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SEQUENTIAL CONFIDENCE INTERVALS BASED ON ROBUST ESTIMATORS

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1. Asymptotic representation of M-estimators and R-estimators

Let $X_1, X_2, \ldots$ be independent random variables, X_i – distributed according to the distribution function $F(x-\theta c_i)$, $i = 1, 2, \ldots$; c_i's are known constants, θ – an unknown parameter; F is a continuous distribution function, generally unspecified; it is assumed that $F \in \mathscr{F}$ where $\mathscr{F}$ is an appropriate system of distribution functions.

The *M-estimator* $T_M^{(n)} = T_M(X_1, \ldots, X_n)$ of θ is defined as

$$T_M^{(n)} = \tfrac{1}{2}(T_M^+ + T_M^-) \tag{1.1}$$

where

$$T_M^+ = \inf\{t\colon \sum_{i=1}^{n} c_i \psi(X_i - tc_i) < 0\}, \tag{1.2}$$

$$T_M^- = \sup\{t\colon \sum_{i=1}^{n} c_i \psi(X_i - tc_i) > 0\} \tag{1.3}$$

with ψ being a nondecreasing (skew-symmetric) function on $\boldsymbol{R}^1$.

The R-estimator $T_R^{(n)} = T_R(X_1, \ldots, X_n)$ of θ is defined as

$$T_R^{(n)} = \tfrac{1}{2}(T_R^+ + T_R^-) \tag{1.4}$$

where

$$T_R^+ = \inf\left\{t\colon \sum_{i=1}^{n} (c_i - \bar{c})\,\varphi\left(\frac{R_{in}^{(t)}}{n+1}\right) < 0\right\}, \tag{1.5}$$

$$T_R^- = \sup\left\{t\colon \sum_{i=1}^{n} (c_i - \bar{c})\,\varphi\left(\frac{R_{in}^{(t)}}{n+1}\right) > 0\right\} \tag{1.6}$$

with $R_{in}^{(t)}$ being the rank of $X_i - tc_i$ among $X_1 - tc_1, \dots, X_n - tc_n$ $(i = 1, \dots, n)$ and φ being a nondecreasing (skew-symmetric) score-generating function defined on (0, 1); $\bar{c} = \bar{c}_n = \frac{1}{n} \sum_{i=1}^{n} c_i$.

We assume that the functions ψ and φ are such that

$$\int_{R^1} \psi^2(x)\, dF(x) < \infty \qquad \text{and} \qquad \int_0^1 \varphi^2(t)\, dt < \infty. \tag{1.7}$$

Jurečková ([11]) proved that if ψ is smooth and F – symmetric with finite Fisher's information then, in the location case ($c_i = 1$, $i = 1, 2, \dots$), we have

$$n^{1/2}(T_M^{(n)} - \theta) = (\gamma_1(\psi, F))^{-1} n^{-1/2} \sum_{i=1}^{n} \psi(X_i - \theta) + O_p(n^{-1/2}) \tag{1.8}$$

while if the function ψ has some jump-discontinuities then (1.8) is replaced by

$$n^{1/2}(T_M^{(n)} - \theta) = (\gamma_1(\psi, F))^{-1} n^{-1/2} \sum_{i=1}^{n} \psi(X_i - \theta) + O_p(n^{-1/4}) \tag{1.9}$$

where

$$\gamma_1(\psi, F) = -\int \psi(x) f'(x)\, dx, \qquad f(x) = \frac{dF}{dx}. \tag{1.10}$$

More precisely, if ψ is nondecreasing, skew-symmetric, absolutely continuous in R^1 and

$$\int (\psi'(x))^2\, dF(x) < \infty, \qquad \int \psi'(x)\, dF(x) \ (= \gamma_1) > 0, \tag{1.11}$$

$$\int (\psi''(x+t))^2\, dF(x) < C \qquad \text{for} \qquad |t| < \delta,\ \delta > 0,\ C > 0, \tag{1.12}$$

then the representation (1.8) holds for the M-estimator $T_M^{(n)}$. Moreover, (1.8) holds also for every nondecreasing skew-symmetric continuous function ψ which has two bounded derivatives in $(-k, k)$ and is constant outside of $(-k, k)$, $k > 0$.

