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Order relations in the set of probability distribution functions and their applications in queueing theory

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CONTENTS

Introduction

This paper deals with order relations in the set of probability distribution functions (p.d.f.'s) or equivalently in the set of random variables (r.v.'s) and their applications in queueing theory. Some desired properties of order relations in the set of p.d.f.'s are defined and families of relations $\leqslant_{1.n}$ and $\leqslant_{2.n}$ are investigated. The relations $\leqslant_{1.n}$, $\leqslant_{2.n}$ are defined for a non-negative integer n; for $n > 1$ they are weaker then the stochastic ordering relation and for $n = 1$ they are equivalent to the stochastic ordering relation. For a given n there exist extremal p.d.f.'s with respect to the relation $\leqslant_{1.n}$ in the set of p.d.f.'s having the first n moments in common.

The first chapter is technical nature. Some families of functions (n-monotonic functions) are defined and their connections with the theory of T-systems [17] are investigated.

In the second chapter order relations in the set of p.d.f.'s are considered and families of relations $\leqslant_{1.n}$, $\leqslant_{2.n}$ are defined. There are given some sufficient conditions for $\leqslant_{1.n}$ or $\leqslant_{2.n}$. Specially interesting cases for $n = 0$ or $n = 1$ are analysed in more detail and conditions for $F \leqslant_{2.1} M$, where M denotes the negative exponential p.d.f., are given. Next, isotonic operators with respect to the relations $\leqslant_{2.0}$ and $\leqslant_{2.1}$ are considered.

The third chapter deals with theorems about order relationships of $GI^{(x)}/G/1$, $GI^{(x)}/M^{(y)}/1$, $M^{(x)}/G^{(y)}/1$ queues.

The fourth chapter deals with bounds for the mean waiting time or the mean queue size for some particular cases of $GI^{(x)}/G^{(y)}/1$ queues. Some other applications of the considerations of chapter 2 in queueing theory are considered.

Finally, in the appendix, there is given an elementary proof of the well-known theorem (see [17]) about extremal p.d.f.'s of the functional $\int_0^\infty \varphi dF$ when the p.d.f.'s run over the set of all p.d.f's (or the set of all lattice p.d.f.'s defined on non-negative integer numbers) with common first or with common first and second moments, respectively.

In the references a large bibliography is given.

For convenience we shall use the following notation:

$m_{F,n}$ — the n-th moment of a p.d.f. F,

m_F — the first moment of a p.d.f. F,
σ_2^F — the variance of a p.d.f. F,
$\chi_A(x)$ — the indicator functions of a set A,
$A*B$ — convolution of p.d.f.'s A and B,
A^{*n} — n-fold convolution of a p.d.f. A,

$$\delta_a(x) = \begin{cases} 0, & x < a, \\ 1, & x \geqslant a. \end{cases}$$

If a p.d.f. F is of the form

$$F(x) = \sum_{i=0}^{\infty} p_i \delta_i(x), \quad p_i \geqslant 0, \quad i = 0, 1, \ldots, \quad \sum_{i=0}^{\infty} p_i = 1,$$

then we put

$$\int_0^{\infty} B^{*x} dF(x) = \sum_{i=0}^{\infty} p_i B^{*i},$$

where $B^{*0} = \delta_0$.

The sequence of p.d.f.'s F_n is said to be weakly convergent to a p.d.f. F if p.d.f.'s F_n are pointwise convergent to the p.d.f. F for all points of continuity of the p.d.f. F (we denote it by $F_n \xrightarrow{\omega} F$ or $\omega - \lim F_n = F$).

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1. n-Monotonic functions on $(-\infty, \infty)$

Let $\mathscr{M}_n^+$ ($\mathscr{M}_n^-$) denote the set of all strictly increasing, differentiable functions G satisfying

$$\int_{-\infty}^{0} |x|^n dG(x) < \infty, \quad G(-\infty) = 0 \tag{1.1}$$

$$\left(\int_0^{\infty} x^n dG(x) < \infty, G(\infty) = a_G < \infty\right),$$

where n is a non-negative integer number.

Condition (1.1) is equivalent to the condition

$$\int_{-\infty}^{0} \int_{-\infty}^{x_n} \ldots \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} G(x_1) dx_1 \ldots dx_n < \infty$$

$$\left(\int_0^{\infty} \int_{x_n}^{\infty} \ldots \int_{x_2}^{\infty} \big(a_G - G(x_1)\big) dx_1 \ldots dx_n < \infty\right).$$

Further, let us define for a positive integer n the following families of functions:

$$(1.2)\quad \mathscr{S}_n^+ = \Big\{\varphi(\cdot):\ \varphi(x) = \int_{-\infty}^{x}\int_{-\infty}^{x_n}\dots\int_{-\infty}^{x_2} G(x_1)\,dx_1\dots dx_n,\ G\in\mathscr{M}_n^-\Big\}$$

$$\Big(\mathscr{S}_n^- = \Big\{\varphi(\cdot):\ \varphi(x) = \int_{x}^{\infty}\int_{x_n}^{\infty}\dots\int_{x_2}^{\infty} \big(a_G - G(x_1)\big)\,dx_1\dots dx_n,\ G\in\mathscr{M}_n^+\Big\}\Big).$$

For $n = 0$ we put

$$\mathscr{S}_0^+ = \mathscr{M}_0^-\big(\mathscr{S}_0^- = \{\varphi(\cdot):\ \varphi(x) = a_G - G(x),\ G\in\mathscr{M}_0^+\}\big).$$

Notice that $\{\mathscr{S}_n^+\}_{n\geqslant 0}$ ($\{\mathscr{S}_n^-\}_{n\geqslant 0}$) is a decreasing family of sets. In the set $\mathscr{S}_n^+(\mathscr{S}_n^-)$ we define the sequence of functions φ_k as being convergent to the function φ in $\xrightarrow{+}(\xrightarrow{-})$ topology if the corresponding functions G_k from (1.2) are weakly convergent to some function G and

$$(1.3)\quad \lim_{k\to\infty}\int_{-\infty}^{0}|x|^n\,dG_k(x) = \int_{-\infty}^{0}|x|^n\,dG(x) < \infty$$

$$\Big(\lim_{k\to\infty}\int_{0}^{\infty} x^n\,dG_k(x) = \int_{0}^{\infty} x^n\,dG(x) < \infty\Big).$$

DEFINITION 1.1. A function φ is said to be an *n-increasing function* (*n-decreasing function*) if it belongs to the closure of the set $\mathscr{S}_n^+(\mathscr{S}_n^-)$ in $\xrightarrow{+}(\xrightarrow{-})$ topology. We denote the set of all n-increasing functions by $\mathscr{K}_n^+$ and the set of all n-decreasing functions by $\mathscr{K}_n^-$.

LEMMA 1.1. (i) *For every $\varphi\in\mathscr{K}_n^+$ we have*

$$(1.4)\quad \varphi(x) = \int_{-\infty}^{x}\int_{-\infty}^{x_n}\dots\int_{-\infty}^{x_2} G(x_1)\,dx_1\dots dx_n = \frac{1}{n!}\int_{-\infty}^{\infty}(x-t)_+^n\,dG(t),$$

where $x_+ = [\max(0,x)]^n$.

(ii) *For every $\varphi\in\mathscr{K}_n^-$ we have*

$$\varphi(x) = \int_{x}^{\infty}\int_{x_n}^{\infty}\dots\int_{x_2}^{\infty}\big(a_G - G(x_1)\big)\,dx_1\dots dx_n = \frac{1}{n!}\int_{-\infty}^{\infty}(t-x)_+^n\,dG(t).$$

Proof. (i) Integrating by parts from A to x, we obtain

$$\frac{1}{n!}\int_{A}^{x}(x-t)_+^n\,dG(t) = G(t)(x-t)^n\Big|_{A}^{x} + \frac{1}{(n-1)!}\int_{A}^{x}(x-t)_+^{n-1}G(t)\,dt.$$

Hence, using condition (1.1), we obtain

$$\frac{1}{n!}\int_{-\infty}^{\infty}(x-t)^n\,dG(t) = \frac{1}{(n-1)!}\int_{-\infty}^{\infty}(x-t)_+^{n-1}G(t)\,dt.$$

Further, integrating $(n-1)$ times, we get (1.4).

(ii) The proof is analogous to that of (i)

EXAMPLE 1.1. For every $t \in R$, $\varphi(x) = (t-x)_+^n \in \mathscr{K}_n^-$ and $\psi(x) = (x-t)_+^n \in \mathscr{K}_n^+$.

Proof. The sequence

$$\varphi_k(x) = \int_{-\infty}^{\infty} (\xi - x)_+^n d\Phi\big(k(\xi - t)\big), \quad \text{where} \quad \Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$$

is convergent to φ in $\xrightarrow{+}$.

Remark. The n-increasing function φ belongs to $C^{n-1}(R)$, $\varphi^{(i)} \geqslant 0$, $i = 1, \ldots, n-1$ and $\varphi^{(n)}$ is an increasing function. The n-decreasing function ψ belongs to $C^{n-1}(R)$, $(-1)^i \psi^{(n)} \geqslant 0$, $i = 1, \ldots, n-1$ and $(-1)^n \psi^{(n)}$ is an increasing function.

Remark. As a consequence of lemma 1.1 we note that $\varphi(\cdot) \in \mathscr{K}_n^+$ if and only if $\varphi(-\cdot) \in \mathscr{K}_n^-$.

LEMMA 1.2. (i) *If*

$$\omega - \lim_{k\to\infty} G_k = G \quad \textit{and} \quad \lim_{k\to\infty} \int_{-\infty}^{0} x^n dG_k(x) = \int_{-\infty}^{0} x^n dG(x) > -\infty,$$

then for every $x \in R$:

$$\lim_k \int_{-\infty}^{\infty} (x-t)_+^n dG_k(t) = \int_{-\infty}^{\infty} (x-t)_+^n dG(t). \tag{1.5}$$

(ii) *If*

$$\omega - \lim_k G_k = G \quad \textit{and} \quad \lim_k \int_0^{\infty} x^n dG_k(x) = \int_0^{\infty} x^n dG(x) < \infty,$$

then for every $x \in R$:

$$\lim_k \int_{-\infty}^{\infty} (t-x)_+^n dG_k(t) = \int_{-\infty}^{\infty} (t-x)_+^n dG(t). \tag{1.6}$$

Proof. (i) We note that

$$\int_{-\infty}^{\infty} (x-t)_+^n dG_k(t) = \sum_{j=0}^{n-1} \binom{n}{j} x^j \int_{-\infty}^{\infty} (-t)^{n-j} \chi_{(-\infty,0]}(t) dG_k(t) + \tag{1.7}$$
$$+ \sum_{j=0}^{n-1} \binom{n}{j} x^j \int_{-\infty}^{\infty} (-t)^{n-j} \chi_{(0,x]}(t) dG_k(t) + x^n \int_{-\infty}^{\infty} \chi_{(-\infty,x]}(t) dG_k(t),$$

where we put $\chi_{(0,x]}(t) = -\chi_{[x,0)}(t)$ for $x \leqslant 0$.

Let x be a continuity point of the function G. The convergence of the respective integrals in (1.7) follows from the properties of weak con-

vergence and theorem 4.5.2 from [8]. Thus we obtain $\lim_k \int_{-\infty}^{\infty} (x-t)_+^n dG_k(t) = \int_{-\infty}^{\infty} (x-t)_+^n dG(t)$ for every continuity point of G. From there follows the final statement since the set of continuity points is dense in R and $\int_{-\infty}^{\infty} (x-t)_+^n dG(t)$ is a continuous function of x (see theorem VII. 6.1, [37]).

(ii) The proof is analogous to that of (i).

The following theorem is to a certain extent a modification of theorem 1 from Karlin and Novikoff's paper [16]. However, the statement holds for the class $\mathscr{K}_n^+$ only and we need not assume that

$$\int_{-\infty}^{\infty} x^k dF_1(x) = \int_{-\infty}^{\infty} x^k dF_2(x), \qquad k = 0, \ldots, n.$$

THEOREM 1.1. *Let F_1, F_2 be non-decreasing, right continuous functions such that*

$$\int_{-\infty}^{0} |x|^n dF_i(x) < \infty, \qquad F_i(-\infty) = 0; \qquad i = 1, 2$$

$$\left(\int_{0}^{\infty} x^n dF_i(x) < \infty, \qquad i = 1, 2\right).$$

If, for every $y \in R$,

$$\int_{-\infty}^{\infty} (x-y)_+^n dF_1(x) \leqslant \int_{-\infty}^{\infty} (x-y)_+^n dF_2(x)$$

(for every $x \in R$: $\int_{-\infty}^{\infty} (x-y)_+^n dF_1(y) \leqslant \int_{-\infty}^{\infty} (x-y)_+^n dF_2(y)$), then for every $\varphi \in \mathscr{K}_n^+$ ($\mathscr{K}_n^-$) we have

$$\int_{-\infty}^{\infty} \varphi(y) dF_1(y) \leqslant \int_{-\infty}^{\infty} \varphi(y) dF_2(y).$$

Proof. Since $\varphi \in \mathscr{K}_n^+$, lemma 1.1 and lemma 1.2 imply that $\varphi(x) = \int_{-\infty}^{\infty} (x-y)_+^n dG(y)$, where G is non-decreasing, right continuous and such that $\int_{-\infty}^{0} |x|^n dG(x) < \infty$. Hence, using Fubini's theorem, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi(x) dF_1(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-y)_+^n dG(y) dF_1(x) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-y)_+^n dF_1(x) dG(y) \\
&\leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-y)_+^n dF_2(x) dG(y) = \int_{-\infty}^{\infty} \varphi(x) dF_2(x).
\end{aligned}$$

Now let us quote a fragment of T-system theory (see [17]).

DEFINITION 1.2. Let $u_0, \ldots, u_n$ denote continuous real-valued function on $[0, \infty)$. These functions form a T-system if

$$U\begin{pmatrix}0, 1, \ldots, n\\ t_0, t_1, \ldots, t_n\end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \ldots & u_0(t_n)\\ u_1(t_0) & u_1(t_1) & \ldots & u_1(t_n)\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ u_n(t_0) & u_n(t_1) & \ldots & u_n(t_n)\end{vmatrix} > 0 \tag{1.8}$$

are stricly positive whenever $0 \leqslant t_0 < \ldots < t_n < \infty$.

We note that the sequence of functions $\{t^j\}_{j=0}^n$ forms a T-system on $[0, \infty)$ since in this case the determinant (1.8) is the familiar Vandermonde determinant.

