POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCLXXV

WEN SONG

Duality in set-valued optimization

WARSZAWA 1998

Wen Song
Department of Mathematics
Harbin Normal University
150080 Harbin, China
and
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: song@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by
Publishing House of the Warsaw University of Technology
ul. Polna 50, 00-644 Warszawa

P R I N T E D I N P O L A N D

ISSN 0012-3862

CONTENTS

1991 *Mathematics Subject Classification*: 90C29, 90C26, 90C30.
Received 3.11.1997; revised version 15.1.1998.

Introduction

In scalar mathematical programming duality means that a maximization problem can be associated with a given minimization problem in such a way that both problems have the same optimal values. Duality theory can also be developed in vector optimization. The aim of this dissertation is to present a systematic study of duality for vector optimization problems with set-valued mappings. We consider the following optimization problems:

$$\text{(P)} \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in X_0, \end{array}$$

where X_0 is a nonempty subset of a topological vector space X, F is a set-valued mapping from X to another topological vector space Y. Problems of type (P) are a natural generalization of the classical vector optimization problems:

$$\text{(VP)} \qquad \begin{array}{l} \min f(x) \\ \text{subject to } x \in X_0 = \{x \in A \mid g(x) \in -Q\}, \end{array}$$

where A is a nonempty subset of X, Q is a convex cone of a topological vector space Z, and $f : X \to Y$, $g : X \to Z$ are vector-valued functions. Together with the classical approaches to duality such as Lagrangian and conjugate duality, we also develop a geometric approach to duality. In [88], [100], [71], [12] and [3], the authors consider duality for the problem (VP) and formulate the dual problems in the form (P), i.e. the dual problem has a set-valued objective. In Fenchel duality for convex vector optimization, Breckner [9] and Gros [31] formulate the dual problems in the form (P). In [51], Kawasaki develops a duality theory where the objectives of both the primal and the dual problem are set-valued mappings. Optimization of set-valued mappings arises, for instance, if the data of a given problem are not known exactly so that it makes sense to replace a vector objective by a set-valued objective representing outcomes.

Duality for linear vector optimization problems has been discussed in [29], [40], [42] and [70]. For nonlinear vector problems with single-valued objectives, duality has been extensively studied by many authors. Lagrangian type duality has been studied in [100], [61], [41], [15], [79], [6], [7], [12], [24], [58], [19] and [87]); Fenchel–Rockafellar type duality has been discussed in [9], [106], [83], [31], [6], [105], [101], [87], [99] and others; on the basis of the duality approaches proposed in [103], [88] or [74], several duality results were also developed in [3], [71], [42] or [69]. An overview of several duality concepts of vector optimization is given in [70].

Since many optimization problems encountered in economics and other fields involve set-valued constraints and set-valued objective mappings, optimization of set-valued mappings has attracted a great deal of attention in recent years. Lagrangian type duality results in an infinite-dimensional setting for convex and/or generalized convex set-valued mappings have been obtained as generalizations of the corresponding results in the single-valued case (see [13], [22], [53], [60], [62], [63], [90], [91], [92], [95], [76]). Fenchel–Rockafeller type duality has been discussed, for instance, in [51], [7], [77], [78], [62], [39], [65], [93], [94]. Wolfe and Mond–Weir type duality theorems for invex set-valued mappings have been proved in [85], [86], [64]. An axiomatic approach to duality is also developed by Luc and Jahn [63].

This dissertation is organized as follows. In Section 1, we present some definitions and properties concerning convexity of sets and set-valued mappings. In Section 2, we give the main concepts of vector optimization such as minimal, weak minimal, proper minimal point (or solution), and we also give characterizations of optimal points (solutions) via scalarization. In Section 3, under some regularity conditions, optimality conditions for vector optimization problems with constraints are formulated by using scalar-valued as well as vector-valued Lagrangian mappings. In Section 4, based on the Lagrangian formalism of Section 3, we consider Lagrangian duality for the weak minimality, minimality and proper minimality, and we formulate the main duality results, Theorems 4.1–4.3. In Section 5, we present a general duality principle for sets. As an application, we study a geometric approach to duality for closure convexlike set-valued mappings and prove the main result (Theorem 5.1) of this section. We also derive a duality result for linear problems by specializing Theorem 5.1. In Section 6, we present a general conjugate duality result (Theorem 6.2) and we also provide some sufficient conditions ensuring the conjugate duality, i.e. stability criteria. This allows us to obtain a conjugate duality result for convexlike set-valued mappings under closedness or boundedness hypotheses (Corollaries 6.3, 6.4). When set-valued mappings are convex, our duality result can be proved without any closedness and boundedness requirements. Finally, a Fenchel type duality result is presented by specializing Theorem 6.2.

The results of Sections 2, 3, 4, 6 are based on the author's papers [90], [91], [92], [93], [94], [95]. The results of Section 5 have not been published elsewhere.

For more motivations and developments concerning vector optimization, we refer the reader to Kuhn and Tucker [56], Hurwicz [38], Stadler [96], Dauer and Stadler [20], Sawaragi, Nakayama and Tanino [87], Jahn [44], Luc [62], and the references therein.

I am very grateful to Professor S. Rolewicz and Dr. E. Bednarczuk for their valuable comments.

1. Preliminaries on convex and set-valued analysis

In this section, we provide some preliminaries concerning the convexity of sets. Also we recall some notions of generalized convexity with respect to a given cone for sets and set-valued mappings and show the relations among these convexity concepts.

1.1. Convexity of sets. Let C be a subset of a real Hausdorff topological vector space Y. Let $\overline{C}$ (resp. $\operatorname{int} C$) denote the closure (resp. the interior) of C.

DEFINITION 1.1. A subset C of Y is said to be:

(i) *convex* if, for any $y_1, y_2 \in C$ and $\alpha \in (0,1)$,

$$\alpha y_1 + (1-\alpha) y_2 \in C;$$

(ii) *nearly convex* if there exists $\alpha \in (0,1)$ such that, for any $y_1, y_2 \in C$,

$$\alpha y_1 + (1-\alpha) y_2 \in C;$$

(iii) *closure convex* if its closure $\overline{C}$ is convex.

In particular, if $\alpha = 1/2$ in (ii), we say that C is *midpoint convex.*

The definition of nearly convexity is due to [47]. It is clear that if C is convex, then it is also nearly convex, but not conversely.

PROPOSITION 1.1. *Let C be a subset of Y. If C is nearly convex, then C is closure convex.*

PROOF. See Proposition 2.1 of [47]. ■

Consequently, if C is closed and nearly convex, then C is convex. In [75], it has been proved that if C is open and nearly convex, then C is also convex.

The following Lemma is a fundamental result on the convexity of sets.

LEMMA 1.1. *Let $C \subset Y$ be a convex subset. Then its closure $\overline{C}$ is convex. If, moreover, $\operatorname{int} C \neq \emptyset$, then $\operatorname{int} C$ is convex and*

$$\overline{C} = \overline{\operatorname{int} C}; \quad \operatorname{int} C = \operatorname{int} \overline{C}.$$

PROOF. See [37]. ■

Among the class of convex sets the convex cones play a very important role in vector optimization and some other fields (see [87], [44], [62], [39] and [54]).

DEFINITION 1.2. A subset S of Y is called a *cone* if, for any $y \in S$ and $\lambda \geq 0$, $\lambda y \in S$. A cone S is *pointed* if $S \cap (-S) = \{0\}$.

Hence, S is a convex cone if and only if $\lambda S \subset S$ for all $\lambda \geq 0$ and $S + S = S$.

Let B be a nonempty subset of Y. The cone

$$\operatorname{cone}(B) = \{\lambda b \mid \lambda \geq 0, b \in B\}$$

is called *the cone generated by B*. By $\overline{\operatorname{cone}}(B)$ we denote the closure of $\operatorname{cone}(B)$.

LEMMA 1.2. *Let $B \subset Y$ be a convex subset and $y_0 \in B$. Then $\overline{\operatorname{cone}}(B - y_0)$ is a closed convex cone.*

PROOF. We begin by stating the following consequence of convexity:

$$\forall y \in \operatorname{cone}(B - y_0),\ \exists h > 0, \forall t \in [0,h], \quad y_0 + ty \in B,$$

since we can write that for any $t \in [0,h]$,

$$y_0 + ty = \left(1 - \frac{t}{h}\right) y_0 + \frac{t}{h}(y_0 + hy)$$

is a convex combination of elements of B.

For any $y_1, y_2 \in \operatorname{cone}(B - y_0)$ and $\alpha \in [0,1]$ there exists $h > 0$ such that $y_0 + \alpha h y_1 \in B$ and $y_0 + (1-\alpha) h y_2 \in B$. Hence $\alpha y_1 + (1-\alpha) y_2 \in \frac{B - y_0}{h} \subset \operatorname{cone}(B - y_0)$. Therefore, $\operatorname{cone}(B - y_0)$ is convex, and so $\overline{\operatorname{cone}}(B - y_0)$ is also convex by Lemma 1.1. ∎

Let B be a subset of Y. For a given point $y_0 \in \overline{B}$, the *sequential tangent cone* $T(B, y_0)$ for the set B at the point y_0 is defined (see [4]) by

$$T(B, y_0) = \{v \in Y \mid v = \lim \lambda_n (y_n - y_0),\ \lambda_n \geq 0,\ y_n \in B,\ y_n \to y_0,\ n \in \mathbb{N}\}.$$

When Y is metrizable, $T(B, y_0)$ is closed, but need not be closed in general spaces (see [4], [5]). By $\overline{T}(B, y_0)$ we denote the closure of the cone $T(B, y_0)$.

LEMMA 1.3. *If* $B \subset Y$ *is a convex subset, then for any* $y_0 \in B$ *we have*

$$\overline{T}(B, y_0) = \overline{\operatorname{cone}}(B - y_0).$$

PROOF. See Theorems 3.43 and 3.44 of [44]. ∎

From Lemmas 1.2 and 1.3, one can see that if Y is metrizable, then $T(B, y_0)$ is a closed convex cone.

Let S be a convex cone of Y. A subset B of S is called a *base* for S if B is convex, $0 \notin \overline{B}$, and $S = \operatorname{cone}(B)$.

The following cone separation theorem has been given in [19].

LEMMA 1.4. *Let* Y *be a locally convex space*, P *and* Q *be cones in* Y, *and* $P \cap Q = \{0\}$. *Assume that one of the following conditions holds.*

(a) P *is weakly closed and* Q *has a weakly compact base*;

(b) P *is closed and* Q *has a compact base.*

Then there is a pointed convex cone K *such that* $Q \setminus \{0\} \subset \operatorname{int} K$ *and* $P \cap K = \{0\}$. *Moreover*, *if* Y *is normable*, *then* K *can be chosen closed and with a closed bounded base.*

DEFINITION 1.3. Let S be a convex cone of Y. A subset C of Y is said to be:

(i) S-*convex* if $C + S$ is convex;

(ii) *nearly* S-*convex* if $C + S$ is nearly convex;

(iii) *closure* S-*convex* if $\overline{C + S}$ is convex.

The definition of S-convex sets has been introduced in [104] and the definitions of (ii) and (iii) can be found in [10]. When $S = \{0\}$, Definition 1.3 reduces to Definition 1.1. It is clear that if C is S-convex, then it is also nearly S-convex and consequently closure S-convex. The following example shows that the converse implication is not generally true.

EXAMPLE 1.1. 1. Let

$$C = \{(y_1, y_2) \in \mathbb{R}^2 \mid -y_1 < y_2 \leq 0,\ 0 < y_1 < 1\} \cup \{(0,0), (1,-1)\}.$$

The set C is closure $\mathbb{R}^2_+$-convex, but it is not nearly $\mathbb{R}^2_+$-convex.

2. Let $C_1 = C \cup \{(q, -q) \mid q \in \mathbb{Q} \cap [0,1]\}$, where $\mathbb{Q}$ denotes the set of all rational numbers. Then C_1 is nearly $\mathbb{R}^2_+$-convex, but it is not $\mathbb{R}^2_+$-convex.

LEMMA 1.5. *Let S be a convex cone in Y with* $\operatorname{int} S \neq \emptyset$ *and let* $C \subset Y$*. Then*

$$\overline{C+S} = \overline{C + \operatorname{int} S}, \tag{1.1}$$

$$\operatorname{int} \overline{C+S} = C + \operatorname{int} S. \tag{1.2}$$

PROOF. By Lemma 1.1, we have

$$\overline{C+S} = \overline{C+\overline{S}} = \overline{C+\overline{\operatorname{int} S}} = \overline{C+\operatorname{int} S}.$$

Let $y \in \operatorname{int} \overline{C+S}$. Then there exists a neighbourhood U of zero such that $y - U \subset \overline{C+S}$. We choose some $s \in \operatorname{int} S$ and a sufficiently small $\alpha > 0$ such that $\alpha s \in U$. Hence $y - \alpha s \in \overline{C+S}$. Consequently, we have $(y - \operatorname{int} S) \cap (C+S) \neq \emptyset$. Thus $y \in C+S+\operatorname{int} S \subset C + \operatorname{int} S$. The arbitrariness of y implies that

$$\operatorname{int} \overline{C+S} \subset C + \operatorname{int} S.$$

Since the converse inclusion is obvious, we obtain (1.2). ■

REMARK 1.1. From Lemma 1.5, one observes that closure S-convexity, closure ($\operatorname{int} S$)-convexity and ($\operatorname{int} S$)-convexity are equivalent.

1.2. Convexity of set-valued mappings. In this subsection, we recall various types of convexity for set-valued mappings (see [62], [90], [95], [57]). For the case of single-valued mappings, the reader is referred to [98]. Let X, Y be real Hausdorff topological vector spaces with topological dual spaces X^*, Y^*, respectively. Let $S \subset Y$ be a convex cone. The *dual cone* S^+ and its *quasi-interior* S^{+i} are defined as

$$S^+ = \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0 \text{ for all } y \in S\},$$
$$S^{+i} = \{y^* \in Y^* \mid \langle y^*, y\rangle > 0 \text{ for all } y \in S \setminus \{0\}\},$$

where $\langle\, ,\, \rangle$ is the canonical bilinear form with respect to the duality between Y^* and Y.

The Krein–Rutman theorem (see [55]) states that, if S is a closed convex and pointed cone of a separable normed space, then $S^{+i} \neq \emptyset$. It is clear that, if $S^{+i} \neq \emptyset$, then S is pointed. Moreover, it is easy to show that if $S^{+i} \neq \emptyset$, then S has a base. Indeed, if $S^{+i} \neq \emptyset$, then for every $y^* \in S^{+i}$, the set $\{y \in S \mid \langle y^*, y\rangle = 1\}$ is a base of S. In locally convex spaces, the converse is true, i.e., if S has a base then the Hahn–Banach theorem implies $S^{+i} \neq \emptyset$.

Let $F : X \to Y$ be a set-valued mapping. Denote by $\operatorname{gr} F$ and $\operatorname{dom} F$ the *graph* and *domain* of F, respectively,

$$\operatorname{gr} F = \{(x,y) \mid y \in F(x)\}, \quad \operatorname{dom} F = \{x \mid F(x) \neq \emptyset\}.$$

If $\operatorname{dom} F = X$, we say that F is *strict*. We denote the *range* of F by $R(F) := \{y \mid y \in F(x) \text{ for some } x \in X\}$. The *inverse set-valued mapping* F^{-1} is defined as follows: $x \in F^{-1}(y)$ if $y \in F(x)$. For any set A in X, $F(A) = \bigcup_{x\in A} F(x)$ so that $R(F) = F(X)$.

DEFINITION 1.4. Let $A \subset X$ be a convex set. A set-valued mapping $F : X \to Y$ is said to be:

(i) S-*convex* on A if, for all $x_1, x_2 \in A$, $y_1 \in F(x_1), y_2 \in F(x_2)$ and $\lambda \in (0,1)$,

$$\lambda y_1 + (1-\lambda) y_2 \in F(\lambda x_1 + (1-\lambda)x_2) + S;$$

(ii) *naturally S-quasiconvex* on A if, for all $x_1, x_2 \in A$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $\eta \in [0,1]$ such that

$$\eta y_1 + (1-\eta)y_2 \in F(\lambda x_1 + (1-\lambda)x_2) + S;$$

(iii) *S-quasiconvex* on A if, for all $x_1, x_2 \in A$, if $y \in F(x_1) + S$ and $y \in F(x_2) + S$, then

$$y \in F(\lambda x_1 + (1-\lambda)x_2) + S \quad \text{for all } \lambda \in (0,1);$$

(iv) *$(*, S)$-quasiconvex* on A if, for every $y^* \in S^+ \setminus \{0\}$, the extended real-valued function $x \to \phi_{y^*}(x) = \inf_{y \in F(x)} \langle y^*, y\rangle$ is quasiconvex, i.e. for every $r \in \mathbb{R}^1$ the lower level set $\{x \in A \mid \phi_{y^*}(x) \le r\}$ is convex;

(v) *S-convexlike* on A if, for all $x_1, x_2 \in A$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$,

$$\lambda y_1 + (1-\lambda)y_2 \in F(A) + S;$$

(vi) *closure S-convexlike* on A if, for all $x_1, x_2 \in A$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$,

$$\lambda y_1 + (1-\lambda)y_2 \in \overline{F(A) + S}.$$

In the definitions (v) and (vi), the set A is not necessarily convex. In the sequel, when no confusion occurs, we omit S- in the above notions of convexity of set-valued mappings.

We shall give characterizations of some notions of convexity for set-valued mappings.

PROPOSITION 1.2. (a) *F is S-convex on A if and only if the epigraph of F*, $\operatorname{epi} F = \{(x,y) \in A \times Y \mid y \in F(x) + S\}$, *is a convex subset of $X \times Y$*;

(b) *F is convexlike* [*resp. closure convexlike*] *on A if and only if $F(A)$ is S-convex* [*resp. closure S-convex*];

(c) *F is quasiconvex if and only if for every $y \in Y$, $A \cap F^{-1}(y - S)$ is convex.*

From Definition 1.4, it is easy to see that the following relations hold among these notions of convexity for set-valued mappings.

PROPOSITION 1.3. (a) *Each S-convex set-valued mapping is also naturally quasiconvex and convexlike*;

(b) *Each naturally quasiconvex set-valued mapping is also quasiconvex and $*$-quasiconvex*;

(c) *Each convexlike set-valued mapping is also closure convexlike.*

PROPOSITION 1.4. *Assume that Y is a locally convex space and $F(x) + S$ is closed and convex for all $x \in A$. If $F : A \to Y$ is a $*$-quasiconvex set-valued mapping*, *then it is also naturally quasiconvex.*

PROOF. See Theorem 2.1 of [57]. ∎

From Propositions 1.4 and 1.3(b), it is easy to see that if F has compact convex images, then $*$-quasiconvexity of F is equivalent to natural quasiconvexity.

DEFINITION 1.5. A set-valued mapping $F : X \to Y$ is said to be *upper semicontinuous at $x_0 \in X$* if, for every open set V containing $F(x_0)$, there exists a neighbourhood U of

x_0 such that

$$F(x) \subset V \quad \text{for all } x \in U.$$

F is called *upper semicontinuous* if F is upper semicontinuous at every point $x \in X$.

PROPOSITION 1.5. *Assume that S is closed and F is upper semicontinuous with convex images. If F is naturally quasiconvex on A, then it is convexlike.*

PROOF. See Theorem 2.2 of [57]. ∎

EXAMPLE 1.2. 1. Let $X = \mathbb{R}^1$, $Y = \mathbb{R}^2$, $A = [-3,3]$, and let $F : A \to \mathbb{R}^2$ be the mapping defined by

$$F(x) = (f_1(x), f_2(x)),$$

where $f_1(x) = \min\{x, 2\}$ and $f_2(x) = \min\left\{\frac{3}{4}x, 1\right\}$. It is obvious that F is not convex on A with respect to $\mathbb{R}^2_+$. However, it is $*$-quasiconvex on A with respect to $\mathbb{R}^2_+$.

2. Let $F : [-1,1] \to \mathbb{R}^2$ be the set-valued mapping defined by

$$F(x) = \begin{cases} [-x,0] \times \{0\} & \text{if } x \in [0,1]; \\ \{0\} \times [x,0] & \text{if } x \in [-1,0). \end{cases}$$

It is easy to verify that F is $\mathbb{R}^2_+$-quasiconvex on $[-1,1]$ and it is not $*$-quasiconvex.

Observe that if Y is one-dimensional (i.e. F is real-valued), then each mapping $F : X \to Y$ is $\mathbb{R}_+$-convexlike.

EXAMPLE 1.3. Let $X = \mathbb{R}^1$, $Y = \mathbb{R}^2$, $A = [-1,1]$, and let $F : A \to \mathbb{R}^2$ be the set-valued mapping defined by

$$F(x) = \begin{cases} \{-x\} \times [0,1] & \text{if } x \in [-1,0]; \\ \{x\} \times (-x,0] & \text{if } x \in (0,1); \\ \{1\} \times [-1,0] & \text{if } x = 1. \end{cases}$$

It is obvious that F is not convex on A, and

$$\begin{aligned} F(A) + \mathbb{R}^2_+ = {} & \{(r_1,r_2) \in \mathbb{R}^2 \mid r_2 \geq 0,\ r_1 = 0\} \\ & \cup \{(r_1,r_2) \in \mathbb{R}^2 \mid r_2 > -r_1,\ 0 < r_1 < 1\} \\ & \cup \{(r_1,r_2) \in \mathbb{R}^2 \mid r_2 \geq -1,\ r_1 \geq 1\} \end{aligned}$$

is not convex, but

$$\begin{aligned} \overline{F(A) + \mathbb{R}^2_+} = {} & \{(r_1,r_2) \in \mathbb{R}^2 \mid r_2 \geq -r_1,\ 0 \leq r_1 \leq 1\} \\ & \cup \{(r_1,r_2) \in \mathbb{R}^2 \mid r_2 \geq -1,\ r_1 > 1\} \end{aligned}$$

is convex. This means that F is closure convexlike on A and it is not convexlike on A.

Now we recall various generalized convexity conditions for vector-valued functions.

DEFINITION 1.6. A single-valued mapping $f : A \to Y$ is said to be *nearly S-subconvexlike* if there exist $s \in \operatorname{int} S$ and $\beta \in (0,1)$ such that for every $\varepsilon > 0$ and $x_1, x_2 \in A$, there exists $x \in A$ such that

$$\varepsilon s + \beta f(x_1) + (1-\beta) f(x_2) - f(x) \in S. \tag{1.3}$$

The mapping f is said to be:

(i) *S-subconvexlike* if (1.3) holds for all $\beta \in (0,1)$;
(ii) *nearly S-convexlike* if (1.3) holds for $\varepsilon = 0$;
(iii) *S-convexlike* if (1.3) holds for $\varepsilon = 0$ and all $\beta \in (0,1)$.

The condition (1.3) is due to Craven, Gwinner and Jeyakumar [18], and the definitions of (i), (ii) and (iii) are due to Jeyakumar [45], König [52], and Fan [26], respectively.

PROPOSITION 1.6. (i) *f is S-convexlike if and only if $f(A)$ is S-convex*;
(ii) *f is S-subconvexlike if and only if $f(A)$ is $(\operatorname{int} S)$-convex*;
(iii) *f is nearly S-convexlike if and only if $f(A)$ is nearly S-convex*;
(iv) *f is nearly S-subconvexlike if and only if $f(A)$ is nearly $(\operatorname{int} S)$-convex.*

PROOF. For the proof of (i), see Theorem 2.11 of [44]. The proof of the "if" part of (ii) is obvious. For the proof of the "only if" part of (ii), see Theorem 2.1 of [46]. The other statements can be proved similarly. ■

From Remark 1.1 and Proposition 1.6 we see that the conditions (1.3) and (i) and (ii) in Definition 1.6 are all equivalent, and each of them implies that f is closure convexlike as a set-valued mapping.

1.3. Closed convex processes and invex set-valued mappings. In this subsection, we assume that X, Y, Z are normed spaces and A is a subset of X. Let $F : X \to Y$ be a set-valued mapping.

DEFINITION 1.7. F is said to be:

- *closed* if $\operatorname{gr} F$ is closed;
- *convex* if $\operatorname{gr} F$ is convex;
- a *process* if $\operatorname{gr} F$ is a cone.

Hence, a closed convex process is a set-valued mapping whose graph is a closed and convex cone.

DEFINITION 1.8. Let $F : X \to Y$ be a process. Its *transpose* F^* is the closed convex process from Y^* to X^* defined by

$$x^* \in F^*(y^*) \Leftrightarrow \forall x \in X,\ \forall y \in F(x),\ \langle x^*, x\rangle \le \langle y^*, y\rangle. \tag{1.4}$$

If $F = T$ is a linear operator (single-valued), then (1.4) becomes

$$\langle T^*(y^*), x\rangle = \langle y^*, T(x)\rangle.$$

In this case T^* is usually called the *adjoint* (*conjugate*) *operator* of T.

If F is a strict closed convex process, then

$$\operatorname{dom} F^* = F(0)^+ = \{y^* \in Y^* \mid \langle y^*, y\rangle \ge 0 \text{ for all } y \in F(0)\}.$$

We denote by $F|_A$ the *restriction* of F to A, defined by

$$F|_A(x) = \begin{cases} F(x) & \text{if } x \in A; \\ \emptyset & \text{otherwise.} \end{cases}$$

Proposition 1.7. *Let X, Y be Banach spaces, $F : X \to Y$ be a closed convex process and $K \subset X$ be a closed convex cone. Assume that $K - \operatorname{dom} F = X$. Then*

$$(F|_K)^*(y^*) = \begin{cases} F^*(y^*) - K^+ & \text{if } y^* \in \operatorname{dom} F^*; \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. See Corollary 2.5.5 of [1]. ∎

Proposition 1.8. *Let X, Y, and Z be reflexive Banach spaces, $F : X \to Y$, $G : X \to Z$ be closed convex processes and $T \in L(Y, Z)$ be a continuous linear operator. If*

$$T^*(\operatorname{dom} G^*) - \operatorname{dom} F^* = Y^*,$$

then the convex process $TF + G$ is closed.

Proof. See Proposition 2.5.10 of [1]. ∎

For a given point $x \in \overline{A}$, the *contingent cone* $T_A(x)$ is defined by

$$T_A(x) = \{v \in X \mid \liminf_{h \downarrow 0} h^{-1} d_A(x + hv) = 0\},$$

where $d_A(x) = \inf_{y \in A} \|x - y\|$. The cone $T_A(x)$ coincides with the sequential tangent cone $T(A, x)$ defined in the previous subsection whenever X is a normed space.

The *Clarke tangent cone* $C_A(x)$ is defined by

$$C_A(x) = \{v \in X \mid \limsup_{\substack{x' \to x \\ h \downarrow 0}} h^{-1} d_A(x' + hv) = 0\}.$$

Clearly, $T_A(x)$ is a closed cone, $C_A(x)$ is a closed and convex cone, and when A is a convex set, $T_A(x) = C_A(x)$ (see [1]).

For $(x, y) \in \operatorname{gr} F$, define the *circatangent derivative* $CF(x, y) : X \to Y$ as follows:

$$\operatorname{gr} CF(x, y) = C_{\operatorname{gr} F}(x, y).$$

When F is single-valued, $CF(x, y) = CF(x, F(x))$. It is clear that $CF(x, y)$ is a closed convex process.

Definition 1.9. A set-valued mapping F is *locally Lipschitz* at $x_0 \in X$ if, for some constant l and some neighbourhood $U \subset \operatorname{dom} F$ of x_0,

$$\varrho(F(x_1), F(x_2)) \le l\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in U,$$

where $\varrho(\cdot, \cdot)$ denotes the Hausdorff distance (see [1]).