On the other hand, if the function ψ could be decomposed into

$$\psi(x) = \psi_1(x) + \psi_2(x), \qquad x \in R^1, \tag{1.13}$$

where ψ_1 is smooth in the above sense and

$$\psi_2(x) = \begin{cases} \alpha_0 & \text{for} \quad -\infty < x \leqslant a_1, \\ \alpha_j & \text{for} \quad a_j < x \leqslant a_{j+1},\ j = 1, \dots, p-1, \\ \alpha_p & \text{for} \quad a_p < x < \infty \end{cases} \tag{1.14}$$

and

$$\sum_{j=1}^{p} (\alpha_j - \alpha_{j-1}) f(a_j) > 0 \tag{1.15}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_p$; $a_1, \ldots, a_p$ are given numbers, $-\infty < a_1 < \ldots < a_p < \infty$, then the stochastic representation (1.9) holds for $T_M^{(n)}$. Both orders in (1.8) and (1.9) are exact. The representations (1.8) and (1.9) could be easily extended to the regression model under some conditions on c_i's (see [12], [13]).

This reveals a qualitative difference between the M-estimators generated by a smooth ψ-function and these generated by possibly discontinuous ψ-function. An analogous difference appears also in the case of R-estimators and in the sequential confidence procedures based on corresponding point estimators. While the first order asymptotic properties are analogous both in the case of smooth and of discontinuous generating function, the rate of convergence to the limiting distribution, which represents the second order property, is in the smooth case typically the square of the rate appearing in the discontinuous case.

Concerning the R-estimators, it follows from [6] that, for F satisfying

$$\left|\frac{f'(F^{-1}(t))}{f(F^{-1}(t))}\right| \leqslant K_1\big(t(1-t)\big)^{-1/2-\varepsilon}, \qquad 0 < t < 1,\ K_1 > 0,\ \varepsilon > 0 \quad (1.16)$$

and

$$\int \big(F(x)(1-F(x)\big)^{-1+\eta}\, dF(x \pm \delta) < \infty \text{ for some } \delta, \eta > 0, \qquad (1.17)$$

and for a nondecreasing function φ: $(0, 1) \to \boldsymbol{R}^1$ with the second derivative φ'' such that

$$|\varphi''(t)| \leqslant K_2\big(t(1-t)\big)^{-2+\varkappa}, \qquad 0 < t < 1,\ K_2 > 0,\ \varkappa > 0 \qquad (1.18)$$

the stochastic representation

$$n^{1/2}(T_R^{(n)} - \theta) = \big(\gamma_2(\varphi, F)\big)^{-1} n^{-1/2} \sum_{i=1}^{n} c_i\, \varphi\big(F(X_i - \theta c_i)\big) + O_p(n^{-1/2}) \quad (1.19)$$

holds for the R-estimator $T_R^{(n)}$ defined in (1.4) where

$$\gamma_2 = \gamma_2(\varphi, F) = -\int \varphi\big(F(x)\big) f'(x)\, dx > 0. \qquad (1.20)$$

On the other hand, if φ could be decomposed into the sum

$$\varphi(t) = \varphi_1(t) + \varphi_2(t), \qquad 0 < t < 1 \qquad (1.21)$$

where φ_1 is smooth in the above sense and

$$\varphi_2(t) = \begin{cases} \beta_0 & \text{for } 0 < t \leqslant b_1, \\ \beta_j & \text{for } b_j < t \leqslant b_{j+1},\ j = 1, \ldots, q-1, \\ \beta_q & \text{for } b_q < t < 1 \end{cases} \qquad (1.22)$$

where $\beta_0, \ldots, \beta_q$ and $b_1, \ldots, b_q$ are real numbers, $0 < b_1 < \ldots < b_q < 1$, and, if

$$\sum_{j=1}^{q} (\beta_j - \beta_{j-1}) f(F^{-1}(b_j)) > 0, \tag{1.23}$$

then it follows from [7] that the stochastic representation

$$n^{1/2}(T_R^{(n)} - \theta) = (\gamma_2(\varphi, F))^{-1} n^{-1/2} \sum_{i=1}^{n} c_i \varphi(F(X_i - \theta c_i)) + O_p(n^{-1/4}) \tag{1.24}$$

holds for $T_R^{(n)}$; both orders in (1.19) and (1.24) are exact.

