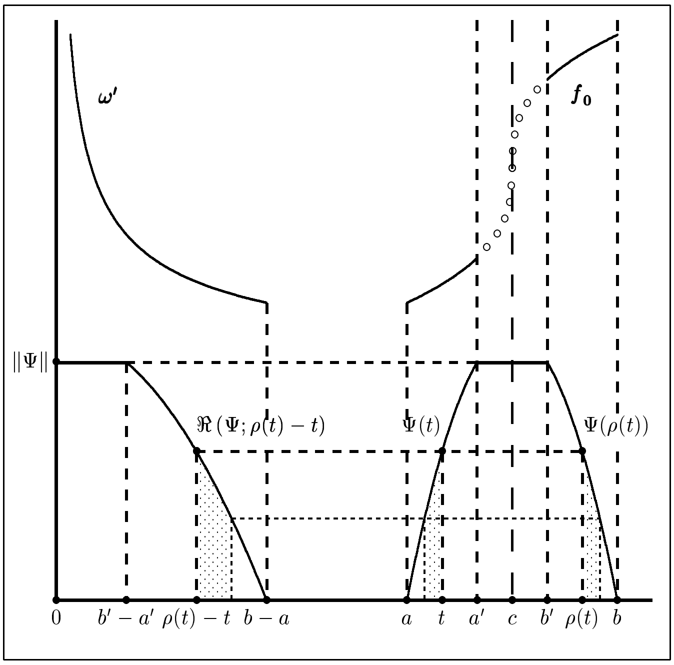

Fig. 1.1.1 Simple kernel Ψ, its rearrangement $\Re(\Psi;\cdot)$ and the function f_0

where $\rho(\cdot)$ *and* $\Re(\Psi;\cdot)$ *are defined in* (1.4) *and* (1.5), *respectively. The upper bound in* (1.6) *is attained for* f_0 *with*

$$\frac{d}{dx}f_0(x) = \begin{cases} \omega'(\rho(x)-x), & a \le x \le c = (a'+b')/2, \\ \omega'(x-\rho^{-1}(x)), & c \le x \le b. \end{cases} \tag{1.7}$$

By Definition 1.1.2, ψ has zero mean on $[a,b]$, so extremal functions of (1.2) are determined up to an additive constant. Therefore,

$$\sup_{h\in H^\omega[a,b]} \int_a^b h(t)\psi(t)\,dt = \sup_{h\in H_a^\omega[a,b]} \int_a^b h(t)\psi(t)\,dt. \tag{1.8}$$

Moreover, it can be observed from (1.4) and (1.7) that $(d/dt)f_0(t)$ is determined uniquely by (1.7) only on the support $[a,a'] \cup [b',b]$ of ψ. We illustrated this phenomenon in Figure 1.1.1 by graphing f_0 with solid lines on the support of ψ and by putting circles along the graph on the zero-interval $[a',b']$ of ψ.

Now we formulate some corollaries from Lemma 1.1.3 employed in Chapters 3 and 7.

Corollary 1.1.4. *If the kernel* ψ *in Lemma* 1.1.3 *is symmetric with respect to the midpoint* $\gamma = \frac{1}{2}(a+b)$ *of* $[a,b]$, *then* $\rho(x)-x = 2(\gamma-x)$, *and*

$$\frac{d}{dx}f_0(x) = \begin{cases} \omega'(2(\gamma-x)), & a \le x \le \gamma, \\ \omega'(2(x-\gamma)), & \gamma \le x \le b. \end{cases} \tag{1.9}$$

Corollary 1.1.5. *Let* $(d/dx)f_0(x)$ *be defined by* (1.7)*. Then* f_0 *has the full modulus of continuity on* $[0,b-a]$: $\omega(f_0;t)=\omega(t)$ *for* $0 \le t \le b-a$, *or, more precisely*,

$$f_0(\rho(t)) - f_0(t) = \omega(\rho(t)-t), \quad 0 \le t \le c. \tag{1.10}$$

1.2. Maximization of functionals in $H^\omega[a,b]$, $-\infty < a < b \le \infty$. Throughout this section we fix an interval $[a,b]$, $-\infty < a < b \le \infty$. Our objective is to characterize extremal functions of the problem

$$\int_a^b h(t)\psi(t)\,dt \to \sup, \quad h \in H_a^\omega[a,b], \tag{2.1}$$

for integrable kernels ψ with a finite number or a countable set of points of sign change on $[a,b]$, accumulating to the endpoint b. In particular, we give the solution of the problem (2.1) on the half-line $\mathbb{R}_+$.

1.2.1. *Notations and definitions.* We adopt the following notations in Section 1.2.

NOTATION 1.2.1. Let $N_1 \in \mathbb{N}$, $N_2 \in \mathbb{N} \cup \{\infty\}$, and $\xi = \{\xi_i = \xi(i)\}_{i=N_1}^{N_2}$ be a collection of points. Then $\alpha \triangleright \xi \blacktriangleright \beta$ if and only if

$$\alpha = \xi(N_1) \le \xi(N_1+1) \le \ldots \le \xi(N_2) = \beta, \quad \text{for } N_2 \in \mathbb{N};$$
$$\alpha = \xi(N_1); \quad \xi(i) \le \xi(i+1),\ i \ge N_1; \quad \lim_{i\to\infty} \xi(i) = \beta; \quad \text{for } N_2 = \infty.$$

Analogously, $\alpha \blacktriangleleft \xi \triangleleft \beta$ if and only if

$$\beta = \xi(N_1) \ge \xi(N_1-1) \ge \ldots \ge \xi(N_2) = \alpha, \quad \text{for } N_2 \in \mathbb{N};$$
$$\beta = \xi(N_1); \quad \xi(i) \ge \xi(i+1),\ i \ge N_1; \quad \lim_{i\to\infty} \xi(i) = \alpha; \quad \text{for } N_2 = \infty.$$

NOTATION 1.2.2. Let $[\alpha,\beta]$ be a finite interval, and $\psi \in \mathbb{L}_1[\alpha,\beta]$. By definition,

$$\operatorname{sign}\psi = 1 \text{ on } [\alpha,\beta] \Leftrightarrow \operatorname{meas}\{t \in [\alpha,\beta] : \psi(t) > 0\} = \beta - \alpha;$$
$$\operatorname{sign}\psi = -1 \text{ on } [\alpha,\beta] \Leftrightarrow \operatorname{sign}(-\psi) = 1 \text{ on } [\alpha,\beta].$$

NOTATION 1.2.3. The notation $I = \boxdot$ will be used for intervals $I = [\gamma,\gamma]$ with coincident endpoints.

NOTATION 1.2.4. Let $E \subset \mathbb{R}$. The function

$$\mathcal{X}(E;t) = \begin{cases} 1, & t \in E, \\ 0, & t \notin E, \end{cases}$$

is called the *indicator* of the set E.

DEFINITION 1.2.1. Let $j \in \{-1, 0, +1\}$, $n \in \mathbb{N} \cup \{\infty\}$, and $\psi \in \mathbb{L}_1[a,b]$. Then $\psi \in \mathcal{M}_n^j[a,b]$ for $n \ge 2$ if and only if $\operatorname{sign}\int_a^b \psi(x)\,dx = j$, and there exist $\alpha = \{\alpha_i\}_{i=0}^n$ such that $\alpha_{i-1} < \alpha_i$ for $i = 1, \ldots, n$, $a \triangleright \alpha \blacktriangleright b$, and

$$\operatorname{sign}\psi = (-1)^i \quad \text{on } [\alpha_{i-1}, \alpha_i], \quad i = 1, \ldots, n.$$

By definition, $\mathcal{M}_1^l[a,b] := \emptyset$ for $l = 0, +1$, and

$$\mathcal{M}_1^{-1}[a,b] := \{\psi \in \mathbb{L}_1[a,b] \mid \operatorname{sign}\psi = -1 \text{ on } [a,b]\}.$$

We also introduce the class

$$\mathbb{M}_n[a,b] := \bigcup_{j=-1}^{1} \mathcal{M}_n^j[a,b] \tag{2.2}$$

of kernels with precisely $n-1$ sign changes on $[a,b]$.

Definition 1.2.2. Let $N \in \mathbb{N} \cup \{\infty\}$. The sets of indices $\{J_i(N)\}_{i\in\mathbb{N}}$, $\{L_i\}_{i\in\mathbb{N}}$ and $\mathcal{P}(N)$ are defined as follows.

(1) For $N = 1, 2, 3$, $J_i(N) = L_i = \emptyset$ for $i = 1, \dots, N$.

(2) For $N \geq 4$,

$$J_i(N) = \emptyset \text{ for } i = N-2, N-1, N;\ L_i = \emptyset \text{ for } i = 1, 2, 3;$$

$$J_i(N) = \{j = i + 1 + 2k,\ k \in \mathbb{N} \mid j \leq N\} \text{ for } 1 \leq i \leq N-3;$$

$$L_i = \{l = i - 1 - 2k,\ k \in \mathbb{N} \mid l \geq 1\} \text{ for } 4 \leq i \leq N;$$

(3) $\mathcal{P}(N) = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq N-3,\ j \in J_i(N)\}$.

1.2.2. *V_n^j-partitions of* $[a, b]$. The structure of extremal functions of the problem (2.1) will be characterized in terms of special partitions of $[a, b]$.

Definition 1.2.3. Let $n \in \mathbb{N} \cup \{\infty\}$. A partition $\mathcal{V} = (\{A_i, B_i, C_i, D_i\}_{i=1}^n; \{B_{ij}, C_{ji}\}_{(i,j)\in\mathcal{P}(N)})$ of $[a, b]$ into subintervals is called a *V_n^j-partition*, $j \in \{-1, 0, +1\}$, if the following conditions are satisfied:

(A) $C_i = [\gamma_{4i-4}, \gamma_{4i-3}]$, $D_i = [\gamma_{4i-3}, \gamma_{4i-2}]$, $B_i = [\gamma_{4i-2}, \gamma_{4i-1}]$, $A_i = [\gamma_{4i-1}, \gamma_{4i}]$ for $i = 1, \dots, n$ and $\gamma = \{\gamma_i\}_{i=0}^{4n}$ such that $a \vartriangleright \gamma \blacktriangleright b$;

(B) $C_i = \boxdot$ for $i = 1, 2, 3$; $B_i = \boxdot$ for $i = n-2, n-1, n$;

(C$_1$) $j = 0 \Rightarrow D_i = \boxdot$ for $i = 1, \dots, n$;

(C$_2$) $j = -1 \Rightarrow D_{2k} = \boxdot$ for $k = 1, \dots, [n/2]$;

(C$_3$) $j = +1 \Rightarrow D_{2k-1} = \boxdot$ for $k = 1, \dots, \lceil n/2 \rceil$;

(D) $B_i = \bigcup_{j\in J_i(n)} B_{ij}$ for $1 \leq i \leq n-3$, where

$$B_{ij} = \left[\xi_i\left(\frac{j-i+1}{2}\right), \xi_i\left(\frac{j-i-1}{2}\right)\right], \quad j \in J_i(n),$$

for $\xi_i = \{\xi_i(k)\}_{k=1}^{|J_i(n)|}$ such that $\gamma_{4i-2} \blacktriangleleft \xi_i \vartriangleleft \gamma_{4i-1}$;

(E) $C_i = \bigcup_{l\in L_i} C_{il}$ for $4 \leq i \leq n$, where

$$C_{il} = \left[\varkappa_i\left(\frac{i-l-1}{2}\right), \varkappa_i\left(\frac{i-l+1}{2}\right)\right], \quad l \in L_i,$$

for $\varkappa_i = \{\varkappa_i(k)\}_{k=1}^{|L_i|}$ such that $\gamma_{4i-4} \vartriangleright \varkappa_i \blacktriangleright \gamma_{4i-3}$.

Remark 1.2.1. We list the atoms of V_n^j-partitions of $[a, b]$ into the intervals $\{A_i, B_i, C_i, D_i\}_{i=1}^n$ in their natural order and without the degenerate intervals A_N, $\{B_i\}_{i=N-2}^N$, $\{C_i\}_{i=1}^3$:

$N = 2:\ D_1A_1D_2$;

$N = 3:\ D_1A_1D_2A_2D_3$;

$N = 4:\ D_1B_1A_1D_2A_2D_3A_3C_4D_4$;

$N = 5:\ D_1B_1A_1D_2B_2A_2D_3A_3C_4D_4A_4C_5D_5$;

$N = 6:\ D_1B_1A_1D_2B_2A_2D_3B_3A_3C_4D_4A_4C_5D_5A_5C_6D_6$,

$N \geq 7:\ D_1B_1A_1D_2B_2A_2D_3B_3,\ A_kC_{k+1}D_{k+1}B_{k+1}$ for $3 \leq k \leq N-4$, $A_{N-3}C_{N-2}D_{N-2}A_{N-2}C_{N-1}D_{N-1}A_{N-1}C_ND_N$.

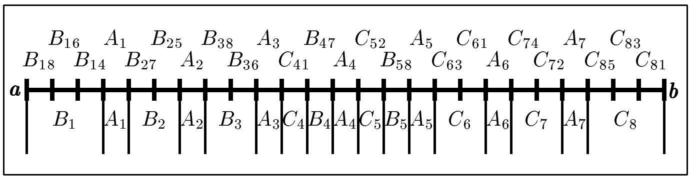

Fig. 1.2.1. V_8^0-partition

Figure 1.2.1 clarifies the order of atoms in a V_8^0-partition.

1.2.3. *Perfect ω-splines.* Given $\psi \in \mathcal{M}_n^j[a,b]$, $j \in \{-1,0,+1\}$, we describe extremal functions of the problem (2.1). In particular, in Theorem 1.2.1 below we show that *if $\psi \in \mathcal{M}_n^0[a,b]$ and $n \geq 2$, then the kernel $\Psi(t) = \int_a^t \psi(x)\,dx$ can be decomposed into the sum of*

(2.3) $$I_n = \begin{cases} n^2/4 & \text{for } n \text{ even}, \\ (n^2-5)/4 & \text{for } n \text{ odd}, \end{cases}$$

simple kernels $\{\Phi_i(\cdot) = \Phi_i(\omega;\cdot)\}_{i=1}^{I_n}$ such that

(2.4) (i) $$\Psi(t) = \sum_{i=1}^{I_n} \Phi_i(t), \quad a \leq t \leq b;$$

(ii) $$\sup_{h \in H^\omega[a,b]} \int_a^b h(t)\psi(t)\,dt = \sum_{i=1}^{I_n} \sup_{h \in H^\omega[a,b]} \int_a^b h(t)\Phi_i'(t)\,dt.$$

In the following theorem we describe the structure of extremal functions of the problem (2.1), called *perfect ω-splines.*

Theorem 1.2.1. *Let $\psi \in \mathcal{M}_n^j[a,b]$, $j \in \{-1,0,+1\}$, and $\{\alpha_i\}_{i=1}^{n-1}$ be the points of sign change of ψ as in Definition* 1.2.1. *Then there exist a solution $x_{\omega,\psi}$ of the problem* (2.1) *and a V_n^j-partition $\mathcal{V}$ of $[a,b]$ with the following properties*:

(A) $\alpha_i \in A_i$ *for* $i = 1,\dots,n-1$;

(B) $\int_{B_{ij}\cup C_{ji}} \psi(t)\,dt = 0$ *for* $(i,j) \in \mathcal{P}(n)$;

(C) $\int_{A_i} \psi(t)\,dt = 0$ *for* $i = 1,\dots,n-1$;

(D_1) $j = -1 \Rightarrow x_{\omega,\psi}(t) = -\omega(t-a)$ *for* $t \in D_{2k-1} \neq \boxdot$ *and* $k = 1,\dots,\lceil n/2 \rceil$;

(D_2) $j = 1 \Rightarrow x_{\omega,\psi}(t) = \omega(t-a)$ *for* $t \in D_{2k} \neq \boxdot$ *and* $k = 1,\dots,[n/2]$;

(E) *for each pair $(i,j) \in \mathcal{P}(n)$, the function $x_{\omega,\psi}$ is extremal for the problem*

(2.5) $$\int_a^b h(t)\psi_{ij}(t)\,dt \to \sup, \quad h \in H^\omega[a,b],$$

where

(2.6) $$\psi_{ij}(t) := \psi(t)\cdot\mathcal{X}(B_{ij}\cup C_{ji};t), \quad t \in [a,b];$$

(F) *for each* $i = 1, \ldots, n-1$, *the function* $x_{\omega,\psi}$ *is extremal for the problem*

$$\int_a^b h(t)\psi_i(t)\,dt \to \sup, \quad h \in H^\omega[a,b], \tag{2.7}$$

where

$$\psi_i(t) := \psi(t) \cdot \mathcal{X}(A_i; t), \quad t \in [a,b]. \tag{2.8}$$

Notice that all kernels

$$\begin{aligned} \Psi_{ij}(t) &= \int_a^t \psi_{ij}(y)\,dy, \quad (i,j) \in \mathcal{P}(n), \\ \Psi_i(t) &= \int_a^t \psi_i(y)\,dy, \quad i = 1, \ldots, n-1, \end{aligned} \tag{2.9}$$

are simple on their respective supports in the sense of Definition 1.1.2. Indeed, from the inclusions $\alpha_i \in A_i = [\gamma_{4i-1}, \gamma_{4i}]$, $i = 1, \ldots, n-1$, and the order of atoms in the V_n^j-partition, shown in Remark 1.2.1, it follows that

$$\operatorname{sign}\psi(t) = \begin{cases} (-1)^i & \text{on} \quad \begin{cases} B_{ij}, \ (i,j) \in \mathcal{P}(n), \\ [\gamma_{4i-1}, \alpha_i], \ i = 1, \ldots, n-1, \end{cases} \\ (-1)^{i+1} & \text{on} \quad \begin{cases} C_{ji}, \ (i,j) \in \mathcal{P}(n), \\ [\alpha_i, \gamma_{4i}], \ i = 1, \ldots, n-1, \end{cases} \end{cases} \tag{2.10}$$

where $[\gamma_{4i-1}, \gamma_{4i}] := A_i$ for $i = 1, \ldots, n-1$, and

$$\operatorname{sign}\psi(t) = \begin{cases} -1 & \text{on } D_{2i-1}, \ i = 1, \ldots, \lceil n/2 \rceil, \\ 1 & \text{on } D_{2i}, \ i = 1, \ldots, \lfloor n/2 \rfloor. \end{cases} \tag{2.11}$$

Therefore, by (2.10) and Theorem 1.2.1(B), (C), the kernels $\{\Psi_{ij}\}_{(i,j)\in\mathcal{P}(n)}$ and $\{\Psi_i\}_{i=1}^{n-1}$ in (2.9) are simple. Then, Korneĭchuk's Lemma 1.1.3 yields

$$\frac{d}{dt} x_{\omega,\psi}(t) = \begin{cases} (-1)^{i+1}\omega'(\rho_{ij}(t) - t), & t \in B_{ij}, \\ (-1)^{i+1}\omega'(t - \rho_{ij}^{-1}(t)), & t \in C_{ji}, \end{cases} \tag{2.12}$$

for all $(i,j) \in \mathcal{P}(n)$, where $\rho_{ij} : B_{ij} \to C_{ji}$ is determined from the equation

$$\Psi_{ij}(t) = \Psi_{ij}(\rho_{ij}(t)), \quad t \in B_{ij}, \ \rho_{ij}(t) \in C_{ji}, \tag{2.13}$$

and

$$\frac{d}{dt} x_{\omega,\psi}(t) = \begin{cases} (-1)^{i+1}\omega'(\rho_i(t) - t), & t \in [\gamma_{3i-1}, \alpha_i], \\ (-1)^{i+1}\omega'(t - \rho_i^{-1}(t)), & t \in [\alpha_i, \gamma_{3i}], \end{cases} \tag{2.14}$$

where $\rho_i : [\gamma_{4i-1}, \alpha_i] \to [\alpha_i, \gamma_{4i}]$ is determined from the equation

$$\Psi_i(t) = \Psi_i(\rho_i(t)), \quad t \in [\gamma_{4i-1}, \alpha_i], \ \rho_i(t) \in [\alpha_i, \gamma_{4i}]. \tag{2.15}$$

The schematic graphs of the extremal function $x_{\omega,\psi}$, $\psi \in \mathcal{M}_n^j[a,b]$, for various values of n and j are displayed in Figures 1.2.2–1.2.4.

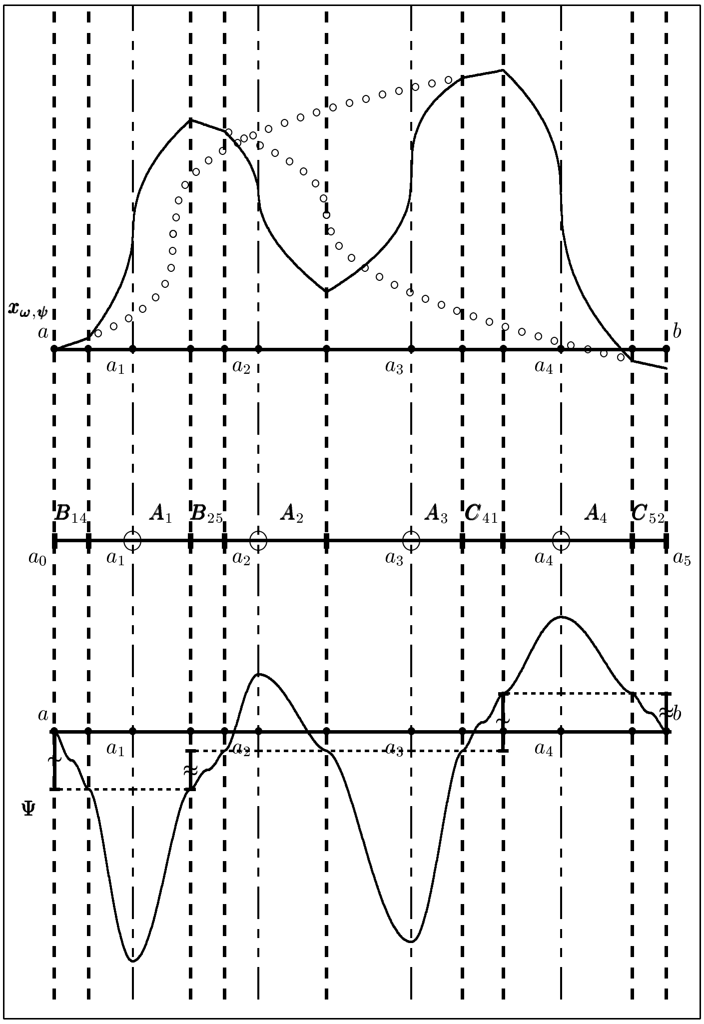

Fig. 1.2.2. V_5^0-partition and the graphs of $x_{\omega,\psi}$, Ψ for $\psi \in \mathcal{M}_5^0[a,b]$

1.2.4. *Structural properties of $x_{\omega,\psi}$.* First of all, the extremal function of the problem (2.1) is *unique*.

COROLLARY 1.2.2. *The derivative of the extremal function $x_{\omega,\psi}$ and the V_n^j-partition $\mathcal{V} = \mathcal{V}(\omega,\psi)$ of the problem* (2.1) *are unique.*

The V_n^j-partition $\mathcal{V}(\omega,\psi)$ from Theorem 1.2.1 is called the *extremal V_n^j-partition* of the problem $(*)$.

REMARK 1.2.2. By (2.12)–(2.15),

$$\text{sign}\,\frac{d}{dt}x_{\omega,\psi}(t) = \begin{cases} (-1)^{i+1}, & t \in A_i,\ i = 1,\dots,n-1, \\ (-1)^{i+1}, & t \in B_{ij}, C_{ji},\ (i,j) \in P(n), \end{cases} \tag{2.16}$$

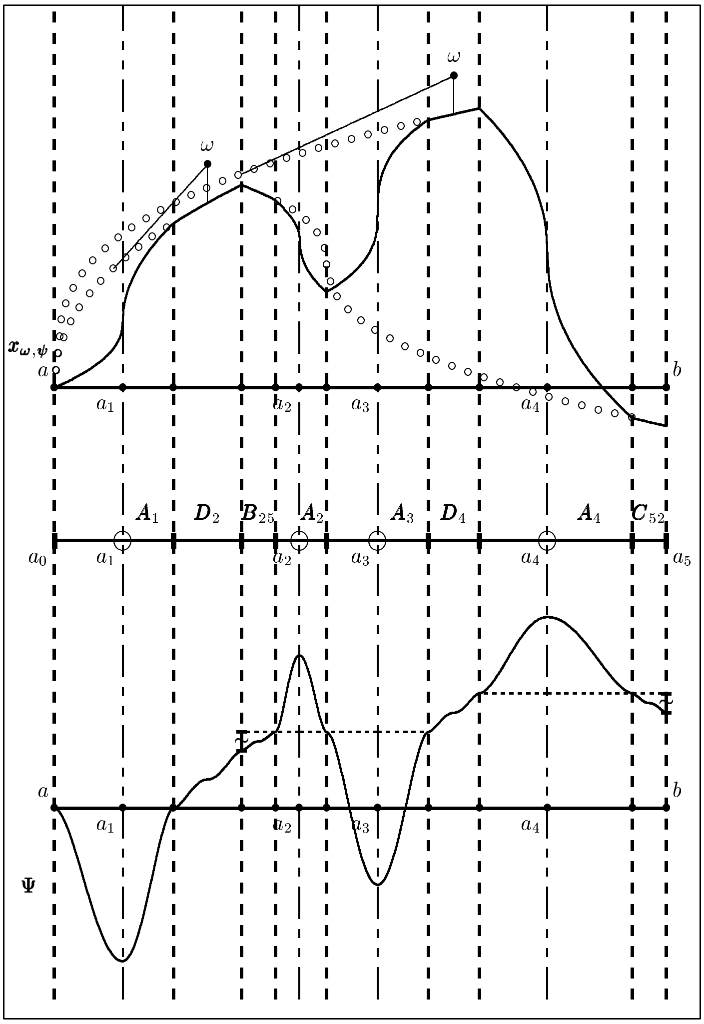

Fig. 1.2.3. V_5^{+1}-partition and the graphs of $x_{\omega,\psi}$, Ψ for $\psi \in \mathcal{M}_5^{+1}[a,b]$

while according to Theorem 1.2.1(D_1), (D_2),

$$\operatorname{sign} \frac{d}{dt} x_{\omega,\psi}(t) = \begin{cases} -1, & t \in D_{2i-1},\ i = 1, \ldots, \lceil n/2 \rceil, \\ 1, & t \in D_{2i},\ i = 1, \ldots, [n/2]. \end{cases} \tag{2.17}$$

Therefore, by (2.16) and (2.17), $(d/dt)x_{\omega,\psi}(t)$ can have at most $n-2$ sign changes in $[a,b]$ if ψ belongs to $\mathcal{M}_n^0[a,b]$ or $\mathcal{M}_n^{+1}[a,b]$, and $n-1$ sign changes in $[a,b]$ if $\psi \in \mathcal{M}_n^{-1}[a,b]$. These are the upper bounds for the number of sign changes of $(d/dt)x_{\omega,\psi}(t)$, because some of the intervals of the extremal V_n^j-partition may degenerate into points. However, the following result shows that in the case of a strictly concave modulus of continuity ω, $(d/dt)x_{\omega,\psi}(t)$ has the maximum possible number of sign changes.

Corollary 1.2.3. *Let ω be a strictly concave modulus of continuity on $[0, b-a]$, $\psi \in \mathcal{M}_n^j[a,b]$, $j \in \{-1, 0, +1\}$, and $\mathcal{V}$ be the extremal V_n^j-partition for the problem* $(*)$.

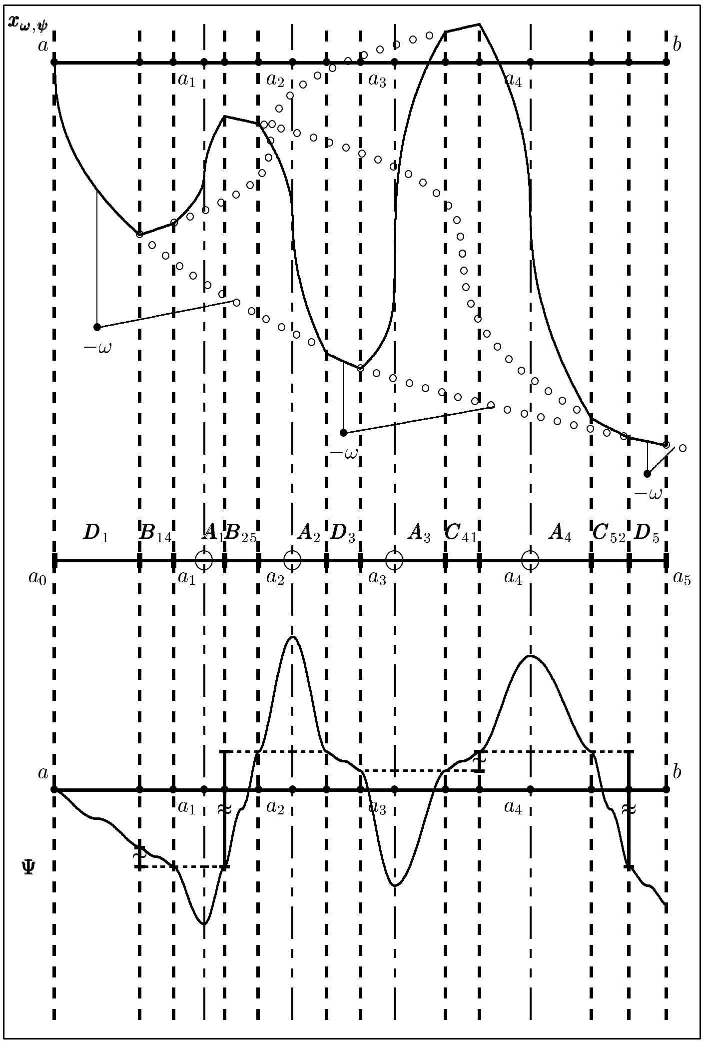

Fig. 1.2.4. V_5^{-1}-partition and the graphs of $x_{\omega,\psi}$, Ψ for $\psi \in \mathcal{M}_5^{-1}[a,b]$

Then:

(I) $A_i \neq \boxdot$ *for* $i = 1, \ldots, n-1$;
(II) *if* $\psi \in \mathcal{M}_n^{-1}[a,b]$, *then* $D_1 \neq \boxdot$.

Now Corollary 1.2.3 and relations (2.16) and (2.17) for $j = -1$ imply that $(d/dt)x_{\omega,\psi}(t)$ has precisely $n-2$ sign changes if $\psi \in \mathcal{M}_n^j[a,b]$, $j = 0, 1$, and exactly $n-1$ sign changes if $\psi \in \mathcal{M}_n^{-1}[a,b]$.

Remark 1.2.3. Remark 1.2.2 and the order of atoms in V_n^j-partitions, specified in Remark 1.2.1, enable us to find the following relations between the points of sign change $\{z_i\}_{i=1}^{n-2}$ of $(d/dt)x_{\omega,\psi}(t)$ and $\{\alpha_i\}_{i=1}^{n-1}$ of $\psi \in \mathcal{M}_n^j[a,b]$, $j = 0, 1$:

(2.18) $$\alpha_i < z_i < \alpha_{i+1}, \quad i = 1, \ldots, n-2.$$

If $\psi \in \mathcal{M}_n^{-1}[a,b]$ and $\{z_i\}_{i=1}^{n-1}$ is the collection of sign change points of $(d/dt)x_{\omega,\psi}(t)$, then

$$\alpha_{i-1} < z_i < a_i, \quad i = 1, \dots, n-1. \tag{2.19}$$

The quantitative solution of the problem (2.1) is given in terms of the extremal rearrangement $\Re_\omega(\Psi; \cdot)$.

Definition 1.2.4. Let $\psi \in \mathbb{M}_n[a,b]$ and $\Psi(t) = \int_b^t \psi(y)\,dy$, $t \in [a,b]$. The *extremal rearrangement* $\Re_\omega(\Psi; \cdot)$ of the kernel Ψ is defined as follows:

$$\Re_\omega(\Psi; t) = \sum_{(i,j)\in\mathcal{P}(n)} \Re(\Psi_{ij}; t) + \sum_{i=1}^{n-1} \Re(\Psi_i; t) + |\Psi(t+a)| \sum_{i=0}^{n} \mathcal{X}(D_i; t) \tag{2.20}$$

for $t \in [0, b-a]$, where $\mathcal{X}(E; \cdot)$ is the indicator function (see Notation 1.2.4).

Corollary 1.2.4. *Let the assumptions and notations be as in Theorem* 1.2.1. *Then*

$$\sup_{h\in H_a^\omega[a,b]} \int_a^b h(t)\psi(t)\,dt = \int_0^{b-a} \Re_\omega(\Psi; x)\omega'(x)\,dx. \tag{2.21}$$

We conclude the section with the identification of the extremal function $x_{\omega,\psi}$ in two trivial cases $\psi \in \pm\mathcal{M}_1^{-1}[a,b]$.

Remark 1.2.4. If $\xi\psi > 0$ on $[a,b]$ for $\xi \in \{-1,1\}$, then $x_{\omega,\psi}(t) = \xi\omega(t-a)$.

1.2.5. *Limiting properties of* $x_{\omega,\psi}$. In this section we touch on some limiting properties of perfect ω-splines.

First, we define the class $M_n[a,b]$ of those integral kernels whose number of sign changes does not exceed $n-1$.