Proposition 1.9. *Let $F : X \to Y$ be a set-valued mapping and A a subset of X. Assume that F is locally Lipschitz at $x_0 \in A$. Then, for any $y_0 \in F(x_0)$,*

$$CF(x_0, y_0)(C_A(x_0)) \subset T_{F(A)}(y_0).$$

Proof. See Proposition 5.3.1 and Definition 5.2.1 of [1]. ∎

When $A = X$, the conclusion of Proposition 1.9 remains true without the requirement of F being locally Lipschitz.

Definition 1.10. A set-valued mapping $F : X \to Y$ is said to be *invex* at $(x_0, y_0) \in \operatorname{gr} F$ if

$$\widehat{F}(X) - y_0 \subset \overline{C\widehat{F}(x_0, y_0)(X)},$$

where $\widehat{F}(x) = F(x) + S$.

For the definitions and some results concerning the invex set-valued mappings, we refer the reader to [84]–[86] and [64].

It is easy to show that if F is S-convex, then it is invex at every point $(x_0, y_0) \in \operatorname{gr} F$. On the other hand, the following example shows that an invex set-valued mapping need not be S-convex.

Example 1.4. 1. Let $X = \mathbb{R}^1$, $Y = \mathbb{R}^2$, $S = \mathbb{R}^2_+$. Define a set-valued mapping as follows:

$$F(x) = \begin{cases} \{y = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 \le x^2\} & \text{if } 0 < x \le 1; \\ \{(-1/2, -1/2), (0, 0)\} & \text{if } x = 0; \\ \emptyset & \text{if } x < 0 \text{ or } x > 1. \end{cases}$$

Clearly, F is not $\mathbb{R}^2_+$-convex. Let $x_0 = 1, y_0 = (-\sqrt{2}/2, -\sqrt{2}/2)$. It is easy to verify that

$$C\widehat{F}(x_0, y_0)(x) = \{y = (\xi_1, \xi_2) \mid \xi_1 + \xi_2 \ge -\sqrt{2}\, x\}$$

and $\widehat{F}(X) - y_0 \subset C\widehat{F}(x_0, y_0)(X)$. Hence, F is invex at (x_0, y_0).

2. Let $X = Y = \mathbb{R}^1$, $S = \mathbb{R}_+$ and $(x_0, y_0) = (0, 0)$. Define a set-valued mapping by

$$F(x) = \begin{cases} \{0\} & \text{if } x = -1; \\ \{-x - k \mid k = 0, 1, 2, \ldots\} & \text{if } x \ge 0; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $\operatorname{epi} F$ is not convex, F is not $\mathbb{R}_+$-convex, but $C_{\operatorname{epi} F}(0, 0) = \{(x, y) \mid x \ge 0\}$, and F is invex at $(0, 0)$.

The second example is taken from [84].

2. Vector optimization problems

In this section, we recall various concepts of the optimal points of a nonempty set and the optimal solutions of a set-valued vector optimization problem, and we also give some characterizations of them.

2.1. Characterization for optimal points of a set. Let Y be a real Hausdorff topological vector space. Let S be a pointed convex cone in Y and C be a subset of Y.

Definition 2.1. A point $y_0 \in C$ is said to be:

- a *minimal point* of C with respect to S if

$$(C - y_0) \cap (-S) = \{0\};$$

- a *weak minimal point* of C with respect to S if

$$(C - y_0) \cap (-\operatorname{int} S) = \emptyset.$$

We denote by $\mathrm{Min}(C,S)$ and $\mathrm{WMin}(C,S)$ the sets of all minimal and all weak minimal points of C with respect to S, respectively.

If $y_0 \in \mathrm{Min}(C,-S)$ [resp. $y_0 \in \mathrm{WMin}(C,-S)$], we say that y_0 is a *maximal* [resp. *weak maximal*] *point* of C, written $y_0 \in \mathrm{Max}(C,S)$ [resp. $y \in \mathrm{WMax}(C,S)$].

DEFINITION 2.2. $y_0 \in C$ is said to be a *Borwein proper minimal point* of C with respect to S (see [4]) if it is a minimal point of C and

$$\overline{T}(C+S,y_0)\cap(-S)=\{0\}.$$

Denote by $\mathrm{Bo}(C,S)$ the set of all Borwein proper minimal points C with respect to S.

DEFINITION 2.3. $y_0 \in C$ is said to be a *Benson proper minimal point* of C with respect to S (see [2]) if

$$\overline{\mathrm{cone}}(C+S-y_0)\cap(-S)=\{0\}.$$

Denote by $\mathrm{Be}(C,S)$ the set of all Benson proper minimal points C with respect to S.

It is easy to show that

$$\mathrm{Min}(C,S)=\mathrm{Min}(C+S,S),\quad \mathrm{WMin}(C,S)\subset\mathrm{WMin}(C+S,S);$$
$$\mathrm{Be}(C,S)\subset\mathrm{Bo}(C,S)\subset\mathrm{Min}(C,S)\subset\mathrm{WMin}(C,S).$$

The following example is taken from [44].

EXAMPLE 2.1. Consider the set

$$C=\{(y_1,y_2)\in[0,2]\times[0,2]\mid y_2\ge 1-\sqrt{1-(1-y_1)^2}\text{ for } y_1\in[0,1]\}$$

in $\mathbb{R}^2$ with the natural ordering cone $\mathbb{R}^2_+$. It is easy to verify that

$$\begin{aligned}\mathrm{Min}(C,S)&=\{\big(y_1,1-\sqrt{1-(1-y_1)^2}\,\big)\mid y_1\in[0,1]\},\\ \mathrm{Be}(C,S)&=\mathrm{Min}(C,S)\setminus\{(0,1),(1,0)\},\\ \mathrm{WMin}(C,S)&=\mathrm{Min}(C,S)\cup\{(0,y_2)\in\mathbb{R}^2\mid y_2\in[1,2]\}\cup\{(y_1,0)\in\mathbb{R}^2\mid y_1\in[1,2]\}.\end{aligned}$$

Consequently, we have $\mathrm{Be}(C,S)\subset\mathrm{Min}(C,S)\subset\mathrm{WMin}(C,S)$.

If C is closure S-convexlike, from Lemma 1.3, we have

$$\overline{T}(\overline{C+S},y_0)=\overline{\mathrm{cone}}(\overline{C+S}-y_0).$$

On the other hand, it is obvious that

$$T(C+S,y_0)=T(\overline{C+S},y_0),\quad \overline{\mathrm{cone}}(\overline{C+S}-y_0)=\overline{\mathrm{cone}}(C+S-y_0).$$

Thus $\overline{T}(C+S,y_0)=\overline{\mathrm{cone}}(C+S-y_0)$, and so, in this case, $\mathrm{Be}(C,S)=\mathrm{Bo}(C,S)$.

In the following we intend to characterize weak minimal points and proper minimal points of a closure S-convex set. First we present an alternative type result.

LEMMA 2.1. *Let S be a convex cone in Y with* $\operatorname{int} S\neq\emptyset$, *let C be a closure S-convex subset of Y, and let $y_0\in Y$. Then exactly one of the following statements holds*:

(i) $(C-y_0)\cap(-\operatorname{int}S)\neq\emptyset$;
(ii) *there exists $y^*\in S^+\setminus\{0\}$ such that $\langle y^*,y\rangle\ge\langle y^*,y_0\rangle$ for all $y\in C$.*

Proof. (i) true $\Rightarrow$ (ii) not true. Suppose that (i) holds. Then there exists $y \in C$ such that $y - y_0 \in -\operatorname{int} S$. Hence, for every $y^* \in S^+ \setminus \{0\}$, $\langle y^*, y - y_0 \rangle < 0$. This proves that (ii) does not hold.

(i) not true $\Rightarrow$ (ii) true. Suppose that $(C - y_0) \cap (-\operatorname{int} S) = \emptyset$. It is easy to show that $(C + S - y_0) \cap (-\operatorname{int} S) = \emptyset$. Since $\operatorname{int} S$ is open, we have

$$(\overline{C + S} - y_0) \cap (-\operatorname{int} S) = \emptyset.$$

By our assumption, $\overline{C + S} - y_0$ is convex. Thus by the standard separation theorem, there exists $y^* \in S^+$ $(y^* \neq 0)$ such that $\langle y^*, y \rangle \geq \langle y^*, y_0 \rangle$ for all $y \in C$. ∎

The proof of Lemma 2.1 is motivated by Lemma 3.1 of [90]. By applying Lemma 2.1, we obtain the following result.

Theorem 2.1. *Let S be a pointed convex cone in Y with $\operatorname{int} S \neq \emptyset$ and let C a closure S-convex subset of Y. Then $y_0 \in C$ is a weak minimal point of C if and only if there exists a $y^* \in S^+ \setminus \{0\}$ such that*

$$\langle y^*, y_0 \rangle = \min_{y \in C} \langle y^*, y \rangle.$$

Theorem 2.1 has also been proved in [10]; it generalizes Corollary 5.28 of [44].

Theorem 2.2. *Let Y be a locally convex space, let S be a pointed convex cone in Y with a weakly compact base, and C be a closure S-convex subset of Y. Then $y_0 \in C$ is a Benson proper minimal point of C if and only if there exists a $y^* \in S^{+i}$ such that*

$$\langle y^*, y_0 \rangle = \min_{y \in C} \langle y^*, y \rangle.$$

Proof. Since C is closure S-convex, by Lemma 1.2, we deduce that

$$\overline{\operatorname{cone}}(C + S - y_0) = \overline{\operatorname{cone}}(\overline{C + S} - y_0)$$

is a weakly closed convex cone. Since y_0 is a Benson proper minimal point of C, we have

$$\overline{\operatorname{cone}}(C + S - y_0) \cap (-S) = \{0\}.$$

By Lemma 1.4, there exists a pointed convex cone $K \subset Y$ such that $S \setminus \{0\} \subset \operatorname{int} K$ and

$$\overline{\operatorname{cone}}(C + S - y_0) \cap (-K) = \{0\}.$$

Hence

$$\overline{\operatorname{cone}}(C + S - y_0) \cap (-\operatorname{int} K) = \emptyset.$$

By the standard separation theorem, there exists $y^* \in K^+ \setminus \{0\}$ such that

$$\langle y^*, y \rangle \geq \langle y^*, y_0 \rangle \quad \text{for all } y \in C.$$

Since $S \setminus \{0\} \subset \operatorname{int} K$, it is easy to show that $y^* \in S^{+i}$.

For the converse assertion, assume the contrary. Then there exist $y_1 \in -S \setminus \{0\}$ and nets $\{\lambda_\alpha\}$ $(\lambda_\alpha > 0)$, $\{y_\alpha\} \subset C$ and $\{s_\alpha\} \subset S$ such that

$$\lambda_\alpha (y_\alpha + s_\alpha - y_0) \to y_1.$$

So,

$$\langle y^*, \lambda_\alpha (y_\alpha + s_\alpha - y_0) \rangle \to \langle y^*, y_1 \rangle.$$

Since $y^* \in S^{+i}$, $\langle y^*, y_1\rangle < 0$. Hence there exists some $\alpha > 0$ such that $\langle y^*, y_\alpha\rangle < \langle y^*, y_0\rangle$, a contradiction. ∎

Similar scalar characterizations for the Hurwicz (for the definition see [38]) and Borwein proper minimal points have been given in [38] (for a correction, see [20]) and [4], [44]. The following lemma gives a sufficient condition under which the ordering cone has a weakly compact base. For a proof, see Lemma 5.8 of [44].

LEMMA 2.2. *Let Y be a reflexive Banach space and S a closed convex cone in Y. Then S has a weakly compact base if and only if there exists a continuous linear functional $y^* \in S^{+i}$ such that the set $\{y \in S \mid \langle y^*, y\rangle = 1\}$ is bounded.*

2.2. Characterization for optimal solutions of an optimization problem. Let X, Y be real Hausdorff topological vector spaces. Let X_0 be a subset of X, let $S \subset Y$ be a pointed convex cone, and let $F : X \to Y$ be a set-valued mapping.

We are concerned with the following vector optimization problem:

$$\text{(2.1)} \qquad \begin{aligned} &\min F(x) \\ &\text{subject to } x \in X_0. \end{aligned}$$

DEFINITION 2.4. A point $x_0 \in X_0$ is said to be a *minimal* (resp. *weak minimal*) *point* for Problem (2.1) if there exists $y_0 \in F(x_0)$ such that $y_0 \in \mathrm{Min}(F(X_0), S)$ (resp. $y_0 \in \mathrm{WMin}(F(X_0), S)$). We say that y_0 is a *minimal* (resp. *weak minimal*) value for Problem (2.1) and say that (x_0, y_0) is a *minimal* (resp. *weak minimal*) *solution* for Problem (2.1).

These definitions are consistent with those of [13], [62], [63], [84], [86].

DEFINITION 2.5. If $(x_0, y_0) \in \mathrm{gr}\, F = \{(x, y) \mid y \in F(x)\}$ and $y_0 \in \mathrm{Bo}(F(X_0), S)$ (resp. $y_0 \in \mathrm{Be}(F(X_0), S)$), we say that (x_0, y_0) is a *Borwein* (resp. *Benson*) *proper minimal solution* for Problem (2.1).

When F is a single-valued function, these definitions are consistent with those of [2] and [4], respectively.

By applying Lemma 2.1, we can easily obtain the following alternative type results.

PROPOSITION 2.1. *Let S be a pointed convex cone in Y with $\mathrm{int}\, S \neq \emptyset$ and let $F : X_0 \to Y$ be closure S-convexlike on X_0. Then exactly one of the following statements holds*:

(i) *there exists $x \in X_0$ such that $F(x) \cap (-\mathrm{int}\, S) \neq \emptyset$;*
(ii) *there exists $y^* \in S^+ \setminus \{0\}$ such that $(y^* \circ F)(X_0) \subset \mathbb{R}_+$.*

Proposition 2.1 generalizes Theorem 3.1 of [45] to the case of set-valued mappings, and the assumption of closure convexlikeness on F is weaker than that used in [45].

PROPOSITION 2.2. *Let S be a pointed convex cone in Y with $\mathrm{int}\, S \neq \emptyset$ and let $X_0 \subset X$ be a convex subset. Assume that $F : X_0 \to Y$ is $*$-quasiconvex with convex images. If, for each $y^* \in S^+ \setminus \{0\}$, the function $\phi_{y^*}(x) = \inf_{y \in F(x)} \langle y^*, y\rangle$ is finite-valued and lower semicontinuous with respect to x, then exactly one of the statements* (i) *and* (ii) *of Proposition* 2.1 *is true.*

Proof. (i) true $\Rightarrow$ (ii) not true. Take $C = F(X_0)$ and $y_0 = 0$. Then apply Lemma 2.1.

(i) not true $\Rightarrow$ (ii) true. Let $B = \{y^* \in S^+ \mid \langle y^*, e\rangle = 1\}$ for fixed $e \in \operatorname{int} S$. Define a function $\phi : X_0 \times B \to \mathbb{R}^1$ by

$$\phi(x, y^*) = \phi_{y^*}(x), \tag{2.2}$$

By our assumptions, the function $y^* \to \phi(x, y^*)$ is concave and upper semicontinuous, and $x \to \phi(x, y^*)$ is lower semicontinuous and quasiconvex.

Since B is w^*-compact, applying the Sion minimax theorem (see [49]) for ϕ, we have

$$\begin{aligned}\text{(ii) is true} &\Leftrightarrow \exists y^* \in B, (y^* \circ F)(X_0) \subset \mathbb{R}_+ \\ &\Leftrightarrow \max_{y^* \in B} \inf_{x \in X_0} \phi(x, y^*) \geq 0 \\ &\Leftrightarrow \inf_{x \in X_0} \max_{y^* \in B} \phi(x, y^*) \geq 0 \\ &\Leftrightarrow \forall x \in X_0,\ \exists y^* \in B,\ \phi(x, y^*) \geq 0.\end{aligned} \tag{2.3}$$

For our purpose, it is sufficient to show that if (i) is not true, then (2.3) is true. Let $x \in X_0$. Since (i) is not true, $F(x) \cap (-\operatorname{int} S) = \emptyset$. Notice that $F(x)$ is convex, and so by the separation theorem there exists $y^* \in S^+ \setminus \{0\}$ such that $\langle y^*, y\rangle \geq 0$ for all $y \in F(x)$. Thus

$$\langle y^*/\langle y^*, e\rangle, y\rangle \geq 0 \quad \text{for all } y \in F(x).$$

This means that $\phi(x, y^*/\langle y^*, e\rangle) \geq 0$. Since $y^*/\langle y^*, e\rangle \in B$, we obtain (2.3). ∎

Proposition 2.2 has been proved by Jeyakumar, Oettli and Natividad [49] when F is a single-valued mapping.

By applying Propositions 2.1 and 2.2, we easily obtain the following characterization of a weak minimal solution of Problem (2.1).

Theorem 2.3. *Under the assumptions of Proposition* 2.1 *or* 2.2, $(x_0, y_0) \in \operatorname{gr} F$ *is a weak minimal solution of Problem* (2.1) *if and only if there exists a* $y^* \in S^+ \setminus \{0\}$ *such that*

$$\langle y^*, y_0\rangle \leq \inf_{y \in F(x)} \langle y^*, y\rangle \quad \textit{for all } x \in X_0.$$

By applying Theorem 2.2, we can obtain the following characterization of proper minimal solutions for Problem (2.1).

Theorem 2.4. *Let* Y *be a locally convex space, let* S *be a pointed convex cone in* Y *with a weakly compact base, and let* $F : X_0 \to Y$ *be closure* S*-convexlike on* X_0. *Then* $(x_0, y_0) \in \operatorname{gr} F$ *is a Benson proper minimal solution of Problem* (2.1) *if and only if there exists a* $y^* \in S^{+i}$ *such that*

$$\langle y^*, y_0\rangle \leq \inf_{y \in F(x)} \langle y^*, y\rangle \quad \textit{for all } x \in X_0.$$

Proof. Take $C = F(X_0)$. Then apply Theorem 2.2.

3. Lagrangian multiplier rule

In this section, we investigate a vector optimization problem with explicit constraints. For this problem, we derive the Lagrangian multiplier rule for weak minimality, minimality and proper minimality.

Let X, Y, Z be real Hausdorff topological vector spaces. Let A be a subset of X, let $S \subset Y$ and $Q \subset Z$ be pointed convex cones, and let $F : X \to Y$ and $G : X \to Z$ be set-valued mappings.

We consider the vector optimization problem:

$$\text{(CP)} \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in X_0 = \{x' \in A \mid G(x') \cap (-Q) \neq \emptyset\}. \end{array}$$

We also say that X_0 is the *feasible set*, F is the *objective mapping* and G is the *constraints mapping*. The definitions of solutions of Problem (CP) are the same as in Definitions 2.4 and 2.5.

3.1. Lagrangian conditions for weak optimality. In this subsection, by applying two alternative type theorems (Propositions 2.1 and 2.2), we derive Lagrangian optimality conditions for weak minimal solutions of Problem (CP) under the generalized Slater condition, when the set-valued mappings are closure convexlike or $*$-quasiconvex. The results obtained generalize the corresponding results of [13] to generalized convex cases.

In the sequel, we let $L^+(Z, Y)$ denote the set of positive continuous linear operators from Z into Y, i.e. the set of all continuous linear operators $\Lambda : Z \to Y$ such that $\Lambda(Q) \subset S$.

We obtain the following optimality conditions.

THEOREM 3.1. *Let* $\operatorname{int} S \neq \emptyset$. *Assume that* $F \times G$ *is closure convexlike on* A. *If* (x_0, y_0) *is a weak minimal solution of* (CP) *and the generalized Slater condition holds*, *i.e.* $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, *then*:

(i) *there exist* $y^* \in S^+ \setminus \{0\}$ *and* $z^* \in Q^+$ *such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq \langle y^*, y_0\rangle \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x),$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\langle z^*, z_0\rangle = 0$;

(ii) *there exists* $\Lambda \in L^+(Z, Y)$ *such that* (x_0, y_0) *is a weak minimal solution of the following problem*:

$$(\overline{\text{P}}) \qquad \begin{array}{l} \min(F(x) + \Lambda G(x)) \\ \text{subject to } x \in A \end{array}$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\Lambda(z_0) = 0$.

PROOF. (i) Since $\overline{(F \times G)(A) + S \times Q}$ is convex, so is $\overline{((F - y_0) \times G)(A) + S \times Q}$. This means that $((F - y_0) \times G)(\cdot)$ is closure convexlike on A. If (x_0, y_0) is a weak minimal solution of (CP), then

$$[((F - y_0) \times G)(A)] \cap [-(\operatorname{int} S \times \operatorname{int} Q)] = \emptyset.$$

Indeed, suppose that there exist $\overline{x} \in A$, $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$ such that $\overline{y} - y_0 \in -\operatorname{int} S$ and $\overline{z} \in -\operatorname{int} Q$. Hence $\overline{x} \in X_0$ and

$$(F(\overline{x}) - y_0) \cap (-\operatorname{int} S) \neq \emptyset.$$

This is a contradiction since (x_0, y_0) is a weak minimal solution of (P).

Applying Proposition 2.1 to $(F - y_0) \times G$, there exist $y^* \in S^+$ and $z^* \in Q^+$, not both zero, such that

$$\langle y^*, y - y_0\rangle + \langle z^*, z\rangle \geq 0 \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x). \tag{3.1}$$

For any $z_0 \in G(x_0) \cap (-Q)(\neq \emptyset)$, if we put $x = x_0$, $y = y_0$, $z = z_0$ in (3.1), then $\langle z^*, z_0\rangle \geq 0$. Since $z_0 \in -Q$, $\langle z^*, z_0\rangle = 0$.

It is easy to prove that $y^* \neq 0$. Indeed, if $y^* = 0$, then $z^* \neq 0$ and

$$\langle z^*, z\rangle \geq 0 \quad \text{for all } x \in A,\ z \in G(x). \tag{3.2}$$

Since $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, there exist $\overline{x} \in A$ and $\overline{z} \in G(\overline{x}) \cap (-\operatorname{int} Q)$. Hence, $\langle z^*, \overline{z}\rangle < 0$. This contradicts (3.2). Therefore, $y^* \neq 0$.

(ii) Fix $e \in \operatorname{int} S$ with $\langle y^*, e\rangle = 1$. Define $\Lambda : Z \to Y$ by

$$\Lambda z = \langle z^*, z\rangle e \quad \text{for all } z \in Z.$$

Then $y^*\Lambda = z^*$, $\Lambda(Q) \subset S$ and $\Lambda z_0 = 0$. Replacing z^* by $y^*\Lambda$ in (3.1), we obtain

$$\langle y^*, y + \Lambda z\rangle \geq \langle y^*, y_0\rangle \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x). \tag{3.3}$$

Since $y_0 \in F(x_0) + \Lambda G(x_0)$ and $y^* \in S^+ \setminus \{0\}$, we conclude from (3.3) that (x_0, y_0) is a weak minimal solution of Problem $(\overline{\mathrm{P}})$. Indeed, otherwise there exist $\overline{x} \in A$, $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$ such that $\overline{y} + \Lambda\overline{z} - y_0 \in -\operatorname{int} S$. Thus, $\langle y^*, \overline{y} + \Lambda\overline{z} - y_0\rangle < 0$, which is a contradiction. ■

When F and G are convex set-valued mappings, Theorem 3.1 has been proved by Corley [13].

THEOREM 3.2. *Let* $\operatorname{int} S \neq \emptyset$ *and let* $A \subset X$ *be a convex set. Assume that* $F : A \to Y$ *and* $G : A \to Z$ *are set-valued mappings with bounded convex images and* $F \times G$ *is* $*$*-quasiconvex on* A. *Suppose that for every* $y^* \in S^+$ *and* $z^* \in Q^+$, *the function*

$$x \to \inf_{(y,z)\in(F\times G)(x)} (\langle y^*, y\rangle + \langle z^*, z\rangle)$$

is lower semicontinuous. If (x_0, y_0) *is a weak minimal solution of* (CP) *and* $G(A) \cap -(\operatorname{int} Q) \neq \emptyset$, *then*:

(i) *there exist* $y^* \in S^+ \setminus \{0\}$ *and* $z^* \in Q^+$ *such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq \langle y^*, y_0\rangle \quad \textit{for all } x \in A,\ y \in F(x) \textit{ and } z \in G(x),$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\langle z^*, z_0\rangle = 0$;

(ii) *there exists* $\Lambda \in L^+(Z, Y)$ *such that* (x_0, y_0) *is a weak minimal solution of Problem* $(\overline{\mathrm{P}})$ *and for every* $z_0 \in G(x_0) \cap (-Q)$, $\Lambda(z_0) = 0$.

PROOF. (i) Since $F \times G$ is $*$-quasiconvex, so is $[(F - y_0) \times G](\cdot)$. If (x_0, y_0) is a weak minimal solution of (CP), then

$$[(F - y_0) \times G](A) \cap [-(\operatorname{int} S \times \operatorname{int} Q)] = \emptyset.$$

By Proposition 2.2, there exist $y^* \in S^+$ and $z^* \in Q^+$, not both zero, such that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \quad \text{for all } x \in A,\ y \in F(x) - y_0 \text{ and } z \in G(x). \tag{3.4}$$

Then we repeat step by step the same arguments as in the proof of Theorem 3.1. ∎

The necessary optimality conditions given in Theorems 3.1 and 3.2 generalize the well-known Lagrangian multiplier. They also extend corresponding results in scalar or vector optimization with generalized convex functions of [11], [14], [25], [34], [48], [49], [66], [38], [4] and [67] to the set-valued case.

3.2. Lagrangian conditions for optimality. In Subsection 3.1, in order to derive the Lagrangian optimality conditions, the generalized Slater condition was assumed. In this subsection, we introduce an image regularity condition and provide a cone separation theorem for two suitable subsets of the image space. By applying the results obtained, we derive a generalized Lagrangian optimality condition for a minimal solution of Problem (CP). In addition, under closure convexlikeness assumptions, we derive the usual Lagrangian optimality conditions. Our results improve the corresponding results of [62], [63].

We introduce some notations. Set $H_{y_0}(x) = [(F - y_0) \times G](x)$, $C = (S \setminus \{0\}) \times Q$ and $R_{y_0} = H_{y_0}(A) + S \times Q$. Then $H_{y_0}(A)$ and R_{y_0} will be called the *image* and *extended image* of Problem (CP), respectively.

Now we present some equivalent conditions for optimality.

PROPOSITION 3.1. *Let* $x_0 \in X_0$ *and* $(x_0, y_0) \in \operatorname{gr} F$. *Then the following statements are equivalent*:

(i) (x_0, y_0) *is a minimal solution of* (CP);
(ii) $H_{y_0}(A) \cap (-C) = \emptyset$;
(iii) $R_{y_0} \cap (-C) = \emptyset$;
(iv) $R_{y_0} \cap [-(S \setminus \{0\}) \times \{0\}] = \emptyset$.

PROOF. (i)⇒(ii). Assume that, on the contrary, there exist $s \in S \setminus \{0\}$, $q \in Q$ and $x \in A$ such that $-(s, q) \in (F(x) - y_0, G(x))$. This means that $x \in X_0$ and $(F(x) - y_0) \cap (-S \setminus \{0\}) = \{-s\}$, a contradiction.

(ii)⇒(iii). Assume that, on the contrary, there exist $s \in S \setminus \{0\}$, $q \in Q$ and $x \in A$ such that $-(s, q) \in (F(x) - y_0, G(x)) + (S \times Q)$. This means that $\emptyset \neq H_{y_0}(x) \cap [-(s, q) - (S \times Q)] = H_{y_0}(x) \cap (-C)$, a contradiction.

