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**Three-valued logic and cut-elimination:
The actual meaning of Takeuti's conjecture**

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Dedicated to Professor
Andrzej Mostowski

Introduction

It is well known that, besides the usual reduction procedure for second order arithmetic G^1LC, there exists in some specific cases elementary normalization procedures (***elementary*** in the sense of being provably normalizing procedures in some system of elementary arithmetic), for instance, if F_0 stands for $\forall x \forall X (x \in X \to x \in X)$, then such a procedure exists for every formula $F_0 \to A$ (see for instance [8]). Furthermore, recent investigations ([1]) have shown that similar procedures exist for more general classes of formulas.

In this paper, we answer the following questions:

(Q_1) What simple syntactic conditions on the formula A do ensure elementary normalization for proofs of A (and more generally, for proofs of sequents $\Gamma \vdash A$)?

(Q_2) What simple semantic conditions on the formula A do ensure elementary normalization for proofs of A (and, more generally for proofs of sequents $\Gamma \vdash A$)?

In order to answer (Q_1) and (Q_2), we essentially need a good description of "elementary" normalizing procedures. This is done by means of a formal system M^1LC which has the two following features:

(i) Cut-free provability in G^1LC canonically corresponds to cut-free provability in M^1LC; moreover, the reduction procedure in M^1LC can be shown to terminate by means of Π^0_2-induction.

(ii) In M^1LC, one can express by means of ordinary sequents the usual metamathematical properties of formulas of G^1LC.

The formulas of M^1LC are the:

$?A$, where A is a formula of G^1LC. The intended meaning of $?A$ is: "A is cut-free provable".

$!A$, where A is a formula of G^1LC. The intended meaning of $!A$ is: "for all B, whenever $A \vdash B$ is cut-free provable, then B is cut-free provable".

The rules of M^1LC are set up in such a way that they clearly preserve at each step this intended meaning; more precisely, define a sequent $\Gamma \vdash \Delta$ to be ***regular*** iff (i) $\Gamma \vdash \Delta = !A_1, \ldots, !A_n \vdash ?B_1, \ldots, ?B_m$, and the sequent is cut-free provable (those sequents canonically correspond to cut-free provable sequents of G^1LC, but not to provability in G^1LC, since from $\Gamma \vdash \Delta, !A$ and $\Gamma, ?A \vdash \Delta$, one cannot infer $\Gamma \vdash \Delta$). (ii) A sequent $\Gamma, ?A \vdash \Delta$ or $\Gamma \vdash \Delta, !B$ must necessarily appear in some cut

$$\frac{\Gamma' \vdash \Delta', ?A \qquad \Gamma, ?A \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \quad \text{or} \quad \frac{\Gamma \vdash \Delta, !B \qquad \Gamma', !B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

and regularity is defined in order to ensure that the conclusion is cut-free provable, provided the auxiliary premises $\Gamma' \vdash \Delta', ?A$ or $\Gamma', !B \vdash \Delta'$ are already cut-free provable. The deduction rules of M^1LC are chosen to preserve regularity of the sequents. Regularity (in a slightly different form) is studied in part B; the proof that every provable sequent is regular can be carried out in the system AA of Π_2^0 induction.

System M^1LC (and its intuitionistic formulation M^1LI) is enough to answer question (Q_1). Two essential concepts can be defined in M^1LC: (a) a formula is said to be ***absorbing*** iff $!A \vdash !F_0$ is provable, (b) a formula is said to be *poor* iff $?(F_0 \to A) \vdash ?A$ is provable in M^1LC. Roughly speaking, A is absorbing iff A behaves like F_0; if A is poor, then we have an elementary reduction procedure (in the sense of AA) for all sequents $\Gamma \vdash A$. The answer to (Q_1) is a list of simple syntactic proverty conditions. These conditions are quite general (for ***analytical*** statements only), they are given in C3.

Moreover, M^1LC can be used to solve other problems:

(a) From the provability of $?\exists x(N(x) \wedge A(x)) \vdash ?\exists x A(x)$, one has a very simple proof of Σ_1^0-reflection for cut-free provability, first proved in [1].

In fact, Σ_1^0 formulas play an essential role in cut-free analysis: these formulas enjoy the property of being equivalent with their own poverty; the full Takeuti conjecture reduces to its particular case for Σ_1^0 formulas (see C4).

(b) The relation $A \equiv B$ (recorded as $\equiv_3$ in [1]) and defined by $?A \vdash ?B$, $!A \vdash !B$, $?B \vdash ?A$, $!B \vdash !A$ provable in M^1LC, is stronger than provable equivalence.

This relation is the good one which permits to transfer the cut-free properties of A to B, and conversely. The Prawitz translations of the intuitionistic connectives [3] and quantifiers are then shown to be inadequate for cut-free provability, since a formula is not necessarily congruent to its Prawitz translation. $\equiv$ behaves in some respect like a synonymity relation. But, it is only an approximation to this notion, since, whenever

A and B are provably equivalent, then $F_0 \wedge (F_0 \to A)$ is congruent to $F_0 \wedge (F_0 \to B)$ (see C2).

The semantics of the modal systems M^1LC and M^1LI is given by means of so-called "three-valued" models, which are, in the intuitionistic case, generalizations of the topological models introduced by Mostowski.

Standard properties s.t. completeness are easily proved for these systems. This is done in A. Classically, a closed formula is absorbing if the only models in which it is necessary are binary models, while A is poor iff it is possible in every proper (non-binary) model; two formulas are congruent iff they take the same value in every model.

It is quite surprising to discover that the 3-valued semantics for M^1LC are nothing more than the "partial valuations" introduced by Schütte [8].

In B4, we show how to "complete" such three valued models. The proof involves the "thinness theorem" of part A. This proof is deeply connected with the proofs of Tait, Takahashi and Prawitz. These old proofs of cut-eliminations are restated in the appendix in our terminology.

The most interesting fact arises in the classical case: if we start from a proper (non-binary) model, then we actually construct a principal model. But a closed analytical formula is valid in a principal model iff it is true (in the standard model). This fact is indeed formalizable in AA, and, using completeness, we get the *poverty theorems* of C5.: if A is a closed analytical formula which is 1-consistent w.r.t. AA (resp. G^1LC), then all equivalents of A in AA (resp. G^1LC) are poor, that is we can prove cut-elimination for proofs of these formulas in AA. This answers question (Q_2); As a corollary, every Gödel sentence for G^1LC is poor, since it is true.

The poverty theorem may be stated as following:

Let A be a closed analytical formula; then

(i) *either there is a false closed Σ^0_1 sentence E_A such that $G^1LC \vdash A \to E_A$,*

(ii) *or for every B such that $G^1LC \vdash A \leftrightarrow B$, $CF(\ulcorner B \urcorner)$ is a theorem of AA ($CF(\ulcorner B \urcorner)$ is the local form of Takeuti's conjecture for the formula B).*

It follows from the answer to question (Q_1) and (Q_2) that it is very doubtful that Takeuti's conjecture for G^1LC might be of some interest in order to prove independence results. (Most of set-theoretical independence proofs involve constructions that leave the integers unchanged; so these proofs are indeed proofs of relative 1-consistency at least; but for analytical statements which are one-consistent, the concepts of provability and cut-free provability are practically the same; that is we do not know much more from the assumption of cut-free provability than from the assumption of mere provability.)

One can argue that the restriction to analytical formulas is criticizable; however, we must say that: 1° Our results extend to formulas which are equivalents of analytical formulas; 2° One does not know any alternative definition of "analytical" that would not enjoy similar "poverty" properties; 3° If we restrict to ω-models, that is if we are in ω-logic, then, no analyticity conditions are necessary in order to prove the corresponding form of the poverty theorem (but poverty is then Π_1^1).

Let us conclude with a description of the way we can obtain an elementary cut-elimination procedure for A, given a proof of poverty of A: suppose we have a proof in M^1LC of $?(F_0 \to A) \vdash ?A$; given any proof of A in G^1LC, there is a canonical way of finding a cut-free proof of $F_0 \to A$; from a cut-free proof of $F_0 \to A$, one gets immediately a (cut-free) proof of $?(F_0 \to A)$. Using a cut, one obtains a proof of $?A$ in M^1LC. Now, we have an elementary normalization procedure in M^1LC, so we get a cut-free proof of $?A$. Now, if we drop all $?$ and $!$ in the proof, we get a cut-free proof of A. Similarly for M^1LI and G^1LI.

Aknowledgements. This paper owes much to Professor Kreisel. System M^1LC was first introduced in order to answer questions of Kreisel concerning the possibility of giving alternative proofs of results already shown to hold in former versions of [1]; also, Kreisel suggested the existence of some relations between M^1LC and partial valuations.

I discussed some points in parts B and C with Professor Feferman. The fact that the poverty theorem can be carried out in such a weak system as AA was discovered in a discussion with him.

Professor Schütte asked me a question concerning the possibility of extending partial valuations for simple type theory (that is 3-valued models) into Henkin general models. I was wrong in answering him this was possible.

However, if we consider the problem of cut-elimination with extensionality axioms, then the corresponding 3-valued models can be "completed" by an extensional one. In the proper 3-valued cases, then this model is principal (see appendix)(1).

A. The three-valued predicate calculus

In the sequel, L is to be a first order language, with a distinguished 0-ary predicate letter $\perp$ for absurdity.

A 1. Three-valued structures; classical case

Three-valued structures are defined in such a way that every closed formula of L takes one of the "*truth values*" F, U, T (false, undeterminated, true). In the general definition of structures, we identify F, U, T with the integers $0, 1, 2$. So $3 = \{0, 1, 2\}$ is the set of the truth values.

(1) Added in print: recent improvements have been found by P. Päppinghaus.

A three-valued structure $\boldsymbol{M}$ for L consists of:

1° A non-void set $M = |\boldsymbol{M}|$, the domain of $\boldsymbol{M}$.

2° For every n-ary function letter f, a function $\boldsymbol{M}(f): M^n \to M$.

3° For every n-ary predicate letter p, a function $\boldsymbol{M}(p): M^n \to 3$.

Moreover, we require that

4° $\boldsymbol{M}(\perp) = 0 = F$.

A.1.1. DEFINITION. $\boldsymbol{M}$ is said to be *binary* if for every n-ary predicate letter p and $c_1, \ldots, c_n$ in M, $\boldsymbol{M}(p)(c_1, \ldots c_n) \neq 1$ $(1 = U)$.

So, what we call binary structures, binary models, ... are exactly structures, models, ... in the standard meaning.

$L(\boldsymbol{M})$, the *language of* $\boldsymbol{M}$, is obtained as usual by adding for every c in M a new constant (0-ary function letter) $\bar{c}$ to L.

A.1.2. Interpretation of $L(\boldsymbol{M})$ in $\boldsymbol{M}$:

— interpretation of closed terms:

$$\boldsymbol{M}(\bar{c}) = c,$$

$$\boldsymbol{M}(ft_1 \ldots t_n) = \boldsymbol{M}(f)\big(\boldsymbol{M}(t_1), \ldots, \boldsymbol{M}(t_n)\big),$$

— interpretation of closed formulas:

$$\begin{aligned}
\boldsymbol{M}(pt_1 \ldots t_n) &= \boldsymbol{M}(p)\big(\boldsymbol{M}(t_1), \ldots, \boldsymbol{M}(t_n)\big), \\
\boldsymbol{M}(A \to B) &= \sup\big(2 - \boldsymbol{M}(A), \boldsymbol{M}(B)\big), \\
\boldsymbol{M}(\forall x A) &= \inf_{c \in M}\big(\boldsymbol{M}(A_x[\bar{c}])\big), \\
\boldsymbol{M}(\neg A) &= 2 - \boldsymbol{M}(A) && \big(= \boldsymbol{M}(A \to \perp)\big), \\
\boldsymbol{M}(A \vee B) &= \sup\big(\boldsymbol{M}(A), \boldsymbol{M}(B)\big) && \big(= \boldsymbol{M}(\neg A \to B)\big), \\
\boldsymbol{M}(A \wedge B) &= \inf\big(\boldsymbol{M}(A), \boldsymbol{M}(B)\big) && \big(= \boldsymbol{M}(\neg(\neg A \vee \neg B))\big), \\
\boldsymbol{M}(\exists x A) &= \sup_{c \in M}\big(\boldsymbol{M}(A_x[\bar{c}])\big) && \big(= \boldsymbol{M}(\neg \forall x \neg A)\big).
\end{aligned}$$

A.1.3. Modal languages. If L is a first-order language, the *modal language* ML associated with L is defined as following:

— the terms of ML are exactly the terms of L;

— the formulas of ML are the formulas $!A$ and $?A$, where A is a *formula of* L. (That is a formula of ML is to begin with ! or ? and no other occurrence of these modalities is to be found inside the formula). ! and ? will be called respectively *necessary* and *possible*.

Interpretation of $ML(\boldsymbol{M})$ in $\boldsymbol{M}$:

$ML(\boldsymbol{M})$ is the modal language associated with $L(\boldsymbol{M})$. The interpretation of the closed terms of $ML(\boldsymbol{M})$ is defined as in A.1.2.; for closed formulas, we define:

$$\boldsymbol{M} \models\, !A \quad \text{iff} \quad \boldsymbol{M}(A) = T,$$

$$\boldsymbol{M} \models\, ?A \quad \text{iff} \quad \boldsymbol{M}(A) \neq F.$$

So $!A$ is *valid* in $\boldsymbol{M}$ iff A is true in $\boldsymbol{M}$, while $?A$ is valid in $\boldsymbol{M}$ iff A is not false in $\boldsymbol{M}$, that is iff A is true or undeterminated in $\boldsymbol{M}$.

A.1.4. Sequents. A *sequent* is an expression $\Gamma \vdash \Delta$, where Γ and Δ are finite sequences of modal formulas, that is of formulas of ML; Δ is always to be non-empty. An *instanciation* of a sequent $\Gamma \vdash \Delta$ is a sequent $\Gamma' \vdash \Delta'$ obtained by substituting some terms $t_1, \ldots, t_n$ for every occurrence of the variables $x_1, \ldots, x_n$ in any formula occurring in the sequent. We suppose that all the trouble with free and bound variables is solved with the help of some technical trick, and so we can speak freely about instanciations. A sequent is closed iff every formula occurring in it is closed. Modal formulas are identified with sequents $\Gamma \vdash \Delta$ such that $\Gamma = \emptyset$ and $\mathrm{card}(\Delta) = 1$. We shall not distinguish between $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ if Γ' (resp. Δ') arises from Γ (resp. Δ) by a permutation.

A closed sequent $\Gamma \vdash \Delta$ of $ML(\boldsymbol{M})$ is *valid* in $\boldsymbol{M}$ iff whenever every formula of Γ is valid in $\boldsymbol{M}$, then one of the formulas of Δ is valid in $\boldsymbol{M}$.

A (not necessarily closed) sequent $\Gamma \vdash \Delta$ of $ML(\boldsymbol{M})$ is *valid* in $\boldsymbol{M}$ iff every closed instanciation $\Gamma' \vdash \Delta'$ of this sequent in $ML(\boldsymbol{M})$ is valid in $\boldsymbol{M}$.

A 2. Three-valued structures: intuitionistic case

A.2.1. Notations. From now on, we shall use the Greek letters $\xi, \eta, \pi, \ldots$ with indices if necessary as syntactical variables for the symbols $?$ and $!$, and we define

$$\bar{\xi} = \begin{cases} ! & \text{if } \xi = ?, \\ ? & \text{if } \xi = !, \end{cases}$$

and so on.

Now, we define three-valued topological structures.