Definition 1.2.5. For $n \in \mathbb{N}$, the class $M_n[a,b]$ is defined as follows:

$$M_n[a,b] := \bigcup_{k=1}^{n} (\pm\mathbb{M}_k[a,b]),$$

where $\mathbb{M}_k[a,b]$ are defined in (2.2), and $-F$ stands for $\{-f \mid f \in F\}$.

The following limiting properties of extremal functions of the problem (2.1) are crucial in various geometrical constructions involving perfect ω-splines.

Corollary 1.2.5. *Let* $\mathbb{S}$ *be a compact set in* $\mathbb{R}^d$, *and let* $\psi_s(t)$, $a_s \le t \le b_s$, $s \in \mathbb{S}$, *be a family of integrable kernels with the following properties*:

(i) *the endpoints* a_s *and* b_s *are continuous functions of* s *on* $\mathbb{S}$, *and* $a_s < a < b < b_s$ *for all* $s \in \mathbb{S}$, *for some* $a < b$;

(ii) *there exists an* $n \in \mathbb{N}$ *such that* $\psi_s \in M_n[a_s, b_s]$ *for all* $s \in \mathbb{S}$;

(iii) *the family* $\{\psi_s\}_{s\in\mathbb{S}}$ *depends continuously on* s *in the integral metric*:

$$\lim_{\mathbb{S}\ni s'\to s} \|\overline{\psi}_{s'} - \overline{\psi}_s\|_{\mathbb{L}_1[a,b]} = 0, \quad s \in \mathbb{S},$$

where $\overline{\psi}_s(x) := \psi_s\big(\frac{b_s-a_s}{b-a}(x-a) + a_s\big)$, $a \le x \le b$, $s \in \mathbb{S}$.

Let z_s be the solution of the problem

$$\int_{a_s}^{b_s} f(t)\psi_s(t)\,dt \to \sup, \quad f \in H^{\omega}[a_s,b_s],\ f(a_s)=0.$$

Then the functions z_s depend continuously on s on $\mathbb{S}$ in the uniform metric, i.e., for all $s \in \mathbb{S}$,

$$\|z_{s'} - z_s\|_{\mathbb{C}[\max\{a_s,a_{s'}\},\min\{b_s,b_{s'}\}]} \to 0 \quad \text{as } \mathbb{S} \ni s' \to s.$$

Corollary 1.2.6. *Let $\{\omega_l\}_{l\in\mathbb{N}}$ be a sequence of concave moduli of continuity on $[0,b-a]$ convergent in $\mathbb{C}[0,b-a]$ to a concave modulus of continuity ω, and let $\psi_l \in M_n[a,b]$ converge in $\mathbb{L}_1[a,b]$ to $\psi \in M_n[a,c]$ which vanishes outside $[a,c] \subset [a,b]$. For each $l \in \mathbb{N}$, let x_l be the solution of the problem*

$$\int_a^b h(t)\psi_l(t)\,dt \to \sup, \quad h \in H_a^{\omega_l}[a,b].$$

Then there exists a subsequence $\{x_{l_k}\}_{k\in\mathbb{N}}$ convergent in $\mathbb{C}[a,c]$ to the solution of the problem

$$\int_a^c h(t)\psi(t)\,dt \to \sup, \quad h \in H_a^{\omega}[a,c].$$

2. Auxiliary results

2.1. General facts. Besides the fundamental Theorem 1.2.1 on the structure of perfect ω-splines, the proof of Theorem 3.3.1 is based on two results of general nature: a topological result known as the *Borsuk Antipodality Theorem* and the *Chebyshev theorem* describing the characteristic properties of the best polynomial approximator of a continuous function.

Theorem 2.1.1. *Let $\mathbb{S}^n = \{\xi : \xi \in \mathbb{R}^{n+1} \mid \|\xi\| = r\}$, where $\|\cdot\|$ is a norm in $\mathbb{R}^{n+1}$, and let $\eta : \mathbb{S}^n \to \mathbb{R}^n$, $\eta(\xi) = \{\eta_1(\xi),\dots,\eta_n(\xi)\}$, be a continuous odd ($\eta(-\xi) = -\eta(\xi)$) vector field on $\mathbb{S}^n$. Then there exists $\overline{\xi} \in \mathbb{S}^n$ such that $\eta(\overline{\xi}) = 0$.*

Theorem 2.1.2. *Let P_n be the linear space of all polynomials of degree n, and $f \in \mathbb{C}[a,b] \setminus P_n$. Then:*

(a) *there exists a unique polynomial $p_f(t) = \sum_{i=0}^n a_i(f)t^i$ of best approximation for f on $[a,b]$ among the polynomials of degree n, i.e.,*

$$\|f - p_f\|_{\mathbb{C}[a,b]} = \min_{p\in P_n} \|f-p\|_{\mathbb{C}[a,b]};$$

(b) *p_f is the best approximator for f among the polynomials from P_n if and only if there exist points $\{x_k\}_{k=1}^{n+2}$, $a \le x_1 < \ldots < x_{n+2} \le b$, such that*

$$(f-p_f)(x_i) = (-1)^i \xi \|f-p_f\|_{\mathbb{C}[a,b]}, \quad i = 1,\dots,n+2, \tag{1.1}$$

for a fixed $\xi = \xi(f) \in \{-1,1\}$.

2.2. Properties of absolutely continuous functions. Recall that $\mathbb{AC}^l[a,b] := \{f \in \mathbb{C}^l[a,b] \mid f^{(l)}$ is absolutely continuous$\}$. In our constructions, employed in the proof of Theorem 3.3.1, we use properties of functions from $\mathbb{AC}^l[a,b]$.

Proposition 2.2.1. *Let $f \in \mathbb{C}^r[a,b]$ have r zeros (counting multiplicities) in $[a,b]$. Then*

$$\|f^{(k)}\|_{\mathbb{C}[a,b]} \le [b-a]^{r-k}\|f^{(r)}\|_{\mathbb{C}[a,b]}, \quad k = 0,\dots,r-1. \tag{2.1}$$

In particular, if $f \in W^rH^\omega[a,b]$ has $r+1$ zeros, then $f^{(r)}$ has a zero in $[a,b]$, and by (2.1),

$$\|f^{(k)}\|_{\mathbb{C}[a,b]} \le [b-a]^{r-k}\omega(b-a), \quad k = 0,\dots,r. \tag{2.2}$$

The following result guarantees the existence of a perfect spline satisfying the zero boundary conditions.

Proposition 2.2.2. *Let $r \in \mathbb{N}$. There exists a perfect spline*

$$Y_r(x) = \frac{x^{r+1}}{(r+1)!} + \frac{2}{(r+1)!}\sum_{j=1}^{r+1}(-1)^j(x-t_j)_+^{r+1} + \sum_{i=0}^{r} a_i x^i \tag{2.3}$$

of degree $r+1$ with $r+1$ knots $\{t_j\}_{j=1}^{r+1}$, $0 < t_1 < \ldots < t_{r+1} < 1$, that vanishes at the endpoints of the interval $[0,1]$:

$$Y_r^{(k)}(0) = Y_r^{(k)}(1) = 0, \quad k = 0,\dots,r. \tag{2.4}$$

Definition 2.2.1. Let $0 < P < M$, $r,n \in \mathbb{N}$, and $f \in \mathbb{AC}^{r-1}[a,b]$. We say that f belongs to the class $\mathcal{F}_{r,n,P}[a,b]$ if there exist points $\{\sigma_i\}_{i=0}^{l}$ with $l \le n$ such that $a = \sigma_0 < \sigma_1 < \ldots < \sigma_l = b$ and

$$(-1)^i f^{(r)}(x) \ge P > 0 \quad \text{for a.e. } x \in [\sigma_{i-1},\sigma_i],\ i = 1,\dots,l. \tag{2.5}$$

Then the function class $\mathcal{D}_{r,n,P,M}[a,b]$ for $P < M$ is introduced as follows:

$$\mathcal{D}_{r,n,P,M}[a,b] := \{f \in \mathcal{F}_{r,n,P}[a,b] \mid \|f^{(r)}\|_{\mathbb{L}_\infty[a,b]} \le M\}. \tag{2.6}$$

Lemma 2.2.3. *Let $f \in \mathcal{F}_{r,n,P}[a,b]$. Then*

$$\|f\|_{\mathbb{L}_1[a,b]} \ge \frac{V_r P(b-a)^{r+1}}{n^r} \tag{2.7}$$

with $V_r := \|Y_r\|_{\mathbb{L}_1[0,1]}$, where Y_r is the perfect spline from Proposition 2.2.2.

Lemma 2.2.4. *Let $n > r$, and let $f \in \mathcal{F}_{r,n-r+1,P}[a,b]$ have precisely n distinct zeros $\{\tau_i\}_{i=1}^n$ and $f^{(r)}$ have exactly $n-r$ points of sign change $\{\sigma_i\}_{i=1}^{n-r}$, arranged in increasing order. If*

(i) $\|f'\|_{\mathbb{C}[a,b]} \le C$,
(ii) $\tau_{i+1} - \tau_i > \delta > 0$ *for* $i = 1,\dots,n-1$,

then there exists a constant $\widehat{\delta} = \widehat{\delta}(n,r,C,P,\delta)$ such that

$$\sigma_i > \tau_i + \widehat{\delta}, \quad i = 1,\dots,n-r. \tag{2.8}$$

Proposition 2.2.5. *Let $f \in \mathcal{D}_{r,n-r+1,P,M}[0,T]$ have n simple zeros $\{t_i\}_{i=1}^n$, and $f^{(r)}$ have $n-r$ points of sign change $\{\sigma_i\}_{i=1}^{n-r}$, where both collections are arranged in increasing order. Then there exists a positive constant $\lambda_r = \lambda(r,n,P,M)$ such that*

$$\sigma_i > \lambda_r t_{i+r}, \quad i = 1, \dots, n-r. \tag{2.9}$$

Proof. By Rolle's theorem, $f^{(k)}$ has precisely $n-k$ points of sign change $\{\eta_i^k\}_{i=1}^{n-k}$ in $[0,T]$ with $0 =: \eta_0^k < \eta_1^k < \eta_2^k < \ldots < \eta_{n-k}^k < \eta_{n-k+1} := T$, for $k = 0, \dots, r$.

By definition, $\{\eta_i^0 := t_i\}_{i=1}^n$ and $\{\eta_i^r := \sigma_i\}_{i=1}^{n-r}$. Also by Rolle's theorem,

$$\eta_i^k < \eta_i^{k+1} < \eta_{i+1}^k, \quad i = 1, \dots, n-k-1, \ k = 0, \dots, r-1. \tag{2.10}$$

Fix a pair of neighboring zeros η_i^k, η_{i+1}^k for $k = 0, \dots, r-1$ and $i = r-k-1, \dots, n-k-1$. The equations $f^{(k)}(\eta_i^k) = f^{(k)}(\eta_{i+1}^k) = 0$ imply that

$$0 = \int_{\eta_i^k}^{\eta_{i+1}^k} f^{(k+1)}(\xi)\,d\xi = \int_{\eta_i^k}^{\eta_i^{k+1}} f^{(k+1)}(\xi)\,d\xi + \int_{\eta_i^{k+1}}^{\eta_{i+1}^k} f^{(k+1)}(\xi)\,d\xi. \tag{2.11}$$

By (2.10), η_i^{k+1} is the only point of sign change of $f^{(k+1)}$ in $[\eta_i^k, \eta_{i+1}^k]$. Thus, by (2.11),

$$\int_{\eta_i^k}^{\eta_i^{k+1}} |f^{(k+1)}(\xi)|\,d\xi = \int_{\eta_i^{k+1}}^{\eta_{i+1}^k} |f^{(k+1)}(\xi)|\,d\xi = \frac{1}{2}\|f^{(k+1)}\|_{\mathbb{L}_1[\eta_i^k, \eta_{i+1}^k]}. \tag{2.12}$$

An application of Lemma 2.2.3 to $f^{(k+1)} \in \mathcal{F}_{r-k-1,n-r+1,P}[\eta_i^{k+1}, \eta_{i+1}^k]$ yields

$$\|f^{(k+1)}\|_{\mathbb{L}_1[\eta_i^{k+1}, \eta_{i+1}^k]} \geq \frac{PV_{r-k-1}}{(n-r+1)^{r-k-1}}(\eta_{i+1}^k - \eta_i^{k+1})^{r-k}. \tag{2.13}$$

On the other hand, $f^{(k+1)}$ has at least $i \geq r-k-1$ zeros $\{\eta_j^k\}_{j=1}^i$ in $[0, \eta_i^{k+1}]$. An application of Proposition 2.2.1 to $f^{(k+1)}$ leads to

$$\begin{aligned} \|f^{(k+1)}\|_{\mathbb{L}_1[\eta_i^k, \eta_i^{k+1}]} &\leq \eta_i^{k+1}\|f^{(k+1)}\|_{\mathbb{C}[0,\eta_i^{k+1}]} \\ &\leq [\eta_i^{k+1}]^{r-k}\|f^{(r)}\|_{\mathbb{L}_\infty[0,\eta_i^{k+1}]} \leq M[\eta_i^{k+1}]^{r-k}. \end{aligned} \tag{2.14}$$

Combining (2.12) with (2.13) and (2.14) implies that

$$\frac{PV_{r-k-1}}{(n-r+1)^{r-k-1}}(\eta_{i+1}^k - \eta_i^{k+1})^{r-k} \leq M[\eta_i^{k+1}]^{r-k}. \tag{2.15}$$

Consequently,

$$\eta_i^{k+1} \geq E_{r,k,n}\eta_{i+1}^k, \tag{2.16}$$

where $E_{r,k,n} = E_{r,k,n}(P,M)$ is defined by

$$\begin{aligned} E_{r,k,n} := &\left[\frac{PV_{r-k-1}}{(n-r+1)^{r-k-1}}\right]^{1/(r-k)} \\ &\times \left(\left[\frac{PV_{r-k-1}}{(n-r+1)^{r-k-1}}\right]^{1/(r-k)} + M^{1/(r-k)}\right)^{-1} \eta_{i+1}^k. \end{aligned} \tag{2.17}$$

From (2.16) we infer that

$$\sigma_i := \eta_i^r \geq E_{r,r-1,n}\eta_{i+1}^{r-1} \geq \ldots \geq \Big(\prod_{i=0}^{r-1} E_{r,i,n}\Big)\eta_{i+r}^0 = \lambda_r t_{i+r}, \quad i = 1, \ldots, r, \tag{2.18}$$

where

$$\lambda_r = \lambda(r, P, M, n) := \prod_{i=0}^{r-1} E_{r,i,n}(P, M). \ \blacksquare \tag{2.19}$$

Corollary 2.2.6. *Let f satisfy the assumptions of Proposition 2.2.5, and $\{\eta_i^k\}_{i=1}^{n-k}$ be the zeros of $f^{(k)}$ arranged in increasing order, for $k = 1, \ldots, r-1$. Then there exists a constant $\lambda_k = \lambda_k(r, n, P, M)$ such that*

$$\eta_{i+r-k}^k > \lambda_k t_{i+r}, \quad k = 1, \ldots, r-1, \ i = 1, \ldots, n-r. \tag{2.20}$$

Proof. From (2.16) it follows that

$$\eta_{i+1}^{r-1} \geq \ldots \geq \Big(\prod_{i=k+1}^{r-1} E_{r,i,n}\Big)\eta_{i+(r-k)}^k \geq \ldots \geq \Big(\prod_{i=1}^{r-1} E_{r,i,n}\Big)\eta_{i+r}^0 \tag{2.21}$$

for all $i = 1, \ldots, n-r$ and $k = 1, \ldots, r-1$, where the constants $\{E_{r,k,n}\}_{k=0}^{r-1}$ are defined in (2.17). Therefore, in (2.20) we can set

$$\lambda_k := \prod_{i=1}^{k} E_{r,i,n}, \quad k = 1, \ldots, r-1. \ \blacksquare \tag{2.22}$$

3. Formulation of the main result

3.0. Chebyshev and Zolotarev perfect polynomial splines

3.0.1. *Tikhomirov's and Karlin's results.* S. Karlin gave a complete description of extremal functions of the Kolmogorov–Landau problem in the Sobolev class $W_\infty^n[0,1] = W^{n-1}H^1[0,1]$:

$$|f^{(m)}(0)| \to \sup, \quad f \in W_\infty^n[0,1], \ \|f\|_{\mathbb{C}[0,1]} \leq B, \tag{0.1}$$

where $n, m \in \mathbb{N}$ with $0 < m < n$, and $B > 0$. The extremal equioscillating perfect polynomial splines of the problem (0.1) can be regarded as generalizations of the classical Chebyshev and Zolotarev polynomials.

Definition 3.0.1. For $n \in \mathbb{Z}_+$ and $-\infty < t_1 < \ldots < t_l < \infty$, the function

$$P(x) = \frac{x^n}{n!} + \frac{2}{n!}\sum_{j=1}^{l}(-1)^j (x - t_j)_+^n + \sum_{i=0}^{n-1} a_i x^i$$

is called a *perfect spline of degree n with l knots* $\{t_i\}_{i=1}^l$.

Theorem 3.0.1. *Let $n \in \mathbb{N}$ and $l \in \mathbb{Z}_+$. There exists a unique perfect spline $T_{n,l}$ of degree n with exactly l knots on $[0,1]$ and $l+n+1$ equioscillation points $\{\nu_i = \nu_i(n,l)\}_{i=0}^{l+n}$,*

$0 = \nu_0 < \nu_1 < \ldots < \nu_{l+n} = 1$, *such that*

$$T_{n,l}(\nu_i) = (-1)^i \|T_{n,l}\|_{\mathbb{C}[0,1]}, \quad i = 0, \ldots, l+n. \tag{0.2}$$

PROPOSITION 3.0.2. *Let* $\mu_{n,-1} := +\infty$ *and* $\mu_{n,l} := \|T_{n,l}\|_{\mathbb{C}[0,1]}$, $n \in \mathbb{N}$, $l \in \mathbb{Z}_+$. *Then*

$$\mu_{n,l-1} > \mu_{n,l}, \quad l \in \mathbb{Z}_+, \quad \textit{and} \quad \lim_{l \to \infty} \mu_{n,l} = 0. \tag{0.3}$$

THEOREM 3.0.3. *For* $n \in \mathbb{N}$, $l \in \mathbb{Z}_+$ *and* $\mu \in (\mu_{l,n}, \mu_{l-1,n})$ *there exists a unique perfect spline* $Z_{n,\mu}$ *of degree* n *with exactly* l *knots and precisely* $l+n$ *points of alternance* $\{\nu_i = \nu_i(\mu, n)\}_{i=0}^{l+n-1}$, $0 = \nu_0 < \nu_1 < \ldots < \nu_{l+n-1} \le 1$, *such that*

$$Z_{n,\mu}(\nu_i) = (-1)^i \|Z_{n,\mu}\|_{\mathbb{C}[0,1]} = (-1)^i \mu, \quad i = 0, \ldots, l+n-1. \tag{0.4}$$

The function

$$T_{n,0}(x) = \frac{2^{-2n+1}}{n!} \cos[n \arccos(2x-1)],$$

defined for $[0,1]$, is the well-known *Chebyshev polynomial* of degree n, while the polynomials $\{Z_{n,\mu}\}_{\mu > \mu_{0,n}}$ constitute the family of classical *Zolotarev polynomials* first introduced in [62] and described in terms of elliptic finctions in [1]. Naturally, the perfect splines $T_{n,l}$, $n \in \mathbb{N}$, $l \in \mathbb{Z}_+$, have received the name of *Chebyshev perfect splines*, and the polynomial perfect splines $Z_{n,\mu}$ were named after E. M. Zolotarev. We give the proof of Theorems 3.0.1 and 3.0.3 in Sections A and B of the Appendix, where we employ slightly different notations.

S. Karlin [30], [31] showed that two methods, the zero counting technique, based on Rolle's theorem, or an argument based on numerical differentiation formulae, could be employed to show the extremality of the function $Z_{n,B}$ for the problem (0.1) for all m with $0 < m < n$. Moreover, both arguments can be exploited to demonstrate the extremality of $(-1)^m Z_{n,B}$ for the extrapolation problem

$$f^{(m)}(\tau) \to \sup, \quad f \in W_\infty^n[\tau, 1], \ \|f\|_{\mathbb{C}[0,1]} \le B, \tag{0.5}$$

for all $\tau < 0$ and m with $0 \le m < n$.

3.0.2. *Kolmogorov–Landau problem on* $\mathbb{R}$ *or* $\mathbb{R}_+$. Let $A > 0$, $m, n \in \mathbb{N}$ with $0 < m < n$, and $\mathcal{Z}_A = \mathcal{Z}_{A,n,B}$ be the Zolotarev spline extremal for the problem

$$|f^{(m)}(\tau)| \to \sup, \quad f \in W_\infty^n[\tau, A], \ \|f\|_{\mathbb{C}[0,A]} \le B. \tag{0.6}$$

S. Karlin [31] established the existence of the pointwise limiting function $\mathcal{Z}$ of the family $\{\mathcal{Z}_{A,n,B}\}_{A>0}$ as $A \to \infty$, and showed the extremality of $\mathcal{Z}$ for the extrapolation problem

$$|f^{(m)}(\tau)| \to \sup, \quad f \in W_\infty^n[\tau, \infty), \ \|f\|_{\mathbb{L}_\infty(\mathbb{R}_+)} \le B. \tag{0.7}$$

Let us introduce the rescaled Chebyshev splines of norm 1:

$$\begin{aligned} \widehat{T}_{n,l}(t) &= \mu_{n,l}^{-1} T_{n,l}(\mu_{n,l}^{1/n} t), & 0 \le t \le \mu_{n,l}^{-1/n}, \\ \widetilde{T}_{n,l}(t) &= \mu_{n,l}^{-1} T_{n,l}(\mu_{n,l}^{1/n} t + 1/2), & -\tfrac{1}{2}\mu_{n,l}^{-1/n} \le t \le \tfrac{1}{2}\mu_{n,l}^{-1/n}. \end{aligned} \tag{0.8}$$

Then the extremal Euler splines of the problem

$$\|f^{(m)}\|_{\mathbb{L}_\infty(I)} \to \sup, \quad f \in W_\infty^n(I), \ \|f\|_{\mathbb{L}_\infty(I)} \le 1, \tag{0.9}$$

for $I=\mathbb{R}$ can be obtained as limiting functions of the sequence $\{\widetilde{T}_{n,l}\}_{n\in\mathbb{N}}$ for $l+n+1=m$ (mod 2). Analogously, the Chebyshev perfect spline, extremal in the problem (0.9) in the case $I=\mathbb{R}_+$, is a limiting function of the sequence $\{\widehat{T}_{n,l}\}_{l\in\mathbb{N}}$.

3.1. Numerical differentiation formulae for $f^{(m)}(0),\ 0<m<r$. Let $m,r,N\in\mathbb{N}$ with $m<r$, $N\ge r$. Let collections of points $\overline{\nu}=\{\nu_i\}_{i=0}^N$, $\overline{\vartheta}=\{\vartheta_i\}_{i=0}^{N-r+2}\in[0,1]$ satisfy

$$\begin{aligned}&0=:\nu_0<\ldots<\nu_N\le 1,\quad 0=:\vartheta_0<\ldots<\vartheta_{N-r+1}<\vartheta_{N-r+2}:=\nu_N,\\ &\nu_{i-1}<\vartheta_i<\nu_{i+r-1},\quad i=1,\ldots,N-r+1.\end{aligned}\tag{1.1}$$

Let the coefficients $\{\alpha_i\}_{i=0}^N$ be derived from the system of linear equations

$$\begin{cases}\displaystyle\sum_{i=0}^N\alpha_i\nu_i^j=m!\delta_{m,j}, & j=0,\ldots,r-1,\\ \displaystyle\sum_{i=0}^N\alpha_i(\nu_i-\vartheta_l)_+^{r-1}=0, & l=1,\ldots,N-r+1.\end{cases}\tag{1.2}$$

Then (1.1) guarantees that the system (1.2) has a unique solution ([32], [50]).

We introduce the spline kernels K and F:

$$K(t)=-\frac{1}{(r-1)!}\sum_{i=0}^N\alpha_i(\nu_i-t)_+^{r-1},\quad F(t)=\frac{1}{r!}\sum_{i=0}^N\alpha_i(\nu_i-t)_+^r.\tag{1.3}$$

Let $f\in W^rH^\omega[0,1]$. Taylor's formula

$$f(\tau)=\sum_{j=0}^{r-1}\frac{f^{(j)}(0)}{j!}\tau^j+\frac{1}{(r-1)!}\int_0^1 f^{(r)}(x)(\tau-x)_+^{r-1}\,dx\tag{1.4}$$

and equations (1.2) lead to

$$f^{(m)}(0)=\sum_{i=0}^j\alpha_i f(\nu_i)+\int_0^1 f^{(r)}(x)K(x)\,dx.\tag{1.5}$$

Thus, for any $f\in W^rH^\omega[0,1]$,

$$|f^{(m)}(0)|\le\sum_{i=0}^N|\alpha_i|\cdot\|f\|_{\mathbb{C}[0,1]}+\sup_{g\in W^rH^\omega[0,1]}\int_0^1 g^{(r)}(x)K(x)\,dx.\tag{1.6}$$

The kernel K enjoys the following properties:

(1.7) (i) $\operatorname{supp}K=[0,\nu_j]$ for some j with $r-1\le j\le N$;
(ii) K has precisely $j-r+1$ simple zeros $\{\vartheta_i\}_{i=1}^{j-r+1}$ in $(0,\nu_j)$;
(iii) $\operatorname{sign}\alpha_i=(-1)^{i+m}$ for $i=0,\ldots,j$;
(iv) $(-1)^{i+r+m}\operatorname{sign}K(t)\ge 0$ for $\vartheta_i\le t\le\vartheta_{i+1}$ and $i=0,\ldots,N-r+1$.

Remark 3.1.1. The verification of properties (1.7) above and (2.5) below is analogous to the proof of Proposition 6.3 in [50].

In addition, assume that K has zero mean on $[0,1]$:

$$\int_0^1 K(t)\,dt=-\frac{1}{r!}\sum_{i=0}^N\alpha_i\nu_i^r=0.\tag{1.8}$$

Then

$$(1.9)\qquad \sup_{g\in W^rH^\omega[0,1]}\int_0^1 g^{(r)}(x)K(x)\,dx = \sup_{g\in W^rH^\omega[0,1]}\int_0^1 [g^{(r)}(x)-g^{(r)}(0)]K(x)\,dx$$
$$= \sup_{h\in H_0^\omega[0,1]}\int_0^1 h(x)K(x)\,dx.$$

Furthermore, (1.8) implies that K does have at least one point of sign change in $[0,\nu_j]$. Therefore, (1.7)(ii) leads to a more precise inequality $j\ge r$ in (1.7)(i).

By (1.7) and (1.8), $(-1)^{r+m+1}K\in\mathcal{M}^0_{j-r+2}[0,\nu_j]$ (see Definition 1.2.1). Theorem 1.2.1 describes the structure of the function realizing the supremum in (1.9). Finally, from (1.6) and Corollary 1.2.4 we derive the estimate

$$(1.10)\qquad |f^{(m)}(0)|\le \sum_{i=0}^{j}|\alpha_i|\cdot\|f\|_{\mathbb{C}[0,1]}+\int_0^{\nu_j}\Re_\omega(F;t)\omega'(t)\,dt$$

for $f\in W^rH^\omega[0,1]$, where $\Re_\omega(F;t)$ is the rearrangement of $F(t)=\int_0^t K(y)\,dy$, defined in (1.2.20).

We mention the following auxiliary result used in the proof of Theorem 3.3.1.

PROPOSITION 3.1.1. *Let* $[0,\nu_j]$ *be the support of the kernel* K *defined by* (1.2), (1.3). *Then*

$$(1.11)\qquad \vartheta_i<\nu_{i+r-2},\quad i=1,\dots,j-r.$$

PROOF. (1.11) follows from Rolle's theorem and the property $K^{(i)}(\nu_j)=0$, $j=0,\dots,r-1$. Indeed, for $i=0,\dots,r-1$, let $\xi_0^i:=\nu_j$, and let $\{\xi_l^i\}_{l=1}^{j-r+i}$ be the $j-r+i$ rightmost (i.e. counting from the left) points of sign change of $K^{(i)}$ in $[0,\nu_j)$ enumerated in decreasing order:

$$\xi^i_{j-r}<\xi^i_{j-r-1}<\dots<\xi^i_0=\nu_j,\quad i=0,\dots,r-1.$$

By Rolle's theorem, for $i=0,\dots,r-2$ we have

$$(1.12)\qquad \xi^i_{l-1}>\xi^{i+1}_l>\xi^i_l,\quad l=1,\dots,j-r+i.$$

In particular, by (1.12) and (1.7)(ii),

$$(1.13)\qquad \nu_{j-l}=:\xi_l^{r-1}>\xi_l^{r-2}>\dots>\xi_l^0:=\vartheta_{j-r+2-l},\quad l=1,\dots,j-r+1,$$

implying (1.11). ∎

MICROLEMMA 3.1.2. *Let* $\{\nu_i\}_{i=0}^N$ *and* $\{\vartheta_i\}_{i=0}^{N-r+2}$ *be as in* (1.1). *Let* $\{\alpha_i\}_{i=0}^N$ *be derived from* (1.2). *If* $\nu_{N-1}\le\vartheta_{N-r+1}$, *then* $\alpha_N=0$.

PROOF. If $t\in[\nu_{N-1},\nu_N]$, then $(\nu_i-t)_+^{r-1}=0$ for $i=0,\dots,N-1$, and $(\nu_N-t)_+^{r-1}=(\nu_N-t)^{r-1}$. Therefore, by (1.2) for $l=N-r+1$, we have, for $\theta_{N-r+1}\in[\nu_{N-1},\nu_N)$,

$$(1.14)\qquad 0=\sum_{i=0}^N\alpha_i(\nu_i-\vartheta_{N-r+1})_+^{r-1}=\alpha_N(\nu_N-\vartheta_{N-r+1})^r,$$

implying that $\alpha_N=0$. ∎

Finally, we remark that the numerical formulae for intermediate derivatives of functions from Sobolev classes are discussed by H. Kallioniemi [28] and S. Karlin [32].

3.2. Numerical differentiation formulae for $f^{(r)}(0)$. Let the collections of points $\overline{\nu} = \{\nu_i\}_{i=0}^{N+1}$ and $\overline{\vartheta} = \{\vartheta_i\}_{i=0}^{N-r+1}$ in $[0,1]$ satisfy the inequalities

$$\begin{aligned} &0 =: \nu_0 < \ldots < \nu_N \le 1, \quad 0 =: \vartheta_0 < \ldots < \vartheta_{N-r} < \vartheta_{N-r+1} := \nu_N, \\ &\nu_{i-1} < \vartheta_i < \nu_{i+r-1}, \quad i = 1, \ldots, N-r. \end{aligned} \tag{2.1}$$

We derive the coefficients $\{\alpha_i\}_{i=0}^{N}$ from the system of linear equations

$$\begin{cases} \sum_{i=0}^{N} \alpha_i \nu_i^j = 0, & j = 0, \ldots, r-1, \\ \sum_{i=0}^{N} \alpha_i (\nu_i - \vartheta_l)_+^{r-1} = 0, & l = 1, \ldots, N-r, \\ \sum_{i=0}^{N} \alpha_i \nu_i^r = r!. \end{cases} \tag{2.2}$$

The last equation in (2.2) is added to normalize the coefficients $\{\alpha_i\}_{i=0}^{n}$, and the inequalities (2.1) guarantee the unique solvability of (2.2) (cf. [32]).

As before, the kernels K and F are defined by

$$K(t) = -\frac{1}{(r-1)!} \sum_{i=0}^{N} \alpha_i (\nu_i - t)_+^{r-1}, \quad F(t) = \int_{\nu_N}^{t} K(y)\,dy = \frac{1}{r!} \sum_{i=0}^{N} \alpha_i (\nu_i - t)_+^{r}. \tag{2.3}$$

Proceeding as in the case $0 < m < r$, we derive the numerical differentiation formula

$$f^{(r)}(0) = \sum_{i=0}^{N} \alpha_i f(\nu_i) + \int_0^d [f^{(r)}(t) - f^{(r)}(0)] K(t)\,dt. \tag{2.4}$$

The kernel K enjoys the properties:

(2.5) (i) $\operatorname{supp} K = [0, \nu_j]$ for some j with $r \le j \le N$;
(ii) K has $j - r$ simple zeros $\{\vartheta_i\}_{i=1}^{j-r}$ in $(0, \nu_j)$;
(iii) $\operatorname{sign} \alpha_i = (-1)^{i+r}$ for $i = 0, \ldots, j$;
(iv) $(-1)^i \operatorname{sign} K(t) \ge 0$ for $\vartheta_i \le t \le \vartheta_{i+1}$ and $i = 0, \ldots, N-r$.

Thus, by (2.5), $K \in M_{j-r+1}^{-1}[0, \nu_j]$ (see Definition 1.2.1). Theorem 1.2.1 provides a formula for the extremal function of the problem

$$\int_0^{\nu_j} h(t) K(t)\,dt \to \sup, \quad h \in H_0^{\omega}[0, \nu_j]. \tag{2.6}$$

The identity (2.4) and Corollary 1.2.4 yield the estimate

$$|f^{(r)}(0)| \le \sum_{i=0}^{j} |\alpha_i| \cdot \|f\|_{\mathbb{C}[0,1]} + \int_0^{\nu_j} \Re_{\omega}(F; t)\, \omega'(t)\,dt \tag{2.7}$$

for $f \in W^r H^{\omega}[0,1]$ (recall Definition 1.2.4 of $\Re_{\omega}(F; \cdot)$).

