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A minorization of the first positive eigenvalue of the scalar laplacian on a compact Riemannian manifold

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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

CONTENTS

Introduction

On a smooth, oriented, compact Riemannian manifold M with a covariant metric tensor field g we have the (positively defined) scalar laplacian

$$\Delta := d^* d = \frac{-1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^\alpha} \sqrt{\det(g_{ij})}\, g^{\alpha\beta} \frac{\partial}{\partial x^\beta},$$

where $(g^{\alpha\beta})$ is dual to the metric tensor $(g_{\alpha\beta})$. To emphasize that Δ depends on g we may write Δ_g instead of Δ.

If $a > 0$, then, obviously, $\Delta_{ag} = (1/a)\Delta_g$, what for the spectra gives

$$(*) \qquad \operatorname{Spec}(ag) = \frac{1}{a} \operatorname{Spec}(g).$$

This answers the question how the dilations in the cone $\mathscr{M}(M)$ of metrics on M influence the spectrum. It is interesting to find both how the spectrum changes under a general transformation in $\mathscr{M}(M)$ and how it depends on a metric itself. However, these are very difficult problems and people confine themselves either to some special Riemannian manifolds or ask more special questions about the spectrum. There are two main classes of such special questions: about the end and about the beginning of the spectrum. In the first class we ask about distribution on $\boldsymbol{R}^1$ of great eigenvalues; mainly the asymptotic expansion (for $t \searrow 0$) of the partition function $Z(t) = \sum_{\lambda \in \operatorname{Spec}} e^{-\lambda s}$, $t > 0$, is considered (see [2], [12]). The second class deals with the first few eigenvalues, e.g. [6], [3], but most efforts have been devoted to the estimation of the first positive eigenvalue λ_1 (several bibliographic references are given in [14]).

Because of the subject of the present paper, it is worthwhile to recall at least those estimations from below of λ_1 which require no assumptions about the curvature of the Riemannian manifold (M, g). The first very beautiful result is due to Cheeger [5]. He proved that for $\dim M = m \geqslant 2$

$$(**) \qquad \lambda_1 \geqslant \frac{I(M)^2}{4} > 0, \quad \text{where} \quad I(M) := \inf_S \frac{\operatorname{vol} S}{\min\{\operatorname{vol} M_1, \operatorname{vol} M_2\}}$$

and the infimum is taken over all $(m-1)$-dimensional submanifolds S

dividing M into two disjoint parts, M_1 and M_2. Let us notice that the estimating constant $\frac{I(M)^2}{4}$ behaves under dilations in the same way as the spectrum, i.e. like (*); moreover, it has a very nice geometrical sense. But on the other hand, it is not practical for calculations in the case of an arbitrary Riemannian manifold. Yau [14] estimated from below Cheeger's isoperimetric constant $I(M)$; the computability of the new minorization of λ_1 is a compensation for the loss of the above-mentioned "good" behaviour under dilations. Earlier, Aubin [1] obtained another computable estimation depending exponentially on the curvature.

This paper contains another minorization (Theorem 16), which is as follows: if $\dim M = m \geqslant 3$, δ is the diameter of (M, g) and V is its volume, then

$$\lambda_1 \geqslant B_3 \frac{\varepsilon(\lambda_1)^{m-1}}{\delta V},$$

where

$$B_3 := \frac{2^{-2(2m+1)}\sqrt{m-1}}{(m+1)\, m^{2(m+2)+1/2}},$$

$\varepsilon(\lambda) := \min\{(H_1 + \lambda H_2)^{-1}, H_3\}$ and the dependence of the constants H_i on the metric g—via the 1st, 2nd and 3rd order derivatives of its components in orthonormal charts and an injectivity radius—is explicitly given. Thus we have (Corollary 17)

$$(*{*}*) \qquad \lambda_1 \geqslant B_3 \frac{H_3^{m-1}}{\delta V} \quad \text{or} \quad \lambda_1 \geqslant \frac{B_3}{\delta V}\left(H_1 + \frac{B_3 H_3^{m-1} H_2}{\delta V}\right)^{1-m}$$

This result has neither a geometrical elegance nor the "good" behaviour, as was the case for (**). But the estimating constants do not depend exponentially on curvature-like quantities, i.e. derivatives of g; this feature could be appreciated by Aubin (cf. his Remark [1], p. 368) and Yau (the case of non-positive curvature).

It is worth mentioning perphaps that in this research I was motivated by the problems and ideas stated and developed in [7] and [8].

The titles of the sections indicate fairly exactly the content of the paper. The first three of them are auxiliary to the fourth one. Besides, an essential use is made of the difference approximations introduced in [9] and [10]. A more extensive summary of the approach presented here is given in [11].

Finally, I should like to express my thanks to Professors T. Bałaban and E. M. Stein, whose advice and hints helped me very much. I am also indebted to Professor K. Maurin for the hopeful—and therefore helpful—encouragements and his interest in this work.

1. A parametrix of the laplacian

Let M be a smooth, oriented, Riemannian manifold of dimension $m \geqslant 3$ and let g denote the covariant metric tensor field on M. So far, instead of compactness we assume that there exists an $r_0 > 0$ such that for every $x \in M$ the ball $K(x, r_0)$ is a normal neighbourhood of the point x. Obviously, if M is compact, such an r_0 exists. By $\overset{m}{\tau}$ we denote the volume m-form on M, compatible with the orientation.

We use the following definition of the Hodge operator:

$$*\omega := \frac{m!}{(m-k)!}\, \tilde{\omega} \;\lrcorner\; \overset{m}{\tau}, \qquad \omega \in \bigwedge^k T^*(M),$$

where $\tilde{}$ is the canonical isomorphism $\bigwedge^k T^*(M) \to \bigwedge^k T(M)$ given by the Riemannian structure and it maps a k-form $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ onto

$$\sum_{j_1, \ldots, j_k = 1}^{m} g^{i_1 j_1} \ldots g^{i_k j_k} \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_k}};$$

the interior product $\lrcorner$ is defined by $\langle \varrho, \tilde{\omega} \;\lrcorner\; \overset{m}{\tau} \rangle = \langle \tilde{\omega} \wedge \varrho, \overset{m}{\tau} \rangle$ where the pairings $\langle \cdot, \cdot \rangle$ between $\bigwedge^k T(M)$ and $\bigwedge^k T^*(M)$ are so normalized that

$$\left\langle \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^k}, dx^1 \wedge \ldots \wedge dx^k \right\rangle = \frac{1}{k!}.$$

Moreover, if $\bigwedge^k T_z^*(M)$ is endowed with the scalar product $(\varrho \mid \omega) := k! \langle \tilde{\varrho}, \omega \rangle$, then $\varrho \wedge *\omega = (\varrho \mid \omega) \overset{m}{\tau}(z)$.

We are interested in the scalar laplacian $\Delta = \delta d$, where $\delta = -*^{-1} d *$. In order to construct its parametrix, which will be our basic tool, we define the smooth function

$$c(t) := \begin{cases} 1, & t \leqslant \frac{1}{2} r_c, \\ F(4t/r_c - 3), & \frac{1}{2} r_c < t < r_c, \\ 0, & t \geqslant r_c, \end{cases} \tag{1}$$

where $0 < r_c \leqslant r_0$ and F is a standard function defined as

$$F(t) := \begin{cases} 1, & t \leqslant -1, \\ a \int_t^1 \exp\left(1/(x^2 - 1)\right) dx, & |t| < 1, \\ 0, & t \geqslant 1, \end{cases} \tag{2}$$

where

$$a := \left[\int_{-1}^{1} \exp\left(\frac{1}{x^2 - 1} \right) dx \right]^{-1} \tag{3}$$

Now, for every $x, y \in M$ such that $x \neq y$ we define

$$p(x, y) := a l(x, y)^{2-m} \operatorname{co} l(x, y), \tag{4}$$

where $l \in C^\infty(M \times M)$ is the distance and

$$a := \frac{1}{2(2-m)} \Gamma(m/2) \pi^{-m/2}. \tag{5}$$

Obviously, for every $x \in M$, the function $p(x, \cdot) \in L^1(M, \mu)$, where μ is the measure on M defined by the canonical form of volume

$$\overset{m}{\tau} := \sqrt{\det(g_{ij})}\, dx^1 \wedge \quad \wedge dx^m;$$

here (x^a) is a positively oriented chart on M and g_{ij} are coordinates of g with respect to this chart. Let us check that p is a parametrix of the laplacian.

PROPOSITION 1. *If for $x, y \in M$, $x \neq y$ we define*

$$e(x, y) := \Delta p(x, \cdot)(y), \tag{6}$$

then e is a smooth function outside the diagonal of $M \times M$, and for every $x \in M$ both the function $e(x, \cdot)$ and its module belong to $L^1(M, \mu)$. Moreover, for every $\varphi \in C^\infty(M)$ and $x \in M$

$$\int_M p(x, \cdot)(\Delta \varphi) \tau = \varphi(x) + \int_M e(x, \cdot) \varphi \overset{m}{\tau}. \tag{7}$$

Proof. Let (x^a) be an orthonormal, positively oriented chart at a point $x \in M$ and let $U \supset K(x, r_0)$ be the domain of this chart; $x^a \in C^\infty(U)$, $g_{ij}(x) = \delta_{ij}$. If $y \in U$, then its coordinates with respect to (x^a) will also be denoted by $y^a := x^a(y)$.

We begin with a few elementary formulae:

$$l(x, \cdot) = \sqrt{\sum_{a=1}^{m} (x^a)^2} \quad \text{on} \quad U,$$

$$dl(x, \cdot) = l(x, \cdot)^{-1} \sum_{a=1}^{m} x^a dx^a \quad \text{on} \quad U \setminus \{x\}, \tag{8}$$

and

$$\begin{aligned} *dx^a &= m g^{a\beta} \frac{\partial}{\partial x^\beta} \lrcorner \overset{m}{\tau} \\ &= (-1)^{\beta-1} \sqrt{\det(g_{ij})}\, g^{a\beta} dx^1 \wedge \overset{\beta}{\ldots} \wedge dx^m. \end{aligned} \tag{9}$$

Since (x^a) is a normal chart, $g_{a\beta|\gamma}(x) = 0 = g^{a\beta}{}_{|\gamma}(x)$ (1), and therefore

(1) We use the notation $|\gamma$ for $\partial/\partial x^\gamma$ and $|\gamma_1 \ldots \gamma_r$ for $\partial^r/\partial x^{\gamma_1} \ldots \partial x^{\gamma_r}$.