(iii)⇒(iv). It is obvious.

(iv)⇒(i). Assume that, on the contrary, there exist $s \in S \setminus \{0\}$ and $x \in X_0$ such that $-s \in F(x - y_0)$. This means that there exists $z \in G(x) \cap (-Q)$ such that $-(s, 0) = (-s, z) + (0, -z) \in [(F(x) - y_0, G(x)) + S \times Q] \cap [-(S \setminus \{0\}) \times \{0\}]$, a contradiction. ∎

Proposition 3.1 has been proved in [11], [21], [30], [67] when Y and Z are finite-dimensional spaces, and F and G are single-valued functions.

We observe that the optimality of a feasible point is equivalent to separation of two suitable subsets of the image space. Separation of two convex sets by a hyperplane is achieved under some conditions. When the sets are not convex, there is no guarantee that separation by a hyperplane is possible. Henig [35] introduced separation by a cone in a finite-dimensional space, which includes the hyperplane separation as a special case. The cone separation was also studied in [21], [43], and [19] in finite-dimensional spaces, normed spaces, and locally convex spaces, respectively.

Note that if R_{y_0} and $-C$ are *cone separated*, i.e., there exists a convex pointed cone K such that $-C \setminus \{(0,0)\} \subset \operatorname{int} K$ and $R_{y_0} \cap K = \{(0,0)\}$, then $(x_0, y_0) \in \operatorname{gr} F$ is a minimal solution of (CP).

We consider the following *image regularity condition*:

$$\overline{\operatorname{cone}}(R_{y_0}) \cap (-S \times \{0\}) = \{(0,0)\}. \tag{3.5}$$

Most of the classical regularity conditions emphasize the role of the constrained mapping. The image regularity condition involves the constrained mapping as well as the objective mapping. The image regularity condition via the image space approach was developed by Giannessi [30] when studying constrained scalar optimization problems, and it was further investigated by other authors studying scalar and vector optimization problems in a finite-dimensional setting (see [21], [67]).

We see that if $(x_0, y_0) \in \operatorname{gr} F$ is a minimal solution of (CP), then $\operatorname{cone}(R_{y_0}) \cap (-S \times \{0\}) = \{(0,0)\}$ by Proposition 3.1. Under the image regularity condition, we obtain the following result.

PROPOSITION 3.2. *If $x_0 \in X_0$ is a feasible point of Problem* (CP) *and* $(x_0, y_0) \in \operatorname{gr} F$, *then the image regularity condition* (3.5) *implies that* (x_0, y_0) *is a Benson proper minimal solution of* (CP).

PROOF. Assume the contrary. Then

$$\overline{\operatorname{cone}}(F(X_0) + S - y_0) \cap (-S) \neq \{0\};$$

i.e., there exist $s \in S \setminus \{0\}$, and nets $\{s_\alpha\} \subset S$, $\{x_\alpha\} \subset A$, and $y_\alpha \in F(x_\alpha)$, $\lambda_\alpha \geq 0$, such that

$$G(x_\alpha) \cap (-Q) \neq \emptyset, \quad \lambda_\alpha(y_\alpha + s_\alpha - y_0) \to -s_0.$$

Hence, there exists $z_\alpha \in G(x_\alpha) \cap (-Q)$ such that $\lambda_\alpha(y_\alpha + s_\alpha - y_0, z_\alpha - z_\alpha) \to (-s_0, 0)$. This contradicts (3.5). ■

THEOREM 3.3. *Let Y be a locally convex space and let $x_0 \in X_0$ and $y_0 \in F(x_0)$. Assume that either of the following conditions holds*:

(a) *S has a compact base*;

(b) *S has a weakly compact base and $\overline{\operatorname{cone}}(R_{y_0})$ is weakly closed.*

Then R_{y_0} and $-C$ are cone separated, i.e., there exists a pointed convex cone K such that

$$-C \setminus \{(0,0)\} \subset \operatorname{int} K, \quad R_{y_0} \cap K = \{(0,0)\},$$

if and only if the image regularity condition (3.5) *holds.*

PROOF. Assume that (3.5) is true. Since S has a compact (resp. weakly compact) base, we see that $S \times \{0\}$ also has a compact (resp. weakly compact) base. By Lemma 1.4, there exists a pointed convex cone P such that

(3.6) $$-(S\times\{0\})\setminus\{(0,0)\}\subset \operatorname{int}P,\quad \overline{\operatorname{cone}}(R_{y_0})\cap P=\{(0,0)\}.$$

Set $K=(-C+P)\cup\{(0,0)\}$. Then K is a convex cone and $\operatorname{int}K=-C+\operatorname{int}P$. For every $s\in S\setminus\{0\}$ and $q\in Q$, since

$$-(s,q)=-(s/2,q)-(s/2,0)\in-(S\setminus\{0\}\times Q)+\operatorname{int}P=\operatorname{int}K,$$

we deduce that $-(C\setminus\{(0,0)\})=-C\subset\operatorname{int}K$.

We now show that K is pointed. Since $x_0\in X_0$, there exists $z_0\in G(x_0)\cap(-Q)$. For every $(s,q)\in S\times Q$, we have

$$(s,q)=(0,z_0)+(s,q-z_0)\in(F(x_0)-y_0,G(x_0))+S\times Q\subset R_{y_0}.$$

This means that $S\times Q\subset R_{y_0}$. Since $\overline{\operatorname{cone}}(R_{y_0})\cap P=\{(0,0)\}$, we have $(S\times Q)\cap P=\{(0,0)\}$ and so $C\cap P=\emptyset$. This, together with the fact that S,Q and P are pointed, implies that K is also pointed.

We need to show $R_{y_0}\cap K=\{(0,0)\}$. Indeed, assume the contrary. Thus there exists $u\in R_{y_0}\cap K$ with $u\neq 0$. Therefore $u=u_1+u_2$ with $u_1\in -C$ and $u_2\in P$. This implies that

$$u_2=u-u_1\in(R_{y_0}-u_1)\cap P\subset\overline{\operatorname{cone}}(R_{y_0})\cap P=\{(0,0)\}.$$

Hence $u=u_1\in(-C)\cap R_{y_0}$. On the other hand, by Proposition 3.1, the condition (3.5) implies $(-C)\cap R_{y_0}=\emptyset$. This is a contradiction.

If there exists a pointed convex cone K such that

$$-C\setminus\{(0,0)\}\subset\operatorname{int}K,\quad R_{y_0}\cap K=\{(0,0)\},$$

then $R_{y_0}\cap\operatorname{int}K=\emptyset$. Since $\operatorname{int}K$ is open, $\overline{\operatorname{cone}}(R_{y_0})\cap\operatorname{int}K=\emptyset$. This, together with $-(S\setminus\{0\})\times\{0\}\subset\operatorname{int}K$, implies (3.5). ■

The scalar result corresponding to Theorem 3.3 can be found in Theorem 2.1 of [21].

A functional $\phi:Y\to\mathbb{R}$ is called *S-increasing* (resp. *strictly S-increasing*) if $y_1,y_2\in Y$, $y_1-y_2\in S$ implies $\phi(y_1)\ge\phi(y_2)$ (resp. $y_1-y_2\in S\setminus\{0\}$ implies $\phi(y_1)>\phi(y_2)$).

$\phi:Y\to\mathbb{R}$ is called *S-decreasing* (resp. *strictly S-decreasing*) if $-\phi:Y\to\mathbb{R}$ is S-increasing (resp. strictly S-increasing).

It is obvious that if $S_1\subset S$ is a convex cone and ϕ is S-increasing, then ϕ is also S_1-increasing.

LEMMA 3.1 ([19]). *Let Y be a topological vector space and let $C\subset Y$ be a pointed convex cone with* $\operatorname{int}C\neq\emptyset$. *If* $y_0\in\operatorname{int}C$, *then*

$$\phi(y):=\inf\{\alpha\in\mathbb{R}\mid y\in\{-\alpha y_0\}+C\}\quad\textit{for } y\in Y$$

defines a continuous sublinear functional on Y which has the following properties:

(a) $\operatorname{int}C=\{y\in Y\mid\phi(y)<0\}$ *and* $\overline{C}=\{y\in Y\mid\phi(y)\le 0\}$;

(b) *ϕ is C-decreasing on Y*;

(c) *if Q is a convex cone such that $Q\setminus\{0\}\subset\operatorname{int}C$, then ϕ is strictly Q-decreasing on Y.*

Now we present a generalized Lagrangian condition.

THEOREM 3.4. *Let Y be a locally convex space and let $(x_0,y_0)\in\operatorname{gr}F$ be a minimal solution of* (CP). *Assume that either of the following conditions holds*:

(a) *S has a compact base*;

(b) *S has a weakly compact base and $\overline{\text{cone}}(R_{y_0})$ is weakly closed.*

Then there exists a continuous sublinear functional $\phi : Y \times Z \to \mathbb{R}^1$ satisfying:

(c) *for each $z \in Z$, $\phi(\cdot, z)$ is strictly S-increasing on Y*;

(d) *for each $y \in Y$, $\phi(y, \cdot)$ is Q-increasing on Z,*

such that

$$0 = \min_{x \in A} \phi(F(x) - y_0 + S, G(x) + Q), \quad \textit{i.e.}, \quad 0 = \min\{\phi(y, z) \mid (y, z) \in R_{y_0}\},$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\phi(0, z_0) = 0$, if and only if the image regularity condition (3.5) *is true.*

Proof. Assume that (3.5) is true. Then by Theorem 3.3, there exists a pointed convex cone K such that

$$-C \setminus \{(0, 0)\} \subset \operatorname{int} K \quad \text{and} \quad R_{y_0} \cap K = \{(0, 0)\}.$$

Take $u_0 \in \operatorname{int} K$ and define

$$\phi(u) := \inf\{\alpha \in \mathbb{R}^1 \mid u \in (\{-\alpha u_0\} + K)\}, \quad u \in Y \times Z.$$

Lemma 3.1 implies that ϕ is a continuous sublinear functional on $Y \times Z$ with

$$\operatorname{int} K = \{(y, z) \mid \phi(y, z) < 0\} \quad \text{and} \quad \overline{K} = \{(y, z) \mid \phi(y, z) \leq 0\}. \tag{3.7}$$

We show that ϕ is the required function.

(c) By the definition of C, for every $y_2 - y_1 \in S \setminus \{0\}$, we have $(y_1 - y_2, 0) \in -C \subset \operatorname{int} K$. Thus, by (3.7), it follows that $\phi(y_1 - y_2, 0) < 0$. By the sublinearity of ϕ, we have

$$\phi(y_1, z) \leq \phi(y_1 - y_2, 0) + \phi(y_2, z) < \phi(y_2, z) \quad \text{for every } z \in Z.$$

(d) By the definition of C, for every $z_2 - z_1 \in Q$, we have $(0, z_1 - z_2) \in -\overline{C} \subset \overline{K}$. Thus, by (3.7), it follows that $\phi(0, z_1 - z_2) \leq 0$. By the sublinearity of ϕ, we have

$$\phi(y, z_1) \leq \phi(0, z_1 - z_2) + \phi(y, z_2) \leq \phi(y, z_2) \quad \text{for every } y \in Y.$$

The condition $R_{y_0} \cap K = \{(0, 0)\}$, together with (3.7), implies that

$$0 = \min_{(y,z) \in R_{y_0}} \phi(y, z), \quad \text{i.e.,} \quad 0 = \min_{x \in A} \phi(F(x) - y_0 + S, G(x) + Q).$$

Now for every $z_0 \in G(x_0) \cap (-Q)$, $(0, z_0) \in R_{y_0} \cap \overline{K}$, hence $\phi(0, z_0) = 0$.

Suppose there exists a continuous sublinear functional $\phi : Y \times Z \to \mathbb{R}^1$ satisfying (c) and (d) such that

$$0 = \min_{x \in A} \phi(F(x) - y_0 + S, G(x) + Q)$$

and $\phi(0, z_0) = 0$ for every $z_0 \in G(x_0) \cap (-Q)$. If (3.5) is not true, then there exists $s \in S \setminus \{0\}$ such that

$$(-s, 0) \in \overline{\text{cone}}(R_{y_0}).$$

Since $\phi(y, z) \geq 0$ for every $(y, z) \in R_{y_0}$ and since ϕ is sublinear and continuous, we can deduce that

$$\phi(y, z) \geq 0 \quad \text{for every } (y, z) \in \overline{\text{cone}}(R_{y_0}),$$

and so $\phi(-s,0) \geq 0$. On the other hand, (c) implies $\phi(-s,0) < \phi(0,0) = 0$, a contradiction. ∎

When F and G are single-valued mappings, a similar necessary condition is given in [19].

If $F \times G$ is closure convexlike, then we obtain the following Lagrangian multiplier theorems.

THEOREM 3.5. *Let Y be a locally convex space and let $(x_0, y_0) \in \operatorname{gr} F$ be a minimal solution of* (CP). *Assume that $F \times G$ is closure convexlike on A, and S has a weakly compact base. Then the following statements are equivalent*:

(i) *the image regularity condition* (3.5) *holds*;

(ii) *there exist $y^* \in S^{+i}$ and $z^* \in Q^+$ such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \quad \textit{for all } (y,z) \in R_{y_0}$$

and

$$\langle z^*, z_0\rangle = 0 \quad \textit{for all } z_0 \in G(x_0) \cap (-Q);$$

(iii) *there exists a continuous linear positive operator $\Lambda \in L^+(Z,Y)$ such that (x_0, y_0) is a Benson proper minimal solution of the problem*

$$(\overline{\mathrm{P}}) \qquad \begin{array}{l} \min(F(x) + \Lambda G(x)) \\ \textit{subject to } x \in A \end{array}$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\Lambda z_0 = 0$.

PROOF. (i)⇒(ii). Since $\overline{R}_{y_0}$ is convex, we deduce, by Lemma 1.2, that so is $\overline{\operatorname{cone}}(R_{y_0}) = \overline{\operatorname{cone}}(\overline{R}_{y_0})$. Hence it is weakly closed. By Theorem 3.3, there exists a pointed convex cone K such that $-C \subset \operatorname{int} K$ and $R_{y_0} \cap K = \{(0,0)\}$. Thus $\overline{R}_{y_0} \cap \operatorname{int} K = \emptyset$. By the standard separation theorem, there exists $(y^*, z^*) \in -K^+$ such that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \quad \text{for all } (y,z) \in R_{y_0} \tag{3.8}$$

and

$$\langle y^*, y\rangle + \langle z^*, z\rangle < 0 \quad \text{for all } (y,z) \in \operatorname{int} K. \tag{3.9}$$

We now show that $y^* \in S^{+i}$ and $z^* \in Q^+$. It follows from (3.9) and $-((S\setminus\{0\})\times Q) \subset \operatorname{int} K$ that

$$\langle y^*, y\rangle + \langle z^*, z\rangle < 0 \quad \text{for all } (y,z) \in -((S \setminus \{0\}) \times Q). \tag{3.10}$$

From (3.10) and $0 \in Q$, we deduce that $y^* \in S^{+i}$. Since $y \in S \setminus \{0\}$ can be taken arbitrarily close to 0 $(\in Y)$, (3.10) implies that $z^* \in Q^+$. For every $z_0 \in G(x_0) \cap (-Q)$, (3.8) and $z^* \in Q^+$ imply that $\langle z^*, z_0\rangle = 0$.

(ii)⇒(iii). By (ii), there exist $y^* \in S^{+i}$ and $z^* \in Q^+$ such that (3.8) holds and

$$\langle z^*, z_0\rangle = 0 \quad \text{for all } z_0 \in G(x_0) \cap (-Q).$$

Fix $e \in S \setminus \{0\}$ such that $\langle y^*, e\rangle = 1$. Define $\Lambda : Z \to Y$ by

$$\Lambda z = \langle z^*, z\rangle e \quad \text{for all } z \in Z.$$

Then $y^*\Lambda = z^*$, $\Lambda(Q) \subset S$ and $\Lambda z_0 = 0$. Replacing z^* by $y^*\Lambda$ in (3.8), we obtain

$$\langle y^*, y + \Lambda z\rangle \geq \langle y^*, y_0\rangle \quad \text{for all } x \in A,\ (y, z) \in (F(x), G(x)). \tag{3.11}$$

Since $y_0 \in F(x_0) + \Lambda G(x_0)$ and $y^* \in S^{+i}$, we can conclude by Theorem 2.4 that (x_0, y_0) is a Benson proper minimal solution of $(\overline{\mathrm{P}})$.

(iii)⇒(i). If (3.5) is not true, then there exist $s \in S\backslash\{0\}$ and nets $\{s_\alpha\} \subset S$, $\{q_\alpha\} \subset Q$, $\{x_\alpha\} \subset A$, $y_\alpha \in F(x_\alpha)$, $z_\alpha \in G(x_\alpha)$, $\lambda_\alpha \geq 0$, such that

$$\lambda_\alpha(y_\alpha + s_\alpha - y_0, z_\alpha + q_\alpha) \to -(s, 0).$$

By the continuity of Λ, we have

$$\lambda_\alpha(y_\alpha + \Lambda z_\alpha + s_\alpha + \Lambda q_\alpha - y_0) \to -s.$$

The definition of Λ implies $\Lambda q_\alpha \in S$. Hence

$$(y_\alpha + \Lambda z_\alpha + s_\alpha + \Lambda q_\alpha - y_0) \in (F + \Lambda G)(A) + S - y_0,$$

and so $-s \in \overline{\mathrm{cone}}(F + \Lambda G)(A) + S - y_0)$. This contradicts the fact that (x_0, y_0) is a proper minimal solution of $(\overline{\mathrm{P}})$. ■

When X, Y and Z are finite-dimensional spaces, and F and G are single-valued mappings, the conclusions of Theorem 3.5 improve Theorems 4.1 and 4.2 of [67]. Theorem 3.5 shows that the image regularity condition and the optimality conditions formulated with a scalar-valued or a vector-valued Lagrangian mapping are equivalent.

THEOREM 3.6. *Let Y be a locally convex space. Assume that $F{\times}G$ is closure convexlike on A. Assume that either of the following conditions holds*:

(a) *S has a weakly compact base and F is closure convexlike on $X_0 = A \cap G^{-1}(-Q)$;*

(b) *S has a compact base.*

If (x_0, y_0) is a Benson proper minimal solution of (CP) *and $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, then*

(i) *there exist $y^* \in S^{+i}$ and $z^* \in Q^+$ such that*

$$\langle y^*, y - y_0\rangle + \langle z^*, z\rangle \geq 0 \quad \textit{for all } x \in A,\ y \in F(x) \textit{ and } z \in G(x),$$

and $\langle z^*, z_0\rangle = 0$ for all $z_0 \in G(x_0) \cap (-Q)$;

(ii) *there exists a continuous linear positive operator $\Lambda \in L^+(Z, Y)$ such that (x_0, y_0) is a Benson proper minimal solution of $(\overline{\mathrm{P}})$ and for every $z_0 \in G(x_0) \cap (-Q)$, $\Lambda z_0 = 0$.*

PROOF. If F is closure convexlike on X_0, then $\overline{F(X_0) + S}$ is convex. Since

$$\overline{F(X_0) + S} - y_0 \subset \overline{\mathrm{cone}}(F(X_0) + S - y_0),$$

by Lemma 1.2, we deduce that

$$\overline{\mathrm{cone}}(F(X_0) + S - y_0) = \overline{\mathrm{cone}}(\,\overline{F(X_0) + S} - y_0)$$

is a weakly closed convex cone. By the definition of the Benson proper minimal solution, we have

$$\overline{\mathrm{cone}}(F(X_0) + S - y_0) \cap (-S) = \{0\}. \tag{3.12}$$

This, together with our assumption, implies that the hypotheses of Lemma 1.4 are satisfied and hence there exists a pointed convex cone $C \subset Y$ such that

$$-S \setminus \{0\} \subset -\operatorname{int} C$$

and

$$\overline{\operatorname{cone}}(F(X_0) + S - y_0) \cap (-C) = \{0\}. \tag{3.13}$$

We claim that

$$\overline{\operatorname{cone}}(R_{y_0}) \cap [-(\operatorname{int} C \times \operatorname{int} Q)] = \emptyset.$$

Since $\operatorname{int} C \times \operatorname{int} Q$ is an open cone, for this we only need to show that

$$R_{y_0} \cap [-(\operatorname{int} C \times \operatorname{int} Q)] = \emptyset.$$

If this is not the case, then there exist $x \in A$, $y \in F(x)$, $z \in G(x)$, $s \in S$ and $q \in Q$ such that

$$y + s - y_0 \in -\operatorname{int} C, \quad z + q \in -\operatorname{int} Q.$$

Thus, $z \in -\operatorname{int} Q - Q \subset -Q$, and $x \in X_0$, $y \in F(X_0)$. Since $y + s - y_0 \in -\operatorname{int} C$, we have

$$\overline{\operatorname{cone}}(F(X_0) + S - y_0) \cap (-C) = \{y + s - y_0\}, \tag{3.14}$$

a contradiction. Therefore

$$\overline{\operatorname{cone}}(R_{y_0}) \cap [-(\operatorname{int} C \times \operatorname{int} Q)] = \emptyset.$$

By a standard separation theorem, there exist $y^* \in C^+$ and $z^* \in Q^+$, not both zero, such that

$$\langle y^*, y - y_0\rangle + \langle z^*, z\rangle \geq 0 \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x). \tag{3.15}$$

For any $z_0 \in G(x_0) \cap (-Q)(\neq \emptyset)$, if we take $x = x_0$, $y = y_0$, $z = z_0$ in (3.15), then $\langle z^*, z_0\rangle \geq 0$. Hence, $\langle z^*, z_0\rangle = 0$, because of $z_0 \in -Q$.

Since $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, it is easy to prove that $y^* \neq 0$. This, together with $S \setminus \{0\} \subset \operatorname{int} C$, implies $y^* \in S^{+i}$.

Then we repeat step by step the same arguments as in the proof of Theorem 3.5. ■

If F is closure convexlike on X_0, then the Benson proper minimal solution coincides with the Borwein proper minimal solution. When F and G are single-valued convex mappings, a similar Lagrangian multiplier theorem has been obtained in [4], [58]. Theorem 3.6 is illustrated by the following example.

EXAMPLE 3.1. Let $X = Z = \mathbb{R}^1$, $Y = \mathbb{R}^2$ and $A = [-1, 1]$. Let $S = \mathbb{R}^2_+$ and $Q = \mathbb{R}_+$. It is obvious that $\mathbb{R}^2_+$ has a compact base and $\operatorname{int} \mathbb{R}_+ \neq \emptyset$. Define set-valued mappings F and G as follows:

$$F(x) = \begin{cases} \{-x\} \times [0, 1] & \text{if } x \in [-1, 0]; \\ \{x\} \times (-x, 0] & \text{if } x \in (0, 1); \\ \{1\} \times [-1, 0] & \text{if } x = 1, \end{cases}$$

$$G(x) = -x \quad \text{for } x \in \mathbb{R}^1.$$

It is clear that $F \times G$ is closure convexlike on A and $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$. Since the feasible point set $X_0 = A \cap G^{-1}(-\mathbb{R}_+) = [0, 1]$, Problem (CP) takes the form

$$(\mathrm{P}_1)\qquad \begin{array}{l}\min F(x)\\ \text{subject to } x \in [0,1].\end{array}$$

Let $x_0 = 1$ and $y_0 = (1,-1)$. Notice that

$$F(X_0) = \{0\} \times [0,1] \cup \{1\} \times [-1,0] \cup \{(r_1,r_2) \in \mathbb{R}^2 \mid -r_1 < r_2 \le 0,\ 0 < r_1 < 1\}.$$

It is easy to show that (x_0,y_0) is a proper minimal solution of (P_1).

Let $y^* = (1,1) \in \operatorname{int}\mathbb{R}^2_+$ and $z^* = 0 \in \mathbb{R}_+$. One can see that

$$\langle y^*, y - y_0\rangle + \langle z^*, z\rangle \ge 0 \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x),$$

and $\langle z^*, G(x_0)\rangle = 0$. Let $e = (1/2,1/2) \in \mathbb{R}^2_+$. It is clear that $\langle y^*, e\rangle = 1$. Define the operator $\Lambda : \mathbb{R} \to \mathbb{R}^2$ by

$$\Lambda z = \langle z^*, z\rangle e = (0,0) \quad \text{for } z \in \mathbb{R}.$$

Then $\Lambda \in L^+(Z,Y)$. Thus, Problem $(\overline{\mathrm{P}})$ is of the form

$$(\overline{\mathrm{P}}_1)\qquad \begin{array}{l}\min F(x)\\ \text{subject to } x \in [-1,1].\end{array}$$

It is evident that (x_0,y_0) is also a proper minimal solution of $(\overline{\mathrm{P}}_1)$.

3.3. Lagrangian conditions for invex set-valued mappings. Invex functions were introduced by Hanson [33], and it was showed that they can replace convex functions in establishing optimality conditions and duality results. Their definition has been extended to smooth and nonsmooth vector functions ([15], [16], [17], [79]), and recently, to set-valued mappings ([84], [85], [64], [86]). The Wolfe and Mond–Weir type duality theorems for invex set-valued mappings were also obtained in [85], [64], [86]. In this subsection, we introduce a regularity condition, a special case of which is the Robinson regularity condition. Under this regularity condition, we derive Lagrangian optimality conditions for weak and proper minimal solutions of Problem (CP) when the set-valued mappings are invex. By using the cone separation theorem from Subsection 3.2, we obtain a Lagrangian optimality condition for minimal solutions.

From now on we assume that X, Y, Z are normed spaces and A is a subset of X. Let $S \subset Y$ and $Q \subset Z$ be pointed closed convex cones, and let $F : X \to Y$ and $G : X \to Z$ be set-valued mappings.

We consider the vector optimization problem (CP) introduced at the beginning of this section:

$$(\mathrm{CP})\qquad \begin{array}{l}\min F(x)\\ \text{subject to } x \in A,\ G(x) \cap (-Q) \ne \emptyset.\end{array}$$

Theorem 3.7. *Let* $\operatorname{int} S \ne \emptyset$, $\operatorname{int} Q \ne \emptyset$ *and let* (x_0,y_0) *be a weak minimal solution of* (CP). *Assume that* F *and* G *are locally Lipschitz at* x_0, $(F \times G)|_A$ *is invex at* (x_0,y_0,z_0) *for some* $z_0 \in G(x_0) \cap (-Q)$, *and*

$$\operatorname{dom} CF(x_0,y_0) \supset \operatorname{dom} CG(x_0,z_0) \cap C_A(x_0).$$

If $0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0,z_0)(C_A(x_0))}$, *then*:

(i) *there exist* $y^* \in S^+ \setminus \{0\}$ *and* $z^* \in Q^+$ *such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \quad \textit{for all } (y,z) \in C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))$$

and $\langle z^*, z_0\rangle = 0$;

(ii) *there exists* $\Lambda \in L^+(Z,Y)$ *such that* (x_0, y_0) *is a weak minimal solution of Problem* $(\overline{\mathrm{P}})$ *and* $\Lambda z_0 = 0$.

PROOF. Set

$$B := C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0)) + (0, z_0). \tag{3.16}$$

Since $C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)$ is a closed convex process and $C_A(x_0)$ is a closed convex cone, B is a closed convex set.