EXAMPLE 1.2. If $\varphi \in \mathscr{S}_n^+$ $(\varphi \in \mathscr{S}_n^-)$, then $(1, \ldots, t^m, \varphi(t))$ $(1, \ldots, t^n, (-1)^{n+1}\varphi(t))$ forms a T-system. Moreover,

$$U = U\begin{pmatrix}0, 1, \ldots, n)\\ t_0, t_1, \ldots, t_n\end{pmatrix} > 0$$

for every system of real number $-\infty < t_0 < t_1 < \ldots < t_n < \infty$.

Proof. Since the $(n+1)$-th derivative of $\psi(t) = \sum_{i=0}^{n} a_i t^i + \varphi(t)$ is greater than zero, the polynomial $\psi(t)$ has at most $n+1$ roots. Hence we obtain $U \neq 0$. To prove that the determinant U is positive let us expand it by its $(n+1)$-th column. Then we have

$$\sum_{i=0}^{n} A_i(t_0, \ldots, t_n) t_{n+1}^i + A_{n+1}(t_0, \ldots, t_n)\varphi(t_{n+1})$$
$$= A_{n+1}(t_0, \ldots, t_n) t_{n+1}^n \left\{\left(\frac{\varphi(t_{n+1})}{t_{n+1}^n} + \frac{A_n(t_0, \ldots, t_n)}{A_{n+1}(t_0, \ldots, t_n)}\right) + \right.$$
$$\left. + t_{n+1}^{-n} \sum_{i=0}^{n-1} \frac{A_i(t_0, \ldots, t_n)}{A_{n+1}(t_0, \ldots, t_n)} t_{n+1}^i \right\}.$$

Since $\varphi \in \mathscr{S}_n^+$, we have

$$\varphi(x) = \int_{-\infty}^{x} \int_{-\infty}^{x_n} \ldots \int_{-\infty}^{x_2} G(x_1)\, dx_1 \ldots dx_n, \qquad G \in \mathscr{M}_n^-$$

and

$$\lim_{t_{n+1}\to\infty} \frac{\varphi(t_{n+1})}{t_{n+1}^n} = G(\infty).$$

Putting $t_0 = t_0$, $t_1 = t_0+h, \ldots, t_n = t_0+nh$, we have

$$\lim_{h\to 0} -\frac{A_n(t_0, \ldots, t_0+nh)}{A_{n+1}(t_0, \ldots, t_0+nh)}$$
$$= \lim_{h\to 0}\frac{\Delta_h^{(n)}\varphi(t_0)}{h^n} = \varphi^{(n)}(t_0) = G(t_0) \leqslant G(\infty),$$

where $\Delta_h\varphi(t_0) = \varphi(t_0+h)-\varphi(t_0)$ and $\Delta_h^{(n)} = \Delta_h\Delta_h^{(n-1)}$. Hence we can choose h so small and t_{n+1} so large that

$$\frac{\varphi(t_{n+1})}{t_{n+1}^n} + \frac{A_n(t_0, \ldots, t_0+nh)}{A_{n+1}(t_0, \ldots, t_0+nh)} > 0,$$

which completes the proof because for large t_{n+1} $t_{n+1}^{-n}\sum_{i=0}^{n-1}\frac{A_n}{A_{n+1}}t_{n+1}^i$ is small.

Now let us consider the following problem. For at measure μ the functional $\int_0^\infty \varphi(x)\,d\mu(x)$ attains its extremal values when μ runs over the following set of measures on $[0, \infty)$ with bounded variation (B.V.):

$$V(c) = V(c_0, \ldots, c_n) = \left\{\mu\colon \int_0^\infty u_i\,d\mu = c_i,\ i = 0, \ldots, n\right\}$$

and $\{u_i\}_{i=0}^n$, $\{u_0, \ldots, u_n, \varphi\}$ form T-systems.

The solution of this problem results from the following theorem (see theorem 5.1 [17]).

THEOREM 1.2. *If* $\{u_i\}_{i=0}^n$ *and* $\{u_0, \ldots, u_n, \varphi\}$ *form a T-system and* $\underline{c} = (c_0, \ldots, c_n) \in \operatorname{int}\{\underline{c}\colon \int_0^\infty u_i\,d\mu = c_i,\ \mu$ *is of B.V. on* $[0, \infty)\}$, *then*

$$I_{\min} = \min_{\mu\in V(c)}\int_0^\infty \varphi(x)\,d\mu(x) = \int_0^\infty \varphi(x)\,d\underline{\mu}(x) \tag{1.9}$$

is uniquely attained for the atomic measure $\underline{\mu}$ *having* $(n+1)/2$ *atoms in* $(0, \infty)$ *whenever n is odd or* $n/2$ *atoms* $(0, \infty)$ *and at* 0 *whenever n is even.*

The upper bound of the integral in (1.9) need not be attained and fact it may even be infinite. However, in the case $u_i(t) = t^i$, $i = 1, \ldots, n$ if we assume $\lim_{t\to\infty}\frac{\varphi(t)}{t^n} = 0$, then there exists a finite value

$$I_{\sup} = \sup_{\mu\in V(c)}\int_0^\infty \varphi\,d\mu = \int_0^\infty \varphi\,d\bar{\mu},$$

where $\bar{\mu} = \omega - \lim_{t\to\infty} \bar{\mu}_t$ and $\bar{\mu}_t$ is the atomic measure having $(n-1)/2$ atoms in the interval $(0, t)$ and at points $0, t$ whenever n is odd or $n/2$ atoms in the interval $(0, t)$ and at t whenever n is even.

An elementary exposition of this problem in the special case where $u_i(t) = t^i$, $i = 0, \ldots, n$, $n = 1$ or 2 is given in the appendix.

2. Order relations in the set of probability distribution functions

2.1. Preliminary concepts. In this chapter we shall investigate some quasi-ordering relations in the set of p.d.f.'s or r.v.'s. A relation is said to be a quasi-ordering relation if it is reflexive and transitive. Some results of this chapter were given in [26] and [27].

Let us consider the set of all p.d.f.'s on $(-\infty, \infty)$ with a given quasi-ordering relation $\dashv$ (we denote them by $(\mathscr{F}, \dashv)$). We shall require that the set $(\{\delta_a\}_{a\in R}, \leqslant)$ which is isomorphic to $(R, \leqslant)$ be isomorphically embedded in the set $(\mathscr{F}, \dashv)$. Such a property of the relation $\dashv$ we call (R), i.e.,

(R) $\qquad a \leqslant b, \quad a, b \in R \quad$ if and only if $\quad \delta_a \dashv \delta_b$.

Some other properties may also be required. For example:

(C) $\qquad F_1 \dashv F_2, \quad G_1 \dashv G_2 \quad$ implies $\quad F_1 * G_1 \dashv F_2 * G_2$,

(A) $\qquad F \dashv G, \quad a \in R \quad$ implies $\quad F(\cdot + a) \dashv G(\cdot + a)$,

(M) $\qquad F \dashv G, \quad c > 0 \quad$ implies $\quad F\left(\frac{\cdot}{c}\right) \dashv G\left(\frac{\cdot}{c}\right)$,

(T) $\qquad F_n \dashv G_n, \quad F_n \xrightarrow{T} F, \quad G_n \xrightarrow{T} G \quad$ implies $\quad F \dashv G$

($\xrightarrow{T}$ denotes convergence in a suitable topology).

LEMMA 2.1. *Let $(\mathscr{F}, \dashv)$ be given and let the relation have properties* (M), (R), (C), (T) *with property* (T) *defined for a topology $\xrightarrow{T}$ such that if $F_k \xrightarrow{T} F$, then $F_k \xrightarrow{\omega} F$.*

If for $F, G \in \mathscr{F}$, $\int_{-\infty}^{\infty} |x|\, dF(x) < \infty$, $\int_{-\infty}^{\infty} |x|\, dG(x) < \infty$,

$$F^{*k}(kx) \xrightarrow{T} \delta_{\int x dF(x)}, \qquad G^{*k}(kx) \xrightarrow{T} \delta_{\int x dG(x)} \tag{2.1}$$

and if $F \dashv G$, then $\int_{-\infty}^{\infty} x dF(x) \leqslant \int_{-\infty}^{\infty} x dG(x)$.

Proof. From properties (M), (C) we infer that, for every non-negative integer k, $F^*(kx) \dashv G^*(kx)$, and hence by property (T) and the law

of large numbers we obtain $\delta_{\int x dF(x)} \dashv \delta_{\int x dG(x)}$ and by property (R) we get $\int_{-\infty}^{\infty} x dF(x) \leqslant \int_{-\infty}^{\infty} x dG(x)$.

Now we introduce an important class of ordering relations in the set of p.d.f.'s $\mathscr{F}$:

DEFINITION 2.1. A relation $\dashv$ in the set $\mathscr{F}$ is said to be defined by a family of functions $\{f_a\}_{a \in A}$ if $F_1 \dashv F_2$ is equivalent to $\int_{-\infty}^{\infty} f_a dF_1 \leqslant \int_{-\infty}^{\infty} f_a dF_2$ for every a such that $\int_{-\infty}^{\infty} |f_a| dF_i < \infty$, $i = 1, 2$, and

$$\{f : f \equiv a,\ a \in R\} \subsetneqq \{f_a : \int |f_a| dF_i < \infty,\ i = 1, 2\}.$$

Notice that the relation defined by a family $\{f_a\}_a$ has property (A) if the cone generated by $\{f_a\}_a$ is invariant under translations, and has property (M) if the cone generated by $\{f_a\}_a$ is invariant under delatations. The second conditions means that for any $f(\cdot) \in \{f_a\}_a$ and a positive $c > 0$ we have $f(c\cdot) \in \{f_a\}_a$. Property (R) easily follows if all functions of the defining family are non-decreasing.

EXAMPLE 2.1. The cones $\mathscr{K}_n^+$, $\{f : -f \in \mathscr{K}_n^-\}$, $n = 0, \ldots,$ of non-decreasing functions are invariant under translations and of delatations.

LEMMA 2.2. *Let the cone generated by a family of functions $\{f_a\}_a$ be invariant under translations. If $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $F_1 * F_2 \dashv G_1 * G_2$.*

Proof. Using the invariant assumption, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_a dF_1 * F_2 &= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f_a(y+z) dF_1(y) \Big) dF_2(z) \\
&\leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_a(y+z) dG_1(y) dF_2(z) \\
&= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f_a(y+z) dF_2(z) \Big) dG_1(y) \\
&\leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_a(y+z) dG_2(z) dG_1(y) \\
&= \int_{-\infty}^{\infty} f_a dG_1 * G_2 \quad \text{for every } a.
\end{aligned}$$

2.2. Relations $\leqslant_{1,n}$, $\leqslant_{2,n}$.

DEFINITION 2.2. (i) For p.d.f.'s F, G such that $\int_{-\infty}^{0} |x|^n dF(x) < \infty$, where n is a non-negative integer number, we define

$$F \leqslant_{1,n} G$$

if, for every function $f \in \mathscr{K}_n^-$ such that $\int_{-\infty}^{\infty} f dF < \infty$, we have

$$\int_{-\infty}^{\infty} f dF \geqslant \int_{-\infty}^{\infty} f dG.$$

(ii) For p.d.f.'s F, G such that $\int_0^{\infty} x^n dG(x) < \infty$, where n is a non-negative integer number, we define

$$F \leqslant_{2.n} G$$

if, for every function $f \in \mathscr{K}_n^+$ such that $\int_{-\infty}^{\infty} f dG < \infty$, we have

$$\int_{-\infty}^{\infty} f dF \leqslant \int_{-\infty}^{\infty} f dG.$$

The basic concepts of the relations $\leqslant_{1.n}$, $\leqslant_{2.n}$ have been studied in [26], [27].

LEMMA 2.3. *$F \leqslant_{1.n} G$ if and only if $1-G(-x-) \leqslant_{2.n} 1-F(-x-)$.*

The proof follows easily from definition 2.2.

Some equivalent forms of $\leqslant_{1.n}$ or $\leqslant_{2.n}$ are given in the following theorem.

THEOREM 2.1. (i) *The following are equivalent conditions:*

(i.1) $F \leqslant_{1.n} G$,

(i.2) *for every $x \in R$:*

$$\infty > \int_{-\infty}^{\infty} (x-t)_+^n dF(t) \geqslant \int_{-\infty}^{\infty} (x-t)_+^n dG(t),$$

(i.3) *for every $x \in R$:*

$$\infty > \int_{-\infty}^{x} \int_{-\infty}^{x_n} \dots \int_{-\infty}^{x_2} F(x_1) dx_1 \dots dx_n \geqslant \int_{-\infty}^{x} \int_{-\infty}^{x_n} \dots \int_{-\infty}^{x_2} G(x_1) dx_1 \dots dx_n.$$

(ii) *The following are equivalent conditions:*

(ii.1) $F \leqslant_{2.n} G$,

(ii.2) *for every $x \in R$:*

$$\int_{-\infty}^{\infty} (t-x)_+^n dF(t) \leqslant \int_{-\infty}^{\infty} (t-x)_+^n dG(t) < \infty,$$

(ii.3) *for every $x \in R$:*

$$\int_{x}^{\infty} \int_{x_n}^{\infty} \dots \int_{x_2}^{\infty} \big(1-F(x_1)\big) dx_1 \dots dx_n \leqslant \int_{x}^{\infty} \int_{x_n}^{\infty} \dots \int_{x_2}^{\infty} \big(1-G(x_1)\big) dx_1 \dots dx_n < \infty.$$

The proof follows immediately from the considerations of chapter 1.

COROLLARY 1. *If* $F \leqslant_{1,n} G$ $(F \leqslant_{2,n} G)$, $\int_{-\infty}^{0} |x|^n dF(x) < \infty$ $(\int_{0}^{\infty} x^n dG(x) < \infty)$ *and* $n \leqslant m$, *then* $F \leqslant_{1,m} G$ $(F \leqslant_{2,m} G)$.