$(\partial/\partial x^\gamma)\sqrt{\det(g_{ij})}g^{\alpha\beta}|_x = 0$. Hence, by the Taylor Formula, for any $y \in U$,

(10) $$\sqrt{\det(g_{ij}(y))}\, g^{\alpha\beta}(y) = \delta^{\alpha\beta} + G^{\alpha\beta}_{\gamma\delta}(y) y^\gamma y^\delta,$$

where

(11) $$G^{\alpha\beta}_{\gamma\delta}(y) := \int_0^1 s\, \frac{\partial^2(\sqrt{\det(g_{ij})}\, g^{\alpha\beta})}{\partial x^\gamma\, \partial x^\delta}(y_{1-s})\, ds$$

and $y_{1-s} := \exp_x(1-s)\overline{x, y}$, i.e. $x^\alpha(y_{1-s}) = (1-s)y^\alpha$ (²). It is seen that each function $G^{\alpha\beta}_{\gamma\delta} \in C^\infty(U)$. By (8) we get

(12) $$dl(x, \cdot)^{2-m} = (2-m)l(x, \cdot)^{-m} \sum_{\alpha=1}^m x^\alpha dx^\alpha.$$

Now, using (9) and (10), we have

(13) $$*dl(x, \cdot)^{2-m}$$
$$= (2-m)l(x, \cdot)^{-m} \sum_{\beta=1}^m \Big(x^\beta + \sum_{\alpha=1}^m G^{\alpha\beta}_{\gamma\delta} x^\alpha x^\gamma x^\delta \Big) (-1)^{\beta-1} dx^1 \wedge \overset{\beta}{\ldots} \wedge dx^m.$$

Hence

(14) $$d*dl(x, \cdot)^{2-m}$$
$$= (2-m)\frac{\partial l(x, \cdot)^{-m}}{\partial x^\beta}\Big(x^\beta + \sum_{\alpha=1}^m G^{\beta\alpha}_{\nu\delta} x^\alpha x^\gamma x^\delta\Big) dx^1 \wedge \ldots \wedge dx^m +$$
$$+ (2-m)l(x, \cdot)^{-m} \sum_{\beta=1}^m \Big(1 + \sum_{\alpha=1}^m G^{\alpha\beta}_{\gamma\delta|\beta} x^\alpha x^\gamma x^\delta\Big) dx^1 \wedge \ldots \wedge dx^m$$
$$= m(m-2)l(x, \cdot)^{-m-2} \sum_{\beta=1}^m \Big((x^\beta)^2 + \sum_{\alpha=1}^m G^{\alpha\beta}_{\gamma\delta} x^\alpha x^\gamma x^\delta\Big) dx^1 \wedge \ldots \wedge dx^m +$$
$$+ m(2-m)l(x, \cdot)^{-m} dx^1 \wedge \quad \wedge dx^m +$$
$$+ (2-m)l(x, \cdot)^{-m} \sum_{\alpha=1}^m G^{\alpha\beta}_{\gamma\delta|\beta} x^\alpha x^\gamma x^\delta dx^1 \wedge \quad \wedge dx^m$$
$$= (2-m)l(x, \cdot)^{2-m} \sum_{\alpha,\beta=1}^m \Big(G^{\alpha\beta}_{\gamma\delta|\beta} - mG^{\alpha\beta}_{\gamma\delta} \frac{x^\beta}{l(x, \cdot)^2}\Big) \frac{x^\alpha x^\gamma x^\delta}{l(x, \cdot)^2}.$$

(²) If y is in a normal neighbourhood of x, then $\overline{x, y} := (\exp_x)^{-1}(y)$.

It follows from (13) that for each point $x \in M$ there exists such a constant $a_1(x) > 0$ that for every $y \in K(x, r_0) \setminus \{x\}$

$$\left\| \left(*dl(x, \cdot)^{2-m} \right)(y) \right\|_{\wedge^{m-1} T_y^*(M)} \leqslant a_1(x) l(x, y)^{1-m}. \tag{15}$$

Since $*\varphi = \varphi \overset{m}{\tau}$ for $\varphi \in C^\infty(M)$,

$$(\Delta\varphi)\overset{m}{\tau} = *\Delta\varphi = *(-*^{-1} d*d\varphi) = -d*d\varphi. \tag{16}$$

Next, we shall use the Stokes Formula and the fact that $\omega \wedge *\varphi = \varphi \wedge *\omega$ for $\omega, \varphi \in \bigwedge^k T^*(M)$. So, by (4) and (16) we have

$$\begin{aligned} \int_M p(x, \cdot)(\Delta\varphi)\overset{m}{\tau} &= a \lim_{\varepsilon \to 0} \int_{CK(x,\varepsilon)} l(x, \cdot)^{2-m} c \circ l(x, \cdot)(\Delta\varphi)\overset{m}{\tau} \\ &= -a \lim_{\varepsilon \to 0} \int_{K(x,\varepsilon)} l(x, \cdot)^{2-m} c \circ l(x, \cdot) d*d\varphi \\ &= -a \lim_{\varepsilon \to 0} \Big[\int_{\partial K(x,\varepsilon)} l^{2-m}(x, \cdot) c \circ l(x, \cdot) *d\varphi - \\ &\quad - \int_{CK(x,\varepsilon)} d\big(l(x, \cdot)^{2-m} c \circ l(x, \cdot)\big) \wedge *d\varphi \Big] \\ &= a \lim_{\varepsilon \to 0} \int_{CK(x,\varepsilon)} d\varphi \wedge *d\big(l(x, \cdot)^{2-m} c \circ l(x, \cdot)\big) \\ &= a \lim_{\varepsilon \to 0} \Big[\int_{\partial K(x,\varepsilon)} \varphi * d\big(l(x, \cdot)^{2-m} c \circ l(x, \cdot)\big) - \\ &\quad - \int_{CK(x,\varepsilon)} \varphi d*d\big(l(x, \cdot)^{2-m} c \circ l(x, \cdot)\big) \Big] \\ &= a \lim_{\varepsilon \to 0} \int_{\partial K(x,\varepsilon)} \varphi * dl(x, \cdot)^{2-m} + \\ &\quad + a \lim_{\varepsilon \to 0} \int_{CK(x,\varepsilon)} \varphi \Delta\big(l(x, \cdot)^{2-m} c \circ l(x, \cdot)\big)\overset{m}{\tau}. \end{aligned} \tag{17}$$

Let us consider separately the above two components.

$$\begin{aligned} &a \lim_{\varepsilon \to 0} \int_{\partial K(x,\varepsilon)} \varphi * dl(x, \cdot)^{2-m} \\ &\quad = a \lim_{\varepsilon \to 0} \int_{\partial K(x,\varepsilon)} \big(\varphi - \varphi(x)\big) * dl(x, \cdot)^{2-m} + a\varphi(x) \lim_{\varepsilon \to 0} \int_{\partial K(x,\varepsilon)} *dl(x, \cdot)^{2-m}. \end{aligned} \tag{18}$$

By the continuity of φ, for each $x \in M$ there exists a constant $a_2(x) > 0$ such that $|\varphi(y) - \varphi(x)| \leqslant a_2(x) l(x, y)$ for every $y \in K(x, r_0)$. Thus, we infer from (15) that for every $y \in K(x, r_0) \setminus \{x\}$

$$\left\| \left[\big(\varphi - \varphi(x)\big) * dl(x, \cdot)^{2-m} \right](y) \right\|_{\wedge^{m-1} T_y^*(M)} \leqslant a_1(x) a_2(x) l(x, y)^{2-m}.$$

Hence $\lim\limits_{\varepsilon\to 0}\int\limits_{\partial K(x,\varepsilon)}(\varphi-\varphi(x))*dl(x,\cdot)^{2-m}=0$. Now, using (13), we see that

$$\lim_{\varepsilon\to 0}\int\limits_{\partial K(x,\varepsilon)}*dl(x,\cdot)^{2-m}=(2-m)\lim_{\varepsilon\to 0}\varepsilon^{-m}\int\limits_{\partial K(x,\varepsilon)}\sum_{a=1}^{m}(-1)^{a-1}x^a dx^1\wedge\quad\wedge dx^m$$

$$=(2-m)\lim_{\varepsilon\to 0}\varepsilon^{-m}\int\limits_{K(x,\varepsilon)}d\sum_{a=1}^{m}(-1)^{a-1}x^a dx^1\wedge\quad\wedge dx^m$$

$$=m(2-m)\lim_{\varepsilon\to 0}\varepsilon^{-m}\int\limits_{K(x,\varepsilon)}dx^1\wedge\quad\wedge dx^m$$

$$=m(2-m)\lim_{\varepsilon\to 0}\varepsilon^{-m}\int\limits_{K(x,\varepsilon)}\overset{m}{\tau}$$

because (x^a) is an orthonormal chart. But $\lim\limits_{\varepsilon\to 0}\varepsilon^{-m}(\text{volume of } K(x,\varepsilon))$ $=$ volume of $K(0,1)$ in $\boldsymbol{R}^m=\dfrac{2\pi^{m/2}}{m\Gamma(m/2)}$. Thus, returning to (18), we obtain

$$(19)\qquad a\lim_{\varepsilon\to 0}\int\limits_{\partial K(x,\varepsilon)}\varphi*dl(x,\cdot)^{2-m}=a\varphi(x)(2-m)\frac{2\pi^{m/2}}{\Gamma(m/2)}=\varphi(x).$$

Now we pass to the second component of (17), which, via (16), equals $a\lim\limits_{\varepsilon\to 0}\int\limits_{CK(x,\varepsilon)}\varphi d*d\big(l(x,\cdot)^{2-m}c\circ l(x,\cdot)\big)$. In order to show that the limit exists, it is sufficient to notice that there exists an $a_3(x)>0$ such that for every $y\in K(x,\frac{1}{2}r_c)\setminus\{x\}$

$$\big\|d*d\big(l(x,\cdot)^{2-m}c\circ l(x,\cdot)\big)(y)\big\|_{\wedge^m T_y^*(M)}\leqslant a_3(x)l(x,y)^{2-m};$$

we made use of (1) and (14). This ensures the absolute integrability of $|e(x,\cdot)|$ on M. The smoothness of e outside the diagonal of $M\times M$ follows directly from the smoothness of the distance l on a normal neighbourhood of the diagonal. ■

Next we shall need the following

LEMMA 2. *If* $y\in K(x,r_0)$, *then*

$$(20)\qquad dl(x,\cdot)(y)=-l(x,y)^{-1}\overline{\overline{y,x}}.$$

Proof. Let (x^a) be an orthonormal chart at the point x; then by (8)

$$(21)\qquad \omega:=dl(x,\cdot)(y)=l(x,y)^{-1}\sum_{a=1}^{m}x^a(y)dx^a(y).$$

The coordinates of the vector $\overline{y,x}\in T_y(M)$, with respect to the chart

(x^a) are

$$\overline{y, x}^a := -x^a(y); \tag{22}$$

indeed, if $t \to p(t)$ is such that $x^a(p_t) = tx^a(y)$, then $\overline{y, x}^a = \frac{d}{dt} x^a(p(t))\Big|_{t=1}$ Obviously, the form ω vanishes on all vectors tangent at y to the sphere $\partial K(x, l(x, y))$. By Gauss' Lemma ([13], p. 201), $\overline{y, x}$ is orthogonal to those vectors and therefore ω must be proportional to $\overline{y, x}$, i.e.

$$\omega = a\overline{y, x}. \tag{23}$$

Let us assume the chart (x^a) chosen so that $x^1(y) = l(x, y)$, $x^a(y) = 0$ for $a \geqslant 2$. By (19) $\omega = dx^1(y)$ and by (21) and (20) we have

$$dx^1(y) = a\,\overline{\overline{y, x}} = -ax^1(y)\overline{\frac{\partial}{\partial x^1}(y)} = -al(x, y)g_{1\beta}(y)dx^\beta(y).$$

Thus $1 = -al(x, y)g_{11}(y) = -al(x, y)\left\|\frac{\partial}{\partial x^1}(y)\right\|^2$. However, $\left\|\frac{\partial}{\partial x^1}(y)\right\| = 1$ because if we take the parallel translation of the unit vector $\frac{\partial}{\partial x^1}(x)$ from x to y along the geodesics $t \to p(t)$, then we obtain the vector $\frac{\partial}{\partial x^1}(y)$, which must also be of the norm 1. Hence $a = -l(x, y)^{-1}$. ■

To state and prove further properties of the parametrix p and the error kernel e, we shall use some new notation.