3.3. Main results

3.3.1. *Sufficient conditions for extremality in the Kolmogorov–Landau problem.* The results of Sections 3.1 and 3.2 enable us to find sufficient conditions for a function $\mathcal{Z} \in W^r H^\omega[0,1]$ to be extremal for the inequalities (1.10) for $0 < m < r$ and (2.7) for $m = r$. The analysis of the formulae (1.5) and (2.4) for $f^{(m)}(0)$ shows that $\mathcal{Z}$ turns inequalities (1.10) for $0 < m < r$ and (2.7) for $m = r$ into equalities if two conditions are met:

(3.1) (1) $$\mathcal{Z}(\nu_i) = (-1)^{i+m}\|\mathcal{Z}\|_{\mathbb{C}[0,1]}, \quad i = 0, \dots, n,$$

(2) $$\sup_{h \in H_0^\omega[0,1]} \int_0^1 h(x)K(x)\,dx = \int_0^1 [\mathcal{Z}^{(r)}(x) - \mathcal{Z}^{(r)}(0)]K(x)\,dx,$$

where K is defined by (1.2), (1.3) for $0 < m < r$, and by (2.2), (2.3) for $m = r$.

3.3.2. *The main theorem*

THEOREM 3.3.1. *Let $B > 0$, $m, r, n \in \mathbb{N}$ with $0 < m \le r$ and $n \ge r$, and ω be a concave modulus of continuity on $[0,1]$. Then there exist an integer $N = N(B,n,r,m,\omega)$ with $r \le N \le n$, collections of points $\overline{\nu} = \overline{\nu}(B,n,r,m,\omega) = \{\nu_i\}_{i=0}^N$ and $\overline{\vartheta} = \overline{\vartheta}(B,n,r,m,\omega) = \{\vartheta_i\}_{i=0}^{N-r+2}$ as in (1.1) for $0 < m < r$, or as in (2.1) for $m = r$, and a function $\mathcal{Z}_n = \mathcal{Z}_{B,n,r,m,\omega}$ with the following properties:*

(I)

$$\sup_{h \in H_0^\omega[0,1]} \int_0^1 h(x)K(x)\,dx = \int_0^1 [\mathcal{Z}_n^{(r)}(x) - \mathcal{Z}_n^{(r)}(0)]K(x)\,dx, \tag{3.2}$$

where the coefficients $\{\alpha_i\}_{i=0}^N$ of the kernel

$$K(t) = -\frac{1}{(r-1)!}\sum_{i=0}^N \alpha_i(\nu_i - t)_+^{r-1}$$

satisfy equations (2.2), (2.8) *for $0 < m < r$, and* (2.12) *for $m = r$.*

(II)(A) *If $N < n$, then*

$$\mathcal{Z}_n(\nu_i) = (-1)^{i+m}\|\mathcal{Z}_n\|_{\mathbb{C}[0,1]} = (-1)^{i+m}B, \quad i = 0, \dots, N. \tag{3.3}$$

(II)(B) *If $N = n$, then*

$$\begin{aligned} &\mathcal{Z}_n(\nu_i) = (-1)^{i+m}\|\mathcal{Z}_n\|_{\mathbb{C}[0,\nu_n]} = (-1)^{i+m}B, \quad i = 0, \dots, n, \\ &\frac{d}{dt}\mathcal{Z}_n(\nu_n) = 0. \end{aligned} \tag{3.4}$$

In other words, if $N < n$, then $\{\nu_i\}_{i=0}^N$ are the points of alternance of $\mathcal{Z}_n$ in the whole interval $[0,1]$ (see Figure 3.3.1, (A)). Therefore, according to the sufficient conditions (3.1), $\mathcal{Z}_n$ is extremal for the Kolmogorov–Landau problem

$$f^{(m)}(0) \to \sup, \quad f \in W^r H^\omega[0,1],\ \|f\|_{\mathbb{C}[0,1]} \le B. \tag{3.5}$$

If $N = n$, then $\{\nu_i\}_{i=0}^n$ are the points of alternance of $\mathcal{Z}_n$ only in $[0,\nu_n] \subset [0,1]$ (see Figure 3.3.1, (B)). Consequently, $\mathcal{Z}_n$ is an extremal function of the problem

$$f^{(m)}(0) \to \sup, \quad f \in W^r H^\omega[0,1],\ \|f\|_{\mathbb{C}[0,\nu_n]} \le B. \tag{3.6}$$

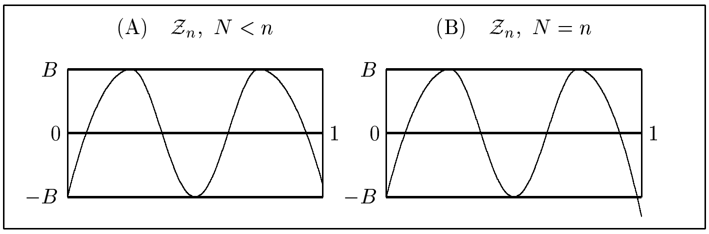

Fig. 3.3.1. Schematic graphs of $\mathcal{Z}_n$ for $N < n$ and $N = n$

However, the following result guarantees that if n is large enough, then N in the statement of Theorem 3.3.1 does not exceed $n-1$, i.e. Case (II)(A) holds and $\mathcal{Z}_n$ is extremal for (3.5).

3.3.3. *Extremality of $\mathcal{Z}_n$ in the global Kolmogorov–Landau problem*

Corollary 3.3.2. *Let $N = N(B, n, r, m, \omega)$ with $r \le N \le n$ be the integer whose existence is proclaimed in Theorem* 3.3.1. *Then* $N \le \max\{r+1, \omega(1)/(2B)\}$.

Proof. In the proof of Theorem 3.3.1 in Chapter 4 we will show that $\{\nu_i\}_{i=1}^N$ are simple (of multiplicity one) zeros of $(d/dt)\mathcal{Z}_n(t)$ in $[0, \nu_N]$. Thus,

$$(3.7)\qquad 2NB = \sum_{i=1}^{N} (-1)^{i+m}[\mathcal{Z}_n(\nu_i) - \mathcal{Z}_n(\nu_{i-1})] = \sum_{i=1}^{N} \int_{\nu_{i-1}}^{\nu_i} \left[(-1)^{i+m}\frac{d}{dt}\mathcal{Z}_n(t)\right] dt$$
$$= \sum_{i=1}^{N} \int_{\nu_{i-1}}^{\nu_i} \left|\frac{d}{dt}\mathcal{Z}_n(t)\right| dt = \|\mathcal{Z}_n'\|_{\mathbb{L}_1[0,\nu_N]}.$$

If $N \ge r+1$, then $\mathcal{Z}_n' \in W^{r-1}H^{\omega}[0,1]$ has at least r zeros at the interior points $\{\nu_i\}_{i=1}^r$ of alternance of $\mathcal{Z}_n$. Using (2.2.2), we obtain

$$(3.8)\qquad \|\mathcal{Z}_n'\|_{\mathbb{L}_1[0,\nu_n]} \le \|\mathcal{Z}_n'\|_{\mathbb{C}[0,1]} \le \omega(1).$$

Combining (3.7), (3.8), we infer that if $N \ge r+1$, then

$$(3.9)\qquad N \le \omega(1)/(2B). \ \blacksquare$$

The result of Corollary 3.3.2 explains that it is sufficient to pick $n = r+2+[\omega(1)/(2B)]$ to ensure the extremality of $\mathcal{Z}_n$ for the problem (3.5).

Let $\mathcal{Z}_{B,\Gamma}$ be an extremal function of the problem

$$f^{(m)}(0) \to \sup, \quad f \in W^r H^{\omega}[0,\Gamma],\ \|f\|_{\mathbb{C}[0,\Gamma]} \le B.$$

An application of a limiting procedure to any sequence $\{\mathcal{Z}_{B,\Gamma_k}\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\Gamma_k = \infty$ produces a pointwise limiting function extremal in the Schoenberg–Cavaretta case of the Kolmogorov problem

$$\|f^{(m)}\|_{\mathbb{L}_\infty(\mathbb{R}_+)} \to \sup, \quad f \in W^r H^{\omega}(\mathbb{R}_+),\ \|f\|_{\mathbb{L}_\infty(\mathbb{R}_+)} \le B.$$

One can consult [10] for details.

3.3.4. *Zolotarev and Chebyshev ω-polynomials of the Kolmogorov–Landau problem*

Case $0 < m < r$. In [6] we showed the existence of a constant $\widehat{B} = \widehat{B}(r, m, \omega)$ such that the generating kernel $K(t) = K(B, r, m, \omega; t)$ has exactly one point of sign change in $[0, 1]$ for all $B \ge \widehat{B}$. Therefore, for such B's the kernel $F(t) = \int_0^t K(x)\,dx$ is simple in the sense of Definition 1.1.2. Then, by Corollary 1.1.5, the extremal function $\mathcal{Z}$ has the full modulus of continuity ω on $[0, 1]$, i.e. $\omega(\mathcal{Z}^{(r)}; t) = \omega(t)$ for $t \in [0, 1]$, and Theorem 3.3.1 guarantees the existence of $r + 1$ alternance points in $[0, 1]$. In view of this inheritance of the properties of classical Zolotarev polynomials of degree $r + 1$, we call the functions $\mathcal{Z}$ for $B > \widehat{B}$ the *Zolotarev ω-polynomials.*

The standard Zolotarev polynomials of degree $r + 1$ constitute the set of extremal functions in the problem (0.1) for $n = r+1$ and all B exceeding the norm $2^{-2r-1}/(r + 1)!$ of the Chebyshev polynomial of degree $r + 1$. In the special case of the Hölder moduli of continuity $\omega(t) = t^\alpha$ for $0 < \alpha < 1$, the constant $\widehat{B}$ was also shown in [6] to coincide with the norm of the Chebyshev ω-polynomial, i.e., the extremal function $\mathcal{Z}$ with the complete alternance at $r + 2$ points and the full modulus of continuity on $[0, 1]$.

Case $m = r$. The Chebyshev Theorem 2.1.2 guarantees the existence of a unique function T_r with the properties

$$T_r^{(r+1)}(x) = -\omega'(x), \quad 0 \le x \le 1,$$
$$T_r(\nu_i) = (-1)^{i+r}\|T_r\|_{\mathbb{C}[0,1]}, \quad i = 0, \dots, r + 1,$$

for some $\{\nu_i\}_{i=0}^{r+1}$ with $0 = \nu_0 < \nu_1 < \ldots < \nu_r < \nu_{r+1} = 1$.

Let $N_r(\omega) := \|T_r\|_{\mathbb{C}[0,1]}$. Then any extremal function $\mathcal{Z}$ of the problem (3.5) for $m = r$ and $B > N_r(\omega)$ has exactly r points of alternance in $[0, 1]$, and $\mathcal{Z}^{(r+1)}(t) = -\omega'(t)$ (see [6] for details).

4. Proof of the main result

4.0. Preliminary assumptions and the choice of absolute constants. Theorem 3.3.1 will be proved first for a strictly concave modulus of continuity ω on $[0, 1]$ with the additional properties

$$\lim_{t\to 0+} \omega'(t) < \infty, \quad \lim_{t\to 1-} \omega'(t) > 0. \tag{0.1}$$

Then we define

$$O_\omega := \lim_{t\to 0+} \omega'(t), \quad L_\omega := \tfrac{1}{2} \lim_{t\to 1-} \omega'(t). \tag{0.2}$$

Notice that the extension

$$\omega(t) := \omega(1) - L_\omega/2 + L_\omega\sqrt{t^2 - 3/4}, \quad t > 1, \tag{0.3}$$

of ω from $[0, 1]$ to $\mathbb{R}_+$ also satisfies the inequality $\omega'(t) > L_\omega$, $t > 0$.

4.0.1. *Mutual inequalities for positive constants $A, C, \Lambda, \Pi, \varepsilon$.* Let $\{\lambda_i := \lambda_i(r, 2n - r + 2, L_\omega, O_\omega)\}_{i=1}^r$, where the constants $\{\lambda_i(r, n, P, M)\}_{i=1}^r$ are defined in (2.2.17), (2.2.19) and (2.2.22). Let also the constants $\{V_i\}_{i\in\mathbb{N}}$ be defined in Proposition 2.2.2 and Lemma 2.2.3, and

$$M_\omega := L_\omega[C\lambda_{r-1}/2-1], \quad \delta := \frac{2B}{\omega(A)A^{r-1}}. \tag{0.4}$$

The constants $A, C, \Lambda, \Pi, \varepsilon$ are chosen to satisfy the following inequalities:

(C.1) $C > 4$,

(C.2) $\max\{\lambda_{r-1}, \lambda_r\}C > 25$,

(C.3) $M_\omega V_{r-1} > 4rB$;

(A.1) $A \geq 12\left[\frac{2nB(2n-r+2)^r}{L_\omega V_r}\right]^{1/(r+1)}$,

(A.2) $A > 12C$;

(Λ.1) $\Lambda < \delta$;

(Π.1) $\Pi < A/5$,

(Π.2) $\Pi < nB[\omega(A)A^{r-1} + 12nB/A]^{-1}$;

(ε.1) $\varepsilon < \Pi/(2n)$,

(ε.2) $\varepsilon < \Lambda/(20n)$,

(ε.3) $\varepsilon < 1/(2n)$.

4.1. Borsuk's mapping $\varkappa : \mathbb{S}^{2n-r+1} \to \mathbb{R}^{2n-r+1}$

4.1.1. *Definitions of collections of points.* Let A satisfy (A.1), (A.2), and $\mathbb{S}$ be the $(2n-r+1)$-dimensional sphere of radius A in l_1^{2n-r+2}:

$$\mathbb{S} := \Big\{ s = (s_1, \ldots, s_{2n-r+2}) \in \mathbb{R}^{2n-r+2} \;\Big|\; \sum_{i=1}^{2n-r+2} |s_i| = A \Big\}. \tag{1.1}$$

Fix ε satisfying (ε.1)–(ε.3). For each $s = (s_1, \ldots, s_{2n-r+2}) \in \mathbb{S}$ we introduce collections of points $\{t_i = t_i(s)\}_{i=0}^{2n-r+2}$, $\{T_i = T_i(s)\}_{i=1}^{n}$, $\{\overline{t}_i = \overline{t}_i(s)\}_{i=0}^{n}$ and $\{\gamma_i = \gamma_i(s)\}_{i=0}^{2n-r+2}$ by

$$\begin{aligned}
&t_0(s) := 0, \quad t_i(s) = \sum_{j=1}^{i} |s_j|, && i = 1, \ldots, 2n-r+2,\\
&T_i(s) := \frac{t_i(s) + \varepsilon i}{1 + n\varepsilon}, && i = 1, \ldots, n,\\
&\overline{t}_i(s) = \min\{t_i(s), 1\}, && i = 0, \ldots, n,\\
&\gamma_i(s) = A - t_{2n-r+2-i}(s), && i = 0, \ldots, 2n-r+2.
\end{aligned} \tag{1.2}$$

We define the collection of points $\{\overline{\beta}_i = \overline{\beta}_i(s)\}_{i=0}^{n-r+1}$ as follows: $\overline{\beta}_0(s) := 0$, and for $i = 1, \ldots, n-r+1$,

$$\overline{\beta}_i(s) = \begin{cases} \overline{t}_{i-1}(s), & \text{if } \gamma_i(s) \leq \overline{t}_{i-1}(s),\\ \gamma_i(s), & \text{if } \overline{t}_{i-1}(s) < \gamma_i(s) < \overline{t}_{i+r-1}(s),\\ \overline{t}_{i+r-1}(s), & \text{if } \gamma_i(s) \geq \overline{t}_{i+r-1}(s). \end{cases} \tag{1.3}$$

We have

$$\overline{t}_{i-1}(s) \le \overline{\beta}_i(s) \le \overline{t}_{i+r-1}(s), \quad i = 1, \dots, n-r+1, \ s \in \mathbb{S}. \tag{1.4}$$

By (1.2) and (1.3),

$$\overline{t}_i \in [0,1], \quad i = 0, \dots, n; \quad \overline{\beta}_i \in [0,1], \quad i = 0, \dots, n-r+1. \tag{1.5}$$

The set $\{\zeta_i = \zeta_i(s)\}_{i=0}^{2n-r+1}$ is the union of $\{\overline{t}_i(s)\}_{i=0}^{n}$ and $\{\overline{\beta}_i(s)\}_{i=1}^{n-r+1}$ with the following order:

$$\begin{cases} \zeta_{2i}(s) := \overline{t}_i(s), & i = 0, \dots, n-r+1, \\ \zeta_{2i-1}(s) := \overline{\beta}_i(s), & i = 1, \dots, n-r+1, \\ \zeta_{i+n+1-r}(s) := \overline{t}_i(s), & i = n-r+2, \dots, n, \end{cases} \tag{1.6}$$

i.e., $(\zeta_0, \dots, \zeta_{2n-r+1}) = (\overline{t}_0, \overline{\beta}_1, \overline{t}_1, \overline{\beta}_2, \overline{t}_2, \overline{\beta}_3, \dots, \overline{\beta}_{n-r+1}, \overline{t}_{n-r+1}, t_{n-r+2}, \dots, \overline{t}_n)$.

Let

$$\xi_i(s) = \frac{\zeta_i(s) + \varepsilon i}{1 + \varepsilon(2n-r+2)}, \quad i = 0, \dots, 2n-r+1. \tag{1.7}$$

Next, we introduce the collections $\{\tau_i = \tau_i(s)\}_{i=0}^{n}$ by

$$\tau_i(s) := \begin{cases} \xi_{2i}(s), & i = 0, \dots, n-r+1, \\ \xi_{i+n+1-r}(s), & i = n-r+2, \dots, n. \end{cases} \tag{1.8}$$

and $\{\beta_i = \beta_i(s)\}_{i=0}^{n-r+2}$ by

$$\beta_0(s) := 0; \quad \beta_i(s) = \xi_{2i-1}(s), \quad i = 1, \dots, n-r+1; \quad \beta_{n-r+2}(s) := \tau_n(s). \tag{1.9}$$

According to (1.4) and (1.6)–(1.9), we have

$$\tau_{i-1} = \frac{\overline{t}_{i-1} + 2(i-1)\varepsilon}{1 + \varepsilon(2n-r+2)} \le \frac{\overline{\beta}_i + 2(i-1)\varepsilon}{1 + \varepsilon(2n-r+2)} < \frac{\overline{\beta}_i + (2i-1)\varepsilon}{1 + \varepsilon(2n-r+2)} = \beta_i \tag{1.10}$$

and

$$\beta_i = \frac{\overline{\beta}_i + (2i-1)\varepsilon}{1 + \varepsilon(2n-r+2)} \le \frac{\overline{t}_{i+r-1} + (2i-1)\varepsilon}{1 + \varepsilon(2n-r+2)} < \tau_{i+r-1} \tag{1.11}$$

for all $i = 1, \dots, n-r+1$. Summarizing, for all $s \in \mathbb{S}$ we have the strict inequalities

$$\tau_{i-1}(s) < \beta_i(s) < \tau_{i+r-1}(s), \quad i = 1, \dots, n-r+1. \tag{1.12}$$

By (1.6)–(1.9) and (1.5), all points $\{\tau_i(s)\}_{i=1}^{n}$ and $\{\beta_i(s)\}_{i=0}^{n-r+2}$ belong to $[0,1]$. Moreover, from (1.7)–(1.9) it follows that $\{\tau_i(s)\}_{i=1}^{n}$ and $\{\beta_i(s)\}_{i=1}^{n-r+2}$ are *uniformly separated*: for all $s \in \mathbb{S}$,

$$\begin{aligned} |\tau_i(s) - \tau_{i-1}(s)| &\ge \frac{\varepsilon}{1 + \varepsilon(2n-r+2)}, \quad i = 1, \dots, n. \\ |\beta_i(s) - \beta_{i-1}(s)| &\ge \frac{\varepsilon}{1 + \varepsilon(2n-r+2)}, \quad i = 1, \dots, n-r+2. \end{aligned} \tag{1.13}$$

In our construction we also use the collection

$$\mathcal{B}_i(s) := \frac{\gamma_i(s) + (2i-1)\varepsilon}{1 + \varepsilon(2n-r+2)}, \quad i = 1, \dots, n-r+2. \tag{1.14}$$

Comparing (1.11) and (1.14) we observe that

$$\{\mathcal{B}_i(s) = \beta_i(s) \Leftrightarrow \overline{t}_i(s) \le \gamma_i(s) \le \overline{t}_{i+r-1}(s)\}, \quad i = 1, \dots, n-r+1. \tag{1.15}$$

4.1.2. *Kernels* K_s. For each $l = r, \ldots, n$, we define $\{\alpha_i^l(s)\}_{i=0}^l$ as solutions of

$$(1.16)\qquad \begin{cases} \sum_{i=0}^{l} \alpha_i^l(s)[\tau_i(s)]^j = m!\delta_{m,j}, & j = 0, \ldots, r-1, \\ \sum_{i=0}^{l} \alpha_i^l(s)(\tau_i(s) - \beta_l(s))_+^{r-1} = 0, & l = 1, \ldots, l-r+1. \end{cases}$$

Inequalities (1.12) guarantee the unique solvability of (1.16) (consult [32]).

Next, we introduce the kernels

$$(1.17)\qquad K_s^l(t) = -\frac{1}{(r-1)!} \sum_{i=0}^{l} \alpha_i^l(s)(\tau_i(s) - t)_+^{r-1}, \quad l = r, \ldots, n.$$

Define the function $\Gamma : \mathbb{S} \to [0,1]$ by the formula

$$(1.18)\qquad \Gamma(s) = \begin{cases} 1, & t_r(s) \geq 1, \\ 1 - \min_{r \leq i \leq n} |1 - t_i(s)|, & t_r(s) \leq 1 \ \wedge \ t_n(s) \geq 1, \\ t_n(s), & t_n(s) \leq 1. \end{cases}$$

Let the constants C and Λ satisfy inequalities (C.1)–(C.3) and (Λ.1), respectively.

Then, for each $s \in \mathbb{S}$, we define the function $\psi_s : \mathbb{R} \to \mathbb{R}_+$ by

$$(1.19)\qquad \psi_s(t) = \begin{cases} 0, & t \in \mathbb{R} \setminus [\Gamma(s) - \Lambda, C], \\ \dfrac{t - (\Gamma(s) - \Lambda)}{\Lambda}, & \Gamma(s) - \Lambda \leq t \leq \Gamma(s), \\ \dfrac{C - t}{C - \Gamma(s)}, & \Gamma(s) \leq t \leq C, \end{cases}$$

i.e., $\operatorname{supp} \psi_s = [\Gamma(s) - \Lambda, C]$, ψ_s is linear on $[\Gamma(s) - \Lambda, \Gamma(s)]$ and $[\Gamma(s), C]$, and $\psi_s(\Gamma(s)) = 1$.

We introduce the kernels

$$(1.20)\qquad K_s(t) = \sum_{l=r}^{n} \psi_s(t_l(s))(\mathcal{B}_{l-r+2}(s) - \tau_l(s))_+ K_s^l(t)$$

and

$$(1.21)\qquad \widehat{K}_s(t) = \operatorname{sign} s_{2n-r+3-i}(|K_s(t)| + \varepsilon), \quad \gamma_{i-1}(s) < t < \gamma_i(s), \ i = 1, \ldots, 2n-r+2,$$

where the function ψ_s is defined in (1.19), the kernels $\{K_s^l\}_{l=r}^n$ in (1.16), (1.17), and the points $\{\mathcal{B}_i(s)\}_{i=1}^{n-r+2}$ in (1.14).

From (1.21) it follows that $\widehat{K}_s$ can have at most $2n - r + 1$ sign changes in $[0, A]$. Consequently, $\widehat{K}_s \in M_{2n-r+2}[0, T_n(s)]$ in the notations of Definition 1.2.5. Set

$$(1.22)\qquad I_s = I(\widehat{K}_s) := \int_0^{T_n(s)} \widehat{K}_s(t)\, dt.$$

4.1.3. *Functions* G_s, W_s, U_s *and* X_s. We define G_s on $[0, T_n(s)]$ as the extremal function of the problem

$$(1.23)\qquad \int_0^{T_n(s)} h(t)\widehat{K}_s(t)\, dt \to \sup, \quad h \in H_0^\omega[0, T_n(s)].$$

The structure of G_s is described in Theorem 1.2.1. We extend G_s to the interval $[T_n(s), A]$ using the continuity at $T_n(s)$ and the formula

$$\frac{d}{dt}G_s(t) = \operatorname{sign} s_{2n-r+3-i} \cdot \omega'(t), \quad t \in [\gamma_{i-1}(s), \gamma_i(s)] \cap [T_n(s), A], \tag{1.24}$$

where $i = 1, \ldots, 2n-r+2$. Consecutive applications of Lemma 1.1.2 enable us to conclude that $G_s \in H^\omega[0, A]$.

The function W_s is defined as the $(r-1)$st integral of G_s:

$$W_s(x) = \frac{1}{(r-2)!}\int_0^A G_s(t)(x-t)_+^{r-2}\, dt, \quad x \in [0, A], \tag{1.25}$$

vanishing at the origin with all its $r-2$ derivatives $\{W_s^{(k)}\}_{k=1}^{r-2}$.

Let $q_s(t) = \sum_{i=0}^{r-1} a_i(s)t^i$ be the polynomial of degree $r-1$, interpolating W_s at the points $\{T_i(s)\}_{i=1}^r$ introduced in (1.2):

$$q_s(T_i(s)) = W_s(T_i(s)), \quad i = 1, \ldots, r. \tag{1.26}$$

Set

$$U_s(t) = W_s(t) - p_s(t), \quad 0 \le t \le A. \tag{1.27}$$

We mention the following properties of the function $U_s^{(r)} = W_s^{(r)}$. If

$$l_s := \max\{1 \le j \le 2n-r+1 \mid t_j(s) < T_n(s)\},$$

then the kernel $\widehat{K}_s$ from (1.21) may have at most l_s points of sign change in $[0, T_n(s)]$. In Remark 1.2.2 we explained that $U_s^{(r)}|_{[0,T_n(s)]} = W_s^{(r)}|_{[0,T_n(s)]}$ can exhibit at most l_s points of sign change in $[0, T_n(s))$. On the other hand, by (1.24), $U_s^{(r)} = W_s^{(r)}$ changes sign in $[T_n(s), A]$ no more than $2n-r+1-l_s$ times. Therefore, the total number of points of sign change of $U_s^{(r)}$ in $[0, A]$ does not exceed $2n-r+1$. Moreover, from (1.24) and (1.2.12), (1.2.14) we derive the following expression for $U_s^{(r)}$ at all but a finite number of points $x \in [0, A]$:

$$U_s^{(r)}(x) = \omega'(e_x) \quad \text{for some } e_x \in [0, A]. \tag{1.28}$$

In particular, by our extension of ω to the entire half-line $\mathbb{R}_+$,

$$\text{(1.29)}\quad \text{(i)}\quad |U_s^{(r)}(t)| \ge \inf_{t\in(0,A]} \omega'(t) =: L_\omega \quad \text{for } t \in [0, A];$$

$$\text{(ii)}\quad \|U_s^{(r)}\|_{\mathbb{L}_\infty[0,A]} \le \sup_{t\in(0,A]} \omega'(t) =: O_\omega.$$

We define a piecewise continuous function X_s on $[0, A]$ by

$$X_s(t) = \operatorname{sign} s_i |U_s(t)|, \quad t_{i-1}(s) \le t \le t_i(s),\ i = 1, \ldots, 2n-r+2. \tag{1.30}$$

4.1.4. *The trap.* The piecewise continuous function Z_s is defined on $[0, A-(C-1)]$ by

$$Z_s(t) = \begin{cases} X_s(t), & 0 \le t \le 1, \\ X_s(t+C-1), & 1 \le t \le A-(C-1), \end{cases} \tag{1.31}$$

i.e. the graph of Z_s on $[0, A-(C-1)]$ consists of pieces of the graph of X_s on $[0, 1]$ and $[C, A]$ shifted by $C-1$ to the left, while the part of the graph of X_s between 1 and C is omitted.

We also define the collection of points $\{\mathcal{T}_i(s)\}_{i=0}^{2n-r+2}$ by

$$\mathcal{T}_i(s) = \begin{cases} t_i(s), & 0 \le t_i(s) < 1, \\ 1, & 1 \le t_i(s) \le C, \\ t_i(s) - C + 1, & C < t_i(s) \le A. \end{cases} \tag{1.32}$$

The interval $[1, C]$ is called the *trap*.

4.1.5. *The mapping* $\varkappa : \mathbb{S} \to \mathbb{R}^{2n-r+1}$. Let

$$\widetilde{H}_s(t) = \int_0^t Z_s(\tau)\,d\tau, \quad 0 \le t \le A - (C-1). \tag{1.33}$$

Let the constant $\Pi > 0$ satisfy (Π.1), (Π.2), and

$$\Theta_s := \max\{\Pi, \mathcal{T}_n(s)\}. \tag{1.34}$$

Let $p_s(t) = \sum_{i=0}^{n-1} b_i(s)t^i$ be the polynomial of best approximation for $\widetilde{H}_s$ on $[0, \Theta_s]$:

$$\|\widetilde{H}_s - p_s\|_{\mathbb{C}[0,\Theta_s]} = \min_{p \in P_{n-1}} \|\widetilde{H}_s - p\|_{\mathbb{C}[0,\Theta_s]}, \tag{1.35}$$

where P_{n-1} is the linear space of polynomials of degree $n-1$. Set

$$H_s(t) = \widetilde{H}_s(t) - p_s(t), \quad 0 \le t \le A - (C-1). \tag{1.36}$$

Define

$$D_i(s) := \begin{cases} |\widetilde{H}_s(\mathcal{T}_i(s)) - \widetilde{H}_s(\mathcal{T}_{i-1}(s))| - \dfrac{2nB}{A}(t_i(s) - t_{i-1}(s)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, n, \\ -\dfrac{2nB}{A}(t_i(s) - t_{i-1}(s)), \qquad\qquad i = n+1, \dots, 2n-r+2, \end{cases} \tag{1.37}$$

$$D(s) := \sum_{i=1}^{2n-r+2} (-1)^{i+m} \operatorname{sign} s_i D_i(s). \tag{1.38}$$

Finally, we define the mapping $\varkappa : \mathbb{S} \to \mathbb{R}^{2n-r+1}$ by

$$\varkappa(s) := \{U_s(T_i(s))\}_{i=r+1}^{n} \times \{b_i(s)\}_{i=1}^{n-1} \times I_s \times D(s), \tag{1.39}$$

where the function U_s, collection $\{T_i(s)\}_{i=1}^n$, mappings I_s and $D(s)$ are introduced in (1.27), (1.2), (1.22) and (1.38), respectively, and $\{b_i(s)\}_{i=0}^{n-1}$ are the coefficients of the polynomial p_s with the property (1.35).

4.2. Continuity of the mapping $\varkappa$ on the sphere $\mathbb{S}_A^{2n-r+2}$. In the following lemma we prove the continuity of the mapping $s \mapsto \varkappa(s)$ defined in (1.39).

LEMMA 4.2.1. *The mapping* $\varkappa : \mathbb{S} \to \mathbb{R}^{2n-r+1}$ *is continuous.*

PROOF. First, let us establish the continuity of $s \mapsto G_s$ in $\mathbb{C}[0, A]$.

MICROLEMMA 4.2.2. *The mapping* $s \mapsto G_s$ *is continuous on* $\mathbb{S}$.

PROOF. From (1.12) it follows that the Fredholm determinants of the system (1.16) never vanish on $\mathbb{S}$. Cramer's formula coupled with the continuity of the mappings $s \mapsto \{\tau_i(s)\}_{i=0}^n$ and $s \mapsto \{\beta_i(s)\}_{i=1}^{n-r+1}$ implies the continuous dependence of the coefficients

$\{\alpha_i^l(s)\}_{i=0}^l$ on s for all $l = r, \dots, n$. The continuity of $s \mapsto \Gamma(s)$ and $s \mapsto K_s^l$, $s \mapsto \psi_s$ in $\mathbb{C}[0, A]$ then implies the continuity of $s \mapsto K_s$ in $\mathbb{C}[0, A]$ and $s \mapsto \widehat{K}_s$ in $\mathbb{L}_1[0, A]$.

As we remarked in §4.1.2, $\widehat{K}_s \in M_{2n-r+2}[0, T_n(s)]$ for all $s \in \mathbb{S}$. By (1.2), $T_n(s) \geq \varepsilon/(1 + \varepsilon n) =: c$ for all $s \in \mathbb{S}$. Define the dilation $\widetilde{K}_s$ of $\widehat{K}_s$ on $[0, c]$ by

$$\widetilde{K}_s(t) = \widehat{K}_s(tT_n(s)/c), \quad t \in [0, c], \ s \in \mathbb{S}.$$

The continuity of $s \mapsto \widehat{K}_s$ in $\mathbb{L}_1[0, A]$ implies the continuity of $\{\widehat{K}_s\big|_{[0,T_n(s)]}\}_{s\in\mathbb{S}}$ in the integral metric: for all $s \in \mathbb{S}$, $\|\widetilde{K}_{s'} - \widetilde{K}_s\|_{\mathbb{L}_1[0,c]} \to 0$ as $\mathbb{S} \ni s' \to s$.