COROLLARY 2. *If* $F \leqslant_{1,n} G$, $G(0-) = 0$, *then for every* $s > 0$

$$\int_{0}^{\infty} e^{-st} dF(t) \geqslant \int_{0}^{\infty} e^{-st} dG(t).$$

Some fur therproperties of the relations $\leqslant_{1,n}$ $(\leqslant_{2,n})$ are given in the following

THEOREM 2.2. (i) *The relation* $\leqslant_{2,n}$ $(\leqslant_{1,n})$ *has properties* (R), (A), (M) *and* (C). *If we assume that* T_n *is the following topology*: $F_k \xrightarrow{T_n} F$ *if* $F_k \xrightarrow{\omega} F$,

$$\int_{-\infty}^{\infty} x_+^n dF_k(x) \to \int_{-\infty}^{\infty} x_+^n dF(x) \qquad \left(\int_{-\infty}^{0} |x|^n dF_k(x) \to \int_{-\infty}^{0} |x|^n dF(x)\right),$$

then property (T) *is valid.*

(ii) *If* $\int_{-\infty}^{\infty} |x| dF(x) < \infty$, $\int_{-\infty}^{\infty} |x| dG(x) < \infty$,

$$\int_{0}^{\infty} x_+^{n+\delta} dF(x) < \infty, \qquad \int_{0}^{\infty} x_+^{n+\delta} dG(x) < \infty, \qquad \delta > 0, \tag{2.2}$$

then

$$F \leqslant_{2,n} G \ (F \leqslant_{1,n} G) \quad \textit{implies} \quad \int_{-\infty}^{\infty} x dF(x) \leqslant \int_{-\infty}^{\infty} x dG(x). \tag{2.3}$$

Proof. (i) Since the set $\mathscr{K}_n^+$ $(-\mathscr{K}_n^-)$ is invariant under translations and delatations and contains increasing functions only, we get properties (R), (A), (M) and by lemma 2.2 we get property (C). Property (T) follows immediately from lemma 1.2.

(ii) To prove (2.3) we use lemma 2.1. First we have to show that for the topology T_n conditions (2.1) hold, i.e.,

$$F^{*k}(kx) \xrightarrow{\omega} \delta_{\int t dF(t)}(x), \qquad G^{*k}(kx) \xrightarrow{\omega} \delta_{\int t dG(t)}(x), \tag{2.4}$$

$$\begin{aligned} &\int_{0}^{\infty} x^n dF^{*k}(kx) \to \max\left(0, \int_{-\infty}^{\infty} t^n dF(t)\right), \\ &\int_{0}^{\infty} x^n dG^{*k}(kx) \to \max\left(0, \int_{-\infty}^{\infty} t^n dG(t)\right). \end{aligned} \tag{2.5}$$

By the law of large numbers we infer that (2.4) is fulfilled. Since the function $x_+^{n+\delta}$ is convex, we obtain

$$\int_{-\infty}^{\infty} x_+^{n+\delta} dF^{*k}(kx) \leqslant \frac{1}{k} \sum_{j=1}^{k} \int_{-\infty}^{\infty} x_+^{n+\delta} dF(x) = \int_{-\infty}^{\infty} x_+^{n+\delta} dF(x)$$

and hence by the assumption (2.2) we get for all k: $\int_{-\infty}^{\infty} x_+^{n+\delta} dF^{*k}(kx) \leqslant c < \infty$, which implies (2.5). The proof of the second part of (2.5) is the same.

Remark. In [26], [27] it was proved that for p.d.f.'s F, G such that $\int_{-\infty}^{\infty} |x|^n dF(x) < \infty$ and $\int_{-\infty}^{\infty} |x|^n dG(x) < \infty$, $n \geqslant 1$, and if

$$F \leqslant_{1,n} G \ (F \leqslant_{2,n} G), \quad \text{then} \quad \int_{-\infty}^{\infty} x dF(x) \leqslant \int_{-\infty}^{\infty} x dG(x).$$

The following theorem gives us the sufficient conditions for $F \leqslant_{1,n} G$ ($F \leqslant_{2,n} G$) (see [16]).

THEOREM 2.3. *Let F, G be p.d.f.'s such that $\int_{-\infty}^{\infty} x^j dF(x) = \int_{-\infty}^{\infty} x^j dG(x)$, $j = 0, \ldots, n$. If the function $F-G$ ($(-1)^n(F-G)$) changes its sign exatly n times and in some interval after the last change of the sign the function*

$$F-G \quad ((-1)^n (F-G)) \tag{2.6}$$

is greater than zero for n even and is less than zero for n odd, then $F \leqslant_{1,n} G$ ($F \leqslant_{2,n} G$).

Proof. Let x be a fixed non-negative real number. We show that $\int_{-\infty}^{\infty} (x-t)_+^n d\big(F(t)-G(t)\big) \geqslant 0$. First we note that

$$\int_{-\infty}^{\infty} (x-t)_+^n d\big(F(t)-G(t)\big) = n \int_{-\infty}^{\infty} (x-t)_+^{n-1} \big(F(t)-G(t)\big) dt.$$

Let $t_1 < t_2 < \ldots < t_n$ be consecutive points at which the function $F-G$ changes its sign. Since $\varphi_x(t) = (x-t)_+^n$ is the limit of functions from $\mathscr{K}_n^-$ (see example 1.1) and the determinant is a continuous function of its arguments, by example 1.2 we have

$$(-1)^i W(t) = \begin{vmatrix} 1 & t & \ldots & t^{n-1} & (-1)^n \varphi_x(t) \\ 1 & t_1 & \ldots & t_1^{n-1} & (-1)^n \varphi_x(t_1) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & t_n & \ldots & t_n^{n-1} & (-1)^n \varphi_x(t_n) \end{vmatrix} \geqslant 0, \tag{2.7}$$

whenever $t_i \leqslant t \leqslant t_{i+1}$, $t_0 = -\infty$, $t_{n+1} = \infty$, $i = 0, 1, \ldots, n$. Expanding the determinant (2.7) by its first row, we obtain

$$W(t) = A_0\varphi(t) + \sum_{i=0}^{n-1} A_i t^i \quad \text{and} \quad A_0 \geqslant 0. \tag{2.8}$$

Since for every t: $W(t)\big(F(t)-G(t)\big) \geqslant 0$, by (2.6), (2 8) we obtain

$$\int_{-\infty}^{\infty} \Big(A_0\varphi_x(t) + \sum_{i=0}^{n-1} A_j t^i\Big)\big(F(t)-G(t)\big)\,dt = A_0 \int_{-\infty}^{\infty} \varphi_x(t)\big(F(t)-G(t)\big)\,dt \geqslant 0,$$

which proves the theorem.

The following example is related to theorem A.1 and to its proof.

EXAMPLE 2.2. Let $\{G_t\}_{0\leqslant t<m_1}$ be a family of step p.d.f.'s with first jump at t, $0 \leqslant t < m_1$, having exactly two jumps. All p.d.f.'s G_t have the first and the second moments equal to m_1 and m_2, respectively. If $0 \leqslant t_1 \leqslant t_2 < m_1$, then $G_{t_1} \leqslant_{1,2} G_{t_2}$.

Proof. The proof follows from theorem 2.3 and from the following representation of the family $\{G_t\}_{0\leqslant t<m_1}$:

$$G_t(x) = \begin{cases} 0, & x < t, \\ \dfrac{m_2 - m_1^2}{m_2 + t^2 - 2m_1 t}, & t \leqslant x < \dfrac{m_2 - m_1 t}{m_1 - t}, \\ 1, & \dfrac{m_2 - m_1 t}{m_1 - t} \leqslant x. \end{cases} \tag{2.9}$$

2.3. Extremal probability distribution functions. Let $m_1, \ldots, m_n$ be a non-degenerate sequence of moments and

$$\mathscr{F}^{(m_1,\ldots,m_n)} = \{F\colon F \text{ is p.d.f.},\ F(0-) = 0,\ \int_0^{\infty} x^i\,dF(x) = m_i,\ i = 1, \ldots, n\}.$$

Let $F_{m_1,\ldots,m_n} \in \mathscr{F}^{(m_1,\ldots,m_n)}$ be the p.d.f. of the measure μ defined in theorem 1.2. In order to find p.d.f. $F_{m_1,\ldots,m_n}$ we have to solve the following system of equations:

$$\begin{aligned} &\sum_{i=1}^{r} p_i = 1, \\ &\sum_{i=1}^{r} p_i x_i = m_1, \\ &\vdots \\ &\sum_{i=1}^{r} p_i x_i^n = m_n, \qquad p_i \geqslant 0;\ 1, \ldots, r,\ 0 \leqslant x_1 \leqslant \ldots \leqslant x_r < \infty, \end{aligned}$$

where

$$r = \begin{cases} \frac{1}{2}n+1 & \text{for } n \text{ even}, \\ (n+1)/2 & \text{for } n \text{ odd}. \end{cases}$$

In the above system of equations p_i denotes the jump of $F_{m_1,\ldots,m_n}$ at the point x_i, $i = 1, \ldots, r$. For $n = 1$ we have

$$F_{m_1}(x) = \begin{cases} 0, & x < m_1, \\ 1, & x \geqslant m_1. \end{cases}$$

For $n = 2$ we have

$$F_{m_1,m_2}(x) = \begin{cases} 0, & x < 0, \\ 1 - \dfrac{m_1^2}{m_2}, & 0 \leqslant x < \dfrac{m_2}{m_1}, \\ 1, & \dfrac{m_2}{m_1} \leqslant x. \end{cases}$$

In [12] there is given an explicit form p.d.f. F_{m_1,m_2,m_3}. Write

$$H^- = H^-_{m_1,\ldots,m_n} = \begin{cases} F_{m_1,\ldots,m_n} & \text{for } n \text{ odd}, \\ F_{m_1,\ldots,m_{n-1}} & \text{for } n \text{ even}, \end{cases}$$

and

$$H^+ = H^+_{m_1,\ldots,m_n} = \begin{cases} F_{m_1,\ldots,m_{n-1}} & \text{for } n \text{ odd}, \\ F_{m_1,\ldots,m_n} & \text{for } n \text{ even}. \end{cases}$$

We note that $H^-_{m_1,\ldots,m_n} \in \mathscr{F}^{(m_1,\ldots,m_n)}$ for n even and $H^+_{m_1,\ldots,m_n} \in \mathscr{F}^{(m_1,\ldots,m_n)}$ for n odd.

THEOREM 2.4. *If $F \in \mathscr{F}^{(m_1,\ldots,m_n)}$, then*

$$H^+ \leqslant_{1.n} F \leqslant_{1.n} H^-$$

and the inequalities are sharp. (For $n = 0$ we put $H^+ = \delta_0$.)

Proof. Since for $\varphi \in \mathscr{S}_n^-$ the function $1, x, \ldots, x^n$, $(-1)^n\varphi(x)$ form a T-system, by theorem 1.2 we have

$$\int_0^\infty \varphi dH^- \leqslant \int_0^\infty \varphi dF \leqslant \int_0^\infty \varphi dH^+$$

and after simple limit consideration we have $H^+ \leqslant_{1.n} F \leqslant_{1.n} H^-$.

2.4. Relations $\leqslant_{2.0}, \leqslant_{2.1}$. Now we give more information about relations $\leqslant_{2.0}$, $\leqslant_{2.1}$. The relation $\leqslant_{2.0}$ or the equivalent relation $\leqslant_{1.0}$ is sometimes called the *stochastic ordering relation* (see [19], [20]). We restrict its here to one-dimensional p.d.f.'s. For the multidimensional case

see for example papers [19], [41]. Let us recall the definition of the relation $\leqslant_{2.0}$, quoting an equivalent form of it (see theorem 2.1 (ii.3)).

DEFINITION 2.3. For p.d.f.'s F, G we define $F \leqslant_{2.0} G$ (F is stochastic less then G) if for all real x we have $F(x) \geqslant G(x)$.

The relation $\leqslant_{2.0}$ has the following representation in the set of r.v.'s.

THEOREM 2.5. *Let $\{X_k\}_{k\geqslant 0}$, $\{Y_k\}_{k\geqslant 0}$ be two sequences of identically distributed r.v.'s with a common p.d.f. equal to F_{X_0} and F_{Y_0}, respectively. In each sequence the r.v.'s are independent. If $F_{X_0} \leqslant_{2.0} F_{Y_0}$, then there exists a probability space (Ω', B', P') and two sequences of r.v.'s $\{X'_k\}_{k\geqslant 0}$, $\{Y'_k\}_{k\geqslant 0}$ on it, being in each sequence independent, identically distributed with a common p.d.f. equal to F_{X_0}, F_{Y_0} respectively, and such that for all $\omega \in \Omega'$ and for every non-negative integer number k:*

$$X'_k(\omega) \leqslant Y'_k(\omega).$$

The proof is given in [14].

Further properties of the relation $\leqslant_{2.0}$ will be given together with information about the relation $\leqslant_{2.1}$. Although the relation $\leqslant_{2.1}$ does not have as intuitive an interpretation as the relation $\leqslant_{2.0}$, the relation $\leqslant_{2.1}$ contains the relation $\leqslant_{2.0}$ and it has also many useful properties. Some of them will be given further. Let us recall

DEFINTION 2.4. For all p.d.f.'s F, G, we define $F \leqslant_{2.1} G$ if $\int_0^\infty x dG(x) < \infty$ and, for all real x, $\int_x^\infty (1-F(t))dt \leqslant \int_x^\infty (1-G(t))dt$.

The relation $\leqslant_{2.1}$ has been used in some papers; for example in [6], [7], [30], [31], [32], [33].

EXAMPLE 2.3. If $\{X_n, \mathscr{F}_n\}_n$ is a submartingale, then $X_1 \leqslant_{2.1} X_2 \leqslant_{2,1} \cdots$

Proof. Since each function $\varphi \in \mathscr{K}_1^+$ is increasing and $X_n \leqslant E(X_{n+1}|\mathscr{F}_n)$, we have $\varphi(X_n) \leqslant \varphi(E(X_{n+1}|\mathscr{F}_n))$ with pr. 1 and by Jensen's inequality we get $\varphi(X_n) \leqslant E(\varphi(X_{n+1})|\mathscr{F}_n)$. Hence we get, for every $\varphi \in \mathscr{K}_1^+$, $E\varphi(X_n) \leqslant E\varphi(X_{n+1})$, and the proof is complete.

Now let us consider the following interesting special case. Let

$$M_{m_1}(x) = \begin{cases} 0, & x < 0, \\ 1-e^{-\frac{x}{m_1}}, & x \geqslant 0 \end{cases}$$

(in the sequel the negative exponential p.d.f. with the first moment equal to m_1 will be denoted by $M_{m_1}(x)$). If F is p.d.f. such that $F(0-) = 0$, $\int_0^\infty x dF(x)$

$= m_1$, then $F \leqslant_{2.1} M_{m_1}$ ($F \leqslant_{1.1} M_{m_1}$) if and only if

$$\forall x \in R^1: \frac{1}{m_1}\int_x^\infty \big(1-F(\xi)\big)d\xi \leqslant (\geqslant) e^{-x/m_1}. \tag{2.10}$$

If F is a non-lattice p.d.f., then (2.10) means that the equilibrium p.d.f. of F is stochastically less than M_{m_1} (the equilibrium p.d.f. of F is equal to $\frac{1}{m_1}\int_0^x \big(1-F(t)\big)dt$, where $m_1 = \int_0^\infty x dF(x)$ — see e.g. [3]).