Let H be a non-empty topological space, let $\boldsymbol{H}$ be the algebra

$$\boldsymbol{H} = (H', \cup, \cap, \Rightarrow, \bigcup, \bigwedge),$$

where H' is the set of open subsets of H, and the operations $\cup$, $\cap$, $\Rightarrow$ are defined as follows: for any $\alpha, \beta \in H'$, $\alpha \cup \beta$, $\alpha \cap \beta$ are the set-theoretical union and intersection, $\alpha \Rightarrow \beta = \mathrm{Int}(\complement \alpha \cup \beta)$, where Int is the interior, and $\complement$ is the complementary. $\bigcup$ and $\bigwedge$ act on arbitrary families of open sets $(\alpha_i)_{i \in I}$: $\bigcup_{i \in I} (\alpha_i)$ is just the set-theoretical union of the family, and $\bigwedge_{i \in I} (\alpha_i)$

$=\mathrm{Int}\big(\bigcap_{i\in I}(a_i)\big)$ is the interior of the set-theoretical intersection of the family $(a_i)_{i\in I}$.

A.2.2. Definition. Let H be a non-empty topological space. We define H'' to be the set of couples of open sets (α, β) such that $\alpha\subset\beta$.

A.2.3. Definition. A (3-valued) topological structure $\boldsymbol{M}$ for L consists of:

(i) a non-void topological space H,

(ii) a non-void set $M = |\boldsymbol{M}|$,

(iii) for every n-ary function letter f of L, a function $\boldsymbol{M}(f)\colon M^n\to M$,

(iv) for every n-ary predicate letter p of L, a function $\boldsymbol{M}(p)\colon M^n\to H''$, where H'' has been defined in A.2.2. We require $\boldsymbol{M}(\perp)$ to be $(\emptyset,\emptyset)$.

A.2.4. Definition. The topological structure $\boldsymbol{M}$ is said to be *binary* iff $\boldsymbol{M}(p)$ maps M^n into the set of couples (α, α), $\alpha\in H'$.

Given a binary (3-valued) topological structure $\boldsymbol{M}$, one can define an (usual) topological structure $\boldsymbol{M}'$ for L, by putting $\boldsymbol{M}'(p)(c_1, \ldots, c_n) = \alpha$ iff $\boldsymbol{M}(p)(c_1, \ldots, c_n) = (\alpha, \alpha)$. For the definitions and properties of topological structures, see [7], Chapter X.

We define $L(\boldsymbol{M})$, ML, $ML(\boldsymbol{M})$ exactly as we did in A1.

A.2.5. Interpretation of $ML(\boldsymbol{M})$ in $\boldsymbol{M}$. Every closed term of $ML(\boldsymbol{M})$ is interpreted in $\boldsymbol{M}$ by a member of M, as in A1. Every closed formula of $L(\boldsymbol{M})$ is interpreted in $\boldsymbol{M}$ by a pair (α, β) of open subsets of H, such that $\alpha\subset\beta$. If $\boldsymbol{M}(A) = (\alpha, \beta)$, then we put

$$\boldsymbol{M}(!A) = \alpha \quad\text{and}\quad \boldsymbol{M}(?A) = \beta.$$

For greater readibility, we define only the interpretation of the closed (modal) formulas of $ML(\boldsymbol{M})$.

If $\boldsymbol{M}(p)\big(\boldsymbol{M}(t_1), \ldots, \boldsymbol{M}(t_n)\big) = (\alpha, \beta)$, then

$$\begin{aligned}
\boldsymbol{M}(!pt_1\ldots t_n) &= \alpha \quad\text{and}\quad \boldsymbol{M}(?pt_1\ldots t_n) = \beta,\\
\boldsymbol{M}\big(\xi(A\vee B)\big) &= \boldsymbol{M}(\xi A)\cup\boldsymbol{M}(\xi B),\\
\boldsymbol{M}\big(\xi(A\wedge B)\big) &= \boldsymbol{M}(\xi A)\cap\boldsymbol{M}(\xi B),\\
\boldsymbol{M}\big(\xi(A\to B)\big) &= \boldsymbol{M}(\bar\xi A) \Rightarrow \boldsymbol{M}(\xi B),\\
\boldsymbol{M}(\xi\neg A) &= \text{interior of } \complement\big(\boldsymbol{M}(\bar\xi A)\big) = \boldsymbol{M}\big(\xi(A\to\perp)\big),\\
\boldsymbol{M}(\xi\forall xA) &= \bigwedge_{c\in M}\boldsymbol{M}(\xi A_x[\bar c]),\\
\boldsymbol{M}(\xi\exists xA) &= \bigcup_{c\in M}\boldsymbol{M}(\xi A_x[\bar c]).
\end{aligned}$$

An *intuitionistic sequent* is a sequent $\Gamma\vdash\Delta$ such that $\mathrm{card}(\Delta) = 1$.

The validity in $\boldsymbol{M}$ of a closed (not necessarily intuitionistic) sequent is defined as follows: $\Gamma \vdash \Delta$ is valid in $\boldsymbol{M}$ iff $\bigcap_{\xi A \in \Gamma} \boldsymbol{M}(\xi A) \subset \bigcup_{\eta B \in \Delta} \boldsymbol{M}(\eta B)$.

A sequent is valid in $\boldsymbol{M}$ iff every closed instanciation in $ML(\boldsymbol{M})$ of this sequent is valid in $\boldsymbol{M}$.

Classical structures are particular topological structures: suppose H is a one-point topological space; then H'' has exactly three members: $(\emptyset, \emptyset)$, $(\emptyset, H)$ and (H, H), which can be identified respectively with F, U and T. It is trivial to verify that, modulo this identification, the definitions of validity given in A1 and A2 coincide in that case.

Remarks. 1° In order to formalize the use of topological structures, one may think of a topological space as being given by a fundamental system of open sets, and this system will be denumerable in general; interiors are definable arithmetically in this fundamental system.

2° Other usual classical semantics for intuitionism such that Kripke structures can be reformulated in the 3-valued case, with essentially the same properties as topological structures. We have chosen the topological interpretation because it is simpler to handle, but in an earlier abstract, we have used 3-valued Kripke structures.

A 3. Theories; models

The definitions we are giving now are those of classical and intuitionistic theories and models. We give a common definition for the classical and intuitionistic cases; these definitions are to be read as follows:

— *classically*: structures are defined as in A1, and the sequents are arbitrary ones,

— *intuitionistically*: structures are topological structures, and the sequents are intuitionistic.

A.3.1. Definition. A *theory* T, with *underlying language* ML is a set of sequents of ML closed under instanciation. These sequents are the *proper axioms* of T.

A.3.2. Definition. Let T be a theory, ML its underlying language. A *model* of T is a structure $\boldsymbol{M}$ for ML such that every proper axiom of T is valid in $\boldsymbol{M}$.

A.3.3. Definition. Let T be a theory, ML its underlying language. A sequent of ML is *valid* in T iff it is valid in any model of T.

A.3.4. Definition. If T is a theory with a void set of axioms, then T is called a *modal predicate calculus*. Modal predicate calculi are denoted by MLI in the intuitionistic version, and MLC in the classical one.

Now, we shall set up a deduction system for the sequents which are valid in theories. The hardcore of this system is the deduction system of *MLI* (or classically *MLC*). The description we give apply to both systems, classical and intuitionistic. However, the rules are to be read differently, according we are in the classical or in the intuitionistic case: we use Γ, Δ as syntactical variables ranging over finite sequences of formulas; since we are not interested in the ordering of these sequences, these sequences are just finite sets of formulas, with possibly repetitions. Δ is supposed to be non-empty. In the intuitionistic case, $\mathrm{card}(\Delta) = 1$. We define Δ^- to be Δ in the classical case, and to be $\emptyset$ in the intuitionistic case.

If ξA is a modal formula, $\Gamma, \xi A$ is the "union" of Γ and $\{\xi A\}$, for instance, if $\Gamma = \{\xi A, !B\}$, then $\Gamma, \xi A$ is $\{\xi A, \xi A, !B\}$.

Similarly, $\xi A, \Delta^-$ is the "union" of $\{\xi A\}$ and Δ^-; in the intuitionistic case $\xi A, \Delta^-$ is equal to $\{\xi A\}$.

Axioms

$\Gamma \vdash \Delta$ (if $\Gamma \vdash \Delta$ is a proper axiom of T),

$\xi A \vdash \xi A$

$!P \vdash ?P$ for P atomic (1),

$?\bot \vdash \xi P$ for P atomic.

Logical rules

Introductions	Eliminations
$\dfrac{\Gamma \vdash \xi A_i, \Delta^-}{\Gamma \vdash \xi(A_1 \vee A_2), \Delta^-}$ $(i = 1, 2)$,	$\dfrac{\Gamma, \xi A_1 \vdash \Delta \quad \Gamma, \xi A_2 \vdash \Delta}{\Gamma, \xi(A_1 \vee A_2) \vdash \Delta}$
$\dfrac{\Gamma \vdash \xi A_1, \Delta^- \quad \Gamma \vdash \xi A_2, \Delta^-}{\Gamma \vdash \xi(A_1 \wedge A_2), \Delta^-}$,	$\dfrac{\Gamma, \xi A_i \vdash \Delta}{\Gamma, \xi(A_1 \wedge A_2) \vdash \Delta}$ $(i = 1, 2)$,
$\dfrac{\Gamma, \bar{\xi} A \vdash \xi B, \Delta^-}{\Gamma \vdash \xi(A \to B), \Delta^-}$,	$\dfrac{\Gamma \vdash \bar{\xi} A, \Delta^- \quad \Gamma, \xi B \vdash \Delta}{\Gamma, \xi(A \to B) \vdash \Delta}$,
$\dfrac{\Gamma \vdash \xi A, \Delta^-}{\Gamma \vdash \xi \forall x A, \Delta^-}$ (2),	$\dfrac{\Gamma, \xi A_x[t] \vdash \Delta}{\Gamma, \xi \forall x A \vdash \Delta}$,
$\dfrac{\Gamma \vdash \xi A_x[t], \Delta^-}{\Gamma \vdash \xi \forall x A, \Delta^-}$,	$\dfrac{\Gamma, \xi A \vdash \Delta}{\Gamma, \xi \exists x A \vdash \Delta}$ (2).

(1) Variant: $!A \vdash !A$ for any A. This is equivalent.

(2) Restriction on variables: x not free in Γ, Δ (resp. Γ, Δ^-).

Structural rules

Weakening

$$\frac{\Gamma \vdash \Delta}{\Gamma, \xi A \vdash \Delta}, \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \xi A, \Delta} \quad \text{(classical only)}.$$

Contraction

$$\frac{\Gamma, \xi A, \xi A \vdash \Delta}{\Gamma, \xi A \vdash \Delta}, \quad \frac{\Gamma \vdash \xi A, \xi A, \Delta}{\Gamma \vdash \xi A, \Delta} \quad \text{(classical)}.$$

Cut

$$\frac{\Gamma \vdash \xi A, \Delta^- \quad \Gamma, \xi A \vdash \Delta}{\Gamma \vdash \Delta}.$$

Remarks. (i) The terminology used here (introductions, eliminations, cut...) is familiar from the literature. See for instance Kleene's *Introduction to metamathematics* (North Holland 1952).

(ii) If one drops all modalities in the above system, one get sequential axioms and rules for intuitionistic (or classical) predicate calculus. There are only minor differences between the (binary) system obtained and the usual ones. One checks easily that the well-known "cut-elimination theorem" still works for this particular formulation of the usual sequent calculus.

(iii) In usual (binary) systems there is no need to take as axioms sequents, because every sequent can be replaced by a formula (namely, replace $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ by $(A_1 \wedge \ldots \wedge A_n) \rightarrow (B_1 \wedge \ldots \wedge B_m)$. Now, when considered as an axiom, every formula can be replaced by its universal closure (the same holds for modal systems; if ξA is considered as an axiom, then it can be replaced by ξ (univ. Clos(A))). In 3-valued systems, a sequent cannot be replaced by a formula. More precisely, the best we can do is to assume that all axioms are of the form $\xi A \vdash \xi B$ (see Lemma 1 of A.4.1).

(iv) In the above formal system, the rules for absurdity are not given, since $\neg A$ is to be an abbreviation for $(A \rightarrow \perp)$.

A.3.5. Definition. A sequent is *provable* (or *derivable*) in T iff it can be obtained by use of the above axioms and rules.

A.3.6. Proposition. *If a sequent is provable in T, then it is valid in T.*

(Proof. Trivial induction on the length of derivations.)

A 4. The completeness theorem

A.4.1. Theorem. *A sequent is valid in T iff it is provable in T.*

Theorem A.4.1 is a theorem of elementary analysis. We shall reduce it to the usual completeness theorem for (binary) predicate calculi. On the other hand, observe that the definition of validity coincides with the usual one in the case of binary models, and also that these models are exactly models of the sequents $?P \vdash !P$ for every atomic P; so A.4.1 implies the (usual) completeness theorem.

Proof of A.4.1. Clearly, every provable sequent is valid.

Conversely, given a modal language ML, define J to be the (usual) language with the same set of terms as ML, and, for each n-ary predicate letter p of ML, two n-ary predicate letters $p^?$ and $p^!$.

A formula A of J is said to be *positive* (resp. *negative*) if all positive occurrences of a predicate letter are occurrences of predicates $p^?$ (resp. $p^!$), and all negative occurrences of a predicate letter are occurrences of some $p^!$ (resp. $p^?$).

Now, we give the inductive definition of the J-translation of a formula of ML, which is to be a formula of J:

$$(\xi pt_1 \dots t_n)^J = p^\xi t_1 \dots t_n; \qquad \big(\xi(A \wedge B)^J\big) = (\xi A)^J \wedge (\xi B)^J;$$
$$\big(\xi(A \vee B)\big)^J = (\xi A)^J \vee (\xi B)^J; \qquad (\xi QxA)^J = Qx(\xi A)^J \qquad (Q = \forall, \exists);$$
$$\big(\xi(A \to B)\big)^J = (\bar{\xi} A)^J \to (\xi B)^J.$$

Clearly, $(\xi A)^J$ is positive (resp. negative) if $\xi = ?$ (resp. $\xi = !$) and conversely, every positive (resp. negative) formula of J is the J-translation of some $?A$ (resp. $!A$).

We choose $\perp^?$ to denote the absurdity in J.

If M is a structure for ML, then one can define a structure M^J for J by putting $|M^J| = |M|$, $M^J(t) = M(t)$ for t closed term, and

$$M^J(p^\xi t_1 \dots t_n) = M(\xi pt_1 \dots t_n). \tag{1}$$

(Of course, M and M^J take their values into the same space.) Observe that M^J satifies all sequents

$$p^! t_1 \dots t_n \vdash p^? t_1 \dots t_n \tag{2}$$

and

$$\perp^? \vdash p^\xi t_1 \dots t_n. \tag{3}$$

Conversely, every (ordinary) structure for J satisfying (2) and (3) induces a structure for ML, with the help of (1).

For $\Delta = (\xi_1 A_1, \dots, \xi_n A_n)$ define Δ^J to be $\big((\xi_1 A_1)^J, \dots, (\xi_n A_n)^J\big)$. Clearly, $M \vDash \Gamma \vdash \Delta$ iff M^J satisfies $\Gamma^J \vdash \Delta^J$.

One may suppose that all proper axioms of T are of the form $\xi A \vdash \xi B$; this is a trivial consequence of the following

LEMMA 1. *Let $\Gamma \vdash \Delta$ be a sequent of ML; then there are two formulas ξA and ξB (with the same modality ξ) such that:*

(i) *$\Gamma \vdash \Delta$ is provable from $\xi A \vdash \xi B$ in MLI (or MLC if* $\text{card}(\Delta) > 1$),

(ii) *$\xi A \vdash \xi B$ is provable from $\Gamma \vdash \Delta$ in MLI (or MLC if* $\text{card}(\Delta) > 1$),

(iii) *the two sequents have the same class of models.*

Proof of the lemma. (iii) is a trivial consequence of (i) and (ii).

(i) and (ii): let $\Gamma \vdash \Delta$ be $!A_1^1, \ldots, !A_n^1, \ ?A_1^2, \ldots, ?A_m^2 \vdash !B_1^1, \ldots, !B_p^1, ?B_1^2, \ldots, ?B_q^2$.

Classical case: replace this sequent by $!A^1, ?A^2 \vdash !B^1, ?B^2$, with A^i = conjunction of the A_j^i (if there are no A_j^i, take $A^i = \bot \to \bot$), B^i = disjunction of the B_j^i (if there are no B_j^i, take $B^i = \bot$).