Put

$$\begin{aligned} a_{s_1,s_2} &:= \min\{T_n(s_1), T_n(s_2)\}, \\ b_{s_1,s_2} &:= \max\{T_n(s_1), T_n(s_2)\}, \end{aligned} \quad (s_1, s_2) \in \mathbb{S} \times \mathbb{S}. \tag{2.1}$$

An application of Corollary 1.2.5 to $\{\widehat{K}_s\big|_{[0,T_n(s)]}\}_{s\in\mathbb{S}}$ shows that the extremal function $G_s\big|_{[0,T_n(s)]}$ of the problem (1.23) depends continuously on s: for all $s_1 \in \mathbb{S}$,

$$\|G_{s_1} - G_{s_2}\|_{\mathbb{C}[0,a_{s_1,s_2}]} \to 0 \quad \text{as } \mathbb{S} \ni s_2 \to s_1. \tag{2.2}$$

The continuity of $G_s|_{[T_n(s),A]}$ follows immediately from (2.2), (1.24) and the continuity of $s \mapsto \{t_i(s)\}_{i=n}^{2n-r+2}$: for all $s_1 \in \mathbb{S}$,

$$\|G_{s_1} - G_{s_2}\|_{\mathbb{L}_1[b_{s_1,s_2},A]} \to 0 \quad \text{as } \mathbb{S} \ni s_2 \to s_1. \tag{2.3}$$

Pick any pair $(s_1, s_2) \in \mathbb{S} \times \mathbb{S}$ and $x \in [a_{s_1,s_2}, A]$. Recalling the inclusion $G_s \in H^\omega[0, A]$, we derive

$$\begin{aligned} |G_{s_1}(x) - G_{s_2}(x)| &\leq |G_{s_1}(a_{s_1,s_2}) - G_{s_2}(a_{s_1,s_2})| + |G_{s_1}(b_{s_1,s_2}) - G_{s_1}(a_{s_1,s_2})| \\ &\quad + |G_{s_2}(b_{s_1,s_2}) - G_{s_2}(a_{s_1,s_2})| + \int_{b_{s_1,s_2}}^{A} |G'_{s_1}(t) - G'_{s_2}(t)|\,dt \\ &\leq \|G_{s_1} - G_{s_2}\|_{\mathbb{C}[0,a_{s_1,s_2}]} + 2\omega(|T_n(s_1) - T_n(s_2)|) \\ &\quad + \|G_{s_1} - G_{s_2}\|_{\mathbb{L}_1[b_{s_1,s_2},A]}. \end{aligned} \tag{2.4}$$

Combination of (2.2)–(2.4) completes the proof of Microlemma 4.2.2. ■

By Microlemma 4.2.2 and (1.25), the mapping $s \mapsto W_s$ is continuous in $\mathbb{C}[0, A]$. From the separation property (1.13) and the Lagrange formula for the interpolating polynomial $q_s(t)$ we deduce the continuity of the coefficients $\{a_i(s)\}_{i=0}^{r-1}$ and the mapping $s \mapsto U_s$ in $\mathbb{C}[0, A]$. Consequently, we have established the continuity of $s \mapsto X_s$, $s \mapsto Z_s$ and $s \mapsto \widetilde{H}_s$ in $\mathbb{L}_1[0, A]$, $\mathbb{L}_1[0, A - (C - 1)]$ and $\mathbb{C}[0, A - (C - 1)]$, respectively.

The continuity of the coefficients $\{b_i(s)\}_{i=0}^n$ of the polynomial p_s of best approximation for $\widetilde{H}_s$ on $[0, \Theta_s]$ follows from the uniqueness of p_s and separation of the length Θ_s of the interval $[0, \Theta_s]$ from zero: $\Theta_s \geq \Pi$, $s \in \mathbb{S}$.

It remains to prove the continuity of the mapping $s \mapsto D(s)$ defined in (1.37), (1.38). By (1.26), (1.27), U_s has at least r distinct zeros $\{T_i(s)\}_{i=1}^r$ in $[0, A]$. We have also shown in §4.1.3 that $U_s^{(r-1)} = G_s + \lambda \in H^\omega[0, A]$. Applying Proposition 2.2.1, we derive

$$\|U_s^{(k)}\|_{\mathbb{C}[0,A]} \leq A^{r-1-k}\omega(A), \quad s \in \mathbb{S}. \tag{2.5}$$

A discontinuity of $D(s)$ may arise at some point $\widehat{s} = (\widehat{s}_1, \dots, \widehat{s}_{2n-r+2})$ if and only if $\widehat{s}_k = 0$ and $D_k(\widehat{s}) \neq 0$ for some $k = 1, \dots, 2n-r+2$. However, (1.37) along with (2.5) for $k = 0$ show that

$$|D_i(s)| \leq E|s_i|, \quad i = 1, \dots, 2n-r+2, \tag{2.7}$$

for some constant E independent of s. Therefore, the entries s_i tend to zero with the corresponding factors $D_i(s)$. ∎

4.3. Signs of entries of solutions of the equation $\varkappa(s) = 0$. The mappings $s \mapsto \{U_s(T_i(s))\}_{i=r+1}^n$, $s \mapsto \{b_i(s)\}_{i=0}^n$ and $s \mapsto I(\widehat{K}_s)$ are odd on the sphere $\mathbb{S}$, since so are $s \mapsto U_s$, $s \mapsto \widetilde{H}_s$ and $s \mapsto \widehat{K}_s$, respectively.

Summarizing, we showed that $\varkappa$ is both *continuous* and *odd* on $\mathbb{S}$. The Borsuk Theorem 2.1.1 ensures the existence of a vector $s^* \in \mathbb{S}$ such that $\varkappa(s^*) = 0$, or, equivalently,

(3.1) (A) $\quad U_{s^*}(T_i(s^*)) = 0 \quad \text{for } i = r+1, \dots, n;$

(B) $\quad b_i(s^*) = 0 \quad \text{for } i = 1, \dots, n-1;$

(C) $\quad I_{s^*} = I(\widehat{K}_{s^*}) = 0;$

(D) $\quad D(s^*) = 0.$

We introduce a more convenient notation for collections of points at $s = s^*$.

NOTATIONS. All collections $(\{t_i(s)\}_{i=0}^{2n-r+2}, \{\tau_i(s)\}_{i=0}^n, \dots)$ at the point $s = s^*$ are marked with an asterisk, e.g.,

$$\{t_i^* := t_i(s^*)\}_{i=0}^{2n-r+2}, \quad \{\tau_i^* := \tau_i(s^*)\}_{i=0}^n, \quad \{\mathcal{T}_i^* = \mathcal{T}_i(s^*)\}_{i=0}^n, \ \dots \tag{3.2}$$

By (3.1)(B) and (1.36),

$$H_{s^*}(t) = \widetilde{H}_{s^*}(t) - b_0(s^*), \quad 0 \leq t \leq A. \tag{3.3}$$

4.3.1. *The identification of the interval* $[0, \Theta(s^*)]$. In the next stage of our analysis we show that $\Theta_{s^*} = \mathcal{T}_n^*$, which, by (1.34), is equivalent to verifying the inequality $\mathcal{T}_n^* \geq \Pi$. As the following lemma shows, this property results from the incompatibility of the inequality $\mathcal{T}_n^* \leq \Pi$ and the property $D(s^*) = 0$.

LEMMA 4.3.1. *Let the constant* Π *satisfy* (Π.1), (Π.2), *and* $\{\mathcal{T}_i^*\}_{i=1}^n$ *be defined in* (3.2). *Then*

$$\mathcal{T}_n^* > \Pi.$$

PROOF. Assume that, on the contrary,

$$\mathcal{T}_n^* \leq \Pi < 1. \tag{3.4}$$

In particular, $\mathcal{T}_n^* = t_n^*$ by (1.32). Then, by (1.2) and (3.4), we have

$$T_i^* = \frac{t_i^* + \varepsilon i}{1 + n\varepsilon} \leq \frac{\Pi + \varepsilon i}{1 + n\varepsilon} \leq 2\Pi, \quad i = 1, \dots, n, \tag{3.5}$$

for $\varepsilon < \Pi/n$, i.e. all points $\{T_i^*\}_{i=1}^n$ lie in $[0, 2\Pi]$.

From (3.4) and (Π.1) we also observe that

$$\gamma_{2n-r+2-i}^* := A - t_i^* \geq A - t_n^* \geq A - \Pi \geq 2\Pi, \quad i = 1, \dots, n, \tag{3.6}$$

i.e. the points $\{\gamma_i^*\}_{n-r+2}^{2n-r+1}$ lie outside $[0, T_n^*] \subset [0, 2\Pi]$.

By (3.6) and (1.21), $\widehat{K}_{s^*}$ can have at most $n-r+1$ sign changes in $[0, T_n^*]$ at $\{\gamma_i^*\}_{i=1}^{n-r+1}$. In addition, by (3.1)(C), $\int_0^{T_n^*} \widehat{K}_{s^*}(t)\,dt = 0$. Therefore,

$$\widehat{K}_{s^*} \in \mathcal{M}_l^0[0, T_n^*] \quad \text{for some } l \le n-r+2$$

(see Definition 1.2.1). By Remark 1.2.2 and Corollary 1.2.3, the number of points of sign change of $G'_{s^*}(t) = U_{s^*}^{(r)}(t)$ is one less than the number of sign changes of $\widehat{K}_s$ in $[0, T_n^*]$, namely, $l-2 \le n-r$.

On the other hand, U_{s^*} has n distinct zeros $\{T_i^*\}_{i=1}^n$ in $[0, T_n^*]$. Therefore, by Rolle's theorem, $U_{s^*}^{(r)}$ has at least $n-r$ sign changes in $[0, T_n^*]$.

This argument shows that $\widehat{K}_s$ has precisely $n-r+1$ points $\{\gamma_i^*\}_{i=1}^{n-r+1}$ of sign change inside $[0, T_n^*] \subset [0, 2\Pi]$:

$$\gamma_i^* = A - t^*_{2n-r+2-i} \le 2\Pi, \quad i = 1, \dots, n-r+1, \tag{3.7}$$

or, equivalently,

$$t_i^* \ge A - 2\Pi, \quad i = n+1, \dots, 2n-r+1. \tag{3.8}$$

By (3.1)(D) and (1.38)

$$(-1)^{n+1} \operatorname{sign} s^*_{n+1} D_{n+1}(s^*) = -\sum_{\substack{i=1\\ i\ne n+1}}^{2n-r+2} (-1)^i \operatorname{sign} s_i^* D_i(s^*). \tag{3.9}$$

But from (3.4), (3.8), and (Π.2), we can show that the left-hand side of (3.9) is substantially greater than the right-hand side.

Indeed, invoking (1.33) and (2.5) for $k = 0$, we have, for $i = 1, \dots, n$,

$$\begin{aligned} |D_i(s^*)| &= \left| |\widetilde{H}_{s^*}(t_i^*) - \widetilde{H}_{s^*}(t^*_{i-1})| - \frac{2nB}{A}(t_i^* - t^*_{i-1}) \right| \\ &\le \left(\|H'_{s^*}\|_{\mathbb{L}_\infty[0, A-C+1]} + \frac{2nB}{A} \right)(t_i^* - t^*_{i-1}) \\ &\le \left(\|U_{s^*}\|_{\mathbb{C}[0,A]} + \frac{2nB}{A} \right)(t_i^* - t^*_{i-1}) \le \left(A^{r-1}\omega(A) + \frac{2nB}{A} \right)(t_i^* - t^*_{i-1}). \end{aligned} \tag{3.10}$$

Therefore, by (3.10) and (3.4) for $\mathcal{T}_n^* = t_n^*$,

$$\sum_{i=1}^n |D_i(s^*)| \le \left(\omega(A)A^{r-1} + \frac{2nB}{A} \right) t_n^* \le \left(\omega(A)A^{r-1} + \frac{2nB}{A} \right)\Pi. \tag{3.11}$$

On the other hand, (3.8) implies that

$$\sum_{i=n+2}^{2n-r+2} |D_i(s^*)| = \sum_{i=n+2}^{2n-r+2} \frac{2nB}{A}(t_i^* - t^*_{i-1}) = \frac{2nB}{A}(A - t^*_{n+1}) \le \frac{2nB}{A} 2\Pi. \tag{3.12}$$

Combining (3.11) and (3.12), we derive an estimate from above for the absolute value of the right-hand side of (3.9):

$$\Big| \sum_{\substack{i=0\\ i\ne n+1}}^{2n-r+2} \operatorname{sign} s_i^* D_i(s^*) \Big| \le \left[\omega(A)A^{r-1} + \frac{6nB}{A} \right]\Pi. \tag{3.13}$$

But (3.4) and (3.8) imply that $|s^*_{n+1}| = t^*_{n+1} - t^*_n \geq A - 3\Pi$, so we can find an estimate from below for the absolute value of the left-hand side of (3.9):

$$|D_{n+1}(s^*)| \geq \frac{2nB}{A}(A - 3\Pi). \tag{3.14}$$

Combining (3.9), (3.13) and (3.14) leads to

$$\frac{2nB}{A}(A - 3\Pi) \leq \left[\omega(A)A^{r-1} + \frac{6nB}{A}\right]\Pi,$$

or, equivalently,

$$\Pi \geq 2nB\left[\omega(A)A^{r-1} + \frac{12nB}{A}\right]^{-1}, \tag{3.15}$$

contrary to (Π.2). ■

4.3.2. *The sign alternance property of entries of $s^* = (s^*_1, \ldots, s^*_{2n-r+2})$.* By Lemma 4.3.1, $\Theta_{s^*} = \max\{\mathcal{T}^*_n, \Pi\} = \mathcal{T}^*_n$. Then Chebyshev's Theorem 2.1.2 shows that $H_{s^*} = \widetilde{H}_{s^*} - b_0(s^*)$ has $n + 1$ points of alternance in $[0, \mathcal{T}^*_n] = [0, \Theta_{s^*}]$. In particular, $(d/dt)H_{s^*}(t) = (d/dt)\widetilde{H}_{s^*}(t)$ has at least $n - 1$ points of sign change at the points of alternance inside $(0, \mathcal{T}^*_n)$.

On the other hand, by (1.30)–(1.33) and (3.3), $\{\mathcal{T}^*_i\}_{i=1}^{n-1}$ are the only possible points of sign change of $Z_{s^*}(t) = (d/dt)\widetilde{H}_{s^*}(t) = (d/dt)H_{s^*}(t)$ in $[0, \mathcal{T}^*_n]$. Therefore, the points of alternance of H_{s^*} in $[0, \mathcal{T}^*_n]$ coincide with $\{\mathcal{T}^*_i\}_{i=0}^{n}$, i.e.,

$$H_{s^*}(\mathcal{T}^*_i) = (-1)^i \xi \|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}^*_n]}, \quad i = 0, \ldots, n,\ \xi \in \{1, -1\} \text{ fixed}, \tag{3.16}$$

and $(d/dt)H_{s^*}(t) = Z_{s^*}(t)$ does change sign at $\{\mathcal{T}^*_i\}_{i=1}^{n-1}$, i.e.,

$$\operatorname{sign} s^*_i = (-1)^i \xi, \quad i = 1, \ldots, n. \tag{3.17}$$

Notice also that *there can be at most one point t^*_i, $i = 1, \ldots, n-1$, trapped in the interval* $[1, C]$. Otherwise, the following lemma shows that $H'_{s^*} := Z'_{s^*}$ would have less than the required $n - 1$ sign changes in $[0, \mathcal{T}^*_n]$.

LEMMA 4.3.2. *Let $t^*_n > 1$, and*

$$j_1 := \max\{0 \leq i \leq n \mid t^*_i < 1\}, \quad j_2 := \max\{0 \leq i \leq n \mid t^*_i \leq C\}.$$

Then $j_2 - j_1 \leq 1$.

PROOF. By (1.31), $\mathcal{Z}_{s^*}$ can only change sign in $[0, \mathcal{T}^*_n]$ at the points $\{t^*_i\}_{i=1}^{j_1}$, possibly at $t = 1$, and at $\{\mathcal{T}^*_i = t^*_i - C + 1\}_{i=j_2+1}^{n-1}$, i.e. it has at most $j_1 + n - j_2$ points of sign change. Then the presence of $n - 1$ points of sign change implies that $j_1 + n - j_2 \leq n - 1$, i.e., $j_2 - j_1 \leq 1$. ■

Lemma 4.3.2 will be crucial in the identification of the kernel K_{s^*} in Section 4.5.

The mapping $\varkappa$ is odd. Therefore, replacing s^* with $-s^*$ if necessary, we can assume that $\xi = (-1)^m$ in (3.16), (3.17). In particular, by (1.30) and (1.33),

$$\begin{aligned} &\frac{d}{dt}X_{s^*}(t) = (-1)^{i+m}|U_{s^*}(t)|, \quad t^*_{i-1} < t < t^*_i, \quad i = 1, \ldots, n, \\ &\frac{d}{dt}H_{s^*}(t) = (-1)^{i+m}|Z_{s^*}(t)|, \quad \mathcal{T}^*_{i-1} < t < \mathcal{T}^*_i,\ i = 1, \ldots, n. \end{aligned} \tag{3.18}$$

Consequently, by (3.18), (3.16) for $\xi = (-1)^m$ and (1.30)–(1.33), we have

$$(3.19)\qquad 2n\|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}_n^*]} = \sum_{i=1}^{n}(-1)^{i+m}[H_{s^*}(\mathcal{T}_i^*) - H_{s^*}(\mathcal{T}_{i-1}^*)]$$
$$= \sum_{i=1}^{n}\int_{\mathcal{T}_{i-1}^*}^{\mathcal{T}_i^*}(-1)^{i+m}X_{s^*}(t)\,dt = \sum_{i=1}^{n}\int_{\mathcal{T}_{i-1}^*}^{\mathcal{T}_i^*}(-1)^{i+m}\frac{d}{dt}\widetilde{H}_{s^*}(t)\,dt$$
$$= \int_0^{\min\{1,t_n^*\}}|U_{s^*}(t)|\,dt + \int_C^{\max\{C,t_n^*\}}|U_{s^*}(t)|\,dt.$$

On the other hand, from (1.38) and $D(s^*) = 0$ we infer that

$$(3.20)\qquad 2n\|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}_n^*]} = \sum_{i=1}^{n}(-1)^{i+m}\operatorname{sign} s_i^*|\widetilde{H}_{s^*}(\mathcal{T}_i^*) - \widetilde{H}_{s^*}(\mathcal{T}_{i-1}^*)|$$
$$= \sum_{i=1}^{2n-r+2}(-1)^{i+m}\operatorname{sign} s_i^*(t_i^* - t_{i-1}^*)\frac{2nB}{A}.$$

In particular, from (3.20) and (3.19) it follows that

$$(3.21)\qquad \|U_{s^*}\|_{\mathbb{L}_1[C,\max\{C,t_n^*\}]} \le \left|\sum_{i=1}^{2n-r+2}\operatorname{sign} s_i^*(t_i^* - t_{i-1}^*)\frac{2nB}{A}\right| \le 2nB.$$

By (1.29) and the fact that $U_{s^*}^{(r)}$ has at most $2n-r+1$ sign changes, the restriction of U_s to any interval $[\alpha,\beta] \subset [0,A]$ belongs to $\mathcal{F}_{r,2n-r+2,L_\omega}[\alpha,\beta]$ (see Definition 2.2.1). Therefore, if $t_n^* > C$, then applying Lemma 2.2.3 to $U_{s^*}|_{[C,t_n^*]}$ and using (3.21), we arrive at

$$\frac{V_r L_\omega (t_n^* - C)^{r+1}}{(2n-r+2)^r} \le \|U_{s^*}\|_{\mathbb{L}_1[t_n^*,C]} \le 2nB.$$

Consequently, according to (A.1), we have

$$(3.22)\qquad t_n^* - C < \left[\frac{2nB(2n-r+2)^r}{L_\omega V_r}\right]^{1/(r+1)} \le \frac{A}{12}.$$

Then, by (A.2),

$$(3.23)\qquad t_n^* < \frac{A}{12} + C \le \frac{A}{6},$$

and, consequently,

$$(3.24)\qquad T_n^* = \frac{t_n^* + \varepsilon n}{1 + n\varepsilon} \le \frac{A}{3}$$

for $\varepsilon \le A/(6n)$. The inequality (3.23) implies that

$$(3.25)\qquad \gamma_{2n-r+2-i}^* := A - t_i^* \ge A - t_n^* \ge 2A/3, \quad i = 1,\dots,n.$$

The comparison of (3.24) with (3.25) and the definition (1.21) show that the points $\{\gamma_i^*\}_{i=n-r+2}^{2n-r+1}$ do not participate in generating the knots of $\widehat{K}_{s^*}$ in $[0,T_n^*] \subset [0,A/3]$. Thus, $\widehat{K}_{s^*}$ can have at most $n-r+1$ points of sign change in $[0,T_n^*]$ at $\{\gamma_i^*\}_{i=1}^{n-r+1}$. By (3.1)(C) and Remark 1.2.2, $(d/dt)G_{s^*}(t) = U_{s^*}^{(r)}(t)$ can have at most $n-r$ sign changes in $[0,T_n^*]$.

On the other hand, U_{s^*} vanishes at $\{T_i^*\}_{i=1}^n$ in $[0,T_n^*]$. Therefore, by Rolle's theorem, $U_{s^*}^{(r)}$ has at least $n-r$ sign changes in $[0,T_n^*]$. This shows that $\widehat{K}_{s^*}$ exhibits exactly $n-r+1$ sign changes in $[0,T_n^*]$ at the distinct points $\{\gamma_i^*\}_{i=1}^{n-r+1}$, and $U_{s^*}^{(r)}$ has precisely $n-r$ sign changes in $[0,\mathcal{T}_n^*]$. Recalling (3.18), we observe that the sign changes of $\widehat{K}_{s^*}$ at $\{\gamma_i^*\}_{i=1}^{n-r+1}$ are possible if and only if

$$\text{(3.26)} \qquad \operatorname{sign} s_i^* = (-1)^i\sigma, \quad i=n+1,\dots,2n-r+2,\ \sigma\in\{1,-1\} \text{ fixed.}$$

The juxtaposition of (3.17) for $\xi=(-1)^m$ and (3.26) shows that two possibilities can arise.

CASE 1: $\sigma=(-1)^m$. In this case,

$$\text{(3.27)} \qquad \operatorname{sign} s_i^* = (-1)^{i+m}, \quad i=1,\dots,2n-r+2.$$

CASE 2: $\sigma=(-1)^{m+1}$. In this case,

$$\text{(3.28)} \qquad \operatorname{sign} s_i^* = \begin{cases} (-1)^{i+m}, & i=1,\dots,n,\\ (-1)^{i+m+1}, & i=n+1,\dots,2n-r+2.\end{cases}$$

In Case 1, the property $D(s^*)=0$, (3.17) with $\xi=(-1)^m$ and (3.27) lead to

$$\text{(3.29)} \qquad 0 = \sum_{i=1}^n (-1)^{i+m}\operatorname{sign} s_i^*\left[|H_{s^*}(\mathcal{T}_i^*)-H_{s^*}(\mathcal{T}_{i-1}^*)| - (t_i^*-t_{i-1}^*)\frac{2nB}{A}\right]$$
$$- \sum_{i=n+1}^{2n-r+2}\operatorname{sign} s_i^*(-1)^{i+m}(t_i^*-t_{i-1}^*)\frac{2nB}{A}$$
$$= 2n\|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}_n^*]} - \frac{2nB}{A}\sum_{i=1}^{2n-r+2}(t_i^*-t_{i-1}^*) = 2n\|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}_n^*]} - 2nB.$$

Therefore,

$$\text{(3.30)} \qquad \|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}_n^*]} = B.$$

Analogously, in Case 2, the property $D(s^*)=0$ and (3.28) lead to

$$\text{(3.31)} \qquad \|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}_n^*]} = t_n^* - (A-t_n^*) = 2t_n^*-A>0,$$

implying that $t_n^*\ge A/2$. This contradiction with (3.23) shows that only Case 1 can hold. In particular, (3.27) and (1.21) enable us to derive a formula for $\widehat{K}_{s^*}$ on $[0,A]$:

$$\text{(3.32)} \qquad \widehat{K}_{s^*}(t) = (-1)^{r+1+i+m}[|K_{s^*}(t)|+\varepsilon], \quad \gamma_{i-1}^*<t<\gamma_i^*,\ i=1,\dots,2n-r+2.$$

4.4. Properties of the points of alternance $\{\mathcal{T}_i^*\}_{i=0}^n$. At this point we remind the reader the dependence $s^*=s^*(\varepsilon)$ of the solution of the equation $\varkappa(s)=0$ on the parameter ε satisfying (ε.1)–(ε.3).

4.4.1. *Separation of alternance points and its corollaries.* The following lemma shows that the alternance points $\{\mathcal{T}_i^*=\mathcal{T}_i^*(\varepsilon)\}_{i=0}^n$ are uniformly separated.

LEMMA 4.4.1. *There exists a constant $\delta=\delta(\omega,r,n)$ such that*

$$\text{(4.1)} \qquad \mathcal{T}_i^*(\varepsilon)-\mathcal{T}_{i-1}^*(\varepsilon)>\delta, \quad i=1,\dots,n,$$

for all ε satisfying (ε.1)–(ε.2).

PROOF. Fix i with $1 \le i \le n$. By (3.16), (3.30) and (3.18),

$$(4.2) \qquad 2B = |H_{s^*(\varepsilon)}(\mathcal{T}_i^*(\varepsilon)) - H_{s^*(\varepsilon)}(\mathcal{T}_{i-1}^*(\varepsilon))| = \|Z_{s^*(\varepsilon)}\|_{\mathbb{L}_1[\mathcal{T}_{i-1}^*(\varepsilon),\mathcal{T}_i^*(\varepsilon)]},$$

while by (1.30), (1.31) and (2.5) for $k = 0$,

$$(4.3) \qquad \|Z_{s^*(\varepsilon)}\|_{\mathbb{L}_1[\mathcal{T}_{i-1}^*(\varepsilon),\mathcal{T}_i^*(\varepsilon)]} \le \|U_{s^*(\varepsilon)}\|_{\mathbb{C}[0,A]}[\mathcal{T}_i^*(\varepsilon) - \mathcal{T}_{i-1}^*(\varepsilon)] \le \omega(A)A^{r-1}[\mathcal{T}_i^*(\varepsilon) - \mathcal{T}_{i-1}^*(\varepsilon)].$$

Combination of (4.2) and (4.3) gives (4.1) with $\delta := 2B/(\omega(A)A^{r-1})$. ∎

In §4.3 we showed that U_{s^*} has precisely n simple zeros $\{T_i^*\}_{i=1}^n$ in $[0, T_n^*]$, and $U_{s^*}^{(r)}$ changes sign exactly $n - r$ times in $[0, T_n^*]$. By the Rolle theorem, each derivative $U_{s^*(\varepsilon)}^{(l)}$, $l = 0, \dots, r$, has precisely $n - l$ points of sign change $\{\eta_i^l(\varepsilon)\}_{i=1}^{n-l}$. The Rolle theorem also yields the following inequalities between the points of sign change of the consecutive derivatives:

$$(4.4) \qquad \eta_i^l(\varepsilon) < \eta_i^{l+1}(\varepsilon) < \eta_{i+1}^l(\varepsilon), \quad i = 1, \dots, n-l-1, \ l = 0, \dots, r-1.$$

Notice that $\eta_i^0(\varepsilon) = T_i^*(\varepsilon)$ for $i = 1, \dots, n$.

COROLLARY 4.4.2. *Let $\{\eta_i^r(\varepsilon)\}_{i=1}^{n-r}$ be the points of sign change of $U_{s^*(\varepsilon)}^{(r)}$. Then there exist constants $\widehat{\delta} > 0$ and $E > 0$ such that for all $0 < \varepsilon < E$,*

$$(4.5) \qquad T_i^*(\varepsilon) + \widehat{\delta} < \eta_i^r(\varepsilon), \quad i = 1, \dots, n-r.$$

PROOF. By (1.2), we have

$$(4.6) \qquad |T_i^*(\varepsilon) - t_i^*(\varepsilon)| \le An\varepsilon, \quad i = 1, \dots, n.$$

Thus, taking $\varepsilon < E := \delta/(4An)$ we infer from (4.6) and (4.2) that

$$(4.7) \qquad T_i^*(\varepsilon) - T_{i-1}^*(\varepsilon) \ge \tfrac{1}{2}\delta, \quad i = 1, \dots, n.$$

Let us show that we can apply Lemma 2.2.4 to the function $U_{s^*(\varepsilon)}$, its simple zeros $\{T_i^*(\varepsilon)\}_{i=1}^n$, and the points $\{\eta_i^r(\varepsilon)\}_{i=1}^{n-r}$ of sign change of $U_{s^*(\varepsilon)}^{(r)}$. Indeed, by (2.5) and (1.29)(i),

$$(4.8) \qquad \|U_{s^*(\varepsilon)}\|_{\mathbb{C}[0,A]} \le \omega(A)A^{r-1}, \quad |U_{s^*(\varepsilon)}(t)| \ge L_\omega, \quad t \in [0, A].$$

Lemma 2.2.4 now yields the existence of a constant $\widehat{\delta} = \widehat{\delta}(n, r, \omega)$ such that

$$(4.9) \qquad \eta_i^r(\varepsilon) - T_i^*(\varepsilon) = \eta_i^r - \eta_i^0 \ge \widehat{\delta}, \quad i = 1, \dots, n. \ ∎$$

The following result gives us inequalities between the zeros $\{T_i^*(\varepsilon)\}_{i=1}^n$ of $U_{s^*(\varepsilon)}$ and the points $\{\gamma_i^*(\varepsilon)\}_{i=1}^{n-r+1}$ of sign change of $\widehat{K}_{s^*(\varepsilon)}$ in $[0, \mathcal{T}_n^*]$.

COROLLARY 4.4.3. *Let $\widehat{\delta}, E$ be as in Corollary 4.4.2. Then, for all $0 < \varepsilon < E$,*

$$(4.10) \qquad \gamma_i^*(\varepsilon) \ge T_{i-1}^*(\varepsilon) + \widehat{\delta}, \quad i = 1, \dots, n-r+1.$$

PROOF. By Remark 1.2.3, we have the following inequalities between $\{\gamma_i^*(\varepsilon)\}_{i=1}^{n-r+1}$ and the points $\{\eta_i^r\}_{i=1}^{n-r}$ of sign change of $U_{s^*(\varepsilon)}^{(r)}$:

$$(4.11) \qquad \gamma_i^*(\varepsilon) < \eta_i^r(\varepsilon) < \gamma_{i+1}^*(\varepsilon), \quad i = 1, \dots, n-r.$$

Now (4.11) and (4.5) lead to

$$T^*_{i-1}(\varepsilon) + \hat{\delta} < \eta^r_{i-1}(\varepsilon) < \gamma^*_i(\varepsilon), \quad i = 1, \dots, n-r+1. \ \blacksquare \tag{4.12}$$

4.4.2. *The estimate for the number of alternance points in* $[0,1]$. By (1.29), the constants L_ω and O_ω from (0.2) are lower and upper bounds for $U^{(r)}_{s^*}(t)$, respectively. In our analysis we need the following reformulation of Proposition 2.2.5 and Corollary 2.2.6 in the particular case of $f(t) = U_{s^*}(t)$.

PROPOSITION 4.4.4. *Let* $\{\eta^k_i\}^{n-k}_{i=1}$, $0 < \eta^k_1 < \ldots < \eta^k_{n-k} \le T^*_n$, *be the points of sign change of* $U^{(k)}_{s^*}$ *in* $[0, T^*_n]$ *for* $k = 0, \dots, r$. *There exist constants* $\{\lambda_k = \lambda_k(r, 2n-r+2, L_\omega, O_\omega)\}^r_{k=0}$ *such that for all* $i = 1, \dots, n-r$,

$$\eta^k_{i+r-k} > \lambda_k T^*_{i+r}, \quad k = 1, \dots, r. \tag{4.13}$$

Let

$$L = \min\{\, 0 \le i \le 2n-r+2 \mid \mathcal{T}^*_i \ge 1 \,\}. \tag{4.14}$$

Let us show that $L \ge r+1$, i.e., H_{s^*} has at least $r+1$ points of alternance $\{\mathcal{T}^*_i = t^*_i\}^r_{i=0}$ in $[0,1]$.

LEMMA 4.4.5. *Let* L *be defined in* (4.14). *Then* $L \ge r+1$.

PROOF. Suppose, on the contrary, $\mathcal{T}^*_r > 1$. Then, by (1.32), $t^*_r := \mathcal{T}^*_r + C - 1 \ge C$, and, consequently,

$$T^*_r = \frac{t^*_r + \varepsilon r}{1 + n\varepsilon} > \frac{C}{2} \quad \text{for } \varepsilon < \frac{1}{n}.$$

By (4.4) for $i = 1$ and $l = r$ and (4.13) for $k = r-1$, we have the following estimate for the first (leftmost) points η^{r-1}_1, η^r_1 of sign change of $U^{(r-1)}_{s^*}$ and $U^{(r)}_{s^*}$, respectively:

$$\eta^r_1 > \eta^{r-1}_1 > \lambda_{r-1} t^*_r \ge \lambda_{r-1} C/2, \tag{4.15}$$

where λ_{r-1} is defined in §4.0 and (2.2.22). According to (C.2), we have $\lambda_{r-1}C/2 > 1$ in (4.15), i.e., η^r_1 and η^{r-1}_1 lie outside $[0,1]$. Therefore, neither $U^{(r-1)}_{s^*}$ nor $U^{(r)}_{s^*}$ changes sign in $[0,1]$. Then, by (4.15) and (4.8), for all $t \in [0,1]$ we have

$$|U^{(r-1)}_{s^*}(t)| = \int_t^{\eta^{r-1}_1} |U^{(r)}_{s^*}(\xi)|\, d\xi \ge L_\omega(\eta^{r-1}_1 - t) \ge L_\omega[C\lambda_{r-1}/2 - 1] =: M_\omega. \tag{4.16}$$

Applying Lemma 2.2.3 to the function U_{s^*}, whose $(r-1)$st derivative does not change sign in $[0,1]$ and satisfies (4.16), we obtain the estimate

$$\|U_{s^*}\|_{\mathbb{L}_1[0,1]} \ge M_\omega V_{r-1}, \tag{4.17}$$

where V_{r-1} is introduced in Lemma 2.2.3.

