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CATEGORICAL EQUIVALENCES OF VARIETIES GENERATED BY ALGEBRAS $\mathfrak{A}$ AND $\mathfrak{A}^r$

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1. Introduction

Let L and K be two varieties which are equivalent as categories by functors $G: K \to L$ and $H: L \to K$. Then many properties of K transfer to L. For instance, subdirect products and homomorphic images are preserved under category equivalence. Hence there is a one-to-one correspondence between the isomorphism classes of irreducibles in the two varieties. The lattices of subvarieties of L and K are isomorphic. If L is generated by an algebra $\mathfrak{P}$, then K is the variety generated by $\mathfrak{P}H$. There is a mapping t from the n-ary terms of L to the n-ary terms of K such that

(a) $\mathfrak{x}_i t = \mathfrak{x}_i$,

(b) if α and β are self-mappings of $\{1, \ldots, n\}$ and L satisfies $x_{1\alpha} \ldots x_{n\alpha}\, \mathfrak{p} = x_{1\beta} \ldots x_{n\beta}\, \mathfrak{q}$, then K satisfies $x_{1\alpha} \ldots x_{n\alpha}(\mathfrak{p}t) = x_{1\beta} \ldots x_{n\beta}(\mathfrak{q}t)$.

Consider the variety $V(\mathfrak{A})$ generated by a finite algebra $\mathfrak{A}$ and assume that each algebra of $V(\mathfrak{A})$ is isomorphic to a subdirect power of $\mathfrak{A}$. Let $\mathfrak{B}(\mathfrak{X}) \in V(\mathfrak{A})$ be the free algebra freely generated by $X = \{x_1, \ldots, x_n\}$ and let $\mathfrak{p}$, $\mathfrak{q}$ be two terms of $\mathfrak{B}(\mathfrak{X})$. Then $\mathfrak{p}$ and $\mathfrak{q}$ are identical if for all homomorphisms $h: \mathfrak{B}(\mathfrak{X}) \to \mathfrak{A}$ we have $\mathfrak{p}h = \mathfrak{q}h$. Let K be category equivalent to $V(\mathfrak{A})$. If we have $(x_1, \ldots, x_n\, \mathfrak{p})h = (x_1 \ldots x_n\, \mathfrak{q})h$ for all homomorphisms $h: \mathfrak{B}(\mathfrak{X}) \to \mathfrak{A}$, then we obtain

$$x_{1\alpha} \ldots x_{n\alpha}(\mathfrak{p}t) = x_{1\beta} \ldots x_{n\beta}(\mathfrak{q}t) \quad \text{in} \quad K.$$

Denote by $\mathbf{2}$ the two-element Boolean algebra. $V(\mathbf{2})$ is the variety of Boolean algebras and every Boolean algebra is isomorphic to a subdirect power of $\mathbf{2}$: $V(\mathbf{2}) = IP_s\mathbf{2}$. If a variety K is category equivalent to $V(\mathbf{2})$, then K is generated by a primal algebra ([4]). The fact described above is meaningful in the complexity theory of Boolean functions. By category equivalence consequences for the complexity theory of primal algebras can be derived.

In order to generalize the well-known equivalence between $V(\mathbf{2})$ and any variety generated by a primal algebra varieties were determined which are equivalent to varieties generated by two-element preprimal algebras in [2]. These varieties are generated by preprimal algebras. A finite algebra $\mathfrak{A} = (A; F)$ is said to be *preprimal* if the clone generated by F is covered by the clone $\mathfrak{F}(A)$ of all functions defined on the set A.

In this paper we consider a category equivalence between varieties generated by algebras with carriers A and A^r $(r > 1)$. If A is finite the preprimality of the generating algebras is preserved under this equivalence, i.e., $\mathfrak{A}$ is preprimal if and only if $\mathfrak{A}^r$ is preprimal.

2. The equivalence S

We use the following definition of the equivalence of categories ([1]):

DEFINITION 2.1. Two categories $\boldsymbol{L}$ and $\boldsymbol{K}$ are *equivalent* if and only if there are functors S: $\boldsymbol{K} \to \boldsymbol{L}$ and R: $\boldsymbol{L} \to \boldsymbol{K}$ and for each $\mathfrak{A} \in \mathrm{ob}\,\boldsymbol{K}$ and $\mathfrak{B} \in \mathrm{ob}\,\boldsymbol{L}$ there are isomorphisms $\alpha_{\mathfrak{A}}$: $\mathfrak{A} \to (\mathfrak{A}S)R$ and $\beta_{\mathfrak{B}}$: $\mathfrak{B} \to (\mathfrak{B}R)S$ such that for each h: $\mathfrak{A} \to \mathfrak{A}'$ in $\boldsymbol{K}$ and each g: $\mathfrak{B} \to \mathfrak{B}'$ in $\boldsymbol{L}$ the following diagrams commute:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{h} & \mathfrak{A}' \\ {\scriptstyle\alpha_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle\alpha_{\mathfrak{A}'}} \\ (\mathfrak{A}S)R & \xrightarrow{(hS)R} & (\mathfrak{A}'S)R \end{array} \qquad \begin{array}{ccc} \mathfrak{B} & \xrightarrow{g} & \mathfrak{B}' \\ {\scriptstyle\beta_{\mathfrak{B}}}\downarrow & & \downarrow{\scriptstyle\beta_{\mathfrak{B}'}} \\ (\mathfrak{B}R)S & \xrightarrow{(gR)S} & (\mathfrak{B}'R)S \end{array}$$

The functor S is called *equivalence*.