In reliability theory some families of p.d.f.'s are defined.

$\boldsymbol{IFR} = \{F\colon F(0) = 0,\ F(x) = 1-e^{-\psi(x)},\ \psi$ is convex and increasing$\}$;
$\boldsymbol{DFR} = \{F\colon F(0-) = 0,\ F(x) = 1-e^{-\varphi(x)},\ \varphi$ is concave and increasing to infinity$\}$;

$$\boldsymbol{IFR_d} = \left\{F\colon F(x) = \sum_{i=0}^\infty p_i\,\delta_i(x),\ \frac{p_i}{\sum\limits_{n=i}^\infty p_n} \leqslant \frac{p_{i+1}}{\sum\limits_{n=i+1}^\infty p_n}\right.$$
$$\left.\text{for all integer number } i \text{ such that } \sum_{n=i}^\infty p_n \neq 0\right\};$$

$$\boldsymbol{DFR_d} = \left\{F\colon F(x) = \sum_{i=0}^\infty p_i\,\delta_i(x),\ \frac{p_i}{\sum\limits_{n=i}^\infty p_n} \geqslant \frac{p_{i+1}}{\sum\limits_{n=i+1}^\infty p_n}\right.$$
$$\left.\text{for all integer numbers } i \text{ such that } \sum_{n=i}^\infty p_n \neq 0\right\};$$

$$\boldsymbol{NBU} = \big\{F\colon F(0) = 0,\ 1-F(x+y-) \leqslant \big(1-F(x-)\big)\big(1-F(y-)\big)$$
$$\text{for all } x, y \geqslant 0\big\};$$

$$\boldsymbol{NBUE} = \left\{F\colon F(0) = 0,\ \int_0^\infty x dF(x) \geqslant \int_0^\infty \frac{1-F(x+t-)}{1-F(t-)}\,dx\right.$$
$$\left.\text{for all } t \text{ such that } F(t) < 1\right\}.$$

If the inequalities in the defining relations of families ***NBU*** (***NBUE***) are reversed, then we obtain families ***NWU*** (***NWUE***).

Notice for example that for p.d.f.'s from ***NBUE*** we have the following interpretation. If F is interpreted as a life p.d.f., then the expected life length of a new unit is greater than the expected remaining life of a used but unfailed unit. There are also similar interpretations for the remaining

sets of p.d.f.'s. In [23] the following chains of inclusions was given:

$$\boldsymbol{IFR} \subset \boldsymbol{NBU} \subset \boldsymbol{NBUE}, \tag{2.11}$$

$$\boldsymbol{DFR} \cap \Big\{F\colon \int_0^\infty x dF(x) < \infty\Big\} \subset \boldsymbol{NWU} \cap \Big\{F\colon \int_0^\infty x dF(x) < \infty\Big\} \subset \boldsymbol{NWUE}.$$

For a detailed study of the above-mentioned sets see for example papers [1], [2], [3], [22].

Now we state the following:

THEOREM 2.6. *If $F \in \boldsymbol{NBUE}$ ($\boldsymbol{NWUE}$) and m_1 is the mean of F, then*

$$\forall x \geqslant 0\colon\ e^{-x/m_1} \geqslant \frac{1}{m_1}\int_x^\infty \big(1-F(t)\big)dt \Big(e^{-x/m_1} \leqslant \frac{1}{m_1}\int_x^\infty \big(1-F(t)\big)dt\Big).$$

Proof. The proof is given in [23].

COROLLARY. *If $F \in \boldsymbol{NBUE}$ ($\boldsymbol{NWUE}$) and $\int_0^\infty x dF(x) \leqslant m_1$ ($\int_0^\infty x dF(x) \geqslant m_1$), then*

$$F \leqslant_{2.1} M_{m_1} \quad (M_{m_1} \leqslant_{2.1} F).$$

THEOREM 2.7. *If $F \in \boldsymbol{IFR}_d$ ($\boldsymbol{DFR}_d$), then $F \in \boldsymbol{NBU}$ ($\boldsymbol{NWU}$) and $F \leqslant_{2.1} M_{m_F}$ ($M_{m_F} \leqslant_{2.1} F$).*

Proof. First we show that $F \in \boldsymbol{IFR}_d$ if and only if the sequence $a_k = -\ln\big(\sum_{i=k}^\infty p_i\big)$, $p_i = F(i) - F(i-)$, is a convex sequence, i.e., for all non-negative integer numbers k we have

$$a_k \leqslant \tfrac{1}{2}(a_{k-1} + a_k). \tag{2.12}$$

Transforming (2.12), we get $\big(\sum_{i=k}^\infty p_i\big)^2 \geqslant \big(\sum_{i=k-1}^\infty p_i\big)\big(\sum_{j=k+1}^\infty p_j\big)$, which is equivalent to $0 \geqslant p_{k-1}\sum_{i=k}^\infty p_i - p_{k-1}p_k - p_k\sum_{i=k}^\infty p_i$ and hence

$$\frac{p_k}{\sum_{i=k}^\infty p_i} \geqslant \frac{p_{k-1}}{\sum_{i=k-1}^\infty p_i}.$$

Now we show that $a_k + a_l \leqslant a_{k+l}$. Since the sequence $\{a_k\}$ is convex and $a_0 = 0$, we have

$$a_k = a_{(k/(k+l))(k+l)+(l/(k+l))0)} \leqslant k/(k+l)\, a_{k+l}$$

and hence $a_k + a_l \leqslant a_{k+l}$ for all non-negative integer k, l. Using the formula for a_k, we obtain $\sum_{i=k+l}^\infty p_i \leqslant \sum_{i=k}^\infty p_i \sum_{i=l}^\infty p_i$, which means that $F \in \boldsymbol{NBU}$. Hence by using theorem 2.6 the assertion is proved.

The following theorem gives a necessary condition for $F \leqslant_{2.1} M$:

THEOREM 2.8. *If* $F \leqslant_{2.1} M_{m_1}$ *and* $m_n = \int_0^\infty x^n dF(x)$, *then*

$$\frac{1}{n!} m_n \leqslant m_1^n, \quad n = 1, \ldots \tag{2.13}$$

Proof. Since $x_+^n \in \mathscr{K}_1^+$, $n = 1, \ldots$, by theorem 2.1

$$m_n = \int_0^\infty x^n dF(x) \leqslant \int_0^\infty x^n \frac{1}{m_1} e^{-x/m_1} dx = n! m_1^n.$$

Some equivalent forms of definitions 2.3 and 2.4 in the case where p.d.f.'s are lattices are given in the following

THEOREM 2.9. *For lattice p.d.f.'s with a common set of increase points* $\{a+kb\}_k$ *we have* $F \leqslant_{2.0} G$ $(F \leqslant_{2.1} G)$ *if and only if for all integer numbers* k

$$\sum_{i=k}^\infty p_i \leqslant \sum_{i=k}^\infty \Pi_i \Big(\sum_{i=1}^\infty i p_{k+i-1} = \sum_{j=k}^\infty \sum_{i=j}^\infty p_i \leqslant \sum_{j=k}^\infty \sum_{i=j}^\infty \Pi_i = \sum_{i=j}^\infty i \Pi_{k+i-1}\Big),$$

where $p_j = F(a+bj) - F(a+bj-)$, $\Pi_j = G(a+bj) - G(a+bj-)$.

Proof. We prove the non-trivial case of $F \leqslant_{2.1} G$ being equivalent to $\sum_{j=k}^\infty \sum_{i=j}^\infty p_i \leqslant \sum_{j=k}^\infty \sum_{i=j}^\infty \Pi_i$. Since F, G are lattice p.d.f.'s, we can write $F(x) = \sum_{k=-\infty}^\infty p_k \delta_{a+kb}(x)$, $G(x) = \sum_{k=-\infty}^\infty \Pi_k \delta_{a+kb}(x)$. Now we put $a_0 = 0, b = 1$. Then for all real x

$$\int_x^\infty \big(1 - F(t)\big) dt = \sum_{i=[x]+2}^\infty \sum_{j=i}^\infty p_j + (1 - \{x\}) \sum_{j=[x]+1}^\infty p_j$$
$$\leqslant \sum_{i=[x]+2}^\infty \sum_{j=i}^\infty \Pi_j + (1 - \{x\}) \sum_{j=[x]+1}^\infty \Pi_j = \int_x^\infty \big(1 - G(t)\big) dt,$$

where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$. The above inequality follows from the fact that the functions $\int_x^\infty \big(1 - F(t)\big) dt$, $\int_x^\infty \big(1 - G(t)\big) dt$ are convex and linear in $[i, i+1)$, $i = 0, 1, \ldots$

To prove the general case it is enough to note that $aX + b \leqslant_{2.1} aY + b$, $a \geqslant 0$, is equivalent to $X \leqslant_{2.1} Y$.

Remark. In this section we do not consider the relation $\leqslant_{1.1}$ but similar theorem to that can be given.

2.5. Isotonic operators. A special interesting problem in this theory are isotonic operators with respect to relations $\leqslant_{2.0}, \leqslant_{2.1}$. Such operators were among those introduced and investigated in [9], [33].

DEFINITION 2.5. An operator $\mathfrak{F}: \mathscr{F} \to \mathscr{F}$ is said to be *isotonic* with respect to the relation $\leqslant_{2.0}$ ($\leqslant_{2.1}$) if $F, G \in \mathscr{F}$, $F \leqslant_{2.0} G$ ($F \leqslant_{2.1} G$) implies $\mathfrak{F}(F) \leqslant_{2.0} \mathfrak{F}(G)$ ($\mathfrak{F}(F) \leqslant_{2.1} \mathfrak{F}(G)$).

DEFINITION 2.6. A multilinear operator $\mathfrak{F}: \mathscr{F} \times \ldots \times \mathscr{F} \to \mathscr{F}$, where $\times$ denotes the Cartesian product, is isotonic with respect to $\leqslant_{2.0}$ ($\leqslant_{2.1}$) if $F_i' \leqslant_{2.0} F_i''$ ($F_i' \leqslant_{2.1} F_i''$), $i = 1, \ldots, n$, implies

$$\mathfrak{F}(F_1, \ldots, F_i', \ldots, F_n) \leqslant_{2.0} \mathfrak{F}(F_1, \ldots, F_i'', \ldots, F_n)$$
$$(\mathfrak{F}(F_1, \ldots, F_i', \ldots, F_n) \leqslant_{2.1} \mathfrak{F}(F_1, \ldots, F_i'', \ldots, F_n)).$$

THEOREM 2.10. *Multilinear operators*

$$\mathfrak{C}: F \times \ldots \times F \ni (F_1, \ldots, F_n) \to F_1 * \ldots * F_n \in \mathscr{F},$$
$$\mathfrak{M}: \mathscr{F} \times \ldots \times \mathscr{F} \ni (F_1, \ldots, F_n) \to F_1 \cdot \ldots \cdot F_n \in \mathscr{F},$$
$$\mathfrak{S}: \mathscr{F} \times \ldots \times \mathscr{F} \ni (F_1, \ldots, F_n) \to \sum_{i=1}^{n} c_i F_i \in \mathscr{F},\ c_i \geqslant 0,\ \sum_{i=1}^{n} c_i = 1$$

are isotonic with respect to the relation $\leqslant_{2.0}$ *or* $\leqslant_{2.1}$.

The proof is a formal checking of definition 2.6.

Some applications of isotonic multilinear operators with respect to $\leqslant_{2.1}$ in the reliability theory were given in [22].

Now let us consider the following class of linear operators:

$$\mathfrak{F}(G)(y) = \int_{\infty}^{\infty} F(x, y)\, dG(x), \tag{2.14}$$

where $F(x, \cdot)$ are p.d.f.'s for all real x and $F(\cdot, y)$ is a measurable function for all real y.

THEOREM 2.11 (Stoyan [35]). *An operator* $\mathfrak{F}$ *of form* (2.14) *is isotonic with respect to the relation* $\leqslant_{2.0}$ ($\leqslant_{2.1}$) *if and only if* $\int_{-\infty}^{\infty} f(y)\, d_y F(x, y) \in \mathscr{K}_0^+$ ($\mathscr{K}_1^+$) *for all* $f \in \mathscr{K}_0^+$ ($\mathscr{K}_1^+$).

If $\mathfrak{F}_1(G)(x) = \int_{-\infty}^{\infty} F_1(x, y)\, dG(x)$, $\mathfrak{F}_2(G)(x) = \int_{-\infty}^{\infty} F_2(x, y)\, dG(y)$ *are isotonic with respect to the relation* $\leqslant_{2.0}$ *or* ($\leqslant_{2.1}$) *and* $F_1(x, \cdot) \leqslant_{2.0} F_2(x, \cdot)$ ($F_1(x, \cdot) \leqslant_{2.1} F_2(x, \cdot)$) *for all real* x, *then* $G \leqslant_{2.0} H$ ($G \leqslant_{2.1} H$) implies $\mathfrak{F}_1(G) \leqslant_{2.0} \mathfrak{F}_2(H)$ ($\mathfrak{F}_1(G) \leqslant_{2.1} \mathfrak{F}_2(H)$).

The proof is given in [35].

Remark. For the relation $\leqslant_{2.0}$ this theorem was given by Daley [9].

Remark. It is possible that a family $\{F(x, y)\}_x$ is numbered by some subset A of real numbers. Such a family generates an operator from the set of p.d.f.'s with the set of increase points equal to A in the set of p.d.f.'s $\mathscr{F}$. Then $\int_{-\infty}^{\infty} f(y)\, dy F(\cdot, y)$ ought to be interpreted as a continuous function

having values $\int_{-\infty}^{\infty} f(y)\,d_y F(x, y)$ for $x \in A$ which is linear in $\bar{A}^c$.

EXAMPLE 2.4. An operator $\mathfrak{F}(G)(y) = \int_{-\infty}^{\infty} \chi_{(-\infty,y]}\big(g(x)\big)\,dG(x)$, where $g(x)$ is a convex increasing function, is isotonic with respect to $\leqslant_{2.0}$ or $\leqslant_{2.1}$.

Proof. The proof follows from the identity

$$\int_{-\infty}^{\infty} f(y)\,d_y \chi_{(-\infty,y]}\big(g(x)\big) = f\big(g(x)\big).$$

EXAMPLE 2.5. An operator

$$\mathfrak{F}(F)(x) = \begin{cases} \int_{-\infty}^{\infty} K(x-y)\,dF(y), & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

where K is p.d.f., is isotonic with respect to $\leqslant_{2.0}$ or $\leqslant_{2.1}$.

Proof. The proof follows from theorem 2.9 and example 2.4 with $g(x) = x_+$.