On the other hand, by (3.16), (3.30) and (3.19),

$$\begin{aligned} 2LB = 2L\|H_{s^*}\|_{\mathbb{C}[0,\mathcal{T}^*_n]} &= \sum_{i=0}^{L} (-1)^{i+m} (H_{s^*}(\mathcal{T}^*_i) - H_{s^*}(\mathcal{T}^*_{i-1})) \\ &= \|H'_{s^*}\|_{\mathbb{L}_1[0,\mathcal{T}^*_L]} \ge \|H'_{s^*}\|_{\mathbb{L}_1[0,1]} := \|U_{s^*}\|_{\mathbb{L}_1[0,1]}, \end{aligned} \tag{4.18}$$

Our assumption $L \le r$ and (4.17) and (4.18) together imply that

$$M_\omega V_{r-1} \le 2LB \le 2rB. \tag{4.19}$$

This clearly contradicts (C.3). ∎

4.5. The form of the kernel K_{s^*}. Recall definitions (1.16)–(1.21) of the kernels $\{K_s^l\}_{l=r}^n$ and K_s. In this chapter we show that

$$K_{s^*}(t) \equiv \kappa K_{s^*}^N(t) \tag{5.1}$$

for some $N \in \{r, \dots, n\}$ and $\kappa > \mathcal{K}$, where the constant $\mathcal{K} > 0$ is independent of ε.

We consider three different cases.

Case 1: $\mathcal{T}_n^* < 1$. By (1.32), in this case $\mathcal{T}_n^* = t_n^*$, and by (3.16), $\{\mathcal{T}_i^* = t_i^*\}_{i=1}^n$ are the points of alternance of H_{s^*} in $[0, t_n^*]$:

$$H_{s^*}(t_i^*) = (-1)^{i+m}\|H_{s^*}\|_{\mathbb{C}[0,t_n^*]} = (-1)^i B, \quad i = 0, \dots, n. \tag{5.2}$$

By (1.18), $\Gamma(s^*) = t_n^*$ in Case 1, and (1.19) implies that $\psi_{s^*}(t_n^*) = 1$ and $\operatorname{supp}\psi_{s^*} = [t_n^* - \Lambda, C]$. By Lemma 4.4.1 and ($\Lambda$.1),

$$t_{n-1}^* \le t_n^* - \delta < t_n^* - \Lambda.$$

Therefore, $\psi_{s^*}(t_i^*) = 0$ for $i = 1, \dots, n-1$. Then from (1.20) it follows that in Case 1,

$$K_{s^*}(t) = (\mathcal{B}_{n-r+2}^* - \tau_n^*)_+ K_{s^*}^n(t). \tag{5.3}$$

By (1.14),

$$\mathcal{B}_{n-r+2}^* > \frac{\gamma_{n-r+2}^*}{1 + (2n-r+2)\varepsilon}.$$

As we showed in (3.6) for $i = n$, the knot γ_{n-r+2}^* of K_{s^*} lies in $[2A/3, A]$. In particular, $\mathcal{B}_{n-r+2}^* > A/4$ for $\varepsilon \le 1/(2n-r+2)$, while all $\{\tau_i\}_{i=1}^n$ lie in $[0,1]$. Consequently, the coefficient of $K_{s^*}^n$ in (5.3) exceeds $A/4 - 1$. Case 1 corresponds to Case (II)(A) of Theorem 3.3.1.

It remains to consider cases where $t_n^* \ge 1$, or, in the notations of (4.14), $L \le n$.

Case 2: $L \le n$ and $t_L^* \ge C/2$. By (4.14), the $n+1$ points of alternance of H_{s^*} in Case 2 are $\{\mathcal{T}_i^* = t_i^*\}_{i=1}^{L-1}$ in $[0,1)$ and $\{\mathcal{T}_i = \max\{1, t_i^* - C + 1\}\}_{i=L}^n$ in $[1, t_n^* - C + 1]$.

By (1.18), (1.19) and our condition $C/2 > 2$ for C in Case 2 we have

$$\Gamma(s^*) = \max\{t_{L-1}^*, 2 - t_L^*\} = t_{L-1}^*, \quad \psi_{s^*}(t_{L-1}^*) = 1, \quad \operatorname{supp}\psi_{s^*} = [t_{L-1}^* - \Lambda, C]. \tag{5.4}$$

From (4.1) for $\{\mathcal{T}_i^* = t_i^*\}_{i=1}^{L-1}$ and (Λ.1) we observe that

$$t_1^* < \ldots < t_{L-2}^* < t_{L-1}^* - \delta < t_{L-1}^* - \Lambda. \tag{5.5}$$

By (5.4) and (5.5), $\{t_i\}_{i=1}^{L-2}$ lie outside the support of ψ_{s^*}. On the other hand, Lemma 4.3.2 implies that there may exist at most one point $\mathcal{T}_i^*$ "trapped" inside $[1, C]$, or, equivalently, $\mathcal{T}_i^* = 1$. By our assumptions, $t_{L-1}^* < 1$ and $t_L^* \ge C/2$, so this point may only be t_L^*. Therefore,

$$C < t_{L+1}^* < t_{L+2}^* < \ldots < T_n^*. \tag{5.6}$$

Employing (5.4)–(5.6), we find points which definitely lie outside the support of ψ_{s^*}:

$$\psi_{s^*}(t_i^*) = 0, \quad i = 1, \dots, L-2, L+1, \dots, n. \tag{5.7}$$

By (1.20), (5.7) and (5.4),

$$K_{s^*}(t) = [\mathcal{B}_{L-r+1}^* - \tau_{L-1}^*]_+ F_{L-1,s^*}(t) + (\psi_{s^*}(t_L^*)[\mathcal{B}_{L-r+2}^* - \tau_L^*]_+) K_{L,s^*}(t). \tag{5.8}$$

By our assumption and (1.2),

$$T_L^* = \frac{t_L^* + L\varepsilon}{1 + n\varepsilon} \geq C/4 \quad \text{ for } \varepsilon < \frac{1}{n}.$$

From this inequality, Proposition 4.4.4 and (C.2) we obtain an estimate for the $(L-r)$th knot of $U_{s^*}^{(r)}$: $\eta_{L-r}^r \geq \lambda_r C/4 > 1$. As we pointed out in 4.3, the property (3.1)(C) and the presence of $n-r+1$ sign changes $\{\gamma_i^*\}_{i=1}^{n-r+1}$ in $[0, T_n^*]$ imply that $\widehat{K}_{s^*} \in \mathcal{M}_{n-r+2}^0[0, T_n^*]$. Remark 1.2.3 enables us to get

$$\gamma_{L-r+1}^* > \eta_{L-r} \geq \lambda_r C/4 > 1. \tag{5.9}$$

Therefore,

$$\mathcal{B}_{L-r+1} := \frac{\gamma_{L-r+1} + [2(L-r+1)-1]\varepsilon}{1 + (2n-r+2)\varepsilon} \geq \lambda_r C/8 \tag{5.10}$$

for all $\varepsilon < 1/(2n-r+2)$.

Case 2 subdivides into two subcases.

Subcase 2.1: $t_L^* > C$. In this case t_L^* lies outside the support $[t_{L-1}^* - \Lambda, C]$ of ψ_{s^*}. Consequently, $\psi_{s^*}(t_L^*) = 0$, and

$$K_{s^*}(t) = (\mathcal{B}_{L-r+1} - \tau_{L-1}^*)_+ K_{s^*}^{L-1}(t). \tag{5.11}$$

By the inclusions $\tau_i \in [0,1]$, $i = 1, \dots, n$, and the estimate (5.10) for $\mathcal{B}_{L-r+1}$, the coefficient of $K_{s^*}^{L-1}$ in (5.11) exceeds $\lambda_r C/8 - 1$.

Subcase 2.2: $t_L^* \leq C$. Then $\beta_{L-r+1}^* > \tau_{L-1}^*$. Indeed, the point t_L^* is now "trapped" inside $[1, C]$, so $\mathcal{T}_L^* = 1$. By Lemma 4.4.1 and (Λ.1) we have

$$t_{L-1}^* = \mathcal{T}_{L-1}^* < \mathcal{T}_L^* - \delta < \mathcal{T}_L^* - \Lambda = 1 - \Lambda. \tag{5.12}$$

Thus, by (5.12),

$$\tau_{L-1}^* := \frac{t_{L-1}^* + 2(L-1)\varepsilon}{1 + (2n-r)\varepsilon} \leq 1 - \Lambda + 2n\varepsilon \leq 1 - \frac{\Lambda}{2}. \tag{5.13}$$

On the other hand, by the definition of $\{\beta_i\}_{i=1}^{n-r+1}$,

$$\beta_{L-r+1}^* = \frac{\overline{\beta}_{L-r+1}^* + [2(L-r+1)-1]\varepsilon}{1 + (2n-r+2)\varepsilon}, \tag{5.14}$$

where

$$\overline{\beta}_{L-r+1}^* = \begin{cases} \overline{t}_{L-r}^*, & \text{if } \gamma_{L-r+1}^* < \overline{t}_{L-r}^*, \\ \gamma_{L-r+1}^*, & \text{if } \overline{t}_{L-r}^* \leq \gamma_{L-r+1}^* \leq \overline{t}_L^*, \\ \overline{t}_L^*, & \text{if } \gamma_{L-r+1}^* > \overline{t}_L^*. \end{cases} \tag{5.15}$$

By (5.12), $\overline{t}^*_{L-r} \leq \overline{t}^*_{L-1} = t^*_{L-1} \leq 1 - \Lambda$, while $\overline{t}^*_L = \max\{1, t^*_L\} = 1$ and by (5.9), $\gamma^*_{L-r+1} > 1$. Therefore, $\overline{\beta}^*_{L-r+1} = 1$, and

$$\beta^*_{L-r+1} \geq \frac{1}{1+(2n-r+1)\varepsilon} \geq 1 - 2n\varepsilon \geq 1 - \frac{\Lambda}{10} \tag{5.16}$$

for $\varepsilon < \Lambda/(20n)$. From (5.13) and (5.16) we infer that $\beta^*_{L-r+1} > \tau^*_{L-1}$.

Recall definitions (1.16), (1.17) of the kernel

$$K^L_{s^*}(t) = \frac{1}{(r-1)!}\sum_{i=0}^{L} \alpha^L_i (\tau^*_i - t)^{r-1}_+, \tag{5.17}$$

where the coefficients $\{\alpha^L_i\}^L_{i=0}$ are derived from the equations

$$\begin{cases} \displaystyle\sum_{i=0}^{L} \alpha^L_i \tau^j_i = m!\delta_{j,m}, & j = 0, \ldots, r-1, \\ \displaystyle\sum_{i=0}^{L} \alpha^L_i (\tau_i - \beta_l)^r_+ = 0, & l = 1, \ldots, L-r+1. \end{cases} \tag{5.18}$$

According to Microlemma 3.1.2, if $\beta^*_{L-r+1} > \tau^*_{L-1}$, then $\alpha^L_L = 0$. Consequently, upon elimination of α^L_L, (5.18) become precisely the equations

$$\begin{cases} \displaystyle\sum_{i=0}^{L-1} \alpha^{L-1}_i \tau^j_i = m!\delta_{j,m}, & j = 0, \ldots, r-1, \\ \displaystyle\sum_{i=0}^{L-1} \alpha^{L-1}_i (\tau_i - \beta_l)^r_+ = 0, & l = 1, \ldots, L-r, \end{cases}$$

defining the coefficients $\{\alpha^L_i = \alpha^{L-1}_i\}^{L-1}_{i=0}$ of $K^{L-1}_{s^*}$. Thus, $K^L_{s^*}(t) \equiv K^{L-1}_{s^*}(t)$, and so

$$K_{s^*}(t) = K^{L-1}_{s^*}(t)[\psi_{s^*}(t^*_L)(\mathcal{B}^*_{L-r+2} - \tau^*_L)_+ + (\mathcal{B}^*_{L-r+1} - \tau^*_{L-1})_+]. \tag{5.19}$$

By (5.10) and (C.2), the coefficient of $K^{L-1}_{s^*}$ in (5.19) now exceeds (in view of $\{\tau^*_i\}^n_{i=1} \subset [0,1]$) $\lambda_r C/8 - 1 > 2$, an absolute constant independent of $\varepsilon > 0$.

Case 3: $1 \leq t^*_L < C/2$. In this case, (1.19) and (1.18) imply that

$$\psi_{s^*}(t^*_L) \geq \psi_{s^*}(C/2) = \frac{C}{2(C - \Gamma_{s^*})} \geq \frac{C}{2(C-(1-\Lambda))}. \tag{5.20}$$

Using (4.14) and proceeding as in Case 2, we can establish (5.5)–(5.7), yielding

$$\begin{aligned} K_{s^*}(t) = {} & (\psi_{s^*}(t^*_{L-1})[\mathcal{B}^*_{L-r+1} - \tau^*_{L-1}]_+)F_{L-1,s^*}(t) \\ & + (\psi_{s^*}(t^*_L)[\mathcal{B}^*_{L-r+2} - \tau^*_L]_+)K_{L,s^*}(t). \end{aligned} \tag{5.21}$$

By the inequality $t^*_{L+1} > C$ from (5.6) and (1.2),

$$T^*_{l+1} > \frac{t^*_{L+1}}{1+(2n-r+2)} \geq \frac{C}{2} \quad \text{for } \varepsilon < \frac{1}{2n-r+2}.$$

Then Proposition 4.4.4 implies $\eta^r_{L-r+1} \geq \lambda_r T^*_{L+1} \geq \lambda_r C/2 \geq 1$. Remark 1.2.3 then shows

$$\gamma^*_{L-r+2} > \eta^r_{L-r+1} \geq \lambda_r C/2 > 1. \tag{5.22}$$

Therefore, for $\varepsilon < 1/(2n-r+2)$,

$$\mathcal{B}^*_{L-r+2} \geq \frac{\gamma^*_{L-r+2}}{1+(2n-r+2)\varepsilon} > \frac{\gamma^*_{L-r+2}}{2} \geq \frac{\lambda_r C}{4}. \tag{5.23}$$

Therefore, we get the following estimate for the coefficient of $K^L_{s^*}$ in (5.21):

$$\psi_{s^*}(t^*_L)(\mathcal{B}^*_{L-r+2} - \tau^*_L)_+ \geq \frac{C[C\lambda_r - 1]}{4(C+\Lambda-1)}. \tag{5.24}$$

We divide Case 3 into two subcases.

Subcase 3.1: $\mathcal{B}^*_{L-r+1} < \tau^*_{L-1}$. Then $(\mathcal{B}^*_{L-r+1} - \tau^*_{L-1})_+ = 0$, and (5.21) yields

$$K_{s^*}(t) \equiv K^L_{s^*}(t)\psi_{s^*}(t^*_L)(\mathcal{B}^*_{L-r+2} - \tau^*_L)_+. \tag{5.25}$$

The inequality (5.24) provides an estimate from below for the coefficient of $K^L_{s^*}$ in (5.25).

Subcase 3.2: $\mathcal{B}^*_{L-r+1} > \tau^*_{L-1}$.

Lemma 4.5.1. *In Subcase* 3.2,

$$\beta^*_{L-r+1} > \tau^*_{L-1}. \tag{5.26}$$

Proof. Let us compare the definitions of $\mathcal{B}^*_{L-r+1}$ in (5.10) and (5.14), and (5.15) of β^*_{L-r+1}. First of all, our assumption $1 \leq t^*_L$ of Case 3 implies that $\overline{t}^*_L := \min\{t^*_L, 1\} = 1$. Next, from the assumption

$$\frac{\gamma^*_{L-r+1} + [2(L-r+1)-1]\varepsilon}{1+(2n-r+2)\varepsilon} =: \mathcal{B}_{L-r+1} > \tau^*_{L-1} := \frac{\overline{t}^*_{L-1} + 2(L-1)\varepsilon}{1+(2n-r+2)\varepsilon}$$

(recall that $r \geq 2$) it follows that

$$\gamma^*_{L-r+1} \geq \overline{t}^*_{L-1} > \overline{t}^*_{L-r}. \tag{5.27}$$

Therefore, by (5.15) and (5.27), β^*_{L-r+1} has the following simple representation:

$$\beta^*_{L-r+1} = \frac{\min\{1, \gamma^*_{L-r+1}\} + [2(L-r+1)-1]\varepsilon}{1+(2n-r+2)\varepsilon}. \tag{5.28}$$

The juxtaposition of (5.10) and (5.28) implies that $\mathcal{B}^*_{L-r+1} = \beta^*_{L-r+1}$ if $\gamma^*_{L-r+1} \leq 1$. Therefore, we only need to prove (5.26) for $\gamma^*_{L-r+1} \geq 1$. In this case, by (5.28),

$$\beta^*_{L-r+1} = \frac{1+[2(L-r+1)-1]\varepsilon}{1+(2n-r+2)\varepsilon} \geq 1-(2n-r+2)\varepsilon \geq 1 - \frac{\Lambda}{5} \tag{5.29}$$

for $\varepsilon \leq \Lambda/(5(2n-r+2))$. On the other hand, by (5.5),

$$\tau^*_{L-1} \leq \frac{t^*_{L-1} + 2(n-r+2)\varepsilon}{1+(2n-r+2)\varepsilon} \leq 1 - \Lambda + 2n\varepsilon \leq 1 - \frac{\Lambda}{2} \tag{5.30}$$

for $\varepsilon \leq \Lambda/(4n)$. The inequalities (5.29) and (5.30) prove (5.26). ∎

According to Microlemma 3.1.2, the inequality $\beta^*_{L-r+1} > \tau^*_{L-1}$ implies that $\alpha^L_L = 0$ in (5.18) and $K^L_{s^*}(t) \equiv K^{L-1}_{s^*}(t)$. Therefore, (5.21) yields

$$K_{s^*}(t) = K^{L-1}_{s^*}(t)[\psi_{s^*}(t^*_L)(\mathcal{B}^*_{L-r+2} - \tau^*_L)_+ + \psi_{s^*}(t^*_{L-1})(\mathcal{B}^*_{L-r+1} - \tau^*_{L-1})_+]. \tag{5.31}$$

The coefficient of $K^{L-1}_{s^*}$ is estimated in (5.24).

Having completed the analysis of all cases, we summarize the results of this chapter. It was shown that the kernel K_{s^*} is expressed by the formula

$$K_{s^*}(t) = \kappa K^N_{s^*}(t) = \kappa \sum_{i=0}^{N} \alpha_i^N(s^*)(\tau_i^* - t)_+^{r-1}, \quad N \in \{r, \dots, n\}, \tag{5.32}$$

where the coefficients $\{\alpha_i^N\}_{i=0}^N$ solve (1.16) for $l = N$ and $s = s^*$. By (5.25), in Subcase 3.1 the index N coincides with the integer L introduced in (4.14). By (5.3), (5.11), (5.19) and (5.31), $N = L - 1$ in all remaining cases.

Further, in any case we have L points of alternance of H_{s^*} in $[0, \min\{t_n^*, 1\}]$, lying in $[0, 1)$. In particular, by (1.30),

$$\mathcal{T}_i^* = t_i^* = \bar{t}_i^*, \quad i = 0, \dots, L-1. \tag{5.33}$$

Now (1.6)–(1.8) and (5.33) imply that

$$\tau_i^* = \mathcal{T}_i^* + O(\varepsilon) = t_i^* + O(\varepsilon), \quad i = 0, \dots, L-1. \tag{5.34}$$

In Subcase 3.1, $1 < t_L^* < C/2$, so by (1.32), $\mathcal{T}_L^* = \bar{t}_L^* := \min\{t_L^*, 1\}$. Therefore, the following refinement of (5.33), (5.34) holds in all cases:

$$\mathcal{T}_i^* = \bar{t}_i^*, \quad \tau_i^* = \mathcal{T}_i^* + O(\varepsilon), \quad i = 0, \dots, N. \tag{5.35}$$

4.6. The limiting procedure as $\varepsilon \to 0$. Let ε satisfy (ε.1)–(ε.3). As shown in (2.5), the families of functions $\{U^{(k)}_{s^*(\varepsilon)}\}_{k=0}^{r-1}$ are uniformly bounded:

$$\|U^{(k)}_{s^*(\varepsilon)}\|_{\mathbb{C}\,[0,A]} \le \omega(A)A^{r-1-k}, \quad k = 0, \dots, r-1.$$

These inequalities also imply that $U^{(k)}_{s^*(\varepsilon)}$ have a common majorizing modulus of continuity on $[0, A]$:

$$\omega(U^{(k)}_{s^*(\varepsilon)}; t) \le \omega(A)A^{r-1}t, \quad 0 \le t \le A,\ k = 0, \dots, r-2. \tag{6.1}$$

We also showed in §4.1.3 that $U^{(r-1)}_{s^*(\varepsilon)} = G_{s^*(\varepsilon)} + \lambda \in H^\omega[0, A]$.

The Arzelà–Ascoli theorem enables us to extract a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \varepsilon_k = 0$ such that the following sequences converge: $\{t_i^*(\varepsilon_k)\}_{i=0}^{2n-r+2}$, $k \in \mathbb{N}$, $\{U_{s^*(\varepsilon_k)}\}_{k\in\mathbb{N}}$ in $\mathbb{C}^{r-1}[0, A]$, $\{X_{s^*(\varepsilon_k)}\}_{k\in\mathbb{N}}$ in $\mathbb{L}_1[0, A]$ and $\{Z_{s^*(\varepsilon_k)}\}_{k\in\mathbb{N}}$ in $\mathbb{L}_1[0, A-C+1]$. Set

$$U := \lim_{k\to\infty} U_{s^*(\varepsilon_k)}, \quad X := \lim_{k\to\infty} X_{s^*(\varepsilon_k)}, \quad Z := \lim_{k\to\infty} Z_{s^*(\varepsilon_k)}, \tag{6.2}$$

where the convergence is understood in the spaces specified above. For $i = 0, \dots, 2n-r+2$, we also set

$$t_i := \lim_{k\to\infty} t_i^*(\varepsilon_k), \quad \gamma_i := \lim_{k\to\infty} \gamma_i^*(\varepsilon_k), \quad \mathcal{T}_i := \lim_{k\to\infty} \mathcal{T}_i^*. \tag{6.3}$$

By the inherited properties (3.18),

$$X(t) = (-1)^{i+m}|U(t)|, \quad t_{i-1} < t < t_i,\ i = 1, \dots, n, \tag{6.4}$$

$$Z(t) = (-1)^{i+m}|U(t)|, \quad \mathcal{T}_{i-1} < t < \mathcal{T}_i,\ i = 1, \dots, n. \tag{6.5}$$

Then (5.35) implies that

$$\lim_{k\to\infty} \tau_i^*(\varepsilon_k) = \bar{t}_i = \mathcal{T}_i, \quad i = 0, \dots, N. \tag{6.6}$$

By Lemma 4.4.1, the limiting points $\{\mathcal{T}_i\}_{i=0}^n$ remain uniformly separated:

$$\mathcal{T}_i - \mathcal{T}_{i-1} \geq \delta, \quad i = 1, \dots, n. \tag{6.7}$$

From Corollary 4.4.3 we infer that

$$\gamma_i \geq t_{i-1} + \widehat{\delta}, \quad i = 1, \dots, n-r+1. \tag{6.8}$$

By (5.32), there exists $N_\varepsilon \in \{r, \dots, n\}$ and $\kappa_\varepsilon > 0$ such that $K_{s^*(\varepsilon)} \equiv \kappa_\varepsilon K_{s^*(\varepsilon)}^{N_\varepsilon}$. Without loss of generality we assume that $N_{\varepsilon_k} \equiv N$ and $\lim_{k\to\infty} \kappa_{\varepsilon_k} = \kappa > 0$. In particular,

$$K_{s^*(\varepsilon_k)} = \kappa_{\varepsilon_k} \sum_{i=0}^{N} \alpha_i^N(\varepsilon_k)(\tau_i^*(\varepsilon_k)) - t)_+^{r-1}, \tag{6.9}$$

where $\{\alpha_i^N(\varepsilon_k)\}_{i=0}^N$ satisfy (1.16) for $l = N$ and $s = s^*(\varepsilon_k)$. According to (3.1.7)(i),

$$\operatorname{supp} K_{s^*(\varepsilon_k)} = [0, t_{j_k}^*(\varepsilon_k)] \quad \text{for some } j_k,\ r-1 \leq j_k \leq N,\ k \in \mathbb{N}. \tag{6.10}$$

Without loss of generality (if necessary, by extracting a subsequence $\{\varepsilon_{k_l}\}_{l\in\mathbb{N}}$), we can assume that $j_k \equiv j$, $\forall k \in \mathbb{N}$, for some j with $r-1 \leq j \leq N$.

Lemma 4.6.1. *Let*

$$\beta_i = \lim_{k\to\infty} \beta_i^*(\varepsilon_k), \quad i = 0, \dots, n-r+1, \tag{6.11}$$

where the points $\{\beta_i(s)\}_{i=0}^{n-r+1}$ are defined in (1.9). *Then*

$$\beta_i = \gamma_i, \quad i = 1, \dots, j-r+1. \tag{6.12}$$

Proof. By Proposition 3.1.1, we have the following relations between the knots $\{\tau_i^*(\varepsilon_k)\}_{i=1}^j$ and zeros $\{\beta_i^*(\varepsilon_k)\}_{i=1}^{j-r+1}$ of the kernel $K_{s^*(\varepsilon)}$ lying inside the support of $K_{s^*(\varepsilon)}$:

$$\beta_i^*(\varepsilon_k) < \tau_{i+r-2}^*(\varepsilon_k), \quad i = 1, \dots, j-r+1. \tag{6.13}$$

Taking the limit in (6.13) and using (6.6) and inequalities (6.7) for the alternance points $\{\overline{t}_i = \mathcal{T}_i\}_{i=0}^j$, we find

$$\beta_i \leq \overline{t}_{i+r-1} - \delta, \quad i = 1, \dots, j-r+1. \tag{6.14}$$

Recall the definitions

$$\beta_i^*(\varepsilon_k) = \frac{\overline{\beta}_i^*(\varepsilon_k) + [2i-1]\varepsilon_k}{1 + \varepsilon_k(2n-r+2)}, \quad i = 1, \dots, n-r+1, \tag{6.15}$$

where $\overline{\beta}_i^*(\varepsilon_k) = \min\{\max\{\gamma_i^*(\varepsilon_k); \overline{t}_{i-1}^*(\varepsilon_k)\}; \overline{t}_{i+r-1}^*(\varepsilon_k)\}$, $i = 1, \dots, n-r+1$.

Taking the limit in (6.15) and employing (6.8), we have

$$\beta_i = \lim_{k\to\infty} \beta_i^*(\varepsilon_k) = \min\{\max\{\gamma_i; \overline{t}_{i-1}\}; \overline{t}_{i+r-1}\} = \min\{\gamma_i; \overline{t}_{i+r-1}\} \tag{6.16}$$

for all $i = 1, \dots, n-r+1$. The combination of (6.14) and (6.16) proves (6.12). ∎

Lemma 6.1 and (6.8) and (6.14) imply that

$$\overline{t}_{i-1} + \widehat{\delta} < \gamma_i = \beta_i \leq \overline{t}_{i+r-1} - \delta, \quad i = 1, \dots, j-r+1. \tag{6.17}$$

Let the coefficients $\{\alpha_i\}_{i=0}^{j}$ be derived from the system of linear equations

$$\left\{\begin{aligned} &\sum_{i=0}^{j}\alpha_i\overline{t}_i^{p} = m!\delta_{m,j}, \qquad p = 0,\ldots,r-1,\\ &\sum_{i=0}^{j}\alpha_i(\overline{t}_i-\beta_l)_+^{r-1} = 0, \quad l = 1,\ldots,j-r+1. \end{aligned}\right. \tag{6.18}$$

Inequalities (6.17) ensure the unique solvability of (6.18). Put

$$K(t) = -\kappa\frac{1}{(r-1)!}\sum_{i=0}^{j}\alpha_i(\overline{t}_i-t)_+^{r-1} \tag{6.19}$$

and

$$\widehat{K}(t) = (-1)^{i+r+m-1}|K(t)|, \quad t\in[\gamma_{i-1},\gamma_i],\ i = 1,\ldots,j-r+2. \tag{6.20}$$

Comparing the definitions (6.9) of K_{s^*} and (3.32) of $\widehat{K}_{s^*}$ with the definitions (6.19), (6.20) of K and $\widehat{K}$ and invoking the limiting relations

$$\lim_{k\to\infty}\beta_i^*(\varepsilon_k) = \beta_i = \gamma_i = \lim_{k\to\infty}\gamma_i^*(\varepsilon_k), \quad i = 1,\ldots,j-r+1; \quad \lim_{k\to\infty}\kappa_{s^*(\varepsilon_k)} = \kappa$$

of Lemma 4.6.1 and (6.6), we find that

$$K = \lim_{k\to\infty}K_{s^*(\varepsilon_k)}, \quad \widehat{K} = \lim_{k\to\infty}\widehat{K}_{s^*(\varepsilon_k)} \quad \text{in } \mathbb{C}\,[0,1]. \tag{6.21}$$

By (3.1.7)(iv),

$$(-1)^{i+r+m-1}K(t)\geq 0, \quad t\in[\gamma_{i-1},\gamma_i],\ i = 1,\ldots,j-r+1. \tag{6.22}$$

Juxtaposing (6.20) and (6.22), we finally conclude that for $t\in[\gamma_{i-1},\gamma_i]$,

$$\widehat{K}(t) := (-1)^{i+r+m-1}|K(t)| = (-1)^{r+i+m-1}\cdot(-1)^{r+i+m-1}K(t) = K(t) \tag{6.23}$$

for all $i = 1,\ldots,N-r+1$. In addition, by the inherited property (3.1)(C), the limiting kernel $K(t) = \widehat{K}(t)$ has zero mean on $[0,\overline{t}_j] =: \operatorname{supp} K\subset[0,\overline{t}_n]$. Therefore, as remarked in Section 3.1, the support of K contains $[0,t_r]$.

Next, $U_{s^*(\varepsilon_k)}^{(r-1)}$ is extremal for the problem (1.23) for $s = s^*(\varepsilon_k)$. The convergence (6.2) and Corollary 1.2.6 show that the limiting function $U^{(r-1)}$ is extremal for the problem

$$\int_0^{\overline{t}_j} h(t)K(t)\,dt\to\sup, \quad h\in H^\omega[0,\overline{t}_j]. \tag{6.24}$$

Remark 4.6.1. At this point we make a trivial observation. Any multiple $K_v(t) = vK(t)$ of the kernel K for $v>0$ generates the same extremal function of the problem

$$\int_0^{\overline{t}_j} h(t)K_v(t)\,dt\to\sup, \quad h\in H^\omega[0,\overline{t}_j],$$

as the kernel K in (6.24). In particular, the function $U^{(r-1)}$ is generated by the kernel

$$\mathcal{K}(t) = -\frac{1}{(r-1)!}\sum_{i=0}^{j}\alpha_i(\overline{t}_i-t)_+^{r-1}$$

proportional to $K(t)$ from (6.19).

Put

$$H := \lim_{k\to\infty} H_{s^*(\varepsilon_k)} \quad \text{in } \mathbb{C}[0,1]. \tag{6.25}$$

Recalling (5.33) and comparing (6.4) with (6.5), we infer that

$$\frac{d}{dt}H(t) := Z(t) \equiv X(t) = (-1)^{i+m}|U(t)|, \quad \overline{t}_{i-1} < t < \overline{t}_i,\ i = 1,\dots,N. \tag{6.26}$$

On the other hand, $U_{s^*}^{(r)}$ has $n-r$ sign changes in $[0, T_n^*]$, so all zeros $\{T_i^*(\varepsilon_k)\}_{i=1}^n$ of $U_{s^*(\varepsilon_k)}$ are simple:

$$\operatorname{sign} U_{s^*(\varepsilon_k)}(t) = (-1)^{i+m}, \quad T_{i-1}^*(\varepsilon_k) < t < T_i^*(\varepsilon_k),\ i = 1,\dots,n. \tag{6.27}$$

From (6.2), (6.6) and the juxtaposition of (6.26) and (6.27) we infer that

$$\frac{d}{dt}H(t) = U(t), \quad 0 \le t \le 1. \tag{6.28}$$

Finally, the limiting function H inherits the properties (3.16) for $\xi = (-1)^m$ and the norm (3.30) of the functions $\{H_{s^*(\varepsilon_k)}\}_{k\in\mathbb{N}}$:

$$H(\overline{t}_i) = (-1)^{i+m}\|H\|_{\mathbb{C}[0,1]} = (-1)^{i+m}B, \quad i = 0,\dots,N. \tag{6.29}$$

It remains to rename H, $\{\overline{t}_i\}_{i=0}^N$, and $\{\gamma_i\}_{i=0}^{N-r+2}$:

$$\begin{aligned} &\mathcal{Z}(t) := H(t), \quad 0 \le t \le 1;\\ &\{\vartheta_i := \gamma_i\}_{i=0}^{N-r+1}, \quad \vartheta_{N-r+2} := \overline{t}_N; \quad \{\nu_i := \overline{t}_i\}_{i=0}^N. \end{aligned} \tag{6.30}$$

The proof of Theorem 3.3.1 is thus complete for strictly concave moduli of continuity ω on $[0,1]$ with the property (0.1).