We show that $B \cap [-(\operatorname{int} S \times \operatorname{int} Q)] = \emptyset$. Otherwise there exist $u \in C_A(x_0)$, $y \in -\operatorname{int} S$ and $z \in -\operatorname{int} Q$ such that

$$(y, z - z_0) \in C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(u).$$

Thus, for every $h_n \to 0^+$ there exist $u_n \to u, \overline{u}_n \to u$, $y_n \to y$ and $z_n \to z - z_0$ such that for all $n \geq 1$,

$$x_0 + h_n \overline{u}_n \in A, \quad y_0 + h_n y_n \in \widehat{F}(x_0 + h_n u_n), \quad z_0 + h_n z_n \in \widehat{G}(x_0 + h_n u_n).$$

The Lipschitz properties of F and G imply that there exist $\overline{y}_n \to y$ and $\overline{z}_n \to z - z_0$ such that

$$y_0 + h_n \overline{y}_n \in \widehat{F}(x_0 + h_n \overline{u}_n), \quad z_0 + h_n \overline{z}_n \in \widehat{G}(x_0 + h_n \overline{u}_n).$$

Since $y \in -\operatorname{int} S$ and

$$\overline{y}_n = \frac{v_n - y_0}{h_n} \to y,$$

where $v_n = y_0 + h_n \overline{y}_n$, there exists N_1 such that $v_n - y_0 \in -\operatorname{int} S$ for $n \geq N_1$. Similarly, there exists N_2 such that

$$z_0 + \frac{w_n - z_0}{h_n} \in -\operatorname{int} Q \quad \text{for } n \geq N_2,$$

where $w_n = z_0 + h_n \overline{z}_n$. Moreover, there exists $N \geq \{N_1, N_2\}$ such that $h_N < 1$ and $(1 - h_N) z_0 \in -Q$. Thus

$$w_N = h_N \left(z_0 + \frac{w_N - z_0}{h_N} \right) + (1 - h_N) z_0 \in -\operatorname{int} Q. \tag{3.17}$$

Thus we have established that

$$x_N = x_0 + h_N \overline{u}_N \in A, \quad v_N \in \widehat{F}(x_N), \quad w_N \in \widehat{G}(x_N) \cap (-\operatorname{int} Q), \quad v_N - y_0 \in -\operatorname{int} S.$$

It is easy to show that $G(x_N) \cap (-Q) \neq \emptyset$ and $(F(x_N) - y_0) \cap (-\operatorname{int} S) \neq \emptyset$. This leads to a contradiction. Therefore

$$B \cap [-(\operatorname{int} S \times \operatorname{int} Q)] = \emptyset.$$

By a standard separation theorem, there exist $y^* \in S^+$ and $z^* \in Q^+$, not both zero, such that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \quad \text{for all } (y,z) \in B. \tag{3.18}$$

From (3.18), we get $\langle z^*, z_0\rangle \geq 0$. On the other hand, since $z_0 \in -Q$ and $z^* \in Q^+$, $\langle z^*, z_0\rangle \leq 0$. Thus, $\langle z^*, z_0\rangle = 0$. Consequently, (3.18) implies that

$$(3.19)\quad \langle y^*, y\rangle + \langle z^*, z\rangle \ge 0$$
$$\text{for all } u \in C_A(x_0) \cap \operatorname{dom} C(\widehat{F}\times\widehat{G})(x_0,y_0,z_0),\ (y,z) \in C(\widehat{F}\times\widehat{G})(x_0,y_0,z_0)(u).$$

We next show that $y^* \neq 0$. Assume that $y^* = 0$. Then $z^* \neq 0$ and

$$(3.20)\qquad \langle z^*, z + z_0\rangle \ge 0 \quad \text{for all } u \in C_A(x_0) \cap \operatorname{dom} C\widehat{G}(x_0,z_0),\ z \in C\widehat{G}(x_0,z_0)(u).$$

This, together with $0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0,z_0)(C_A(x_0))}$, implies that $z^* = 0$, a contradiction.

Since $(\widehat{F}\times\widehat{G})|_A$ is invex at (x_0,y_0,z_0), we deduce that

$$(3.21)\qquad (\widehat{F}\times\widehat{G})(A) - (y_0,z_0) \subset \overline{C(\widehat{F}\times\widehat{G})|_A(x_0,y_0,z_0)(X)} \subset \overline{C(\widehat{F}\times\widehat{G})(x_0,y_0,z_0)(C_A(x_0))}.$$

Hence

$$\langle y^*, y\rangle + \langle z^*, z\rangle \ge \langle y^*, y_0\rangle + \langle z^*, z_0\rangle = \langle y^*, y_0\rangle \quad \text{for all } (y,z) \in (\widehat{F}\times\widehat{G})(A).$$

Then we can repeat step by step the argument as in Theorem 3.1. ■

The condition $0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0,z_0)(C_A(x_0))}$ in Theorem 3.7 under which the functional $y^* \neq 0$ is called the *regularity condition*.

When $G(x) = g(x)$ is a continuously differentiable single-valued mapping and A is a closed and convex set, since $C\widehat{G}(x_0,z_0)(u) = g'(x_0)(u) + Q$, one can see that the above regularity condition reduces to the so-called *Robinson regularity condition* (see [80]), i.e.

$$0 \in \operatorname{int}[g(x_0) + g'(x_0)(C_A(x_0)) + Q].$$

If $G|_A$ is invex at (x_0,y_0) and $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, then

$$0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0,z_0)(C_A(x_0))}.$$

The necessary optimality condition (i) in Theorem 3.7 extends the so-called Karush–Kuhn–Tucker condition.

When $A = X$, the optimality condition in Theorem 3.7 has been obtained in [84] and [85] under a slightly stronger assumption.

Now we present a necessary optimality condition for (x_0,y_0) to be a Benson proper minimal solution of Problem (CP).

THEOREM 3.8. *Let (x_0,y_0) be a Benson proper minimal solution of* (CP). *Suppose that F and G are locally Lipschitz at x_0, $(F\times G)|_A$ is invex at (x_0,y_0,z_0) for some $z_0 \in G(x_0) \cap (-Q)$, and*

$$\operatorname{dom} CF(x_0,y_0) \supset \operatorname{dom} CG(x_0,z_0) \cap C_A(x_0).$$

Assume that either S has a weakly compact base and F is closure convexlike on X_0, or S has a compact base. If $0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0,z_0)(C_A(x_0))}$ and $\operatorname{int} Q \neq \emptyset$, then:

(i) *there exist $y^* \in S^{+i}$ and $z^* \in Q^+$ such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \ge 0 \quad \textit{for all } (y,z) \in C(\widehat{F}\times\widehat{G})(x_0,y_0,z_0)(C_A(x_0))$$

and $\langle z^*, z_0\rangle = 0$;

(ii) *there exists $\Lambda \in L^+(Z,Y)$ such that (x_0,y_0) is a proper minimal solution of* $(\overline{\mathrm{P}})$ *and $\Lambda z_0 = 0$.*

PROOF. From the proof of Theorem 3.6, there exists a pointed closed convex cone $C \subset Y$ such that $-S \setminus \{0\} \subset -\operatorname{int} C$ and

$$\overline{\operatorname{cone}}(R_{y_0}) \cap [-(\operatorname{int} C \times \operatorname{int} Q)] = \emptyset. \tag{3.22}$$

Since F and G are locally Lipschitz at x_0, by Proposition 1.9, we deduce that

$$C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0)) \subset T_{(\widehat{F}\times\widehat{G})(A)}(y_0, z_0). \tag{3.23}$$

We next show that

$$(0, z_0) + T_{(\widehat{F}\times\widehat{G})(A)}(y_0, z_0) \subset \overline{\operatorname{cone}}(R_{y_0}). \tag{3.24}$$

Let $(u, v) \in T_{(\widehat{F}\times\widehat{G})(A)}(y_0, z_0)$. Then there exist $h_n \to 0^+$, $(u_n, v_n) \to (u, v)$ and $x_n \in A$ such that for any n,

$$(y_0, z_0) + h_n(u_n, v_n) \in (\widehat{F} \times \widehat{G})(x_n).$$

Since $z_0 \in G(x_0) \cap (-Q)$, we have

$$h_n(v_n + z_0) = z_0 + h_n v_n - (1 - h_n)z_0 \in \widehat{G}(x_n) + Q \subset \widehat{G}(x_n).$$

Hence $(y_0, 0) + h_n(u_n, v_n + z_0) \in (\widehat{F} \times \widehat{G})(A)$ and so

$$(u, v + z_0) \in \overline{\operatorname{cone}}[(\widehat{F} \times \widehat{G})(A) - (y_0, 0)] = \overline{\operatorname{cone}}(R_{y_0}).$$

Therefore, it follows from (3.22)–(3.24) that

$$[(0, z_0) + C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))] \cap (-(\operatorname{int} C \times \operatorname{int} Q)) = \emptyset. \tag{3.25}$$

Since $C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)$ is a closed convex process and $C_A(x_0)$ is a closed convex cone, $C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))$ is convex.

By standard separation arguments, there exist $y^* \in C^+$ and $z^* \in Q^+$, not both zero, such that

$$\langle y^*, y\rangle + \langle z^*, z + z_0\rangle \ge 0 \quad \text{for all } (y, z) \in C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0)). \tag{3.26}$$

Since $z_0 \in G(x_0) \cap (-Q)$ and $(0, 0) \in C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))$, the inequality (3.26), with $z^* \in Q^+$, implies that $\langle z^*, z_0\rangle = 0$. Hence

$$\langle y^*, y\rangle + \langle z^*, z\rangle \ge 0 \quad \text{for all } (y, z) \in C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0)). \tag{3.27}$$

We only need to show that $y^* \in S^{+i}$. From the proof of Theorem 3.7, we see that $y^* \neq 0$. Hence $\langle y^*, y\rangle > 0$ for all $y \in \operatorname{int} C$. Since $S \setminus \{0\} \subset \operatorname{int} C$, we obtain $y^* \in S^{+i}$.

Moreover, since $(\widehat{F} \times \widehat{G})|_A$ is invex at (x_0, y_0, z_0), we deduce that

$$\begin{aligned}(\widehat{F} \times \widehat{G})(A) - (y_0, z_0) &\subset \overline{C(\widehat{F} \times \widehat{G})|_A(x_0, y_0, z_0)(X)} \\ &\subset \overline{C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))}.\end{aligned} \tag{3.28}$$

Hence, (3.26) implies that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \ge \langle y^*, y_0\rangle + \langle z^*, z_0\rangle = \langle y^*, y_0\rangle \quad \text{for all } (y, z) \in (\widehat{F} \times \widehat{G})(A). \tag{3.29}$$

Fix $e \in S \setminus \{0\}$ such that $\langle y^*, e\rangle = 1$ (such an e exists, since $y^* \in S^{+i}$). Define $\Lambda : Z \to Y$ by

$$\Lambda z = \langle z^*, z\rangle e \quad \text{for all } z \in Z.$$

Then $y^*\Lambda = z^*$, $\Lambda(Q) \subset S$ and $\Lambda z_0 = 0$. Replacing z^* by $y^*\Lambda$ in (3.29), we obtain

$$\langle y^*, y + \Lambda z\rangle \geq \langle y^*, y_0\rangle \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x). \tag{3.30}$$

Since $y_0 \in F(x_0) + \Lambda G(x_0)$ and $y^* \in S^{+i}$, by Theorem 2.4 we conclude that (x_0, y_0) is a proper minimal solution of Problem $(\overline{\mathrm{P}})$. ■

From the proof of Theorem 3.7 or Theorem 3.8, it is easy to see that when $A = X$, we need only assume that either F is locally Lipschitz at x_0 or G is locally Lipschitz at x_0.

EXAMPLE 3.2. Let $X = Z = \mathbb{R}^1$, $Y = \mathbb{R}^2$, $S = \mathbb{R}^2_+$, $Q = \mathbb{R}_+$ and $A = [0,1]$. Define set-valued mappings F and G as follows:

$$F(x) = \begin{cases} \{y = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 \leq x^2\} & \text{if } 0 < x \leq 1;\\ \{(-1/2, -1/2), (0,0)\} & \text{if } x = 0;\\ \emptyset, & \text{if } x < 0 \text{ or } x > 1, \end{cases}$$

$$G(x) = 1/2 - x.$$

Consider the vector optimization problem

$$(\mathrm{P}_1) \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in [0,1],\ G(x) \in -\mathbb{R}_+. \end{array}$$

Let $x_0 = 1$, $y_0 = (-\sqrt{2}/2, -\sqrt{2}/2)$ and $z_0 = G(x_0) = -1/2$. Then F and G are locally Lipschitz at x_0. Since $C\widehat{G}(x_0, z_0)(x) = -x + \mathbb{R}_+$ and $C_A(x_0) = \{x \in \mathbb{R}^1 \mid x \leq 0\}$, we have

$$\overline{z_0 + C\widehat{G}(x_0, z_0)(C_A(x_0))} = [-1/2, \infty).$$

Hence,

$$0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0, z_0)(C_A(x_0))}.$$

Notice that

$$X_0 = A \cap G^{-1}(-\mathbb{R}_+) = [1/2, 1], \quad F(X_0) = \{y = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 \leq 1\}.$$

It is easy to show that (x_0, y_0) is a proper minimal solution of (P_1).

We can easily verify that

$$C_{\operatorname{epi} F}(x_0, y_0) = C_{\operatorname{gr} F}(x_0, y_0) = \{(x, y) \mid y = (\xi_1, \xi_2),\ \xi_1 + \xi_2 \geq -\sqrt{2}\, x\}.$$

By the definition of $C\widehat{F}(x_0, y_0)(\cdot)$, we have

$$C\widehat{F}(x_0, y_0)(x) = \{y = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1 + \xi_2 \geq -\sqrt{2}\, x\}.$$

Hence $\operatorname{dom} C\widehat{F}(x_0, y_0) = \operatorname{dom} C\widehat{G}(x_0, z_0) = \mathbb{R}^1$ and

$$C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0)) = \{(y, z) \mid y = (\xi_1, \xi_2),\ \xi_1 + \xi_2 \geq 0,\ z \geq 0\}.$$

Let $y^* = (1,1) \in \operatorname{int} \mathbb{R}^2_+$ and $z^* = 0 \in \mathbb{R}_+$. One can see that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \quad \text{for all } (y, z) \in C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))$$

and $\langle z^*, z_0\rangle = 0$.

It is easy to verify that

$$(F \times G)(A) - (y_0, z_0) \subset C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0)).$$

Thus, $(F \times G)|_A$ is invex at (x_0, y_0, z_0).

Let $e = (1/2, 1/2) \in \mathbb{R}^2_+$. It is clear that $\langle y^*, e\rangle = 1$. Define the operator $\Lambda : \mathbb{R}^1 \to \mathbb{R}^2$ by

$$\Lambda z = \langle z^*, z\rangle e = (0,0) \quad \text{for } z \in \mathbb{R}.$$

Then $\Lambda \in L^+(Z, Y)$. Thus, Problem $(\overline{\mathrm{P}})$ is of the form

$$(\overline{\mathrm{P}}_1) \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in [0,1]. \end{array}$$

It is evident that (x_0, y_0) is also a proper minimal solution of $(\overline{\mathrm{P}}_1)$.

THEOREM 3.9. *Let $(x_0, y_0) \in \operatorname{gr} F$ be a minimal solution of* (CP). *Assume that F, G are locally Lipschitz at x_0 and $(F \times G)|_A$ is invex at (x_0, y_0, z_0) for some $z_0 \in G(x_0) \cap (-Q)$. Assume that S has a weakly compact base. Then the following statements are equivalent*:

(i) *the image regularity condition* (3.5) *holds*;

(ii) *there exist $y^* \in S^{+i}$ and $z^* \in Q^+$ such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \ge 0 \quad \textit{for all } (y,z) \in R_{y_0},$$

and $\langle z^*, z_0\rangle = 0$;

(iii) *there exists a continuous linear positive operator $\Lambda \in L^+(Z, Y)$ such that (x_0, y_0) is a Benson proper minimal solution of the problem*

$$(\overline{\mathrm{P}}) \qquad \begin{array}{l} \min(F(x) + \Lambda G(x)) \\ \textit{subject to } x \in A \end{array}$$

and $\Lambda z_0 = 0$.

PROOF. Since $(\widehat{F} \times \widehat{G})|A$ is invex at (x_0, y_0, z_0), we deduce that

$$\begin{aligned}(\widehat{F} \times \widehat{G})(A) - (y_0, z_0) &\subset \overline{C(\widehat{F} \times \widehat{G})|_A(x_0, y_0, z_0)(X)} \\ &\subset \overline{C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))}.\end{aligned}$$

From the proof of Theorem 3.8, we have

$$(0, z_0) + \overline{C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))} \subset \overline{\operatorname{cone}} R_{y_0}.$$

Hence

$$\overline{\operatorname{cone}} R_{y_0} = \overline{\operatorname{cone}}[(\widehat{F} \times \widehat{G})(A) - (y_0, 0)] = (0, z_0) + \overline{C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))}.$$

Since $C(\widehat{F} \times \widehat{G})(x_0, y_0, z_0)(C_A(x_0))$ is convex, $\overline{\operatorname{cone}} R_{y_0}$ is convex and so weakly closed.

Then we can repeat step by step the same arguments as in the proof of Theorem 3.5. ∎

4. Lagrangian duality

In this section, based on the Lagrangian formalism of Section 3, the Lagrangian duality for weak minimality, minimality and proper minimality is presented.

4.1. Duality for weak optimality. We are concerned with the vector optimization problem (CP) introduced in Section 3, i.e.,

$$\text{(CP)} \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in A,\ G(x) \cap (-Q) \neq \emptyset. \end{array}$$

Define a set-valued mapping $\Phi : L^+(Z,Y) \to Y$ by

$$\Phi(\Lambda) = \mathrm{WMin}((F + \Lambda G)(A), S),$$

where $(F + \Lambda G)(A) = \bigcup_{x\in A} F(x) + \Lambda G(x)$, and consider the following maximization problem:

$$\text{(LD)} \qquad \begin{array}{l} \max \Phi(\Lambda) \\ \text{subject to } \Lambda \in L^+(Z,Y). \end{array}$$

We say that (Λ_0, y_0) is a *weak maximal solution* of Problem (LD) if $\Lambda_0 \in L^+(Z,Y)$, $y_0 \in \Phi(\Lambda_0)$, and there is no $\Lambda \in L^+(Z,Y)$ such that

$$(y_0 - \Phi(\Lambda)) \cap (-\operatorname{int} S) \neq \emptyset. \tag{4.1}$$

We have the following duality results.

THEOREM 4.1. *Let* $\operatorname{int} S \neq \emptyset$.

(i) *If* Λ_0 *is a feasible point of Problem* (LD) *and* x_0 *is a feasible point of Problem* (CP), *then*

$$\Phi(\Lambda_0) \cap (F(x_0) + \operatorname{int} S) = \emptyset. \tag{4.2}$$

(ii) *Assume that either of the following conditions holds*:

(a) $F \times G$ *is closure convexlike on* A;

(b) $A \subset X$ *is a convex set*, $F : A \to Y$ *and* $G : A \to Z$ *are set-valued mappings with bounded convex images*, $F \times G$ *is* $*$*-quasiconvex on* A, *and for every* $y^* \in S^+$ *and* $z^* \in Q^+, x \to \inf_{(y,z)\in(F\times G)(x)}(\langle y^*, y\rangle + \langle z^*, z\rangle)$ *is a lower semicontinuous function.*

If (x_0, y_0) *is a weak minimal solution of* (CP) *and* $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, *then there exists* $\Lambda_0 \in L^+(Z,Y)$ *such that* (Λ_0, y_0) *is a weak maximal solution of* (LD).

PROOF. (i) Suppose that Λ_0 is a feasible point for (LD). By the definition, for every $y \in \Phi(\Lambda_0)$, we have

$$[(F + \Lambda_0 G)(A) - y] \cap (-\operatorname{int} S) = \emptyset.$$

We show that

$$y \notin F(x_0) + \operatorname{int} S.$$

Indeed, otherwise there is $y_0 \in F(x_0)$ such that $y_0 - y \in -\operatorname{int} S$. Since x_0 is a feasible point of (CP), there exists $z_0 \in G(x_0) \cap (-Q)$. Since $\Lambda_0 \in L^+(Z,Y)$, it follows that $\Lambda_0 z_0 \in -S$. Thus,

$$y_0 + \Lambda_0 z_0 - y \in -\operatorname{int} S + \Lambda_0 z_0 \subset -\operatorname{int} S.$$

This means that $[(F + \Lambda_0 G)(A) - y] \cap (-\operatorname{int} S) \neq \emptyset$, a contradiction.

Hence $y \notin F(x_0) + \operatorname{int} S$. Since $y \in \Phi(\Lambda_0)$ is arbitrary, we obtain (4.2).

(ii) Suppose that (x_0, y_0) is a weak minimal solution of (CP) and $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$. By Theorem 3.1 or Theorem 3.2, there exists $\Lambda_0 \in L^+(Z, Y)$ such that (x_0, y_0) is a weak minimal solution of $(\overline{\mathrm{P}})$ corresponding to Λ_0. Hence Λ_0 is feasible for (LD) and $y_0 \in \Phi(\Lambda_0)$. From (i), it follows that for every feasible point Λ for Problem (LD), we have

$$(y_0 - \Phi(\Lambda)) \cap (-\operatorname{int} S) = \emptyset.$$

Therefore, (Λ_0, y_0) is a weak maximal solution of (LD). ■

Theorem 4.1 under assumption (a) generalizes Theorems 4.2 and 4.3 of [13] to the generalized convex case.

4.2. Duality for optimality. First we present a general duality theorem. We let Ψ denote the set of all continuous sublinear functionals ψ defined on $Y \times Z$ which satisfy:

(c) for each $z \in Z$, $\psi(\cdot, z)$ is strictly S-increasing on Y;
(d) for each $y \in Y$, $\psi(y, \cdot)$ is Q-increasing on Z.

Define a set-valued mapping $\Phi' : \Psi \to Y$ by

$$\Phi'(\psi) = \{y \in Y \mid 0 = \min_{x \in A} \psi(F(x) - y, G(x))\}.$$

The maximization problem

$$(\mathrm{LD}') \qquad \begin{array}{l} \max \Phi'(\psi) \\ \text{subject to } \psi \in \Psi \end{array}$$

will be called the *dual* of Problem (CP).

We say that (ψ_0, y_0) is a *maximal solution* of problem (LD′) if $\psi_0 \in \Psi$, $y_0 \in \Phi'(\psi_0)$, and there is no $\psi \in \Psi$ such that

$$(y_0 - \Phi'(\psi)) \cap (-S \setminus \{0\}) \neq \emptyset.$$

THEOREM 4.2. (i) *If x_0 is a feasible point of* (CP) *and ψ_0 is a feasible point of* (LD′), *then*

$$\Phi'(\psi_0) \cap (F(x_0) + S \setminus \{0\}) = \emptyset.$$

(ii) *Let Y be a locally convex space. Assume that either S has a weakly compact base and $\overline{\operatorname{cone}}(R_{y_0})$ is weakly closed, or S has a compact base. If (x_0, y_0) is a minimal solution of* (CP) *and the image regularity condition* (3.5) *holds, then there exists a $\psi_0 \in \Psi$ such that (ψ_0, y_0) is a maximal solution of* (LD′).

PROOF. (i) If ψ_0 is a feasible point of (LD′), then by the definition, for every $y \in \Phi'(\psi_0)$, we have

$$0 = \min_{x \in A} \psi_0(F(x) - y, G(x)).$$

We show that

$$y \notin F(x_0) + S \setminus \{0\}.$$

Assume the contrary; i.e., there exists $s_0 \in S \setminus \{0\}$ such that $-s_0 \in F(x_0) - y$. Since x_0 is a feasible point of (CP), there is $z_0 \in G(x_0) \cap (-Q)$. Hence, $0 \leq \psi(-s_0, z_0)$. Conditions (c) and (d) imply that $\psi(-s_0, z_0) < \psi(0, z_0) \leq \psi(0, 0) = 0$. This is a contradiction. Therefore, $\Phi'(\psi_0) \cap (F(x_0) + S \setminus \{0\}) = \emptyset$.

(ii) If (x_0, y_0) is a minimal solution of (CP), then by Theorem 3.4 there exists $\psi_0 \in \Psi$ such that

$$0 = \min_{\xi \in A} \psi_0(F(\xi) - y_0, G(\xi)).$$

This means that ψ_0 is a feasible point of (LD′) and $y_0 \in \Phi'(\psi_0)$. By (i), for every feasible point ψ of (LD′), we have $(y_0 - \Phi'(\psi)) \cap (-S \setminus \{0\}) = \emptyset$. Thus, (ψ_0, y_0) is a maximal solution of (LD′). ■

Luc and Jahn [63] obtain a similar result which is based on the Henig proper minimal solution (see [36]) under a stronger condition. Under the hypotheses of Theorem 3.5 or Theorem 3.6, we can obtain Lagrangian duality theorems. Define a set-valued mapping $\overline{\Phi} : L^+(Z, Y) \to Y$ by

$$\overline{\Phi}(\Lambda) = \mathrm{Be}((F + \Lambda G)(A), S).$$

Consider the maximization problem

$$(\mathrm{LD}'') \qquad \begin{array}{l} \max \overline{\Phi}(\Lambda) \\ \text{subject to } \Lambda \in L^+(Z, Y). \end{array}$$

Using an argument similar to the above, we can prove the following duality results:

Theorem 4.3. (i) *If Λ_0 is a feasible point of Problem* (LD″) *and x_0 is a feasible point of Problem* (CP), *then*

$$\overline{\Phi}(\Lambda_0) \cap (F(x_0) + S \setminus \{0\}) = \emptyset.$$

(ii) *Let Y be a locally convex space. Assume that $F \times G$ is closure convexlike on A and S has a weakly compact base. If (x_0, y_0) is a minimal point of* (CP) *and the image regularity condition* (3.5) *holds*, *then there exists a $\Lambda_0 \in L^+(Z, Y)$ such that (Λ_0, y_0) is a maximal solution of* (LD″).

Theorem 4.4. *Let Y be a locally convex space. Assume that $F \times G$ is closure convexlike on A. Assume that either S has a weakly compact base and F is closure convexlike on $X_0 = A \cap G^{-1}(-Q)$, or S has a compact base. If (x_0, y_0) is a proper minimal solution of* (CP) *and $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, then there exists a $\Lambda_0 \in L^+(Z, Y)$ such that (Λ_0, y_0) is a maximal solution of* (LD″).

A similar duality result for Henig proper minimal solutions has been proved in [63] under a slightly stronger assumption.

4.3. Duality for invex set-valued mappings. In this subsection, we assume that X, Y, Z are normed spaces and A is a subset of X. Consider the dual problem (LD) introduced in Subsection 4.1.

A similar argument to that for Theorem 4.1 gives the following theorem:

Theorem 4.5. *Let $\operatorname{int} S \neq \emptyset$, $\operatorname{int} Q \neq \emptyset$ and let (x_0, y_0) be a weakly minimal solution of* (CP). *Suppose that F and G are locally Lipschitz at x_0, $(F \times G)|_A$ is invex at (x_0, y_0, z_0) for some $z_0 \in G(x_0) \cap (-Q)$, and*

$$\operatorname{dom} CF(X_0, y_0) \supset \operatorname{dom} CG(x_0, z_0) \cap C_A(x_0).$$

If (x_0, y_0) *is a weak minimal solution of* (CP) *and*

$$0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0, z_0)(C_A(x_0))},$$

then there exists $\Lambda_0 \in L^+(Z, Y)$ *such that* (Λ_0, y_0) *is a weak maximal solution of* (LD).