Let E^i be a vector bundle over a manifold X_i, $i = 1, 2$; E^i_x is the fibre of E^i over a point $x \in X_i$. Then $E^1 \boxtimes E^2$ is the canonically defined vector bundle over $X_1 \times X_2$ whose fibre over a point $(x, y) \in X_1 \times X_2$ is $(E^1 \boxtimes E^2)_{(x,y)} := E^1_x \otimes E^2_y$. If $\omega_i \in \Gamma(E^i)$, $i = 1, 2$, then we define $\omega_1 \boxtimes \omega_2 \in \Gamma(E^1 \boxtimes E^2)$ as

$$(\omega_1 \boxtimes \omega_2)(x, y) := \omega_1(x) \otimes \omega_2(y). \tag{24}$$

We are interested in the case where $E^1 = \bigwedge^r T^*(M)$, $E^2 = \bigwedge^s T^*(M)$, $r, s \geqslant 0$.

Let (x^a) (resp. (y^a)) be a chart at a point $x_0 \in M$ (resp. $y_0 \in M$). Then each section $f \in \Gamma(E^1 \boxtimes E^2)$ has the form

$$f(x, y) = \sum_{I,J} f_{I,J}(x, y)\,dx^I(x) \otimes dy^J(y) \tag{25}$$

for (x, y) belonging to a respective neighbourhood of (x_0, y_0);

here

$$
\begin{aligned}
I &= (i_1, \dots, i_r), \quad 1 \leqslant i_1 < \dots < i_r \leqslant m,\\
J &= (j_1, \dots, j_s), \quad 1 \leqslant j_1 < \dots < j_s \leqslant m,\\
&dx^I := dx^{i_1} \wedge \dots \wedge dx^{i_r},\\
&dx^J := dx^{j_1} \wedge \dots \wedge dx^{j_s},
\end{aligned} \tag{26}
$$

and $f_{IJ} \in C^\infty(M \times M)$. If $r = 0$, then $dx^I := 1$ and analogously for the case $s = 0$. There is a natural definition of operators d_1 and d_2 which on a section f of the form (25) are defined as:

$$d_1 f := \sum_{I,J} \sum_{i=1}^{m} \frac{\partial f_{IJ}}{\partial x^i} (dx^i \wedge dx^I) \boxtimes dy^J, \tag{27}$$

$$d_2 f := \sum_{I,J} \sum_{i=1}^{m} \frac{\partial f_{IJ}}{\partial y^i} dx^I \boxtimes (dy^i \wedge dy^J). \tag{28}$$

Analogously, "partial" Hodge star operators may be introduced:

$$*_2 f := \sum_{I,J} f_{IJ} dx^I \boxtimes (* dy^J); \tag{29}$$

$*_1 f$ is defined in a similar (obvious) way.

In these terms the definition (6) takes the form

$$e = - *_2 d_2 *_2 d_2 p; \tag{30}$$

notice that $*$ and $*^{-1}$ coincide on m-forms.

If $f \in C^\infty(M \times M)$, then $d_1 f \in \Gamma\big(\bigwedge^1 T^*(M) \boxtimes \bigwedge^0 T^*(M)\big) = \Gamma\big(T^*(M) \boxtimes (M \times \boldsymbol{R})\big)$. The bundle $T^*(M) \boxtimes (M \times \boldsymbol{R}) \to M \times M$ is canonically isomorphic to the bundle $T^*(M) \times M \to M \times M$ whose fibre over a point (x, y) is $T_x^*(M)$. Thus $d_1 f$ may be considered as a section of the bundle $T^*(M) \times \times M \to M \times M$. So, if a point $x \in M$ is fixed, then $d_1 f(x, \cdot)$ is a smooth mapping $M \to T_x^*(M)$. Now we can prove

PROPOSITION 3. *For each $x \in M$ the smooth mapping*

$$d_1 e(x, \cdot)\colon\ M \setminus \{x\} \to T_x^*(M) \tag{31}$$

determines an integrable mapping $M \to T_x^*(M)$. Moreover, if (x^a) is a positively oriented orthonormal chart at x, then for every $(x, y) \in M \times M$ such *that* $y \in K(x, r_0)\setminus\{x\}$

$$d_1 p = a f_1 \sum_{\sigma=1}^{m} dx^\sigma \boxed{\times} x^\sigma, \tag{32}$$

$$\begin{aligned} d_1 e = -a\Big\{&[(m+2)f_2 + l^2 f_3] \sum_{\sigma=1}^{m} dx^\sigma \boxed{\times} (\det(g_{ij}))^{-1/2} x^\sigma + \\ &+ f_1 \sum_{\sigma=1}^{m} dx^\sigma \boxed{\times} (\det(g_{ij}))^{-1/2} (G^{\sigma\beta}_{\gamma\delta|\beta} x^\delta + 2G^{\sigma\beta}_{\gamma\beta}) x^\gamma + \\ &+ f_2 \sum_{\sigma=1}^{m} dx^\sigma \boxed{\times} (\det(g_{ij}))^{-1/2} \Big[\Big(G^{\alpha\beta}_{\gamma\delta|\beta} x^\sigma x^\alpha + 2G^{\alpha\sigma}_{\gamma\delta} x^\alpha + \\ &+ \sum_{\alpha=1}^{m} G^{\alpha\alpha}_{\gamma\delta} x^\sigma\Big) x^\gamma x^\delta + 2G^{\alpha\beta}_{\gamma\beta} x^\sigma x^\alpha x^\gamma\Big] + \\ &+ f_3 \sum_{\sigma,\alpha,\beta=1}^{m} dx^\sigma \boxed{\times} (\det(g_{ij}))^{-1/2} G^{\alpha\beta}_{\gamma\delta} x^\sigma x^\alpha x^\beta x^\gamma x^\delta\Big\}, \end{aligned} \tag{33}$$

where a *and* $G^{\alpha\beta}_{\gamma\delta}$ *were defined in* (5) *and* (11), *respectively, and*

$$\begin{aligned} &f_1 := (m-2) l^{-m} c\circ l - l^{1-m} c'\circ l, \\ &f_2 := -m(m-2) l^{-m-2} c\circ l + (2m-3) l^{-m-1} c'\circ l - l^{-m} c''\circ l, \\ &f_3 := m(m^2-4) l^{-m-4} c\circ l - 3(m^2-m-1) l^{-m-3} c'\circ l + \\ &\qquad + 3(m-1) l^{-m-2} c''\circ l - l^{-m-1} c'''\circ l. \end{aligned} \tag{34}$$

Proof. The function l is symmetric. Thus, by Lemma 2 we have at the point (x, y)

$$d_1(l^{2-m} c\circ l) = [(m-2) l^{-m} c\circ l - l^{1-m} c'\circ l] \sum_{\sigma=1}^{m} dx^\sigma \boxed{\times} x^\sigma, \tag{35}$$

i.e. (32) is valid.

We are going to use the following formula: if $0 \neq k \in N$, $f \in C^\infty(\boldsymbol{R})$, $\omega_i \in \Gamma(\bigwedge^{r_i} T^*(M))$, $i = 1, 2$, and the chart (x^α) is defined on a neighbourhood $\mathcal{O}$ of x, then for such (x, y) that $y \in \mathcal{O}\setminus\{x\}$ we have

$$\begin{aligned} &d_2(l^k f\circ l\, \omega_1 \boxed{\times} \omega_2) \\ &\quad = (l^{k-2} f\circ l + l^{k-1} f'\circ l) \sum_{\alpha=1}^{m} \omega_1 \boxed{\times} x^\alpha (dx^\alpha) \wedge \omega_2 + l^k f\circ l\, \omega_1 \boxed{\times} d\omega_2. \end{aligned} \tag{36}$$

This formula is a direct consequence of (8). Now, applying (36) to (35), we get

$$*_2 d_2 d_1 (l^{2-m} c\circ l) = f_2 \sum_{\sigma,\alpha=1}^{m} dx^\sigma \boxed{\times} x^\sigma x^\alpha * dx^\alpha + f_1 \sum_{\sigma=1}^{m} dx^\sigma \boxed{\times} * dx^\sigma.$$

Using (36) once more, we obtain

$$
\begin{aligned}
d_2 *_2 d_2 d_1 (l^{2-m} c \circ l) &= f_3 \sum_{\sigma,\alpha,\beta=1}^{m} dx^\sigma \boxtimes x^\sigma x^\alpha x^\beta (dx^\beta) \wedge * dx^\alpha + \\
&\quad + f_2 \sum_{\sigma,\alpha=1}^{m} dx^\sigma \boxtimes [(x^\alpha dx^\sigma + x^\sigma dx^\alpha) \wedge * dx^\alpha + x^\sigma x^\alpha d * dx^\alpha] + \\
&\quad + f_2 \sum_{\sigma,\alpha=1}^{m} dx^\sigma \boxtimes x^\alpha (dx^\alpha) \wedge * dx^\sigma + f_1 \sum_{\sigma=1}^{m} dx^\sigma \boxtimes d * dx^\sigma \\
&= f_1 \sum_{\sigma=1}^{m} dx^\sigma \boxtimes d * dx^\sigma + \\
&\quad + f_2 \sum_{\sigma,\alpha=1}^{m} dx^\sigma \boxtimes [(2x^\alpha dx^\sigma + x^\sigma dx^\alpha) \wedge * dx^\alpha + x^\sigma x^\alpha d * dx^\alpha] + \\
&\quad + f_3 \sum_{\sigma,\alpha,\beta=1}^{m} dx^\sigma \boxtimes x^\sigma x^\alpha x^\beta (dx^\beta) \wedge * dx^\alpha .
\end{aligned}
$$

By (9) and (10) we know that

$$
* dx^\alpha = (-1)^{\alpha-1} dx^1 \wedge \qquad \wedge dx^m + \sum_{\beta=1}^{m} (-1)^{\beta-1} G^{\alpha\beta}_{\gamma\delta} x^\gamma x^\delta dx^1 \wedge \overset{\beta}{\dots} \wedge dx^m ,
$$

and therefore, using the notation $dx^M := dx^1 \wedge \ldots \wedge dx^m$,

(37) $\quad d_2 *_2 d_2 d_1 (l^{2-m} c \circ l)$

$$
\begin{aligned}
&= f_1 \sum_{\sigma=1}^{m} dx^\sigma \boxtimes \frac{\partial}{\partial x^\beta} (G^{\alpha\beta}_{\gamma\delta} x^\gamma x^\delta) dx^M + f_2 \sum_{\sigma=1}^{m} dx^\sigma \boxtimes (m+2) x^\sigma dx^M + \\
&\quad + f_2 \sum_{\sigma,\alpha=1}^{m} dx^\sigma \boxtimes [2x^\alpha G^{\alpha\sigma}_{\gamma\delta} x^\gamma x^\delta + x^\sigma G^{\alpha\alpha}_{\gamma\delta} x^\gamma x^\delta + x^\sigma x^\alpha \frac{\partial}{\partial x^\beta} (G^{\alpha\beta}_{\gamma\delta} x^\gamma x^\delta)] dx^M + \\
&\quad + f_3 \sum_{\sigma,\alpha=1}^{m} dx^\sigma \boxtimes x^\sigma (x^\alpha)^2 dx^M + f_3 \sum_{\sigma,\alpha,\beta=1}^{m} dx^\sigma \boxtimes x^\sigma x^\alpha x^\beta G^{\alpha\beta}_{\gamma\delta} x^\gamma x^\delta dx^M \\
&= [(m+2) f_2 + l^2 f_3] \sum_{\sigma=1}^{m} dx^\sigma \boxtimes x^\sigma dx^M + \\
&\quad + f_1 \sum_{\sigma=1}^{m} dx^\sigma \boxtimes (G^{\sigma\beta}_{\gamma\delta|\beta} x^\gamma x^\delta + 2 G^{\sigma\beta}_{\gamma\beta} x^\gamma) dx^M +
\end{aligned}
$$

$$+f_2 \sum_{\sigma,\alpha=1}^{m} dx^\sigma \boxed{\times} [(2G^{\alpha\sigma}_{\gamma\delta}x^\alpha + G^{\alpha\alpha}_{\gamma\delta}x^\sigma + G^{\alpha\beta}_{\gamma\delta|\beta}x^\sigma x^\alpha)x^\gamma x^\delta + 2G^{\alpha\beta}_{\gamma\beta}x^\sigma x^\alpha x^\gamma] dx^M +$$

$$+f_3 \sum_{\sigma,\alpha,\beta=1}^{m} dx^\sigma \boxed{\times} G^{\alpha\beta}_{\gamma\delta}x^\sigma x^\alpha x^\beta x^\gamma x^\delta dx^M.$$

By (30) we have $d_1 e = -*_2 d_2 *_2 d_2 d_1(al^{2-m} \circ l) = -a *_2 d_2 *_2 d_2 d_1(l^{2-m} \circ l)$. So we must apply $-a*_2$ to (37). If we do this and make use of the fact that $*dx^M = (\det(g_{ij}))^{-1/2}$, then we obtain (33). The last three components on the right-hand side of (33) behave—when y is close to x—like $l(x, y)^{k-m}$, $k \geqslant 1$, and therefore they are integrable. Meanwhile, when $l(x, y) \leqslant \frac{1}{2}r_c$, $(m+2)f_2 + l^2 f_3 = 0$, which obviously ensures the integrability of the first component on the right-hand side of (33). ∎

Let us formulate the following trivial

COROLLARY 4. *For every* $u(x) \in T_x(M)$ *the function*

$$M \setminus \{x\} \ni y \to \langle u(x), d_1 e(x, y)\rangle \in \mathbf{R} \tag{38}$$

and its module belongs to $L^1(M, \mu)$.