4.7. Conclusion of the proof for arbitrary ω

4.7.1. *Transition from strictly concave to concave ω.* So far, we have proved Theorem 3.3.1 for strictly concave moduli of continuity ω enjoying the special property (0.1). In the general case, we approximate a given concave modulus of continuity ω by a sequence $\{\omega_l\}_{l\in\mathbb{N}}$ of moduli of continuity with the property (0.1), convergent in $\mathbb{C}[0,1]$ to ω. Then, for each $l \in \mathbb{N}$, Theorem 3.3.1 enables us to construct a family of functions $X_l(t) = \mathcal{Z}_{B,n,r,m,\omega_l}(t)$, $t \in [0, d_l]$, with the properties (3.2)–(3.4) for $\omega = \omega_l$. Arguing as in the proof of Theorem 3.3.1, we can show that for all $l \in \mathbb{N}$,

$$\omega(X_l^{(k)}; t) \le (\omega_l(1) + 1)t \le At, \quad 0 \le t \le 1,\ k = 0,\dots,r-1, \tag{7.1}$$

where $A := \sup_{l\in\mathbb{N}} \omega_l(1) + 1$. Therefore, by the Arzelà–Ascoli theorem, we can extract a subsequence $\{l_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} X_{l_k} = \mathcal{Z}$ in $\mathbb{C}^r[0,1]$. Clearly, $\mathcal{Z}$ inherits the properties (3.3) or (3.4) of $\{X_{l_k}\}_{k\in\mathbb{N}}$, i.e. the alternance at some points $\{\nu_i\}_{i=0}^n$. Corollary 1.2.6 shows that $\mathcal{Z}$ enjoys the property (3.2) as well. The proof of Theorem 3.1.1 for $0 < m < r$ is complete.

4.7.2. *Modifications in the proof for $m = r$.* In the case $0 < m < r$ the definition (3.1.2), (3.1.3) of the kernel K (for $N = n$) involves $2n-r+1$ essential parameters $\{\nu_i\}_{i=1}^n$ and $\{\vartheta_i\}_{i=1}^{n-r+1}$. This explains the use of a $(2n-r+1)$-dimensional sphere $\mathbb{S} = \mathbb{S}^{2n-r+1}$ in our proof of Theorem 3.3.1 for $0 < m < r$.

However, in the case $m = r$ the kernel K is defined in (3.2.2), (3.2.3) by $2n - r$ parameters $\{\nu_i\}_{i=1}^n$ and $\{\vartheta_i\}_{i=1}^{n-r}$. Therefore, proving Theorem 3.3.1, we construct the corresponding odd and continuous mapping $\varkappa$ on the $(2n-r)$-dimensional sphere $\mathbb{S} = \mathbb{S}^{2n-r}$ by replacing the formulas (3.1.2), (3.1.3) for the kernels K_s^l in (1.17) with their analogs in (3.2.2), (3.2.3). Since $\widehat{K}_s$ is no longer required to have zero mean ($I_s = I(\widehat{K}_s) = 0$), we then have a mapping from $\mathbb{S}^{2n-r}$ to $\mathbb{R}^{2n-r}$ defined as follows:

$$\varkappa(s) = \{U_s(T_i(s)\}_{i=r+1}^n \times \{b_i(s)\}_{i=1}^{n-1} \times D(s).$$

This modification is the only difference in the constructions and arguments in the proofs of Theorem 3.3.1 for $0 < m < r$ and $m = r$.

5. The extrapolation problem

Let $\tau < 0$, $m, r, n \in \mathbb{N}$ with $0 \le m \le r$ and $n \ge r$. Let ω be a concave modulus of continuity on $[0, 1 + |\tau|]$. We describe the family $\{\mathcal{Z}_B = \mathcal{Z}_{B,r,m,\omega,\tau}\}_{B \ge 0}$ of Zolotarev ω-splines extremal for the problem

$$(\mathbb{E}) \qquad |f^{(m)}(\tau)| \to \sup, \quad f \in W^r H^\omega[\tau, 1], \ \|f\|_{\mathbb{C}[0,1]} \le B.$$

Notice that the problem $(\mathbb{E})$ can be posed for $m = 0$ and even $B = 0$.

5.1. Numerical differentiation formulae

Case 1: $0 \le m < r$. Let the collections of points $\overline{\nu} = \{\nu_i\}_{i=0}^n$ and $\overline{\vartheta} = \{\vartheta_i\}_{i=0}^{n-r+1}$ in $[0,1]$ be as in (3.1.1), and $\{\alpha_i = \alpha_i(m, r, \overline{\nu}, \overline{\vartheta}, \tau)\}_{i=0}^n$ be determined from the linear system

$$(1.1) \qquad \begin{cases} \sum_{i=0}^n \alpha_i (\nu_i - \tau)^j = m!\delta_{m,j}, & j = 0, \dots, r-1, \\ \sum_{i=0}^n \alpha_i (\nu_i - \vartheta_l)_+^{r-1} = 0, & l = 1, \dots, n-r+1. \end{cases}$$

Case 2: $m = r$. Let $\overline{\nu} = \{\nu_i\}_{i=0}^n$ and $\overline{\vartheta} = \{\vartheta_i\}_{i=0}^{n-r}$ in $[0,1]$ satisfy (3.2.1). Let $\{\alpha_i = \alpha_i(r, \overline{\nu}, \overline{\vartheta}, \tau)\}_{i=0}^n$ be the solutions of the linear system

$$(1.2) \qquad \begin{cases} \sum_{i=0}^n \alpha_i (\nu_i - \tau)^j = 0, & j = 0, \dots, r-1, \\ \sum_{i=0}^n \alpha_i (\nu_i - \vartheta_l)_+^{r-1} = 0, & l = 1, \dots, n-r, \\ \sum_{i=0}^n \alpha_i (\nu_i - \tau)^r = r!. \end{cases}$$

It can be shown (see, e.g., [50]) that the signs of the solutions $\{\alpha_i = \alpha_i(m)\}_{i=0}^n$ of systems (1.1) for $0 \le m < r$ and (1.2) for $m = r$ alternate: $(-1)^{i+m}\alpha_i \ge 0$. Put

$$(1.3) \qquad K(t) = -\frac{1}{(r-1)!} \sum_{i=0}^n \alpha_i (\nu_i - t)_+^{r-1}, \quad F(t) = \int_{\nu_n}^t K(y)\, dy = \frac{1}{r!} \sum_{i=0}^n \alpha_i (\nu_i - t)_+^r,$$

where $\{\alpha_i\}_{i=0}^n$ are derived from (1.1) for $0 \le m < r$ and from (1.2) for $m = r$. As in Sections 3.1 and 3.2, we find numerical differentiation formulae for $f^{(m)}(\tau)$:

$$f^{(m)}(\tau) = \sum_{i=0}^{n} \alpha_i f(\nu_i) + \int_{\tau}^{1} [f^{(r)}(x) - f^{(r)}(\tau)] K(x)\, dx. \tag{1.4}$$

Therefore, for any $f \in W^r H^\omega[\tau, 1]$,

$$|f^{(m)}(\tau)| \le \sum_{i=0}^{n} |\alpha_i| \cdot \|f\|_{\mathbb{C}\,[0,1]} + \sup_{h \in H^\omega_\tau[\tau,1]} \int_{\tau}^{1} h(x) K(x)\, dx, \tag{1.5}$$

where $H^\omega_\tau[\tau, 1]$ is the subset of functions in $H^\omega[\tau, 1]$ vanishing at τ.

5.2. Zolotarev ω-splines of the extrapolation problem. The description of Zolotarev ω-splines of the problem ($\mathbb{E}$) for $\tau < 0$ is analogous to characterization of those for $\tau = 0$. In the proof of the following theorem we make use of the formulas (1.1) for $0 < m < r$ or (1.2) for $m = r$ in defining the coefficients of the kernel K. Due to the complete analogy in the properties of K for $\tau < 0$ and $\tau = 0$ the rest of the verification of Theorem 5.2.1 proceeds along the lines of the proof of Theorem 3.3.1.

THEOREM 5.2.1. *Let ω be a concave modulus of continuity on $[0, 1+|\tau|]$, and $0 < m \le r$. There exist an index $n = n(B, r, m, \omega, \tau)$, collections of points $\overline{\nu}$ and $\overline{\vartheta}$ as in* (3.1.1) *for $0 \le m < r$ and as in* (3.2.1) *for $m = r$, and a function $\mathcal{Z}_B = \mathcal{Z}_{B,r,m,\omega,\tau}$ such that*:

(2.1) (A) $$\sup_{h \in H^\omega_\tau[\tau,1]} \int_{\tau}^{1} h(x) K(x)\, dx = \int_{\tau}^{1} [\mathcal{Z}_B^{(r)}(x) - \mathcal{Z}_B^{(r)}(\tau)] K(x)\, dx,$$ *where K is defined in* (1.1), (1.3) *for $0 \le m < r$ and in* (1.2), (1.3) *for $m = r$;*

(B) $\mathcal{Z}_B(\nu_i) = (-1)^{i+m} \|\mathcal{Z}_B\|_{\mathbb{C}[0,1]}$ *for $i = 0, \ldots, n$.*

REMARK 5.2.1. As pointed out in Section 3.0, the Zolotarev perfect spline of degree $r+1$ of norm B is extremal for the problem ($\mathbb{E}$) if $\omega(t) = t$ for all $m \in \{0, 1, \ldots, r\}$ and all $\tau \le 0$. By Theorem 5.2.1, the extremal functions of the problem ($\mathbb{E}$) for nonlinear ω do depend on τ and m.

REMARK 5.2.2. Let us describe the solution of ($\mathbb{E}$) for $B = 0$. If $\|f\|_{\mathbb{C}[0,1]} = 0$, then $f^{(i)}(0) = 0$ for $i = 0, \ldots, r$. Thus,

$$f(x) = \frac{(-1)^r}{(r-1)!} \int_{\tau}^{0} f^{(r)}(t)(t-x)_+^{r-1}\, dt = \frac{(-1)^r}{(r-1)!} \int_{\tau}^{0} [f^{(r)}(t) - f^{(r)}(0)](t-x)_+^{r-1}\, dt. \tag{2.2}$$

Therefore, for all $f \in W^r H^\omega[\tau, 1]$ with $\|f\|_{\mathbb{C}[0,1]} = 0$,

$$|f^{(m)}(\tau)| \le \frac{1}{(r-1-m)!} \int_{\tau}^{0} \omega(t)(t-\tau)^{r-1-m}\, dt. \tag{2.3}$$

and equality holds in (2.3) only for the functions $\{\widehat{f}, -\widehat{f}\}$, where

$$\widehat{f}(x) = \frac{1}{(r-1)!} \int_{\tau}^{0} \omega(t)(t-x)_+^{r-1}\, dt, \quad \tau \le x \le 1. \tag{2.4}$$

Remark 5.2.3. Explicit formulas for extremal functions of the problem ($\mathbb{E}$) in the elementary case $m = 0$ and $r = 1$ are given in [10].

6. Maximization of functionals in $H^\omega[a_1, a_2]$, $-\infty \le a_1 < a_2 \le \infty$

6.0. Formulation of the extremal problem. We describe extremal functions and rearrangements of the problem

$$(\bigstar) \qquad \int_{a_1}^{a_2} h(t)\psi(t)\,dt \to \sup, \quad h \in H_0^\omega[a_1, a_2],$$

where $a_1 < 0 < a_2$, and the kernel ψ has a finite number or a countable monotonically ordered set of points of sign change in $[a_1, a_2]$, $-\infty \le a_1 < a_2 \le \infty$. In particular, we give the solution of the problem ($\bigstar$) in the case of the entire line $[a_1, a_2] = \mathbb{R}$.

6.1. Definitions. Recall Definition 1.2.1 of the function classes $\mathbb{M}_n[\alpha, \beta]$, $n \in \mathbb{N}$.

Definition 6.1.1. Let $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$, $j \in \{-1, 0, +1\}$, and $a_1 < 0 < a_2$. Let also $\psi \in \mathbb{L}_1[a_1, a_2]$, $\psi_m(x) := \psi((-1)^m x)$, $x \in [0, (-1)^m a_m]$, for $m = 1, 2$. Then,

$$\psi \in \mathcal{M}^j_{n_1,n_2}[a_1, a_2] \Leftrightarrow \begin{cases} \psi_m \in \mathbb{M}_{n_m}[0, (-1)^m a_m], \quad m = 1, 2, \\ \operatorname{sign} \int_{a_1}^{a_2} \psi(x)\,dx = j. \end{cases}$$

We also introduce the class

$$\mathbb{M}_{n_1,n_2}[a_1, a_2] := \bigcup_{j=-1}^{1} \mathcal{M}^j_{n_1,n_2}[a_1, a_2].$$

In addition to the sets of indices $\{J_i = J_i(N)\}_{i=1}^N$, $\{L_i\}_{i=1}^N$, and pairs of indices $\mathcal{P} = \mathcal{P}(N)$, introduced in Definition 1.2.2, we will need a collection of indices $\{K_i = K_i(N)\}_{i=1}^N$ and pairs of indices $\mathbb{P}(N_1, N_2)$.

Definition 6.1.2. Let $N, N_1, N_2 \in \mathbb{N} \cup \{\infty\}$. Then

(1) $K_i(N) := \{k = i \pmod 2) + 2l - 1,\ l \in \mathbb{N} \mid k \le N\}$,

(2) $\mathbb{P}(N_1, N_2) := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \le i \le N_1,\ j \in K_i(N_2)\}$.

Definition 6.1.3. Let $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$. A partition $\mathcal{W}$ of $[a_1, a_2]$ with atoms $\{A_i^m, B_i^m, C_i^m, E_i^m, D_i^m\}_{i=1}^{n_m}$ and subatoms $\{B_{ij}^m, C_{ji}^m\}_{(i,j)\in\mathcal{P}(n_m)}$ and $\{E_{ik}^m\}_{(i,k)\in\mathbb{P}(n_m,n_{3-m})}$ for $m = 1, 2$ is called a W_{n_1,n_2}*-partition* of $[a_1, a_2]$ if the following holds for $m = 1, 2$:

(A) $C_i^m = [(-1)^m\gamma^m_{5i-5}, (-1)^m\gamma^m_{5i-4}]$, $\quad D_i^m = [(-1)^m\gamma^m_{5i-4}, (-1)^m\gamma^m_{5i-3}]$,

$E_i^m = [(-1)^m\gamma^m_{5i-3}, (-1)^m\gamma^m_{5i-2}]$, $\quad B_i^m = [(-1)^m\gamma^m_{5i-2}, (-1)^m\gamma^m_{5i-1}]$,

$A_i^m = [(-1)^m\gamma^m_{5i-1}, (-1)^m\gamma^m_{5i}]$,

for $i = 1, \ldots, n_m$ and $\gamma^m = \{\gamma_i^m\}_{i=0}^{5n_m}$ such that $0 \triangleright \gamma^m \blacktriangleright (-1)^m a_m$;

(B) $B_i^m = \boxdot$ for $i = n_m - 2, n_m - 1, n_m$; $C_i^m = \boxdot$ for $i = 1, 2, 3$; $A^m_{n_m} = \boxdot$;

(C) $B_i^m = \bigcup_{j\in J_i(n_m)} B_{ij}^m$ for $1 \le i \le n_m - 3$, where

$$B_{ij}^m = \left[(-1)^{m+1}\xi_i\left(\frac{j-i+1}{2}\right), (-1)^{m+1}\xi_i\left(\frac{j-i-1}{2}\right)\right], \quad j \in J_i(n_m),$$

for $\xi_i^m = \{\xi_i^m(k)\}_{k=1}^{|J_i(n_m)|}$ such that $\gamma_{5i-2}^m \blacktriangleleft \xi_i^m \triangleleft \gamma_{5i-1}^m$;

(D) $C_i^m = \bigcup_{l\in L_i} C_{il}^m$ for $4 \le i \le n_m$, where

$$C_{il}^m = \left[(-1)^m \varkappa_i^m\left(\frac{i-l-1}{2}\right), (-1)^m \varkappa_i^m\left(\frac{i-l+1}{2}\right)\right], \quad l \in L_i,$$

for $\varkappa_i^m = \{\varkappa_i^m(k)\}_{k=1}^{|L_i|}$ such that $\gamma_{5i-5}^m \triangleright \varkappa_i^m \blacktriangleright \gamma_{5i-4}^m$;

(E) $E_i^m = \bigcup_{k\in K_i(n_{3-m})} E_{ik}^m$ for $1 \le i \le n_m$, where

$$E_{ik}^m = \left[(-1)^{m+1} e_i\left(\frac{k+1-i \pmod 2}{2}\right), (-1)^{m+1} e_i\left(\frac{k+3-i \pmod 2}{2}\right)\right],$$

for $e_i^m = \{e_i^m(l)\}_{l=1}^{|K_i(n_{3-m})|}$ such that $\gamma_{5i-3}^m \triangleright e_i^m \blacktriangleright \gamma_{5i-2}^m$.

6.2. Structure of perfect ω-splines

THEOREM 6.2.1. *Let $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$ and $\psi \in \mathcal{M}_{n_1,n_2}^k[a_1, a_2]$, where $k \in \{-1, 0, 1\}$. Let $\alpha_0^m := 0$, $\alpha_{n_m}^m := a_i$, $m = 1, 2$, and $\alpha^m = \{\alpha_i^m\}_{i=1}^{n_m-1}$ be distinct points of sign change of ψ in $[0, a_m]$ such that*

$$a_1 \blacktriangleleft \alpha^1 \triangleleft 0, \quad 0 \triangleright \alpha^2 \blacktriangleright a_2.$$

I. *There exist a solution $y_{\omega,\psi}$ of the problem ($\bigstar$) and a W_{n_1,n_2}-partition $\mathcal{W}$ with the following properties holding for $m = 1, 2$:*

(A) $\alpha_i^m \in A_i^m$ *for $i = 1, \dots, n_m - 1$;*

(B) $\int_{B_{ij}^m \cup C_{ji}^m} \psi(t)\,dt = 0$ *for $(i, j) \in \mathcal{P}(n_m)$;*

(C) $\int_{A_i^m} \psi(t)\,dt = 0$ *for $i = 1, \dots, n_m - 1$;*

(D) $y_{\omega,\psi} = -\omega(|t|)$ *for $t \in D_{2k-1}^m \ne \boxdot$ and $k = 0, \dots, \lceil n_m/2 \rceil$;*

(E) $y_{\omega,\psi} = \omega(|t|)$ *for $t \in D_{2k}^m \ne \boxdot$ and $k = 0, \dots, [n_m/2]$;*

(F_1) *for $(i, j) \in \mathcal{P}(n_m)$, the function $y_{\omega,\psi}$ is extremal for the problem*

$$\int_{a_1}^{a_2} h(t)\psi_{ij}^m(t)\,dt \to \sup, \quad h \in H^\omega[a_1, a_2], \tag{2.1}$$

where

$$\psi_{ij}^m(t) = \psi(t) \cdot \mathcal{X}(B_{ij}^m \cup C_{ji}^m; t), \quad t \in [a_1, a_2]; \tag{2.2}$$

(F_2) *for $(i, j) \in \mathbb{P}(n_1, n_2)$, the function $y_{\omega,\psi}$ is extremal for the problem*

$$\int_{a_1}^{a_2} h(t)\phi_{ij}(t)\,dt \to \sup, \quad h \in H^\omega[a_1, a_2], \tag{2.3}$$

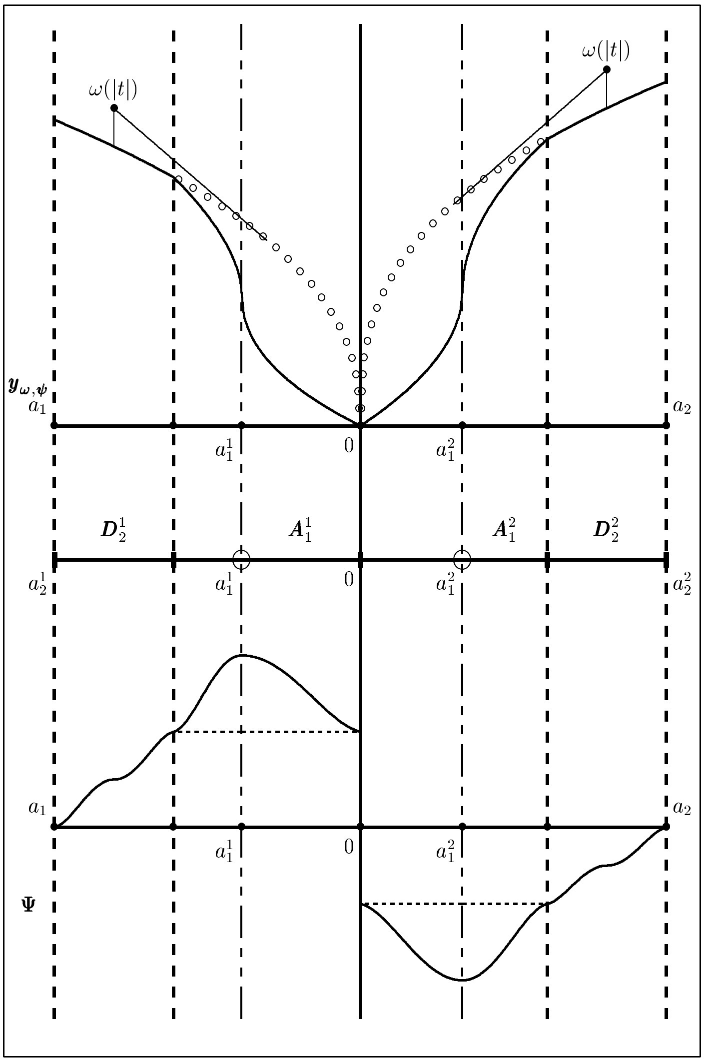

Fig. 6.3.1. $W_{2,2}$-partition and the graphs of $y_{\omega,\psi}$, Ψ for $\psi \in \mathcal{M}_{2,2}^{+1}[a_1, a_2]$

where

(2.4) $$\phi_{ij}(t) = \psi(t) \cdot \mathcal{X}(E_{ij}^1 \cup E_{ji}^2; t), \quad t \in [a_1, a_2];$$

(F_3) *for* $i = 1, \ldots, n_m - 1$, *the function* $y_{\omega,\psi}$ *is extremal for the problem*

(2.5) $$\int_{a_1}^{a_2} h(t)\psi_i^m(t)\, dt \to \sup, \quad h \in H^\omega[a_1, a_2],$$

where

(2.6) $$\psi_i^m(t) = \psi(t) \cdot \mathcal{X}(A_i^m; t), \quad t \in [a_1, a_2].$$

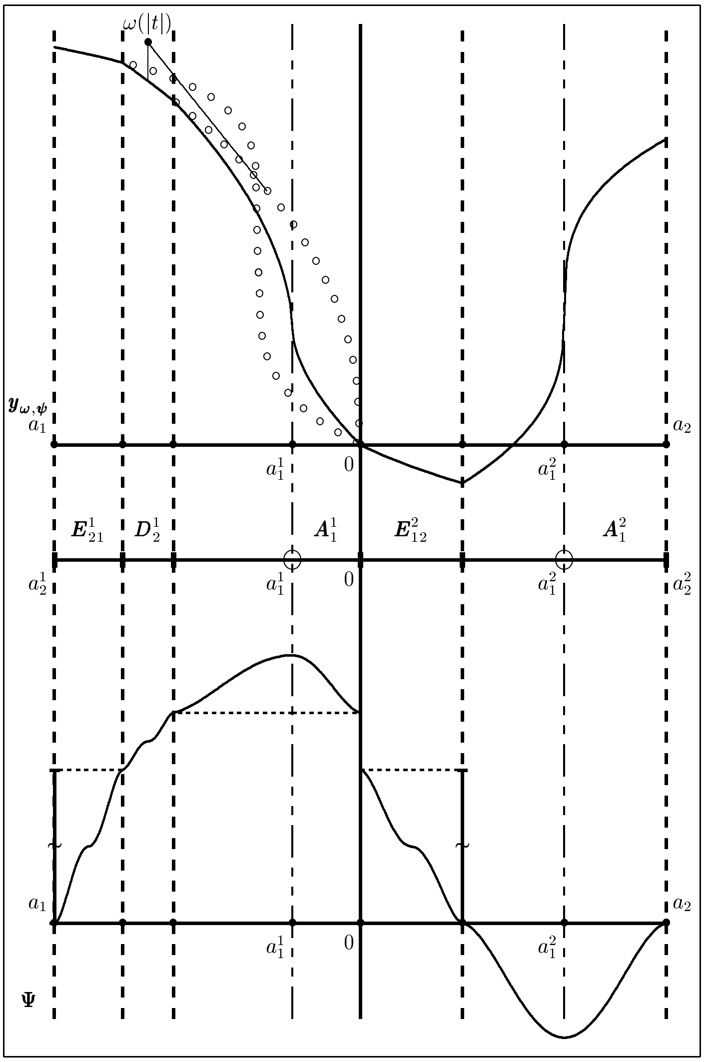

Fig. 6.3.2. $W_{2,2}$-partition and the graphs of $y_{\omega,\psi}$, Ψ for $\psi \in \mathcal{M}_{2,2}^{+1}[a_1, a_2]$

II.

$$k = 1 \Rightarrow D_{2i-1}^m = \boxdot \quad \textit{for } i = 1, \dots, \lceil n_m/2 \rceil \textit{ and } m = 1, 2;$$
$$k = -1 \Rightarrow D_{2i}^m = \boxdot \quad \textit{for } i = 1, \dots, [n_m/2] \textit{ and } m = 1, 2;$$
$$k = 0 \Rightarrow D_i^m = \boxdot \quad \textit{for } i = 1, \dots, n_m, \textit{ and } m = 1, 2.$$

Corollary 6.2.2. *The extremal function $y_{\omega,\psi}$ and W_{n_1,n_2}-partitions of the problem (★) are unique.*

6.3. Kernels $\Psi(\cdot)$ and their rearrangements $\Re_\omega(\Psi;\cdot)$. Put

$$\Psi(t) = \int_{a_m}^{t} \psi(y)\, dy, \quad t \in [0, a_m],\ m = 1, 2. \tag{3.1}$$

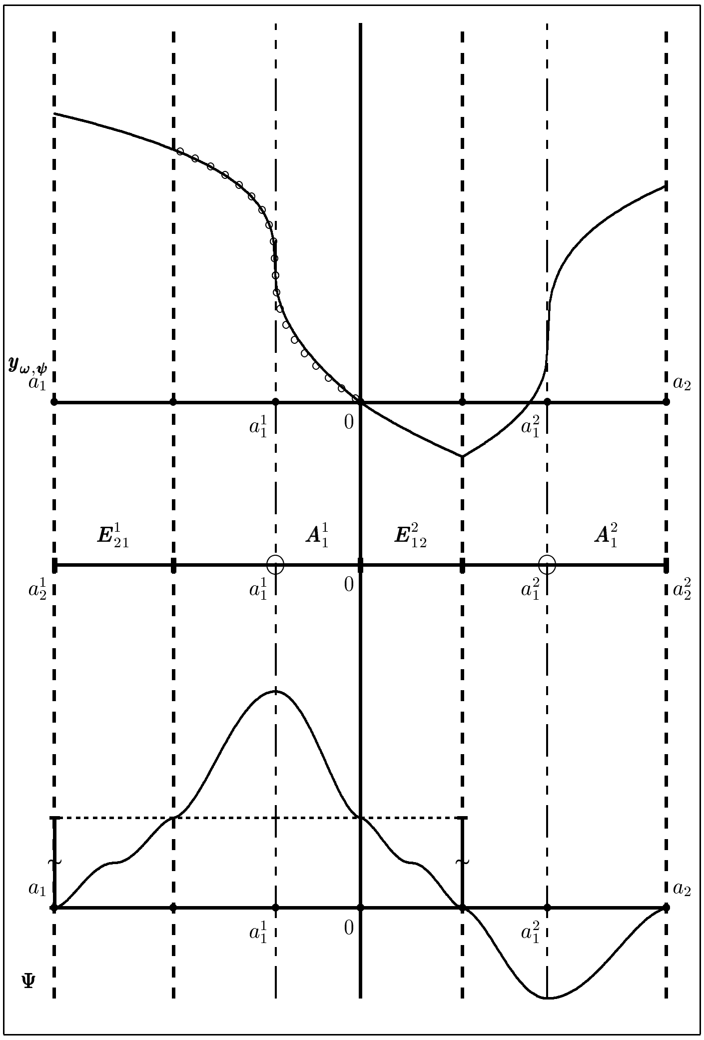

Fig. 6.3.3. $W_{2,2}$-partition and the graphs of $y_{\omega,\psi}$, Ψ for $\psi \in \mathcal{M}^0_{2,2}[a_1, a_2]$

and for $t \in [a_1, a_2]$ and $m = 1, 2$, let

(3.2)
$$\begin{aligned} \Psi^m_{ij}(t) &= \int_{a_1}^{t} \psi^m_{ij}(y)\,dy, \quad (i,j) \in \mathcal{P}(n_m), \\ \Psi^m_{i}(t) &= \int_{a_1}^{t} \psi^m_{i}(y)\,dy, \quad i = 1, \dots, n_m - 1, \\ \Phi_{ij}(t) &= \int_{a_1}^{t} \phi_{ij}(y)\,dy, \quad (i,j) \in \mathbb{P}(n_1, n_2). \end{aligned}$$

As in Chapter 1.2, Theorem 6.2.1 implies that all kernels in (3.2) are simple. Figures 6.3.1–6.3.4 illustrate the graphs of the functions $y_{\omega,\psi}$ for $\psi \in \mathbb{M}_{2,2}[a_1, a_2]$.

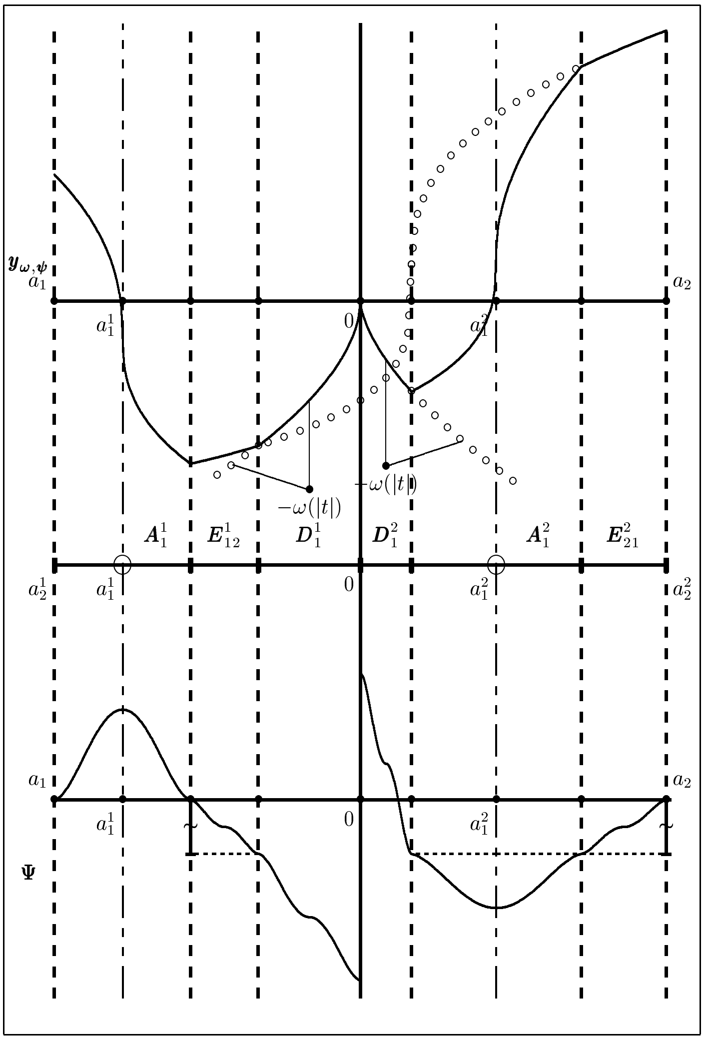

Fig. 6.3.4. $W_{2,2}$-partition and the graphs of $y_{\omega,\psi}$, Ψ for $\psi \in \mathcal{M}_{2,2}^{-1}[a_1, a_2]$

Definition 6.3.1. Let $\psi \in \mathbb{M}_{n_1,n_2}[a_1, a_2]$, and $\Psi(t)$ be defined by (3.1). The *extremal rearrangement* $\Re_\omega(\Psi; \cdot)$ of the kernel Ψ is defined as follows:

$$(3.3) \quad \Re_\omega(\Psi; t) := \sum_{m=1}^{2} \sum_{(i,j)\in\mathcal{P}(n_m)} \Re(\Psi_{ij}^m; t) + \sum_{m=1}^{2} \sum_{i=1}^{n_m-1} \Re(\Psi_i^m; t) + \sum_{m=1}^{2} \sum_{(i,j)\in\mathbb{P}(n_1,n_2)} \Re(\Psi_{ij}^m; t) + |\Psi(|t|)| \cdot \mathcal{X}(D; t), \quad t \in [0, a_2 - a_1],$$

where $D := \bigcup_{i=1}^{n_1} D_i^1 \cup \bigcup_{i=1}^{n_2} D_i^2$, and the rearrangements $\Re(\Phi; \cdot)$ of simple kernels Φ are defined by (1.1.5).