For the dual problem (LD″) introduced in Subsection 4.2, we can prove the following duality results:

THEOREM 4.6. *Let* (x_0, y_0) *be a proper minimal solution of* (CP). *Suppose that* F *and* G *are locally Lipschitz at* x_0, $(F \times G)|_A$ *is invex at* (x_0, y_0, z_0) *for some* $z_0 \in G(x_0) \cap (-Q)$, *and*

$$\operatorname{dom} CF(x_0, y_0) \supset \operatorname{dom} CG(x_0, z_0) \cap C_A(x_0).$$

Assume that either S *has a weakly compact base and* F *is closure convexlike on* X_0, *or* S *has a compact base. If*

$$0 \in \operatorname{int} \overline{z_0 + C\widehat{G}(x_0, z_0)(C_A(x_0))}, \quad \operatorname{int} Q \neq \emptyset,$$

then there exists $\Lambda_0 \in L^+(Z, Y)$ *such that* (Λ_0, y_0) *is a maximal solution of* (LD″).

THEOREM 4.7. *Let* (x_0, y_0) *be a minimal solution of* (CP). *Suppose that* F *and* G *are locally Lipschitz at* x_0 *and* $(F \times G)|_A$ *is invex at* (x_0, y_0, z_0) *for some* $z_0 \in G(x_0) \cap (-Q)$. *Suppose that* S *has a weakly compact base. If the image regularity condition* (3.5) *holds*, *then there exists* $\Lambda_0 \in L^+(Z, Y)$ *such that* (Λ_0, y_0) *is a maximal solution of* (LD″).

5. Geometric duality

In the previous section, we have obtained the duality theorems for the primal problem (CP) and dual problem (LD) or (LD″), i.e., each optimal value of the primal problem is also an optimal value of the dual problem. This duality theory is not complete since the converse duality is not necessarily true. In this section, we investigate a geometric approach to duality in vector optimization introduced by Jahn [44] and Nakayama [69], which is based on the duality theory of [88], and we show that both the duality and converse duality hold under some weaker assumptions.

5.1. A general duality principle for sets. Let Y be a Hausdorff topological vector space, let S be a pointed convex cone in Y, and let C be a nonempty subset of Y.

Let $\operatorname{int} S \neq \emptyset$. Define $D_0 = Y \setminus \overline{C + S}$, $D_1 = Y \setminus (C + \operatorname{int} S)$ and

$$D_1' = \{y_0 \in Y \mid \text{there exists } y^* \in S^+ \setminus \{0\} \text{ such that } \langle y^*, y\rangle \geq \langle y^*, y_0\rangle \text{ for all } y \in C\}.$$

It is obvious that $D_0 \subset D_1$ and $D_1' \subset D_1$. On the other hand, since $C + \operatorname{int} S = \operatorname{int} \overline{C + S}$ (see Lemma 1.5), we have

$$D_1 = Y \setminus \operatorname{int} \overline{C + S} \subset \overline{Y \setminus \overline{C + S}} = \overline{D}_0.$$

We couple the primal problem

$$\min \overline{C + S} \tag{5.1}$$

with the dual problem

$$\max D_1. \tag{5.2}$$

Then we have the following results.

PROPOSITION 5.1. *Let* $\operatorname{int} S \neq \emptyset$. *Then*:

(i) $D_1 \cap (\overline{C+S} + \operatorname{int} S) = \emptyset$;

(ii) *if* $y \in \overline{C+S} \cap D_1$, *then* $y \in \operatorname{WMin}(\overline{C+S}, S)$ *and* $y \in \operatorname{WMax}(D_1, S)$, *and if* $y \in C \cap D_1$, *then* $y \in \operatorname{WMin}(C, S)$ *and* $y \in \operatorname{WMax}(D_1, S)$;

(iii) $\operatorname{WMin}(C, S) \subset \operatorname{WMin}(\overline{C+S}, S) = \operatorname{WMax}(D_1, S)$;

(iv) *if* C *is closure* S-*convex*, *then* $D_1 = D_1'$, *consequently*, $\operatorname{WMax}(D_1', S) = \operatorname{WMax}(D_1, S) = \operatorname{WMin}(\overline{C+S}, S)$.

PROOF. (i) Assume on the contrary that there exist $y_0 \in \overline{C+S}$ and $y_1 \in D_1$ such that $y_1 - y_0 \in \operatorname{int} S$. Hence, $y_1 \in \overline{C+S} + \operatorname{int} S$. By Lemma 1.5,

$$\overline{C+S} + \operatorname{int} S = \operatorname{int} \overline{\overline{C+S} + S} = \operatorname{int} \overline{C+S} = C + \operatorname{int} S.$$

Thus $y_1 \in C + \operatorname{int} S$, a contradiction.

(ii) follows directly from (i).

(iii) The first inclusion is obvious. If $y \in \operatorname{WMin}(\overline{C+S}, S)$, then

$$y \in \overline{C+S} \setminus \operatorname{int} \overline{C+S} = \overline{C+S} \setminus (C + \operatorname{int} S).$$

Hence $y \in D_1$. It follows from (ii) that $y \in \operatorname{WMax}(D_1, S)$. If $y \in \operatorname{WMax}(D_1, S)$, then y is a boundary point of D_1. Since $D_0 \subset D_1 \subset \overline{D}_0$ and D_0 is open, we have $y \in \overline{C+S}$. It follows from (ii) that $y \in \operatorname{WMin}(\overline{C+S}, S)$.

(iv) By Lemma 2.1, we have $D_1 = D_1'$. ∎

Let $S^{+i} \neq \emptyset$. Define $D_2 = Y \setminus (C + S \setminus \{0\})$ and

$$D_2' = \{y_0 \in Y \mid \text{there exists } y^* \in S^{+i} \text{ such that } \langle y^*, y\rangle \geq \langle y^*, y_0\rangle \text{ for all } y \in C\}.$$

It is obvious that $D_2' \subset D_2$.

The following general duality principle has been introduced by Jahn [44]. We couple the primal problem

$$\min C \tag{5.3}$$

with the dual problem

$$\max D_2. \tag{5.4}$$

Then we have the following results.

PROPOSITION 5.2. (i) $D_2 \cap (C + S \setminus \{0\}) = \emptyset$;

(ii) *if* $y \in C \cap D_2$, *then* $y \in \operatorname{Min}(C, S)$ *and* $y \in \operatorname{Max}(D_2, S)$;

(iii) $\operatorname{Min}(C, S) \subset \operatorname{Max}(D_2, S)$;

(iv) *if* $C + S$ *is closed and convex*, *then* $\operatorname{Max}(D_2, S) \subset \operatorname{Min}(C, S)$;

(v) *let* Y *be a locally convex space*; *if* S *has a weakly compact base in* Y *and* C *is closure* S-*convex*, *then* $\operatorname{Be}(C, S) \subset \operatorname{Max}(D_2', S)$, *and the converse inclusion holds if*, *in addition*, $C + S$ *is closed and convex.*

PROOF. (i), (ii) and (iii) can be proved similarly to Proposition 5.1.

(iv) Let $y \in \mathrm{Max}(D_2, S)$, and assume that $y \notin C + S$. Since $Y \setminus (C + S)$ is open, there exists a 0-neighbourhood U in Y such that $y + U \subset Y \setminus (C + S)$. Take $s \in S \setminus \{0\}$ and a sufficiently small $\lambda > 0$ such that $\lambda s \in U$. Then $y + \lambda s \in D_2$, which contradicts the maximality of y. Consequently, $y \in C + S$, and since $y \notin C + S \setminus \{0\}$, we conclude that $y \in C$. By (i), we get $y \in \mathrm{Min}(C, S)$.

(v) Let $y_0 \in \mathrm{Be}(C, S)$. By Theorem 2.2, we have $y_0 \in D_2'$. From $D_2' \subset D_2$ and (i), we get $y_0 \in \mathrm{Max}(D_2', S)$.

Let $y_0 \in \mathrm{Max}(D_2', S)$, and assume that $y_0 \notin \overline{C + S}$. By a separation theorem there exists a linear continuous functional $y_0^* \in Y^* \setminus \{0\}$ and a number α with

$$\langle y_0^*, y_0\rangle < \alpha \le \langle y_0^*, y\rangle \quad \text{for all } y \in \overline{C + S}.$$

From $\overline{C + S} + S = \overline{C + S}$, we deduce that $y_0^* \in S^+ \setminus \{0\}$. Since $y_0 \in D_2'$, there exists $y_1^* \in S^{+i}$ such that

$$\langle y_1^*, y_0\rangle \le \langle y_1^*, y\rangle \quad \text{for all } y \in C.$$

Next we define a continuous linear functional

$$y^* = \tfrac{1}{2} y_1^* + \tfrac{1}{2} y_0^*$$

which belongs to S^{+i}. Then we obtain

$$\langle y^*, y_0\rangle < \tfrac{1}{2}\langle y_1^*, y_0\rangle + \tfrac{1}{2}\alpha \le \langle y^*, y\rangle \quad \text{for all } y \in C + S.$$

Hence,

$$\langle y^*, y_0\rangle < \inf_{y \in \overline{C+S}} \langle y^*, y\rangle.$$

Take $s \in S \setminus \{0\}$ and $\lambda > 0$ sufficiently small such that

$$\langle y^*, y_0 + \lambda s\rangle \le \inf_{y \in \overline{C+S}} \langle y^*, y\rangle.$$

Then $y_0 + \lambda s \in D_2'$, which contradicts the maximality of y_0. Consequently, $y_0 \in \overline{C + S} = C + S$, and since $D_2' \subset D_2$ and (i), we conclude that $y_0 \in C$. By Theorem 2.2, we get $y_0 \in \mathrm{Be}(C, S)$. ■

5.2. A geometric approach to duality. In this subsection, we introduce geometric dual problems for the primal problem (CP) and show that both the duality and converse duality theorems hold for weak and proper minimal solutions.

We are concerned with the vector optimization problem (CP) introduced in Section 3, i.e.,

$$\text{(CP)} \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in X_0 = \{x' \in A \mid G(x') \cap (-Q) \ne \emptyset\}. \end{array}$$

For the weak minimality of (CP), we define

$$D_1 = Y \setminus (F(X_0) + \operatorname{int} S),$$

$$\begin{aligned} D_1'' = \{y_0 \in Y \mid {} & \text{there exist } y^* \in S^+ \setminus \{0\} \text{ and } z^* \in Q^+ \text{ such that} \\ & \langle y^*, y\rangle + \langle z^*, z\rangle \ge \langle y^*, y_0\rangle \text{ for all } x \in A,\ y \in F(x) \text{ and } z \in G(x)\}, \end{aligned}$$

and a set-valued mapping $\Phi_1 : L^+(Z, Y) \to Y$ by

$$\Phi_1(\Lambda) = \{y \in Y \mid [(F + \Lambda G)(A) - y] \cap (-\operatorname{int} S) = \emptyset\},$$

where $(F+\Lambda G)(A)=\bigcup_{x\in A}(F(x)+\Lambda G(x))$. Consider the following maximization problem:

$$(\mathrm{D}_1)\qquad \begin{array}{l}\max \Phi_1(\Lambda)\\ \text{subject to } \Lambda\in L^+(Z,Y).\end{array}$$

For the proper minimality, we define

$$D_2=Y\setminus(F(X_0)+S\setminus\{0\}),$$

$$D_2'=\{y_0\in Y \mid \text{there exists } y^*\in S^{+i} \text{ such that } \langle y^*,y\rangle\geq\langle y^*,y_0\rangle \text{ for all } y\in F(X_0)\},$$

$$D_2''=\{y_0\in Y \mid \text{there exist } y^*\in S^{+i} \text{ and } z^*\in Q^+ \text{ such that } \langle y^*,y\rangle+\langle z^*,z\rangle\geq\langle y^*,y_0\rangle \text{ for all } x\in A,\ y\in F(x) \text{ and } z\in G(x)\},$$

and a set-valued mapping $\Phi_2: L^+(Z,Y)\to Y$ by

$$\Phi_2(\Lambda)=\{y\in Y \mid \overline{\mathrm{cone}}[(F+\Lambda G)(A)+S-y]\cap(-S)=\{0\}.$$

Consider the following maximization problem:

$$(\mathrm{D}_2)\qquad \begin{array}{l}\max \Phi_2(\Lambda)\\ \text{subject to } \Lambda\in L^+(Z,Y).\end{array}$$

LEMMA 5.1. (i) *Let* $\operatorname{int} S\neq\emptyset$. *Then* $D_1''\subset\Phi_1(L^+(Z,Y))\subset D_1$. *If, in addition,* $F\times G$ *is closure convexlike on* A *and* $G(A)\cap(-\operatorname{int} Q)\neq\emptyset$, *then* $D_1''=\Phi_1(L^+(Z,Y))=D_1$.

(ii) *Let* Y *be a locally convex space and let* $S^{+i}\neq\emptyset$. *Then* $D_2''\subset\Phi_2(L^+(Z,Y))\subset D_2$. *If, in addition,* S *has a weakly compact base in* Y, $F\times G$ *is closure convexlike on* A *and* $G(A)\cap(-\operatorname{int} Q)\neq\emptyset$, *then* $D_2'=D_2''=\Phi_2(L^+(Z,Y))$.

PROOF. (i) From the proof of Theorem 3.1(ii), it is easy to see that $D_1''\subset\Phi_1(L^+(Z,Y))$. Let $y_1\in\Phi_1(\Lambda)$ for some $\Lambda\in L^+(Z,Y)$, and assume that $y_1\notin D_1$. Then $y_1\in F(X_0)+\operatorname{int} S$. Thus there exist $x\in X_0$, $y\in F(x)$ such that $y-y_1\in-\operatorname{int} S$. Since $\Lambda\in L^+(Z,Y)$ and $G(x)\cap(-Q)\neq\emptyset$, there exists $z\in G(x)$ such that $\Lambda z\in -S$. Consequently, $y+\Lambda z-y_1\in-\operatorname{int} S-S\subset-\operatorname{int} S$, which contradicts $y_1\in\Phi_1(\Lambda)$.

Let $y_1\in D_1$. Then $(F(X_0)-y_1)\cap(-\operatorname{int} S)=\emptyset$. We claim that

$$[(F\times G)(A)-(y_1,0)]\cap[-(\operatorname{int} S\times\operatorname{int} Q)]=\emptyset.$$

Indeed, assume that there exist $s\in\operatorname{int} S$ and $q\in\operatorname{int} Q$ such that

$$(-s,-q)\in(F\times G)(A)-(y_1,0).$$

It then follows that there exists $x\in X_0$ such that $-s\in F(x)-y_1$, which is a contradiction.

By using a similar argument to the proof of Theorem 3.1(i), we obtain $y_1\in D_1''$.

(ii) From the proof of (ii)⇒(iii) in Theorem 3.5, it is easy to see that the first inclusion holds. The second inclusion can be proved similarly to the counterpart of (i).

Let $y_2\in\Phi_2(\Lambda)$ for some $\Lambda\in L^+(Z,Y)$. From the proof of (iii)⇒(i) in Theorem 3.5, we obtain

$$\overline{\mathrm{cone}}[(F\times G)(A)+S\times Q-(y_2,0)]\cap(-S\setminus\{0\}\times\{0\})=\emptyset.$$

From the proof of (i)⇒(ii) in Theorem 3.5, we conclude that there exist $y^*\in S^{+i}$ and $z^*\in Q^+$ such that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq \langle y^*, y_2\rangle \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x).$$

This proves that $y_2 \in D_2''$. It is clear that $D_2'' \subset D_2'$.

Let $y_2 \in D_2'$. Then there exists $y^* \in S^{+i}$ such that

$$[(y^* \circ F)(X_0) - \langle y^*, y_2\rangle] \cap (-\operatorname{int} \mathbb{R}_+) = \emptyset.$$

By using similar arguments to the proof of the second part of (i) above, we can show that

$$[((y^* \circ F) \times G)(A) - (\langle y^*, y_2\rangle, 0)] \cap [-(\operatorname{int} \mathbb{R}_+ \times \operatorname{int} Q)] = \emptyset.$$

Note that $y^*(S) = \{\langle y^*, s\rangle \mid s \in S\} \subset \mathbb{R}_+$. It is easy to show that

$$\overline{[((y^* \circ F) \times G)(A) - (\langle y^*, y_2\rangle, 0) + y^*(S) \times Q]} \cap [-(\operatorname{int} \mathbb{R}^+ \times \operatorname{int} Q)] = \emptyset.$$

Since $\overline{(F \times G)(A) + S \times Q}$ is convex, it is obvious that

$$\overline{[(y^* \circ F) \times G](A) - (\langle y^*, y_2\rangle, 0) + y^*(S) \times Q}$$

is also convex. By the standard separation theorem, there exist $r \in \mathbb{R}_+$ and $z^* \in Q^+$, not both zero, such that

$$\langle ry^*, y\rangle + \langle z^*, z\rangle \geq \langle ry^*, y_2\rangle \quad \text{for all } x \in A,\ y \in F(x) \text{ and } z \in G(x).$$

Since $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, it is easy to show that $r > 0$, and so $ry^* \in S^{+i}$. Hence $y_2 \in D_2''$. ■

It can be similarly proved that the assertion of Lemma 5.1(i) remains true under the assumptions of Theorem 3.2. By applying Propositions 5.1, 5.2 and Lemma 5.1, we obtain the following results.

THEOREM 5.1. (I) *Assume that* $\operatorname{int} S \neq \emptyset$.

(i) *If* $x \in X_0$ *and* $\Lambda \in L^+(Z, Y)$, *then*

$$\Phi_1(\Lambda) \cap (F(x) + \operatorname{int} S) = \emptyset.$$

(ii) *If* $y \in F(x) \cap \Phi_1(\Lambda)$, $x \in X_0$ *and* $\Lambda \in L^+(Z, Y)$, *then* (x, y) *is a weak minimal solution of* (CP) *and* (Λ, y) *is a weak maximal solution of* (D_1).

(iii) *If* $F \times G$ *is closure convexlike on* A *and* $G(A) \cap (-\operatorname{int} Q) \neq \emptyset$, *then*

$$\begin{aligned}\operatorname{WMin}(F(X_0), S) \subset \operatorname{WMin}(\overline{F(X_0) + S}, S) &= \operatorname{WMax}(D_1'', S)\\ &= \operatorname{WMax}(\Phi_1(L^+(Z, Y)), S).\end{aligned}$$

(II) *Assume that* $S^{+i} \neq \emptyset$.

(i) *If* $x \in X_0$ *and* $\Lambda \in L^+(Z, Y)$, *then*

$$\Phi_2(\Lambda) \cap (F(x) + S \setminus \{0\}) = \emptyset.$$

(ii) *If* $y \in F(x) \cap \Phi_2(\Lambda)$, $x \in X_0$ *and* $\Lambda \in L^+(Z, Y)$, *then* (x, y) *is a Benson proper minimal solution of* (CP) *and* (Λ, y) *is a maximal solution of* (D_2).

(iii) *Let* Y *be a locally convex space. If* S *has a weakly compact base in* Y, $F \times G$ *is closure convexlike on* A *and* $F(X_0) + S$ *is closed and convex*, *then*

$$\operatorname{Be}(F(X_0), S) = \operatorname{Max}(D_2'', S) = \operatorname{Max}(\Phi_2(L^+(Z, Y)), S).$$

From Theorem 5.1, we see that both the duality and converse duality for problems (CP) and (D_2), $\min \overline{F(X_0) + S}$ and (D_1) hold.

Remark 5.1. 1. The results of (I)(iii) of Theorem 5.1 remain true under the assumptions of Theorem 3.2. The duality results

$$\mathrm{WMin}(\overline{F(X_0)+S}, S) = \mathrm{WMax}(D_2'', S), \quad \mathrm{Be}(F(X_0), S) = \mathrm{Max}(D_2'', S)$$

extend the corresponding results in [42], [44] and [69] to the generalized convex and set-valued case, and the second result also refines the corresponding result in [53].

2. The set $R_0 = (F \times G)(A) + S \times Q$ is called the *extended image* of Problem (CP). It is easy to show that, if R_0 is closed, then $F(X_0) + S$ is closed. Some conditions ensuring the closedness of R_0 can be found in [8], [32], [53]. When F and G are single-valued mappings, the closedness and the convexity of the extended image R_0 play an important role in duality, stability, and existence of solutions for optimization problems (see [8], [32], [48], [53], [59], [74], [103], [68], [102]).

5.3. Linear optimization problems. In this subsection we specialize Theorem 5.1 to linear problems. Let X, Y and Z be Banach spaces. Let $K \subset X$, $S \subset Y$ and $Q \subset Z$ be convex pointed cones.

Let $C : X \to Y$ and $A : X \to Z$ be closed convex processes, and let $b \in Z$ be a fixed vector.

Consider the problem

$$(\mathrm{CP}_1) \qquad \begin{array}{l} \min C(x) \\ \text{subject to } x \in X_0 = \{x' \in K \mid (A(x') - b) \cap Q \neq \emptyset\}. \end{array}$$

In this special case, the two dual sets D_1'' and D_2'' in the previous section can be formalized as:

$$\begin{aligned} D_1'' = \{\overline{y} \in Y \mid {} & \text{there exist } y^* \in S^+ \setminus \{0\} \text{ and } z^* \in Q^+ \text{ such that} \\ & \langle y^*, y\rangle - \langle z^*, z\rangle \geq \langle y^*, \overline{y}\rangle - \langle z^*, b\rangle \text{ for all } x \in K,\ y \in C(x) \text{ and } z \in A(x)\}, \\ D_2'' = \{\overline{y} \in Y \mid {} & \text{there exist } y^* \in S^{+i} \text{ and } z^* \in Q^+ \text{ such that} \\ & \langle y^*, y\rangle - \langle z^*, z\rangle \geq \langle y^*, \overline{y}\rangle - \langle z^*, b\rangle \text{ for all } x \in K,\ y \in C(x) \text{ and } z \in A(x)\}. \end{aligned}$$

Since K is a cone and C, A are closed convex processes, it follows that

$$\begin{aligned} D_1'' = \{\overline{y} \in Y \mid {} & \text{there exist } y^* \in S^+ \setminus \{0\} \text{ and } z^* \in Q^+ \text{ such that} \\ & \langle y^*, \overline{y}\rangle \leq \langle z^*, b\rangle,\ \langle y^*, y\rangle - \langle z^*, z\rangle \geq 0 \text{ for all } x \in K,\ y \in C(x) \text{ and } z \in A(x)\}, \\ D_2'' = \{\overline{y} \in Y \mid {} & \text{there exist } y^* \in S^{+i} \text{ and } z^* \in Q^+ \text{ such that} \\ & \langle y^*, \overline{y}\rangle \leq \langle z^*, b\rangle,\ \langle y^*, y\rangle - \langle z^*, z\rangle \geq 0 \text{ for all } x \in K,\ y \in C(x) \text{ and } z \in A(x)\}. \end{aligned}$$

Define

$$\begin{aligned} \overline{D}_1'' = \{\overline{y} \in Y \mid {} & \text{there exist } y^* \in S^+ \setminus \{0\} \text{ and } z^* \in Q^+ \text{ such that} \\ & \langle y^*, \overline{y}\rangle = \langle z^*, b\rangle,\ \langle y^*, y\rangle - \langle z^*, z\rangle \geq 0 \text{ for all } x \in K,\ y \in C(x) \text{ and } z \in A(x)\}, \\ \overline{D}_2'' = \{\overline{y} \in Y \mid {} & \text{there exist } y^* \in S^{+i} \text{ and } z^* \in Q^+ \text{ such that} \\ & \langle y^*, \overline{y}\rangle = \langle z^*, b\rangle,\ \langle y^*, y\rangle - \langle z^*, z\rangle \geq 0 \text{ for all } x \in K,\ y \in C(x) \text{ and } z \in A(x)\}. \end{aligned}$$

Clearly we have $\overline{D}_1'' \subset D_1''$ and $\overline{D}_2'' \subset D_2''$.

Proposition 5.3. (i) *Assume that* $\operatorname{int} S \neq \emptyset$. *Then*

$$\mathrm{WMax}(\overline{D}_1'', S) = \mathrm{WMax}(D_1'', S).$$

(ii) *Assume that* $S^{+i} \neq \emptyset$. *Then*

$$\mathrm{Max}(\overline{D}_2'', S) = \mathrm{Max}(D_2'', S).$$

Proof. We only prove (i); the proof of (ii) is analogous.

Let $\overline{y} \in \mathrm{WMax}(D_1'', S)$. Then there exist $y^* \in S^+ \setminus \{0\}$ and $z^* \in Q^+$ such that

$$\langle y^*, \overline{y}\rangle \le \langle z^*, b\rangle,$$
$$\langle y^*, y\rangle - \langle z^*, z\rangle \ge 0 \quad \text{for all } x \in K,\ y \in C(x) \text{ and } z \in A(x).$$

If $\langle y^*, \overline{y}\rangle < \langle z^*, b\rangle$, then there exists $y_1 \in \operatorname{int} S$ such that

$$\langle y^*, \overline{y} + y_1\rangle = \langle z^*, b\rangle.$$

Hence $\overline{y} + y_1 \in \overline{D}_1'' \subset D_1''$ with the same y^* and z^*. The inequality $\overline{y} + y_1 > \overline{y}$ contradicts $\overline{y} \in \mathrm{WMax}(D_1'', S)$. So $\overline{y} \in \overline{D}_1''$. Since $\overline{D}_1'' \subset D_1''$, we get $\overline{y} \in \mathrm{WMax}(\overline{D}_1'', S)$.

Conversely, let $\overline{y} \in \mathrm{WMax}(\overline{D}_1'', S)$. Suppose that there exists $y_1 \in D_1'' \cap (\overline{y} + \operatorname{int} S)$ with $y^* \in S^+ \setminus \{0\}$ and $z^* \in Q^+$ given by the definition. If $\langle y^*, y_1\rangle < \langle z^*, b\rangle$, then there exists $y_2 \in \operatorname{int} S$ such that

$$\langle y^*, y_1 + y_2\rangle = \langle z^*, b\rangle.$$

Hence $y_1 + y_2 \in \overline{D}_1''$ with the same y^* and z^*. The inequality $y_1 + y_2 > \overline{y}$ contradicts $\overline{y} \in \mathrm{WMax}(\overline{D}_1'', S)$. So $y_1 \in \overline{D}_1''$, which again contradicts $\overline{y} \in \mathrm{WMax}(\overline{D}_1'', S)$. Thus $\overline{y} \in \mathrm{WMax}(D_1'', S)$. ∎

The assertion of (ii) has been proved by Kouada [53]. Now we introduce another two dual sets:

$$\widehat{D}_1'' = \{\overline{y} \in Y \mid \text{there exist } y^* \in S^+ \setminus \{0\} \text{ and } T \in L(Z, Y) \text{ with}$$
$$\overline{y} = T(b),\ T^*(y^*) \in Q^+,\ 0 \in (C - TA)^*(y^*) - K^+\},$$
$$\widehat{D}_2'' = \{\overline{y} \in Y \mid \text{there exist } y^* \in S^{+i} \text{ and } T \in L(Z, Y) \text{ with}$$
$$\overline{y} = T(b),\ T^*(y^*) \in Q^+,\ 0 \in (C - TA)^*(y^*) - K^+\}.$$

Lemma 5.2. *Let* X, Y *and* Z *be reflexive Banach spaces*, $K \subset X$ *be closed*, *and let* $K - \operatorname{dom} C \cap \operatorname{dom} A = X$, $\operatorname{dom} A^* = Z^*$. *Then*:

(i) $\widehat{D}_1'' \subset \overline{D}_1''$ *and* $\widehat{D}_2'' \subset \overline{D}_2''$;

(ii) *if* $b \neq 0$, *then* $\widehat{D}_1'' = \overline{D}_1''$ *and* $\widehat{D}_2'' = \overline{D}_2''$.

The proof of Lemma 5.2 is based on the following known result.