Let A be a non-empty set. We denote by $\mathfrak{F}_n(A)$ the set of all n-ary functions defined on A $(n = 0, 1, \dots)$ and by $\mathfrak{F}(A)$ the set of all on A defined functions, i.e., $\mathfrak{F}(A) = \bigcup_{n=0}^{\infty} \mathfrak{F}_n(A)$.

Consider algebras $\mathfrak{A}_0 = (A; F)$ and $\mathfrak{B}_0 = (A^r; G)$, where $A^r = A \times \dots \times A$ is the rth cartesian power of A $(r > 1)$. All term functions of an algebra can be derived by superposition from the fundamental operations of the algebra. Superposition of functions can be described by the following operations on $\mathfrak{F}(A)$:

$$x_1 \dots x_n(f\xi) = x_2 x_3 \dots x_n x_1 f,$$

$$x_1 \dots x_n(f\tau) = x_2 x_1 x_3 \dots x_n f, \quad \text{for} \quad n \geqslant 2$$

$$x_1 \dots x_{n-1}(f\Delta) = x_1 x_1 x_2 \dots x_{n-1} f,$$

$$f\xi = f\tau = f\Delta = f \quad \text{for} \quad n = 1,$$

$$x_1 \dots x_m x_{m+1} \dots x_{m+n-1}(f * g) = x_1 \dots x_m g x_{m+1} \dots x_{m+n-1} f$$

for an n-ary function f and an m-ary function g, $x_1 x_2 e = x_1$ (e projection).

The set of all functions from $\mathfrak{F}(A)$ arising from the fundamental operations of the algebra $\mathfrak{A}$ using $*$, ξ, τ, Δ, e is called *clone* of term functions $T(\mathfrak{A})$ of the algebra $\mathfrak{A}$. $T(\mathfrak{A})$ can be regarded as an algebra of type $(2, 1, 1, 1, 0)$: $\mathfrak{T}(\mathfrak{A}) := (T(\mathfrak{A}); *, \xi, \tau, \Delta, e)$.

Any variable ranging over A^r may be considered as $X_i = (x_{i1}, \dots, x_{ir})$, where $x_{i1}, \dots, x_{ir}$ are variables ranging over A. Since the values of an n-ary function $f \in \mathfrak{F}_n(A^r)$ are elements of A^r, f may be regarded as an r-tuple of (nr)-ary functions from $\mathfrak{F}(A)$. Thus we have

$$X_1 \dots X_n f = (x_{11} \dots x_{1r}) \dots (x_{n1} \dots x_{nr}) f$$
$$= (x_{11} \dots x_{1r} \dots x_{n1} \dots x_{nr} f_1, \dots, x_{11} \dots x_{1r} \dots x_{n1} \dots x_{nr} f_r).$$

We say that $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are *related by* S' if the following is satisfied:

$$S'\colon\ T(\mathfrak{A}_0) = \{f_i \mid f \in T(\mathfrak{B}_0),\ i = 1, 2, \dots, r\} \quad \text{and} \quad T(\mathfrak{B}_0)$$
$$= \{(f_1, \dots, f_r) \mid f_i \in T(\mathfrak{A}_0),\ i = 1, 2, \dots, r\}$$

([7]). (Clearly, if f is an n-ary term function of $\mathfrak{B}_0$, then the f_i are (nr)-ary term functions of $\mathfrak{A}_0$.)

There holds the following Equivalence Theorem:

THEOREM 2.2. *Let $\mathfrak{A}_0 = (A; F)$ and $\mathfrak{B}_0 = (A^r; G)$ be two algebras which are related by S'. Then the varieties $V(\mathfrak{A}_0)$ and $V(\mathfrak{B}_0)$ are equivalent by functors $S\colon V(\mathfrak{A}_0) \to V(\mathfrak{B}_0)$, $R\colon V(\mathfrak{B}_0) \to V(\mathfrak{A}_0)$, and ontoness is preserved.*