Define $A = A^1 \wedge \neg B^2$, $B = A^2 \to B^1$. The sequent $!A \vdash !B$ satisfies (i) and (ii).

Intuitionistic case: the right-hand side is of the form $\Delta = C$; define the A^i's as above, and let A be A^1 if $\xi = !$, A^2 otherwise, and B be $A^2 \to C$ if $\xi = !$, $A^1 \to C$ otherwise; then the sequent $\xi A \vdash \xi B$ satisfies (i) and (ii).

If all axioms of T are of the form $\xi A \vdash \xi B$, define T^J to be the theory (with underlying language J), and proper axioms the axioms $(\xi_i A_i)^J \to (\xi_i B_i)^J$ for $\xi_i A_i \vdash \xi_i B_i$ proper axiom of T, together with the axioms $\big(?(P \to P)\big)^J$ for P atomic; we may assume that this set of axioms is closed under substitution. $\bot^?$ stands for the absurdity of T^J.

LEMMA 2. *If $\Gamma^J \vdash \Delta^J$ is provable in T^J, then $\Gamma \vdash \Delta$ is provable in T.*

Proof of the lemma. If $\Gamma^J \vdash \Delta^J$ is provable in T^J, then there are formulas $C_1, \ldots, C_r$ such that $C_1, \ldots, C_r, \Gamma^J \vdash \Delta^J$ is provable in the logical sequent calculus, and each C_i is obtained from a proper axiom of T^J by applying a finite number of universal quantifications (let us call such C_i's universalisations of axioms).

1st case. No C_i is a universalisation of an axiom $(!P)^J \to (?P)^J$ for P atomic. Then the C_i's are neither positive nor negative. Consider a cut-free proof of $C_1, \ldots, C_r, \Gamma^J \vdash \Delta^J$; since all subformulas of positive or negative formulas are positive or negative, and all strict subformulas of proper axioms of T^J are positive or negative, and since every positive or negative formula is the J-translation of some formula of ML, it is clear that all sequents occurring in the proof are of the form $D_1, \ldots, D_n, \Lambda^J \vdash \Pi^J$, where $\Lambda \vdash \Pi$ is a sequent of ML, and the D_i's are universalisations of axioms of T^J, distinct from the $(!P \vdash ?P)$.

Now, we prove by induction over the length of a cut-free derivation of $D_1, \ldots, D_n, \Lambda^J \vdash \Pi^J$ that $\Lambda \vdash \Pi$ is provable in T.

The proof splits into two cases:

(i) the new formula occurring in the conclusion of the last rule of the proof is one of the D_i's, for instance D_1. Except for the case of a weakening (which is trivial) the last rule must have the form

(a) $$\frac{D_1', \ldots, D_n, \Lambda^J \vdash \Pi^J}{\forall x D_1', \ldots, D_n, \Lambda^J \vdash \Pi^J}$$

or

(b) $$\frac{B_1, D_1', \ldots, D_n', \Lambda^J \vdash \Pi^J \qquad D_1'', \ldots, D_n'', \Lambda^J \vdash (\Pi^J)^-, A_1}{A_1 \to B_1, D_2, \ldots, D_n, \Lambda^J \vdash \Pi^J}$$

with, in case (a) $D_1 = \forall x D_1'$, in case (b), $D_1 = A_1 \to B_1$.

In case (a), since $\Lambda \vdash \Pi$ is already provable by induction hypothesis, there is nothing to prove. In case (b) let $A_1 = (\xi A)^J$, $B_1 = (\xi B^J)$; then $\xi A \vdash \xi B$ is a proper axiom of T; by induction hypothesis ξB, $\Lambda \vdash \Pi$ and $\Pi \vdash (\Lambda)^-$, ξA are provable in T; applying the cut twice, one gets that $\Lambda \vdash \Pi$ is provable in T.

(ii) otherwise the last rule applies as well to the subsequents of the sequents involved in the rule obtained by dropping everything which is not a J-translation of a formula of ML; for instance, if the last rule is of the form

$$\frac{D_1, \ldots, D_n, (\Lambda')^J \vdash (\Pi')^J}{D_1, \ldots, D_n, \Lambda^J \vdash \Pi^J},$$

then by induction hypothesis, $\Lambda' \vdash \Pi'$ is provable in T; now, by looking to all the possible cases, one checks that

$$\frac{\Lambda' \vdash \Pi'}{\Lambda \vdash \Pi}$$

is a rule of the modal predicate calculus. Similarly for two or zero premises.

2nd case. Some C_i's are universalisations of proper axioms $(!P)^J \to (?P)^J$ of T^J, with P atomic. Such C_i's are positive; we can apply the first case of the proof to the sequent $C_1', \ldots, C_p', (\Gamma')^J \vdash \Delta^J$, where $C_1', \ldots, C_p'$ are those C_i's which are not universalisations of $(!P)^J \to (?P)^J$, and $(\Gamma')^J$ is the sum of Γ^J and of the other C_i's. Then we get that $\Gamma' \vdash \Delta$ is provable in T. Now observe that every formula occurring in Γ' and not in Γ is of the form $?\forall x_1 \ldots \forall x_n (P \to P)$; since $!P \vdash ?P$ is an axiom of T, $\Gamma \vdash \Delta$ is provable in T.

Now, take any $\Gamma \vdash \Delta$ valid in T; then $\Gamma^J \vdash \Delta^J$ is valid in T^J; using the completeness theorem: "*if A is valid in all models of T^J, then A is provable in T^J*" (for a proof in the intuitionistic case, with topological models, see [7], pp. 422–423), it follows that $\Gamma^J \vdash \Delta^J$ is provable in T^J. Now, by Lemma 2 $\Gamma \vdash \Delta$ is provable in T. Q.E.D.

A 5. The thinness theorem

Roughly speaking, the thinness theorem states that if we start with a given 3-valued model, M, and form a new model M' by giving some determinated values to atomic statements which are not determinated in M, then, the value of a formula computed in M' does not contradict its value computed in M. We state the theorem only for topological structures, since classical structures are a particular case.

Let H be a non-void topological space, and let H'' be as in A2. We define the binary relation $<$ on H'' by $(\alpha, \beta) < (\alpha', \beta')$ iff $\alpha \subset \alpha'$ and $\beta' \subset \beta$. This relation is an order relation, and can be read as "is coarser than", while the opposite relation can be read as "is thinner than".

Let M, M' be structures for ML with the same topological space H. A binary relation $\leqslant$ between $|M|$ and $|M'|$ is called a *thinness relation* iff

(i) $c_i \leqslant d_i$ $(c_i \in |M|,\ d_i \in |M'|)$ $(i = 1, \ldots, n)$ implies

$$M(f\bar{c}_1 \ldots \bar{c}_n) \leqslant M'(f\bar{d}_1 \ldots \bar{d}_n) \quad \text{for every function letter } f,$$

and

$$M(p\bar{c}_1 \ldots \bar{c}_n) < M'(p\bar{d}_1 \ldots \bar{d}_n) \quad \text{for every predicate letter } p.$$

(ii) For every c in $|M|$, there is a d in $|M'|$ such that $c \leqslant d$

and

for every d in $|M'|$, there is a c in $|M|$ such that $c \leqslant d$

A.5.1. Theorem. *Let M, M' be structures for ML, and let $\leqslant$ be a thinness relation between M and M'; let $A(x_1, \ldots, x_n)$ be a formula of ML; then, for any c_i in M and d_i in M' such that $c_i \leqslant d_i$ $(i = 1, \ldots, n)$,*

$$M\big(A(\bar{c}_1, \ldots, \bar{c}_n)\big) < M'\big(A(\bar{d}_1, \ldots, \bar{d}_n)\big),$$

that is

$$M\big(!A(\bar{c}_1, \ldots, \bar{c}_n)\big) \subset M'\big(!A(\bar{d}_1, \ldots, \bar{d}_n)\big)$$

and

$$M'\big(?A(\bar{d}_1, \ldots, \bar{d}_n)\big) \subset M\big(?A(\bar{c}_1, \ldots, \bar{d}_n)\big).$$

Proof. By induction over the complexity of A: let us note $\underline{c}$, $\underline{d}$, the sequences $(\bar{c}_i)$, $(\bar{d}_i)$.

(1) A is atomic; the result follows by (i).

(2) A is a conjunction or a disjunction; trivial.

(3) A is $B \to C$; by induction hypothesis, we know that:

$$M\big(!B(\underline{c})\big) \subset M'\big(!B(\underline{d})\big) \quad \text{and} \quad M'\big(?B(\underline{d})\big) \subset M\big(?B(\underline{c})\big),$$

so

$$\complement M'\big(!B(\underline{d})\big) \subset \complement M\big(!B(\underline{c})\big) \quad \text{and} \quad M\big(?B(\underline{c})\big) \subset M'\big(?B(\underline{d})\big),$$

and also that

$$M(!C(\underline{c})) \subset M'(!C(\underline{d})) \quad \text{and} \quad M'(?C(\underline{d})) \subset M(?C(\underline{c})).$$

So

$$\left(\complement M(?B(\underline{c})) \cup M(!C(\underline{c}))\right) \subset \left(\complement M'(?B(\underline{d})) \cup M(!C(\underline{d}))\right)$$

and

$$\left(\complement M'(!B(\underline{d})) \cup M'(?C(\underline{d}))\right) \subset \left(\complement M(!B(\underline{c})) \cup M(?C(\underline{c}))\right).$$

Similar inclusions hold for the interiors, so, recalling the interpretation of implication, we get the result for $A = B \to C$.

(4) A is $\forall x B$; suppose $e \in |M|$, $f \in |M'|$, $e \leqslant f$; then

$$M(!B(\underline{e}, \bar{e})) \subset M'(!B(\underline{d}, \bar{f})) \quad \text{and} \quad M'(?B(\underline{d}, \bar{f})) \subset M(?B(\underline{c}, \bar{e})).$$

Now, by property (ii), similar inequalities hold for the intersections $\bigcap_{e \in M} M(B(\underline{c}, \bar{e}))$ and $\bigcap_{d \in M'} M'(B(\boldsymbol{d}, \bar{f}))$, and also for the interiors.

So the result is true for $A = \forall x B$.

(5) A is $\exists x B$; symmetrical to (4).

B. Three-valued analysis and cut elimination

B 1. The syntax

B.1.1. The syntax of second order arithmetic. The language L^1 of second order arithmetic is built up from the symbols

– first order variables $x, y, z, \ldots$,

– function and predicate letters $f, g, \ldots, p, q, \ldots$ which represent primitive recursive functions and predicates. We suppose that the absurdity, equality, the zero, successor, sum, product, ... are among those functions and predicates which are called *PR* functions and predicates,

– second order variables $X, Y, Z, \ldots$,

– the binary predicate ϵ.

The *first order terms* and the *formulas* are given by the following inductive definition:

– a first order variable is a first order term,

– whenever $t_1, \ldots, t_n$ are first order terms, so is $ft_1, \ldots, t_n$ (f is n-ary),

– whenever p is a n-ary *PR* predicate letter and $t_1, \ldots, t_n$ are first order terms, then $pt_1, \ldots, t_n$ is a formula, called a *PR-formula*,

– if t is a term and X is a second order variable, then $t \in X$ is a formula,

– if A and B are formulas, so are $A \wedge B$, $A \vee B$, $A \to B$,

— if A is a formula, x (resp. X) a first (resp. second) order variable, then $\forall xA$, $\exists xA$, $\forall XA$, $\exists XA$ are formulas.

B.1.2. Abstraction terms. We define the *abstraction terms* of L^1 to be the expressions λxA (1), where A is a formula of L^1, and x is a first order variable. A *second order term* is either a second order variable or an abstraction term. If A is a formula of L^1, T is an abstraction term, then the result of substituting every occurrence of some second order variable X by an occurrence of T is not in general a formula of L^1, but if we replace every atomic part $t\epsilon T$ of the expression obtained by the formula $B(t)$ '$T = \lambda xB(x)$', then we get a formula of L^1. This formula is denoted by $A(T)$. (Recall that T cannot occur in $A(T)$.)

If U is any second order term, and $A(X)$ any formula, then $A(U)$ is to be:

— $A(T)$ as defined above, if U is an abstraction term T,

— $A(Y)$, defined as usual if U is the second order variable Y.

B.1.3. The modal calculi M^1LC and M^1LI. Define a logic-free calculus of intuitionistic sequents Q, as follows: the sequents are of the form $P_1, \ldots, P_n \vdash R$, for $P_1, \ldots, P_n$, R PR-formulas.

AXIOMS OF Q. The axioms of Q are all the instanciations of the following:

$$P \vdash P \quad \text{for } P \ PR\text{-formula},$$

— defining equations for PR functions $ft_1 \ldots t_n = u$,

— defining equations for PR predicates $\begin{cases} t(x_1, \ldots, x_n) = 0 \vdash px_1 \ldots x_n, \\ px_1 \ldots x_n = 0 \vdash t(x_1, \ldots, x_n) = 0, \end{cases}$

— equality and successor axioms

$$\vdash x = x, \quad x = y,\ P(x) \vdash P(y),$$

$$Sx = 0 \vdash P, \quad Sx = Sy \vdash x = y \quad (P\ PR\text{-formula}) \quad \bot \vdash Sx = 0.$$

RULES OF Q. The rules of Q are the following:

— contraction, cut,

— left weakening on a PR-formula.

The modal language ML^1 associated with L^1 is defined as in A. (That is the formulas of ML^1 are the $!A$ and the $?A$, for A formula of L^1.)

Now, we give the definitions of systems M^1LC and M^1LI. M^1LC is the classical version of M^1LI ("I" for intuitionistic). The symbols Γ, Δ, Δ^- are used in the same sense that in part A3. $\xi < \eta$ will stand for ($\xi = ?$ or $\eta = !$).

(1) λxA stands for "the set of x such that $A(x)$".

Structural rules. The same that in A3, but for formulas of ML^1. The right weakenings and contractions are permitted only in the classical system M^1LC.

Axioms.

(AX 1) $$\xi A \vdash \xi A$$

(AX 2) $$\xi_1 P_1, \ldots, \xi_n P_n \vdash \eta R$$

for any $P_1, \ldots, P_n$, R which are PR formulas such that $P_1, \ldots, P_n \vdash R$ is provable in Q, and any $\xi_1, \ldots, \xi_n, \eta$.

(AX 3) $$\xi_1 P_1, \ldots, \xi_n P_n, \xi(t \epsilon X) \vdash \eta(u \epsilon X)$$

whenever $P_1, \ldots, P_n$, are PR-formulas such that $P_1, \ldots, P_n \vdash t = u$ is provable in Q, $\xi_1, \ldots, \xi_n$ are arbitrary, and $\eta < \xi$.

(AX 4) $$\xi_1 P_1, \ldots, \xi_n P_n \vdash \eta(t \epsilon X)$$

where the PR-formulas $P_1, \ldots, P_n$ are such that $P_1, \ldots, P_n \vdash \perp$ is provable in Q, and $\xi_1, \ldots, \xi_n$, are any modalities.

(AX 5) $!(t \epsilon X) \vdash ?(t \epsilon X)$ (particular case of (AX 3)).

Logical rules. The logical rules of M^1LC (or M^1LI) are exactly the rules of A3 for $\vee, \wedge, \rightarrow$. The logical rules for 1st order quantification are exactly those of A3 (if the variable x of those rules is understood as being a first order variable). We write out the logical rules for 2nd order quantifications.

Introduction	Elimination
$\dfrac{\Gamma \vdash \xi A(X), \Delta^-}{\Gamma \vdash \xi \forall X A(X), \Delta^-}$ (1),	$\dfrac{\Gamma, \xi A(T) \vdash \Delta}{\Gamma, \xi \forall X A(X) \vdash \Delta}$ (2),
$\dfrac{\Gamma \vdash \xi A(T), \Delta^-}{\Gamma \vdash \xi \exists X A(X), \Delta^-}$ (2),	$\dfrac{\Gamma, \xi A(X) \vdash \Delta}{\Gamma, \xi \exists X A(X) \vdash \Delta}$ (1).