EXAMPLE 2.6. Let G be a lattice p.d.f. with non-negative integer increase points with $p_i = G(i) - G(i-)$ and let F be a fixed p.d.f. such that $F(0) = 0$. Then the operator $\mathfrak{F}(G)(x) = \sum_{n=0}^{\infty} p_n F^{*n}(x)$ is isotonic with respect to $\leqslant_{2.1}$ ($\leqslant_{2.0}$, $\leqslant_{1.1}$).

Proof. From theorem 2.11 it follows that we must check the convexity of the sequence $a_i = \int_{-\infty}^{\infty} f(x)\,dF^{*i}(x)$ for all $f \in \mathscr{K}_1^+$. Let $\varphi(y) = \int_{-\infty}^{\infty} f(x+y)\,dF^{*n}(x)$. Notice that $\varphi \in \mathscr{K}_1^+$. Further, for all convex functions φ and for all $x, y \geqslant 0$ we have $\varphi(x+y) - \varphi(x) - \varphi(y) + \varphi(0) \geqslant 0$. This follows from the inequalities

$$\varphi(x) = \varphi\left(\frac{x}{x+y}(x+y) + \frac{y}{x+y}\cdot 0\right) \leqslant \frac{x}{x+y}\varphi(x+y) + \frac{y}{x+y}(\varphi)\,0,$$

$$\varphi(y) = \varphi\left(\frac{y}{x+y}(x+y) + \frac{x}{x+y}\cdot 0\right) \leqslant \frac{y}{x+y}\varphi(x+y) + \frac{x}{x+y}\varphi(0).$$

Hence we get

$$\int_{-\infty}^{\infty} \varphi(x+y)\,dF(y)\,dF(x) - 2\int_{-\infty}^{\infty} \varphi(x)\,dF(x) + \varphi(0) \geqslant 0,$$

which implies that

$$a_n \leqslant \frac{a_{n-1} + a_{n+1}}{2}.$$

Hence and from the remark after theorem 2.11 it follows that $\int_{-\infty}^{\infty} f(y)\, d_y F^{*x}(y) \in \mathscr{K}_1^+$, and the proof of the example is complete.

Remark. From example 2.6 it follows that if $F_1 \leqslant_{2.0} F_2$ $(F_1 \leqslant_{k.1} F_2)$, $G_1 \leqslant_{2.0} G_2$ $(G_1 \leqslant_{k.1} G_2)$, where G_1, G_2 are lattice p.d.f.'s with jumps at an integer point k equal to p_k, Π_k respectively, then

$$\sum_{i=0}^{\infty} p_i F_1^{*i} \leqslant_{2.0} \sum_{i=0}^{\infty} \Pi_i F_2^{*i} \quad \Big(\sum_{i=0}^{\infty} p_i F_1^{*i} \leqslant_{k.1} \sum_{i=0}^{\infty} \Pi_i F_2^{*i}\Big), \qquad k = 1, 2.$$

This remark has been proved in [7] but our proof is different.

EXAMPLE 2.7. Let

$$F_a(y\,|\,x) = \sum_{i=0}^{[y]} \frac{(ax)^k}{k!} e^{-ax}, \qquad a > 0.$$

If $a_1 \leqslant a_2$ and $G_1 \leqslant_{2,1} G_2$ $(G_1 \leqslant_{i,j} G_2,\ G_1(0) = G_2(0) = 0,\ i = 1, 2, j = 0, 1)$ then

$$\int_0^{\infty} F_{a_1}(y\,|\,x)\, dG_1(x) \leqslant_{2.1} \int_0^{\infty} F_{a_2}(y\,|\,x)\, dG_2(x)$$

$$\Big(\int_0^{\infty} F_{a_1}(y\,|\,x)\, dG_1(x) \leqslant_{i.j} \int_0^{\infty} F_{a_2}(y\,|\,x)\, dG_2(x), \qquad i = 1, 2,\ j = 0, 1\Big).$$

Proof. By theorem 2.11 it suffices to verify that for all $f \in \mathscr{K}_1^+$ we have

$$g_a(x) = \sum_{k=0}^{\infty} e^{-ax} \frac{(ax)^k}{k!} f(k) \in \mathscr{K}_1^+ \tag{2.15}$$

and

$$\sum_{i=0}^{[\cdot]} e^{-a_1 x} \frac{(a_1 x)^i}{i!} \leqslant_{2.0} \sum_{i=0}^{[\cdot]} e^{-a_2 x} \frac{(a_2 x)^i}{i!}.$$

The last inequality results from the following:

$$\frac{\partial}{\partial a} \sum_{i=0}^{n} e^{-ax} \frac{(ax)^i}{i!} = -a e^{-ax} \frac{(ax)^n}{n!} \leqslant 0.$$

To prove (2.15) we note that

$$\frac{\partial}{\partial x} g_a(x) = a e^{-ax} \sum_{k=0}^{\infty} \frac{(ax)^k}{k!} \big(f(k+1) - f(k)\big) \geqslant 0$$

and

$$\frac{\partial^2}{\partial x^2} g_a(x) = a^2 e^{-ax} \sum_{k=0}^{\infty} \frac{(ax)^k}{k} \big(f(k+2) - 2f(k+1) + f(k)\big) \geqslant 0.$$

2.6. Remarks about quasi-ordering relations in the set of random variables. In applications of order relations in queuing theory, reliability theory, etc., sometimes it is more convenient to operate with r.v.'s than with their p.d.f.'s. For example many queueing characteristics are given as simple recurrent relations of some r.v.'s.

Let $(\mathscr{F}, \dashv)$ be given. One defines respective quasi-ordering relations in the set of r.v.'s $\mathscr{X}$ in the following way:

DEFINITION 2.7. For r.v.'s X, Y we define $X \dashv Y$ if $F_X \dashv F_Y$.

Let us note that all previous theorems and properties can be stated in terms of relations in the set of r.v.'s. For example property (C) can be written as:

(C) If X_1, Y_1 are independent r.v.'s and X_2, Y_2 are independent r.v.'s, and if $X_1 \dashv X_2$, $Y_1 \dashv Y_2$, then $X_1 + Y_1 \dashv X_2 + Y_2$.

Other families of order relations in the set of r.v.'s can be defined. For example one can find some of them in [5], [42] but we do not consider them.

3. Order relationship between queueing systems

3.1. Preliminary concepts, $GI^{(x)}/G^{(y)}/1$ queues. The general bulk queue is described as follows: Groups of customers arrive at the queueing system and get served in batches. The size of the arriving groups and those of the batches for service are random variables having independent distributions. The time intervals between successive group arrivals are independent and identically distributed r.v.'s; so are also the service times of the different batches. Following Bhat [4] we use the notation $GI^{(x)}/G^{(y)}/1$ to represent the general single server bulk queue, the exponent x and y denoting the sizes of the arriving groups and the service capacity, respectively. We shall suppress these exponents when they are equal to one. Further, we shall assume that the queue discipline is first-come-first-served and that when the arrivals are in groups, the units will be ordered for the purpose of service. Three special cases of general $GI^{(x)}/G^{(y)}/1$ bulk queues will be interesting for us: $GI^{(x)}/G/1$, $M^{(x)}/G^{(y)}/1$ (Poisson arrivals

and arbitrary service), $GI^{(x)}/M^{(y)}/1$ (arbitrary inter-arrival time distribution and negative exponential service time).

The following notation will be used: τ_n, $n \geqslant 1$, $\tau_0 = 0$ — the instant of the n-th arrival; $X_n = \tau_{n+1} - \tau_n$ — the n-th inter-arrival time; $\bar{X}_n$, $n \geqslant 1$ — the batch size the of n-th group; Y_n, $n \geqslant 1$ — the service time of the n-th group, $\bar{Y}_n$, $n \geqslant 1$ — the batch size for service of the n-th group, W_n, $n \geqslant 1$ — the waiting time for the service of the first customer from the n-th group, W — equilibrium waiting time for the service of the first customer from a group.

Next let us denote by $A(x)$ — the d.f. of the inter-arrival times, $\bar{A}(x)$ — the d.f. of size of the arriving group, $B(x)$ — the p.d.f. of the service time, $\bar{B}(x)$ —the p.d.f. of the batch sizes for service, $W_n(x)$ — the p.d.f. of the waiting time of the first customer from the n-th group, $W(x)$ — the p.d.f. of the waiting time of the first customer in a group.

Now let us recall some results from random walk theory (see [7], [28]). Let $\{\xi_i\}_{i \geqslant 0}$ be a sequence of independent, identically distributed (i.i.d.) r.v.'s. If $E\xi_0 < 0$, then the limiting p.d.f. F_η of the sequence of p.d.f.'s F_{η_n}, where

$$\eta_n = \max(0, \eta_{n-1} + \xi_{n-1}) \tag{3.1}$$

exists independently of η_0, $E|\eta_0| < \infty$. If we assume $P(\eta_0 = 0) = 1$, then $\eta_0 \leqslant_{2.1} \eta_1 \leqslant_{2.1} \cdots$

The limiting p.d.f. F_η has a finite n-th moment when $\int_0^\infty x^{n+1} dF_{\xi_0}(x) < \infty$. For the case $\xi_i = Y_i - X_i$, $i \geqslant 0$, where $0 \leqslant X_i$, $i \geqslant 0$, are i.i.d. r.v.'s, $0 \leqslant Y_i$, $i \geqslant 0$, are i.i.d. r.v.'s and the sequences $\{X_i\}_i$, $\{Y_i\}_i$ are independent, we have $\int_0^\infty x^2 dF_{\xi_0}(x) < \infty$ if and only if

$$\int_0^\infty x^2 dF_{X_0}(x) < \infty, \quad \int_0^\infty x dF_{Y_0}(x) < \infty \quad \text{(see [28])}.$$

In the sequel the following lemma will be useful.

LEMMA 3.1. *If X_n, $n \geqslant 1$, are i.i.d. r.v.'s and N is a non-negative integer-valued r.v. independent of $\{X_n\}_{n \geqslant 1}$, then*

$$E\Big(\sum_{i=1}^{N} X_i\Big)^2 = m_{F_N} m_{F_{X_1},2} + m_{F_N,2} m_{F_{X_1}}^2 - m_{F_N} m_{F_{X_1}}^2.$$

The easy proof is omitted.

3.2. $GI^{(x)}/G/1$ **queues.** It was shown in [7] that for $GI^{(x)}/G/1$ queues

$$W_n = \max(0, W_{n-1} + S_{n-1} - X_{n-1}),$$

where

$$S_n = \sum_{i=1}^{\overline{X}_n} Y_{n-1 \sum_{j=1} \overline{X}_{j+i}} \tag{3.2}$$

whenever $P(W_0 = 0) = 1$.

In the following theorem $\dashv$ denotes either the relation $\leqslant_{2.0}$ or $\leqslant_{2.1}$.

THEOREM 3.1 (Borovkov [7]). *Let us consider two $GI^{(x)}/G/1$ queueing systems: Σ' and Σ''. If $Y'_n \dashv Y''_n$, $-X'_n \dashv -X''_n$, $\overline{X}'_n \dashv \overline{X}''_n$, $n \geqslant 1$, $W'_0 = W''_0 = 0$ with* pr. 1, *then*

$$W'_n \dashv W''_n, \quad n \geqslant 1. \tag{3.3}$$

If the queueing system Σ'' is stable ($E\,S''_1 < E\,X''_1$) and $\int_0^\infty x^2 d\overline{A}''(x) < \infty$, $\int_0^\infty x^2 dB''(x) < \infty$, then

$$W' \dashv W''.$$

Proof. The r.v.'s S_n, $n \geqslant 1$, defined in (3.2) are i.i.d. Hence if we show that $S'_1 \dashv S''_1$, then $S'_n \dashv S''_n$ for all $n \geqslant 1$. But the r.v. S_1 has the p.d.f. equal to $\int_0^\infty B^{*x} d\overline{A}(x)$ and hence $S'_1 \dashv S''_1$ follows from the remark after example 2.6. Since the operator $\mathfrak{F}$ equal to

$$\mathfrak{F}(G)(x) = \begin{cases} 0, & x < 0, \\ \int_0^\infty F_{S_1 - X_1}(x-y)\,dG(y), & x \geqslant 0, \end{cases}$$

is isotonic with respect to $\leqslant_{2.0}$ or $\leqslant_{2.1}$, then using the result of example 2.5 and the principle of induction we obtain (3.3).

It is known (see [29]) that the necessary and sufficient conditions for $EW < \infty$ are

$$EX_1 < \infty, \quad ES_1^2 < \infty.$$

By lemma 3.1 and the hypothesis of the theorem we have $ES_1''^2 < \infty$. Since $ES''_1 < EX''_1$, we have $\omega - \lim_n W''_n(x) = W''(x)$ and since $P(W''_0 = 0) = 1$ we obtain

$$W''_1 \leqslant_{2.0} W''_2 \leqslant_{2.0} \cdots$$

(see section 3.1), which implies

$$\int_0^\infty x\,dW''_n(x) \leqslant \int_0^\infty x\,dW''(x).$$

Hence we get

$$\lim_{n\to\infty} \int_0^\infty x\,dW''_n(x) = \int_0^\infty x\,dW''(x).$$

The assumptions of the theorem imply

$$ES_1' < EX_1', \quad \int_0^\infty x^2 d\bar{A}'(x) < \infty,$$

$$\int_0^\infty x^2 dB'(x) < \infty \quad \text{and} \quad \lim_{n\to\infty} \int_0^\infty x dW_n'(x) = \int_0^\infty x dW'(x)$$

and hence by (3.3) and lemma 1.2 we get $W' \dashv W''$, which proves the theorem.

Remark. In the case of $GI/G/1$ systems this theorem has been given by Stoyan and Stoyan [31].

COROLLARY. *Let us consider a $GI^{(x)}/G/1$ stable $(m_{\bar{A}} m_B < m_A)$ queueing system.*

(i) *If the p.d.f. A is fixed, then*

$$\inf_{\substack{\bar{A}\in\mathscr{F}^{(m_{\bar{A}})}\\ B\in\mathscr{F}^{(m_B)}}} EW$$

is reached for $B = \delta_{m_B}$, $\bar{A} = (1-\{m_{\bar{A}}\})\,\delta_{m_{\bar{A}}} + \{m_{\bar{A}}\}\,\delta_{[m_{\bar{A}}]+1}$.

(ii) *If the p.d.f.'s B and $\bar{A}$ are fixed with* $\int_0^\infty x^2 dB(x) < \infty$, $\int_0^\infty x^2 d\bar{A}(x) < \infty$, *then*

$$\inf_{A\in\mathscr{F}^{(m_1)}} EW$$

is reached for $A = \delta_{m_A}$.