Remark. The representation (1.24) was proved in [7] just for the two-sample model with $c_{2i-1} = 1$, $c_{2i} = 0$, $i = 1, 2, \ldots$; but it easily extends to a more general regression model.

2. Bounded-length confidence intervals

A qualitative difference appears also in the second order properties of sequential confidence intervals based on M- and R-estimators with smooth and with discontinuous score-generating functions, respectively.

Let $I_n = I_n(X_1, \ldots, X_n)$ be a $(1-\alpha)$-confidence interval for θ and let L_n denote its length. We desire to construct a confidence interval for θ with the coverage probability $(1-\alpha)$ and satisfying $L_n \leqslant 2d$ $(d > 0)$ over a class $\mathscr{F}$ of distributions. This cannot be attained by a finite-sample procedure; to ensure the bounded length for all $F \in \mathscr{F}$, we have to take the observations $X_1, X_2, \ldots$ sequentially.

Various bounded-length sequential confidence intervals were suggested by many authors ([2], [3], [4], [5] among others). In the location case, we may take the interval

$$I_{N(d)} = (T_{N(d)} - d,\ T_{N(d)} + d) \tag{2.1}$$

where $T_{N(d)}$ is an appropriate point estimator of θ based on $X_1, X_2, \ldots, X_{N(d)}$; $N(d)$ is the random sample size (stopping rule), dependent on α and on the asymptotic variance (as $n \to \infty$) of $\sqrt{n}(T_n - \theta)$. Typically, we have

(i) $N(d) < \infty$ with probability 1 for every $d > 0$;

(ii) $N(d) \uparrow \infty$ a.s. as $d \downarrow 0$;

(iii) $P_\theta(\theta \in I_{N(d)}) \to 1-\alpha$ as $d \downarrow 0$.

Moreover, in all cases under consideration,

(iv)

$$d\sqrt{N(d)} \to \tau_{\alpha/2}\sigma_F \qquad \text{as} \quad d \downarrow 0 \tag{2.2}$$

with $\tau_{\alpha/2} = \Phi^{-1}(1-\alpha/2)$, Φ being the standard normal distribution and σ_F^2 being the asymptotic variance (as $n \to \infty$) of $\sqrt{n}(T_n - \theta)$. The convergence in

(2.2) is that in probability but could be extended to the convergence in the mean or to the almost sure convergence under additional assumptions.

The qualitative differences appear in the rates of convergence (2.2) corresponding to different score-generating functions. It could be proved, in the case of both M-estimators and R-estimators, that

$$a(d)\left(d\sqrt{N(d)}-\tau_{\alpha/2}\sigma_F\right) \tag{2.3}$$

is, with a proper $a(d)$, asymptotically normally distributed as $d \downarrow 0$. While $a(d) = d^{-1}$ if T_n is generated by a smooth ψ or φ function, $a(d) = d^{-1/2}$ only if ψ or φ has some jump-discontinuities. As a consequence, we get

$$d\sqrt{N(d)}-\tau_{\alpha/2}\sigma_F = O_p(d) \text{ as } d \downarrow 0 \tag{2.4}$$

in the case of a smooth score-generating function, while

$$d\sqrt{N(d)}-\tau_{\alpha/2}\sigma_F = O_p(d^{1/2}) \text{ as } d \downarrow 0 \tag{2.5}$$

if the score-generating function has at least one jump-discontinuity.