2. An estimation of the differential of an eigenfunction of the laplacian

Let $\varphi \in C^\infty(M)$ be an eigenfunction of the laplacian, i.e.

$$\Delta\varphi = \lambda\varphi.$$

Then by Proposition 1

$$\varphi(x) = \int_M (\lambda p - e)(x, \cdot)\varphi \overset{m}{\tau}, \tag{39}$$

and therefore, for an $u(x) \in T_x(M)$,

$$\langle u(x), d\varphi(x)\rangle = \int_M \langle u(x), (\lambda d_1 p - d_1 e)(x, \cdot)\varphi\rangle \overset{m}{\tau}. \tag{40}$$

Our goal in this section is to estimate $|\langle u(x), d\varphi(x)\rangle|$. The formulae (32) and (33) of Proposition 3 suggest starting with estimations of the functions $G^{\alpha\beta}_{\gamma\delta}$ and their first derivatives. We shall follow this suggestion.

Let us begin with the following definition: if $x \in M$ and $\varkappa = (x^\alpha)$ is an orthonormal chart at x, then we write

$$\begin{aligned} (41) \qquad \gamma_i &:= \gamma_i(x, \varkappa) \\ &:= \sup\{|\Gamma_{\alpha\beta\gamma|\delta_1\ldots\delta_i}(z)|:\ z \in K(x, r_0);\ \alpha, \beta, \gamma, \delta_1, \ldots, \delta_i = 1, \ldots, m\}. \end{aligned}$$

LEMMA 5. *If* $\varkappa \doteq (x^a)$ *is an orthonormal chart at a point* $x \in M$ *and* $0 < r \leqslant r_0$ *is such that*

$$(42)\qquad r^2 \leqslant \frac{\sqrt[m]{\frac{3}{2}}-1}{m^2\gamma_1}, \quad \gamma_1 = \gamma_1(x, \varkappa),$$

then on the ball $K(x, r)$ *we have*

$$(43)\qquad |G^{\alpha\beta}_{\gamma\delta}| \leqslant 30\, m^4\gamma_1,$$

$$(44)\qquad |G^{\alpha\beta}_{\gamma\delta|\theta}| \leqslant (180\, m^3 r + \tfrac{3}{2}) m^4\gamma_1^2 + \tfrac{1}{3} m^2\gamma_2.$$

Proof. I. Since $g^{\alpha\sigma}g_{\sigma\varrho} = \delta^\alpha_\varrho = \text{const}$ on $K(x, r_0)$,

$$0 = \frac{\partial}{\partial x^\gamma}(g^{\alpha\sigma}g_{\sigma\varrho}) = g^{\alpha\sigma}|_\gamma g_{\sigma\varrho} + g^{\alpha\sigma}g_{\sigma\varrho}|_\gamma.$$

Multiplying on the right by $g^{\varrho\beta}$ and summing over ϱ's, we get

$$(45)\qquad g^{\alpha\beta}{}_{|\gamma} = -g^{\alpha\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\beta}.$$

Differentiating both sides and using (45), we obtain

$$(46)\qquad g^{\alpha\beta}{}_{|\gamma\delta} = g^{\alpha\xi}g_{\xi\eta|\delta}g^{\eta\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\beta} - g^{\alpha\sigma}g_{\sigma\varrho|\gamma\delta}g^{\varrho\beta} + g^{\alpha\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\xi}g_{\xi\eta|\delta}g^{\eta\beta}.$$

Further differentiation gives

$$(47)\qquad \begin{aligned} g^{\alpha\beta}{}_{|\gamma\delta\theta} = {} & -g^{\alpha\varphi}g_{\varphi\psi|\theta}g^{\psi\xi}g_{\xi\eta|\delta}g^{\eta\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\beta} + g^{\alpha\xi}g_{\xi\eta|\delta\theta}g^{\eta\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\beta} - \\ & -g^{\alpha\xi}g_{\xi\eta|\delta}g^{\eta\varphi}g_{\varphi\psi|\theta}g^{\psi\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\beta} + g^{\alpha\xi}g_{\xi\eta|\delta}g^{\eta\sigma}g_{\sigma\varrho|\gamma\theta}g^{\varrho\beta} - \\ & -g^{\alpha\xi}g_{\xi\eta|\delta}g^{\eta\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\varphi}g_{\varphi\psi|\theta}g^{\psi\beta} + g^{\alpha\xi}g_{\xi\eta|\theta}g^{\eta\sigma}g_{\sigma\varrho|\gamma\delta}g^{\varrho\beta} - \\ & -g^{\alpha\sigma}g_{\sigma\varrho|\gamma\delta\theta}g^{\varrho\beta} - g^{\alpha\sigma}g_{\sigma\varrho|\gamma\delta}g^{\varrho\xi}g_{\xi\eta|\theta}g^{\eta\beta} - \\ & -g^{\alpha\varphi}g_{\varphi\psi|\theta}g^{\psi\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\xi}g_{\xi\eta|\delta}g^{\eta\beta} + g^{\alpha\sigma}g_{\sigma\varrho|\gamma\theta}g^{\varrho\xi}g_{\xi\eta|\delta}g^{\eta\beta} - \\ & -g^{\alpha\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\varphi}g_{\varphi\psi|\theta}g^{\psi\xi}g_{\xi\eta|\delta}g^{\eta\beta} + g^{\alpha\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\xi}g_{\xi\eta|\delta\theta}g^{\eta\beta} - \\ & -g^{\alpha\sigma}g_{\sigma\varrho|\gamma}g^{\varrho\xi}g_{\xi\eta|\delta}g^{\eta\varphi}g_{\varphi\psi|\theta}g^{\psi\beta}. \end{aligned}$$

Let us introduce auxiliary notation,

$$(48)\qquad h_i(y) := \max\{|g_{\alpha\beta|\gamma_1\ldots\gamma_i}(y)|:\ \alpha, \beta, \gamma_1, \ldots, \gamma_i = 1, \ldots, m\}$$

and

$$(49)\qquad H(y) := \max\{|g^{\alpha\beta}(y)|:\ \alpha, \beta = 1, \ldots, m\}$$

for every $y \in K(x, r_0)$; later we shall estimate these functions by γ_i's. Thus (45), (46) and (47) tell us that on $K(x, r_0)$

$$(50)\qquad \begin{gathered} |g^{\alpha\beta}{}_{|\gamma}| \leqslant m^2H^2h_1, \qquad |g^{\alpha\beta}{}_{|\gamma\delta}| \leqslant m^2H^2(2m^2Hh_1^2 + h_2), \\ |g^{\alpha\beta}{}_{|\gamma\delta\theta}| \leqslant m^2H^2(6m^2H^2h_1^3 + 6m^2Hh_1h_2 + h_3). \end{gathered}$$

II. Now, we shall make similar estimates for the first, second and third order derivatives of the function

$$\det(g_{ij}) = \frac{1}{m!}\sum_{\pi\in\Pi(m)} \operatorname{sgn}\pi\, g_{1\pi(1)}\cdots g_{m\pi(m)}.$$

Since

$$\frac{\partial}{\partial x^\gamma}\det(g_{ij}) = \frac{1}{m!}\sum_{\pi\in\Pi(m)}\operatorname{sgn}\pi\sum_{i=1}^{m} g_{i\pi(i)|\gamma}\, g_{1\pi(1)} \overset{i}{\cdots} g_{m\pi(m)},$$

$$\frac{\partial^2}{\partial x^\delta\,\partial x^\gamma}\det(g_{ij}) = \frac{1}{m!}\sum_{\pi\in\Pi(m)}\operatorname{sgn}\pi\sum_{i=1}^{m}\Big[g_{i\pi(i)|\gamma\delta}\, g_{1\pi(1)} \overset{i}{\cdots} g_{m\pi(m)} + $$

$$+ g_{i\pi(i)|\gamma}\sum_{\substack{j=1\\ j\neq i}}^{m} g_{j\pi(j)|\delta}\, g_{1\pi(1)} \overset{i\,j}{\cdots} g_{m\pi(m)}\Big],$$

$$\frac{\partial^3}{\partial x^\theta\,\partial x^\delta\,\partial x^\gamma}\det(g_{ij}) = \frac{1}{m!}\sum_{\pi\in\Pi(m)}\operatorname{sgn}\pi\sum_{i=1}^{m}\Big\{g_{i\pi(i)|\gamma\delta\theta}\, g_{1\pi(1)} \overset{i}{\cdots} g_{m\pi(m)} +$$

$$+ g_{i\pi(i)|\gamma\delta}\sum_{\substack{j=1\\ j\neq i}}^{m} g_{j\pi(j)|\theta}\, g_{1\pi(1)} \overset{i\,j}{\cdots} g_{m\pi(m)} +$$

$$+ g_{i\pi(i)|\gamma\theta}\sum_{\substack{j=1\\ j\neq i}}^{m} g_{j\pi(j)|\delta}\, g_{1\pi(1)} \overset{i\,j}{\cdots} g_{m\pi(m)} +$$

$$+ g_{i\pi(i)|\gamma}\sum_{\substack{j=1\\ j\neq i}}^{m}\Big[g_{j\pi(j)|\delta\theta}\, g_{1\pi(1)} \overset{i\,j}{\cdots} g_{m\pi(m)} +$$

$$+ g_{j\pi(j)|\delta}\sum_{\substack{k=1\\ k\neq i,j}}^{m} g_{k\pi(k)|\theta}\, g_{1\pi(1)} \overset{i\,j\,k}{\cdots} g_{m\pi(m)}\Big]\Big\},$$

we have on $K(x, r_0)$

$$\left|\frac{\partial}{\partial x^\gamma}\det(g_{ij})\right| \leqslant m h_1 h_0^{m-1},$$

$$\left|\frac{\partial^2}{\partial x^\delta\,\partial x^\gamma}\det(g_{ij})\right| \leqslant m[h_2 h_0^{m-1} + (m-1)h_1^2 h_0^{m-2}], \tag{51}$$

$$\left|\frac{\partial^3}{\partial x^\theta\,\partial x^\delta\,\partial x^\gamma}\det(g_{ij})\right| \leqslant m[h_3 h_0^{m-1} + 3(m-1)h_2 h_1 h_0^{m-2} + (m-1)(m-2)h_1^3 h_0^{m-3}].$$

III. Let us estimate the functions $H, h_0, \ldots, h_3$ by expressions involving only γ_i's; the result applied to (50) and (51) will give us new estimations leading directly to (43) and (44).