The numerical value of the maximum in the problem (★) is expressed in terms of the extremal rearrangement as follows:

$$\sup_{h\in H_0^\omega[a_1,a_2]} \int_{a_1}^{a_2} h(t)\psi(t)\,dt = \int_0^{a_2-a_1} \Re_\omega(\Psi;t)\omega'(t)\,dt. \tag{3.4}$$

REMARK 6.3.1. If $\chi\psi(x) > 0$ for a.e. $x \in [a_1, a_2]$ and a fixed $\chi \in \{-1, 1\}$, then

$$y_{\omega,\psi}(t) = \chi\omega(|t|), \quad t \in [a_1, a_2]. \tag{3.5}$$

7. Euler ω-splines on the finite interval

Let $m, r \in \mathbb{N}$ with $0 < m \le r$, and $\Gamma, B > 0$. In this chapter we describe the family of ω-splines extremal for the problem

$$|f^{(m)}(0)| \to \sup, \quad f \in W^r H^\omega[-\Gamma,\Gamma],\ \|f\|_{\mathbb{C}[-\Gamma,\Gamma]} \le B. \tag{$\star$}$$

7.1. Numerical differentiation formulae for $f^{(m)}(0)$, $f \in W^r H^\omega[-\Gamma,\Gamma]$, $0 < m < r$. Fix $N \in \mathbb{N}$, $N \ge r$, such that $N = m + 1 \pmod 2$. Let the collections of points $\overline{\nu} = \{\nu_i\}_{i=0}^{N+1}$ and $\overline{\vartheta} = \{\vartheta_i\}_{i=1}^{n-r+1}$ in $[-\Gamma,\Gamma]$ be such that

(1.1) (A) $-\Gamma =: \nu_0 < \nu_1 < \ldots < \nu_{N+1} := \Gamma$; $\quad \vartheta_1 < \ldots < \vartheta_{N-r+1}$;
(B) $\nu_i < \vartheta_i < \nu_{i+r-1}$ for $i = 1, \ldots, N - r + 1$;
(C) $\nu_i = \nu_{N+1-i}$ for $i = 0, \ldots, N + 1$; $\quad \vartheta_i = -\vartheta_{N-r+2-i}$ for $i = 1, \ldots, N - r + 1$.

Let $\{\alpha_i\}_{i=0}^N$ be determined from the system of linear equations

$$\begin{cases} \sum_{i=0}^N \alpha_i(\nu_i+\Gamma)^j = 0, & j = 0,\ldots,m-1,\\ \sum_{i=0}^N \alpha_i(\nu_i+\Gamma)^j = \dfrac{j!}{(j-m)!}, & j = m,\ldots,r-1,\\ \sum_{i=0}^N \alpha_i(\nu_i-\vartheta_l)_+^{r-1} = \dfrac{(r-1)!}{(r-1-m)!}(-\vartheta_l)_+^{r-1-m}, & l = 1,\ldots,N-r+1. \end{cases} \tag{1.2}$$

Inequalities (1.1)(B) guarantee that (1.2) has a unique solution ([50]).

Let $f \in \mathbb{C}^r[-\Gamma,\Gamma]$. From the Taylor formula

$$f(\tau) = \sum_{j=0}^{r-1} \beta_j(\tau+\Gamma)^j + \frac{1}{(r-1)!}\int_{-\Gamma}^{\Gamma} f^{(r)}(x)(\tau-x)_+^{r-1}\,dx, \tag{1.3}$$

where $\beta_j = f^{(j)}(-\Gamma)/j!$, $\tau \in [-\Gamma,\Gamma]$, it follows that

$$f^{(m)}(\tau) = \sum_{j=m}^{r-1} \frac{j!}{(j-m)!}\beta_j(\tau+\Gamma)^{j-m} + \frac{1}{(r-1-m)!}\int_{-\Gamma}^{\Gamma} f^{(r)}(x)(\tau-x)_+^{r-1-m}\,dx. \tag{1.4}$$

(1.2)–(1.4) imply that

$$f^{(m)}(0) = \sum_{i=0}^{N} \alpha_i f(\nu_i) + \int_{-\Gamma}^{\Gamma} f^{(r)}(x)K(x)\,dx, \tag{1.5}$$

where

$$K(x) = \frac{1}{(r-1-m)!}(-x)_+^{r-1-m} - \frac{1}{(r-1)!}\Big[\sum_{i=1}^{N} \alpha_i(\nu_i - x)_+^{r-1} + \alpha_0(\nu_0 - x)^{r-1}\Big]. \tag{1.6}$$

The following property of K is a consequence of (1.2) and the symmetry (1.1)(C) of the collections $\{\nu_i\}_{i=0}^{N+1}$ and $\{\vartheta_i\}_{i=1}^{N-r+1}$ with respect to the origin (cf. [10]).

LEMMA 7.1.1. *Let* $\{\nu_i\}_{i=0}^{N+1}$ *and* $\{\vartheta_i\}_{i=1}^{N-r+1}$ *be as in* (1.1), *and* $K(t)$ *be defined by* (1.2), (1.6). *Then* $\alpha_0 = 0$, *and*

$$K(-t) = (-1)^{r-m}K(t). \tag{1.7}$$

Assume that $\{\nu_i\}_{i=0}^{N}$ and $\{\vartheta_i\}_{i=1}^{N-r+1}$ are chosen in such a way that K has zero mean on $[-\Gamma,\Gamma]$. This is equivalent to an additional equation for $\{\alpha_i\}_{i=0}^{N}$:

$$\sum_{i=1}^{N} \alpha_i(\nu_i + \Gamma)^r = \frac{r!}{(r-m)!}\Gamma^{r-m}. \tag{1.8}$$

As a consequence,

$$\sup_{h\in H^\omega[-\Gamma,\Gamma]} \int_{-\Gamma}^{\Gamma} h(t)K(t)\,dt = \sup_{g\in H_0^\omega[-\Gamma,\Gamma]} \int_{-\Gamma}^{\Gamma} g(t)K(t)\,dt.$$

The verification of the following properties of K proceeds along the lines of the argument in Proposition 6.3 of [50]:

(1.9) (i) $\operatorname{supp} K = [\nu_k, \nu_{N+1-k}]$ for some k with $1 \le k \le [(N+2-r)/2]$;
(ii) the kernel K has precisely $N+3-2k-r$ simple zeros $\{\vartheta_i\}_{i=k}^{N+2-k-r}$ in (ν_k, ν_{N+1-k});
(iii) $\operatorname{sign}\alpha_i = (-1)^{i+m}$ for $i = k+1,\dots,N+1-k$;
(iv) $(-1)^{i+r+m}\operatorname{sign}K(t) \ge 0$ and $\vartheta_i \le t \le \vartheta_{i+1}$ for $i = k-1,\dots,N+2-k-r$.

By (1.8) and (1.9), $K \in \pm\mathcal{M}^0_{N+4-2k-r}[\nu_k,\nu_{N+1-k}]$ (see Definition 1.2.1).

Hence, for every $f \in W^rH^\omega[-\Gamma,\Gamma]$,

$$|f^{(m)}(0)| \le \sum_{i=1}^{N} |\alpha_i|\cdot\|f\|_{\mathbb{C}[-\Gamma,\Gamma]} + \int_0^{2\Gamma} \Re_\omega(F;t)\omega'(t)\,dt, \tag{1.10}$$

where $\Re_\omega(F;t)$ is the rearrangement of $F(t) = \int_{\nu_N}^{t} K(y)\,dy$ (see (1.2.20)).

7.2. An elementary case of the pointwise Landau problem. Before proceeding with the derivation of numerical differentiation formulae for $f^{(r)}(0)$, we describe solutions of the problem

$$f'(0) \to \sup, \quad f \in W^1H^\omega[-1,1],\ \|f\|_{\mathbb{C}[-1,1]} \le B. \tag{2.1}$$

For any $A > 0$ and $\xi \in (0,1)$, we introduce the function $\mathcal{F}_{\xi,A}$ uniquely defined by the following conditions (see Figure 7.2.1):

$$\frac{d}{dx}\mathcal{F}_{\xi,A}(x) := \begin{cases} (A-\omega(\xi-x))_+, & x \in [0,\xi], \\ (A-\omega(x-\xi))_+, & x \in [\xi,1], \end{cases}$$
$$\mathcal{F}_{\xi,A}(1) = -\mathcal{F}_{\xi,A}(0) = \|\mathcal{F}_{\xi,A}\|_{\mathbb{C}[0,1]}.$$

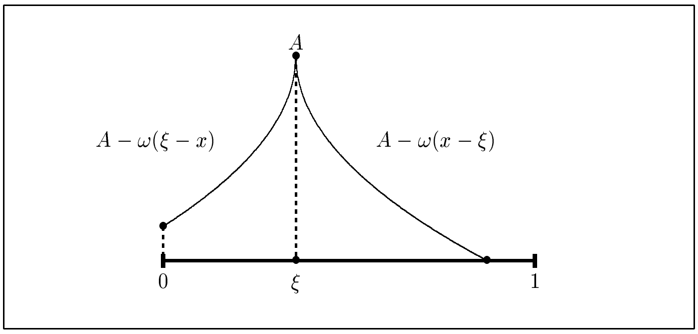

Fig. 7.2.1. The function $\frac{d}{dx}F_{\xi,A}(x)$

Proposition 7.2.1. *Let $f \in W^1H^\omega[0,1]$. Then*

$$f'(\xi) > A \;\Leftrightarrow\; \|f\|_{\mathbb{C}[0,1]} < \|\mathcal{F}_{\xi,A}\|_{\mathbb{C}[0,1]}.$$

Proof. Let $\mathbb{K} := \operatorname{supp} \mathcal{F}'_{\xi,A}$. Since $f' \in H^\omega[0,1]$ and $f'(\xi) > A$ it follows that

$$f'(x) > \mathcal{F}'_{\xi,A}(x) > 0, \quad x \in \mathbb{K}. \tag{2.2}$$

Thus, by (2.2),

$$\|f\|_{\mathbb{C}[0,1]} \geq \|f\|_{\mathbb{C}(\mathbb{K})} \geq \tfrac{1}{2}\|f'\|_{\mathbb{L}_1(\mathbb{K})} \geq \tfrac{1}{2}\|\mathcal{F}'_{\xi,A}\|_{\mathbb{L}_1(\mathbb{K})} = \|\mathcal{F}_{\xi,A}\|_{\mathbb{C}[0,1]}. \tag{2.3}$$ ■

For any fixed $\xi \in [0,1]$, the function $B_{\xi,A}$ strictly increases from 0 to ∞, as A increases from 0 to ∞. Therefore, the following corollary of Proposition 7.2.1 describes solutions of the problem (2.1) for all $B > 0$.

Corollary 7.2.2. *The function $\mathcal{F}_{\xi,A}$ is extremal for the problem*

$$f'(\xi) \to \sup, \quad f \in W^1H^\omega[0,1], \; \|f\|_{\mathbb{C}[0,1]} \leq B_{\xi,A}. \tag{2.4}$$

7.3. Numerical differentiation formulae for $f^{(r)}(0)$. Notice that in this case N is even for r odd and odd for r even.

Let the collections of points $\overline{\nu} = \{\nu_i\}_{i=0}^{N+1}$ and $\overline{\vartheta} = \{\vartheta_i\}_{i=1}^{N-r-1}$ on $[-\Gamma,\Gamma]$ be such that

(3.1) (A) $-\Gamma =: \nu_0 < \nu_1 < \ldots < \nu_{N+1} := \Gamma;\ \vartheta_1 < \vartheta_2 < \ldots < \vartheta_{N-r-1}$;

(B) $\nu_i < \vartheta_i < \nu_{i+r}$ for $i = 1,\ldots,N-r-1$;

(C) $\nu_i = \nu_{N+1-i}$ for $i = 0,\ldots,N+1$; $\vartheta_i = -\vartheta_{N-r-i}$ for $i = 1,\ldots,N-r-1$.

Let

$$K(t) = -\frac{1}{(r-1)!}\sum_{i=1}^{N} \alpha_i(\nu_i - t)_+^{r-1}, \tag{3.2}$$

where the coefficients $\{\alpha_i\}_{i=0}^N$ are determined from the linear equations

$$\begin{cases} \sum_{i=0}^{N} \alpha_i(\nu_i+\Gamma)^j = 0, & j=0,\dots,r-1, \\ \sum_{i=0}^{N} \alpha_i(\nu_i-\vartheta_l)_+^{r-1} = 0, & l=1,\dots,N-r, \\ \sum_{i=0}^{N} \alpha_i(\nu_i+\Gamma)_+^{r} = -r!. \end{cases} \tag{3.3}$$

REMARK 7.3.1. The last equation in (3.3) is added to get the normalization $\int_{-\Gamma}^{\Gamma} K(x)\,dx = 1$.

Put

$$F(t) = \begin{cases} \int\limits_{-\Gamma}^{t} K(y)\,dy, & t\in[-\Gamma,0], \\ \int\limits_{\Gamma}^{t} K(y)\,dy, & t\in[0,\Gamma]. \end{cases} \tag{3.4}$$

From (1.3) and (3.3) we derive the numerical differentiation formula

$$f^{(r)}(0) = \sum_{i=0}^{N} \alpha_i f(\nu_i) + \int\limits_{-\Gamma}^{\Gamma} [f^{(r)}(x) - f^{(r)}(0)]K(x)\,dx. \tag{3.5}$$

As in Lemma 7.1.1, we can show that $\alpha_0 = 0$ and K is even. The reader is referred to Proposition 6.3 of [50] for the proof of the following properties of K:

(3.6) (i) $\operatorname{supp} K = [\nu_k, \nu_{N+1-k}]$ for some k with $1 \le k \le [(N+1-r)/2]$;

(ii) K has precisely $N+1-2k-r$ simple zeros $\{\vartheta_i\}_{i=k}^{n-k-r}$ in (ν_k, ν_{N+1-k});

(iii) $\operatorname{sign} \alpha_i = (-1)^{i+r}$ for $i = k+1,\dots,N+1-k$;

(iv) $(-1)^i \operatorname{sign} K(t) \ge 0$ and $\vartheta_i \le t \le \vartheta_{i+1}$ for $i = k-1,\dots,N+1-k-r$.

In addition, it follows from the type of normalization in (3.3) that

$$F(0-) = \tfrac{1}{2}, \quad F(0+) = -\tfrac{1}{2}, \tag{3.7}$$

where $F(0-)$ and $F(0+)$ are the left-hand and right-hand limits of F at the origin.

By (3.6), (3.7),

$$K \in -\mathcal{M}_{l,l}^{-1}[\nu_k, \nu_{N+1-k}], \quad l = \tfrac{1}{2}(N+3-2k-r) \tag{3.8}$$

(see Definition 6.1.1). The reader is referred to Theorem 6.2.1 for a description of the function extremal for the problem $\int_{\nu_k}^{\nu_{N+1-k}} h(t)K(t)\,dt \to \sup,\ h \in H_\xi^\omega[\nu_k, \nu_{N+1-k}]$.

Hence, for all $f \in W^r H^\omega[-\Gamma,\Gamma]$,

$$|f^{(r)}(0)| \le \sum_{i=1}^{N} |\alpha_i| \cdot \|f\|_{\mathbb{C}[-\Gamma,\Gamma]} + \int\limits_{0}^{2\Gamma} \Re_\omega(F;t)\omega'(t)\,dt, \tag{3.9}$$

where $\Re_\omega(F;t)$ is the rearrangement introduced in Definition 6.3.1.

7.4. Euler perfect ω-splines. First, we formulate an analog of Theorem 3.3.1 on the structure of perfect ω-splines of a fixed norm, whose number of alternance points does not exceed a fixed integer.

7.4.1. *The main theorem*

THEOREM 7.4.1. *Let $B, \Gamma > 0$, $m, r, n \in \mathbb{N}$ with $0 < m \le r$ and $n \ge r$, and ω be a concave modulus of continuity on $[0, 2\Gamma]$. Then there exist an integer $N = N(B, n, r, m, \omega)$ with $r \le N \le n$, collections of points $\overline{\nu} = \overline{\nu}(B, n, r, m, \omega) = \{\nu_i\}_{i=1}^N$ and $\overline{\vartheta} = \overline{\vartheta}(B, n, r, m, \omega) = \{\vartheta_i\}_{i=0}^{N-r+2}$ as in* (1.1) *for $0 < m < r$, and $\overline{\vartheta} = \overline{\vartheta}(B, n, r, \omega) = \{\vartheta_i\}_{i=1}^{N-r-1}$ as in* (3.1) *for $m = r$, and a function $\mathcal{Y}_n = \mathcal{Y}_{B,n,r,m,\omega}$, with the following properties*:

(I)

$$\sup_{h \in H_\xi^\omega[-\Gamma,\Gamma]} \int_{-\Gamma}^{\Gamma} h(x)K(x)\,dx = \int_{-\Gamma}^{\Gamma} [\mathcal{Y}_n^{(r)}(x) - \mathcal{Y}_n^{(r)}(0)]K(x)\,dx, \tag{4.1}$$

where the coefficients $\{\alpha_i\}_{i=1}^N$ satisfy (1.2), (1.8) *for $0 < m < r$, and* (3.3) *for $m = r$, and the kernel K is defined in* (1.6) *for $0 < m < r$ and in* (3.2) *for $m = r$.*

(II)(A) *If $N < n$, then*

$$\mathcal{Y}_n(\nu_i) = (-1)^{i+m}\|\mathcal{Y}_n\|_{\mathbb{C}[-\Gamma,\Gamma]} = (-1)^{i+m}B, \quad i = 1, \dots, N. \tag{4.2}$$

(II)(B) *If $N = n$, then*

$$\begin{aligned} &\mathcal{Y}_n(\nu_i) = (-1)^{i+m}\|\mathcal{Y}_n\|_{\mathbb{C}[\nu_1,\nu_n]} = (-1)^{i+m}B, \quad i = 1, \dots, n, \\ &\frac{d}{dt}\mathcal{Y}_n(\nu_k) = 0 \quad \text{if } \nu_k \in (-\Gamma, \Gamma), \quad k = 1, n. \end{aligned} \tag{4.3}$$

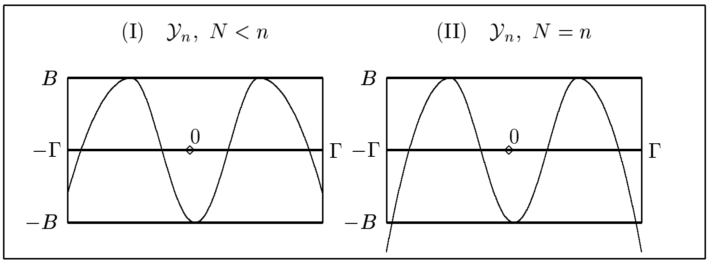

Fig. 7.4.1 Schematic graphs of $\mathcal{Y}_n$ for $N < n$ and $N = n$

Therefore, if $N < n$, then $\{\nu_i\}_{i=1}^N$ are the points of alternance of $\mathcal{Y}_n$ in the whole interval $[-\Gamma, \Gamma]$ (see Figure 7.4.1, (I)). Therefore, according to the sufficient conditions (3.1), $\mathcal{Y}_n$ is extremal for the Kolmogorov–Landau pointwise problem

$$f^{(m)}(\xi) \to \sup, \quad f \in W^r H^\omega[-\Gamma, \Gamma], \ \|f\|_{\mathbb{C}[-\Gamma,\Gamma]} \le B. \tag{4.4}$$

If $N = n$, then $\{\nu_i\}_{i=1}^n$ are the points of alternance of $\mathcal{Y}_n$ only in $[\nu_1, \nu_n] \subset [-\Gamma, \Gamma]$ (see Figure 7.4.1, (II)). Consequently, $\mathcal{Y}_n$ is an extremal function of the problem

$$f^{(m)}(\xi) \to \sup, \quad f \in W^r H^\omega[-\Gamma, \Gamma], \ \|f\|_{\mathbb{C}[\nu_1,\nu_n]} \le B. \tag{4.5}$$

The argument in the proof of Corollary 3.3.2 shows that $\mathcal{Y}_n$ can oscillate between B and $-B$ only a limited number of times dependent only on ω, r, B.

Corollary 7.4.2. *If* $n > r + 2 + [\omega(2\Gamma)/(2B)]$, *then* $\mathcal{Y}_n$ *is extremal for the problem* (4.5).

Let $\mathcal{Z}_{B,\Gamma}$ be an extremal function of the problem (4.4). An application of a limiting procedure to any sequence $\{\mathcal{Z}_{B,\Gamma_k}\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\Gamma_k = \infty$ produces a pointwise limiting function extremal for the original Kolmogorov problem

$$\|f^{(m)}\|_{\mathbb{L}_\infty(\mathbb{R})} \to \sup, \quad f \in W^r H^\omega(\mathbb{R}),\ \|f\|_{\mathbb{L}_\infty(\mathbb{R})} \le B.$$

The reader is referred to [10] for details.

7.4.2. *Extrema of the Landau problem for large norms B*

Case $0 < m < r$, $m = r + 1 \pmod 2$. As in the case of the Kolmogorov–Landau problem for the endpoint of a finite interval, the generating kernel K has exactly one point of sign change 0 in $[-\Gamma,\Gamma]$ for all sufficiently large B in (4.4). According to (1.7), the kernel K is odd. Then Corollary 1.1.4 gives the formula (1.1.9) for the rth derivative of the extremal function $\mathcal{Z}$ of (4.4) for all sufficiently large B:

$$\mathcal{Z}^{(r)}(t) = J(t) := \begin{cases} \frac{1}{2}\omega(2t), & 0 \le t \le \Gamma, \\ -\frac{1}{2}\omega(-2t), & -\Gamma \le t \le 0. \end{cases} \tag{4.6}$$

Therefore, if B is large enough, this derivative is independent of B and is given by (4.6).

The Borsuk theorem guarantees the existence of a unique function C_r with the properties

$$\begin{aligned} &\frac{d^r}{dx^r}C_r(x) = J(x), \quad -\Gamma \le x \le \Gamma, \\ &C_r(x_i) = (-1)^{i+r+1}\|C_r\|_{\mathbb{C}[-\Gamma,\Gamma]}, \quad i = 0,\dots,r+1, \end{aligned} \tag{4.7}$$

for some points $-\Gamma = x_0 < x_1 < \ldots < x_{r+1} = \Gamma$. The following result can also be proved by an application of the Borsuk theorem and differentiation formulae.

Proposition 7.4.3. *Let* $\widehat{\mathcal{N}}_r(\omega) := \|C_r\|_{\mathbb{C}[-\Gamma,\Gamma]}$. *For any* $B > \widehat{\mathcal{N}}_r(\omega)$ *there exists a function* $\mathcal{Z}_B$ *and exactly* r *points* $\{x_i = x_i(r,\omega,B)\}_{i=1}^r$, $-\Gamma < x_1 < \ldots < x_r < \Gamma$, *with the properties*

$$\begin{aligned} &\frac{d^r}{dx^r}\mathcal{Z}_B(x) = J(x), \quad -\Gamma \le x \le \Gamma, \\ &\mathcal{Z}_B(x_i) = (-1)^{i+r+1}\|\mathcal{Z}_B\|_{\mathbb{C}[-\Gamma,\Gamma]} = (-1)^{i+r+1}B, \quad i = 1,\dots,r. \end{aligned} \tag{4.8}$$

Remark 7.4.1. The function C_r inherits the well-known characteristic property of the classical Chebyshev polynomials of degree $r + 1$:

$$\|C_r\|_{\mathbb{C}[-\Gamma,\Gamma]} = \min_{p\in P_r}\|C_r - p\|_{\mathbb{C}[-\Gamma,\Gamma]} = \max_{x\in W^r H^\omega[-\Gamma,\Gamma]}\ \min_{p\in P_r}\|x - p\|_{\mathbb{C}[-\Gamma,\Gamma]}, \tag{4.9}$$

where P_r is the space of polynomials of degree r.

Case $m = r$. The generating kernel K is negative on the entire interval $(-\Gamma,\Gamma)$. For $m = r$ and all sufficiently large B Remark 6.3.1 gives

$$\mathcal{Z}^{(r+1)}(t) = \mathcal{J}(t) := -\operatorname{sign} t \cdot \omega'(|t|), \quad -\Gamma < t < \Gamma. \tag{4.10}$$

Let $\mathcal{C}_r$ be a function such that $\mathcal{C}_r^{(r+1)} = \mathcal{J}$ and $\mathcal{C}_r$ has the maximum possible number $r+3$ of alternance points. Let also $\widetilde{\mathcal{N}}_r(\omega) := \|\mathcal{C}_r\|_{\mathbb{C}[-\Gamma,\Gamma]}$. Then any extremal function $\mathcal{Z}$ of

(4.4) for $m = r$ and $B > \widetilde{\mathcal{N}}_r(\omega)$ has exactly $r+1$ points of alternance and satisfies (4.10). This feature of extremal functions of (4.4) is proved either by referring to numerical differentiation formulae or by a standard zero counting argument ([30], [44]).

Appendix A
Construction of Chebyshev splines

In this section we remind the reader the topological construction of classical polynomial Chebyshev splines.

The following proof of the existence of polynomial perfect splines with a given number of alternance points proceeds along the lines of the construction in [44].

LEMMA A.1.1. *Let $r, m, n \in \mathbb{N}$ with $m \le r$ and $n \ge r$. Then there exist a collection of points $\{\nu_i\}_{i=0}^{n+1}$, $0 = \nu_0 < \nu_1 < \ldots < \nu_{n+1} = 1$, and a perfect spline $T_{n,r}(x)$ of degree $r+1$ with $n-r$ knots $\{\vartheta_i\}_{i=1}^{n-r}$ in $[0,1]$ such that*

$$T_{n,r}(\nu_i) = (-1)^{m+i}\|T_{n,r}\|_{\mathbb{C}[0,1]}, \quad i = 0, \ldots, n+1. \tag{1.1}$$

PROOF. The proof is based on the Borsuk Theorem 2.1.1. Let

$$\mathbb{S}^{n-r} = \Big\{s = (s_1, \ldots, s_{n-r+1}) \in \mathbb{R}^{n-r+1} \;\Big|\; \sum_{i=1}^{n-r+1} |s_i| = 1\Big\}.$$

Given an $s \in \mathbb{S}^{n-r}$, we generate a partition $\{t_i(s)\}_{i=0}^{n-r+1}$ of $[0,1]$:

$$t_0(s) = 0, \quad t_j(s) = \sum_{i=1}^{j} |s_i|, \quad j = 1, \ldots, n-r+1. \tag{1.2}$$

Put

$$g_s(x) = \operatorname{sign} s_j, \quad x \in (t_{j-1}(s), t_j(s)), \quad j = 1, \ldots, n-r+1. \tag{1.3}$$

Let

$$f_s(x) = \frac{1}{r!}\int_0^1 (x-t)_+^r g_s(t)\,dt, \quad 0 \le x \le 1. \tag{1.4}$$

Define $P_s(x) = \sum_{i=0}^{n} a_i(s)x^i$ to be the polynomial of best approximation for f_s on $[0,1]$. Put

$$T_s(x) = f_s(x) - P_s(x), \quad 0 \le x \le 1. \tag{1.5}$$

Consider the mapping $\varkappa : \mathbb{S}^{n-r} \to \mathbb{R}^{n-r}$ defined as follows:

$$\varkappa(s) = (a_{r+1}(s), \ldots, a_n(s)), \quad s \in \mathbb{S}^{n-r}. \tag{1.6}$$

Clearly, $\varkappa$ is continuous and odd on $\mathbb{S}^{n-r}$. Therefore, by the Borsuk Theorem 2.1.1 there exists an $s^* \in \mathbb{S}^{n-r}$ such that $\varkappa(s^*) = 0$. This means the degree of the polynomial P_{s^*} does not exceed r, implying that $T_{s^*}^{(r+1)} = f_{s^*}^{(r+1)} = g_{s^*}$. By the Chebyshev Theorem 2.1.2, T_{s^*} exhibits $n+2$ points of alternance in $[0,1]$. Therefore, the derivative T'_{s^*} has at least

n distinct zeros at the interior points of alternance in $(0,1)$. By the Rolle theorem, $T_{s^*}^{(r+1)}$ exhibits at least $n-r$ distinct sign changes in $(0,1)$:

$$\mu(T_{s^*}^{(r+1)};[0,1]) \geq n-r. \tag{1.7}$$

On the other hand, by (1.3), the step function $T_{s^*}^{(r+1)} = g_{s^*}$ can have at most $n-r$ sign changes at the points $\{t_i(s^*)\}_{i=1}^{n-r}$:

$$\mu(T_{s^*}^{(r+1)};[0,1]) \leq n-r. \tag{1.8}$$

The juxtaposition of (1.7) and (1.8) leads to the conclusion that $g_{s^*} = T_{s^*}^{(r+1)}$ changes sign precisely $n-r$ times in $[0,1]$ at $\{t_i(s^*)\}_{i=1}^{n-r}$. Thus, by (1.3), the entries $\{s_i^*\}_{i=1}^{n-r+1}$ of the vector s^* have alternating signs:

$$\operatorname{sign} s_j^* = (-1)^j\chi, \quad j=1,\dots,n-r+1,\ \chi = 1 \text{ or } -1, \text{ fixed}. \tag{1.9}$$

Next, $\varkappa(-s^*) = -\varkappa(s^*) = 0$, since $\varkappa$ is an odd mapping on $\mathbb{S}^{n-r}$, so we can assume without loss of generality that $\chi = (-1)^{m+r+1}$ in (1.9):

$$\operatorname{sign} T_{s^*}^{(r+1)}(x) = (-1)^{j+m+r+1}, \quad t_{j-1}(s^*) < x < t_j(s^*),\ j=1,\dots,n-r+1. \tag{1.10}$$

By the Chebyshev Theorem, T_{s^*} has $n+2$ alternance points $\{\nu_i\}_{i=0}^{n+1}$ with $0 \leq \nu_0 < \nu_1 < \ldots < \nu_{n+1} \leq 1$:

$$T_{s^*}(\nu_i) = (-1)^i\eta\|T_{s^*}\|_{\mathbb{C}[0,1]}, \quad i=0,\dots,n+1,\ \eta = 1 \text{ or } -1 \text{ fixed}. \tag{1.11}$$

Since the derivative vanishes at the interior extremal points, $T'_{s^*}(\nu_i) = 0$ for all $i = 1,\dots,n$. By (1.8), T'_{s^*} can have at most n zeros in $[0,1]$. Thus, $\{\nu_i\}_{i=1}^{n}$ exhausts the set of zeros of T'_{s^*}. Therefore, $\nu_0 = 0$ and $\nu_{n+1} = 1$. The juxtaposition of (1.10) and (1.11) enables us to conclude that $\eta = (-1)^m$ in (1.11). It remains to rename the function T_{s^*} and the knots $\{t_i(s^*)\}_{i=1}^{n-r}$:

$$T_{n,r} := T_{s^*}, \quad \vartheta_i := t_i(s^*), \quad i=1,\dots,n-r. \ \blacksquare \tag{1.12}$$

LEMMA A.1.2. *Let $T_{n,r}$ be the Chebyshev spline of degree $r+1$ with $n+2$ alternance points and $n-r$ knots in $[0,1]$. Then*

$$\|T_{n,r}\|_{\mathbb{C}[0,1]} = \min \|T\|_{\mathbb{C}[0,1]}, \tag{1.13}$$

where the minimum in (1.13) *is taken over all perfect splines T of degree $r+1$ with at most $n-r$ knots in* $[0,1]$.

PROOF. Assume that, on the contrary, there exists a perfect spline T_* of degree $r+1$ with k knots ($k \leq n-r$) such that $T_*^{(r+1)}(x) \equiv T_{n,r}(x)$, $x \in (-\infty, 0]$, and

$$\|T_*\|_{\mathbb{C}[0,1]} < \|T_{n,r}\|_{\mathbb{C}[0,1]}. \tag{1.14}$$

Consider the spline $S_* = \frac{1}{2}(T_{n,r} - T_*)$ of the form

$$S_*(x) = \sum_{i=0}^{r} c_i x^i + \sum_{j=1}^{N} \xi_j (x-t_j)_+^{r+1}, \tag{1.15}$$

where $t_1 \leq \ldots \leq t_N$ and $\xi_j = \pm 1$. In our case $N = n-r+k \leq 2(n-r)$. Note that since $\xi_j = \pm 1$, $j=1,\dots,N$, the function $S_*^{(r)}$ can change sign at most $[(n-r+k+1)/2] \leq$

$n-r$ times in $[0,1]$. However, by (1.14) and (1.1),

$$(-1)^{m+i} S_*(x_i) > 0, \quad i = 0, \dots, n+1. \tag{1.16}$$

Therefore, S_* has at least $n+1$ zeros. By the Rolle theorem, $S_*^{(r)}$ has to have at least $n-r+1$ sign changes. This contradiction proves the result. ∎

Remark A.1.1. Using a refinement of the argument in the proof of Lemma A.1.2 one can show (cf. [44]) that

$$\|T_{n,r}\|_{\mathbb{C}[0,1]} < \|T\|_{\mathbb{C}[0,1]} \tag{1.17}$$

for any perfect spline T of degree $r+1$ with at most $n-r$ knots such that $T \not\equiv T_{r,m}$ or $T \not\equiv -T_{r,m}$.

Definition A.1.1. Put

$$\gamma_{-1,r} := \infty, \quad \gamma_{i,r} := \|T_{r+i,r}\|_{\mathbb{C}[0,1]}, \quad i \in \mathbb{Z}_+.$$

According to Lemma A.1.2,

$$\gamma_{i-1,r} \geq \gamma_{i,r}, \quad i \in \mathbb{Z}_+. \tag{1.18}$$

while Remark A.1.1 implies that strict inequalities hold everywhere in (1.18).