In the subsequent text, we shall illustrate some recent results of this type. For the simplicity of notation, the procedures will be illustrated on the following *two-sample model*:

$X_1, X_2, \ldots$ are independent random variables, X_i distributed according to a distribution function $F(x-\theta c_i)$ with $c_{2i-1} = 1$, $c_{2i} = 0$, $i = 1, 2, \ldots$ (2.6)

2.1. Sequential confidence intervals based on M-estimators with smooth ψ-function ([12]). Let $T_M^{(n)}$ be the M-estimator defined in (1.1)–(1.3) where ψ is a nonconstant, nondecreasing, skew-symmetric function on $\boldsymbol{R}^1$ such that

$$\gamma_1 = \int \psi'(x)\,dF(x) > 0 \qquad \left(\psi' = \frac{d\psi}{dx}\right), \tag{2.7}$$

$$\sigma_0^2 = \int \psi^2(x)\,dF(x) < \infty, \qquad \int \psi^4(x)\,dF(x) < \infty, \tag{2.8}$$

$$\sigma_1^2 = \int \psi'^2(x)\,dF(x) - \gamma_1^2 < \infty \tag{2.9}$$

and

$$\lim_{t\to 0} \int (\psi'(x+t)-\psi'(x))^2\,dF(x) = 0. \tag{2.10}$$

Assume that F is absolutely continuous, $F(x)+F(-x) = 1$, $x \in \boldsymbol{R}^1$ and that F has finite Fisher information $I(F)$.

Then $\sqrt{n}(T_M^{(n)}-\theta)$ is, as $n \to \infty$, asymptotically normally $N(0, \sigma_F^2)$ distributed with

$$\sigma_F^2 = 2\sigma_0^2/\gamma_1^2. \tag{2.11}$$

Denote

$$S_n(t) = \sum_{i=1}^{n} c_i \psi(X_i - tc_i), \tag{2.12}$$

and

$$s_n^2 = \frac{1}{n} \sum_{i=1}^{n} \psi^2(X_i - T_M^{(n)} c_i), \qquad n = 2, 4, 6, \ldots \tag{2.13}$$

Notice that $S_n(t)$ is nonincreasing in t. Define

$$\begin{aligned} \hat{\theta}_n^+ &= \inf\{t\colon\ S_n(t) < -\tau_{\alpha/2} s_n \sqrt{n/2}\}, \\ \hat{\theta}_n^- &= \sup\{t\colon\ S_n(t) > \tau_{\alpha/2} s_n \sqrt{n/2}\}, \end{aligned} \qquad n = 2, 4, 6, \ldots \tag{2.14}$$

and introduce the stopping rule $N(d)$ as

$$N(d) = \min\{n \geqslant n_0,\ n \text{ even: } \hat{\theta}_n^+ - \hat{\theta}_n^- \leqslant 2d\} \tag{2.15}$$

with n_0 being an initial sample size. We then suggest

$$I_{N(d)} = (\hat{\theta}_{N(d)}^-,\ \hat{\theta}_{N(d)}^+) \tag{2.16}$$

as a confidence interval for θ. Besides the properties (i), (ii), (iii) mentioned above, it is proved in [12] that

$$d\sqrt{N(d)} \xrightarrow{P} \tau_{\alpha/2}\sigma_F \qquad \text{as} \qquad d \downarrow 0 \tag{2.17}$$

and

$$\mathscr{L}\{d^{-1}(d\sqrt{N(d)} - \tau_{\alpha/2}\sigma_F)\} \to N(0, \sigma^{*2}) \tag{2.18}$$

with

$$\sigma^{*2} = 2\sigma_1^2\gamma_1^{-2} - (\sigma_0^{-2}\gamma_1^{-1} \int \psi^2(x)\psi'(x)\,dF(x) - 1) + \\ + \frac{1}{4}(\sigma_0^{-4} \int \psi^4(x)\,dF(x) - 1). \tag{2.19}$$