Remark. In axioms (AX 2,3,4), the allowed choice of any modalities for PR-formulas indicates that for purely 1st order formulas, we make no difference between ! and ?; that is the 1st order part of M^1LC or M^1LI is binary.

The binary logical calculi G^1LC and G^1LI are obtained from M^1LC and M^1LI by dropping all ? and ! in the rules and axioms of these theories. Or equivalently, since a three-valued theory is binary when $?A = !A$

(1) Restriction on variables; X not free in (Γ, Δ^- or Γ, Δ).

(2) T is a second order term; see the definition of $A(T)$ in B.1.2.

for all A, by adding the axioms $?(t \in X) \vdash !(t \in X)$, which ensure $?A \vdash !A$ for any A (recall that $!A \vdash ?A$ is provable for all A in modal systems).

Remark. The set of sequents $A_1, \ldots, A_n \vdash B$, where $\xi_1 A_1, \ldots, \xi_n A_n \vdash \eta B$ is provable in M^1LI for some $\xi_1, \ldots, \xi_n$ (one can always take $\xi_i = !$, $\eta = ?$) is included in the set of theorems of G^1LI; the reverse inclusion is by no means elementary, as we shall see; similar remark for M^1LC and G^1LC.

B 2. The semantics

We speak only about the intuitionistic semantics, since the classical one is a particular case.

B.2.1. Structures. A structure M for G^1LI consists of

— a non-void topological space H,

— a *binary* structure $\boldsymbol{M}_0$ (with values into H) for the PR functions and predicate letters; let $M_0 = |\boldsymbol{M}_0|$,

— a non-empty set M_1 for the interpretation of second order variables,

— a function $\underline{\epsilon}$, with domain $M_0 \times M_1$, such that, for every $c \in M_0$, $C \in M_1$, $\underline{\epsilon}(c, C) = (\alpha, \beta)$, where α and β are open subsets of H, and $\alpha \subset \beta$.

Clearly, if $\underline{\epsilon}(c, C) = (\alpha, \beta)$, then $\boldsymbol{M}(?\ c \in C) = \alpha$, while $\boldsymbol{M}(!\ c \in C) = \beta$. Every rule of M^1LI is valid in a structure except possibly

(i) the axioms (AX 2) and (AX 3),

(ii) the second order rules of $\forall$-elim and $\exists$-introd.

B.2.2. Models. A model of G^1LI is a structure satisfying

(i) the axioms (AX 2) and (AX 3),

(ii) for every closed formula $\forall x A(x)$ of $L^1(\boldsymbol{M})$, there exists C in M_1, such that for every $c \in \boldsymbol{M}_0$, and modality ξ, $\boldsymbol{M}(\xi A(\bar{c})) = \boldsymbol{M}(\xi \bar{c} \in \bar{C})$.

(Clearly, if $\lambda x A$ is a closed abstraction term of $L^1(\boldsymbol{M})$, we may choose a particular C satisfying (ii), in order to extend the interpretation to this term.)

B.2.3. Completeness theorem.

THEOREM B.2.3.

(i) *The sequent $\Gamma \vdash \Delta$ is provable in M^1LI (resp. M^1LC) iff it is valid in every topological model (resp. classical model) of M^1LI (resp. M^1LC).*

(ii) *$\Gamma \vdash \Delta$ is provable in G^1LI (resp. G^1LC) iff it is valid in every binary topological model (resp. binary classical model) of G^1LI (resp. G^1LC).*

Proof. (ii) is trivially deducible from (i), since binary models are models of $?(t \in X) \vdash !(t \in X)$, or equivalently of $!\forall x \forall X\ (x \in X \to x \in X)$.

(i) follows from the completeness theorem of A.4.1, or better of its straightforward adaptation to many-sorted theories. Now we show how to eliminate the abstraction terms in order to reduce the problem to many-sorted completeness. The abstraction terms can be represented as functions letters of a certain kind: if λxA is an abstraction term with free variables $y_1, \ldots, y_n, Y_1, \ldots, Y_m$, introduce the (n, m)-ary function letter $F_{\lambda xA}$, with values in type 1 objects together with the axioms

$$\xi(t \in F_{\lambda xA} t_1 \ldots t_n T_1 \ldots T_m) \vdash \xi A(t_1 \ldots t_n T_1 \ldots T_m, t)$$

and

$$\xi(A(t_1 \ldots t_n T_1 \ldots T_m, t)) \vdash \xi(t \in F_{\lambda xA} t_1 \ldots t_n T_1 \ldots T_m)$$

(here the terms $T_1 \ldots T_n$ are not abstraction terms, but are built with the help of variables and the new function letters).

Every model of M^1LI can be extended to the language containing the new function letters (property (ii) of models), and satisfying the associated axioms. Now, from a proof containing the $F_{\lambda xA}$, one can get a proof using abstraction terms by a straightforward argument. Similarly for M^1LC. Q.E.D.

B.2.4. Classical case. Let us describe briefly classical models and structures.

$\boldsymbol{M} = (\boldsymbol{M}_0, \boldsymbol{M}_1, \underline{\epsilon})$ is a (classical) structure for M^1LC iff:

(i) $\boldsymbol{M}_0$ is a classical binary structure for the PR predicates and functions,

(ii) M_1 is a non-void set,

(iii) $\underline{\epsilon}$ is a mapping from $M_0 \times M_1$ into $3 = \{0, 1, 2\}$,

$\boldsymbol{M}$ is said to be a *model* if $\boldsymbol{M}$ is a structure and the following hold:

(iv) $\boldsymbol{M}$ is a model for the axioms (AX 2) and (AX 3). (So we may take the interpretation of $=$ to be the identity.)

(v) for any closed $\forall x A(x) \in L^1(\boldsymbol{M})$, there is C in M_1 such that for all c in $\boldsymbol{M}_0$, and ξ, $\boldsymbol{M} \models \xi(\bar{c} \in \bar{C})$ iff $\boldsymbol{M} \models \xi A(\bar{c})$.

Remark. Essentially, a classical 3-valued model for G^1LC does not differ from a partial valuation in Schütte's sense ([8]).

B 3. Cut-free provability

A sequent of M^1LC, M^1LI, G^1LC, G^1LI is said to be *cut-free provable* iff there is a derivation $\boldsymbol{D}$ of this sequent in the system which does not use the cut rule.

B.3.1. Definition. If $\Gamma \vdash \Delta$ is a sequent of G^1LC, let $\Gamma^* \vdash \Delta^*$ be the sequent of M^1LC defined by $\Gamma^* = \{!A; A \in \Gamma\}$, $\Delta^* = \{?A; A \in \Delta\}$.

B.3.2. THEOREM. *$\Gamma \vdash \Delta$ is provable cut-free in G^1LC (resp. G^1LI) iff $\Gamma^* \vdash \Delta^*$ is provable cut-free in M^1LC (resp. M^1LI).*

Proof. Trivial induction on cut-free proofs.

The next theorem is the main result of this section.

B.3.3. THEOREM. *$\Gamma \vdash \Delta$ is cut-free provable in G^1LC (resp. G^1LI) iff $\Gamma^* \vdash \Delta^*$ is provable in M^1LC (resp. M^1LI).*

COROLLARY. *A is cut-free provable in G^1LC (resp. G^1LI) iff $?A$ is valid in any classical (resp. topological) model.*

(Compare with [8]: "*A is cut-free provable in G^1LC iff A is never false in any partial valuation*".)

The corollary follows from B.2.3 and B.3.3.

B.3.3 is consequence of the following theorem and of B.3.1.

B.3.4. THEOREM. *If $\Gamma \vdash \Delta$ is provable in M^1LC (resp. M^1LI), then $\Gamma \vdash \Delta$ is cut-free provable in M^1LC (resp. M^1LI).*

Theorem B.3.4 will be proved in 1st order arithmetic, with induction restricted to Π_2^0 formulas.

The following lemmas are stated without proof:

LEMMA 1. *For all A, $!A \vdash ?A$ is cut-free provable.*

LEMMA 2. *If $P_1, \ldots, P_n \vdash t = u$ is provable in Q ($P_1, \ldots, P_n$ PR-formulas) and if $\Gamma(t) \vdash \Delta(t)$ is cut-free provable, then, for any $\xi_1, \ldots, \xi_n$, $\Gamma(u)$, $\xi_1 P_1, \ldots, \xi_n P_n \vdash \Delta(u)$ is cut-free provable.*

LEMMA 3. *Every instanciation of a cut-free provable sequent is cut-free provable.*

(The non-trivial contents of Lemma 3 concern 2nd order instanciation; it suffices to prove that every instanciation of an axiom is cut-free provable; this is easy from Lemmas 1 and 2.)

B.3.5. DEFINITION. The *principal formula* of a logical rule (and also of weakenings and contractions) is that occurence of a formula which is written explicitely in the conclusion of the rule (as the rules are written in A3 and B1). The principal formula of a $\mathscr{K}$-introd. or $\mathscr{K}$-elim is always to begin with $!\mathscr{K}$ or $?\mathscr{K}$ ($\mathscr{K} = \wedge, \vee, \rightarrow, \forall, \exists$).

B.3.6. DEFINITION. Let $?A$ be a formula; a *$?A$-test* is a sequent $\Gamma \vdash \Delta$ such that $\Gamma \vdash \Delta^-, ?A$ is cut-free provable. A *weak $?A$-test* is a sequent $\Gamma \vdash \Delta$ such that there is a cut-free proof $\boldsymbol{D}$ of $\Gamma \vdash \Delta^-, ?A$, and either $\boldsymbol{D}$ is obtained by weakenings from an axiom, or the last rule of $\boldsymbol{D}$ is an introd. with $?A$ as principal formula.

Every weak $?A$-test is an $?A$-test; if $?A \in \Gamma$, then $\Gamma \vdash \Delta$ is a weak $?A$-test.

B.3.7. DEFINITION. Let $!A$ be a formula; a *$!A$-test* is a sequent $\Gamma \vdash \Delta$ such that $\Gamma, !A \vdash \Delta$ is cut-free provable. A *weak $!A$-test* is a sequent $\Gamma \vdash \Delta$

such that there is a cut-free proof $\boldsymbol{D}$ of $\Gamma, !A \vdash \Delta$ and either $\boldsymbol{D}$ is obtained from an axiom by weakenings, or the last rule of $\boldsymbol{D}$ is an elim. with principal formula $!A$.

Every weak $!A$-test is an $!A$-test; if $!A \in \Delta$, then $\Gamma \vdash \Delta$ is a weak $!A$-test.

B.3.8. Definition. Let Λ be a finite set of formulas (with repetitions); we define $\xi\Lambda$ to be $\xi\Lambda = \{\xi A;\ \xi A \in \Lambda\}$.

B.3.9. Definition. Let $\Gamma \vdash \Delta$ be a sequent. A sequent $\Lambda \vdash \Pi$ is said to be a *test* (resp. *weak test*) for $\Gamma \vdash \Delta$ iff $!\Gamma \subset \Lambda$, $?\Delta \subset \Pi$, and for every $?A$ in Γ, $\Lambda \vdash \Pi$ is an $?A$-test (resp. a weak $?A$-test) and for every $!A$ in Δ, $\Lambda \vdash \Pi$ is an $!A$-test (resp. a weak $!A$-test).

Every weak test for $\Gamma \vdash \Delta$ is a test for $\Gamma \vdash \Delta$. If $\Gamma \subset \Lambda$, $\Delta \subset \Pi$, then $\Lambda \vdash \Pi$ is a weak test for $\Gamma \vdash \Delta$..

B.3.10. Definition. The sequent $\Gamma \vdash \Delta$ is said to be *regular* (resp. *weakly regular*) iff for any test (resp. weak test) $\Lambda \vdash \Pi$ for $\Gamma \vdash \Delta$, then $\Lambda \vdash \Pi$ is cut-free provable.

B.3.11. Proposition.

(i) *Every weakly regular sequent is cut-free provable.*

(ii) *Every weakly regular sequent is regular* (*and conversely*).

Proof. (i) Trivial, since a sequent is a weak test for himself.

(ii) Let $\Gamma \vdash \Delta$ be a weakly regular sequent, and let $\Lambda \vdash \Pi$ be a test for $\Gamma \vdash \Delta$. Let $?\Gamma = \{?A_1, \ldots, ?A_n\}$, $!\Delta = \{!B_1, \ldots, !B_m\}$ (intuitionistic case: $m = 0$ or 1) Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ (resp. $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_m$) be cut-free proofs of $\Lambda \vdash \Pi^-, ?A_i$ (resp. of Λ, $!B_j \vdash \Pi$). Let us assume that the $\boldsymbol{D}_i$'s and $\boldsymbol{E}_j$'s have the following properties:

(a) the only rules before a weakening are weakenings or axioms·

(b) the immediate rule before a contraction of principal formula ξA is a logical rule with principal formula ξA.

Clearly, given any cut-free proof of a sequent, one can get another cut-free proof of it satisfying (a) and (b).

For $i = 1, \ldots, n$, $j = 1, \ldots, m$, define functions l_i and l'_j from proofs satisfying (a) and (b) and cut-free into $\boldsymbol{N}$:

– if the last rule of $\boldsymbol{D}$ is an axiom, a weakening, or an introd. with principal formula $?A_i$ (resp. an elim. with principal formula $!B_j$), then $l_i(\boldsymbol{D}) = 0$ (resp. $l'_j(\boldsymbol{D}) = 0$);

– otherwise, if $\boldsymbol{D}$ arises from $\boldsymbol{D}'$ by a rule, then $l_i(\boldsymbol{D}) = l_i(\boldsymbol{D}')+1$ (resp. $l'_j(\boldsymbol{D}) = l'_j(\boldsymbol{D}')+1$). Similarly, if $\boldsymbol{D}$ arises from $\boldsymbol{D}'$ and $\boldsymbol{D}''$, then $l_i(\boldsymbol{D}) = (\sup(l_i(\boldsymbol{D}'), l_i(\boldsymbol{D}''))+1$ (resp. $l'_j(\boldsymbol{D}) = (\sup(l'_j(\boldsymbol{D}'), l'_j(\boldsymbol{D}''))+1)$.

Now, we prove by induction over $\sum_i (l_i(\boldsymbol{D}_i) + \sum_j l'_j(\boldsymbol{E}_j))$ that $\Lambda \vdash \Pi$ is cut-free provable.

Basis: $l_i(\boldsymbol{D}_i) = 0$, $l'_j(\boldsymbol{E}_j) = 0$ for all i, j; then $\Lambda \vdash \Pi$ is a weak test for $\Gamma \vdash \Delta$; then $\Lambda \vdash \Pi$ is cut-free provable.

Induction step: $l_1(\boldsymbol{D}_1) \neq 0$ (the case $l'_1(\boldsymbol{E}_1) \neq 0$ is symmetrical).

1° The last rule of $\boldsymbol{D}_1$ is a contraction with principal formula $?A_1$. By property (b), this contraction follows an introd. (R) with principal formula $?A_1$. Let $\Lambda' \vdash \Pi'$, $?A_1$ and $\Lambda'' \vdash \Pi''$, $?A_1$ be the premises of this introd. (in the general case of a rule with two premises), $\boldsymbol{D}'_1$ and $\boldsymbol{D}''_1$ the subderivations of $\boldsymbol{D}_1$ ending with these sequents. $l_1(\boldsymbol{D}'_1), l_1(\boldsymbol{D}''_1) < l_1(\boldsymbol{D}_1)$. By using appropriate weakenings, one can find proofs $\boldsymbol{D}'_i$ and $\boldsymbol{D}''_i$ of $\Lambda, \Lambda' \vdash \Pi, \Pi'$, $?A_i$ and $\Lambda, \Lambda'' \vdash \Pi, \Pi''$, $?A_i$, with the same l_i's as $\boldsymbol{D}_i$ $(i \neq 1)$, and similarly for $\boldsymbol{E}_j$. Applying the induction hypothesis, one gets cut-free proofs of $\Lambda, \Lambda' \vdash \Pi, \Pi'$ and $\Lambda, \Lambda'' \vdash \Pi, \Pi''$. Applying introduction (R), one gets a cut-free proof of $\Lambda, \Lambda \vdash \Pi, \Pi$, $?A_1$. Using appropriate contractions, one gets a cut-free proof $\boldsymbol{D}'''_1$ of $\Lambda \vdash \Pi$, $?A_1$, and $l_1(\boldsymbol{D}'''_1) = 0$. Applying once more the induction hypothesis to $\boldsymbol{D}'''_1, \boldsymbol{D}_2, \ldots, \boldsymbol{D}_n$, $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_m$, one concludes that $\Lambda \vdash \Pi$ is cut-free provable.