(iii) *If the p.d.f.'s A and $\bar{A}$ are fixed with* $\int_0^\infty x^2 d\bar{A}(x) < \infty$, *then*

$$\inf_{B\in\mathscr{F}^{(m_B)}} EW$$

is reached for $B = \delta_{m_B}$.

Proof. (i) Using theorem A.I (or equivalently theorem 1.2) and theorem A.2 (see Appendix), we have for $f \in \mathscr{K}_1$ and for all $\bar{A} \in \mathscr{F}^{(m_{\bar{A}})}$, $B \in \mathscr{F}^{(m_B)}$:

$$\int f dB \geqslant \int f d\delta_{m_B},$$

$$\int f d\bar{A} \geqslant \int f d\big((1-\{m_{\bar{A}}\})\,\delta_{[m_{\bar{A}}]} + \{m_{\bar{A}}\}\delta_{[m_{\bar{A}}]+1}\big)$$

and hence, using the remark after example 2.6, we get

$$\int_0^\infty B^{*x}(\cdot)\,d\bar{A}(x) \geqslant_{2.1} \int_0^\infty \delta_{m_B}^{*x}(\cdot)\,d\big((1-\{m_{\bar{A}}\})\delta_{[m_{\bar{A}}]} + \{m_{\bar{A}}\}\,\delta_{[m_{\bar{A}}]+1}\big)$$

and the assertion follows at once.

Remark. Theorem 3.1 and the corollary were given in [7].

3.3. Order relationship between $GI^{(x)}/M^{(y)}/1$ and $M^{(x)}/G^{(y)}/1$ queues. The following additional assumption about the queueing discipline is made. The service channel is such that when there is a vacancy in the group being served, the arriving customers will join the group immediately, until its capacity is reached. The rest of the group of arrivals will wait for the next service. In [4] and [7] were given the following recurrence relations for the queue size Q_n, immediately before the arrival of the n-th group in the case of $GI^{(x)}/M^{(y)}/1$ and the queue size Q_n^* immediately after the departure of the n-th group in the case of $M^{(x)}/G^{(y)}/1$ queues:

$$Q_n = \max(0, Q_{n-1}+\bar{X}_{n-1}-\bar{Y}_{c_{n-1}}-\ldots-\bar{Y}_{c_{n-1}+N_{n-1}-1}), \tag{3.4}$$

where c_n denotes the number of the group which will be served immediately after the instant τ_n and N_n is the size of the group which were taken for service during the time X_n,

$$Q_n^* = \max(0, Q_{n-1}^*+\bar{X}_{b_{n-1}}+\ldots+\bar{X}_{b_{n-1}+M_{n-1}}-\bar{Y}_n),$$

where b_n denotes the number of the group which come into the system after the n-th departure and M_n is the number of the group which arrive at the system during the time Y_n. One can see that

$$P(N_n = k \mid X_n = u) = \frac{(m_B^{-1}u)^k}{k!}e^{-m_B^{-1}u},$$

and

$$P(M_n = k \mid Y_n = u) = \frac{(m_A^{-1}u)^k}{k!}e^{-m_A^{-1}u}.$$

The queue $GI^{(x)}/M^{(y)}/1$ is stable if $m_{\bar{A}}-m_B^{-1}m_A m_{\bar{B}} < 0$ holds.

The queue $M^{(x)}/G^{(y)}/1$ is stable if $m_{\bar{B}}-m_A^{-1}m_B m_{\bar{A}} > 0$ holds.

THEOREM 3.2. (i) *Let us consider two $GI^{(x)}/M^{(y)}/1$ queueing systems Σ', Σ'' and let Σ'' be stable with $\int_0^\infty x^2 d\bar{A}''(x) < \infty$.*

If

$$\begin{aligned} &\bar{X}_1' \leqslant_{2.1} \bar{X}_1'' \quad (\bar{X}_1' \leqslant_{2.0} \bar{X}_1''), \\ &\bar{Y}_1'' \leqslant_{2.1} \bar{Y}_1' \quad (\bar{Y}_1' \leqslant_{2.0} \bar{Y}_1''), \\ &X_1'' \leqslant_{1.1} X_1' \quad (X_1'' \leqslant_{1.0} X_1'), \\ &m_{B'} \leqslant m_{B''}, \end{aligned} \tag{3.5}$$

then

$$Q' \leqslant_{2.1} Q'' \quad (Q' \leqslant_{2.0} Q'').$$

(ii) *Let us consider two $M^{(x)}/G^{(y)}/1$ queuing systems Σ', Σ'' and let*

Σ'' be stable with $\int_0^\infty x^2 dB(x) < \infty$, $\int_0^\infty x^2 d\bar{A}(x) < \infty$.
If

$$\bar{X}_1' \leqslant_{2.1} \bar{X}_1'' \qquad (\bar{X}_1' \leqslant_{2.0} \bar{X}_1''),$$
$$-\bar{Y}_1' \leqslant_{2.1} -\bar{Y}_1'' \qquad (-\bar{Y}_1' \leqslant_{2.0} -\bar{Y}_1''),$$
$$Y_1' \leqslant_{2.1} Y_1'' \qquad (Y_1' \leqslant_{2.0} Y_1''),$$
$$m_{A'} \geqslant m_{A''},$$

then

$$Q' \leqslant_{2.1} Q'' \qquad (Q' \leqslant_{2.0} Q'').$$

Proof. (i) From (3.4) we have

$$Q_n^* = \max(0, Q_{n-1} + \bar{X}_{n-1} + Z_{n-1}),$$

where $Z_n = \sum_{i=1}^{N_n} (-\bar{Y}_i)$, Z_n are independent r.v.'s and $\{Z_n\}_n$ are independent of $\{X_n\}_n$. By example 2.7 we have $N_n'' \leqslant_{1.1} N_n'$, and hence, by the remark after example 2.6, we get

$$Z_n' \leqslant_{2.1} Z_n''. \tag{3.6}$$

From the asumption that Σ'' is stable and from (3.5) it follows that Σ' is stable since $\bar{X}_1' + Z_1' \leqslant_{2.1} \bar{X}_1'' + Z_1''$ implies that $E(X_1' + Z_1') < 0$. Hence we obtain $Q_n' \underset{\omega}{\to} Q'$ and $Q_n'' \underset{\omega}{\to} Q''$ independently of Q_0', Q_0'' and we may assume that $P(Q_0' = 0) = 1$ and $P(Q_0'' = 0) = 1$. From this if follows that $Q_n' \leqslant_{2.1} Q''$ and $Q_n' \leqslant_{2.1} Q''$ (see section 3.1) and from the finiteness (by Wald's identity and the assumptions of the theorem) of EQ' and EQ'' we have

$$\int_0^\infty x dQ_n'(x) \to \int_0^\infty x dQ'(x),$$
$$\int_0^\infty x dQ_n''(x) \to \int_0^\infty x dQ''(x).$$

Since the operator

$$\mathfrak{F}(F_{Q_n}, F_{\bar{X}_n + Z_n}) = \begin{cases} \int_0^\infty F_{\bar{X}_n + Z_n}(x-y)\, dF_{Q_n}(y), & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

is isotonic with respect to the relation $\leqslant_{2.1}$ (see example 2.5) we have by the principle of mathematical induction for all n:

$$Q_n' \leqslant_{2.1} Q_n''$$

and hence from lemma 1.2 we have

$$Q' \leqslant_{2.1} Q''.$$

(ii) It is enough to prove that $EU_1^2 < \infty$, where $U_1 = \sum_{i=0}^{M_0} \bar{X}_i$. The rest of the proof is similar to that of (i).

Since $E\,\bar{X}_1^2 < \infty$, by lemma 3.1 we only have to show that $EM_1^2 < \infty$. Let

$$g_a(u) = \sum_{k=0}^{\infty} k^2 \frac{(au)^k}{k!} e^{-au}, \qquad a > 0.$$

The second derivative of $\frac{\partial^2}{u^2} g_a(u)$ is equal to $2a^2$ and hence

$$\int_0^\infty x dF_{M_1}(x) = \int_0^\infty \sum_{k=0}^{\infty} k^2 \frac{(m_A^{-1}u)^k}{k!} e^{-(m_A^{-1}u)} dF_{Y_1}(u) < \infty.$$

This completes the proof of the theorem.

As an example of utilization of theorem 3.2 we have the following corollary. Similar corollaries to that can easily be given by changing the fixed p.d.f.'s.

COROLLARY. *Let us consider an $M^{(x)}/G^{(v)}/1$ stable queueing system. If $\bar{A}_1$ is a fixed p.d.f. with $\int_0^\infty x^2 d\bar{A}_1(x) < \infty$, then*

$$\inf_{B \in \mathscr{F}^{(m_B)}, \bar{B} \in \mathscr{F}^{(m_{\bar{B}})}} EW$$

is reached for $B = \delta_{m_B}$,

$$\bar{B} = (1 - \{m_{\bar{B}}\})\, \delta_{[m_{\bar{B}}]} + \{m_{\bar{B}}\}\, \delta_{[m_{\bar{B}}+1]}.$$

The proof of this corollary is similar to that of the corollary after theorem 3.1.

4. Bounds for $GI^{(x)}G^{(v)}/1$ queues

4.1. Introduction. Now we show some applications of the results of the preceding chapters. We give some bounds for EW (EQ) in case of various types of $GI^{(x)}/G^{(v)}/1$ queues. Similar problems have been considered

for example in papers [18], [24], [29], etc. In [18], [24] can be found

$$E\eta \leqslant \frac{\operatorname{Var}\xi 1}{-2E\xi_1}, \tag{4.1}$$

where F_η is the limiting p.d.f. of the random walk η_n defined in (3.1). Stoyan [34] has shown that this bound in some cases can be improved but even this new bound is not sharp in many cases. It seems to be interesting to find for $GI^{(x)}/G^{(v)}/1$ queues

$$\overline{W} = \sup_{\substack{A\in\mathscr{F}_1, B\in\mathscr{F}_2\\ \bar{A}\in\mathscr{F}_3, \bar{B}\in\mathscr{F}_4}} EW,$$

where $\mathscr{F}_1, \ldots, \mathscr{F}_4$ are some families of p.d.f.'s.

Some special cases of this problem will be considered in the next section. In section 4.3 an application of bound (4.1) to $GI^{(x)}/M^{(v)}/1$, $M^{(x)}/G^{(v)}/1$ queues will be given and in the last section various applications of the relations $\leqslant_{1.n}$, $\leqslant_{2.n}$ in the queueing theory will be considered.

4.2. Bounds for $GI^{(x)}/G^{(v)}/1$ queues. In Takacs' book [38] it is proved that the size Q of the queueing system at the moment immediately before the arrival of a customer to a $GI/M/1$ system under stationary conditions has the geometrical distribution

$$P(Q = k) = (1-\nu_A)\nu_A^k, \quad k = 0, \ldots$$

The parametr ν_A fulfils the equation

$$\nu_A = \Phi_A\big(\mu(1-\nu_A)\big), \tag{4.2}$$

where Φ_A denotes the Laplace transform of the distribution function A of the inter-arrival time and $1/\mu$ is the mean of the service. It is assumed here that $r = m_A\cdot\mu > 1$. Now we shall prove the following (see [26]).

THEOREM 4.1. *We have*

$$\inf_{A\in\mathscr{F}(m_1,m_2)} \nu_A = l, \tag{4.3}$$

$$\max_{A\in\mathscr{F}(m_1,m_2)} \nu_A = \nu_{F_{m_1,m_2}} = 1+\frac{m_1^2}{m_2}(l-1), \tag{4.4}$$

where l is the root of the equation

$$l = \exp(-r+rl). \tag{4.5}$$

Proof. Notice that the function $\varphi_A(x) = \Phi_A\big(\mu(1-x)\big)-x$ is convex and

$$\varphi_A(0) > 0, \quad \varphi_A(1) = 0. \tag{4.6}$$

From theorem A.2 (or theorem 1.2) it follows that

$$\Phi_A\big(\mu(1-x)\big)-x \leqslant \Phi_{F_{m_1,m_2}}\big(\mu(1-x)\big)-x, \tag{4.7}$$

where F_{m_1,m_2} is defined in section 2.3.

First we show that $\max\limits_{A\in\mathcal{F}^{(m_1,m_2)}} \nu_A = \nu_{F_{m_1,m_2}}$.

Let us assume, to the contrary, that there exists an $A'\in\mathcal{F}^{(m_1,m_2)}$ such that $\nu_{A'} > \nu_{F_{m_1,m_2}}$. Formula (4.7) implies $0 \leqslant \varphi_{F_{m_1,m_2}}(\nu_A)$. Convexity and (4.6) implies $\varphi_{F_{m_1,m_2}}(\nu_{A'}) = 0$. Hence $\varphi_{F_{m_1,m_2}}(\nu_{A'}) = 0$, which is impossible because $\nu_{F_{m_1,m_2}}$ is the only root of this equation (see [38]).

From the above considerations it follows that the greatest root we search for is the root of the equation

$$z = p+q\exp[(z-1)r/q],$$

where $p = 1-m_1^2/m_2$, $q = m_1^2/m_2$.

Let l be the root of equation (4.5), i.e., $l = \exp[(l-1)r]$. Hence

$$l = \exp[(lq-q)r/q].$$

By a suitable transformation of this equality we achieve

$$1-\frac{m_1^2}{m_2}+\frac{m_1^2}{m_2}l = 1-\frac{m_1^2}{m_2}+\frac{m_1^2}{m_2}\exp\left[\left(l\frac{m_1^2}{m_2}-\frac{m_1^2}{m_2}\right)\frac{r}{q}\right].$$

Hence we can see that

$$1+\frac{m_1^2}{m_2}(l-1)$$

is the root of the equation $\Phi_{F_{m_1,m_2}}\big(\mu(1-x)\big) = x$.

The lower bound (4.3) can be proved in a similar way.

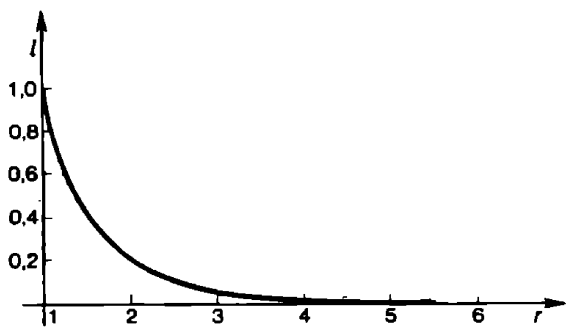

Fig. 1

Equation (4.5) has a unique solution for any $r > 1$. Values of the root l for some values of r are given in Table 1. In Fig. 1 there is shown the dependence of l on r.