Lemma 5.3. *Let* X, Y *be Banach spaces*, *and let* $x \in X$, $x^* \in X^*$, $y \in Y$, $y^* \in Y^*$.

(i) *If there exists a linear mapping* $T : X \to Y$ *with* $y = T(x)$ *and* $x^* = T^*(y^*)$, *then* $\langle y^*, y\rangle = \langle x^*, x\rangle$.

(ii) *If* $x \neq 0$, $y^* \neq 0$ *and* $\langle y^*, y\rangle = \langle x^*, x\rangle$, *then there exists a continuous linear mapping* $T \in L(X, Y)$ *with* $y = T(x)$ *and* $x^* = T^*(y^*)$.

Proof. See Theorem 2.3 of [44]. ∎

PROOF OF LEMMA 5.2. We prove the inclusion $\widehat{D}_1'' \subset \overline{D}_1''$ and the equality $\widehat{D}_1'' = \overline{D}_1''$. The other assertions can be proved similarly.

(i) Let $y_1 \in \widehat{D}_1''$. Then there exist $y^* \in S^+ \setminus \{0\}$ and $T \in L(Z,Y)$ with $y_1 = T(b)$, $T^*(y^*) \in Q^+$ and $0 \in (C - TA)^*(y^*) - K^+$. With Lemma 5.3 we get for $z^* := T^*(y^*)$ the equality

$$\langle y^*, y_1 \rangle = \langle z^*, b \rangle.$$

Since $0 \in (C - TA)^*(y^*) - K^+$, there exists $x^* \in K^+$ with $x^* \in (C - TA)^*(y^*)$. By the definition of $(C - TA)^*$, we have

$$\langle y^*, y - T(z) \rangle \geq \langle x^*, x \rangle \geq 0 \quad \text{for all } x \in K,\ y \in C(x) \text{ and } z \in A(x),$$

and so

$$\langle y^*, y \rangle - \langle z^*, z \rangle \geq 0 \quad \text{for all } x \in K,\ y \in C(x) \text{ and } z \in A(x).$$

Hence $y_1 \in \overline{D}_1''$. Therefore $\widehat{D}_1'' \subset \overline{D}_1''$.

(ii) Let $b \neq 0$ and $y_1 \in \overline{D}_1''$. Then there exist $y^* \in S^+ \setminus \{0\}$ and $z^* \in Q^+$ such that

$$\langle y^*, y_1 \rangle = \langle z^*, b \rangle, \tag{5.5}$$

$$\langle y^*, y \rangle - \langle z^*, z \rangle \geq 0 \quad \text{for all } x \in K,\ y \in C(x) \text{ and } z \in A(x). \tag{5.6}$$

By Lemma 5.3(ii), there exists a continuous linear mapping $T \in L(Z,Y)$ with $y_1 = T(b)$ and $z^* = T^*(y^*)$. From (5.6), we get

$$\langle y^*, y - T(z) \rangle \geq 0 \quad \text{for all } x \in K,\ y \in C(x) \text{ and } z \in A(x).$$

Since $K - \operatorname{dom} C \cap \operatorname{dom} A = X$ and $\operatorname{dom} A^* = Z^*$, by Propositions 1.7 and 1.8, we get

$$0 \in (C - TA)^*(y^*) - K^+.$$

Hence $y_1 \in \widehat{D}_1''$. So $\widehat{D}_1'' = \overline{D}_1''$. ■

PROPOSITION 5.4. *Let X, Y and Z be reflexive Banach spaces, $K \subset X$ be closed, and let $K - \operatorname{dom} C \cap \operatorname{dom} A = X$, $\operatorname{dom} A^* = Z^*$.*

(I) *Assume that* $\operatorname{int} S \neq \emptyset$. *Then*:

(i) $\mathrm{WMax}(\widehat{D}_1'', S) \subset \mathrm{WMax}(\overline{D}_1'', S)$;

(ii) *if, in addition*, $b \neq 0$, *then* $\mathrm{WMax}(\widehat{D}_1'', S) = \mathrm{WMax}(\overline{D}_1'', S)$.

(II) *Assume that* $S^{+i} \neq \emptyset$. *Then*:

(i) $\mathrm{Max}(\widehat{D}_2'', S) \subset \mathrm{Max}(\overline{D}_2'', S)$;

(ii) *if, in addition*, $b \neq 0$, *then* $\mathrm{Max}(\widehat{D}_2'', S) = \mathrm{Max}(\overline{D}_2'', S)$.

PROOF. If $b \neq 0$, from Lemma 5.2(ii), the assertions of (I)(ii) and (II)(ii) are clearly true. For the case $b = 0$, we only prove the assertion of (I)(i). The proof of (II)(i) is analogous.

Assume that $b = 0$. In this case we have $\widehat{D}_1'' = \{0\} \subset Y$. By Lemma 5.2(i) we get $0 \in \overline{D}_1''$. If we assume that $0 \notin \mathrm{WMax}(\overline{D}_1'', S)$, then there exists some $y_1 \in \operatorname{int} S \cap \overline{D}_1''$ with $y_1^* \in S^+ \setminus \{0\}$ and $z_1^* \in Q^+$ given by the definition. But then it follows that $\langle y_1^*, y_1 \rangle > 0$, which contradicts the inequality $\langle y_1^*, y_1 \rangle \leq \langle z_1^*, b \rangle = 0$. Consequently, $0 \in \mathrm{WMax}(\overline{D}_1'', S)$. ■

Therefore, from Theorem 5.1 and the statements above, we can formalize the following duality results.

THEOREM 5.2. *Let* X, Y *and* Z *be reflexive Banach spaces*, $K \subset X$ *be closed*, *and let* $K - \operatorname{dom} C \cap \operatorname{dom} A = X$ *and* $\operatorname{dom} A^* = Z^*$.

(i) *Assume that* $\operatorname{int} S \neq \emptyset$ *and* $b \neq 0$. *If* $A(K) \cap (b + \operatorname{int} Q) \neq \emptyset$, *then*

$$\mathrm{WMin}(\overline{C(X_0) + S}, S) = \mathrm{WMax}(\overline{D}_1'', S) = \mathrm{WMax}(\widehat{D}_1'', S).$$

(ii) *Assume that* $S^{+i} \neq \emptyset$ *and* $b \neq 0$. *If* S *has a weakly compact base and* $C(K \cap A^{-1}(b + Q)) + S)$ *is closed*, *then*

$$\mathrm{Be}(C(X_0), S) = \mathrm{Max}(\overline{D}_1'', S) = \mathrm{Max}(\widehat{D}_1'', S).$$

When C and A are continuous linear mappings, the primal problem (CP) takes the form

$$(\mathrm{CP}_1) \qquad \begin{array}{l} \min C(x) \\ \text{subject to } x \in K,\ A(x) - b \in Q, \end{array}$$

while the dual sets $\widehat{D}_1''$ and $\widehat{D}_2''$ take the form

$$\widehat{D}_1'' = \{T(b) \mid \text{there exist } y^* \in S^+ \setminus \{0\} \text{ and } T \in L(Z, Y) \text{ with } T^*(y^*) \in Q^+,\ (C - TA)^*(y^*) \in K^+\},$$

$$\widehat{D}_2'' = \{T(b) \mid \text{there exist } y^* \in S^{+i} \text{ and } T \in L(Z, Y) \text{ with } T^*(y^*) \in Q^+,\ (C - TA)^*(y^*) \in K^+\}.$$

In this case, the above duality results have been studied by Isermann [40] and Nakayama [70] in a finite-dimensional setting, and by Jahn [42], [44] in an infinite-dimensional setting.

6. Conjugate duality

In this section, a general conjugate duality result is presented and some sufficient conditions ensuring the duality are provided. As an application, conjugate duality results for vector optimization problems with convexlike set-valued mappings are obtained under closedness or boundedness hypotheses; when the set-valued mappings considered are convex, the duality result can be proved without any closedness and boundedness requirements. Finally, a Fenchel type duality is obtained by applying the general duality result. The results of this section concern weak minimality. The conjugate duality for minimality has been studied in [101], [77], [78] (see also [87], [62] and [39]).

6.1. Conjugate mappings and subdifferentials. Let Y be a real topological vector space which is partially ordered by a pointed closed convex cone S with a nonempty interior $\operatorname{int} S$ in Y. We use the following notations:

$$y \geq y' \quad \text{iff} \quad y - y' \in S \setminus \{0\},$$
$$y > y' \quad \text{iff} \quad y - y' \in \operatorname{int} S.$$

We add two imaginary points $+\infty$ and $-\infty$ to Y and denote the extended space by $\overline{Y}$. As a matter of fact, we suppose that for any $y \in Y$, $-\infty < y < +\infty$. We extend the addition and scalar multiplication of Y to $\overline{Y}$ using the following conventions:

$$(\pm\infty) + y = y + (\pm\infty) = \pm\infty \quad \text{for all } y \in Y, \quad (\pm\infty) + (\pm\infty) = \pm\infty,$$
$$\lambda(\pm\infty) = \pm\infty \quad \text{for } \lambda > 0 \quad \text{and} \quad \lambda(\pm\infty) = (\mp\infty) \quad \text{for } \lambda < 0;$$

the sum $+\infty + (-\infty)$ is not considered here, since we can avoid it.

Given a subset $C \subset \overline{Y}$, we define the subset $S_>(C)$ of $\overline{Y}$ by

$$S_>(C) = \{y \in \overline{Y} \mid y > y' \text{ for some } y' \in C\}.$$

Let C be a nonempty subset of $\overline{Y}$ such that $C \neq \{+\infty\}$.

Definition 6.1. A point $p \in \overline{Y}$ is said to be a *weak infimal point* of C if there is no $y \in C$ such that $y < p$ and if the relation $y' > p$ implies the existence of some $y \in C$ such that $y' > y$. The set of all *weak infimal* points of C is called the *weak infimum* of C and is denoted by $\operatorname{WInf} C$. Whenever $C = \emptyset$ and $C = \{+\infty\}$, we define $\operatorname{WInf} C = \{+\infty\}$. The *weak supremum* of C, $\operatorname{WSup} C$, is defined similarly.

Definition 6.2. A point $p \in C$ is said to be a *weak minimal point* of C if there is no $y \in C$ such that $y < p$. The set of all weak minimal points of C is called the *weak minimum* of C and is denoted by $\operatorname{WMin} C$. The *weak maximum* of C, $\operatorname{WMax} C$, is defined similarly.

When C is a subset of Y, Definition 6.2 coincides with Definition 2.1.

As a matter of fact, $-\operatorname{WInf}(-C) = \operatorname{WSup} C$, $-\operatorname{WMin}(-C) = \operatorname{WMax} C$, and $\operatorname{WMin} C = C \cap \operatorname{WInf} C$.

For these definitions we refer to [99]. One observes that the infimum and minimum here are defined for weak Pareto minimality (using the order $>$). The definitions of the infimum for Pareto minimality (using the order $\geq$) have been introduced in [22], [39], [77], [78], [76]. Let F be a set-valued mapping from X to $\overline{Y}$. We define the *effective domain* of F by

$$\operatorname{edom} F = \{x \in X \mid F(x) \neq \emptyset,\ F(x) \neq \{+\infty\}\}.$$

We can admit that $F(x_0) = \emptyset$ for some $x_0 \in X$, if we adopt the convention that for any set $B \subset \overline{Y}$ and any $\lambda \in \mathbb{R}$, $\emptyset + B = \emptyset$ and $\lambda\emptyset = \emptyset$.

The following results are given in [99].

Proposition 6.1. (i) $\operatorname{WInf} C = \{-\infty\}$ *if and only if* $S_>(C) = Y \cup \{+\infty\}$;

(ii) $S_>(C) = S_>(\operatorname{WInf} C)$;

(iii) $C \cap Y \subset \overline{C \cap Y + \operatorname{int} S} \subset \operatorname{WInf} C \cup S_>(C)$;

(iv) *if* F_1 *and* F_2 *are set-valued mappings from a space* X *to* $\overline{Y}$, *then*

$$\operatorname{WSup} \bigcup_{x \in X} [F_1(x) + F_2(x)] = \operatorname{WSup} \bigcup_{x \in X} [F_1(x) + \operatorname{WSup} F_2(x)],$$

where the sum $+\infty + (-\infty)$ *is assumed not to occur.*

Now, we define the conjugate mapping and subdifferential of a set-valued mapping. Let X be another real topological vector space and let $L(X, Y)$ be the space of all continuous linear operators from X to Y.

Definition 6.3. The set-valued mapping F^* from $L(X,Y)$ to $\overline{Y}$ defined by

$$F^*(T) = \mathrm{WSup} \bigcup_{x\in X} [Tx - F(x)] \quad \text{for } T \in L(X,Y)$$

is called the *conjugate mapping* of F. Moreover, the set-valued mapping F^{**} from X to $\overline{Y}$ defined by

$$F^{**}(x) = \mathrm{WSup} \bigcup_{T\in L(X,Y)} [Tx - F^*(T)] \quad \text{for } x \in X$$

is called the *biconjugate mapping* of F.

When f is a single-valued mapping from X to $\overline{Y}$, its conjugate mapping and biconjugate mapping can be defined by identifying it with the set-valued mapping $x \to \{f(x)\}$. The concept of the conjugate mapping introduced by Luc [62] is based on Pareto maximality rather than the supremum.

We observe that if there exists $x_0 \in X$ such that $-\infty \in F(x_0)$, then $F^* \equiv +\infty$; conversely, if there exists $T \in L(X,Y)$ such that $F^*(T) = \{-\infty\}$, then $F \equiv \emptyset$ or $F \equiv +\infty$. Thus, in the sequel, we shall only consider the case when $\operatorname{edom} F \neq \emptyset$.

Definition 6.4. Let $x_0 \in X$ and $y_0 \in F(x_0)$. An operator $T \in L(X,Y)$ is called a *subgradient* of F at (x_0, y_0) if

$$Tx_0 - y_0 \in \mathrm{WMax} \bigcup_{x\in X} [Tx - F(x)].$$

The set of all subgradients of F at (x_0,y_0) is called the *subdifferential* of F at (x_0,y_0) and is denoted by $\partial F(x_0,y_0)$. Moreover, we let $\partial F(x_0) = \bigcup_{y\in F(x_0)} \partial F(x_0,y)$. When $\partial F(x_0,y_0) \neq \emptyset$ for every $y_0 \in F(x_0)$, F is said to be *subdifferentiable* at x_0.

As direct consequences of the definitions of subgradient and conjugate mapping, we have the following results.

Proposition 6.2. *A point $y_0 \in F(x_0)$ is in* $\mathrm{WMin} \bigcup_{x\in X} F(x)$ *if and only if* $0 \in \partial F(x_0,y_0)$.

Proposition 6.3. *Let $y_0 \in F(x_0)$ for some $x_0 \in X$. Then $T \in \partial F(x_0,y_0)$ if and only if $Tx_0 - y_0 \in F^*(T)$.*

The following relationship between a mapping and its biconjugate has been proved by Tanino [99].

Proposition 6.4. *If F is subdifferentiable at x_0, then $F(x_0) \subset F^{**}(x_0)$. Moreover, if, in addition, $F(x_0) = \mathrm{WInf}\, F(x_0)$, then $F(x_0) = F^{**}(x_0)$.*

For a given set-valued mapping F from X to $\overline{Y}$, we define its *epigraph* by

$$\operatorname{epi} F = \{(x,y) \in X \times Y \mid y \in S_>(F(x)) \cup (F(x) + S)\}.$$

Clearly, if F is a set-valued mapping from X to $Y \cup \{+\infty\}$, then

$$\operatorname{epi} F = \{(x,y) \in X \times Y \mid y \in F(x) + S\}.$$

Definition 6.5. A set-valued mapping F from X to $\overline{Y}$ is said to be *S-convex* (resp. *midpoint S-convex*) if its epigraph is convex (resp. midpoint convex).

Clearly, if F is a set-valued mapping from X to $Y\cup\{+\infty\}$, then F is S-convex (resp. midpoint S-convex) if and only if for all $t\in[0,1]$ and $x_1,x_2\in X$,

$$tF(x_1)\cap Y+(1-t)F(x_2)\cap Y\subset F(tx_1+(1-t)x_2)\cap Y+S$$
$$(\text{resp. } \tfrac{1}{2}[F(x_1)\cap Y+F(x_2)\cap Y]\subset F\big(\tfrac{1}{2}(x_1+x_2)\big)\cap Y+S);$$

if F is a set-valued mapping from X to Y, Definition 6.5 coincides with Definition 1.4(i).

We shall show that a convexity assumption guarantees the subdifferentiability of F under some additional assumptions.

Proposition 6.5. *Suppose that F is an S-convex set-valued mapping from X to $Y\cup\{+\infty\}$ with* $\operatorname{int}(\operatorname{epi}F)\neq\emptyset$. *If $x_0\in\operatorname{int}(\operatorname{edom}F)$ and $F(x_0)\subset\operatorname{WInf}F(x_0)$, then F is subdifferentiable at x_0.*

This proposition has been proved by Tanino [99]. However, the assumption of $\operatorname{int}(\operatorname{epi}F)\neq\emptyset$ is missing in Proposition 4.3 of [99]. When X and Y are finite-dimensional, this assumption can be removed. For the infinite-dimensional case, the assumption is needed to ensure that each boundary point of $\operatorname{epi}F$ is a support point, and even if $Y=\mathbb{R}$ and F is an extended real-valued function, the assumption that $x_0\in\operatorname{int}(\operatorname{edom}F)$ is not sufficient (see [23], [82]). Thus it is essential to assume a certain type of continuity or boundedness of the set-valued mapping F, which guarantees $\operatorname{int}(\operatorname{epi}F)\neq\emptyset$. In the sequel, we shall present some such conditions.

Definition 6.6. A set-valued mapping F from X to $Y\cup\{+\infty\}$ is said to be *S-Hausdorff upper continuous* (resp. *S-Hausdorff lower continuous*) at $x_0\in X$ if, for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x)\subset F(x_0)+V+S$$
$$(\text{resp. } F(x_0)\subset F(x)+V+S)$$

for all $x\in(x_0+U)\cap\operatorname{edom}F$. We say that F is *S-Hausdorff continuous* at x_0 if it is both S-Hausdorff upper continuous and S-Hausdorff lower continuous at x_0.

For these definitions, we refer to [72], [97].

Definition 6.7. F is said to be *weakly S-upper bounded* on a set $A\subset X$ if there exists a point $b\in Y$ such that $(x,b)\in\operatorname{epi}F$ for all $x\in A$.

If F is a set-valued mapping from X to $Y\cup\{+\infty\}$, then F is weakly S-upper bounded on a set $A\subset X$ if and only if $F(x)\cap(b-S)\neq\emptyset$ for all $x\in A$.

Theorem 6.1. *Let F be a set-valued mapping from X to $Y\cup\{+\infty\}$. Then the following statements* (i) *and* (ii) *are equivalent*:

(i) $\operatorname{int}(\operatorname{epi}F)\neq\emptyset$;

(ii) *there exists $x_0\in\operatorname{int}(\operatorname{edom}F)$ such that F is weakly S-upper bounded on some neighbourhood of x_0.*

Moreover, if F is S-Hausdorff lower continuous on $\operatorname{int}(\operatorname{edom}F)$, *then* (i) *and* (ii) *hold.*

Proof. (i)⇒(ii). Since $\operatorname{int}(\operatorname{epi} F) \neq \emptyset$, there exist $(x_0, y_0) \in \operatorname{int}(\operatorname{epi} F)$ and a neighbourhood U of zero in X and a neighbourhood V of zero in Y such that

$$(x, y) \in \operatorname{epi} F \quad \text{for all } x \in x_0 + U,\ y \in y_0 + V.$$

In particular, $(x, y_0) \in \operatorname{epi} F$ for all $x \in x_0 + U$. Thus, F is weakly S-upper bounded on $x_0 + U$.

(ii)⇒(i). Suppose that there exist a neighbourhood U of zero in X and a point $b \in Y$ such that

$$F(x) \cap (b - S) \neq \emptyset \quad \text{for all } x \in x_0 + U.$$

This implies that for every $x \in x_0 + U$, $b \in F(x) + S$.

Take any $s_0 \in \operatorname{int} S$. Then there exists a neighbourhood V of zero in Y such that $s_0 + V \subset S$. Thus, for every $x \in x_0 + U$ and $y \in b + s_0 + V$, we have

$$y \in F(x) + s_0 + V + S \subset F(x) + S.$$

This means that $(x_0, b + s_0) \in \operatorname{int}(\operatorname{epi} F)$.

Suppose that F is S-Hausdorff lower continuous at $x_0 \in \operatorname{int}(\operatorname{edom} F)$. For fixed $s_0 \in \operatorname{int} S$, there exists a neighbourhood V of zero in Y such that $s_0 + V \subset S$. Since F is S-Hausdorff lower continuous at x_0, there exists a neighbourhood U of zero in X such that $x_0 + U \subset \operatorname{edom} F$ (since $x_0 \in \operatorname{int}(\operatorname{edom} F)$) and

$$F(x_0) \subset F(x) + V + S \quad \text{for all } x \in x_0 + U.$$

Let $y_0 \in F(x_0)$ $(y_0 \neq +\infty)$. We have, for every $x \in x_0 + U$,

$$y_0 + s_0 \subset F(x) + s_0 + V + S \subset F(x) + S.$$

This means that (ii) holds. ∎

It has been proved by Nikodem [72] that if F is midpoint S-convex and has bounded images and if (ii) is satisfied, then F is S-Hausdorff lower continuous on $\operatorname{int}(\operatorname{edom} F)$.

The following result is a consequence of Theorem 1 of [73]. For completeness we shall give a modified proof.

Proposition 6.6. *Let X and Y be real topological vector spaces with X a Baire space. Let F be a set-valued mapping from X to $Y \cup \{+\infty\}$. If its epigraph is closed and* $\operatorname{int}(\operatorname{edom} F) \neq \emptyset$, *then there exists $x_0 \in \operatorname{int}(\operatorname{edom} F)$ such that F is weakly S-upper bounded on some neighbourhood of x_0.*

Proof. Take any $s_0 \in \operatorname{int} S$. Consider the sets

$$A_n = \{x \in \operatorname{int}(\operatorname{edom} F) \mid F(x) \cap (ns_0 - S) \neq \emptyset\}, \quad n \in \mathbb{N},$$

where $\mathbb{N}$ denotes the set of all positive integers.

We will show that these sets are closed in $\operatorname{int}(\operatorname{edom} F)$. For this purpose, fix an $n \in \mathbb{N}$ and take a point $\overline{x} \in \operatorname{int}(\operatorname{edom} F) \setminus A_n$. Then $F(\overline{x}) \cap (ns_0 - S) = \emptyset$. Hence $(\overline{x}, ns_0) \cap \operatorname{epi} F = \emptyset$.

Since the epigraph of F is closed, there exist a neighbourhood U of zero in X and a neighbourhood V of zero in Y such that

$$[(\overline{x}, ns_0) + U \times V] \cap \operatorname{epi} F = \emptyset.$$

In particular, for every $x \in U_{\bar{x}} = (\overline{x} + U) \cap \operatorname{int}(\operatorname{edom} F)$, we have

$$F(x) \cap (ns_0 - S) = \emptyset.$$

This implies that $U_{\bar{x}} \subset \operatorname{int}(\operatorname{edom} F) \setminus A_n$ and proves that A_n is closed in $\operatorname{int}(\operatorname{edom} F)$. Since $Y = \bigcup_{n\geq 1}(ns_0 - S)$, by the definition of $\operatorname{edom} F$, we have

$$\bigcup_{n\geq 1} A_n = \operatorname{int}(\operatorname{edom} F).$$

The set $\operatorname{int}(\operatorname{edom} F)$, as a nonempty open subset of a Baire space, is a Baire space. Therefore, there exists an $n \in \mathbb{N}$ such that $\operatorname{int}_1 \operatorname{cl}_1 A_n \neq \emptyset$, where int_1 and cl_1 denote the relative interior and closure in $\operatorname{int}(\operatorname{edom} F)$, respectively. Since A_n is closed in $\operatorname{int}(\operatorname{edom} F)$ and $\operatorname{int}(\operatorname{edom} F)$ is open, we deduce that $\operatorname{int} A_n \neq \emptyset$. So, there exist $x_0 \in \operatorname{int}(\operatorname{edom} F)$ and a neighbourhood U of zero in X such that $x_0 + U \subset A_n$. By the definition of A_n, F is weakly S-upper bounded on $x_0 + U$. ■

The following condition ensuring the closedness of $\operatorname{epi} F$ has been given by Gwinner in [32].

Proposition 6.7. *If a set-valued mapping F from X to $Y \cup \{+\infty\}$ is S-Hausdorff upper continuous on* $\operatorname{edom} F$ *with $F(x) + S$ closed for $x \in \operatorname{edom} F$, then the epigraph of F is closed.*

6.2. A general conjugate duality. Let X, Y be real Hausdorff topological vector spaces, and $S \subset Y$ be a closed convex pointed cone with nonempty interior. Let F be a set-valued mapping from X to $Y \cup \{+\infty\}$ with $\operatorname{edom} F \neq \emptyset$ and consider a vector optimization problem

$$\text{(P)} \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in X. \end{array}$$

Solving this problem means to find the set

$$\operatorname{WMin}(\mathrm{P}) = \operatorname{WMin} F(X) \quad \text{or} \quad \operatorname{WInf}(\mathrm{P}) = \operatorname{WInf} F(X).$$

We introduce a perturbation parameter $z \in Z$ and imbed the primal problem (P) into a family of vector optimization problems, where Z is another Hausdorff topological vector space. Let Φ be a set-valued mapping from $X \times Z$ to $Y \cup \{+\infty\}$ such that

$$\Phi(x, 0) = F(x) \quad \text{for all } x \in X.$$

Then the perturbed problem is

$$(\mathrm{P}_z) \qquad \begin{array}{l} \min \Phi(x, z) \\ \text{subject to } x \in A. \end{array}$$

Definition 6.8. The set-valued mapping W from Z to $\overline{Y}$ defined by

$$W(z) = \operatorname{WInf}(\mathrm{P}_z) = \operatorname{WInf} \bigcup_{x\in X} \Phi(x, z)$$

is called the *value mapping* of the perturbed problem (P_z). It is clear that $W(0) = \operatorname{WInf}(\mathrm{P})$.

In accordance with the definition, the conjugate mapping of Φ is the set-valued mapping Φ^* from $L(X,Y)\times L(Z,Y)$ to $\overline{Y}$ defined by

$$\Phi^*(T,\Lambda)=\mathrm{WSup}\bigcup_{(x,z)\in X\times Z}(Tx+\Lambda z-\Phi(x,z))\quad\text{for } T\in L(X,Y),\ \Lambda\in L(Z,Y).$$

Therefore,

$$-\Phi^*(0,\Lambda)=-\mathrm{WSup}\bigcup_{(x,z)\in X\times Z}(\Lambda z-\Phi(x,z))=\mathrm{WInf}\bigcup_{(x,z)\in X\times Z}(\Phi(x,z)-\Lambda z).$$

We define the dual problem to (P) as

$$\text{(D)}\qquad \max_{\Lambda\in L(Z,Y)}-\Phi^*(0,\Lambda).$$

We can prove the following weak duality results.