2.2. Sequential confidence intervals based on M-estimators with discontinuous ψ-function ([13]). Let us start again with the model (2.6) with F symmetric and such that $I(F) < \infty$. Let $T_M^{(n)}$ be the M-estimator defined in (1.1)–(1.3) where the function ψ could be decomposed into the sum

$$\psi(x) = \psi_1(x) + \psi_2(x), \qquad x \in \mathbf{R}^1 \tag{2.20}$$

where both ψ_1 and ψ_2 are nondecreasing and skew-symmetric (i.e., $\psi_j(-x) = -\psi_j(x)$, $x \in \mathbf{R}^1$, $j = 1, 2$). Assume that ψ_1 is a smooth function satisfying (2.7)–(2.10) and that ψ_2 is the step-function given in (1.14) and (1.15). Then, again, $\sqrt{n}(T_M^{(n)} - \theta)$ is, as $n \to \infty$, asymptotically normally $N(0, \sigma_F^2)$ distributed with

$$\sigma_F^2 = 2 \int \psi^2(x)\, dF(x)/\gamma^2, \tag{2.21}$$

$$\gamma = \int \psi_1'(x)\, dF(x) + \sum_{j=1}^{p} (\alpha_j - \alpha_{j-1}) f(a_j). \tag{2.22}$$

Define $S_n(t)$, s_n, $\hat{\theta}_n^+$ and $\hat{\theta}_n^-$, $N(d)$ and $I_{N(d)}$ as in (2.12)–(2.16). Then, besides the properties (i)–(iii), it is proved in [13] that

$$d\sqrt{N(d)} \xrightarrow{P} \tau_{\alpha/2}\sigma_F \qquad \text{as} \qquad d \downarrow 0 \tag{2.23}$$

and

$$\mathscr{L}\{d^{-1/2}(d\sqrt{N(d)} - \tau_{\alpha/2}\sigma_F)\}$$
$$\to N\Big(0,\, 2\gamma^{-2} \sum_{j=1}^{p} (\alpha_j - \alpha_{j-1})^2 f(a_j)\Big) \qquad \text{as} \qquad d \downarrow 0. \tag{2.24}$$

2.3. Sequential confidence intervals based on R-estimators with smooth φ-function ([10], [6]). Let $T_R^{(n)}$ be the R-estimator of θ defined in (1.4)–(1.6) generated by a nondecreasing function defined on (0, 1). The model under consideration is again that given in (2.6). If φ is a bounded smooth function (e.g., the Wilcoxon test) then it suffices to assume that F has finite Fisher's information. In the general case, we assume that F satisfies the conditions

$$|f'(F^{-1}(t))/f(F^{-1}(t))| \leqslant K_1\big(t(1-t)\big)^{-1/2-\varepsilon} \tag{2.25}$$

for $0 < t < 1$ with $\varepsilon > 0$, $K_1 > 0$, and

$$\int [F(x)(1-F(x))]^{-1+\eta}\, dF(x \pm \delta) < \infty, \qquad \eta, \delta > 0. \tag{2.26}$$

We assume that φ is twice differentiable with the second derivative φ'' satisfying

$$|\varphi''(t)| \leqslant K_2\big(t(1-t)\big)^{-2}, \qquad 0 < t < 1,\ K_2 > 0. \tag{2.27}$$

Then $\sqrt{n}(T_R^{(n)} - \theta)$ is, as $n \to \infty$, asymptotically normally $N(0, \sigma_F^2)$ distributed (see, e.g., [9]) with

$$\sigma_F^2 = 4\gamma^{-2}\Big(\int_0^1 \varphi^2(t)\, dt - \bar{\varphi}^2\Big), \qquad \bar{\varphi} = \int_0^1 \varphi(t)\, dt, \tag{2.28}$$

where

$$\gamma = \gamma(\varphi, F) = -\int_{R^1} \varphi(F(x)) f'(x)\, dx > 0. \tag{2.29}$$

Denote

$$S_n(t) = 2n^{-1/2} \sum_{i=1}^{n} (c_i - \tfrac{1}{2})\, \varphi\Big(\frac{R_{in}^{(t)}}{n+1}\Big), \qquad n = 2, 4, \ldots \tag{2.30}$$

with $R_{in}^{(t)}$ being the rank of $X_i - tc_i$ among $X_1 - tc_1, \ldots, X_n - tc_n$, $i = 1, \ldots, n$. It is known (see, e.g., [8]) that $S_n(t)$ is nonincreasing in t with probability 1. Define