Proof. At first we construct $\mathfrak{B} = \mathfrak{A}S$ for any algebra $\mathfrak{A} \in V(\mathfrak{A}_0)$ as follows: Let A^r be the carrier of $\mathfrak{B}$. Each r-tuple consisting of (nr)-ary term functions of $\mathfrak{A}$ is an n-ary term function of $\mathfrak{B}$: $f^B = (f_1^A, \dots, f_r^A)$, where $f_1^A, \dots, f_r^A$ are interpretations of term functions $f_1, \dots, f_r$ of $\mathfrak{A}_0$. Thus we can write:

$$T(\mathfrak{B}) = \{(f_1^A, \dots, f_r^A) \mid f_i^A \in T(\mathfrak{A}),\ i = 1, \dots, r\}.$$

Let $\varphi = \psi$ be an identity in $\mathfrak{B}_0$. Since φ and ψ are term functions of $\mathfrak{B}_0$ we have $\varphi = (\varphi_1, \dots, \varphi_r)$, $\psi = (\psi_1, \dots, \psi_r)$ with $\varphi_1, \dots, \varphi_r, \psi_1, \dots, \psi_r \in T(\mathfrak{A}_0)$. Clearly, $(\varphi_1, \dots, \varphi_r) = (\psi_1, \dots, \psi_r)$ iff $\varphi_i = \psi_i$ for all $i = 1, \dots, r$. It follows $\varphi_i^A = \psi_i^A$ ($i = 1, \dots, r$) for the interpretations of φ_i and ψ_i in $\mathfrak{A}$. Therefore, $(\varphi_1^A, \dots, \varphi_r^A) = (\psi_1^A, \dots, \psi_r^A)$ and $\varphi^B = \psi^B$ is an identity in $\mathfrak{B}$. From $\operatorname{Id}\mathfrak{B}_0 \subset \operatorname{Id}\mathfrak{B}$ for the sets of identities of $\mathfrak{B}_0$ and $\mathfrak{B}$ it follows $\mathfrak{B} \in V(\mathfrak{B}_0)$.

Now we extend S to homomorphisms so that it is a functor. Let $h\colon \mathfrak{A} \to \mathfrak{A}'$ be a morphism in the category $V(\mathfrak{A}_0)$. The image $hS\colon S(\mathfrak{A}) \to S(\mathfrak{A}')$ is defined by

$$(a_1, \ldots, a_r)(hS) := (a_1 h, \ldots, a_r h) \quad \text{for all} \quad (a_1, \ldots, a_r) \in B = A^r.$$

Let f be an n-ary term function of $\mathfrak{B}$. Then we have:

$$\begin{aligned}((a_{11}, \ldots, a_{1r}) \ldots (a_{n1}, \ldots, a_{nr}) f)(hS) &= (a_{11} \ldots a_{nr} f_1^A, \ldots, a_{11} \ldots a_{nr} f_r^A)(hS)\\ &= ((a_{11} \ldots a_{nr} f_1^A) h, \ldots, (a_{11} \ldots a_{nr} f_r^A) h)\\ &= ((a_{11} h) \ldots (a_{nr} h) f_1^A, \ldots, (a_{11} h) \ldots (a_{nr} h) f_r^A)\\ &= ((a_{11} h) \ldots (a_{1r} h)) \ldots ((a_{n1} h) \ldots (a_{nr} h)) f\\ &= (a_{11}, \ldots, a_{1r})(hS) \ldots (a_{n1}, \ldots, a_{nr})(hS) f.\end{aligned}$$

From the definition of hS it follows that S maps surjective homomorphisms to surjective homomorphisms. Further we have:

$$\begin{aligned}(a_1, \ldots, a_r)((h_1 \circ h_2) S) &= (a_1 (h_1 \circ h_2), \ldots, a_r (h_1 \circ h_2))\\ &= ((a_1 h_1) h_2, \ldots, (a_r h_1) h_2) = ((a_1, \ldots, a_r)(h_1 S))(h_2 S)\\ &= (a_1, \ldots, a_r)(h_2 S \circ h_1 S)\end{aligned}$$

for all $(a_1, \ldots, a_r) \in B = A^r$, and

$$(a_1, \ldots, a_r)(1_A S) = (a_1, \ldots, a_r) = (a_1, \ldots, a_r) 1_{S(A)}.$$

Therefore S is a functor S: $V(\mathfrak{A}_0) \to V(\mathfrak{B}_0)$.