Proposition 6.8. *For any $x\in X$ and $\Lambda\in L(Z,Y)$,*

$$-\Phi^*(0,\Lambda)\cap S_>(\Phi(x,0))=\emptyset,$$

and hence

$$S_>(\mathrm{WInf(P)})\cap\mathrm{WSup(D)}=\emptyset.$$

Proof. Suppose the contrary. Then there exist $y_1\in\Phi(x,0)$ and $y_2\in-\Phi^*(0,\Lambda)$ such that $y_1<y_2$. But this contradicts the fact that

$$-\Phi^*(0,\Lambda)=\mathrm{WInf}\bigcup_{(x,z)\in X\times Z}(\Phi(x,z)-\Lambda z),$$

and $y_1\in\bigcup_{(x,z)\in X\times Z}(\Phi(x,z)-\Lambda z)$. ■

Corollary 6.1.

$$\mathrm{WMin}\bigcup_{x\in X}\Phi(x,0)\cap\mathrm{WMax}\bigcup_{\Lambda\in L(Z,Y)}-\Phi^*(0,\Lambda)\neq\emptyset$$

if and only if there exist $x_0\in\mathrm{edom}\,F$ and $\Lambda_0\in L(Z,Y)$ such that

$$0\in\Phi(x_0,0)+\Phi^*(0,\Lambda_0)\quad\textit{or equivalently}\quad(0,\Lambda_0)\in\partial\Phi(x_0,0).$$

Proof. The "only if" part is trivial.

For the converse, let $y_0\in\Phi(x_0,0)\cap(-\Phi^*(0,\Lambda_0))$. If $y_0\notin\mathrm{WMin}\bigcup_{x\in X}\Phi(x,0)$, then there exist $x_1\in X$ and $y_1\in\Phi(x_1,0)$ such that $y_0>y_1$. This contradicts the result of Proposition 6.8. Similarly, we can prove that $y_0\in\mathrm{WMax}\bigcup_{\Lambda\in L(Z,Y)}-\Phi^*(0,\Lambda)$. ■

Similar weak duality assertions are given in [62] for Pareto minimality.

It is easy to show that $W^*(\Lambda)=\Phi^*(0,\Lambda)$. Indeed,

$$\begin{aligned}W^*(\Lambda)&=\mathrm{WSup}\bigcup_{z\in Z}(\Lambda z-W(z))=\mathrm{WSup}\bigcup_{z\in Z}\Big(\Lambda z-\mathrm{WInf}\bigcup_{x\in X}\Phi(x,z)\Big)\\&=\mathrm{WSup}\bigcup_{z\in Z}\mathrm{WSup}\bigcup_{x\in X}(\Lambda z-\Phi(x,z))\\&=\mathrm{WSup}\bigcup_{(x,z)\in X\times Z}(\Lambda z-\Phi(x,z))\quad\text{(Proposition 6.1(iv))}\\&=\Phi^*(0,\Lambda).\end{aligned}$$

In view of this equality, we can rewrite WSup(D) as

$$\mathrm{WSup(D)} = \mathrm{WSup} \bigcup_{\Lambda \in L(Z,Y)} (-W^*(\Lambda)) = W^{**}(0).$$

Since WInf(P) $= W(0)$, the relationship between the primal problem WInf(P) and the dual problem WSup(D) is nothing but the relationship between $W(0)$ and $W^{**}(0)$.

Definition 6.9. The primal problem (P) is said to be *stable* if the value mapping W is subdifferentiable at 0.

By Definition 6.9 and Proposition 6.4, we can obtain the exact duality for this class of problems.

Theorem 6.2. *If Problem* (P) *is stable*, *then*

$$\mathrm{WInf(P)} = \mathrm{WSup(D)} = \mathrm{WMax(D)}.$$

Proof. WInf(P) = WSup(D) is clear from Definition 6.9 and Proposition 6.4.

Since Problem (P) is stable, for any $\overline{y} \in$ WInf(P), there exists $\overline{\Lambda} \in L(Z,Y)$ such that $\overline{\Lambda} \in \partial W(0, \overline{y})$. Thus

$$-\overline{y} \in \mathrm{WMax} \bigcup_{z \in Z} [\overline{\Lambda} z - W(z)] \subset W^*(\overline{\Lambda}) = \Phi^*(0, \overline{\Lambda}).$$

If $\overline{y} \notin$ WMax(D), then there exist $\Lambda_0 \in L(Z,Y)$ and $y_0 \in -\Phi^*(0, \Lambda_0)$ such that $\overline{y} < y_0$. From the definition of W, there exist $x_1 \in X$ and $y_1 \in \Phi(x_1, 0)$ such that $y_1 < y_0$. This contradicts Proposition 6.8. Thus WInf(P) $\subset$ WMax(D), which completes the proof. ∎

The first equality in Theorem 6.2 has been proved by Tanino [99] when F and Φ are single-valued mappings. Similar duality results for scalar or vector optimization problems can be found in [81], [82], [23], [106], [6] and [105]. Duality assertions for Pareto minimality under similar assumptions are also given by Luc [62] and Isac–Postolică [39].

Corollary 6.2. *If Problem* (P) *is stable*, *then for each* $y_0 \in$ WMin(P), *there exist* $x_0 \in \operatorname{edom} F$ *and* $\Lambda_0 \in L(Z,Y)$ *such that*

$$y_0 \in \Phi(x_0, 0) \cap (-\Phi^*(0, \Lambda_0)) \quad \text{and hence} \quad y_0 \in \mathrm{WMax(D)}.$$

Proof. Let $y_0 \in$ WMin(P). Then $y_0 \in W(0)$ and there exists $x_0 \in \operatorname{edom} F$ such that $y_0 \in \Phi(x_0, 0)$. From the proof of Theorem 6.2, there exists $\Lambda_0 \in L(Z,Y)$ such that $-y_0 \in \Phi^*(0, \Lambda_0)$ and $y_0 \in$ WMax(D). ∎

Theorem 6.3. *The following conditions on an element* $\overline{\Lambda}$ *of* $L(Z,Y)$ *are equivalent*:

(i) *there exists* $\overline{y} \in -\Phi^*(0, \overline{\Lambda})$ *such that* $\overline{y} \in$ WMax(D) $\cap$ WInf(P);

(ii) $-\Phi^*(0, \overline{\Lambda}) \cap$ WInf(P) $\neq \emptyset$;

(iii) $\overline{\Lambda} \in \partial W(0)$.

Proof. (i)⇒(ii) is obvious. (iii)⇒(i) is clear from the proof of Theorem 6.2.

For (ii)⇒(iii), let $\overline{y} \in -\Phi^*(0, \overline{\Lambda}) \cap$ WInf(P). Then $\overline{y} \in W(0) \cap (-W^*(\overline{\Lambda}))$. Proposition 6.3 implies that $\overline{\Lambda} \in \partial W(0)$. ∎

We define the Lagrangian mapping $L : X \times L(Z, Y) \to \overline{Y}$ for Problem (P) by

$$-L(x, \Lambda) = \mathrm{WSup} \bigcup_{z \in Z} [\Lambda z - \Phi(x, z)] \quad \text{for } x \in X,\ \Lambda \in L(Z, Y).$$

If $\Lambda \in L(Z, Y)$ is such that

$$\mathrm{WInf} \bigcup_{x \in X} F(x) \cap \mathrm{WInf} \bigcup_{x \in X} L(x, \Lambda) \neq \emptyset,$$

we call Λ a *Lagrangian multiplier* for Problem (P).

Since

$$\begin{aligned} \mathrm{WInf} \bigcup_{x \in X} L(x, \Lambda) &= -\mathrm{WSup} \bigcup_{x \in X} \mathrm{WSup} \bigcup_{z \in Z} (\Lambda z - \Phi(x, z)) \\ &= -\mathrm{WSup} \bigcup_{(x,z) \in X \times Z} (\Lambda z - \Phi(x, z)) = -\Phi^*(0, \Lambda), \end{aligned}$$

we deduce from Theorem 6.3 that

$$\partial W(0) = \{\Lambda \in L(Z, Y) \mid \Lambda \text{ is Lagrangian multiplier for (P)}\}.$$

Now let us present some conditions ensuring the stability of the primal problem.

DEFINITION 6.10. A set-valued mapping F from X to $Y \cup \{+\infty\}$ is said to be S-*convexlike* on $A \subset X$ if for every $y_1, y_2 \in F(A) \cap Y$ and $\alpha \in (0, 1)$, there exists $x_3 \in A$ such that

$$\alpha y_1 + (1 - \alpha) y_2 \in F(x_3) + S. \tag{6.1}$$

If (6.1) holds for $\alpha = 1/2$, we say that Φ is *midpoint* S-*convexlike* on A.

When F is a set-valued mapping from X to Y, this definition reduces to Definition 1.4(v). It is easy to show that F is S-convexlike on A if and only if $F(A) \cap Y + S$ is convex. Clearly, if F is S-convex (resp. midpoint S-convex), then F is S-convexlike (resp. midpoint S-convexlike).

DEFINITION 6.11. Let Φ be a set-valued mapping from $X \times Z$ to $Y \cup \{+\infty\}$. If for every $z_i \in Z,\ y_i \in \Phi(X, z_i) \cap Y,\ i = 1, 2$, and $\alpha \in (0, 1)$, there exists $x_3 \in X$ such that

$$\alpha y_1 + (1 - \alpha) y_2 \in \Phi(x_3, \alpha z_1 + (1 - \alpha) z_2) + S, \tag{6.2}$$

we say that Φ is S-*convexlike-convex*. In particular, if (6.2) holds for $\alpha = 1/2$, we say that Φ is *midpoint* S-*convexlike-convex*.

Note that if $X = \{0\}$, i.e., Φ does not depend on x, then S-convexlike-convexity coincides with S-convexity. If $Z = \{0\}$, then S-convexlike-convexity reduces to S-convexlikeness.

Define a set-valued mapping Ψ from Z to $Y \cup \{+\infty\}$ as

$$\Psi(z) = \Phi(X, z) = \bigcup_{x \in X} \Phi(x, z).$$

Clearly, Φ is S-convexlike-convex (midpoint S-convexlike-convex) if and only if Ψ is S-convex (midpoint S-convex).

Proposition 6.9. *Let Ψ be an S-convex (resp. midpoint S-convex) set-valued mapping from Z to $Y \cup \{+\infty\}$. Then the value mapping $W(\cdot)$ is S-convex (resp. midpoint S-convex).*

Proof. Let $(z_i, y_i) \in \operatorname{epi} W$, $i = 1, 2$. From the definition of infimum and epigraph, for every $\varepsilon \in \operatorname{int} S$, there exist $\overline{y}_i \in \Psi(z_i) \cap Y$, $i = 1, 2$, such that $y_i + \varepsilon > \overline{y}_i, i = 1, 2$. Since Ψ is S-convex, for every $\alpha \in (0, 1)$,

$$\alpha\overline{y}_1 + (1-\alpha)\overline{y}_2 \in \Psi(\alpha z_1 + (1-\alpha)z_2) \cap Y + S.$$

Hence

$$\alpha y_1 + (1-\alpha)y_2 + \varepsilon \in \Psi(\alpha z_1 + (1-\alpha)z_2) \cap Y + \operatorname{int} S.$$

By Proposition 6.1(ii), it follows that

$$\alpha y_1 + (1-\alpha)y_2 + \varepsilon \in W(\alpha z_1 + (1-\alpha)z_2) \cap Y + \operatorname{int} S.$$

Since ε is arbitrary, by Proposition 6.1(iii), (iv),

$$\begin{aligned}\alpha y_1 + (1-\alpha)y_2 &\in \overline{W(\alpha z_1 + (1-\alpha)z_2) \cap Y + \operatorname{int} S} \\ &\subset W(\alpha z_1 + (1-\alpha)z_2) \cup A(W(\alpha z_1 + (1-\alpha)z_2)).\end{aligned}$$

Thus $(\alpha z_1 + (1-\alpha)z_2, \alpha y_1 + (1-\alpha)y_2) \in \operatorname{epi} W$ and so W is S-convex. For the midpoint S-convex case, we only need to put $\alpha = 1/2$ in the above proof. ∎

Proposition 6.9 generalizes Proposition 5.2 of [99] where Φ was assumed to be an S-convex single-valued mapping.

We note that Ψ and W satisfy the following relation:

Proposition 6.10. $\operatorname{epi} \Psi \subset \operatorname{epi} W \subset \overline{\operatorname{epi} \Psi}$.

Proof. Let $(z, y) \in \operatorname{epi} \Psi$. By Proposition 6.1(ii), (iii), we have

$$y \in \Psi(z) + S \subset (W(z) + S) \cup S_>(W(z)).$$

This means that $\operatorname{epi} \Psi \subset \operatorname{epi} W$.

For the second inclusion, let $(z, y) \in \operatorname{epi} W$. If $y \in S_>(W(z))$, then $y \in S_>(\Psi(z))$ and then $(z, y) \in \operatorname{epi} \Psi$. If $y \in W(z) \cap Y + S \setminus \operatorname{int} S$, then there exist $s \in S$ and a net $\{s_\alpha\}$ in $\operatorname{int} S$ such that $y - s \in W(z)$ and $\{y - s + s_\alpha\}$ converge to y. From the above proof, we conclude that $(z, y) \in \overline{\operatorname{epi} \Psi}$. ∎

Theorem 6.4. *Suppose that Ψ is an S-convex set-valued mapping and that there exists $\overline{z} \in \operatorname{int}(\operatorname{edom} \Psi)$ such that Ψ is weakly S-upper bounded on some neighbourhood of $\overline{z}$. If $0 \in \operatorname{int}(\operatorname{edom} \Psi)$, then the primal problem* (P) *is stable.*

Proof. By Proposition 6.9, the value mapping W is S-convex. If $W(0) = \{-\infty\}$, then $W^* \equiv \{+\infty\}$ and so W is subdifferentiable at 0. So we may assume $W(0) \neq \{-\infty\}$.

We claim that $W(z) \neq \{-\infty\}$ for all $z \in \operatorname{edom} W$.

Suppose there is $z_0 \in \operatorname{edom} W$ such that $W(z_0) = \{-\infty\}$. Since $0 \in \operatorname{int}(\operatorname{edom} \Psi)$, $W(z) \neq \{+\infty\}$ for all z in some neighbourhood of 0. Hence $0 \in \operatorname{int}(\operatorname{edom} W)$. Thus there exists $\varepsilon > 0$ such that

$$z_1 = -\varepsilon z_0 \in \operatorname{edom} W \quad \text{and} \quad 0 = \frac{1}{1+\varepsilon} z_1 + \frac{\varepsilon}{1+\varepsilon} z_0.$$

Since $W(z_0) = \{-\infty\}$, we have $(z_0, y) \in \operatorname{epi} W$ for all $y \in Y$. Take any $y_1 \in Y$ such that $(z_1, y_1) \in \operatorname{epi} W$. By the S-convexity of W,

$$\left(0, \frac{1}{1+\varepsilon} y_1 + \frac{\varepsilon}{1+\varepsilon} y\right) \in \operatorname{epi} W \quad \text{for all } y \in Y.$$

Hence $\{(0, y) \mid y \in Y\} \subset \operatorname{epi} W$ and hence $S_>(W(0)) = Y \cup \{+\infty\}$. By Proposition 6.1(i), $W(0) = \operatorname{WInf} W(0) = \{-\infty\}$. This is a contradiction. Therefore, $W(\cdot)$ is a set-valued mapping from X to $Y \cup \{+\infty\}$.

Since Ψ is weakly S-upper bounded on some neighbourhood of $\overline{z} \in \operatorname{int}(\operatorname{edom} \Psi)$, by Theorem 6.1, $\operatorname{int}(\operatorname{epi} \Psi) \neq \emptyset$. Hence $\operatorname{int}(\operatorname{epi} W) \neq \emptyset$ by Proposition 6.10. Since $W(0) = \operatorname{WInf} W(0)$ and $0 \in \operatorname{int}(\operatorname{edom} W)$, we can apply Proposition 6.5 to prove the assertion. ∎

THEOREM 6.5. *Let X, Y and Z be real Hausdorff topological vector spaces with Z a Baire space. Suppose that Φ is a midpoint S-convexlike-convex set-valued mapping and* $\operatorname{epi} \Psi$ *is closed. If* $0 \in \operatorname{int}(\operatorname{edom} \Psi)$, *then the primal problem* (P) *is stable.*

PROOF. Since Φ is a midpoint S-convexlike-convex, Ψ is midpoint S-convex. This, with the closedness of $\operatorname{epi} \Psi$, implies that Ψ is S-convex by Proposition 1.1. Since $\operatorname{epi} \Psi$ is closed and $0 \in \operatorname{int}(\operatorname{edom} \Psi)$, by Proposition 6.6, Ψ is weakly S-upper bounded on some neighbourhood of $z_0 \in \operatorname{int}(\operatorname{edom} \Psi)$. Therefore, we can apply Theorem 6.4. ∎

6.3. Duality in vector optimization with constraints. In this subsection, we consider the set-valued vector optimization problem

$$\text{(CP)} \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in A,\ G(x) \cap (-Q) \neq \emptyset, \end{array}$$

where $F : X \to Y \cup \{+\infty\}$ and $G : X \to Z$ are set-valued mappings with $\operatorname{edom} F \neq \emptyset$ (the spaces X, Y and Z are the same as in Subsection 6.2), A is a subset of X, and Q is a closed, pointed and convex cone in Z. Define the set-valued mapping

$$\Phi(x, z) = \begin{cases} F(x) & \text{if } x \in A,\ G(x) \cap (z - Q) \neq \emptyset; \\ \emptyset & \text{otherwise.} \end{cases}$$

The perturbed problem (CP_z) corresponding to this case is of the form

$$(\mathrm{CP}_z) \qquad \begin{array}{l} \min F(x) \\ \text{subject to } x \in A,\ G(x) \cap (z - Q) \neq \emptyset. \end{array}$$

Now we calculate the conjugate mapping of Φ:

$$\Phi^*(T, \Lambda) = \operatorname{WSup} \bigcup_{(x,z) \in X \times Z} (Tx + \Lambda z - \Phi(x, z)) \quad \text{for } T \in L(X, Y),\ \Lambda \in L(Z, Y).$$

Thus,

$$-\Phi^*(0, \Lambda) = \operatorname{WInf} \bigcup_{x \in A} \bigcup_{z \in G(x)+Q} [F(x) - \Lambda z] = \operatorname{WInf} \bigcup_{x \in A} (F(x) - \Lambda G(x) - \Lambda Q).$$

Therefore, the dual problem corresponding to the primal problem (CP) can be written as

$$\text{(CD)} \qquad \max_{\Lambda \in L(Z, Y)} \operatorname{WInf} \bigcup_{x \in A} (F(x) + \Lambda G(x) + \Lambda Q).$$

Whenever there exists $q \in Q$ such that $\Lambda q \in -\operatorname{int} S$, it is easy to show that $-\Phi^*(0, \Lambda) = \{-\infty\}$. Indeed, in this case,

$$Y = \bigcup_{\alpha > 0} \alpha(S + \Lambda q).$$

So,

$$S_> \Big(\bigcup_{x \in A} (F(x) + \Lambda G(x) + \Lambda Q) \Big) = Y \cup \{+\infty\}.$$

Thus, the feasible points Λ in the dual problem (CD) are among those operators in $L(Z, Y)$ for which there are no $q \in Q$ such that $\Lambda q \in -\operatorname{int} S$.

It is interesting to note that, if the dual problem is defined as

$$\text{(CD}'\text{)} \qquad \max_{\Lambda \in L(Z,Y)} \operatorname{WInf} \bigcup_{x \in A} (F(x) + \Lambda G(x)),$$

instead of (CD), then weak duality does not hold for Problem (CP) and Problem (CD′). The following example (see [3]) illustrates this point.

Example 6.1. Let $X = Y = Z = \mathbb{R}^2$ and $S = Q = \mathbb{R}^2_+$. Let $F : X \to Y$ and $G : X \to Z$ be the mappings defined by

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \quad G(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix},$$

where $f_1(x) = x_1$, $f_2(x) = x_2$, $g_1(x) = 1 - x_1 - x_2$, $g_2(x) = x_1 + x_2 - 2$.

It is not difficult to show that $x^0 = \binom{1}{1}$ is such that $g(x^0) \in -Q$ and

$$y^0 = \begin{pmatrix} 3 \\ 11 \end{pmatrix} \in (F(x^0) + \Lambda^0 G(x^0) \cap \operatorname{WInf} \bigcup_{x \in A} (F(x) + \Lambda^0 G(x)) \quad \text{for } \Lambda^0 = \begin{pmatrix} -1 & 1 \\ 1 & 6 \end{pmatrix}.$$

However, $F(x^0) = x^0 < y^0$, so that weak duality does not hold.

It is an interesting question whether the Lagrangian dual problem for (CP), i.e.,

$$\text{(LD)} \qquad \max_{\Lambda \in L^+(Z,Y)} \operatorname{WInf} \bigcup_{x \in A} (F(x) + \Lambda G(x))$$

can be derived from the dual problem (CD).

Proposition 6.11. *If $F \times G$ is $S \times Q$-convexlike (resp. midpoint $S \times Q$-convexlike) on A, then Φ is S-convexlike-convex (resp. midpoint S-convexlike-convex).*

Proof. By the definition of Φ, for all $z_i \in Z$ and $y_i \in \Phi(X, z_i) \cap Y$, $i = 1, 2$, there exist $x_i \in A$, $i = 1, 2$, such that $y_i \in F(x_i) \cap Y$ and $z_i \in G(x_i) + Q$, $i = 1, 2$. Since $F \times G$ is $S \times Q$-convexlike on A, for every $\alpha \in (0, 1)$, there exists $x_3 \in A$ such that

$$\alpha y_1 + (1-\alpha) y_2 \in F(x_3) + S, \quad \alpha z_1 + (1-\alpha) z_2 \in G(x_3) + Q + Q = G(x_3) + Q.$$

By the definition of Φ, this implies that

$$\alpha y_1 + (1-\alpha) y_2 \in \Phi(x_3, \alpha z_1 + (1-\alpha) z_2) + S.$$

Putting $\alpha = 1/2$ in the proof above, we obtain the second statement. ∎

It is easy to show that, in this case, $\Psi(z) = F(A \cap G^{-1}(z-Q))$ and $\operatorname{epi}\Psi = (G \times F)(A) \cap (Z \times Y) + Q \times S$. The set $R_0 = \operatorname{epi}\Psi$ is called the *extended image* of Problem (CP).

COROLLARY 6.3. *Suppose that $F \times G$ is $S \times Q$-convexlike on A. If $0 \in \operatorname{int}[G(A \cap \operatorname{edom} F) + Q]$ and F is weakly S-upper bounded on $A \cap \operatorname{edom} F$, in particular, if $G(A \cap \operatorname{edom} F) \cap (-\operatorname{int} Q) \neq \emptyset$, then Problem* (CP) *is stable. Consequently*,

$$\operatorname{WInf}(\mathrm{CP}) = \operatorname{WMax}(\mathrm{CD}).$$

PROOF. Since $0 \in \operatorname{int}[G(A \cap \operatorname{edom} F) + Q]$, there exists a neighbourhood V of zero in Z such that

$$\begin{aligned} z \in G(A \cap \operatorname{edom} F) + Q \quad &\text{for all } z \in V \\ \Leftrightarrow G^{-1}(z-Q) \cap A \cap \operatorname{edom} F \neq \emptyset \quad &\text{for all } z \in V \\ \Leftrightarrow F(A \cap G^{-1}(z-Q)) \cap Y \neq \emptyset \quad &\text{for all } z \in V \\ \Leftrightarrow 0 \in \operatorname{int}(\operatorname{edom}\Psi). & \end{aligned}$$

Now we use Proposition 6.11 and Theorem 6.4 to finish the proof. ∎

COROLLARY 6.4. *Let X, Y and Z be real Hausdorff topological vector spaces with Z a Baire space, and let A be a subset of X. Suppose that $F \times G$ is midpoint $S \times Q$-convexlike on A and the extended image R_0 is closed. If $0 \in \operatorname{int}[G(A \cap \operatorname{edom} F) + Q]$, in particular, $G(A \cap \operatorname{edom} F) \cap (-\operatorname{int} Q) \neq \emptyset$, then Problem* (CP) *is stable. Consequently*,

$$\operatorname{WInf}(\mathrm{CP}) = \operatorname{WMax}(\mathrm{CD}).$$

Some conditions ensuring the closedness of R_0 can be found in [8], [32], [53].

Finally, we present a very general stability criterion for (CP) by using a result on automatic openness of convex set-valued mappings from [8].

DEFINITION 6.12. A set-valued mapping $H : X \to Z$ is said to be *open* at $z \in R(H)$ if for each $x \in H^{-1}(z)$, we have $z \in \operatorname{int} H(x+U)$ for any neighbourhood U of zero in X.

DEFINITION 6.13. Let $Q \subset Z$ be a convex cone. We say that a set-valued mapping $H : X \to Z$ is *Q-invariant* if for $x \in X$,

$$H(x) - Q \subset H(x).$$

For these definitions we refer to [8].

Now we recall that a convex cone Q in Z is *generating* if $Q - Q = Z$. The *core* of a set A, $\operatorname{core}(A)$, is the set of points $a_0 \in A$ such that for every $x \in X$, there exists $\overline{\lambda} > 0$ with $a_0 + \lambda x \in A$ for all $\lambda \in [0, \overline{\lambda}]$. For A convex, we have $\operatorname{core} A = \operatorname{int} A$ if $\operatorname{int} A \neq \emptyset$. If $0 \in \operatorname{core}(A)$ we say A is *absorbing*.

THEOREM 6.6. *Let X, Y and Z be Hausdorff topological vector spaces with Z complete and metrizable. Let $A \subset X$ be a convex set, $S \subset Y$ be a closed convex pointed cone with nonempty interior, and $Q \subset Z$ be a closed convex pointed and generating cone. Let $F : X \to Y \cup \{+\infty\}$ be S-convex and $G : X \to Z$ be Q-convex. Suppose that $0 \in \operatorname{core}(G(A)+Q) \cap [G(A \cap \operatorname{core}(\operatorname{edom} F)) + Q]$. Then Problem* (CP) *is stable. Consequently*,

$$\operatorname{WInf}(\mathrm{CP}) = \operatorname{WMax}(\mathrm{CD}).$$

The proof will be based on the following lemma:

LEMMA 6.1. *Let X, Z and Q be as in Theorem 6.6 and let $H : X \to Z$ be a set-valued mapping such that $\operatorname{gr} H$ is convex. If H is $(-Q)$-invariant and $y_0 \in \operatorname{core} R(H)$, then*

$$y_0 \in \operatorname{int} H(x_0 + U)$$

for any $x_0 \in H^{-1}(y_0)$ and any absorbing set U in X.

PROOF. See Remark 2.3 of [8] ■

PROOF OF THEOREM 6.6. Let

$$H(x) = \begin{cases} G(x) + Q & \text{if } x \in A; \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\operatorname{gr} H = \{(x, z) \in X \times Z \mid z \in H(x)\}$ is convex and H is $(-Q)$-invariant. Since $0 \in \operatorname{core}(G(A)+Q))$, it follows that $0 \in \operatorname{core} R(H)$. By Lemma 6.1, we have $0 \in \operatorname{int} H(x_0 + U)$ for any $x_0 \in H^{-1}(0)$ and any absorbing set U in X.