$$\begin{aligned} \hat{\theta}_n^- &= \sup\{t\colon\ S_n(t) > \tau_{\alpha/2}\big(\int_0^1 \varphi^2(t)\,dt - \bar{\varphi}^2\big)^{1/2}\}, \\ \hat{\theta}_n^+ &= \inf\{t\colon\ S_n(t) < -\tau_{\alpha/2}\big(\int_0^1 \varphi^2(t)\,dt - \bar{\varphi}^2\big)^{1/2}\} \end{aligned} \tag{2.31}$$

for $n = 2, 4, \ldots$ Introduce the stopping rule $N(d)$ as

$$N(d) = \min\{n \geqslant n_0,\ n \text{ even: } \hat{\theta}_n^+ - \hat{\theta}_n^- \leqslant 2d\}. \tag{2.32}$$

We then suggest the interval

$$I_{N(d)} = (\hat{\theta}_{N(d)}^-, \hat{\theta}_{N(d)}^+) \tag{2.33}$$

as the confidence interval for θ. Besides the properties (i), (ii), (iii), it is proved in [6] that

$$d\sqrt{N(d)} \to \tau_{\alpha/2}\sigma_F \qquad \text{a.s. as} \quad d \downarrow 0 \tag{2.34}$$

and

$$\mathscr{L}\{d^{-1}\big(d\sqrt{N(d)} - \tau_{\alpha/2}\sigma_F\big)\} \to N(0, \sigma^2/\gamma^2) \qquad \text{as} \quad d \downarrow 0 \tag{2.35}$$

with

$$\sigma^2 = \operatorname{var}\{\varphi'(U_1) f\big(F^{-1}(U_1)\big) + \int_{-\infty}^{\infty} \big(I[F(x) > U_1] - F(x)\big)\varphi\big(F(x)\big) f'(x)\,dx\} \tag{2.36}$$

where U_1 denotes the random variable with uniform (0, 1) distribution and $I[A]$ denotes the indicator of A.

2.4. Sequential confidence intervals based on R-estimators with discontinuous φ-function. Suppose that $X_1, X_2, \ldots$ satisfy the model (2.6). Let $T_R^{(n)}$ be the R-estimator of θ defined in (1.4)–(1.6) generated by a nondecreasing score-function φ defined on (0, 1) which can be decomposed as

$$\varphi(t) = \varphi_1(t) + \varphi_2(t), \qquad 0 < t < 1 \tag{2.37}$$

where φ_1 is twice differentiable on (0, 1) with the second derivative satisfying

$$|\varphi_1''(t)| \leqslant K\big(t(1-t)\big)^{-9/4+\varepsilon}, \qquad 0 < t < 1,\ K > 0,\ \varepsilon > 0 \tag{2.38}$$

and φ_2 is the step-function given in (1.22). We assume that the underlying distribution function F is twice differentiable with the derivatives f and f'

being bounded in neighbourhoods of $F^{-1}(b_1), \ldots, F^{-1}(b_p)$ and such that

$$\sum_{j=1}^{p} (\beta_j - \beta_{j-1}) f(F^{-1}(b_j)) > 0, \tag{2.39}$$

$$\int (f'^2(x+t)/f(x))\,dx \leqslant C \qquad \text{for} \quad |t| \leqslant \delta, \quad \delta > 0 \tag{2.40}$$

and

$$\int f^2(x)[F(x)(1-F(x))]^{-2}\,dF(x+t) \leqslant C \qquad \text{for} \quad |t| \leqslant \delta,\ \delta > 0,\ C > 0. \tag{2.41}$$

Then $\sqrt{n}(T_R^{(n)} - \theta)$ is asymptotically normally $N(0, \sigma_F^2)$ distributed (see, e.g., [9] for the proof) with

$$\sigma_F^2 = 4\gamma^{-2} \int_0^1 (\varphi(t) - \bar{\varphi})^2\,dt, \tag{2.42}$$

$$\gamma = -\int_{\mathbb{R}^1} \varphi(F(x)) f'(x)\,dx. \tag{2.43}$$

Define $S_n(t)$, $\hat{\theta}_n^-$, $\hat{\theta}_n^+$ as in (2.30), (2.31), $N(d)$ and $I_{N(d)}$ as in (2.32), (2.33). Then, besides (i), (ii), (iii),