Table 1

r	1.01	1.02	1.05	1.10	1.50	2.50	3.50	4.50	5.00
l	.9802	.9610	.9063	.8239	.4172	.1074	.0340	.0117	.0069

COROLLARY 1. *For the queueing system $GI/M/1$, if*

$$A \in \mathscr{F}^{(m_1,m_2)}$$

and $1/\mu$ is the mean time of service, then

$$\frac{1}{\mu(1-l)} < EW \leqslant \frac{\nu_{F_{m_1,m_2}}}{\mu(1-\nu_{F_{m_1,m_2}})},$$

$$\frac{l}{1-l} < EQ \leqslant \frac{\nu_{F_{m_1,m_2}}}{1-\nu_{F_{m_1,m_2}}}.$$

Proof. The proof follows easily from theorem 4.1 since for a $GI/M/1$ queue we have

$$E\,W = \frac{\nu_A}{\mu(1-\nu_A)}, \qquad EQ = \frac{\nu_A}{1-\nu_A}.$$

COROLLARY 2. (i) *Let Σ be a $GI^{(x)}/G/1$ queueing system. If*

$$A \in \mathscr{F}^{(m_1,m_2)}, \quad B \leqslant_{2.1} M_{1/\mu}, \quad \bar{A} \leqslant_{2.1} (1-p)\sum_{i=1}^{\infty} p^{i-1}\delta_i,$$

and if $\dfrac{1}{\mu}\dfrac{1}{1-p} < m_1$, *then*

$$EW \leqslant \frac{(1-p)^{-1} - \dfrac{m_1^2}{m_2} + \dfrac{m_1^2}{m_2}l}{\dfrac{\mu m_1^2}{m_2}(1-l)},$$

where l is given in (4.5) *with* $r = (1-p)\mu m_1$.

(ii) *Let Σ be a $GI/G/1$ queueing system. If* $A \in \mathscr{F}^{(m_1,m_2)}$, $B \leqslant_{2.1} M_{1/\mu}$, *then*

$$EW \leqslant \frac{1 - \dfrac{m_1^2}{m_2} + \dfrac{m_1^2}{m_2}l}{\dfrac{\mu m_1^2}{m_2}(1-l)},$$

where l is given in (4.5).

Proof. By hypothesis and theorems 3.1, 3.2 (i) we have

$$EW \leqslant EW',$$

where W' is the equilibrium waiting time in $GI^{(x)}/M/1$ queueing system in which

$$A'(x) = A(x), \quad B'(x) = M_{1/\mu}(x), \quad \bar{A}'(x) = (1-p)\sum_{i=1}^{\infty} p^{i-1}\delta_i(x).$$

The input stream is equivalent to the renewal process with the p.d.f.

$$A^*(x) = p\delta_0(x) + (1-p)A(x).$$

Now we can apply the result of the previous corollary to the system $GI/M/1$, putting $A = A^*$, and hence

$$EW' \leqslant \frac{(1-p)^{-1} - \dfrac{m_1^2}{m_2} + \dfrac{m_1^2}{m_2} l}{\dfrac{\mu m_1^2}{m_2}(1-l)}.$$

(ii) The proof is similar to that of (i). The queueing system $GI/G/1$ is a special case of $G^{(x)}/G/1$ with $p = 0$.

4.3. Bounds for $GI^{(x)}/M^{(y)}/1$, $M^{(x)}/G^{(y)}/1$ queues. Now we give some bounds for EQ, EQ^* in $GI^{(x)}/M^{(y)}/1$, $M^{(x)}/G^{(y)}/1$ queues respectively. From the considerations of chapter 3.3 we can see that Q_n and Q_n^* form random walks and hence we can use inequality (4.1).

Let us consider a stable $M^{(x)}/G^{(y)}/1$ queueing system $(m_{\bar{B}} - m_{\bar{A}}^{-1} m_B m_{\bar{A}} > 0)$ with

$$\int_0^\infty x^2 d\bar{A}(x) < \infty, \quad \int_0^\infty x^2 dB(x) < \infty. \tag{4.8}$$

The expected value of $U_1 = \sum_{i=1}^{M_0} \bar{X}_i$ is equal, by Wald's identity, to $EU_1 = m_{\bar{A}} \cdot EM_0$ and

$$EM_0 = \int_0^\infty \sum_{n=0}^{\infty} k e^{-u/m_A} \frac{\left(\dfrac{u}{m_A}\right)^n}{n!} dB(u) = m_{\bar{A}}^{-1} m_B.$$

The variance of U_1 is equal to

$$\operatorname{Var} U_1$$

$$= \int_0^\infty \sum_{n=0}^{\infty} \int_0^\infty \ldots \int_0^\infty (x_1 + \ldots + x_n)^2 d\bar{A}(x_1) \ldots d\bar{A}(x_n) e^{-(u/m_A)} \cdot \frac{\left(\dfrac{u}{m_A}\right)^n}{n!} dB(u) -$$

$$- (m_{\bar{A}}^{-1} m_B m_{\bar{A}})^2 = m_{\bar{A},2} m_B m_{\bar{A}}^{-1} + m_{B,2} m_{\bar{A}}^2 m_{\bar{A}}^{-2} - m_B^2 m_{\bar{A}}^2 m_{\bar{A}}^{-2}$$

$$= \sigma_B^2 m_{\bar{A}}^2 m_{\bar{A}}^{-2} + m_{\bar{A},2} m_B m_{\bar{A}}^{-1}.$$

Hence, using Kingman's inequality (4.1), we obtain

THEOREM 4.2. *For $M^{(x)}/G^{(y)}/1$ stable queueing systems with p.d.f.'s $\bar{A}$, B fulfilling* (4.8) *we have*

$$EQ^* \leqslant \frac{m_{\bar{A}}^{-1} m_{\bar{A},2} m_B + m_{\bar{A}}^{-2} m_{\bar{A}}^2 \sigma_B^2 + \sigma_{\bar{B}}^2}{-m_{\bar{A}}^{-1} m_B m_{\bar{A}} + m_{\bar{B}}}.$$

Also for $GI^{(x)}/M^{(y)}/1$ stable queueing systems with

$$m_{\bar{A},2} < \infty, \tag{4.9}$$

we have

THEOREM 4.3. *For $GI^{(x)}/M^{(y)}/1$ stable queueing $(m_{\bar{B}} \cdot m_B^{-1} m_A - m_{\bar{A}} > 0)$ systems with p.d.f.'s fulfilling* (4.9) *we have*

$$EQ \leqslant \frac{m_{\bar{B}}^{-1} m_{\bar{B},2} m_A + m_{\bar{B}}^{-2} m_{\bar{B}}^2 \sigma_A^2 + \sigma_{\bar{A}}^2}{-m_{\bar{A}} + m_{\bar{B}}^{-1} m_A m_{\bar{B}}}.$$

The proof is similar to that of theorem 4.2.

4.4. Application of the relations $\leqslant_{1,n}$, $\leqslant_{2,n}$ in queues. The n-th semi-invariant S_n of the equilibrum waiting time in $GI/G/1$ queues with the p.d.f. K of the difference between the service and inter-arrival times are given by

$$S_n = (-i)^k \frac{d}{dt} \ln Ee^{itw}|_{t=0} = \sum_{j=1}^{\infty} \frac{1}{j} \int_0^{\infty} x^n dK^{*j}(x), \qquad n = 1, 2, \ldots$$

THEOREM 4.4. *Let Σ', Σ'' be stable $GI/G/1$ queueing systems with the p.d.f. of the difference between the service and the inter-arrival times equal to K' and K'', respectively. If $K' \leqslant_{2,n} K''$, then for the corresponding n-th semi-invariants S_k', S_k'', $k \geqslant n$, we have*

$$S_k' \leqslant S_k''.$$

Proof. The proof follows easily since $x_+^n \in \mathscr{K}_n^+$ and the relation $\leqslant_{2,n}$ has property (C).

Remark. Bergman [5] has proved a similar theorem using more complicated ordering relations. In this form theorem 4.4 has been given in [27].

[26] contains the following:

THEOREM 4.5. *Let A', A'' be the p.d.f.'s of the inter-arrival times in two systems Σ', Σ'' of type $GI/M/1$ and $1/\mu$ the common mean service time. If $A'' \leqslant_{1,n} A'$, then*

$$F_{W'}(x) \geqslant F_{W''}(x), \quad 0 \leqslant x, \tag{4.10}$$

$$F_{Q'}(x) \geqslant F_{Q''}(x), \quad 0 \leqslant x. \tag{4.11}$$

Proof. The p.d.f. of the waiting time is given by

$$W(x) = \begin{cases} 1-\nu_A \exp\big(-(1-\nu_A)\big)\mu x, & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

where $\nu_A = \Phi_A\big(\mu(1-\nu_A)\big)$ (see Takacs [38]). Since $A'' \leqslant_{l.n} A'$, we have

$$\int_0^\infty \exp(-sx)\,dA'(x) \leqslant \int_0^\infty \exp(-sx)\,dA''(x), \qquad s > 0.$$

Hence $\nu_{A'} \leqslant \nu_{A''}$ and since

$$W_x(\nu) = 1-\nu\exp\big(-(1-\nu)\mu x\big)$$

is a monotonous function of ν, we obtain (4.10). The proof of (4.11) is omitted.

Some other characteristics of $GI/M/N$, $1 \leqslant N \leqslant \infty$, queues are monotonous functionals of the p.d.f. A with respect of the relation $\leqslant_{l.n}$. For example the equilibrum mean time of the distance between consecutive lost calls S in the system $GI/M/N$, $1 \leqslant N < \infty$, with losses, is equal to (see Takacs [38])

$$S = m_A \sum_{i=1}^{N} \binom{N}{j} \prod_{i=1}^{j} \frac{\Phi_A(i\mu)}{1-\Phi_A(i\mu)}$$

and the equilibrium n-th binomial moment of numbers of units in a $GI/M/\infty$ queueing system is equal to

$$B_n = \prod_{i=1}^{n} \frac{\Phi_A(\mu i)}{1-\Phi_A(\mu i)}.$$

Appendix

Here will be given an elementary proof of a theorem about extremal p.d.f.'s of a functional $\int \varphi dF$ when the p.d.f.'s run over the sets

$$\mathscr{F}^{(m_1)}(\mathscr{F}^{(m_1,m_2)}) = \Big\{F\colon F(0) = 0, \int_0^\infty x^i dF(x) = m_i,\ i = 1\,(i = 1, 2)\Big\},$$

$$\mathscr{F}_{a,b}^{(m_1)}(\mathscr{F}_{a,b}^{m_1,m_2}) = \Big\{F\colon F(x) = \sum_{i=0}^{\infty} p_i\,\delta_{a+bi}(x), \int_0^\infty x^i dF(x) = m_i,$$
$$i = 1\ (i = 1, 2)\Big\}.$$

For the particular case where $\varphi_s(x) = e^{-sx}$, $s > 0$, a similar proof to that of theorem A.1 has been given by Vasylev and Kozlov [39]. A utilization of such results has been given for example in [17]. In [40] an application of the result of theorem A.1 is given

THEOREM A.1. (i) *For every bounded, convex (and hence decreasing function) we have*

$$\min_{F\in\mathscr{F}^{(m_1)}} \int_0^\infty \varphi dF = \varphi(m_1), \tag{A.1}$$

$$\sup_{F\in\mathscr{F}^{(m_1)}} \int_0^\infty \varphi dF = \varphi(0). \tag{A.2}$$

(ii) *For every function φ of form $\varphi(x) = \int_x^\infty \psi(t)\,dt$, where ψ is convex and decreasing, we have*

$$\max_{F\in\mathscr{F}^{(m_1,m_2)}} \int_0^\infty \varphi dF = \left(1-\frac{m_1^2}{m_2}\right)\varphi(0)+\frac{m_1^2}{m_2}\varphi\left(\frac{m_2}{m_1}\right), \tag{A.3}$$

$$\inf_{F\in\mathscr{F}^{(m_1,m_2)}} \int_0^\infty \varphi dF = \varphi(m_1). \tag{A.4}$$

Proof. (i) (A.1) follows from Jensen's inequality (see [8]). In order to prove (A.2) we note that $\int_0^\infty \varphi dF \leqslant \varphi(0)$. To complete the proof it is enough to consider the family of step p.d.f.'s from $\mathscr{F}^{(m_1)}$ with jumps equal to p, $1-p$ at points 0, $m_1/(1-p)$ respectively. For this family the supremum in (A.2) is attained.

(ii) The proof of (A.3) is divided into 2 steps. First we show, as in Hoeffding's paper (see [13]), that it is enough to restrict considerations to the set $\mathscr{F}_d^{(m_1,m_2)}$ of step p.d.f.'s from $\mathscr{F}^{(m_1,m_2)}$ having a finite number of jumps, i.e., we show that $\mathscr{F}_d^{(m_1,m_2)}$ is dense in $\mathscr{F}^{(m_1,m_2)}$ in the weak topology. Let $F\in\mathscr{F}^{(m_1,m_2)}$ be given. For any $\delta > 0$ let $0 = a_0 < a_1 < \ldots < a_m < a_{m+1} = \infty$ be such that $p_r = F(a_{r+1}) - F(a_r) < \delta$, $r = 0, \ldots, m$. If $p_r \neq 0$ let

$$F_r(x) = \begin{cases} 0, & x < a_r, \\ p_r^{-1}[F(x)-F(a_r)], & a_r \leqslant x < a_{r+1}, \\ 1, & a_{r+1} \leqslant x. \end{cases}$$

It is obvious that there exists a p.d.f. F_r^* which is a step function with a finite number of jumps such that

$$\int_{[a_r, a_{r+1})} x^i dF_r^*(x) = \int_{[a_r, a_{r+1})} x^i dF_r(x), \quad i = 1, 2.$$

Let

$$F^*(x) = \begin{cases} 0, & x < 0, \\ F(a_r) + p_r F_r^*(x), & a_r \leqslant x < a_{r+1}, \ r = 0, \ldots, m, \end{cases}$$

where $p_r F_r^*(x) = 0$ if $p_r = 0$.

It is easy to verify that $\int_0^\infty x^i dF(x) = m_i$, $i = 1, 2$. Since δ is an arbitrary number, the set $\mathscr{F}_d^{(m_1, m_2)}$ is dense in $\mathscr{F}^{(m_1, m_2)}$. Hence it follows that we can restrict ourselves only to the set $\mathscr{F}_d^{(m_1, m_2)}$ because the function φ is bounded and continuous.