By the Taylor Formula we have for $y \in K(x, r_0)$

$$\begin{aligned} g_{\alpha\beta}(y) &= \delta_{\alpha\beta} + \tfrac{1}{2} g_{\alpha\beta|\gamma\delta}(y') x^{\gamma}(y) x^{\delta}(y) \\ &= \delta_{\alpha\beta} + \tfrac{1}{2}\big(\Gamma_{\alpha\beta\gamma|\delta}(y') + \Gamma_{\beta\alpha\gamma|\delta}(y')\big) x^{\gamma}(y) x^{\delta}(y) \end{aligned}$$

for some point y' belonging to the geodesic interval $[x, y]$. Thus, on $K(x, r)$, $r \leqslant r_0$, we have

$$|g_{\alpha\beta} - \delta_{\alpha\beta}| \leqslant m^2 \gamma_1 r^2. \tag{52}$$

Analogously we find that on $K(x, r)$

$$|g_{\alpha\beta|\gamma}| \leqslant 2m\gamma_1 r. \tag{53}$$

Finally, just from the relation $g_{\gamma\beta|\gamma} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma}$, we get

$$|g_{\alpha\beta|\gamma\delta}| \leqslant 2\gamma_1, \qquad |g_{\alpha\beta|\gamma\delta\theta}| \leqslant 2\gamma_2. \tag{54}$$

Thus, making also use of (42), we have

$$h_0 \leqslant 1 + m^2\gamma_1 r^2 \leqslant \sqrt[m]{\tfrac{3}{2}}, \qquad h_1 \leqslant 2m\gamma_1 r, \qquad h_2 \leqslant 2\gamma_1, \qquad h_3 \leqslant 2\gamma_2. \tag{55}$$

So, we have yet to estimate the function H. For this purpose we shall need the following fact: if (a_{ij}) and (b_{ij}) are $n \times n$-matrices, then

$$\begin{aligned} &|\det(a_{ij}) - \det(b_{ij})| \\ &\qquad \leqslant \frac{1}{n!} \sum_{\pi \in \Pi(n)} \sum_{j=1}^{n} \Big[\sum_{i=0}^{j-1} |a_{i\pi(i)}|\Big] |a_{j\pi(j)} - b_{j\pi(j)}| \Big[\sum_{i=j+1}^{n} |b_{i\pi(i)}|\Big], \end{aligned} \tag{56}$$

where $a_{0\pi(0)} := 1 =: b_{n+1,\pi(n+1)}$ for every $\pi \in \Pi(n)$.

Now, from (56) and (52) we infer that on $K(x, r)$

$$|\det(g_{ij}) - 1| \leqslant \sum_{j=1}^{m} (1 + m^2\gamma_1 r^2)^{j-1} m^2 \gamma_1 r^2 = (1 + m^2\gamma_1 r^2)^m - 1.$$

Thus

$$\Big(r^2 \leqslant \frac{\sqrt[m]{\frac{3}{2}} - 1}{m^2 \gamma_1}\Big) \Rightarrow \big(\tfrac{1}{2} \leqslant \det(g_{ij}) \leqslant \tfrac{3}{2} \text{ on } K(x, r)\big). \tag{57}$$

Let us use the following (standard) convention: if (a_{ij}) is an $m \times m$-matrix, then A^{ij} denotes the $(m-1) \times (m-1)$-minor obtained by cancelling the ith row and the jth column, i.e. $(-1)^{i+j} \det A^{ij}$ is the algebraic complement of the element a_{ij}, in the matrix (a_{ij}). By (56), (52) and

(42) we get

$$|\delta^{\alpha\beta}-\det G^{\alpha\beta}| = |\det G^{\alpha\beta}-\det \Delta^{\alpha\beta}| \leqslant \sum_{j=1}^{m-1}(1+m^2\gamma_1 r^2)^{j-1}m^2\gamma_1 r^2$$
$$= (1+m^2\gamma_1 r^2)^{m-1}-1 \leqslant \tfrac{1}{2}.$$

Hence, using (57),

$$\left|g^{\alpha\beta}-\frac{\delta^{\alpha\beta}}{\det(g_{ij})}\right| = \left|\frac{(-1)^{\alpha+\beta}\det G^{\alpha\beta}}{\det(g_{ij})}-\frac{(-1)^{\alpha+\beta}\delta^{\alpha\beta}}{\det(g_{ij})}\right|$$
$$= \frac{1}{\det(g_{ij})}\,|\delta^{\alpha\beta}-\det G^{\alpha\beta}| \leqslant 1.$$

Therefore, $1-2\delta^{\alpha\beta}\leqslant g^{\alpha\beta}\leqslant 1+2\delta^{\alpha\beta}$ and

(58) $$|g^{\alpha\beta}|\leqslant 1+2\delta^{\alpha\beta},$$

i.e.

(59) $$H\leqslant 3 \quad \text{on} \quad K(x,r).$$

IV. Applying (55), (59) and —possibly— (42), we obtain

$$|g^{\alpha\beta}{}_{|\gamma}|\leqslant 18m^3\gamma_1 r,$$

(60) $$|g^{\alpha\beta}{}_{|\gamma\delta}|\leqslant 18m^2(12m^4\gamma_1 r^2+1)\gamma_1\leqslant 18m^2(2m^2+1)\gamma_1,$$

$$|g^{\alpha\beta}{}_{|\gamma\delta\theta}|\leqslant 18m^2[36m^3(6m^4\gamma_1 r^2+1)\gamma_1^2 r+\gamma_2]\leqslant 18m^2[36m^3(m^2+1)\gamma_1^2 r+\gamma_2]$$

on $K(x,r)$.

In (51) partial derivatives of the function $\det(g_{ij})$ have been estimated. Since we are interested in $G^{\alpha\beta}_{\gamma\delta}$, it is necessary to have similar estimations for $\sqrt{\det(g_{ij})}$. This can be achieved by combining (51), (55) and (57). Then we find that on $K(x,r)$

(61) $$\left|\frac{\partial}{\partial x^\gamma}\sqrt{\det(g_{ij})}\right|\leqslant\frac{1}{2}\det(g_{ij})^{-1/2}\left|\frac{\partial}{\partial x^\gamma}\det(g_{ij})\right|$$
$$\leqslant\frac{m}{\sqrt{2}}h_1 h_0^{m-1}\leqslant\frac{3}{\sqrt{2}}m^2\gamma_1 r$$

and similarly

(62) $$\left|\frac{\partial^2}{\partial x^\delta\partial x^\gamma}\sqrt{\det(g_{ij})}\right|\leqslant\frac{m^2}{\sqrt{2}}h_1^2h_0^{2(m-1)}+\frac{m}{\sqrt{2}}[h_2h_0^{m-1}+(m-1)h_1^2h_0^{m-2}]$$
$$\leqslant\frac{3}{\sqrt{2}}m\gamma_1[1+(5m-2)m^2\gamma_1 r^2]\leqslant\frac{5}{2\sqrt{2}}m^2\gamma_1,$$

where the last inequality follows from the fact that

(63) $$\big((42)\text{ and } m\geqslant 3\big)\Rightarrow(m^2\gamma_1 r^2\leqslant\tfrac{1}{6});$$

finally —using also (63) —we obtain

$$(64)\qquad \left|\frac{\partial^3}{\partial x^\theta \partial x^\delta \partial x^\gamma}\sqrt{\det(g_{ij})}\right| \leqslant \frac{3m^3}{\sqrt{2}} h_1^3 h_0^{3(m-1)} + \frac{2m}{\sqrt{2}} h_1 h_0^{m-1} +$$

$$+\frac{m^2}{\sqrt{2}} h_1 h_0^{m-1}[h_2 h_0^{m-1} + (m-1)h_1^2 h_0^{m-2}] +$$

$$+\frac{m}{\sqrt{2}}[h_3 h_0^{m-1} + 3(m-1)h_2 h_1 h_0^{m-2} + (m-1)(m-2)h_1^3 h_0^{m-3}]$$

$$\leqslant \frac{3}{\sqrt{2}} m^2\gamma_1[(31m^2-36m+8)m^2\gamma_1^2 r^3 + 3(3m-2)\gamma_1 r + 6(m-1)m^3\gamma_1^2 r^2 + r]$$

$$+\frac{3}{\sqrt{2}} m\gamma_2$$

$$\leqslant \frac{1}{2\sqrt{2}} m^2\gamma_1[(31m^2+18m-28)\gamma_1 r + 6(m-1)m\gamma_1 + 6r] + \frac{3}{\sqrt{2}} m\gamma_2.$$

V. Equipped with (60), (61), (62) and (64), we are ready to pass to the final step of the proof, i.e. to the estimation of functions $G_{\gamma\delta}^{\alpha\beta}$ and their first derivatives. Let us recall the definition (11):

$$G_{\gamma\delta}^{\alpha\beta}(y) = \int_0^1 s\frac{\partial^2\left(\sqrt{\det(g_{ij})}\; g^{\alpha\beta}\right)}{\partial x^\gamma \partial x^\delta}(y_{1-s})ds, \qquad y \in K(x, r_0),$$

and the coordinates of the point y_{1-s} are $x^\alpha(y_{1-s}) = (1-s)x^\alpha(y)$. It is easily seen that for functions φ, ψ on $K(x, r_0)$

$$\left|\frac{\partial^k(\varphi\psi)}{\partial x^{\gamma_1}\dots\partial x^{\gamma_k}}\right|$$

$$\leqslant \sum_{i=0}^{k}\binom{k}{i}\left[\max_{\delta_1,\dots,\delta_i}\left|\frac{\partial^i\varphi}{\partial x^{\delta_1}\dots\partial x^{\delta_i}}\right|\right]\left[\max_{\delta_1,\dots,\delta_{k-i}}\left|\frac{\partial^{k-i}\psi}{\partial x^{\delta_1}\dots\partial x^{\delta_{k-i}}}\right|\right].$$

Hence, using also (57), (59) and (63), we find that on $K(x, r)$

$$\left|\frac{\partial^2\left(\sqrt{\det(g_{ij})}\; g^{\alpha\beta}\right)}{\partial x^\gamma \partial x^\delta}\right| \leqslant 3m^2\gamma_1\left(18\sqrt{2}m^3\gamma_1 r^2 + 6\sqrt{6}m^2 + 9\sqrt{6} + \frac{5}{2\sqrt{2}}\right)$$

$$\leqslant 3m^2\gamma_1\left(6\sqrt{6}m^2 + 3\sqrt{2}m + 9\sqrt{6} + \frac{5}{2\sqrt{2}}\right),$$

$$\left|\frac{\partial^3\big(\sqrt{\det(g_{ij})}\; g^{\alpha\beta}\big)}{\partial x^\gamma\,\partial x^\delta\,\partial x^\theta}\right| \leqslant \frac{3}{\sqrt{2}}\, m^2\gamma_1^2 r\Big[108(2\sqrt{3}\,m+1)m^4+9(24\sqrt{3}+5)m^3+ + \tfrac{139}{2}\, m^2+9m-11\Big] \leqslant \frac{9}{\sqrt{2}}\, m\big[(m-1)m^2\gamma_1^2+(2\sqrt{3}\,m+1)\gamma_2\big].$$