Appendix B
Construction of Zolotarev splines

In this section we describe the structure of the one-parameter Zolotarev perfect splines $\{Q_B\}_{B>0}$ as extremal functions of the problems

$$f^{(m)}(0) \to \sup, \quad f \in W_\infty^{r+1}[0,1], \ \|f\|_{\mathbb{C}[0,1]} \leq B, \tag{0.0}$$

for all $m \in \mathbb{N}$ with $0 < m \leq r$ and $B > 0$.

B.1. Auxiliary results. In our construction we use the following result known as the *fundamental theorem of algebra for perfect splines* due to S. Karlin ([44], pp. 51–55). Fix $n, r \in \mathbb{N}$.

Lemma B.1.1. *Let the interpolating points* $\overline{x} = \{x_i\}_{i=1}^{n+r}$ *satisfy*

$$0 \leq x_1 < \dots < x_{n+r}. \tag{1.1}$$

Then there exists a unique perfect spline

$$T(\overline{x}; t) = \sum_{i=0}^{r-1} a_i(\overline{x}) t^i + \frac{t^r}{r!} + \frac{2}{r!} \sum_{j=1}^{n} (-1)^j (t - t_j(\overline{x}))_+^r, \tag{1.2}$$
$$x_1 < t_1(\overline{x}) < \dots < t_n(\overline{x}) < x_{n+r},$$

that satisfies

$$T(\overline{x}; x_i) = 0, \quad i = 1, \dots, n+r. \tag{1.3}$$

Definition B.1.1. Let $\sigma > 0$. Then

$$\mathcal{D}_\sigma := \{(x_1, \dots, x_{n+r}) \in \mathbb{R}_+^{n+r} \mid x_{i+1} - x_i \geq \sigma \ \text{ for } i = 1, \dots, n+r-1\}. \tag{1.4}$$

The following result is an immediate consequence of the uniqueness of the interpolating perfect spline $T(\overline{x},\cdot)$.

COROLLARY B.1.2. *For* $\overline{x} \in \mathcal{D}_\sigma$, *let* $T(\overline{x},t)$ *be the spline of the form* (1.2) *with the properties* (1.3). *Then the polynomial coefficients* $\{a_i(\overline{x})\}_{i=0}^{r-1}$ *and the knots* $\{t_j(\overline{x})\}_{j=1}^{n}$ *of* $T(\overline{x};t)$ *are continuous functions of* $\overline{x}$ *on* $\mathcal{D}_\sigma$.

B.2. Structure and properties of Zolotarev polynomials. Recall Definition A.1.1 of the sequence $\{\gamma_i\}_{i=-1}^{\infty}$ of the norms of Chebyshev splines of degree $r=1$.

THEOREM B.2.1. *Let* $r \in \mathbb{Z}_+$. *For all* $n \in \mathbb{Z}_+$ *and* $B < \gamma_{n-1,r}$ *there exist collections of points* $\{\theta_i = \theta_i(B)\}_{i=1}^{n}$ *with* $0 < \theta_1 < \ldots < \theta_n < 1$, *and* $\{t_i = t_i(B)\}_{i=0}^{n+r}$ *with* $0 = t_0 < t_1 < \ldots < t_{n+r} \le 1$, *and a perfect spline*

$$Q_B(x) = \sum_{i=0}^{r} a_i x^i + (-1)^{r+1}\frac{x^{r+1}}{(r+1)!} + \frac{2}{(r+1)!}\sum_{j=1}^{n}(-1)^{r+j+1}(x-\theta_j)_+^{r+1}$$

with the property

$$Q_B(t_i) = (-1)^i \|Q_B\|_{\mathbb{C}[0,t_{n+r}]} = (-1)^i B, \quad i = 0,\ldots,n+r.$$

If $t_{n+r} < 1$, *then* $(d/dt)Q_B(t_{n+r}) = 0$.

COROLLARY B.2.2. *If* $B \in [\gamma_{n,r},\gamma_{n-1,r})$, *then* $\{t_i = t_i(B)\}_{i=0}^{n+r}$ *is the set of alternance points of* Q_B *in* $[0,1]$:

$$Q_B(t_i) = (-1)^i \|Q_B\|_{\mathbb{C}[0,1]} = (-1)^i B, \quad i = 0,\ldots,n+r.$$

COROLLARY B.2.3. *For* $B \in [\gamma_{n-1,r},\gamma_{n,r})$ *the function* Q_B *is extremal for the problem* (0.0).

B.3. Proof of Theorem B.2.1. Fix $A > 2$, $n \in \mathbb{N}$, and $B \in (0,\gamma_{n-1,r})$. Set

$$\mathbb{S}_A^{n+r} := \Big\{ s = (s_1,\ldots,s_{n+r+1}) \in \mathbb{R}^{n+r+1} \;\Big|\; \sum_{i=1}^{n+r+1} |s_i| = A \Big\}.$$

Fix $\varepsilon > 0$. We generate three sets of points $\{t_i(s)\}_{i=0}^{n+r+1}, \{\overline{t}_i(s)\}_{i=0}^{n+r+1}$, $\{\tau_i(s)\}_{i=0}^{n+r}$ by

$$\begin{aligned}
&t_0(s) = 0, \quad t_j(s) = \sum_{i=1}^{j} |s_i|, && j = 1,\ldots,n+r+1,\\
&\overline{t}_0(s) = 0, \quad \overline{t}_j(s) = \min\{t_j(s);1\}, && j = 1,\ldots,n+r+1,\\
&\tau_0(s) = 0, \quad \tau_j(s) = \frac{t_j(s)+\varepsilon j}{1+(n+r)\varepsilon}, && j = 1,\ldots,n+r.
\end{aligned} \tag{3.1}$$

The points $\{\tau_i(s)\}_{i=0}^{n+r}$ are uniformly separated:

$$\tau_{i+1}(s) - \tau_i(s) \ge \frac{\varepsilon}{1+(n+r)\varepsilon}, \quad i = 1,\ldots,n+r-1, \ \forall s \in \mathbb{S}_A^{n+r}. \tag{3.2}$$

The inequalities (3.2) show that $(\tau_1(s),\ldots,\tau_{n+r}(s)) \in \mathcal{D}_\sigma$ for $\sigma := \varepsilon/(1+(n+r)\varepsilon)$ for all $s \in \mathbb{S}_A^{n+r}$, where the set $\mathcal{D}_\sigma$ is introduced in Definition B.1.1.

Lemma B.1.1 ensures the existence of a unique perfect spline W_s of degree r with n knots,

$$W_s(t) = \sum_{j=0}^{r-1} \alpha_j(s) t^j + \frac{t^r}{r!} + \frac{2}{r!} \sum_{j=1}^{n} (-1)^j (t - \theta_j(s))_+^r, \tag{3.3}$$

satisfying

$$W_s(\tau_i(s)) = 0, \quad i = 1, \dots, n+r. \tag{3.4}$$

According to Lemma B.1.1 and Corollary B.1.2, W_s depends continuously on $\{\tau_i = \tau_i(s)\}_{i=1}^{n+r}$, and, consequently, on s. The mapping $s \mapsto W_s(\cdot)$ is even on $\mathbb{S}_A^{n+r}$.

Next, we introduce the piecewise continuous function

$$U_s(t) := (\operatorname{sign} s_i)|W_s(t)|, \quad t_{i-1}(s) \le t \le t_i(s),\ i = 1, \dots, n+r+1, \tag{3.5}$$

and set

$$\widetilde{H}_s(t) = \int_0^t U_s(x)\, dx, \quad 0 \le t \le A. \tag{3.6}$$

The family $\{U_s\}_{s \in \mathbb{S}_A^{n+r}}$ depends continuously on s in the norm of $\mathbb{L}_1[0, A]$, which implies the continuity of $s \mapsto \widetilde{H}_s$ in the uniform norm of $\mathbb{C}[0, A]$. The mappings $s \mapsto U_s$ and $s \mapsto \widetilde{H}_s$ are odd on $\mathbb{S}_A^{n+r}$.

Next, we define the constant

$$\Theta = \Theta(A, B) := \min\left\{\frac{1}{4} \frac{B}{A^r + B/A}; 1\right\}, \tag{3.7}$$

and let

$$p_s(t) = \sum_{i=0}^{n+r-1} a_i(s) t^i \tag{3.8}$$

be the polynomial of degree $n+r-1$ of best approximation for $\widetilde{H}_s$ on $[0, \max\{\Theta, \overline{t}_{n+r}\}]$. The continuity of $s \mapsto \widetilde{H}_s$, coupled with the uniqueness property of polynomials of best approximation, entails the continuity of the coefficients $\{a_i(s)\}_{i=0}^{n+r-1}$ as functions of s. We also remark that the mapping $s \mapsto \{a_i(s)\}_{i=0}^{n+r-1}$ is odd, since so is $s \mapsto \widetilde{H}_s$. Put

$$H_s(t) = \widetilde{H}_s(t) - p_s(t), \quad 0 \le t \le A, \tag{3.9}$$

and set

$$\begin{aligned} D(s) = {} & \sum_{i=1}^{n+r} (-1)^i \operatorname{sign} s_i \left[|\widetilde{H}_s(\overline{t}_i(s)) - \widetilde{H}_s(\overline{t}_{i-1}(s))| - \frac{2(n+r)B}{A}(t_i(s) - t_{i-1}(s)) \right] \\ & - (-1)^{n+r+1} \operatorname{sign} s_{n+r+1} \cdot \frac{2(n+r)B}{A}(A - t_{n+r}(s)). \end{aligned} \tag{3.10}$$

Finally, we define $\varkappa : \mathbb{S}_A^{n+r} \to \mathbb{R}^{n+r}$ by

$$\varkappa(s) = (a_1(s), \dots, a_{n+r-1}(s), D(s)), \quad s \in \mathbb{S}_A^{n+r}. \tag{3.11}$$

The examination of the expression for $D(s)$ and our earlier remark on the oddity of $s \mapsto \{a_i(s)\}_{i=0}^{n+r-1}$ guarantee that $\varkappa(s)$ is an odd function of s on $\mathbb{S}_A^{n+r}$.

In view of the continuity of $s \mapsto \{a_i(s)\}_{i=0}^{n+r-1}$, the proof of the continuity of $s \mapsto \varkappa(s)$ on $\mathbb{S}_A^{n+r}$ reduces to the verification of the continuity of $D(s)$, which is the subject of the following lemma.

Lemma B.3.1. *For any $i = 1, \dots, n+r$,*

$$\widetilde{H}_s(\overline{t}_i(s)) - \widetilde{H}_s(\overline{t}_{i-1}(s)) \to 0 \quad \text{as } s_i \to 0. \tag{3.12}$$

Proof. From the elementary inequalities

$$|\widetilde{H}_s(\overline{t}_i(s)) - \widetilde{H}_s(t_{i-1}(s))| \le \|\widetilde{H}_s'\|_{\mathbb{L}_\infty[0,A]}|s_i|, \quad i = 1, \dots, n+r, \tag{3.13}$$

it follows that (3.12) will be proven once the boundedness of $\|\widetilde{H}_s'\|_{\mathbb{L}_\infty[0,A]}$ is established.

From (3.5), (3.6) we infer that $\|\widetilde{H}_s'\|_{\mathbb{L}_\infty[0,A]} = \|W_s\|_{\mathbb{L}_\infty[0,A]}$. The function W_s is a perfect spline of degree r with $n+r$ distinct zeros $\{\tau_i(s)\}_{i=1}^{n+r}$ in $[0, A]$. By Rolle's theorem, each derivative $W_s^{(k)}$ has at least $n+r-k \ge 1$ zeros in $[0, A]$ for $k = 0, \dots, r-1$. Thus,

$$\|\widetilde{H}_s'\|_{\mathbb{L}_\infty[0,A]} := \|W_s\|_{\mathbb{L}_\infty[0,A]} \le A\|W_s'\|_{\mathbb{L}_\infty[0,A]} \le \dots \le A^r\|W_s^{(r)}\|_{\mathbb{L}_\infty[0,A]} = A^r, \tag{3.14}$$

which, in view of (3.13), completes the proof of the lemma. ∎

Theorem 2.1.1 yields the existence of an $s^*(A, \varepsilon) \in \mathbb{S}_A^{n+r}$ such that

$$\varkappa(s^*) = 0. \tag{3.15}$$

Notation B.3.1. We shall use more convenient notations for collections of points at $s = s^*$:

$$\{t_i^* := t_i(s^*)\}_{i=0}^{n+r+1}; \quad \{\overline{t}_i^* = \overline{t}_i(s^*)\}_{i=0}^{n+r+1}; \quad \{\tau_i^* := \tau_i(s^*)\}_{i=0}^{n+r}.$$

First of all, let us show that $t_{n+r}^* > \Theta$, and, consequently, $\overline{t}_{n+r}^* \ge \Theta$.

Lemma B.3.2. *For any solution $s^* = (s_1^*, \dots, s_{n+r}^*)$ of* (3.15),

$$t_{n+r}^* := \sum_{i=1}^{n+r} |s_i^*| \ge \Theta.$$

Proof. If we assume that $t_{n+r}^* < \Theta$, then by (3.7),

$$t_{n+r}^* \le \frac{1}{4}\frac{B}{A^r + B/A}. \tag{3.16}$$

Then we use the fact that

$$D(s^*) := \sum_{i=1}^{n+r}(-1)^i \operatorname{sign} s_i^*\Big[|\widetilde{H}_{s^*}(\overline{t}_i^*) - \widetilde{H}_s(\overline{t}_{i-1}^*)| - \frac{2(n+r)B}{A}(t_i^* - t_{i-1}^*)\Big] \\ - (-1)^{n+r+1}\operatorname{sign} s_{n+r+1}^* \cdot \frac{2(n+r)B}{A}(A - t_{n+r}^*) = 0, \tag{3.17}$$

or, equivalently,

$$\sum_{i=1}^{n+r}(-1)^i \operatorname{sign} s_i^*\Big[(\widetilde{H}_{s^*}(\overline{t}_i^*) - \widetilde{H}_{s^*}(\overline{t}_{i-1}^*)) - \frac{2(n+r)B}{A}(t_i^* - t_{i-1}^*)\Big] \\ = (-1)^{n+r+1}\operatorname{sign} s_{n+r+1}^*\frac{2(n+r)B}{A}(A - t_{n+r}^*). \tag{3.18}$$

By (3.13), (3.14), for $i = 1, \dots, n+r$,

$$\sum_{i=1}^{n+r} |\widetilde{H}_{s^*}(\bar{t}_i^*) - \widetilde{H}_{s^*}(\bar{t}_{i-1}^*)| \le \|\widetilde{H}'_{s^*}\|_{\mathbb{C}[0,\bar{t}^*_{n+r}]}\bar{t}^*_{n+r} \le A^r \bar{t}^*_{n+r}. \tag{3.19}$$

Thus, the left-hand side of (3.18) can be estimated as follows:

$$\Big|\sum_{i=1}^{n+r}(-1)^i \operatorname{sign} s_i^*\Big[|\widetilde{H}_{s^*}(\bar{t}_i^*) - \widetilde{H}_{s^*}(\bar{t}_{i-1}^*)| - \frac{2(n+r)B}{A}(t_i^* - t_{i-1}^*)\Big]\Big| < 2(n+r)t^*_{n+r}[A^r + B/A]. \tag{3.20}$$

The juxtaposition of (3.18) and (3.20) leads to

$$2(n+r)\frac{B}{A}(A - t^*_{n+r}) < 2(n+r)t^*_{n+r}\Big[A^r + \frac{B}{A}\Big],$$

or, equivalently,

$$t^*_{n+r} > \frac{B}{A^r + 2B/A} > \frac{1}{2}\frac{B}{A^r + B/A}, \tag{3.21}$$

contradicting (3.16). ■

Thus, we have proved that $\bar{t}^*_{n+r} > \Theta$. Therefore, by the Chebyshev Theorem 2.1.2, H_{s^*} has $n + r + 1$ points of alternance in $[0, \bar{t}^*_{n+r}]$.

Since $\varkappa(s^*) = 0$, the coefficients $\{a_i(s^*)\}_{i=1}^{n+r-1}$ are zeros. Therefore, by (3.9),

$$H_{s^*}(t) = \widetilde{H}_{s^*}(t) - a_0(s), \quad 0 \le t \le A, \tag{3.22}$$

and, consequently, by (3.6) and (3.9),

$$\frac{d}{dt}H_{s^*}(t) = \frac{d}{dt}\widetilde{H}_{s^*}(t) := U_{s^*}(t), \quad 0 \le t \le A. \tag{3.23}$$

However, as (3.5) shows, $(d/dt)H_{s^*} = U_{s^*}$ can change sign in $[0, \bar{t}^*_{n+r}]$ only at $\{t_i^*\}_{i=1}^{n+r-1}$. Therefore, the $n + r + 1$ points of alternance of H_{s^*} on $[0, t^*_{n+r}]$ are $\{t_i^*\}_{i=0}^{n+r}$,

$$0 = t_0^* < t_1^* = \bar{t}_1^* < t_2^* = \bar{t}_2^* < \ldots < t^*_{n+r-1} = \bar{t}^*_{n+r-1} < \bar{t}^*_{n+r}, \tag{3.24}$$

and $(d/dt)H_{s^*}$ does change sign at $\{t_i^*\}_{i=1}^{n+r-1}$. Therefore,

$$\operatorname{sign} s_i^* = (-1)^i\xi, \quad i = 1, \dots, n+r, \ \xi = \{\pm 1\} \text{ fixed}. \tag{3.25}$$

Thus, by (3.5) and (3.23),

$$\frac{d}{dt}H_{s^*}(t) = (-1)^i\xi|W_{s^*}(t)|, \quad t^*_{i-1} \le t \le t_i^*, \ i = 1, \dots, n+r. \tag{3.26}$$

We observe that all $n + r$ zeros $\{\tau_i^*\}_{i=1}^{n+r}$ of W_{s^*} are simple (otherwise, by the Rolle theorem, W_{s^*} would have more than n knots), i.e.,

$$\operatorname{sign} W_{s^*}(t) = (-1)^i\gamma, \quad \tau^*_{i-1} < t < \tau_i^*, \ i = 1, \dots, n+r, \tag{3.27}$$

where $\gamma = 1$ or -1 is fixed.

Notice that the mapping $s \mapsto U_s$ is even, while $s \mapsto H_s$ is odd on $\mathbb{S}_A^{n+r}$. Because $\varkappa$ is odd on $\mathbb{S}_A^{n+r}$, $\varkappa(s^*) = 0$ implies that $\varkappa(-s^*) = -\varkappa(s^*) = 0$. Therefore, we can choose s^* in such a way (replacing s^* by $-s^*$ if necessary) that the γ in (3.27) equals the ξ in (3.26).

Next, (3.22) implies that $\widetilde{H}_{s^*}(\overline{t}_i^*) - \widetilde{H}_{s^*}(\overline{t}_{i-1}^*) = H_{s^*}(t_i^*) - H_{s^*}(t_{i-1}^*)$. Therefore,

$$D(s^*) = \sum_{i=1}^{n+r}(-1)^i \operatorname{sign} s_i^* \Big[|H_{s^*}(\overline{t}_i^*) - H_{s^*}(\overline{t}_{i-1}^*)| - \frac{2(n+r)B}{A}(t_i^* - t_{i-1}^*)\Big] - (-1)^{n+r+1} \operatorname{sign} s_{n+r+1}^* \cdot \frac{2(n+r)B}{A}(A - t_{n+r}^*). \tag{3.28}$$

At this point we remind the reader that the solutions s^* of (3.15) depend on both $\varepsilon > 0$ and A. We apply the Arzelà–Ascoli theorem to extract a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, with $\varepsilon_k \downarrow 0$ as $k \uparrow \infty$, such that

$$\begin{aligned} H_{s^*(\varepsilon_k,A)} &\to H_A \quad \text{in } \mathbb{C}[0,A] \text{ as } k\to\infty,\\ W_{s^*(\varepsilon_k,A)} &\to W_A \quad \text{in } \mathbb{C}^r[0,A] \text{ as } k\to\infty. \end{aligned} \tag{3.29}$$

Notation B.3.2. Set

$$s^*(A) = \lim_{k\to\infty} s^*(\varepsilon_k, A), \quad D_A = \lim_{k\to\infty} D(s^*(\varepsilon_k, A)),$$

and

$$\begin{aligned} \theta_i^*(A) &= \lim_{k\to\infty} \theta_i^*(\varepsilon_k, A), \quad i = 1,\dots,n+r+1,\\ t_i^*(A) &= \lim_{k\to\infty} t_i^*(\varepsilon_k, A), \quad \overline{t}_i^*(A) = \min\{t_i^*(A), 1\}, \quad i = 0,\dots,n+r+1. \end{aligned} \tag{3.30}$$

By (3.1)

$$\tau_i^*(\varepsilon_k, A) := \frac{t_i^*(\varepsilon_k, A) + \varepsilon_k i}{1 + (n+r)\varepsilon_k} \to t_i^*(A) \quad \text{as } \varepsilon_k \downarrow 0,\ i = 1,\dots,n+r. \tag{3.31}$$

Then, the juxtaposition of (3.26) and (3.27) ($\gamma = \xi$ by our choice) leads to

$$\frac{d}{dt}H_A(t) = W_A(t), \quad 0 \le t \le A. \tag{3.32}$$

Therefore, H_A is a true polynomial perfect spline of degree $r+1$ with n distinct knots $\{\theta_i^*(A)\}_{i=1}^n$, $0 < \theta_1^* < \ldots < \theta_n^* < A$, and $n+r+1$ points of alternance:

$$H_A(\overline{t}_i^*) = (-1)^i \xi \|H_A\|_{\mathbb{C}[0,\overline{t}_{n+r}^*]}, \quad i = 0,\dots,n+r. \tag{3.33}$$

From the fact that $\{t_i^*\}_{i=1}^{n+r}$ remain distinct (see (3.33)) and our choice of the sign of $\{s_i^*(A,\varepsilon_k)\}_{i=1}^{n+r}$ we infer that

$$\operatorname{sign} s_i^*(A) = (-1)^i \xi, \quad i = 1,\dots,n+r. \tag{3.34}$$

Our next goal is to show that

$$\operatorname{sign} s_{n+r+1}^*(A) = (-1)^{n+r+1}\xi \tag{3.35}$$

for all sufficiently large values of A.

Assume that, on the contrary,

$$\operatorname{sign} s_{n+r+1}^*(A) = (-1)^{n+r}\xi. \tag{3.36}$$

By (3.28), (3.33)–(3.35), in this case we have

$$\begin{aligned}(3.37)\qquad 0 = D_A &:= \sum_{i=1}^{n+r}(-1)^i \operatorname{sign} s_i^* |H_A(\bar{t}_i^*) - H_A(\bar{t}_{i-1}^*)| \\ &\quad - \sum_{i=1}^{n+r+1}(-1)^i \operatorname{sign} s_i^* \frac{2(n+r)B}{A}(t_i^* - t_{i-1}^*) \\ &= 2(n+r)\xi\left(\|H_A\|_{\mathbb{C}[0,\bar{t}_{n+r}^*]} - \frac{B}{A}(2t_{n+r}^* - A)\right).\end{aligned}$$

Therefore,

$$\|H_A\|_{\mathbb{C}[0,\bar{t}_{n+r}^*]} = \frac{B}{A}(2t_{n+r}^*(A) - A). \tag{3.38}$$

As a result, we obtain

$$t_{n+r}^*(A) \geq A/2 \tag{3.39}$$

and

$$\|H_A\|_{\mathbb{C}[0,\bar{t}_{n+r}^*]} \leq B, \tag{3.40}$$

since $t_{n+r}^* \leq A$. Therefore, H_A' has $n+r-1$ zeros $\{t_i^*\}_{i=1}^{n+r-1}$ in $[0,1]$ and a zero $t_{n+r}^* \geq A/2$. By the Rolle theorem, $H_A^{(k)}$ has precisely $n+r+1-k$ zeros in $[0,A]$ for $k = 1,\dots,r$. We enumerate these zeros $\{\eta_i^k\}_{i=1}^{n+r+1-k}$ in such a way that

$$0 < \eta_{n+r+1-k}^k < \eta_{n+r-k}^k < \ldots < \eta_1^k < A. \tag{3.41}$$

Also by the Rolle theorem, at least $n+r-k$ of these zeros, namely, $\{\eta_j^k\}_{j=2}^{n+r+1-k}$, belong to $[0,1]$.

We wish to apply Corollary 2.2.6 to show that for any $k = 1,\dots,r$, and for all sufficiently large values of A, the rightmost zero η_1^k of $H_A^{(k)}$ belongs to $[1,A]$. The assumptions are satisfied, since H_A' is a perfect spline of degree r with $n+r$ simple zeros $\{t_i\}_{i=1}^{n+r}$ and n knots in $[0,A]$. Therefore, an application of Corollary 2.2.6 in combination with (3.39) leads to

LEMMA B.2.3. *For any $k = 1,\dots,r$, there exists a constant $E_{r,k,n} = E(r,k,n) > 0$, independent of A, and a constant $A_0 > 0$, such that for all $A \geq A_0$,*

$$\eta_1^k > E_{r,k,n}A, \quad k = 1,\dots,r, \tag{3.42}$$

The function $H_A^{(r)}$ is a perfect spline of degree 1 with n knots $\{\theta_i^*\}_{i=1}^n$ and $n+1$ zeros $\{\eta_i\}_{i=1}^{n+1}$. Therefore,

$$\theta_i^* = \frac{\eta_{n-i+1}^r + \eta_{n-i+2}^r}{2}, \quad i = 1,\dots,n, \tag{3.43}$$

where $\{\eta_i^r\}_{i=1}^{n+1}$ are the $n+1$ zeros of $H_A^{(r)}$ enumerated in decreasing order.

According to Lemma B.2.3, $\theta_n^*(A) = (\eta_2^r + \eta_1^r)/2 > E_{r,k,n}A/2$. Therefore, the knot $\theta_n^*(A)$ lies outside $[0,A]$ for all sufficiently large A (definitely, for $A > 2A_0$, where A_0 is defined in Lemma B.2.3). Therefore, H_A is a perfect spline of degree $r+1$ with $n-1$ knots in $[0,1]$. Lemma A.1.2 then shows that for all such A,

$$\|H_A\|_{\mathbb{C}[0,1]} \geq \gamma_{n-1,r}. \tag{3.44}$$

However, from (3.4), we infer that

$$\|H_A\|_{\mathbb{C}[0,1]} = \|H_A\|_{\mathbb{C}[0,\bar{t}^*_{n+r}]} = \frac{B}{A}(2t^*_{n+r} - A) \le \frac{B}{A}(2A - A) = B. \tag{3.45}$$

On the other hand, we picked B to be less than $\gamma_{n-1,r}$. This contradiction with our assumption that $\operatorname{sign} s^*(A) = (-1)^{n+r}\xi$ proves that $\operatorname{sign} s^*(A) = (-1)^{n+r+1}\xi$, and the computation of $D(A)$ (as in (3.38)) leads to the desired property

$$\|H_A\|_{\mathbb{C}[0,\bar{t}^*_{n+r}]} = B, \quad \forall A > 2A_0. \tag{3.46}$$

Moreover, the inequality $\|H_A\|_{\mathbb{C}[0,\bar{t}^*_{n+r}]} = B < \gamma_{n-1,r}$ coupled with Lemma A.1.2 implies that all n knots $\{\theta^*_i\}_{i=1}^n$ lie in $(0,1)$.

It remains to pick an A^* exceeding $2A_0$, e.g., $A^* = 3A_0$, and rename the points of alternance of H_{A^*}:

$$Q_B(t) = H_{A^*}(t),\ t \in [0,1];\ t_i := t^*_i(A^*),\ i = 0,\dots,n+r-1,\ t_{n+r} := \bar{t}^*_{n+r}(A^*). \ \blacksquare$$

B.4. Proof of Corollary B.2.2. Let us show that if $\gamma_{n,r} \le B < \gamma_{n-1,r}$, then a stricter property holds:

$$Q_B(t_i) = (-1)^i \|Q_B\|_{\mathbb{C}[0,1]} = (-1)^i B, \quad i = 0,\dots,r+n, \tag{4.1}$$

i.e., the collection $\{t_i\}_{i=0}^{r+n}$ is the set of alternance points of Q_B in the *entire* interval $[0,1]$, not just in the subinterval $[0,t_{r+n}] \subset [0,1]$.

Suppose that, on the contrary, there exists a $\widehat{B} \in [\gamma_{n,r}, \gamma_{n-1,r})$ such that

$$\|Q_{\widehat{B}}\|_{\mathbb{C}[0,1]} > \|Q_{\widehat{B}}\|_{\mathbb{C}[0,t_{n+r}(\widehat{B})]}. \tag{4.2}$$

Then there exists a $t_{n+r+1} \in [t_{n+r}, 1)$ such that

$$Q_{\widehat{B}}(t_{n+r+1}) = (-1)^{n+r+1}\widehat{B}.$$

The function $Q_{\widehat{B}}^{(r+1)}$ has no more than n points of sign change in $[0,1]$. Thus, by the Rolle theorem, the points $\{t_i\}_{i=1}^{n+r}$ exhaust the set of zeros of $Q'_{\widehat{B}}$. Therefore, $Q_{\widehat{B}}$ is monotone in $[t_{n+r}, 1]$, and

$$Q_{\widehat{B}}(t_i) = (-1)^i \|Q_{\widehat{B}}\|_{\mathbb{C}[0,t_{n+r+1}]} = (-1)^i \widehat{B}, \quad i = 0,\dots,n+r+1. \tag{4.3}$$

Let

$$T(x) = t_{n+r+1}^{-(r+1)} Q_{\widehat{B}}(t_{n+r+1}\cdot x), \quad 0 \le x \le 1, \tag{4.4}$$

and define

$$\tau_i := t_{n+r+1}^{-1}\cdot t_i, \quad i = 0,\dots,n+r+1. \tag{4.5}$$

By the normalization (4.4), T is a polynomial spline of degree $r+1$ with n knots and $n+r+1$ points of alternance $\{\tau_i\}_{i=0}^{n+r}$ in $[0,1]$. By Lemma A.1.2 and Remark A.1.1,

$$\|T\|_{\mathbb{C}[0,1]} = \gamma_{n,r}. \tag{4.6}$$

Then, by (4.4)

$$\widehat{B} = \|Q_{\widehat{B}}\|_{\mathbb{C}[0,t_{n+r}]} = t_{n+r+1}^{r+1}\|T\|_{\mathbb{C}[0,1]} = t_{n+r+1}^{r+1}\cdot\gamma_{n,r} < \gamma_{n,r}, \tag{4.7}$$

contrary to our assumption that $\widehat{B} \ge \gamma_{n,r}$. $\blacksquare$

B.5. Proof of Corollary B.2.3. Let points $\overline{t} = \{t_i\}_{i=0}^{n+r}$ with $0 = t_0 < t_1 < \ldots < t_{r+n} \le 1$, and $\overline{\theta} = \{\theta_l\}_{l=1}^{n}$ with $0 < \theta_1 < \ldots < \theta_n < 1$ satisfy

$$t_i < \theta_i < t_{i+r}, \quad i = 1, \ldots, n. \tag{5.1}$$

Fix m with $0 < m \le r$. Let coefficients $\{\alpha_j\}_{j=0}^{r+n}$ be derived from the linear system

$$\begin{cases} \sum_{j=0}^{r+n} \alpha_j t_j^i = (-1)^m m! \cdot \delta_{im}, & i = 0, \ldots, r, \\ \sum_{j=0}^{r+n} \alpha_j (t_j - \theta_l)_+^r = 0, & l = 1, \ldots, n. \end{cases} \tag{5.2}$$

As usual, $[t_0]^0 := 1$. The total positivity nature of the system $\{t^j\}_{j=0}^{r} \cup \{(t-\theta_l)_+^r\}_{l=1}^{n}$ guarantees that (5.2) is uniquely solvable once (5.1) is satisfied (cf. [32]).

From Taylor's formula and (5.2) we obtain the identity

$$(-1)^m f^{(m)}(0) = \sum_{j=0}^{r+n} \alpha_j f(t_j) + \int_0^{t_{r+n}} f^{(r+1)}(u) F_m(u)\, du, \tag{5.3}$$

where

$$F_m(u) := -\frac{1}{r!} \sum_{j=0}^{r+n} \alpha_j (t_j - u)_+^r. \tag{5.4}$$

The kernel F_m enjoys the following properties (cf. [10]):

$$\begin{aligned} &(-1)^i \operatorname{sign} \alpha_i \ge 0, \quad i = 0, \ldots, r+n, \\ &(-1)^{r+i} \operatorname{sign} F_m(u) \ge 0, \quad u \in (\theta_{i-1}, \theta_i), \ i = 1, \ldots, n+1, \end{aligned} \tag{5.5}$$

where $\theta_0 := 0$, $\theta_{n+1} := t_{r+n}$. Now we can specify our choice of collections $\overline{t}$ and $\overline{\theta}$: let $\{t_i\}_{i=0}^{n+r}$ be the points of alternance of Q_B, and $\{\theta_l\}_{l=1}^{n}$ be the set of knots of Q_B. That (5.1) holds is an immediate consequence of Rolle's theorem.

The formula (5.3) then leads to

$$|f^{(m)}(0)| \le \sum_{i=0}^{r+n} |\alpha_i| B + \int_0^{t_{r+n}} |F_m|\, du \tag{5.6}$$

for all $f \in W_\infty^{r+1}[0,1]$ such that $\|f\|_{\mathbb{C}[0,1]} \le B$.

The properties of Q_B enable us to conclude that Q_B is an extremal function for the inequality (5.6):

$$Q_B^{(m)}(0) = \sum_{i=0}^{r+n} |\alpha_i| B + \int_0^{t_{r+n}} |F_m(u)|\, du. \ \blacksquare \tag{5.7}$$

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