Let $0 \in H(x_0)$ with $x_0 \in A \cap \operatorname{core}(\operatorname{edom} F)$. Since $Y = \bigcup_{n\geq 1}(ns_0 - S)$, for any fixed $s_0 \in \operatorname{int} S$ there exists an $n_1 \geq 1$ such that

$$F(x_0) \cap (n_1 s_0 - S) \neq \emptyset.$$

Let $U_1 = \{x \in X \mid F(x + x_0) \cap ((n_1 + 1)s_0 - S) \neq \emptyset\}$. We show that U_1 is absorbing in X. Indeed, since $x_0 \in \operatorname{core}(\operatorname{edom} F)$, for any $x \in X$ there exists a $\lambda \neq 0$ such that

$$\frac{x}{\lambda} + x_0 \in \operatorname{edom} F.$$

So, there exists $n_2 \geq 1$ such that

$$F\left(\frac{x}{\lambda} + x_0\right) \cap (n_2 s_0 - S) \neq \emptyset.$$

Thus

$$\left[\frac{1}{n_2}F\left(\frac{x}{\lambda} + x_0\right) + \left(1 - \frac{1}{n_2}\right)F(x_0)\right] \cap \left(s_0 + \left(1 - \frac{1}{n_2}\right)n_1 s_0 - S\right) \neq \emptyset$$

and so

$$\left[\frac{1}{n_2}F\left(\frac{x}{\lambda} + x_0\right) + \left(1 - \frac{1}{n_2}\right)F(x_0)\right] \cap ((n_1 + 1)s_0 - S) \neq \emptyset.$$

The S-convexity of F implies that

$$\begin{aligned} \frac{1}{n_2}F\left(\frac{x}{\lambda} + x_0\right) \cap Y + \left(1 - \frac{1}{n_2}\right)F(x_0) \cap Y &\subset F\left[\frac{1}{n_2}\left(\frac{x}{\lambda} + x_0\right) + \left(1 - \frac{1}{n_2}\right)x_0\right] + S \\ &= F\left(\frac{x}{n_2\lambda} + x_0\right) + S. \end{aligned}$$

Hence, we deduce that

$$F\left(\frac{x}{n_2\lambda} + x_0\right) \cap ((n_1 + 1)s_0 - S) \neq \emptyset.$$

The arbitrariness of $x \in X$ implies that U_1 is absorbing. Thus, $0 \in \operatorname{int} H(x_0 + U_1) =: V$. The definition of U_1 implies that for any $z \in V$,

$$F(H^{-1}(z)) \cap ((n_1 + 1)s_0 - S) \neq \emptyset.$$

This means that $\Psi(z) = F(H^{-1}(z))$ is weakly S-upper bounded on V and $0 \in \operatorname{int}(\operatorname{edom} \Psi)$. Since F is S-convex and G is Q-convex, $\Psi(z)$ is S-convex. Therefore, by Theorem 6.4, (CP) is stable. ∎

6.4. The Fenchel type duality. In this subsection, we consider the vector optimization problem of the form

$$\text{(FP)} \qquad \begin{array}{l} \min(F(x) + G(x)) \\ \text{subject to } x \in X, \end{array}$$

where F and G are set-valued mappings from X to $Y \cup \{+\infty\}$, and the spaces X and Y are the same as in Subsection 6.1. In this case, we take the parameter space $Z = X$ and the set-valued mapping

$$\Phi(x, z) = F(x) + G(x - z).$$

Hence the value mapping is

$$W(z) = \operatorname{WInf} \bigcup_{x \in X} (F(x) + G(x - z)).$$

We now calculate the conjugate mapping of Φ:

$$\Phi^*(T, \Lambda) = \operatorname{WSup} \bigcup_{(x,z) \in X \times Z} [Tx + \Lambda z - \Phi(x, z)] \quad \text{for } T \in L(X, Y),\ \Lambda \in L(Z, Y).$$

Thus,

$$\begin{aligned} \Phi^*(0, \Lambda) &= \operatorname{WSup} \bigcup_{(x,z) \in X \times X} [\Lambda z - F(x) - G(x - z)] \\ &= \operatorname{WSup} \bigcup_{x \in X} \operatorname{WSup} \bigcup_{z \in X} [\Lambda z - F(x) - G(x - z)] \quad \text{(by Proposition 6.1(iv))}. \end{aligned}$$

Setting $q = x - z$ for fixed x, we have

$$\begin{aligned} \Phi^*(0, \Lambda) &= \operatorname{WSup} \bigcup_{x \in X} \operatorname{WSup} \bigcup_{q \in X} [\Lambda x - \Lambda q - F(x) - G(q)] \\ &= \operatorname{WSup} \bigcup_{x \in X} \operatorname{WSup} \Big[(\Lambda x - F(x)) + \operatorname{WSup} \bigcup_{q \in X} (\Lambda q - G(q))\Big] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 6.1(iv))} \\ &= \operatorname{WSup} \bigcup_{x \in X} [\Lambda x - F(x) + G^*(-\Lambda)] \\ &= \operatorname{WSup} \Big[G^*(\ \Lambda) + \operatorname{WSup} \bigcup_{x \in X} (\Lambda x - F(x)\Big]. \\ &= \operatorname{WSup}(F^*(\Lambda) + G^*(-\Lambda)) \end{aligned}$$

Thus, the dual problem corresponding to (FP) can be written as

$$\text{(FD)} \qquad \max_{\Lambda \in L(X,Y)} - \operatorname{WSup}(F^*(\Lambda) + G^*(-\Lambda)).$$

If $Y = \mathbb{R}$ and F and G are convex extended real-valued functions, it has been shown by Fenchel ([27], [28]) for $X = \mathbb{R}^n$ and by Rockafellar [82] for X a Banach space that both problems (FP) and (FD) have the same optimal value. Generalizations of the Fenchel duality to vector optimization has been obtained by Zowe [106], Zălinescu [105], Breckner [9], Gros [31] and Borwein [7] for single-valued and set-valued mappings. Postolică [78] has studied the duality for minimality of (FP) by using the concept of nuclear cone. Also Malivert [65] considered the duality for weak minimality of problem (FP). However, in his framework, both the classical weak and strong duality results are no longer true.

We derive a Fenchel duality result by specializing Theorem 6.2. Observe that if F and G are S-convex, then $\Phi(x,z) = F(x) + G(x-z)$ and $\Psi(z) = \bigcup_{x\in X}(F(x)+G(x-z))$ are also S-convex.

Theorem 6.7. *Suppose that F and G are S-convex set-valued mappings. If there exists $x_0 \in \operatorname{edom} F \cap \operatorname{int}(\operatorname{edom} G)$ such that G is weakly S-upper bounded on some neighbourhood of x_0, in particular, G is S-Hausdorff lower continuous at x_0, then the primal problem* (FP) *is stable. Consequently,*

$$\operatorname{WInf}(\mathrm{FP}) = \operatorname{WMax}(\mathrm{FD}).$$

Proof. Since G is weakly S-upper bounded on some neighbourhood of x_0, there exist a balanced neighbourhood U of zero in X and a point $b \in Y$ such that

$$G(x_0 - z) \cap (b - S) \neq \emptyset \quad \text{for all } z \in U.$$

Take any $y_0 \in F(x_0) \cap Y$. Then

$$(y_0 + G(x_0 - z)) \cap (y_0 + b - S) \neq \emptyset \quad \text{for all } z \in U$$

and so

$$(F(x_0) + G(x_0 - z)) \cap (y_0 + b - S) \neq \emptyset \quad \text{for all } z \in U.$$

Hence, Ψ is weakly S-upper bounded on U and $0 \in \operatorname{int}(\operatorname{edom}\Psi)$. Thus, we can apply Theorem 6.4 to finish the proof. ∎

When $Y = \mathbb{R}$ and F, G are real-valued functionals, Theorem 6.7 reduces to the classical duality result of [82].

Theorem 6.7 is illustrated by the following example.

Example 6.2. Let $X = \mathbb{R}^1$, $Y = \mathbb{R}^2$ and $S = \mathbb{R}^2_+$. Define mappings F and G by

$$F(x) = G(x) = \begin{cases} (x,0) & \text{if } x \in [0,1]; \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear that F and G are S-convex and G is weakly S-upper bounded on $[0,1]$.

It is easy to show that

$$\bigcup_{x\in\mathbb{R}^1} (F(x) + G(x)) = \{(2x,0) \mid x \in [0,1]\}$$

and

$$\operatorname{WInf}(\mathrm{FP}) = \operatorname{WInf}\{(2x,0) \mid x \in [0,1]\} = \{(x,0) \mid x \geq 0\} \cup \{(0,y) \mid y \geq 0\}.$$

On the other hand, since every linear continuous operator $T \in L(\mathbb{R}^1, \mathbb{R}^2)$ can be represented as

$$Tx = (\alpha,\beta)x, \quad (\alpha,\beta) \in \mathbb{R}^2,$$

one has

$$F^*(T) = \mathrm{WSup}\{(\alpha-1,\beta)x \mid x \in [0,1]\},$$
$$G^*(-T) = \mathrm{WSup}\{(-\alpha-1,-\beta)x \mid x \in [0,1]\}.$$

To calculate the dual problem (FD), since $F = G$, we only need to consider the case when $\alpha \geq 0$.

1. $\alpha \geq 1,\ \beta \geq 0$:

$$\begin{aligned} F^*(T) &= \mathrm{WSup}\{(\alpha-1,\beta)x \mid x \in [0,1]\} \\ &= \{(x,\beta) \mid x \leq \alpha-1\} \cup \{(\alpha-1,y) \mid y \leq \beta\}, \\ G^*(-T) &= \mathrm{WSup}\{(-\alpha-1,-\beta)x \mid x \in [0,1]\} \\ &= \{(x,0) \mid x \leq 0\} \cup \{(0,y) \mid y \leq 0\}. \end{aligned}$$

One has

$$\mathrm{WSup}(F^*(T)+G^*(-T)) = \{(x,\beta) \mid x \leq \alpha-1\} \cup \{(\alpha-1,y) \mid y \leq \beta\}.$$

2. $\alpha \geq 1,\ \beta < 0$:

$$\begin{aligned} F^*(T) &= \{(x,0) \mid x \leq 0\} \cup \{(\alpha-1,y) \mid y \leq \beta\} \\ &\quad \cup \left\{(x,y) \,\middle|\, y = \frac{\beta}{\alpha-1}x,\ 0 \leq x \leq \alpha-1\right\}, \\ G^*(-T) &= \{(x,-\beta) \mid x \leq -\alpha-1\} \cup \{(0,y) \mid y \leq 0\} \\ &\quad \cup \left\{(x,y) \,\middle|\, y = \frac{\beta}{\alpha+1}x,\ -\alpha-1 \leq x \leq 0\right\}. \end{aligned}$$

One has

$$\begin{aligned} \mathrm{WSup}(F^*(T)+G^*(-T)) &= \{(x,-\beta) \mid x \leq -\alpha-1\} \cup \{(\alpha-1,y) \mid y \leq \beta\} \\ &\quad \cup \left\{(x,y) \,\middle|\, y = \frac{\beta}{\alpha-1}x,\ 0 \leq x \leq \alpha-1\right\} \\ &\quad \cup \left\{(x,y) \,\middle|\, y = \frac{\beta}{\alpha+1}x,\ -\alpha-1 \leq x \leq 0\right\}. \end{aligned}$$

3. $0 \leq \alpha < 1,\ \beta \geq 0$:

$$\begin{aligned} F^*(T) &= \{(x,\beta) \mid x \leq \alpha-1\} \cup \{(0,y) \mid y \leq 0\} \\ &\quad \cup \left\{(x,y) \,\middle|\, y = \frac{\beta}{\alpha-1}x,\ \alpha-1 \leq x \leq 0\right\}, \\ G^*(-T) &= \{(x,0) \mid x \leq 0\} \cup \{(0,y) \mid y \leq 0\}. \end{aligned}$$

One has

$$\mathrm{WSup}(F^*(T)+G^*(-T)) = F^*(T).$$

4. $0 \leq \alpha < 1,\ \beta < 0$:

$$\begin{aligned} F^*(T) &= \{(x,0) \mid x \leq 0\} \cup \{(0,y) \mid y \leq 0\}, \\ G^*(-T) &= \{(x,-\beta) \mid x \leq -\alpha-1\} \cup \{(0,y) \mid y \leq 0\} \\ &\quad \cup \left\{(x,y) \,\middle|\, y = \frac{\beta}{\alpha+1}x,\ -\alpha-1 \leq x \leq 0\right\}. \end{aligned}$$

One has

$$\mathrm{WSup}(F^*(T) + G^*(-T)) = G^*(-T).$$

It is easy to show that

$$\bigcup_{T \in L(\mathbb{R}^1, \mathbb{R}^2)} - \mathrm{WSup}(F^*(T) + G^*(-T)) = \{(x, y) \mid (x, y) \notin \mathrm{int}\, \mathbb{R}^2_+\}.$$

Hence WInf(FP) = WMax(FD).

By applying Corollary 6.1, we obtain

Theorem 6.8.

$$\mathrm{WMin} \bigcup_{x \in X} (F(x) + G(x)) \cap \mathrm{WMax} \bigcup_{\Lambda \in L(X,Y)} - \mathrm{WSup}(F^*(\Lambda) + G^*(-\Lambda)) \neq \emptyset$$

if and only if there exist $x_0 \in \mathrm{edom}\, F$ *and* $\Lambda_0 \in L(X, Y)$ *such that*

$$0 \in F(x_0) + G(x_0) + \mathrm{WSup}(F^*(\Lambda_0) + G^*(-\Lambda_0)).$$

References

[1] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Basel, 1990.

[2] H. P. Benson, *An improved definition of proper efficiency for vector minimization with respect to cones*, J. Math. Anal. Appl. 71 (1979), 232–241.

[3] G. R. Bitran, *Duality for nonlinear multiple-criteria optimization problems*, J. Optim. Theory Appl. 35 (1981), 367–401.

[4] J. M. Borwein, *Proper efficient points for maximizations with respect to cones*, SIAM J. Control Optim. 15 (1977), 57–63.

[5] —, *The geometry of Pareto efficiency over cones*, Math. Oper. Statist. Ser. Optim. 11 (1980), 235–248.

[6] —, *Convex relations in optimization and analysis*, in: Generalized Concavity in Optimization and Economics, Academic Press, 1981, 335–377.

[7] —, *A Lagrange multiplier theorem and a sandwich theorem for convex relations*, Math. Scand. 48 (1981), 189–204.

[8] —, *Automatic continuity and openness of convex relations*, Proc. Amer. Math. Soc. 99 (1987), 49–55.

[9] W. W. Breckner, *Dualität bei Optimierungsaufgaben in halbgeordneten topologischen Vektorräumen* (*I*), Rev. Anal. Numér. Théor. Approx. 1 (1972), 5–35.

[10] W. W. Breckner and G. Kassay, *A systematization of convexity concepts for sets and functions*, J. Convex Anal. 4 (1997), 109–127.

[11] A. Cambini, *Nonlinear separation theorems, duality and optimality conditions*, in: Optimization and Related Fields, R. Conti, E. De Giorgi, and F. Giannessi (eds.), Springer, Berlin, 1986, 57–93.

[12] H. W. Corley, *Duality theory for maximizations with respect to cones*, J. Math. Anal. Appl. 84 (1981), 560–568.

[13] —, *Existence and Lagrangian duality for maximizations of set-valued functions*, J. Optim. Theory Appl. 54 (1987), 489–501.

[14] B. D. Craven, *Mathematical Programming and Control Theory*, Chapman and Hall, London, 1978.

[15] —, *Nonsmooth multiobjective programming*, Numer. Funct. Anal. Optim. 10 (1989), 64–78.

[16] B. D. Craven, *A modified Wolfe dual for weak vector minimization*, Numer. Funct. Anal. Optim. 10 (1989), 899–907.

[17] B. D. Craven and B. Glover, *Invex functions and duality*, J. Austral. Math. Soc. Ser. A 39 (1985), 1–20.

[18] B. D. Craven, J. Gwinner and V. Jeyakumar, *Nonconvex theorems of the alternative and minimization*, Optimization 18 (1987), 151–163.

[19] J. P. Dauer and O. A. Saleh, *A characterization of proper minimal points as solutions of sublinear optimization problems*, J. Math. Anal. Appl. 178 (1993), 227–246.

[20] J. P. Dauer and W. Stadler, *A survey of vector optimization in infinite-dimensional spaces, Part 2*, J. Optim. Theory Appl. 51 (1986), 205–241.

[21] P. H. Dien, G. Mastroeni, M. Pappalardo and P. H. Quang, *Regularity conditions for constrained extremum problems via image space*, ibid. 80 (1994), 19–37.

[22] S. Dolecki and C. Malivert, *General duality for vector optimization*, Optimization 27 (1993), 97–119.

[23] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.

[24] K. H. Elster and A. Göpert, *Recent results on duality in vector optimization*, in: Recent Advances and Historical Development of Vector Optimization, J. Jahn and W. Krabs (eds.), Springer, New York, 1987, 129–136.

[25] K. H. Elster and R. Nehse, *Optimality Conditions for Some Nonconvex Problems*, Lecture Notes in Control and Information Sci. 23, Springer, 1980.

[26] K. Fan, *Minimax theorems*, Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 42–47.

[27] W. Fenchel, *On conjugate convex functions*, Canad. J. Math. 1 (1949), 73–77.

[28] —, *Convex Cones, Sets and Functions*, Lecture Notes, Princeton University, 1953.

[29] D. Gale, H. W. Kuhn and A. W. Tucker, *Linear programming and theory of games*, in: Activity Analysis of Production and Allocation, T. C. Koopmann (ed.), Wiley, New York, 1951, 317–329.

[30] F. Giannessi, *Theorems of the alternative and optimality conditions*, J. Optim. Theory Appl. 42 (1984), 331–365.

[31] C. Gros, *Generalization of Fenchel's duality theorem for convex vector optimization*, European J. Oper. Res. 2 (1978), 368–376.

[32] J. Gwinner, *Closed images of convex multivalued mappings in linear topological spaces with applications*, J. Math. Anal. Appl. 60 (1977), 75–86.

[33] M. A. Hanson, *On sufficiency of the Kuhn–Tucker conditions*, ibid. 80 (1981), 545–550.

[34] M. Hayashi and H. Komiya, *Perfect duality for convexlike programs*, J. Optim. Theory Appl. 38 (1982), 179–189.

[35] M. I. Henig, *A cone separation theorem*, ibid. 36 (1982), 451–455.

[36] —, *Proper efficiency with respect to cones*, ibid. 36 (1982), 387–407.

[37] R. B. Holmes, *Geometric Functional Analysis and its Applications*, Springer, New York, 1975.

[38] L. Hurwicz, *Programming in linear spaces*, in: Studies in Linear and Nonlinear Programming, K. J. Arrow, L. Hurwicz, and H. Uzawa (eds.), Stanford Univ. Press, Stanford, Calif., 1958; 2nd printing, Oxford Univ. Press, London, 1958, 38–102.

[39] G. Isac and V. Postolică, *The Best Approximation and Optimization in Locally Convex Spaces*, Verlag Peter Lang, 1993.

[40] H. Isermann, *On some relations between a dual pair of multiple objective linear programs*, Z. Oper. Res. 22 (1978), 33–41.

[41] E. H. Ivanov and R. Nehse, *Some results on dual vector optimization*, Optimization 16 (1985), 505–517.

[42] J. Jahn, *Duality in vector optimization*, Math. Programming 25 (1983), 343–353.

[43] J. Jahn, *A characterization of properly minimal elements of a set*, SIAM J. Control Optim. 23 (1985), 649–656.

[44] —, *Mathematical Vector Optimization in Partially Ordered Linear Spaces*, Verlag Peter Lang, Frankfurt, 1986.

[45] V. Jeyakumar, *Convexlike alternative theorems and mathematical programming*, Optimization 16 (1985), 643–652.

[46] —, *A generalization of a minimax theorem of Ky Fan via a theorem of the alternative*, J. Optim. Theory Appl. 48 (1986), 525–533.

[47] V. Jeyakumar and J. Gwinner, *Inequality systems and optimization*, J. Math. Anal. Appl. 159 (1991), 51–71.

[48] V. Jeyakumar, W. Oettli, and M. Natividad, *A solvability theorem for a class of quasiconvex mappings with applications to optimization*, ibid. 179 (1993), 537–546.

[49] V. Jeyakumar and H. Wolkowicz, *Zero duality gaps in infinite-dimensional programming*, J. Optim. Theory Appl. 67 (1990), 87–108.

[50] H. Kawasaki, *Conjugate relations and weak subdifferentials*, Math. Oper. Res. 6 (1981), 593–607.

[51] —, *A duality theorem in multiobjective nonlinear programming*, ibid. 7 (1982), 95–110.

[52] H. König, *On certain applications of the Hahn–Banach and minimax theorems*, Arch. Math. (Basel) 21 (1970), 583–591.

[53] I. Kouada, *Upper-semicontinuity and cone-concavity of multi-valued vector functions in a duality theory for vector optimization*, Z. Oper. Res. 46 (1997), 169–192.

[54] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[55] M. G. Krein and M. A. Rutman, *Linear operator leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl. no. 26, Providence, R.I., 1950.

[56] H. W. Kuhn and A. W. Tucker, *Nonlinear programming*, in: Proc. Second Berkeley Sympos. Math. Statist. Probab., J. Neyman (ed.), Berkeley, Calif., 1951, 481–492.

[57] D. Kuroiwa, *Convexity for set-valued maps*, Appl. Math. Lett. 9 (1996), 97–101.

[58] H. C. Lai and L. S. Yang, *Strong duality for infinite-dimensional vector-valued programming problems*, J. Optim. Theory Appl. 62 (1989), 449–466.

[59] R. Lehmann and W. Oettli, *The theorem of the alternative, the key-theorem, and the vector-maximum problem*, Math. Programming 8 (1975), 332–344

[60] L. J. Lin, *Optimization of set-valued functions*, J. Math. Anal. Appl. 186 (1994), 30–51.

[61] D. T. Luc, *On duality theory in multiobjective programming*, J. Optim. Theory Appl. 43 (1984), 557–582.

[62] —, *Theory of Vector Optimization*, Lecture Notes in Econom. and Math. Systems 319, Springer, Berlin, 1989.

[63] D. T. Luc and J. Jahn, *Axiomatic approach to duality in optimization*, Numer. Funct. Anal. Optim. 13 (1992), 305–326.

[64] D. T. Luc and C. Malivert, *Invex optimization problems*, Bull. Austral. Math. Soc. 46 (1992), 47–66.

[65] C. Malivert, *Fenchel duality in vector optimization*, in: Advances in Optimization, Lecture Notes in Econom. and Math. Systems 382, Springer, Berlin, 1992.

[66] O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.

[67] L. Martein, *Some results on regularity in vector optimization*, Optimization 20 (1989), 787–798.

[68] G. Mastroeni, M. Pappalardo and N. D. Yen, *Image of a parametric optimization problem and continuity of the perturbation function*, J. Optim. Theory Appl. 81 (1994), 193–202.

[69] H. Nakayama, *Geometric consideration of duality in vector optimization*, ibid. 44 (1984), 625–655.
[70] —, *Some remarks on dualization in vector optimization*, J. Multi-Criteria Decision Anal. 5 (1996), 218–225.
[71] J. W. Nieuwenhuis, *Supremal points and generalized duality*, Math. Oper. Statist. Ser. Optim. 11 (1980), 41–59.
[72] K. Nikodem, *K-convex and K-concave set-valued functions*, Zeszyty Nauk. Politech. Łódz. 559 (Rozprawy Mat. 114), Łódź, 1989.
[73] —, *Remarks on K-midconvex set-valued functions with closed epigraph*, Le Matematiche 45 (1990), 277–281.
[74] W. Oettli, *A duality theorem for the nonlinear vector-maximum problem*, in: Progress in Operations Research (Eger, 1974), A. Prékopa (ed.), Colloq. Math. Soc. János Bolyai 12, North-Holland, 1976, 697–703.
[75] S. Paeck, *Convexlike and concavelike conditions in alternative, minimax, and minimization theorems*, J. Optim. Theory Appl. 74 (1992), 317–332.
[76] D. Pallaschke and S. Rolewicz, *Foundations of Mathematical Optimization*, Math. Appl. 388, Kluwer, 1997.
[77] V. Postolică, *Vectorial optimization programs with multifunctions and duality*, Ann. Sci. Math. Québec 10 (1986), 85–102.
[78] —, *A generalization of Fenchel's duality theorem*, ibid., 199–206.
[79] T. W. Reiland, *Generalized invexity for nonsmooth vector-valued mappings*, Numer. Funct. Anal. Optim. 10 (1989), 1191–1202.
[80] S. M. Robinson, *Stability theory for systems of inequalities in nonlinear programming, part II, differentiable nonlinear systems*, SIAM J. Numer. Anal. 13 (1976), 497–513.
[81] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, N.J., 1970.
[82] —, *Conjugate Duality and Optimization*, CBMS Lecture Note Ser. 16, SIAM, Philadelphia, 1974.
[83] E. E. Rosinger, *Multiobjective duality without convexity*, J. Math. Anal. Appl. 66 (1978), 442–450.
[84] P. H. Sach and B. D. Craven, *Invexity in multifunction optimization*, Numer. Funct. Anal. Optim. 12 (1991), 383–394.
[85] —, —, *Invex multifunctions and duality*, ibid. 12 (1991), 575–591.
[86] P. H. Sach, N. D. Yen, and B. D. Craven, *Generalized invexity and duality theories with multifunctions*, ibid. 15 (1994), 131–153.
[87] Y. Sawaragi, H. Nakayama, and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, New York, 1985.
[88] P. Schönfeld, *Some duality theorems for the non-linear vector maximum problem*, Unternehmensforschung 14 (1970), 51–63.
[89] M. Sion, *On general minimax theorems*, Pacific J. Math. 8 (1958), 171–176.
[90] W. Song, *Lagrangian duality for minimization of nonconvex multifunctions*, J. Optim. Theory Appl. 93 (1997), 167–182.
[91] —, *Duality for vector optimization of set-valued functions*, J. Math. Anal. Appl. 201 (1996), 212–225.
[92] —, *Optimality conditions and Lagrangian duality for vector optimization of invex set-valued functions*, Control and Cybernetics 26 (1997), 68–85.
[93] —, *Conjugate duality in set-valued vector optimization*, J. Math. Anal. Appl. 216 (1997), 265–283.
[94] —, *A generalization of the Fenchel duality in set-valued vector optimization*, Z. Oper. Res., to appear.
[95] —, *Duality for vector optimization of set-valued mappings*, Ph.D. thesis, Inst. Math., Polish Acad. Sci., 1997.

[96] W. Stadler, *A survey of multicriteria optimization or the vector maximum problem, Part I: 1776–1960*, J. Optim. Theory Appl. 29 (1979), 1–52.
[97] A. Sterna-Karwat, *Convexity of the optimal multifunctions and its consequences in vector optimization*, Optimization 20 (1989), 799–808.
[98] T. Tanaka, *Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions*, J. Optim. Theory Appl. 81 (1994), 355–377.
[99] T. Tanino, *Conjugate duality in vector optimization*, J. Math. Anal. Appl. 167 (1992), 84–97.
[100] T. Tanino and Y. Sawaragi, *Duality theory in multiobjective programming*, J. Optim. Theory Appl. 27 (1979), 509–529.
[101] —, —, *Conjugate maps and duality in multiobjective optimization*, ibid. 31 (1980), 473–499.
[102] F. Tardella, *On the image of a constrained extremum problem and some applications to the existence of a minimum*, ibid. 60 (1989), 93–104.
[103] R. M. Van Slyke and R. J.-B. Wets, *A duality theory for abstract mathematical programs with applications to optimal control theory*, J. Math. Anal. Appl. 22 (1968), 679–706.
[104] P. L. Yu, *Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives*, J. Optim. Theory Appl. 14 (1974), 319–377.
[105] C. Zălinescu, *Duality for vectorial nonconvex optimization by convexification and applications*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi 29 (1983), no. 3, 15–34.
[106] J. Zowe, *A duality theorem for a convex programming problem in order complete vector lattices*, J. Math. Anal. Appl. 50 (1975), 273–287.

List of symbols

Index