2° Otherwise, $\Lambda \vdash \Pi^-$, $?A_1$ follows from $\Lambda' \vdash (\Pi')^-$, $?A_1$ and $\Lambda'' \vdash (\Pi'')^-$, $?A_1$ by the rule (R).

As in case 1° one can construct $\boldsymbol{D}'_i$, $\boldsymbol{E}'_j$, and $\boldsymbol{D}''_i$, $\boldsymbol{E}''_j$, and apply the induction hypothesis to conclude that $\Lambda, \Lambda' \vdash \Pi$, $(\Pi')^-$ and $\Lambda, \Lambda'' \vdash \Pi$, $(\Pi'')^-$ are cut-free provable. (The only difference with case (1) is that we have to take care of the intuitionistic case.)

Applying (R) to these sequents, one gets a cut-free proof of Λ, $\Lambda \vdash \Pi, \Pi^-$. Appropriate contractions give a cut-free proof of $\Lambda \vdash \Pi$.

B.3.12. Proposition.

(i) *Regular sequents are closed under cut, weakening, and contraction.*

(ii) *Regular sequents are closed under the logical rules for* $\vee$, $\wedge$ *and* $\rightarrow$.

Proof. (i) The only non-trivial case is that of a cut; let $\Gamma \vdash \Delta$ be obtained from $\Gamma \vdash \Delta^-$, ξA and Γ, $\xi A \vdash \Delta$. We treat the case where $\xi = ?$, the other case being symmetrical. So, suppose $\Gamma \vdash \Delta^-$, $?A$ and Γ, $?A \vdash \Delta$ regular, and let $\Lambda \vdash \Pi$ be a test for $\Gamma \vdash \Delta$. We claim that $\Lambda \vdash \Pi$ is a test for Γ, $?A \vdash \Delta$: since $\Lambda \vdash \Pi$ is already a test for $\Gamma \vdash \Delta$, it suffices to show that $\Lambda \vdash \Pi^-$, $?A$ is cut-free provable; but $\Lambda \vdash \Pi^-$, $?A$ is clearly a test for $\Gamma \vdash \Delta^-$, $?A$, and since this sequent is regular, $\Lambda \vdash \Pi^-$, $?A$ is cut-free provable and the claim is proved. Now, if $\Lambda \vdash \Pi$ is a test for $\Gamma \vdash \Delta^-$, $?A$, $\Lambda \vdash \Pi$ must be cut-free provable.

(ii) The proof splits into two cases:

(a) !-introd or ?-elim. We treat the case of a $!\!\rightarrow$-introd. ($?\!\rightarrow$elim is symmetrical, and the other cases are simpler). Suppose Γ, $?A \vdash !B, \Delta^-$ is regular, and let $\Lambda \vdash \Pi$ be a weak test for $\Gamma \vdash !(A \rightarrow B), \Delta^-$. Let $\boldsymbol{D}$ be a cut-free proof of Π, $!(A \rightarrow B) \vdash \Lambda$, as in Definition B.3.7. Then:

— either $\boldsymbol{D}$ comes by weakenings from an axiom, and we may assume that $!(A \to B)$ is not the principal formula of any of the left weakenings involved, since in this case $\Lambda \vdash \Pi$ is already cut-free provable. Then the axiom involved in $\boldsymbol{D}$ must be $!(A \to B) \vdash !(A \to B)$, so $\Pi = \Pi', !(A \to B)$. On cheecks easily that Λ, $?A \vdash !B, \Pi'$ is a test for Γ, $?A \vdash !B$, Δ^-.

So, Λ, $?A \vdash !B, \Pi'$ (and hence $\Lambda \vdash !(A \to B)$, $\Pi' = \Lambda \vdash \Pi$) is cut-free provable,

— otherwise the last rule of $\boldsymbol{D}$ is $! \to$-elim. So $\Lambda \vdash \Pi^-$, $?A$ and Λ, $!B \vdash \Pi$ are cut-free provable. So $\Lambda \vdash \Pi$ is a test for Γ, $?A \vdash !B$, Δ^-, and since this sequent is regular, $\Lambda \vdash \Pi$ is provable.

So $\Gamma \vdash !(A \to B)$, Δ^- is weakly regular, hence regular by B.3.11 (ii).

(b) $?$-introd or $!$-elim. Let us treat the case of $! \to$-elim. Suppose $\Gamma \vdash ?A$, Δ^- and Γ, $!B \vdash \Delta$ are regular, et let $\Lambda \vdash \Pi$ be a test for Γ, $!(A \to B) \vdash \Delta$.

One checks easily that $\Lambda \vdash \Pi^-$, $?A$ is a test for $\Gamma \vdash ?A$, Δ^-; similarly, Λ, $!B \vdash \Pi$ is a test for Γ, $!B \vdash \Delta$. So $\Lambda \vdash ?A, \Pi^-$ and Λ, $!B \vdash \Pi$ are cut-free provable. It follows that Λ, $!(A \to B) \vdash \Pi$ is cut-free provable. Now observe that, since $\Lambda \vdash \Pi$ is a test for Γ, $!(A \to B) \vdash \Delta$, $!(A \to B) \in \Lambda$. By contraction, get a cut-free proof of $\Lambda \vdash \Pi$.

B.3.13. Definition. The sequent $\Gamma \vdash \Delta$ is said to be *completely regular* iff every instanciation of this sequent is regular.

B.3.14. Proposition.

(i) *Every axiom is completely regular.*

(ii) *Completely regular sequents are closed under the logical rules for* $\forall$ *and* $\exists$.

Proof. (ii) is proved similarly to B.3.12 (ii).

(i) Let us prove that (AX 3) is completely regular. (Other axioms are simpler to handle). It amounts to show that whenever the *PR* formulas $P_1, \ldots, P_n$ are such that $P_1, \ldots, P_n \vdash t = u$ is provable in Q, then $\xi_1 P_1, \ldots, \xi_n P_n$, $\xi A(t) \vdash \eta A(u)$ is regular, for $\xi = !$ or $\eta = ?$. Let $\Lambda \vdash \Pi$ be a weak test for this sequent. We claim that $\Gamma, \Lambda \vdash \Pi$ is cut-free provable, with $\Gamma = \xi_1 P_1, \ldots, \xi_n P_n$:

1° $\xi = ?$; then $\eta = ?$, and $\Pi = \Pi'$, $?A(u)$; also $\Lambda \vdash (\Pi)^-$, $?A(t)$ is cut-free provable and, by Lemma 2, Γ, $\Lambda \vdash (\Pi)^-$, $?A(u)$ is cut-free provable;

2° $\eta = !$, hence $\xi = !$, and $\Lambda = \Lambda'$, $!A(t)$; observe that $\Lambda, !A(u) \vdash \Pi$ is cut-free provable, so, by Lemma 2, $\Lambda, !A(t) \vdash \Pi$ is cut-free provable;

3° $\xi = !$ and $\eta = ?$; then $!A(t) \in \Lambda$ and $?A(u) \in \Pi$; since $\Gamma, !A(t) \vdash ?A(u)$ is cut-free provable (Lemmas 1 and 2), Γ, $\Lambda \vdash \Pi$ is cut-free provable.

Now, if $\xi_i P_i \notin \Lambda$, then $\xi_i = ?$, and $\Lambda \vdash ?P_i, \Pi^-$ is cut-free proved from axioms $\eta_1 Q_1, \ldots, \eta_p Q_p \vdash ?P_i$. For each such axiom we can apply the above proof to $\Gamma' = \xi_1 P_1, \ldots, \xi_{i-1} P_{i-1}, \eta_1 Q_1, \ldots, \eta_p Q_p, \xi_{i+1} P_{i+1}, \ldots, \xi_n P_n$; then we conclude that $\Gamma', \Delta \vdash \Pi$ is cut-free provable; from this we get a cut-free proof of $\Gamma^i, \Delta, \Delta \vdash \Pi$, then of $\Gamma^i, \Delta \vdash \Pi$, with $\Gamma^i = \Delta - \{?A_i\}$.

So, by induction on the number of ξ_i's equal to ?, one shows that $!\Gamma, \Delta \vdash \Pi$ is cut-free provable, and so is $\Delta \vdash \Pi$, since $!\Gamma \subset \Delta$.

B.3.15. THEOREM. *Every provable sequent is completely regular.*

Proof. Straightforward induction on the length of proofs, using B.3.12 and B.3.14.

Theorem B.3.4 is an immediate corollary of B.3.15, since every completely regular sequent is regular, and regular sequents are cut-free provable.

COROLLARY. *The consistency of modal theories M^1LC and M^1LI is a theorem of elementary arithmetic.*

Proof. Consistency means that some sequent is not provable. Clearly $\vdash ?\perp$ is provable iff it is cut-free provable, and this sequent is cut-free provable iff $\vdash \perp$ is a theorem of the consistent calculus Q.

Remark. Consistency of M^1LC does not mean that for no A either $?A$ or $?\neg A$ is not provable; in fact if A is provable formula of G^1LC which is absorbing (see part C), then we have a proof of $?A$ in M^1LC, while the non-derivability of $?\neg A$ implies the consistency of G^1LC.

B 4. Takeuti's conjecture

Takeuti's conjecture for G^1LC and G^1LI is the question: "Is every provable sequent cut-free provable?"

B.4.1. THEOREM. *Takeuti's conjecture holds for G^1LC and G^1LI.*

Proof. Let $\boldsymbol{M} = (\boldsymbol{M}_0, M_1, \epsilon, H)$ be a three-valued topological model for $G'LI$. Let $\boldsymbol{M}' = (\boldsymbol{M}_0, M_1', \epsilon', H)$ be the binary topological model defined as follows: Let H' be the algebra of the open sets of H; then ϵ is a function from $\boldsymbol{M}_0 \times \boldsymbol{M}_1$ into $H' \times H'$. If $S \epsilon M_1$, then define two functions $!S$ and $?S$ from $M_0 = |\boldsymbol{M}_0|$ into H': $!S(a) = \pi_1(\epsilon(a, S))$ and $?S(a) = \pi_2(\epsilon(a, S))$, where π_1 and π_2 are the first and second projections.

Define M_1' to consist of those functions C from M_1 into H' such that:

(i) for all a, b in M_1, $\boldsymbol{M}(?a = b) \subset (C(a) \Rightarrow C(b))$,

(ii) there is an S in M_1 s.t. for all a in $\boldsymbol{M}_0$, $!S(a) \subset C(a) \subset ?S(a)$.

ϵ' is just the application function: $\epsilon'(a, S) = S(a)$.

Now, define a thinness relation $\leqslant$ between $\boldsymbol{M}$ and $\boldsymbol{M}'$:

– for first order objects, $a \leqslant b$ iff a is identical to b;

– for second order objects, $S \leqslant C$ iff for all a in $\boldsymbol{M}_0$, $!S(a) \subset C(a) \subset ?S(a)$.

In order to make sure that $\leqslant$ is a thinness relation, we have to verify

(1) $a \leqslant b$ and $S \leqslant C$ implies $\boldsymbol{M}(a \in S) < \boldsymbol{M}'(b \in C)$. Trivial;

(2) for all C in M_1' there is an S in M_1 s.t. $S \leqslant C$; this is just (ii);

(3) for all S in M_1 there is a C in M_1' s.t. $S \leqslant C$; take $C = !S$.

By the thinness theorem we get the following property:

(TH) *Let* $a_1, \ldots, a_n = \underline{a}$ *in* M_0, $C_1, \ldots, C_m = \underline{C}$ *in* M_1', $S_1, \ldots, S_m = \underline{S}$ *in* M_1 *be such that* $S_i \leqslant C_i$ $(i = 1, \ldots, m)$, *and let* $A(x_1, \ldots, x_n, X_1, \ldots, X_m)$ *be a formula with no free variables other than those listed; then*

$$\boldsymbol{M}\big(!A(\underline{a}, \underline{S})\big) \subset \boldsymbol{M}'\big(A(\underline{a}, \underline{C})\big) \quad \textit{and} \quad \boldsymbol{M}'\big(A(\underline{a}, \underline{C})\big) \subset \boldsymbol{M}\big(?A(\underline{a}, \underline{S})\big),$$

$\boldsymbol{M}'$ is clearly a structure for G^1LI satisfying the PR axioms and equality axioms. It remains only to show that $\boldsymbol{M}'$ satisfies comprehension.

Let $A(\underline{a}, \underline{C}, x)$ be a formula of $L^1(\boldsymbol{M}')$, and let $\underline{S}$ be such that $\underline{S} \leqslant \underline{C}$; then, by (TH), for all b in M_0, $\boldsymbol{M}\big(A(\underline{a}, \underline{S}, b)\big) \leqslant \boldsymbol{M}'\big(A(\underline{a}, \underline{C}, b)\big)$.

Let C_0 be the function defined by $C_0(b) = \boldsymbol{M}'\big(A(a, C, b)\big)$. We claim that C_0 belongs to M_1':

(i) is fulfilled, since $\boldsymbol{M}'$ satisfies the equality axioms,

(ii) holds, because, since $\boldsymbol{M}$ is a model, there is an S_0 such that $\boldsymbol{M}(b \in S_0) = \boldsymbol{M}\big(A(\underline{a}, \underline{C}, b)\big)$ for all b in M_0; clearly $S_0 \leqslant C_0$.

Now, Theorem B.4.1 follows easily:

Intuitionistic case. Suppose A is provable in G^1LI, where A is a closed formula, and let $\boldsymbol{M}$ be a three-valued topological model. Since $\boldsymbol{M}'$ is a binary model, A is true in $\boldsymbol{M}'$, and, by (TH), possible in $\boldsymbol{M}$. So we have proved that A is possible in every 3-valued topological model, and, by completeness, $?A$ is provable in M^1LI.

Classical case. If A is provable in G^1LC, then if $\boldsymbol{M}$ is a 3-valued classical model (that is $\mathrm{Card}(H) = 1$), A is true in $\boldsymbol{M}'$, since $\boldsymbol{M}'$ is a classical model and, by (TH), A is possible in $\boldsymbol{M}$. Applying completeness, we have shown that $?A$ is provable in M^1LC.

So we have proved Takeuti's conjecture for sequents consisting of one closed formula. For the general case it suffices to remark that, in binary logic, $\Gamma \vdash \Delta$ is provable without cut iff the universal closure of $\bigwedge\Gamma \to \bigvee\Delta$ is cut-free provable. Q.E.D.

Remarks. 1° Everything in this proof (except the fact that $\boldsymbol{M}'$ is a model for the full comprehension) can be formalized in quite weak subsystems of analysis.

2° This proof is not essentially different from the proofs of Takahashi and Prawitz, provided we translate these proofs in a 3-valued model theoretic language; this is discussed in the appendix. The proof of Tait is essentially different from these proofs. A simplified version of Tait's proof, which is of independant interest will be given in C5.

C. Applications to the metamathematics of cut-free analysis

In the sequel, we shall let D vary over C and I, when we do not want to specify if we are in the classical or in the intuitionistic case.

Besides the already defined M^1LD, G^1LD, D-models, we shall define a lot of D-concepts, which will become C- or I-concepts when necessary. In C5, C6, only classical notions are needed, and the "C" will be dropped everywhere it is possible.

C.0.1. Definition.

(i) UAD is short for G^1LD.

(ii) $CFAD$ is the cut-free part of UAD $(= G^1LD)$.

C.0.2. Definition.