We construct $\mathfrak{A} = \mathfrak{B}R$ for any algebra $\mathfrak{B} \in V(\mathfrak{B}_0)$ as follows: π_0 with $(b_1, \ldots, b_r)\pi_0 = (b_1, \ldots, b_1)$ is a unary term function of $\mathfrak{B}_0$ since $(b_1, \ldots, b_r)\pi_0 = ((b_1, \ldots, b_r)p_1^r, \ldots, (b_1, \ldots, b_r)p_1^r)$. The projection p_1^r is a term function of $\mathfrak{A}_0$. Let π_0^B be the interpretation of π_0 in $\mathfrak{B}$. Take $\mathfrak{A}$ as the range of π_0^B. To define the term functions of $\mathfrak{A}$ we consider such term functions $\hat{f}$ of $\mathfrak{B}_0$ which can be represented in the form

$$\begin{aligned}&(b_{11}, \ldots, b_{1r}) \ldots (b_{n1}, \ldots, b_{nr}) \hat{f}\\ &\qquad = (b_{11} \ldots b_{1r} \ldots b_{n1} \ldots b_{nr} f^0, \ldots, b_{11} \ldots b_{1r} \ldots b_{n1} \ldots b_{nr} f^0),\end{aligned}$$

where f^0 is an (nr)-ary term function of $\mathfrak{A}_0$. Let $\hat{f}^B$ be the interpretation of $\hat{f}$ in B. The term functions of $\mathfrak{A}$ are the restrictions $\hat{f}^B/A$ to A. Clearly, $\hat{f}^B$ preserves A because $\hat{f}^B$ satisfies the identity $\pi_0 * \hat{f} = \hat{f}$ in $\mathfrak{B}_0$. Let $\varphi = \psi$ be an identity in $\mathfrak{A}_0$, i.e., $\varphi, \psi \in T(\mathfrak{A}_0)$. Then $\hat{\varphi} = (\varphi, \ldots, \varphi) = (\psi, \ldots, \psi) = \hat{\psi}$ is an identity in $\mathfrak{B}_0$ and $\hat{\varphi}^B = \hat{\psi}^B$ is an identity in $B \in V(\mathfrak{B}_0)$. It follows that $\hat{\varphi}^B/A = \hat{\psi}^B/A$ is an identity in $\mathfrak{A}$, i.e., Id $\mathfrak{A}_0 \subseteq$ Id $\mathfrak{A}$ and therefore $\mathfrak{A} \in V(\mathfrak{A}_0)$. We define gR: $\mathfrak{B}R \to \mathfrak{B}'R$ for any homomorphism g: $\mathfrak{B} \to \mathfrak{B}'$ by $(a, \ldots, a)(gR) := (a, \ldots, a)g$ for all $(a, \ldots, a) \in A^r$. Clearly, gR is a homomorphism, R is a functor and R maps surjective homomorphisms to surjective homomorphisms.

What is to be shown next is that there are isomorphisms $\alpha_{\mathfrak{A}}$: $\mathfrak{A}$

$\to (\mathfrak{A}S)R$ and β_B: $\mathfrak{B} \to (\mathfrak{B}R)S$ such that the diagrams in Definition 2.1 are commutative. The carrier of $(\mathfrak{A}S)R$ is $\{(a, \ldots, a) | a \in A\}$, i.e., $\alpha_{\mathfrak{A}}$ is the mapping defined by $a\alpha_{\mathfrak{A}} = (a, \ldots, a)$ for all $a \in A$. Each term function of $\mathfrak{A}$ can be regarded as an intepretation of a term function of $\mathfrak{A}_0$. If $f \in T(\mathfrak{A}_0)$, then each function obtained from f by identifying variables belongs to $T(\mathfrak{A}_0)$, i.e., if $f \in T(\mathfrak{A}_0)$ then $f^0 \in T(\mathfrak{A}_0)$ with $b_1 \ldots b_n f = b_1 \ldots b_n \ldots b_n f^0$. Then we have in $\mathfrak{A}$: $b_1 \ldots b_n f^A = b_1 \ldots b_1 \ldots b_n \ldots b_n f^{0A}$, therefore

$$(b_1 \ldots b_n f^A)\alpha_{\mathfrak{A}} = (b_1 \ldots b_1 \ldots b_n \ldots b_n f^{0A})\alpha_{\mathfrak{A}} = (b_1 \ldots b_1 \ldots b_n \ldots b_n f^{0A}, \ldots$$
$$\ldots, b_1 \ldots b_1 \ldots b_n \ldots b_n f^{0A}) = (b_1 \ldots b_1) \ldots (b_n \ldots b_n) \hat{f}^B = (b_1 \alpha_{\mathfrak{A}} \ldots b_n \alpha_{\mathfrak{A}}) \hat{f}^B,$$

i.e., $\alpha_{\mathfrak{A}}$ is an isomorphism $\alpha_{\mathfrak{A}}$: $\mathfrak{A} \to (\mathfrak{A}S)R$.