$$d\sqrt{N(d)} \to \tau_{\alpha/2}\sigma_F \quad \text{in probability, as} \quad d \downarrow 0 \tag{2.44}$$

and

$$\mathscr{L}\{d^{-1/2}(d\sqrt{N(d)} - \tau_{\alpha/2}\sigma_F)\} \to N(0, \sigma^2/\gamma^2) \quad \text{as} \quad d \downarrow 0 \tag{2.45}$$

where

$$\sigma^2 = \sum_{j=1}^{q} (\beta_j - \beta_{j-1})^2 f(F^{-1}(b_j)). \tag{2.46}$$

We shall give some hints on the proof of (2.44)–(2.46) (not explicitly given in [7]) which is analogous to the proofs of parallel result mentioned in Sections 2.1–2.3. This will also illustrate the method of the proof of other results.

It follows from Theorem 2.1, [7], that the process

$$\{n^{1/4}\sigma^{-1}(S_n(t) - S_n(0) + \tfrac{1}{2}t\gamma):\ 0 \leqslant t \leqslant 1\} \tag{2.47}$$

converges weakly to the standard Wiener process in the Skorokhod topology. It implies (denoting by $L_n = \hat{\theta}_n^+ - \hat{\theta}_n^-$ the length of the confidence interval I_n) that

$$S_n(\hat{\theta}_n^-) - S_n(\hat{\theta}_n^+) - \tfrac{1}{2}\sqrt{n}\,L_n\gamma \xrightarrow{P} 0 \qquad \text{as} \quad n \to \infty$$

so that

$$\sqrt{n}\, L_n \xrightarrow{P} 2\tau_{\alpha/2}\, \sigma_F \qquad \text{as} \quad n \to \infty. \tag{2.48}$$

and this implies (2.44).

It further follows from the weak convergence of the process (2.47) that

$$\mathscr{L}\{n^{1/4}[S_n(\hat{\theta}_n^- + 2\tau_{\alpha/2}\,\sigma_F) - S_n(\hat{\theta}_n^-) + \tau_{\alpha/2}\,\sigma_F\,\gamma]\} \to N(0,\, \sigma^2\,\tau_{\alpha/2}\,\sigma_F) \qquad \text{as} \quad n \to \infty \tag{2.49}$$

and

$$n^{1/4}\big[S_n(\hat{\theta}_n^+) - S_n(\hat{\theta}_n^+ - (L_n - 2\tau_{\alpha/2}\,\sigma_F)) + \tfrac{1}{2}(\sqrt{n}\, L_n - 2\tau_{\alpha/2}\,\sigma_F)\gamma\big] \xrightarrow{P} 0 \qquad \text{as} \quad n \to \infty. \tag{2.50}$$

(2.50) then implies that

$$n^{1/4}|[S_n(\hat{\theta}_n^+) - S_n(\hat{\theta}_n^-) + \tfrac{1}{2}\sqrt{n}\, L_n\, \gamma] - [S_n(\hat{\theta}_n^- + 2\tau_{\alpha/2}\,\sigma_F) - S_n(\hat{\theta}_n^-) + \tau_{\alpha/2}\,\sigma_F\,\gamma]| \xrightarrow{P} 0 \qquad \text{as} \quad n \to \infty. \tag{2.51}$$

(2.49), (2.51) and (2.31) imply that

$$\mathscr{L}\left\{n^{1/4}\left(\frac{\sqrt{n}\, L_n}{2\tau_{\alpha/2}\,\sigma_F} - 1\right)\right\} \to N(0,\, \sigma^2\gamma^{-2}\sigma_F^{-1}\tau_{\alpha/2}^{-1}) \qquad \text{as} \quad n \to \infty. \tag{2.52}$$

(2.45) then follows from (2.52) with the aid of Theorem 1 of Anscombe ([1]); the "uniform continuity in probability" which is an assumption in Anscombe's theorem follows from the martingale property of linear rank statistics (see [4]) and from the Kolmogorov inequality for martingales.

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