(i) We define the predicate $\boldsymbol{N}(x)$ ("x is an integer") by

$$\boldsymbol{N}(x) : \forall x\big(0 \in X \wedge \forall y (y \in X \rightarrow Sy \in X) \rightarrow x \in X\big).$$

(ii) *Numerical* quantifiers $\forall n$, $\exists n$ $(\forall m, \exists m, \ldots)$ are defined by

$$\forall n A(n) : \forall x\big(\boldsymbol{N}(x) \rightarrow A(x)\big),$$
$$\exists n A(n) : \exists x\big(\boldsymbol{N}(x) \wedge A(x)\big).$$

(iii) The formula A is said to be *analytical* if all its 1st order quantifiers are numerical.

(iv) Numerical abstraction λn $(\lambda m, \ldots)$ is defined by

$$\lambda n A(n) : \lambda x\big(\boldsymbol{N}(x) \wedge A(x)\big).$$

(v) Σ_p^0 (resp. Π_p^0) formulas are formulas of the kind $Q_1 n_1 \ldots Q_p n_p A$, where A is quantifier-free, the Q_i's are numerical, $Q_i \neq Q_{i+1}$, and $Q_1 = \exists$ (resp. $Q_1 = \forall$). Σ_p^0 and Π_p^0 formulas are analytical.

Induction is not provable in UAD. However, one can prove $\forall n\big(A(0) \wedge \wedge \forall m\big(A(m) \rightarrow A(Sm)\big) \rightarrow A(n)\big)$, where $\forall n$ and $\forall m$ are numerical. It follows that the induction schema is conservative w.r.t. closed analytical formulas.

C.0.3. Definition. $A^\forall$ (resp. $A^\exists$) will denote the universal (resp. existential) closure of the formula A.

For the next definition, remark that is always possible to choose the set of PR functions and predicates of UAD (with their defining equations) in such a way that any given in advance finite set of recursive primitive predicates (depending possibly on the language and axioms of UAD) is represented in UAD by PR formulas.

C.0.4. Definition. Suppose we have a standard assignment $A \rightarrow \ulcorner A \urcorner$ of Gödel numbers for formulas (and also for terms, proofs...). Then $\mathrm{Prov}_{UAD}(x, y)$ (resp. $\mathrm{Prov}_{CFAD}(x, y)$) is short for a PR formula of language

L^1 which expresses that x is the Gödel number of a proof of the formula with Gödel number y in UAD (resp. $CFAD$). $UAD\vdash\ulcorner A\urcorner$ will be short for the Σ^0_1 statement $\exists n \mathrm{Prov}_{UAD}(n, \ulcorner A\urcorner)$. Similarly for $CFAD$.

C.0.5. Definition. We define CFD by:

$$CFD(\ulcorner A\urcorner) : (UAD\vdash\ulcorner A\urcorner \to CFAD\vdash\ulcorner A\urcorner) \qquad (CFD(x) \text{ is } \Delta^0_2).$$

(CFD is the local form of Takeuti's conjecture for the formula A, when formalized.)

C.0.6. Definition.

(i) F_0 will be short for the formula $\forall x \forall X (x \in X \to x \in X)$;

(ii) A model of M^1LC is *proper* iff it is not binary.

C.0.7. Definition. In the sequel we shall denote by AA the subsystem of UAC obtained by restricting $\forall_2$-elim. and $\exists_2$-introd. to the case where the term T is of the form $\lambda n A$, where A is either atomic, or Π^0_2 without free second order variables.

(Roughly speaking, AA is Peano arithmetic with induction restricted to Π^0_2 formulas); notice that

– cut-elimination for M^1LD is provable in AA,

– one can develop elementary model theory in AA, and prove (2 or 3-ary) completeness for M^1LC and G^1LC. More precisely:

Define a sequence (M^n) of models of M^1LC, with the property that they all have the same domains, definable in AA by a PR formula, and a Δ^0_2 predicate $VAL(n, a)$ which asserts that the modal formula with Gödel number a in $ML^1(M^n)$ is valid in M^n.

Trivially, in AA, one cannot prove that $VAL(n)$ is equivalent with satisfaction in M^n. However, one can manage in order to have:

– for every closed ξA in ML^1, there is a proof in AA of

$$\forall n \big(VAL(n, \ulcorner \xi A\urcorner) \leftrightarrow M_n \vdash \ulcorner \xi A\urcorner\big),$$

– $\forall m \big(M^1LC\vdash m \leftrightarrow \forall n (VAL(n, m))\big)$ is provable in AA.

The second statement is a form of the completeness theorem.

From these remarks, it will be clear that many model theoretic arguments are formalizable in AA.

C 1. Poor and absorbing formulas

C.1.1. Definition. The formula A is said to be *D-absorbing* iff every model of M^1LD which is a model of $!A^3$ is binary.

Examples.

1° If A is refutable in UAD, then A is D-absorbing (see B4).

2° Binary structures for M^1LD are exactly models of $!F_0$. In particular F_0 is D-absorbing.

So, A is D-absorbing iff every model of M^1LD which is a model of $!A^{\exists}$ is a model of $!F_0$; using completeness, A is D-absorbing iff $!A^{\exists}\vdash !F_0$ is provable in M^1LD. Using the rules for existence, one can give a new Σ_1^0 definition of being absorbing:

C.1.2. Definition. A is D-absorbing iff $!A\vdash !F_0$ is provable in M^1LD.

Absorbtion lemma (provable in AA). *Suppose A is absorbing; then for any formula B, $UAD\vdash B\to CFAD\vdash A\to B$.*

Proof. 1° semantic proof (in AA): if $A\to B$ is not cut-free provable, then one must find a 3-valued D-model of M^1LD which is a model of $!A^{\exists}$, and which is not a model of $?B^{\forall}$. But this model is to be binary, and B is not provable.

2° syntactic proof (in AA): G^1LD is obtained from M^1LD by adding the sequents $?t\in X\vdash !t\in X$. Since $!F_0$, $?t\in X\vdash !t\in X$ is provable in M^1LD, $UAD\vdash B\to M^1LD\vdash(!F_0\vdash ?B)$, and, applying cut, $M^1LD\vdash(!A\vdash ?B)$. So $A\to B$ is cut-free provable.

Corollary (provable in AA). *$UAD\vdash B$ iff $CFAD\vdash F_0\to B$.*

Proof. If $CFAD\vdash F_0\to B$, then $UAD\vdash F_0\to B$, and, since $UAD\vdash F_0$, $UAD\vdash B$.

The formula $N(t)$ is D-absorbing: suppose $!N(t)$; then $!\big(F_0\wedge\forall y(F_0\to\to F_0)\to F_0\big)$, and, since $?\big(F_0\wedge\forall y(F_0\to F_0)\big)$, we get $!F_0$.

C.1.3. Definition. A is *D-poor* iff $?(F_0\to A)\vdash ?A$ is provable in M^1LD.

Examples.

1° If A is provable in UAD, then A is D-poor (see B4).

2° Classically, $?(F_0\to A)\vdash ?A$ is provable iff $?(\neg F_0\vee A)\vdash ?A$ is provable, and this sequent is provable iff $?\neg F_0\vdash ?A$ is provable; so A is C-poor iff A is possible in every proper three-valued C-model of UAC. Equivalently, A is C-poor iff $\neg A$ is C-absorbing.

Poverty lemma (provable in AA). *If A is D-poor, then $CFD(A)$ holds.*

Proof. Suppose A is provable in UAD, and A is D-poor; then $?(F_0\to A)$ is provable in M^1LD, and, applying cut, $?A$ is provable in M^1LD, hence A is cut-free provable in UAD, that is A is provable in $CFAD$.

C 2. A candidate for synonymity

C.2.1. Definition.

(i) The relation $A \Rightarrow_D B$ is defined by: $A \Rightarrow_D B$ iff $?A\vdash ?B$ and $!B\vdash !A$ are provable in M^1LD.

(ii) $A \equiv_D B$ is defined by: $A \equiv_D B$ iff $A \Rightarrow_D B$ and $B \Rightarrow_D A$. A and B are said to be *D-congruent* iff $A \equiv_D B$.

LEMMA (provable in AA).

(i) *Let* $A \Rightarrow_D B$, *and let* F' *be obtained from* F *by replacing some occurrences of instanciations of* A *by the corresponding occurrences of the corresponding instanciations of* B. (*For instance*, $F = C \vee \forall x A(Sx)$ *and* $F' = C \vee \forall x B(Sx)$.) *Then* $F \Rightarrow_D F'$.

(ii) *If* $A \Rightarrow_D B$, *then* $A \leftrightarrow B$ *is a theorem of* $CFAD$.

(iii) *If* $A \equiv_D B$, *and* A *is* D*-poor* (*resp.* D*-absorbing*), *then* B *is* D*-poor* (*resp.* D*-absorbing*).

Proof. Trivial.

D-congruence is a way of expressing that two formulas have the same "meaning" from the viewpoint of cut-elimination, since congruent formulas must have the same value in all 3-valued models.

Clearly, D-congruence is strictly stronger than provable equivalence in UAD or $CFAD$.

Now we have in mind a vague notion of *synonymity*, roughly speaking, two formulas are synonymous iff they *literally express* the same property. Clearly, if synonymity makes sense, then this relation must be strictly stronger than provable equivalence. On the other hand, it is reasonable to assume that synonymous formulas have the same behaviour with respect to cut-elimination (from a very elementary viewpoint), that is that synonymous formulas are D-congruent.

In fact, there are many reasons, for instance:

– undecidability of $\equiv_D$,

– the fact that classical cut-elimination is provable in AA for "most" of the formulas (see C6) to be convinced that D-congruence is only an approximation to synonymity. (See also [1].)

PROPERTIES OF $\equiv_D$ (in AA).

1° $\equiv_D$ is an equivalence relation stronger than provable equivalence in UAD or $CFAD$.

2° If F and F' are built up from A and B in the same manner as in the lemma above, then $A \equiv_D B \rightarrow F \equiv_D F'$.

3°

(i) $A \wedge (B \vee C) \equiv_I (A \wedge B) \vee (A \wedge C)$,

(ii) $A \vee (B \wedge C) \equiv_I (A \vee B) \wedge (A \vee C)$,

(iii) $(A \wedge B) \rightarrow C \equiv_I A \rightarrow (B \rightarrow C)$,

(iv) $(A \vee B) \rightarrow C \equiv_I (A \rightarrow C) \wedge (B \rightarrow C)$,

(v) $A \wedge A \equiv_I A \vee A \equiv_I A$,

(vi) $A \rightarrow (B \wedge C) \equiv_I (A \rightarrow B) \wedge (A \rightarrow C)$,

(vii) $A \rightarrow \forall x B \equiv_I \forall x (A \rightarrow B)$; $A \rightarrow \forall X B \equiv_I \forall X (A \rightarrow B)$,

(viii) $\exists x A \rightarrow B \equiv_I \forall x (A \rightarrow B)$; $\exists X A \rightarrow B \equiv_I \forall X (A \rightarrow B)$,

$$\begin{array}{rll}
\text{(ix)} & \forall x A \wedge B & \equiv_I \forall x(A \wedge B); \quad \forall X A \wedge B \equiv_I \forall X(A \wedge B),\\
\text{(x)} & A \wedge B & \equiv_I B \wedge A; \quad A \vee B \equiv_I B \wedge A,\\
\text{(xi)} & \forall x A \to B & \equiv_C \exists x(A \to B); \quad \forall X A \to B \equiv_C \exists X(A \to B),\\
\text{(xii)} & \neg\neg A & \equiv_C A,\\
\text{(xiii)} & (A \wedge B) \to C & \equiv_C (A \to C) \vee (B \to C),\\
\text{(xiv)} & A \to (B \vee C) & \equiv_C (A \to B) \vee (A \to C),\\
\text{(xv)} & \neg(A \wedge B) & \equiv_C \neg A \vee \neg B,\\
\text{(xvi)} & A \to B & \equiv_C \neg A \vee B,\\
\text{(xvii)} & A \to \exists x B & \equiv_C \exists x(A \to B); \quad A \to \exists X B \equiv_C \exists X(A \to B),
\end{array}$$

and so on; remark that (xii)–(xvii) are intuitionistic congruences if one takes the double negations.

In particular, every usual step leading from a formula to its prenex form preserves classical congruence. But it is important to remark that this is not true for prenex analytical transformations, for instance, one may give a counterexample to $\forall n(A(n) \wedge B) \equiv_C \forall n A(n) \wedge B$, taking $A = \perp \to \perp$ and $B = \perp$: the left-hand side is poor, while the right-hand side is false in every 3-valued model for M^1LC.

4° If $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ are sequents of M^1LD, such that $\Gamma' \vdash \Delta'$ is obtained from $\Gamma \vdash \Delta$ by replacing some occurrences of instanciations of A by occurrences of the corresponding instanciations of B, then

$$A \equiv_D B \to \big(M^1LD(\Gamma \vdash \Delta) \leftrightarrow M^1LD(\Gamma' \vdash \Delta')\big).$$

In particular, if F and F' are as in 1°, then $A \equiv_D B \to (CFAD \vdash F \leftrightarrow CFAD \vdash F')$.

The properties of $\Rightarrow_D$ are quite similar; this is a preorder with associated equivalence $\equiv_D$. The main property of $\Rightarrow_D$ is that, whenever $A \Rightarrow_D B$, and F and F' are as in 1°, then $CFAD \vdash F \to CFAD \vdash F'$.

Application to translations of connectives and quantifiers.

1° Classically, any of the usual translations preserves congruence; so it is really correct to work on a fragment of the calculus, since every property of a formula from the view point of cut-elimination can be obtained by means of the similar property of the translation of the formula.

2° One may ask a similar question for the translations of $\wedge$, $\vee$, $\exists$ given essentially by Prawitz in the intuitionistic case. The answer is negative, that is these translations do not preserve congruence: for instance, let $(A \vee B)'$ be $\forall X \forall x\big((A \to x \in X) \to \big((B \to x \in X) \to x \in X\big)\big)$; then clearly $(A \vee B)'$ is I-absorbing, while $A \vee B$ is not necessarily absorbing. However, $A \vee B \Rightarrow_I (A \vee B)'$ and similarly for the other Prawitz translations; so, in AA, we can prove cut-elimination for the Prawitz translation A' of A, under the hypothesis of cut-elimination for A; but we cannot prove the converse (and it is only the converse property which would be interesting): for instance consider the Prawitz translation B of $\neg\exists x(\perp\to\perp)$; the formula B is poor, so we can prove cut-elimination for B in AA. But cut-elimination for $\neg\exists x(\perp\to\perp)$ is not provable in UAC, since it implies consistency. In other words, the "separation principle" does not hold elementarily for Prawitz translations. Prawitz translations do not preserve classical congruence, and we cannot use them for cut-elimination, even in the classical case.

C 3. Syntactic conditions for poverty

Since poverty implies an elementary cut-elimination theorem, it is interesting to list simple syntactic conditions which ensure that a formula is poor. We consider only D-poverty; we get C-poverty by dropping the $\neg\neg$'s.

(i) If A is D-absorbing, then $A\to B$ is D-poor.

(ii) If A is D-poor, $\forall xA$ and $\forall XA$ are D-poor.

(iii) $\forall nA(n)$ is D-poor (proof: by (i) $N(x)\to A$ is poor, so, by (ii) $\forall nA(n)$ is D-poor).

(iv) If A is D-poor, then $\neg\neg\exists xA$, $\neg\neg\exists nA$, $\neg\neg\exists XA$, $\neg\neg(A\vee B)$ are D-poor.

Proof (for $\neg\neg\exists nA$, $\neg\neg(A\vee B)$). Suppose $?(F_0\to A)\vdash ?A$ is provable in M^1LD; then $?\neg F_0\vdash ?A(0)$ is also provable, as well as $\vdash ?N(0)$, so, by $\wedge$ and $\exists$-introd., $?\neg F_0\vdash ?\exists nA(n)$ is provable. Then $?\neg\neg(\neg F_0\vee\exists nA(n))$ $\vdash ?\neg\neg\exists nA(n)$ is provable. Now, $\neg\neg(\neg A\vee B)$ is D-congruent to $\neg\neg(A\to\neg\neg B)$, and $?(F_0\to\neg\neg\exists nA(n))\vdash ?\neg\neg\exists nA(n)$ is provable.