Consider the mapping $\beta_{\mathfrak{B}}$: $\mathfrak{B} \to (\mathfrak{B}R)S$ defined by

$$(b_1, \ldots, b_r)\beta_{\mathfrak{B}} = ((b_1, \ldots, b_r)\pi_1^B, \ldots, (b_1, \ldots, b_r)\pi_r^B).$$

We show that $\beta_{\mathfrak{B}}$ is an isomorphism. Let f be an n-ary term function of $\mathfrak{B}$; thus $f = (f_1^A, \ldots, f_r^A)$, where $f_1^A, \ldots, f_r^A$ are (nr)-ary term functions of $\mathfrak{A}$. Then there holds:

$$((b_{11}, \ldots, b_{1r}) \ldots (b_{n1}, \ldots, b_{nr}) f)\beta_{\mathfrak{B}} = (b_{11} \ldots b_{nr} f_1^A, \ldots, b_{11} \ldots b_{nr} f_r^A)\beta_{\mathfrak{B}}$$
$$= ((b_{11} \ldots b_{nr} f_1^A, \ldots, b_{11} \ldots b_{nr} f_r^A)\pi_1^B, \ldots, (b_{11} \ldots b_{nr} f_1^A, \ldots, b_{11} \ldots b_{nr} f_r^A)\pi_r^B).$$

Let f^* be the image of f by the isomorphism $\beta_{\mathfrak{B}}$. Then f^* has the form $(f_1^*, \ldots, f_r^*)$, where $f_1^*, \ldots, f_r^*$ are certain (nr)-ary term functions of $\mathfrak{B}R$. Each term function of $\mathfrak{B}R$ has the form $\hat{f}^B/\mathfrak{B}R$, where $\hat{f}^B/\mathfrak{B}R$ arises by interpretation in $\mathfrak{B}$ and restriction to $\mathfrak{B}R$ from

$$(b_{11}, \ldots, b_{1r}) \ldots (b_{n1}, \ldots, b_{nr}) \hat{f} = (b_{11} \ldots b_{nr} f_1, \ldots, b_{11} \ldots b_{nr} f_1),$$
$$\hat{f} \in T(\mathfrak{B}_0),\ f_1 \in T(\mathfrak{A}_0).$$

Therefore we have

$$(b_{11}, \ldots, b_{1r}) \ldots (b_{n1}, \ldots, b_{nr}) f^* = ((b_{11} \ldots b_{nr} f_1^*, \ldots, b_{11} \ldots b_{nr} f_r^*)\pi_1^B, \ldots$$
$$\ldots, (b_{11} \ldots b_{nr} f_1^*, \ldots, b_{11} \ldots b_{nr} f_r^*)\pi_r^B).$$

There holds:

$$((b_{11}, \ldots, b_{1r})\beta_{\mathfrak{B}} \ldots (b_{n1}, \ldots, b_{nr})\beta_{\mathfrak{B}}) f^* = ((b_{11}, \ldots, b_{1r})\pi_1^B, \ldots$$
$$\ldots, (b_{11}, \ldots, b_{1r})\pi_r^B) \ldots ((b_{n1}, \ldots, b_{nr})\pi_1^B, \ldots, (b_{n1}, \ldots, b_{nr})\pi_r^B) f^*$$
$$= ((((b_{11}, \ldots, b_{1r})\pi_1^B \ldots (b_{11}, \ldots, b_{1r})\pi_r^B \ldots (b_{n1}, \ldots, b_{nr})\pi_1^B \ldots$$
$$\ldots (b_{n1}, \ldots, b_{nr})\pi_r^B) f_1^*, \ldots, ((b_{11}, \ldots, b_{1r})\pi_1^B \ldots (b_{n1}, \ldots, b_{nr})\pi_r^B) f_r^*)\pi_1^B, \ldots$$
$$\ldots, (((b_{11}, \ldots, b_{1r})\pi_1^B \ldots (b_{n1}, \ldots, b_{nr})\pi_r^B) f_1^*, \ldots$$
$$\ldots, ((b_{11}, \ldots, b_{1r})\pi_1^B \ldots (b_{n1}, \ldots, b_{nr})\pi_r^B) f_r^*)\pi_r^B).$$

f_i^* $(i = 1, \ldots, r)$ is the interpretation of a function $f_i^{*A_0}$ $(i = 1, \ldots, r)$ in $\mathfrak{A}$.

If $f_i \in T(\mathfrak{A}_0)$ then $f_i^0 \in T(\mathfrak{A}_0)$ with

$$b_{11} b_{12} \ldots b_{1r} \ldots b_{n1} \ldots b_{nr} f_i = b_{11} \ldots b_{11} b_{12} \ldots b_{12} \ldots b_{n1} \ldots b_{n1} \ldots b_{nr} \ldots b_{nr} f_i^0$$

$(i = 1, \ldots, r)$.