So, on $K(x,r)$

$$|G^{\alpha\beta}_{\gamma\delta}| \leqslant \tfrac{3}{2}\, m^2\gamma_1\Big(6\sqrt{6}\,m^2+3\sqrt{2}\,m+9\sqrt{6}+\frac{5}{2\sqrt{2}}\Big). \tag{65}$$

Let us notice that

$$G^{\alpha\beta}_{\gamma\delta}|_\theta(y) = \int_0^1 s\,\frac{\partial^3\big(\sqrt{\det(g_{ij})}\; g^{\alpha\beta}\big)}{\partial x^\gamma\,\partial x^\delta\,\partial x^\sigma}\,(y_{1-s})\,\frac{\partial x^\sigma(\cdot_{1-s})}{\partial x^\theta}\,(y)\,ds$$
$$= \int_0^1 s(1-s)\,\frac{\partial^3\big(\sqrt{\det(g_{ij})}\; g^{\alpha\beta}\big)}{\partial x^\gamma\,\partial x^\delta\,\partial x^\theta}\,(y_{1-s})\,ds,$$

because $x^\sigma(\cdot_{1-s})(y) := x^\sigma(y_{1-s}) = (1-s)x^\sigma(y)$. Therefore on $K(x,r)$

$$|G^{\alpha\beta}_{\gamma\delta|\theta}| \leqslant \frac{1}{2\sqrt{2}}\, m^2\gamma_1 r\big[108(2\sqrt{3}\,m+1)m^4+9(24\sqrt{3}+5)m^3+ + \tfrac{139}{2}m^2+9m-11\big] + \frac{3}{2\sqrt{2}}\, m\big[(m-1)m^2\gamma_1^2+(2\sqrt{3}\,m+1)\gamma_2\big]. \tag{66}$$

If we apply to (65) and (66) very rough estimates consisting in: 1° replacing m^k by $m^{k+l}/3^l$ which is not less because $m \geqslant 3$, 2° replacing irrational square roots by greater rationals, then we arrive at much simpler inequalities (43) and (44). ■

The above lemma enables us to prove the following

PROPOSITION 6. *Let $\varkappa = (x^\sigma)$ be a (positively oriented) orthonormal chart at a point $x \in M$. If the constant r_c connected with the function c is such that*

$$r_c^2 \leqslant \frac{\sqrt[m]{\tfrac{3}{2}}-1}{m^2\gamma_1}, \qquad \gamma_1 = \gamma_1(x,\varkappa), \tag{67}$$

then for every $u(x) \in T_x(M)$

$$\int_M |\langle u(x), d_1 p(x,\cdot)\rangle|\,\overset{m}{\tau} \leqslant 5m2^m\|u(x)\|r_c \tag{68}$$

and

$$\int_M |\langle u(x), d_1 e(x,\cdot)\rangle|\,\overset{m}{\tau} \leqslant 530m^{10+1/2}2^m\Big[(\gamma_1^2+\gamma_2)r_c^2+\gamma_1 r_c+\frac{1}{r_c}\Big], \tag{69}$$

where, just as γ_1, also $\gamma_2 = \gamma_2(x,\varkappa)$.

Proof. For the moment we assume the positive orientation of $\varkappa$. But this is irrelevant if we look at our assertions (68) and (69).

Let $u^\sigma := \langle u(x), dx^\sigma\rangle$ be the coordinates of a vector $u(x) \in T_x(M)$ in the basis $\left\{\frac{\partial}{\partial x^\sigma}\right\}_1^m$ Then for $\psi \in C^\infty(M)$ and $y \in K(x, r_0)$

$$\langle u(x), (dx^\sigma \boxed{\times} \psi)(x, y)\rangle = u^\sigma \psi(y).$$

Moreover,

$$\sum_{\sigma=1}^{m} |u^\sigma| \leqslant \sqrt{m}\, \|u(x)\|. \tag{70}$$

It follows from Proposition 3 that for $y \in K(x, \frac{1}{2}r_c)$

$$|\langle u(x), d_1 p(x, y)\rangle| \leqslant |a|(m-2)\|u(x)\| l(x, y)^{1-m}, \tag{71}$$

and for $y \in K(x, r_c) \setminus K(x, \frac{1}{2}r_c)$

$$\begin{aligned} |\langle u(x), d_1 p(x, y)\rangle| &\leqslant |a|\sqrt{m}\, \|u(x)\| \left(\frac{2}{r_c}\right)^m \left(m + \frac{5}{2}\right) r_c \\ &\leqslant 2|a| m^{3/2} \|u(x)\| \left(\frac{2}{r_c}\right)^m r_c. \end{aligned} \tag{72}$$

By (67) and (57) we know that $\frac{1}{\sqrt{2}} \leqslant \sqrt{\det(g_{ij})} \leqslant \sqrt{\frac{3}{2}}$ on $K(x, r_c)$, and therefore for every $r \leqslant r_c$ and $k \geqslant 1$

$$0 \leqslant \int_{K(x,r)} l(x, \cdot)^{k-m} \overset{m}{\tau} \leqslant \frac{\sqrt{6}\, m \pi^{m/2}}{\Gamma(m/2)} \cdot \frac{r^k}{k}. \tag{73}$$

Now, using (71), (72), (73) and (5), we obtain

$$\begin{aligned} \int_M |\langle u(x), d_1 p(x, \cdot)\rangle| \overset{m}{\tau} &= \int_{K(x,r_c)} |\langle u(x), d_1 p(x, \cdot)\rangle| \overset{m}{\tau} \\ &\leqslant |a|(m-2)\|u(x)\| \int_{K(x,\frac{1}{2}r_c)} l(x, \cdot)^{1-m} \overset{m}{\tau} + \\ &\quad + 2|a| m^{3/2} \|u(x)\| \left(\frac{2}{r_c}\right)^m r_c \int_{K(x,r_c)\setminus K(x,\frac{1}{2}r_c)} \overset{m}{\tau} \\ &\leqslant \sqrt{6}\, m \left[\frac{\sqrt{m}\ (2^m - 1)}{m-2} + \frac{1}{4}\right] \|u(x)\| r_c \leqslant 5m\, 2^m \|u(x)\| r_0, \end{aligned}$$

which completes the proof of (68).

Let us pass to the proof of (69). By Lemma 5 we know that for $y \in K(x, r_0)$

$$|G^{\alpha\beta}_{\gamma\delta}(y)| \leqslant 30\, m^4 \gamma_1 =: A \tag{74}$$

and

(75) $$|G^{\alpha\beta}_{\gamma\delta|\theta}(y)| \leqslant 180\, m^7 \gamma_1^2 l(x, y) + m^2\left(\tfrac{3}{2} m^2 \gamma_1^2 + \tfrac{1}{3}\gamma_2\right) =: Bl(x, y) + C.$$

Next, we shall apply the inequality $m^k \leqslant 3^{-l} m^{k+l}$ resulting from $m \geqslant 3$, which was already used in the proof of Lemma 5. If $y \in K(x, \tfrac{1}{2} r_c)$, then $c \circ l(x, y) = 1$, $c^{(i)} \circ l(x, y) = 0$ and $|x^a(y)| \leqslant l(x, y)$. By Proposition 3 and (70) we get

$$\begin{aligned} &|\langle u(x), d_1 e(x, y)\rangle| \\ &\leqslant |a| \sqrt{2}\, m^{3/2}(m-2)\, \|u(x)\| \left[(m^4 + 2m^3 + 5m^2 + 2) A l(x, y)^{1-m} + \right. \\ &\quad \left. + m(m^2+1)\big(Bl(x, y) + C\big) l(x, y)^{2-m}\right] \\ &\leqslant \tfrac{\sqrt{2}}{9} |a|\, m^{3/2}(m-2)\, \|u(x)\| \left[\tfrac{62}{3} m^4 A l(x, y)^{1-m} + \right. \\ &\quad \left. + 10 m^3 \big(Bl(x, y) + C\big) l(x, y)^{2-m}\right]. \end{aligned}$$

Now, using (73), (5), (74), (75) and the analogue of (63)

(76) $$m^2 \gamma_1 r_c^2 \leqslant \tfrac{1}{6}$$

following from (67), we obtain

(77) $$\begin{aligned} &\int\limits_{K(x, \frac{1}{2} r_c)} |\langle u(x), d_1 e(x, \cdot)\rangle| \overset{m}{\tau} \\ &\leqslant \tfrac{\sqrt{3}}{9} m^{5+1/2} \|u(x)\| \left(\tfrac{5}{12} B r_c^2 + \tfrac{5}{4} C r_c + \tfrac{31}{3} m A\right) r_c \\ &= \tfrac{\sqrt{3}}{9} m^{5+1/2} \|u(x)\| \left[75 m^7 \gamma_1^2 r_c^2 + \tfrac{5}{4} m^2 \left(\tfrac{3}{2} m^2 \gamma_1^2 + \tfrac{1}{3}\gamma_2\right) r_c + 310\, m^5 \gamma_1\right] r_c \\ &\leqslant \tfrac{\sqrt{3}}{9} m^{10+1/2} \|u(x)\| \left[\tfrac{645}{2}\gamma_1 + \tfrac{5}{12}\left(\tfrac{3}{2}\gamma_1^2 + \tfrac{1}{27}\gamma_2\right) r_c\right] r_c \\ &\leqslant 36\, m^{10+1/2} [(\gamma_1^2 + \gamma_2) r_c + \gamma_1] r_c, \end{aligned}$$

where the last step consists in very rough simplifying estimations. If $y \in K(x, r_c) \setminus K(x, \tfrac{1}{2} r_c)$, then $0 \leqslant c \circ l(x, y) \leqslant 1$ and all $|c^{(i)} \circ l(x, y)|$, $i = 1, 2, 3$, are estimated when we know that on $\boldsymbol{R}$

(78) $$\begin{aligned} |c'| &\leqslant \frac{4a}{e} \cdot \frac{1}{r_c} < \frac{9}{r_c}, \\ |c''| &\leqslant \frac{2^7 a}{e^2} \cdot \frac{1}{r_c^2} < \frac{96}{r_c^2}, \\ |c'''| &\leqslant \frac{2^{15} a}{e^4} \cdot \frac{1}{r_c^3} < \frac{1725}{r_c^3}; \end{aligned}$$

these inequalities will be proved later. Hence, for the functions f_i,

$i = 1, 2, 3$, defined in (34) we obtain

$$|f_1(x, y)| \leqslant (m+\tfrac{5}{2})\left(\frac{2}{r_c}\right)^m \leqslant \tfrac{11}{6}\, m\left(\frac{2}{r_c}\right)^m,$$

$$|f_2(x, y)| \leqslant (m^2+7m-\tfrac{15}{2})\left(\frac{2}{r_c}\right)^{m+2} \leqslant \tfrac{10}{3}m^2\left(\frac{2}{r_c}\right)^{m+2},$$

$$|f_3(x, y)| \leqslant \tfrac{1}{2}(2m^3+27m^2+108m+\tfrac{45}{4})\left(\frac{2}{r_c}\right)^{m+4} \leqslant \tfrac{281}{24}m^3\left(\frac{2}{r_c}\right)^{m+4},$$

$$|(m+2)f_2(x, y)+l(x, y)^2 f_3(x, y)|$$
$$\leqslant \tfrac{1}{2}(27m^2+135m+280+\tfrac{5}{4})\left(\frac{2}{r_c}\right)^{m+2} \leqslant \tfrac{413}{8}\, m^2\left(\frac{2}{r_c}\right)^{m+2}$$