Similarly, if A is D-poor, then $?\neg F_0\vdash ?A$ is provable in M^1LD; so, $?(\neg F_0\vee(A\vee B))\vdash ?(A\vee B)$ is provable; from this, we get a proof of $?(F_0\to\neg\neg(A\vee B))\vdash ?\neg\neg(A\vee B)$.

(v) If A and B are D-poor, then $A\wedge B$ is D-poor.

The most surprising fact about these conditions is certainly (iv): in the classical system, if A is poor, then $\exists nA$ and $A\vee B$ are poor; however, in the intuitionistic system, the poverty of A does not imply the poverty of $\exists nA$ and $A\vee B$, since every provably recursive function of analysis can be computed from cut-free proofs of existential formulas $\exists nA(n,\bar{p})$, and we can manage to choose A poor (by adding an unneccessary universal numerical quantifier), and, clearly no cut-elimination procedure for such a formula can be carried out in AA, unless the function is provably recursive in AA. Similarly for the disjunction, using provably recursive predicates.

C 4. Takeuti's conjecture and $\sum_1^0$-reflection

Let $A(x) = \exists nP(n, x)$ be a universal Σ_1^0 formula. We define

$\mathrm{Refl}_D(x) =_{\mathrm{def}} UAD \vdash \ulcorner A(\bar{x})\urcorner \to A(x)$ and

$$\mathrm{Refl}_D^-(x) =_{\mathrm{def}} CFAD \vdash \ulcorner A(\bar{x})\urcorner \to A(x),$$

Σ_1^0-Refl_D is the formula $\forall n \mathrm{Refl}_D(n)$, while TCD is Takeuti conjecture $\big(TCD =_{\mathrm{def}} \forall n CFD(n)\big)$.

THEOREM.

(i) $AA \vdash \forall n \mathrm{Refl}_D^-(n)$;

(ii) *There exists a PR function f (which is supposed to be among the PR functions represented in UAD by a function letter) such that*

$$AA \vdash \forall n \big(\mathrm{Refl}_D(fn) \to CFD(n)\big);$$

(iii) $AA \vdash \Sigma_1^0 \mathrm{Refl}_D \leftrightarrow TCD$;

(iv) $AA \vdash TCD \leftrightarrow \forall n CFD\big(\ulcorner A(f\bar{n})\urcorner\big)$.

(i) states that Σ_1^0-reflection for cut-free provability is a theorem of AA.

(ii) states that every particular case of Takeuti's conjecture can be obtained in AA from a specific case of Σ_1^0-reflection.

(iii) states that Takeuti's conjecture is equivalent in AA with Σ_1^0-reflection (recall that Σ_1^0-reflection is exactly 1-consistency of UAD).

(iv) states that the whole Takeuti conjecture is equivalent with the part of this conjecture which is concerned only with Σ_1^0-formulas.

Proof. (i) Suppose $A(\bar{n})$ is cut free provable in UAD; then there is a proof of $?\exists y\big(N(y) \wedge P(y, \bar{n})\big)$ in M^1LD. The sequent $?\exists y(A \wedge B) \vdash ?\exists yB$ is clearly provable in M^1LD, so, applying cut, we get a proof of $?\exists yP(y, \bar{n})$ in M^1LD, and also a cut-free proof of $\exists yP(y, \bar{n})$ in UAD; such a proof must be purely first order, and by Π_2^0-induction, we conclude that $\exists mP(m, n)$ is true.

(ii) let B be a formula with Gödel number n; if B is provable in UAD, then B is provable in a finitely axiomatizable subsystem $UAD^{(p)}$. Now, Theorem B.4.1 can be proved in UAD for the subsystem $UAD^{(p)}$. Summing up these facts, we get $\forall n\big(UAD \vdash n \to UAD \vdash (CFAD \vdash n)\big)$. Let f be such that $CFAD \vdash n \leftrightarrow A(fn)$. Then $\mathrm{Refl}_D\big(A(fn)\big) \to CFAD \vdash n$. So, $\forall n\big(\mathrm{Refl}_D(fn) \to CFD(n)\big)$.

(iii) by (ii) Σ_1^0-$\mathrm{Refl}_D \to TCD$; on the other hand, $TCD \to \forall n\big(UAD \vdash \ulcorner A(\bar{n})\urcorner \to CFAD \vdash \ulcorner A(\bar{n})\urcorner\big)$, so, $TCD \to \forall n\big(\mathrm{Refl}_D^-(n) \to \mathrm{Refl}_D(n)\big)$, and, by (i) $TCD \to \Sigma_1^0$-Refl_D.

(iv) as in (iii), we see that $\forall n CFD\big(\ulcorner A(f\bar{n})\urcorner\big) \to \forall n \mathrm{Refl}_D(fn)$, and we apply (ii).

Remarks. (i) was first proved by Kreisel and Takeuti [1]. The essential step is to pass from a cut-free proof of $\exists nQ(n)$ to a cut-free proof of $\exists xQ(x)$, and, if we use usual methods, this may be quite long in the classical case. The use of modal sequents makes this step completely trivial.

Kreisel and Takeuti have shown also (ii), (iii) and (iv), with TCD' instead of TCD, where TCD' is Takeuti's conjecture for prenex analytical statements.

The results of C5 will give a new proof of (ii), (iii) and (iv), provided we restrict TCD to TCD'', where TCD'' is Takeuti's conjecture for formulas which are equivalent to an analytical formula.

C 5. Cut-elimination in the classical case

From now on, we speak only about classical systems; so we allow the C in the end of CFC, $CFAC$, UAC to be dropped.

First, let us restate the proof of Theorem B.4.1 in the classical case: We start from a given 3-valued model $(\boldsymbol{M}_0, M_1, \epsilon)$, where $\boldsymbol{M}_0$ is binary. For any S in M_1, define two subsets $!S$ and $?S$ of M_0, as follows:

$$\xi S = \{x \in M_0 \colon \boldsymbol{M} \vDash \xi x \in S\}.$$

Now, the binary model $\boldsymbol{M}'$ is defined as follows: $\boldsymbol{M}' = (\boldsymbol{M}_0, M_1', \epsilon)$, where ϵ is the standard ϵ-relation, and M_1' is the subset of $\mathscr{P}(M_0)$ consisting of those subsets C of M_0 which satisfy $!S \subset C \subset ?S$ for some S in M_1. Now, we can prove that $\boldsymbol{M} \vDash !A \rightarrow \boldsymbol{M}' \vDash A$ and $\boldsymbol{M}' \vDash A \rightarrow \boldsymbol{M} \vDash ?A$ for every closed A, and also verify, using full comprehension that $\boldsymbol{M}'$ is a model for UA.

Let us look again to $\boldsymbol{M}'$; in fact two cases occur:

(i) $\boldsymbol{M}$ is already binary; then there is nothing to prove,

(ii) $\boldsymbol{M}$ is a proper model; in that case, we claim that $\boldsymbol{M}'$ is the *principal model with first order part* $\boldsymbol{M}_0$: let T_0 be $\lambda x F_0$; then $!T_0 = \emptyset$ while $?T_0 = \boldsymbol{M}_1$; looking back to the definition of M_1', we see that $M_1' = \mathscr{P}(M_0)$.

Let us call $\boldsymbol{M}_0^*$ the principal model with first order part $\boldsymbol{M}_0$. Then, we have proved that, whenever A is true in $\boldsymbol{M}_0^*$, then A is possible in every proper three-valued model $\boldsymbol{M}$ with first order part $\boldsymbol{M}_0$. Moreover, if A is analytical, A is true in $\boldsymbol{M}_0^*$ iff A is true (in the standard model). So, for closed analytical statements, we have a proof of $A \rightarrow$"A is possible in every proper three-valued model", that is $A \rightarrow$"A is poor".

Now, we formalize this informal proof in AA.

C.5.1. Theorem. *Let A be closed analytical statement; then*

(i) $AA \vdash \forall \boldsymbol{M}$ $(\boldsymbol{M}$ *is a proper model* $\rightarrow$ $(A \rightarrow \boldsymbol{M} \vdash \ulcorner ?A \urcorner))$,

(ii) $AA \vdash A \to \text{Poor}(\ulcorner A \urcorner)$ (*where* Poor *is a* Σ_1^0*-formula which is an arithmetization of* (*classical*) *poverty*).

Proof. (ii) follows from (i) by completeness (locally provable in AA).

(i) Let M be a proper 3-valued model; define $S \leqslant_M X$ (for S in M_1, X second order variable) by $\forall n\big((M \vDash !\bar{n} \in S \to n \in X) \wedge (n \in X \to M \vDash ?\bar{n} \in S)\big)$. If $\underline{S}, \underline{X}$ are the sequences $S_1, \ldots, S_q, X_1, \ldots, X_q$, then $\underline{S} \leqslant_M \underline{X}$ stands for $\bigwedge_{i \leqslant q} (S_i \leqslant_M X_i)$.

Now, for every analytical $A(x_1, \ldots, x_p, \underline{X})$, with no other free variable, one gives a proof in AA of $(TH_A) = \forall M \forall n_1 \ldots n_p \forall \underline{S} \forall \underline{X}$ $\big(M$ is a proper model $\wedge\, \underline{S} \leqslant_M \underline{X} \to \big(M \vDash !A(\bar{n}_1, \ldots, \bar{n}_p, \underline{S}) \to A(n_1, \ldots, n_p, \underline{X})\big) \wedge$ $\wedge \big(A(n_1, \ldots, n_p, \underline{X}) \to M \vDash ?A(n_1, \ldots, n_p, \underline{S})\big)\big)$.

1° A is atomic: $Pt_1 \ldots t_m$; one can prove $\forall n_1 \ldots n_p \exists k_i \big(t_i(n_1, \ldots, n_p) = k_i\big)$; and also $M \vDash ?P\bar{k}_1 \ldots \bar{k}_n \leftrightarrow Pk_1 \ldots k_n$ for all numerical $k_1 \ldots k_n$, that is by induction over $k_1 \ldots k_n$. Now, TH_A follows trivially.

2° A is atomic: $t \in X_i$: similarly to 1°, using the fact that $S_i \leqslant_M X_i$.

3° A is $B \wedge C$; then $TH_B \wedge TH_C \to TH_A$.

4° A is $\neg B$; then $TH_B \to TH_A$, since if $M \vDash !C \to C \to M \vDash ?C$, then $M \vDash !\neg C \to \neg C \to M \vDash ?\neg C$.

5° A is $\forall n B(n)$; let us drop the auxiliary variables in A. First remark that A is poor, so A is possible in M (since M is not binary). On the other hand, suppose $M \vDash !\forall n B(n)$, that is $M \vDash !\forall x\big(N(x) \to B(x)\big)$; remark that $\forall n\big(M \vDash ?N(\bar{n})\big)$ is a theorem of AA, and conclude that $\forall n ! M \vDash B(\bar{n})$; it follows by TH_B that $\forall n B(n)$.

6° A is $\forall X B(X)$; we drop the auxiliary variables; first, if $M \vDash !\forall X B(X)$, then $M \vDash !B(S_0)$, where S_0 is an interpretation of $\lambda x F_0$; since M is proper, $\forall X(S_0 \leqslant_M X)$, and, by TH_B, one concludes that $\forall X B(X)$; on the other hand, suppose that $\forall X B(X)$ holds, and let $S \in M_1$; define $X = \{n; N(n) \wedge$ $\wedge M \vDash (\bar{n} \in S)\}$; the existence of X is provable in AA, and clearly $S \leqslant_M X$; so, by TH_B, $B(X) \to M \vDash ?B(S)$. Conclude now that $\forall X B(X) \to M \vDash ?\forall X B(X)$.

7° other connectives are unnecessary, since one may first prove the theorem for translations, and use the fact that translations preserve congruence.

Theorem C.5.1 may be obtained from a more general result applying to arbitrary formulas. If M is a three-valued model, and A is a formula with parameters in M_0, containing no free 1st order variable, define A^M as follows: $(Pt_1 \ldots t_n)^M = M \vDash ?Pt_1 \ldots t_n$; $(t \in X)^M = M(t) \in X$; $(A \wedge B)^M = A^M \wedge B^M$; $(\neg A)^M = \neg A^M$; $(\forall x A)^M = \forall a\big(a \in M_0 \to (A(a))^M\big)$; $\big(\forall X A(X)\big)^M = \forall X A(X)^M$. Then A^M depends only on M_0.

C.5.2. Lemma. *If A is a closed analytical formula, then $A^M \leftrightarrow A$ is provable in AA.*

The proof is trivial; A^M says that A is true in the principal model M_0^*.

C.5.3. Theorem.

(i) *If A is a closed formula, then* $AA \vdash \forall \boldsymbol{M}\big(\boldsymbol{M}$ *is proper* $\to (A^{\boldsymbol{M}} \to \boldsymbol{M} \models ?A)\big)$.

(ii) *Suppose* $UA \vdash A \leftrightarrow B$, *$B$ is analytical and A is closed; then* $UA \vdash A \to \mathrm{Poor}(\ulcorner A \urcorner)$.

Proof. (i) is proved in the same way as Theorem C.5.1, using an appropriate induction hypothesis.

(ii) one may suppose that B is closed and analytical. By (i), if $\boldsymbol{M}$ is proper, then $A^{\boldsymbol{M}} \to \boldsymbol{M} \models ?A$; now, remark that $UA \vdash C \to UA \vdash C^{\boldsymbol{M}}$ (by induction over proofs of C). So $A^{\boldsymbol{M}} \leftrightarrow B^{\boldsymbol{M}}$ is provable in UA. But, since B is a closed analytical statement, $B^{\boldsymbol{M}} \leftrightarrow B$, so $A^{\boldsymbol{M}} \leftrightarrow B$ is provable, and since $A \leftrightarrow B$ is a theorem, we have a proof in UA of $A^{\boldsymbol{M}} \leftrightarrow A$. So $A \to \boldsymbol{M} \models ?A$ for every proper $\boldsymbol{M}$, and we conclude that $A \to \mathrm{Poor}(\ulcorner A \urcorner)$.

C 6. The poverty theorem

Poverty theorem (1st form). *There exists a PR function which associates to every closed analytical formula A a closed Σ_1^0 formula E_A such that*

(i) $AA \vdash E_A \to CF(\ulcorner A \urcorner)$.

(ii) $AA \vdash A \to E_A$.

(iii) *If A is a closed Σ_1^0 formula, then* $AA \vdash E_A \leftrightarrow A$.

Proof. Take $E_A = \mathrm{Poor}(\ulcorner A \urcorner)$. (i) and (ii) follow respectively from the poverty lemma (C1) and C.5.1.

(iii) If A is closed and Σ_1^0, then we can prove $A \to CFA \vdash \ulcorner A \urcorner$, which is the arithmetization of the fact that every true Σ_1^0 formula is provable in elementary arithmetic, hence provable in CFA. Now, $CFA \vdash \ulcorner A \urcorner \to M^1LC \vdash ?A$, so $CFA \vdash \ulcorner A \urcorner \to \mathrm{Poor}(\ulcorner A \urcorner)$. Conversely suppose A is poor; then we have a proof of $? \neg F_0 \vdash ?A$ in M^1LC. Then one can show that A is provable in UA: either we use Löb's theorem [2] since poverty of A implies reflection for A, or we interprete M^1LC into a binary language L' whose second order domain is the set of pairs $(X^!, X^?)$, with $X^! \subset X^?$ and two ϵ-relations $\epsilon^!$ and $\epsilon^?$, defined by $t \epsilon \xi(X^!, X^?) = t \epsilon X^{\xi}$, as we did for the completeness theorem (A4).

L' is interpretable in UA, and, from a proof of $? \neg F_0 \vdash ?A$, we get a proof of $\neg \forall X^! \forall X^? (X^! \subset X^? \to X^? \to X^!) \vdash \exists x \big(\boldsymbol{N}'(x) \wedge B(x)\big)$, where $B(x)$ is the quantifier-free part of A, and $\boldsymbol{N}'(x) = \forall X^! \forall X^? \big(X^! \subset X^? \to \big(0 \epsilon X^! \wedge \wedge \forall y (y \epsilon X^? \to Sy \epsilon X^!) \to x \epsilon X^?\big)\big)$.