It follows:

$$b_{11} \ldots b_{nr} f_i = ((b_{11}, \ldots, b_{1r}) \pi_1 \ldots (b_{11}, \ldots, b_{1r}) \pi_r \ldots (b_{n1}, \ldots, b_{nr}) \pi_1 \ldots \\ \ldots (b_{n1}, \ldots, b_{nr}) \pi_r) f_i^0 \quad (i = 1, \ldots, r).$$

Then we have:

$$((b_{11} \ldots b_{nr} f_1, \ldots, b_{11} \ldots b_{nr} f_1), \ldots, (b_{11} \ldots b_{nr} f_r, \ldots, b_{11} \ldots b_{nr} f_r)) \\ = ((b_{11} \ldots b_{11} \ldots b_{nr} \ldots b_{nr} f_1^0, \ldots, b_{11} \ldots b_{11} \ldots b_{nr} \ldots b_{nr} f_1^0), \ldots \\ \ldots, (b_{11} \ldots b_{11} \ldots b_{nr} \ldots b_{nr} f_r^0, \ldots, b_{11} \ldots b_{11} \ldots b_{nr} \ldots b_{nr} f_r^0)),$$

i.e., by interpretation in A we get:

$$((b_{11} \ldots b_{nr} f_1^A, \ldots, b_{11} \ldots b_{nr} f_r^A) \pi_1^B, \ldots, (b_{11} \ldots b_{nr} f_1^A, \ldots, b_{11} \ldots b_{nr} f_r^A)) \pi_r^B \\ = (((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_1^{0A}, \ldots \\ \ldots, ((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_r^{0A}) \pi_1^B, \ldots \\ \ldots, (((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_1^{0A}, \ldots \\ \ldots, ((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_r^{0A}) \pi_r^B.$$

It follows:

$$((b_{11} \ldots b_{nr} f_1^A, \ldots, b_{11} \ldots b_{nr} f_r^A) \pi_1^B, \ldots, (b_{11} \ldots b_{nr} f_1^A, \ldots, b_{11} \ldots b_{nr} f_r^A) \pi_r^B) \\ = (((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_1^*, \ldots \\ \ldots, ((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_r^*) \pi_1^B, \ldots \\ \ldots, (((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_1^*, \ldots \\ \ldots, ((b_{11}, \ldots, b_{1r}) \pi_1^B \ldots (b_{n1}, \ldots, b_{nr}) \pi_r^B) f_r^*) \pi_r^B.$$

Thus we have

$$((b_{11}, \ldots, b_{1r}) \ldots (b_{n1}, \ldots, b_{nr}) f) \beta_B = ((b_{11}, \ldots, b_{1r}) \beta_B \ldots (b_{n1}, \ldots, b_{nr}) \beta_B) f^*.$$

The last thing needed in order to show categorical equivalence is that the diagrams in Definition 2.1 are commutative. For all $a \in A$ one has

$$(ah) \alpha_{\mathfrak{A}'} = (ah, \ldots, ah) \quad \text{and} \quad (a\alpha_{\mathfrak{A}})(hS) R = ((a, \ldots, a)(hS)) R = (ah, \ldots, ah).$$

For all $(b_1, \ldots, b_r)$ there holds:

$$((b_1, \ldots, b_r)g)\beta_{\mathfrak{B}'} = (((b_1, \ldots, b_r)g)\pi_1^{B'}, \ldots, ((b_1, \ldots, b_r)g)\pi_r^{B'})$$
$$= (((b_1, \ldots, b_r)\pi_1^{B'})g, \ldots, ((b_1, \ldots, b_r)\pi_r^{B'})g) = ((b_1, \ldots, b_r)\pi_1^{B'}, \ldots$$
$$\ldots, (b_1, \ldots, b_r)\pi_r^{B'})(gS) = ((b_1, \ldots, b_r)\pi_1^{B'}, \ldots, (b_1, \ldots, b_r)\pi_r^{B'})(gR)S$$
$$= ((b_1, \ldots, b_r)\beta_{\mathfrak{B}'})(gR)S.$$

This finishes the proof of Theorem 2.2.

3. Application of the equivalence S

Recall that the lattices of subvarieties of categorically equivalent varieties are isomorphic. To describe subvarieties of the variety $V(\mathfrak{A})$ one often makes use of the clone of $\mathfrak{A}$. The clone $T(\mathfrak{A})$ of an algebra $\mathfrak{A} = (A; F)$ can be regarded as an algebra of type (2, 1, 1, 1, 0): $\mathfrak{T}(\mathfrak{A}) := (T(\mathfrak{A}); *, \xi, \tau, \Delta, e)$. There are three congruences on each clone $T(\mathfrak{A})$: $\varkappa_0$, $\varkappa_a$, $\varkappa_1$, defined by

$$(f, g) \in \varkappa_0 :\Leftrightarrow \{f, g\} \subseteq T(\mathfrak{A}) \wedge f = g,$$
$$(f, g) \in \varkappa_a :\Leftrightarrow \{f, g\} \subseteq T(\mathfrak{A}) \wedge af = ag, \quad \text{where } af \text{ is the arity of } f,$$
$$(f, g) \in \varkappa_1 :\Leftrightarrow \{f, g\} \subseteq T(\mathfrak{A}).$$