Now, by Proposition 3, the fact that $\frac{1}{2}r_c \leqslant x^a(y) \leqslant r_c$, (70), (74) and (75), we get

$$|\langle u(x), d_1 e(x, y)\rangle|$$
$$\leqslant \sqrt{2}\,|a|\, m^{5/2}\|u(x)\| \left\{ \tfrac{413}{4}\left(\frac{2}{r_c}\right)^{m+1} + \tfrac{1}{3}m\left(\frac{2}{r_c}\right)^{m-1} \times \right.$$
$$\left. \times \left[m(80m^2+11)(Br_c+C)r_c+2(562m^4+200m^2+11)A\right]\right\}$$
$$\leqslant \frac{|a|}{\sqrt{2}}\, m^{5/2}\left(\frac{2}{r_c}\right)^m \|u(x)\|\left\{\frac{413}{r_c}+\tfrac{1}{27}m^4\left[731(Br_c+C)r_c+\tfrac{94666}{81}mA\right]r_c\right\}.$$

Applying (73), (5), and the fact that $m^2\gamma_1 r_c^2 \leqslant \frac{1}{6}$ (cf. (63)), we have

$$(79) \qquad \int\limits_{K(x,r_c)\setminus K(x,\frac{1}{2}r_c)} |\langle u(x), d_1 e(x, \cdot)\rangle| \overset{m}{\tau}$$

$$\leqslant \tfrac{\sqrt{3}}{2}\cdot \frac{m^{5/2}(2^m-1)}{m-2}\, \|u(x)\|\left\{\frac{413}{r_c}+\frac{1}{27}m^4\left[731(Br_c+C)r_c+\frac{94666}{81}mA\right]r_c\right\}$$
$$\leqslant \tfrac{3\sqrt{3}}{2}\, m^{3/2}(2^m-1)\,\|u(x)\|\left\{\frac{413}{r_c}+m^9[(14\gamma_1^2+\tfrac{1}{2}\gamma_2)r_c+212\gamma_1]r_c\right\}$$
$$\leqslant 530\, m^{10+\frac{1}{2}}(2^m-1)\left[(\gamma_1^2+\gamma_2)r_c^2+\gamma_1 r_c+\frac{1}{r_c}\right],$$

where the last step consists in very rough, simplifying estimations. Now, adding (77) to (79), we arrive at (69).

To complete the proof we must show the inequalities (78). Let us use the fact that, for $k = 1, 2$,

$$\sup_{|t|\leqslant 1}\left|(t^2-1)^k \exp\left(\frac{1}{t^2-1}\right)\right| = \left(\frac{k}{e}\right)^k.$$

Thus for the function F defined in (2) we have

$$|F'| \leqslant \frac{a}{e}, \quad |F''| \leqslant \frac{2^3 a}{e^2}, \quad |F'''| \leqslant \frac{2^9 a}{e^4}. \tag{80}$$

By the symmetry of the function $\exp\left(\frac{1}{x^2-1}\right)$ we have

$$\int_{-1}^{1} \exp\frac{1}{x^2-1}\,dx \geqslant \exp\left(\frac{1}{x^2-1}\right)\Bigg|_{x=1/\sqrt{2}} \cdot 2\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{e^2}.$$

Hence $a \leqslant \frac{e^2}{\sqrt{2}}$, which together with (80) gives (78). ∎

Let us return to the situation described at the beginning of this section. The last proposition gives most of the arguments necessary to prove

THEOREM 7. *Let $x \in M$ and let $\varkappa$ be an orthonormal chart at x. We denote*

$$\varrho_0 := \varrho_0(x, \varkappa) := \min\left\{r_0, \left[\frac{\sqrt[m]{\frac{3}{2}}-1}{m^2\gamma_1(x,\varkappa)}\right]^{1/2}\right\}. \tag{81}$$

If $\varphi \in C^\infty(M)$ is an eigenfunction of the laplacian, i.e.

$$\Delta\varphi = \lambda\varphi,$$

then for every $u(x) \in T_x(M)$

$$|\langle u(x), d\varphi(x)\rangle| \leqslant 5m2^m[A_1 + \lambda\varrho_0]\,\|u(x)\| \sup_{y\in K(x,\varrho_0)} |\varphi(y)|, \tag{82}$$

where

$$A_1 := A_1(x, \varkappa) := 106\,m^{9+1/2}\left[(\gamma_1^2+\gamma_2)\varrho_0^2 + \gamma_1\varrho_0 + \frac{1}{\varrho_0}\right] \tag{83}$$

(here both γ_i's and ϱ_0 depend on x and $\varkappa$).

Proof. Let us take the already defined parametrix p (cf. (4)) but choose the constant r_c—in the definition (1) of the function c—equal to ϱ_0; this permits us to use Proposition 6. That Proposition and (40)—obtained with Proposition 1—gives (82). ∎

3. A normal chart on a neighbourhood of a geodesic

Let

$$[0, L] \ni s \to q(s) \in M \tag{84}$$

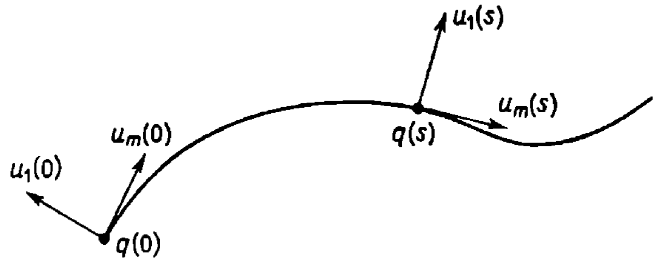
be a geodesic connecting distinct points $q(0)$ and $q(L)$; s is the natural parameter. We are going to investigate the mapping exp from a neighbourhood $\mathcal{O}$ of the zero section of the normal bundle of the geodesic q onto a neighbourhood of this geodesic. It is known that there exists $\mathcal{O}$ so small that $\exp\colon \mathcal{O} \to M$ is a diffeomorphism "into". The neighbourhood $\mathcal{O}$ has the shape of a tube. Our goal is to find, for each minimal geodesic connecting two distinct points of M, a positive constant estimating from below the maximal radius of such a tube.

Let us choose an ordered, orthonormal basis $\mathscr{B}_0 := \big(u_k(0)\big)_1^m$ in $T_{q(0)}(M)$, assuming that its mth vector is tangent to q, i.e.

$$u_m(0) = \frac{d}{ds} q(s)\Big|_{s=0} \tag{85}$$

For every $s \in [0, L]$, $\mathscr{B}_s := \big(u_k(s)\big)_1^m$ is the ordered, orthonormal basis in $T_{q(s)}(M)$, obtained from $\mathscr{B}_0$, by parallel translation along q. Hence

$$\nabla_{u_m(s)} u_k(s) \underset{s}{\equiv} 0. \tag{86}$$



Let

$$Q := K^{m-1}(0, r_0) \times [0, L] \subset \mathbf{R}^m, \tag{87}$$

where $K^{m-1}(0, r_0)$ is the ball in $\mathbf{R}^{m-1}$, of the radius r_0 and the centre at 0. We define a mapping $f\colon Q \to M$ as

$$f(s_1, \ldots, s_m) : = \exp_{q(s_m)}\Big(\sum_{k=1}^{m-1} s_k u_k(s_m)\Big). \tag{88}$$

The assumption $0 \leqslant \sum_{k=1}^{m-1} s_k^2 \leqslant r_0^2$ ensures that the vector $\sum_{k=1}^{m-1} s_k u_k(s_m)$ has the norm not greater than r_0 and therefore it belongs to the domain on which $\exp\colon T_{q(s_m)}(M) \to M$ is a diffeomorphism.

Let $\varkappa = (x^a)$ be a normal chart at a point $x \in M$. We denote

$$\Gamma_i := \Gamma_i(x, \varkappa) := \sup\{|\Gamma^a_{\beta\gamma|\delta_1\ldots\delta_i}(z)| : z \in K(x, r_0);\ a, \beta, \gamma, \delta_1, \ldots, \delta_i = 1, \ldots, m\}. \tag{89}$$

Moreover, we define two constants to be used in the next proposition:

$$A_2 := A_2(x, \varkappa) := \begin{cases} \dfrac{1}{m\sqrt{2(m-1)}}\left[\Gamma_0 + \left(4\Gamma_0 + \dfrac{3}{m}\Gamma_1\right)^{1/2}\right]^{-1}, & \text{if} \quad \Gamma_0 \neq 0, \\ \infty, & \text{if} \quad \Gamma_0 = 0, \end{cases}$$

(90)

$$A_3 := A_3(x, \varkappa) := \begin{cases} \dfrac{1}{m^3}\{2m\Gamma_0^2 + \Gamma_1[1 + 6(m-1)(m\Gamma_0^2 + \Gamma_1)A_2^3\}^{-1}, & \text{if} \quad \Gamma_0 \neq 0, \\ \infty, & \text{if} \quad \Gamma_0 = 0, \end{cases}$$

where $\Gamma_i = \Gamma_i(x, \varkappa)$ and $A_2 = A_2(x, \varkappa)$. If we want to avoid a rather complicated expression defining A_3, we may use

(90)′

$$A'_3 := A'_3(x, \varkappa) := \begin{cases} \dfrac{1}{m^3}\left[2m\Gamma_0^2 + \Gamma_1 + \sqrt{\dfrac{m}{3}\Gamma_1}\right]^{-1}, & \text{if} \quad \Gamma_0 \neq 0, \\ \infty, & \text{if} \quad \Gamma_0 = 0, \end{cases}$$

and then, by simple calculations, we obtain

$$A'_3 \leqslant A_3.$$

This allows us to replace A_3 by A'_3 in requirements of the type: »let ... be less than or equal to A_3«.

The coordinates of points with respect to $\varkappa$, as well as the coordinates of other geometrical objects, are denoted by adding a respective index, e.g. $p^a := x^a(p)$ for a point p from the domain of $\varkappa$.

PROPOSITION 8. *Let $a = (s_1, \ldots, s_m) \in Q$ and let $\varkappa = (x^a)$ be the orthonormal chart at the point $q(s_m)$, spanned by the (orthonormal) basis $\mathscr{B}_{s_m}$, i.e.*

$$\frac{\partial}{\partial x^k}\big(q(s_m)\big) = u_k(s_m), \qquad k = 1, \ldots, m.$$

We write

(91)
$$|a| := \Big(\sum_{k=1}^{m-1} s_k^2\Big)^{1/2};$$

obviously, $|a| \leqslant \|a\|$.

If

(92)
$$|a| \leqslant A_2\big(q(s_m), \varkappa\big)$$

and

(93)
$$|a|^4 < A_3\big(q(s_m), \varkappa\big),$$

then the mapping f'_a: $\boldsymbol{R}^m \to T_{f(a)}(M)$ *has the maximal rank (equal to* n*) and therefore* f *is a diffeomorphism "into" on a neighbourhood of the point* a.

Proof. Let $(e_a)_1^m$ be the canonical ordered basis in $\boldsymbol{R}^m$. The proof consists of few steps.

I. Let us calculate $f'_a e_i$ for $i \neq m$. Obviously $f'_a e_i = \dfrac{d}{d\tau} f(a+\tau e_i)\Big|_{\tau=0}$. But

$$f(a+\tau e_i) = \exp_{q(s_m)}\Big[\sum_{k=1}^{m-1} (s_k+\tau\delta_{ki})\, u_k(s_m)\Big]$$
$$= \exp_{q(s_m)}\Big[\sum_{k=1}^{m-1} (s_k+\tau\delta_{ki}) \frac{\partial}{\partial x^k}\big(q(s_m)\big)\Big].$$

Thus $f(a+\tau e_i)^k = s_k+\tau\delta_{ki}$, i.e. $(f'_a e_i)^k = \delta_{ki}$ and therefore

(94)
$$f'_a e_i = \frac{\partial}{\partial x^i}\big(f(a)\big), \qquad i = 1, \ldots, m-1.$$

Since we would like to find $f'_a e_m$, we are interested in vectors

(95)
$$v(\tau) := \sum_{k=1}^{m-1} s_k u_k(s_m+\tau) \in T_{q(s_m+\tau)}(M).$$

Obviously

(96)
$$f'_a e_m = \frac{d}{d\tau} \exp_{q(s_m)} v(\tau)\Big|_{\tau=0}$$

II. Let us calculate the coordinates $v(\tau)^a$ of the vector $v(\tau)$. Each vector $v(\tau)$ is the result of parallel translation of $v(0)$ along q. Thus

$$\nabla_{\partial_m(s_m+\tau)} v(\tau) \underset{\tau}{\equiv} 0 \qquad \text{(for small } \tau\text{'s)};$$

here we have used the simplifying notation $\partial_\alpha(s) := \frac{\partial}{\partial x^\alpha}\big(q(s)\big)$. This and the fact that $v(\tau) \perp \partial_m(s_m+\tau)$, i.e.