Since the premise of this sequent is provable in UA, $\exists x \big(\boldsymbol{N}'(x) \wedge B(x)\big)$ is provable in UA. Now observe that $\boldsymbol{N}'(x) \to \boldsymbol{N}(x)$, and conclude that $UA \vdash A$.

Since A is provable in UA and A is poor, then A is provable in CFA. By Σ_1^0-reflection for cut-free provability, A is true. So $\mathrm{Poor}(\ulcorner A \urcorner) \to A$.

Let S be a system for analysis; call a formula A *1-consistent* with respect to S iff the following is true: for any closed Σ_1^0 formula B, if $A \to B$ is provable in S, then B is true. If S is 1-consistent, then A is 1-consistent with respect to S iff $S \cup \{A\}$ is 1-consistent.

POVERTY THEOREM (2nd form) (under the assumption of 1-consistency of UA).

(i) *If A is a closed analytical formula which is 1-consistent with respect to AA, then A is poor. In particular, Takeuti's conjecture for A is provable in AA.*

(ii) *If A is a closed formula which is provably equivalent in UA to an analytical formula, and if A is 1-consistent with respect to UA, then A is poor, and Takeuti's conjecture for A is provable in AA.*

Proof. (i) Since A is 1-consistent with respect to AA, from $A \to \mathrm{Poor}(\ulcorner A \urcorner)$, we conclude that $\mathrm{Poor}(\ulcorner A \urcorner)$ is a theorem of PA. Using metamathematical Σ_1^0-reflection, one concludes that A is poor. Now, if A is poor, then this fact is provable in elementary arithmetic, hence in AA. Now apply the poverty lemma.

(ii) is proved similarly.

Remark. If A is a true analytical statement, then A has no false Σ_1^0 consequence in AA or even in UA. So we have a proof of $CF(\ulcorner A \urcorner)$ in AA. For instance take A to be any instance of the axiom of choice, dependant choice (if we assume that these axioms are true; but this does not matter, since we know that they are 1-consistent) then we have an elementary proof of Takeuti's conjecture for A. Since most of relative consistency proofs are indeed 1-consistency proofs, it seems doubtful that Takeuti's conjecture may have some interest in order to make such proofs.

Now, there are plenty of 1-inconsistent statements which are poor, for instance every false Π_1^0-formula. However, one can state the

POVERTY THEOREM (3rd form).

(i) *If A is a closed analytical formula, then A is 1-consistent with respect to AA iff every closed analytical B (resp. every closed B) such that $AA \vdash A \leftrightarrow B$ is poor.*

(ii) *Similarly with UA instead of AA.*

Proof. (i) let A be 1-consistent with respect to AA. If B is any equivalent of A in AA, then B is also 1-consistent, and one may apply one of the preceding theorems, or a slight modification of it. Conversely if every analytical equivalent of A is poor, let A' be a Σ_1^0 consequence of A in AA. Then A is provably equivalent to $(A \wedge A')$; so $(A \wedge A')$ is poor, that is $?\neg F_0 \vdash ?(A \wedge A')$ is provable in M^1LC; as a consequence $?\neg F_0 \vdash ?A'$ is also provable. Now, (Poverty Theorem, 1st form, (iii)) A' is poor iff A'

is true. We conclude that A' is provable in AA. (ii) is proved in the same way.

COROLLARY.

(i) *Gödel sentences for UA and CFA are poor.*

(ii) *every Gödel sentence of UA is a Gödel sentence of CFA and conversely.*

Proof. For the definition of Gödel sentences for a formal system F, see [1].

(i) A Gödel sentence for UA or CFA is necessarily true. So, such a sentence is poor, since a Gödel sentence is provably equivalent to a Π_1^0 formula. (ii) As a consequence, for these Gödel sentences, they are provable iff they are cut-free provable, and this can be shown in AA, hence in UA. So, $(UA\vdash\ulcorner A\urcorner)$ is provably equivalent to $(CFA\vdash\ulcorner A\urcorner)$.

Remark. The corollary appears in [1] in the course of the study of Gödel sentences. Although the authors make an essential use of the poverty theorem (recorded as II.2.1 in [1]), they had previously proved this corollary in the case of Π_1^0 Gödel sentences.

The poverty theorems give a very simple proof of Takeuti's conjecture for formulas which are provably equivalent to a closed analytical formula, from Σ_1^0-reflection: let A be such a formula; then $UA\vdash\big(A\to\text{Poor}(\ulcorner A\urcorner)\big)$, and also $UA\vdash\big(UA\vdash\ulcorner\big(A\to\text{Poor}(\ulcorner A\urcorner)\big)\urcorner\big)$. Now, we conclude that $UA\vdash(UA\vdash\ulcorner A\urcorner\to UA\vdash\ulcorner\text{Poor}(\ulcorner A\urcorner)\urcorner\big)$. So, $UA\vdash\big(\Sigma_1^0\text{-Refl}\wedge UA\vdash\ulcorner A\urcorner\to\text{Poor}(\ulcorner A\urcorner)\big)$, since poverty is Σ_1^0. Now, $\text{Poor}(\ulcorner A\urcorner)\wedge UA\vdash\ulcorner A\urcorner\to CFA\vdash\ulcorner A\urcorner$ is a theorem of UA, so $UA\vdash\Sigma_1^0\text{-Refl}\to CF(\ulcorner A\urcorner)$.

Appendix

Here, we discuss briefly the proofs of Tait, Takahashi and Prawitz. We consider also what happens for type theory and ω-logic.

1. Tait's proof [9]

(Evidently, we discuss the proofs in *our* terminology.)

Most of the proof is contained in the last half-page (p. 983). He starts with a *proper* 3-valued model $\boldsymbol{M}=(\boldsymbol{M}_0, M_1, \epsilon)$, and considers the binary model $\boldsymbol{M}'=(\boldsymbol{M}_0, M_1', \epsilon)$, where ϵ is the standard ϵ-relation, and M_1' is an arbitrary subset of $\mathscr{P}(M_0)$ closed under comprehension, and containing the set M_1^0, defined by $M_1^0=\{!S;\, S\in M_1\}$ (recall that $!S=\{c\in M_0/\boldsymbol{M}\vDash!(c\in S)\}$. But he does not use anything like a thinness relation between $\boldsymbol{M}$ and $\boldsymbol{M}'$, that is he proves that, if $\underline{S}=S_1,\dots,S_n$ and $!\underline{S}=!S_1,\dots,!S_n$ (so $S_i\in M_1$, $!S_i\in M_1'$), then $\boldsymbol{M}\vDash!A(\underline{S})\to\boldsymbol{M}'\vDash A(!\underline{S})\to\boldsymbol{M}\vDash?A(\underline{S})$, by induction on A.

But, since the relation $!S = C$ is not a thinness relation, the induction hypothesis breaks down for $A = \forall X B(X)$, that is he cannot infer directly $M \vDash !\forall X B(X) \to M' \vDash \forall X B(X)$, from $M \vDash !B(S) \to M' \vDash B(!S)$, since there are presumably some C in M_1' which are not of the form $!S$. What he does in p. 981–982 is to replace the formula $B(X)$ by a formula of no greater complexity $B^{X,\forall}$ which does not depend on X; the properties of $B^{X,\forall}$ are the following:

$$(*) \qquad G^1LC \vdash \big(B^{X,\forall} \vdash B(X)\big),$$

$$(**) \qquad M^1LC \vdash \big(?\neg F_0,\ !\forall X B(X) \vdash !B^{X,\forall}\big),$$

and the induction step for $\forall X B(X)$ follows trivially from (*) and (**).

The use of $B^{X,\forall}$ is of course unnecessary if we take a good induction hypothesis; however, the existence of such a formula is of independant interest, and we explain the construction; in fact it is simpler to define also a formula $B^{X,\exists}$, with the two properties:

$$(*)' \qquad G^1LC \vdash \big(B(X) \vdash B^{X,\exists}\big),$$

$$(**)' \qquad M^1LC \vdash \big(?\neg F_0,\ ?B^{X,\exists} \vdash ?\exists X B(X)\big).$$

If A is atomic and distinct from $t \in X$, then $A^{X,Q} = A$, $(t \in X)^{X,\forall} = \perp$; $(t \in X)^{X,\exists} = (0 = 0)$, $(A \wedge B)^{X,Q} = (A^{X,Q}) \wedge (B^{X,Q})$; $(\neg A)^{X,Q} = \neg(A^{X,\bar{Q}})$; $(Q'xA)^{X,Q} = Q'x(A^{X,Q})$ and $(Q'YA)^{X,Q} = Q'Y(A^{X,Q})$ (Y distinct of X, Q' arbitrary quantifier). Then (**) and (**)' are easily obtained from proofs of

$$M^1LC \vdash \big(?\neg F_0,\ !B(\lambda x F_0) \vdash !B^{X,\forall}\big) \quad \text{and} \quad M^1LC \vdash \big(?\neg F_0,\ ?B^{X,\exists} \vdash ?B(\lambda x F_0)\big).$$

Roughly speaking, the existence of $B^{X,\forall}$ says that if a formula $\forall X B(X)$ is necessary in a proper model, then the occurrences of X are not useful in the computation of the formula, and we can remove them, and still have a necessary formula.

2. Prawitz's first proof ([4])

The idea of Prawitz is to start with M_1^0 (as defined above), and define second order domains M_1 by transfinite induction:

$$(*) \qquad M_1^{a+1} = \mathrm{Def}(M_1^a),$$

$$(**) \qquad M_1^\lambda = \bigcup_{a<\lambda} M_1^a \quad \text{for limit } \lambda,$$

where $\mathrm{Def}(M_1^a)$ is the set of subsets C of M_0 s.t. there is a closed term $\lambda x A$ (with parameters in $M_a = (M_0, M_1^a, \epsilon)$) such that $C = \{c \in M_0;\ M_a \vDash A(c)\}$.

Obviously, for some β_0, the construction must be stationary; so M' is defined to be $M\beta_0$, and clearly M' is a model of G^1LC.

Since Prawitz does not use thinness conditions explicitely, he gives names for the objects of M' in M, without changing M significatively, but this amounts exactly to give thinness conditions. Then he shows by induction on α that M and M_α satisfy the thinness theorem, and this also show that for each level α the elements of $\mathrm{Def}(M_1)$ are "between" some $!S$ and $?S$, for an S in M_1, and this is essentially what we need in order to have a thinness relation between M and $M_{\alpha+1}$.

We think that this proof is less informative than Tait's, essentially because the binary model constructed here depends on M_1^0, while in Tait's proof, the binary model can be taken principal.

3. Prawitz's third proof ([6])

This is a proof of Hauptsatz for G^1LI. The proof involves essentially what we would call a *three-valued Kripke model*. (**K** is a *3-valued Kripke model* iff **K** is set of classical 3-valued structures indexed by an ordered set, and satisfying monotonicity conditions for domains, necessity and possibility of atomic formulas.)

4. The ω-rule

The ω-rule obtained by considering only formulas with no free 1st order variables, and, for instance in M^1LC, the $\forall$-introd rule

$$\frac{\ldots \Gamma \vdash \xi A(\bar{n}),\ \Delta \ldots}{\Gamma \vdash \xi \forall x A(x),\ \Delta} \quad (n < \omega)$$

while the $\forall$-elim rule is restricted to closed first order terms.

Now, deducibility in ω-systems ω-G^1LC, ω-M^1LC is Π_1^1, as well as poverty.

The models are the same as in part B, except that M_0 is now the standard structure 3 for the integers. Cut-elimination for ω-M^1LC is a theorem of ω-G^1LC, as well as completeness for ω-M^1LC. Now, if we complete a model for ω-M^1LC, then we get the standard model. Since this proof is easily carried out in ω-G^1LC, we conclude that

$$\omega\text{-}G^1LC \vdash A \rightarrow (\omega\text{-Poor}(\ulcorner A \urcorner)) \quad \text{for every closed formula } A.$$

(Trivially, one may replace ω-G^1LC by an essentially weaker system.) Now, if A has no false Π_1^1-consequence, then ω-Poor$(\ulcorner A \urcorner)$ is true, hence provable in ω-G^1LC, and the statement ω-$CF(\ulcorner A \urcorner)$ is also provable in this system (indeed in a quite weak system of ω-logic).

5. The simple theory of types

Prawitz ([5]) and Takahashi ([10]) gave proofs of cut elimination for the simple theory of types. These proof essentially use

(i) Schütte's theorem about valuations,

(ii) the construction of a binary model of type theory, together with a thinness relation defined at each type. However, the model they construct can be non-extensional (the reason is that for instance there may be an object C^2 of type 2 in the model s.t. $!(\lambda x F_0) \epsilon C^2$, while for a certain C^1 $!\neg(C^1 \epsilon C^2)$. So, they have to introduce new sets at each level, and also keep the information "from which element in the 3-valued model is this set introduced". The relativization of this proof to G^1LC yields essentially the proof we have given in B.4.1.

But, if we consider the problem of cut-elimination in extensional simple type theory, we have to consider the rules

$$\frac{\Gamma,\ !(X^i \epsilon T^{i+1}) \vdash ?(X^i \epsilon U^{i+1}),\ \Delta \qquad \Gamma,\ !(X^i \epsilon U^{i+1}) \vdash ?(X^i \epsilon T^{i+1}),\ \Delta}{\Gamma,\ !(T^{i+1} \epsilon Y^{i+2}) \vdash ?(U^{i+1} \epsilon Y^{i+2}),\ \Delta}$$

which are completely regular, (X^i not free in Γ, Δ, T, U).

Now, if we start with a model of these rules, then, we get an extensional model: for each i, define ξ^i and $S^i \leqslant C^i$ by $!S^0 = S^0$; $S^0 \leqslant C^0 = (S^0 = C^0)$

$$\xi S^{i+1} = \{C^i;\ \exists S^i \text{ in } M,\ S^i \leqslant C^i \text{ and } M \vDash \xi(S^i \epsilon S^{i+1})\},$$
$$S^{i+1} \leqslant C^{i+1} = \forall S^i \forall C^i \big(S^i \leqslant C^i \to M(S^i \epsilon S^{i+1}) \leqslant (C^i \epsilon C^{i+1})\big).$$

Then $\leqslant$ is a thinness relation, if we define the ith domain of M' to consist of those C^i such that $S^i \leqslant C^i$ for S^i in the ith domain of M. The only non-trivial point is to show that for all S^i there is C^i such that $S^i \leqslant C^i$ (this is this point which fails in the non-extensional case): indeed, choose $C^i = !S^i$. Suppose $i = j+1$, and let $M \vdash !(S^j \epsilon S^i)$; then if $S^j \leqslant C^j$, $C^j \epsilon\, !S^i$.

If $M' \vDash (C^j \epsilon S^i)$, that is $C^j \epsilon S^i$, there is a T^j such that $M \vDash !(T^j \epsilon S^i)$ and $T^j \leqslant C^j$. Let $S^j \leqslant C^j$; then $M \vDash ?(T^j = S^j)$ (where $=$ is the extensional equality). So, by the extensionality rule, $M \vDash ?(S^j \epsilon S^i)$.

Now, if we start with a proper model, we construct a principal model, that is level $i+1$ is the power set of level i. Now if we consider analytical statements (in the sense that quantifiers of type i are restricted to $\mathscr{P}\big(\dots(\mathscr{P}(N))\dots\big)$ (i times), a closed analytical statement which is true is poor (and this extends to formulas which contain abstraction terms, provided these terms are restricted to some iterated power set of N), and this result is indeed provable in extensional simple type theory, that is, with obvious notations;

$$E - G^\omega LC \vdash A \to (E - M^\omega LC)\text{-Poor}(\ulcorner A \urcorner) \qquad \text{for any closed } A$$

whose quantifiers and abstraction terms are hereditarily restricted to N.

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