Each congruence on $T(\mathfrak{A})$ with $\varkappa \subseteq \varkappa_a$ is called *arity congruence*. Let $\mathrm{Con}_a T(\mathfrak{A})$ be the lattice of all arity congruences of $T(\mathfrak{A})$. Let $\mathfrak{F}(\mathfrak{X}) \in V(\mathfrak{A})$ be the free algebra freely generated by the countable set X. We are interested in the fully invariant congruences of $\mathfrak{F}(\mathfrak{X})$. (By definition a congruence is fully invariant if it is compatible with every endomorphism of the algebra.) Let $\mathrm{Con}_{\mathrm{inv}}\, \mathfrak{F}(\mathfrak{X})$ be the lattice of all fully invariant congruences of $\mathfrak{F}(\mathfrak{X})$. $\mathrm{Con}_{\mathrm{inv}}\, \mathfrak{F}(\mathfrak{X})$ is antiisomorphic to the lattice of all subvarieties of $V(\mathfrak{A})$ and it is isomorphic to the lattice of all arity congruences of $T(\mathfrak{A})$ ([3]). Using these results we obtain for algebras $\mathfrak{A}_0$ and $\mathfrak{B}_0$ related by S' from Theorem 2.2:

COROLLARY 3.1. *Let $\mathfrak{A}_0$ and $\mathfrak{B}_0$ be two algebras related by S'. Then there holds* $\mathrm{Con}\, \mathfrak{T}(\mathfrak{A}_0) = \mathrm{Con}\, \mathfrak{T}(\mathfrak{B}_0)$.

We remark that the isomorphism $\mathrm{Con}\, \mathfrak{T}(\mathfrak{A}_0) \cong \mathrm{Con}\, \mathfrak{T}(\mathfrak{B}_0)$ is given by $(f, g) \in \hat{\varkappa} \Leftrightarrow (f_1, g_1) \wedge \ldots \wedge (f_r, g_r) \in \varkappa$ with $f, g \in T(\mathfrak{B}_0)$, $f_1, \ldots, f_r, g_1, \ldots, g_r \in T(\mathfrak{A}_0)$, $f = (f_1, \ldots, f_r)$, $g = (g_1, \ldots, g_r)$, $\hat{\varkappa} \in \mathrm{Con}\, \mathfrak{T}(\mathfrak{B}_0)$, $\varkappa \in \mathrm{Con}\, \mathfrak{T}(\mathfrak{A}_0)$.

We consider further properties of the equivalence S. To this end, we introduce the concept of a primal and of a preprimal algebra.

$\mathfrak{A} = (A; F)$ is primal: $\Leftrightarrow$ A is finite and $T(\mathfrak{A}) = \mathfrak{F}(A)$, i.e., each function defined on A is a term function of $\mathfrak{A}$.

$\mathfrak{A} = (A; F)$ is preprimal: $\Leftrightarrow$ A is finite and $T(\mathfrak{A})$ is covered by $\mathfrak{F}(A)$.

It is known ([4]) that for any category equivalence between two varieties $V(\mathfrak{A})$ and $V(\mathfrak{B})$ generated by finite algebras $\mathfrak{A}$ and $\mathfrak{B}$ it follows: $\mathfrak{A}$ is primal iff $\mathfrak{B}$ is primal. Indeed, T. K. Hu has shown that any variety generated by a primal algebra is category equivalent to the variety of Boolean algebras and that a variety which is equivalent to the variety of Boolean algebras is generated by a primal algebra. The Equivalence S given in Theorem 2.2 preserves the preprimality of the generating algebras, i.e., it holds

THEOREM 3.2 ([7]). *Let* $\mathfrak{A}_0$ *and* $\mathfrak{B}_0$ *be two algebras which are related by* S'. *Then* $\mathfrak{A}_0$ *is preprimal iff* $\mathfrak{B}_0$ *is preprimal.*

We state one further consequence of Theorem 2.2. Let ϱ be an h-ary relation on A, i.e., $\varrho \subseteq A^h$. We understand by $\mathrm{Pol}_A\,\varrho$ the class of all on A defined functions which preserve ϱ [(6]). For an n-ary function f this means that from

$$(a_{11}, \ldots, a_{1h}) \in \varrho,$$
$$\ldots\ldots\ldots\ldots$$
$$(a_{n1}, \ldots, a_{nh}) \in \varrho$$

it follows $(f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1h}, \ldots, a_{nh})) \in \varrho$.