In order to show the second step of the proof let us consider the set of p.d.f.'s having jumps at two points. Such p.d.f.'s have the following representation:

$$\text{(A.5)} \qquad G_t(x) = \begin{cases} 0, & x < t, \\ \dfrac{m_2 - m_1^2}{m_2 + t^2 - 2m_1 t}, & t \leqslant x < \dfrac{m_2 - m_1 t}{m_1 - t}, \\ 1, & \dfrac{m_2 - m_1 t}{m_1 - t} \leqslant x. \end{cases}$$

Further we show that the function $\lambda(t) = \int_0^\infty \varphi(x) dG_t(x)$ is decreasing in $[0, m_1)$. This follows from an elementary property of the convex function

$$\frac{1}{b-a} \int_a^b h(t) dt \leqslant \frac{h(b) + h(a)}{2},$$

which implies

$$\frac{d\lambda(t)}{dt} = \frac{m_2 - m_1^2}{m_2 + t_1^2 - 2m_1 t} \left(\frac{2(m_1 - t)}{m_2 + m_1^2 - 2m_1 t} \int_t^{\frac{m_2 - m_1 t}{m_1 - t}} \psi(x) dx - \psi(t) - \right.$$
$$\left. - \psi\left(\frac{m_2 - m_1 t}{m_1 - t} \right) \right) \leqslant 0$$

in $[0, m_1)$.

Now we suppose that the p.d.f. $F = F_{\langle x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n \rangle} \in \mathscr{F}^{(m_1, m_2)}$ has jumps $p_1, \ldots, p_n$ at points $x_1, \ldots, x_n$, $n \geqslant 3$. Since $F \in \mathscr{F}^{(m_1, m_2)}$, we

have

$$\frac{p_1}{1-p_3-\ldots-p_n}+\frac{p_2}{1-p_3-\ldots-p_n}=1,$$

$$\text{(A.6)}\qquad \frac{p_1x_1}{1-p_3-\ldots-p_n}+\frac{p_2x_2}{1-p_3-\ldots-p_n}=\frac{m_1-\sum_{i=3}^{n}p_ix_i}{1-p_3-\ldots-p_n},$$

$$\frac{p_1x_1^2}{1-p_3-\ldots-p_n}+\frac{p_2x_2^2}{1-p_3-\ldots-p_n}=\frac{m_2-\sum_{i=3}^{n}p_ix_i^2}{1-p_3-\ldots-p_n}.$$

Having fixed $x_3,\ldots,x_n,p_3,\ldots,p_n$, and basing ourselves on the fact that $\lambda(0)\geqslant\lambda(t)$, $0\leqslant t<m_1$, we have

$$(1-p_3-\ldots-p_n)\left(\frac{p_1\varphi(x_1)}{1-p_3-\ldots-p_n}+\frac{p_2\varphi(x_2)}{1-p_3-\ldots-p_n}\right)+\sum_{i=3}^{n}p_i\varphi(x_i)$$
$$\leqslant p_1'\varphi(0)+p_2'\varphi(x_1')+\sum_{i=3}^{n}p_i\varphi(x_i)$$

and

$$F_{\langle x_1'=0,x_2',\ldots,x_n,p_1',p_2',\ldots,p_n\rangle}\in\mathscr{F}^{(m_1,m_2)}.$$

Now we fix $x_1'=0$, $x_4,\ldots,x_n$, $p_1',p_4,\ldots,p_n$ and we obtain in a similar way

$$p_1'\varphi(0)+p_2'\varphi(x_2')+p_3\varphi(x_3)+\sum_{i=4}^{n}p_i\varphi(x_i)$$
$$\leqslant p_1'\varphi(0)+p_2'\varphi(0)+p_3'\varphi(x_3')+\sum_{i=4}^{n}p_i\varphi(x_i)$$

and

$$F_{\langle x_1'=0,x_2'=0,x_3',\ldots,x_n,p_1',p_2',p_3',\ldots,p_n\rangle}\in\mathscr{F}^{(m_1,m_2)}.$$

Finally we obtain

$$\sum_{i=1}^{n}p_i\varphi(x_i)\leqslant p^*\varphi(0)+(1-p^*)\varphi(x^*)$$

and

$$F_{\langle x=0,x^*,p^*,1-p^*\rangle}\in\mathscr{F}^{(m_1,m_2)}.$$

Hence by (A.5) we get $p^*=m_1^2/m_2$, $x^*=m_2/m_1$ and the proof is complete.

Now we formulate and prove a modification of theorem A.1 in which we additionally assume that p.d.f.'s belong to $\mathscr{F}_{a,b}^{m_1}$ ($\mathscr{F}_{a,b}^{(m_1,m_2)}$). The esti-

mations given in theorem A.2 have been used for example in [7], [21]. The general theory of such problems has been given in [17]. The proof of theorem A.2 is based on the simplex method (see [15]).

THEOREM A.2. (i) *For every bounded convex function we have*

$$\sup_{F\in\mathscr{F}_{0,1}^{(m_1)}}\int_0^\infty \varphi dF = \varphi(0), \tag{A.7}$$

$$\min_{F\in\mathscr{F}_{0,1}^{(m_1)}}\int_0^\infty \varphi dF = (1-\{m_1\})\varphi([m_1])+\{m_1\}\varphi([m_1]+1), \tag{A.8}$$

where $[x]$ *denotes the integer part of* x *and* $\{x\} = x-[x]$.

(ii) *For every function* φ *of form* $\varphi(x) = \int_x^\infty \psi(t)\,dt$, *where* ψ *is convex and decreasing, we have*

$$\inf_{F\in\mathscr{F}_{0,1}^{(m_1,m_2)}}\int_0^\infty \varphi dF = (1-\{m_1\})\varphi([m_1])+\{m_1\}\varphi([m_1]+1), \tag{A.9}$$

$$\max_{E\in\mathscr{F}_{1,0}^{(m_1,m_2)}}\int_0^\infty \varphi dF \tag{A.10}$$

$$= p_0\varphi(0)+p_{[m_2/m_1]}\left(\left[\frac{m_2}{m_1}\right]\right)+p_{([m_2/m_1]+1)}\,\varphi\left(\left[\frac{m_2}{m_1}\right]+1\right),$$

where

$$p_0 = 1-p_{[m_2/m_1]}-p_{([m_2/m_1]+1)},$$

$$p_{[m_2/m_1]} = \left(m_1\left(1+\left[\frac{m_2}{m_1}\right]\right)-m_2\right)\Big/\left[\frac{m_2}{m_1}\right],$$

$$p_{([m_2/m_1]+1)} = \left(m_2-m_1\left[\frac{m_2}{m_1}\right]\right)\Big/\left(1+\left[\frac{m_2}{m_1}\right]\right).$$

Remark. We note that for the general case $\mathscr{F}_{a,b}^{(m_1)}$ ($\mathscr{F}_{a,b}^{(m_1,m_2)}$) we can find bounds by a simple change of scale and translation.

Proof of theorem A.2. First we note that it is enough to consider the p.d.f.'s from $\mathscr{F}_{0,1}^{(m_1)}$ ($\mathscr{F}_{0,1}^{m_1,m_2}$) having a finite number of jumps. We show it for the set $\mathscr{F}_{0,1}^{(m_1,m_2)}$.

Let $F(x) = \sum_{i=0}^{\infty} p_i\delta_i(x)\in\mathscr{F}_{0,1}^{(m_1,m_2)}$ and N be such that $\sum_{i=N}^{\infty} p_i < \varepsilon$. Then

we have

$$\sum_{n=1}^{N-1} p_n + \sum_{n=N}^{\infty} p_n = 1,$$

$$\sum_{n=1}^{N-1} np_n + \sum_{n=N}^{\infty} np_n = m_1,$$

$$\sum_{n=1}^{N-1} n^2 p_n + \sum_{n=N}^{\infty} n^2 p_n = m_2.$$

One can show that there exist non-negative integers ν_1, ν_2, ν_3 and $\bar{p}_{\nu_1}, \bar{p}_{\nu_2}, \bar{p}_{\nu_3} \geqslant 0$ such that

$$\frac{\sum_{i=1}^{3} \bar{p}_{\nu_i}}{\sum_{n=N}^{\infty} p_n} = 1,$$

$$\sum_{i=1}^{3} \frac{\nu_i \bar{p}_{\nu_i}}{\sum_{n=N}^{\infty} p_n} = \frac{\sum_{n=N}^{\infty} np_n}{\sum_{n=N}^{\infty} p_n} = m_1',$$

$$\sum_{i=1}^{3} \frac{\nu_i^2 \bar{p}_{\nu_i}}{\sum_{n=N}^{\infty} p_n} = \sum_{n=N}^{\infty} \frac{n^2 p_n}{\sum_{n=N}^{\infty} p_n} = m_2'.$$

This results from the following geometrical considerations. The point $(m_1', m_2') \in R^2$ belongs to the closure of the convex hull of the set

$$\left\{(s, t): s = \int_0^{\infty} x d\delta_i(x),\ t = \int_0^{\infty} x^2 d\delta_i(x)\, i = 0, 1, \ldots\right\} = \{(i, i^2): i = 0, 1, \ldots\}.$$

More precisely, it is contained in the triangle with vertices in $(0, 0)$, $([m_1], [m_1]^2)$, (k, k^2), $k > m_1$, $k > m_2$ (i.e., $k > \max(m_1, m_2)$) and it is known that each point of a triangle can be represented as a convex combination of the vertices.

In the sequel we show only the supports of the extremal measures. The probabilities can be computed by using the conditions that the extremal measures belong to $\mathscr{F}_{0,1}^{(m_1)}$ or $\mathscr{F}_{0,1}^{(m_1, m_2)}$.

Now let us prove formulae (A.7)–(A.8). The proof of (A.7) is analogous to that of (A.2). To prove (A.8) let us consider the following program:

$$\min \sum_{i=0}^{n} p_i \varphi(i) \quad (n > [m_1] + 1 \text{ fixed})$$

subject to

$$\sum_{i=0}^{n} p_i = 1, \quad \sum_{i=0}^{n} i p_i = m_1, \quad p_1, \ldots, p_n \geqslant 0.$$

Put $P_k = (1, k)^T$, $k = 0, \ldots, n$, where P^T denotes the transposition of vector P. Let the vectors $P_{[m_1]}, P_{[m_1+1]}$ form a basis and let $\lambda_0 = (0, \ldots, 1-\{m_1\}, \{m_1\}, 0, \ldots, 0)$ be an extreme point. Then, according to the simplex method, λ_0 is optimal if $z_i - c_i \leqslant 0$ for $i = 0, \ldots, n$, but by the convexity assumption we obtain

$$z_i - c_i = ([m_1]+1-j)\varphi([m_1])\varphi(j-[m_1])\varphi([m_1]+1) - \varphi(j) \leqslant 0.$$

To prove (1.10) Let us consider the following program:

$$\max \sum_{i=1}^{n} p_i \varphi(i) \quad \left(n > \left[\frac{m_2}{m_1}\right]+1 \text{ fixed}\right)$$

subject to

$$\sum_{i=1}^{n} p_i = 1, \quad \sum_{i=1}^{n} i\, p_i = m_1, \quad \sum_{i=1}^{n} i^2 p_i = m_2, \quad p_1, \ldots, p_n \geqslant 0.$$

Put $\Pi_k = (1, k, k^2)^T$, $k = 0, \ldots, n$, and assume that the vectors $\Pi_0, \Pi_{[m_2/m_1]}, \Pi_{[m_2/m_1+1]}$ form a basis. Then

$$\varkappa_0 = (p_0, 0, \ldots, 0, p_{[m_2/m_1]}, p_{[m_2/m_1+1]}, \ldots, 0)$$

is an extreme point.

According to the simplex method $\varkappa_0$ is optimal if $z_i - c_i \geqslant 0$ for $i = 0, \ldots, n$. To check this, let us suppose that $\varphi(x) = \int_x^\infty \psi(t)\,dt$ and ψ is strictly convex and decreasing. Then an arbitrary polynomial $\sum_{i=0}^{2} a_i x^i + b\varphi(x)$ has at most two roots (the second derivate is strictly positive) and hence

$$\text{(A.11)} \qquad \begin{vmatrix} 1 & x_1 & x_1^2 & \varphi(x_1) \\ 1 & x_2 & x_2^2 & \varphi(x_2) \\ 1 & x_3 & x_3^2 & \varphi(x_3) \\ 1 & x_4 & x_4^2 & \varphi(x_4) \end{vmatrix} < 0,$$

whenever $0 \leqslant x_1 < \ldots < x_4 < \infty$ (see example 1.2).

Putting $x_0 = 0$, $x_2 = j$, $x_3 = a$, $x_4 = a+1$ and expanding (A.11) by the last column, we obtain, we obtain after simple transformations

$$\left(1 - \left(\frac{1}{a} + \frac{1}{a+1}\right)j + \frac{1}{a(a+1)}\right)j^2\varphi(0) + \left(\frac{a+1}{a}j - \frac{1}{a}j^2\right)\varphi(a) +$$
$$+ \left(\frac{1}{a+1}j^2 - \frac{a}{a+1}j\right)\varphi(a+1) - \varphi(j) > 0,$$

whenever $0 < j < a < a+1$ or $0 < a < a+1 < j$.

For $a = [m_2/m_1]$ the left-hand side of the above inequality is equal to $z_i - c_i$ and hence we get $z_i - c_i > 0$, $i = 0, \ldots, n$.

If φ is such that ψ is not stricly convex, then there exists a sequence ψ_n of strictly convex functions such that $\varphi_n \to \varphi$ and the proof of (A.10) is completed.

To prove (A.9) we note that from (A.8) we have

$$\inf_{F \in \mathscr{F}_{0,1}^{(m_1,m_2)}} \int_0^\infty \varphi dF \geqslant \inf_{F \in \mathscr{F}_{0,1}^{(m_1)}} \int_0^\infty \varphi dF = (1-\{m_1\})\varphi([m_1]) + \{m_1\}\varphi([m_1]+1).$$

Now let us consider a p.d.f. $F_N \in \mathscr{F}_{0,1}^{(m_1,m_2)}$ having jumps p', p'', p_N at $[m_1]$, $[m_1]+1$, N, where $[a-] = \lim_{0<\varepsilon\to 0} [a-\varepsilon]$. If N is sufficiently large such a p.d.f. exists. If $N \to \infty$, then $p_N \to 0$ and

$$F_N(x) \xrightarrow{w} \begin{cases} 0, & x < [m_1], \\ 1-\{m_1-\}, & [m_1] \leqslant x \leqslant [m_1]+1, \\ 0, & x \geqslant [m_1]+1, \end{cases}$$

which completes the proof.

Let us note that one can prove theorem A.2. from theorem A.1 because the set of p.d.f.'s $\bigcup_{a,b} \mathscr{F}_{a,b}^{(m_1,m_2)}$ is dense in $\mathscr{F}^{(m_1,m_2)}$.

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