Assume that $T(\mathfrak{A}_0)$ has the form $\mathrm{Pol}_A\,\varrho$ for $\varrho \subseteq A^h$ and that $\mathfrak{B}_0$ and $\mathfrak{A}_0$ are related by S'. We define an h-ary relation ϱ^r on A^r ($\varrho^r \subseteq (A^r)^h$) as follows:

$$\varrho^r = \{((a_{11}, \ldots, a_{1r}), \ldots, (a_{h1}, \ldots, a_{hr}))\| \; (a_{1i}, \ldots, a_{hi}) \in \varrho \text{ for all } i = 1, \ldots, r\}.$$

Let

$$((a^1_{11}, \ldots, a^1_{1r}), \ldots, (a^1_{h1}, \ldots, a^1_{hr})) \in \varrho^r,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$((a^n_{11}, \ldots, a^n_{1r}), \ldots, (a^n_{h1}, \ldots, a^n_{hr})) \in \varrho^r$$

and let $\hat{f} \in T(\mathfrak{B}_0)$. Then we have

$$((a^1_{11}, \ldots, a^n_{1r}) \ldots (a^n_{11}, \ldots, a^n_{1r})\hat{f}, \ldots, (a^1_{h1}, \ldots, a^1_{h1}) \ldots (a^n_{h1}, \ldots, a^n_{hr})\hat{f})$$
$$= ((a^1_{11} \ldots a^n_{1r} f_1, \ldots, a^1_{11} \ldots a^n_{1r} f_r), \ldots, (a^1_{h1} \ldots a^n_{hr} f_1, \ldots, a^1_{h1} \ldots a^n_{hr} f_r)) \in \varrho^r,$$

because $(a^1_{11}, \ldots, a^1_{h1}) \in \varrho, \ldots, (a^n_{1r}, \ldots, a^n_{hr}) \in \varrho$ and f_i $(i = 1, \ldots, r)$ are term functions of $T(\mathfrak{A}_0) = \mathrm{Pol}_A\,\varrho$. That means: $\hat{f} \in \mathrm{Pol}_{A^r}\,\varrho^r$ and $T(\mathfrak{B}_0) \subseteq \mathrm{Pol}_{A^r}\,\varrho^r$. Clearly, from $\hat{f} \in \mathrm{Pol}_{A^r}\,\varrho^r$ it follows $\hat{f} \in T(\mathfrak{B}_0)$, thus $T(\mathfrak{B}_0) = \mathrm{Pol}_{A^r}\,\varrho^r$.

Let $\mathfrak{A}_\varrho$ be any algebra with $T(\mathfrak{A}_\varrho) = \mathrm{Pol}\,\varrho$. Then $\mathfrak{A}_\varrho$ and $\mathfrak{A}^r_{\varrho^r}$ are related by S'.

From Theorem 2.2 we obtain:

COROLLARY 3.3. *The varieties* $V(\mathfrak{A}_\varrho)$ *and* $V(\mathfrak{A}^r_{\varrho^r})$ *are categorically equivalent.*

COROLLARY 3.4. ([7]). $\mathfrak{A}_\varrho$ *is preprimal if and only if* $\mathfrak{A}^r_{\varrho^r}$ *is preprimal.*

4. Examples

Let us elucidate the above results by a few examples. A primitive h-elementary relation ι_h is the h-ary relation on $A = \{0, \ldots, h-1\}$, $h \geqslant 3$, defined as follows:

$(a_1, \ldots, a_h) \in \iota_h: \Leftrightarrow$ at least two of the a_i are equal ([8], [5]).

An algebra of the form $\mathfrak{A}_\iota$ is preprimal ([8]). Then ι^r $(r > 1)$ is the relation defined on $B = A^r$ as follows:

$$\iota^r = \{((a_{11}, \ldots, a_{1r}), \ldots, (a_{h1}, \ldots, a_{hr})) \text{ for all } j \leqslant r$$
$$\text{at least two of the elements } a_{1j}, a_{2j}, \ldots, a_{hj} \text{ are equal}\}.$$

By Corollary 3.3 $V(\mathfrak{A}_\iota)$ and $V(\mathfrak{A}_{\iota^r})$ are categorically equivalent and by Corollary 3.4 $\mathfrak{A}_{\iota^r}$ is preprimal.

By ([8]) each preprimal algebra has the form $\mathfrak{A}_\varrho$, where ϱ is a certain relation defined on A.

In the second example we consider an algebra $\mathfrak{A}_\varrho$ which is not preprimal. Consider the two-element algebra $\mathfrak{F}_8^2 = (\{0, 1\}, x \wedge Ny, m)$ with $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. The term functions of $\mathfrak{F}_8^2$ are all 1-separating Boolean functions of degree 2. $\mathfrak{F}_8^2$ is generated by the binary relation $\mathfrak{D} = \{(00), (01), (10)\}$, i.e. $\mathfrak{F}_8^2$ has the form $2_{\mathfrak{D}}$. Then D^r $(r > 1)$ is the following relation on $\{0, 1\}^r$:

$$D^r := \{((a_1, \ldots, a_r), (b_1, \ldots, b_r)) \mid (a_i, b_i) \neq (1, 1),\ i = 1, \ldots, r\}.$$

By Corollary 3.2 the varieties $V(2_{\mathfrak{D}})$ and $V(2_{\mathfrak{D}^r})$ are categorically equivalent.

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