(97) $$v(\tau)^m \underset{\tau}{\equiv} 0 \quad \text{and} \quad v(\tau) = \sum_{k=1}^{m-1} v(\tau)^k \partial_k(s_m+\tau),$$

gives

$$\sum_{k=1}^{m-1} \left[\dot v(\tau)^k \partial_k(s_m+\tau) + v(\tau)^k \Gamma^\alpha_{mk}\big(q(s_m+\tau)\big)\partial_\alpha(s_m+\tau)\right] \underset{\tau}{\equiv} 0,$$

where $\dot v(\tau)^k := \frac{d}{d\tau} v(\tau)^k$ and the summation over α runs through $1, \ldots, m$. If we put $\Gamma^\alpha_{mk}(\tau) := \Gamma^\alpha_{mk}\big(q(s_m+\tau)\big)$, then we see that for $k = 1, \ldots, m-1$

$$\dot v(\tau)^k + \sum_{i=1}^{m-1} v(\tau)^i \Gamma^k_{mi}(\tau) \underset{\tau}{\equiv} 0.$$

Next, the summation convention will be understood as follows: summations over Latin indices runs through $1, \ldots, m-1$, while summations over Greek indices runs through $1, \ldots, m$. So we have

$$\dot v(\tau)^k \underset{\tau}{\equiv} -\Gamma^k_{mi}(\tau) v(\tau)^i \quad \text{for} \quad k = 1, \ldots, m-1.$$

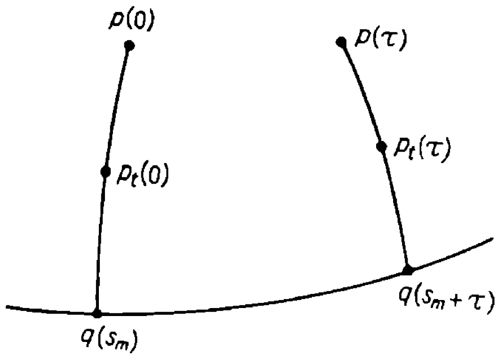
In particular, when $\tau = 0$, it follows from the normality of $\varkappa$ ($\Gamma^k_{mi}(0) = 0$) and from (97) that

(98) $$\dot v(0)^\alpha = 0, \quad \alpha = 1, \ldots, m.$$

III. Let us calculate $f'_a e_m$. Because of (96) we introduce

(99) $$p(\tau) := \exp_{q(s_m+\tau)} v(\tau).$$

By the definition of the basis $\mathscr{B}_{s_m+\tau}$ we see that $d\big(q(s_m+\tau), p(\tau)\big) = \|v(\tau)\| = \big(\sum_{k=1}^{m-1} s_k^2\big)^{1/2} = |a|$. For every τ let a curve $[0, |a|] \ni t \to p_t(\tau) \in M$ be the unique minimal geodesic connecting $q(s_m+\tau)$ and $p(\tau)$, i.e.



(100) $$p_t(\tau) = \exp_{q(s_m+\tau)} t v(\tau).$$

Then obviously $p_0(\tau) = q(s_m+\tau)$ and $p_{|a|}(\tau) = p(\tau)$. If we denote $\dot{p}_t(\tau)^a := \frac{d}{dt} p_t(\tau)^a$, then for every (small) τ we have the equation of geodesic

$$\ddot{p}_t(\tau)^a + \Gamma^a_{\beta\gamma}\big(p_t(\tau)\big)\dot{p}_t(\tau)^\beta \dot{p}_t(\tau)^\gamma \underset{t}{\equiv} 0.$$

Let us notice that $p_0(\tau)^a = q(s_m+\tau)^a = \tau\delta^{am}$ and $\dot{p}_0(\tau)^a = v(\tau)^a$. Thus, by the Taylor Formula, we have

$$\begin{aligned} p(\tau)^a &= p_{|a|}(\tau)^a \\ &= p_0(\tau)^a + |a|\dot{p}_0(\tau)^a + |a|^2\int_0^1 (1-\theta)\ddot{p}_\theta(\tau)^a d\theta \\ &= \tau\delta^{am} + |a|v(\tau)^a - |a|^2\int_0^1 (1-\theta)\Gamma^a_{\beta\gamma}\big(p_\theta(\tau)\big)\dot{p}_\theta(\tau)^\beta\dot{p}_\theta(\tau)^\gamma d\theta. \end{aligned}$$

It follows from (100) that

$$\dot{p}_\theta(0)^a = \frac{d}{dt}\big(\exp_{q(s_m)} tv(0)\big)^a\Big|_{t=\theta} = \frac{d}{dt}\, tv(0)^a\Big|_{t=\theta} = v(0)^a.$$

Using this, (96) and (99), we find that

$$(f'_a e_m)^a = \frac{d}{d\tau}\, p(\tau)^a\Big|_{\tau=0} = \delta^{am} + E^a(a),$$

where for $a = 1, \ldots, m$

(101)

$$\begin{aligned} E^a(a) := |a|^2\int_0^1 (1-\theta)\Big\{&\Gamma^a_{\beta\gamma|\delta}\big(p_{\theta|a|}(0)\big)\Big[\frac{d}{d\tau}\, p_{\theta|a|}(\tau)^\delta\Big]_{\tau=0} v(0)^\beta v(0)^\gamma - \\ &-\Gamma^a_{\beta\gamma}\big(p_{\theta|a|}(0)\big)\big[\Gamma^\beta_{\xi\eta}\big(p_{\theta|a|}(0)\big)\, v(0)^\gamma + \Gamma^\gamma_{\xi\eta}\big(p_{\theta|a|}(0)\big)\, v(0)^\beta\big]\, v(0)^\xi v(0)^\eta\Big\} d\theta. \end{aligned}$$

In other words,

(102) $$f'_a e_m = \frac{\partial}{\partial x^m}\big(f(a)\big) + E^a(a)\frac{\partial}{\partial x^a}\big(f(a)\big).$$

This and (94) give us f'_a.

IV. Now we are going to prove the following

LEMMA 9. *If we assume the hypothesis of Proposition* 8, *then the numbers* $E^a(a)$, $a = 1, \ldots, m$, *defined in* (101) *satisfy*

(103) $$|E^a(a)| \leqslant \frac{|a|^4}{A_3} < 1,$$

where $A_3 = A_3(q(s_m), \varkappa)$, *cf.* (90), *and* $\varkappa$ *is the orthonormal chart spanned by the basis* $\mathscr{B}_{s_m}$.

Proof. From (95) and (97) we have

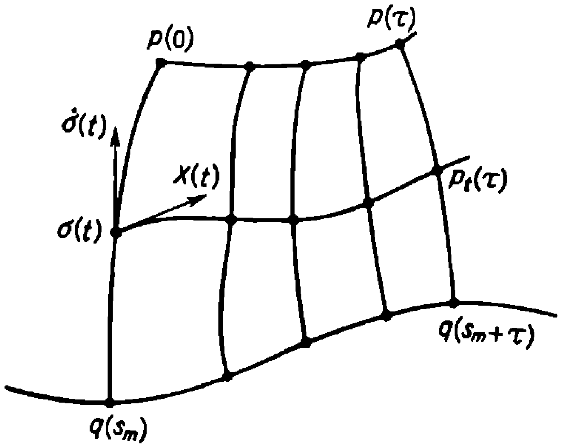
$$|v(0)^a| = |s_a| \leqslant |a|, \quad a = 1, \ldots, m. \tag{104}$$

Looking at (101) we see that the crucial point of the proof is to estimate $\left.\frac{d}{d\tau} p_t(\tau)^a\right|_{\tau=0}$ for $0 \leqslant t \leqslant |a|$. Let us notice that we have a 1-parameter family of geodesics

$$t \to p_t(\tau), \quad \text{where} \quad \tau > 0. \tag{105}$$

We distinguish that for $\tau = 0$, denoting it by

$$t \to \sigma(t) := p_t(0). \tag{106}$$



Let $\dot{\sigma}(t) = \frac{d}{dt}\sigma(t)$. We are interested in the vector field X along the geodesic σ, defined by the family (105) as follows:

$$X(t) := \left.\frac{d}{d\tau} p_t(\tau)\right|_{\tau=0} \in T_{\sigma(t)}(M). \tag{107}$$

Thus X is a Jacobi field along σ; cf. [4], p. 174, Th. 1. So, it satisfies the equation

$$\nabla_{\dot{\sigma}} \nabla_{\dot{\sigma}} X = -R_{\dot{\sigma}X}\dot{\sigma}, \tag{108}$$

where R is the curvature tensor.

Using the coordinates, we have $X(t) = X(t)^a \frac{\partial}{\partial x^a}(\sigma(t))$. By dots we shall denote the derivatives of $X(t)^a$ with respect to t.

Since

$$\dot{\sigma}(t)^a = \frac{d}{dt} p_t(0)^a = v(0)^a = \begin{cases} s_a, & a < m, \\ 0, & a = m, \end{cases} \tag{109}$$

equation (108) takes the form (cf. [4], p. 117)

$$\ddot{X}(t)^a + 2\dot{X}^\beta \sum_{k=1}^{m-1} s_k \Gamma^a_{k\beta}(\sigma(t)) + \\ + X(t)^\beta \sum_{k,l=1}^{m-1} s_k s_l \big[\Gamma^a_{k\beta|l}(\sigma(t)) + \Gamma^\gamma_{k\beta}(\sigma(t))\Gamma^a_{l\gamma}(\sigma(t)) - R^a_{lk\beta}(\sigma(t))\big] \underset{t}{\equiv} 0 \tag{110}$$

for $a = 1, \ldots, m$.

It can be seen that

$$X(0) = \frac{d}{d\tau} p_0(\tau)\Big|_{\tau=0} = \frac{d}{d\tau} q(s_m + \tau)\Big|_{\tau=0} = \frac{\partial}{\partial x^m}\big(q(s_m)\big);$$

moreover, by (98),

$$\dot{X}(0) = \frac{d}{dt}\left(\frac{d}{d\tau} p_t(\tau)\Big|_{\tau=0}\right)\Big|_{t=0} \\ = \frac{d}{d\tau}\left(\frac{d}{dt} p_t(\tau)\Big|_{t=0}\right)\Big|_{\tau=0} = \frac{d}{d\tau} v(\tau)\Big|_{\tau=0} = \dot{v}(0) = 0.$$

Thus the equations (110) are accompanied by the initial conditions:

$$\begin{aligned} X(0)^a &= \delta^{am}, \\ \dot{X}(0)^a &= 0, \end{aligned} \qquad a = 1, \ldots, m. \tag{111}$$

Let us denote

$$G^a_\beta(t) := -2\sum_{k=1}^{m-1} s_k \Gamma^a_{k\beta}(\sigma(t)),$$

$$H^a_\beta(t) := -\sum_{k,l=1}^{m-1} s_k s_l \big[\Gamma^a_{k\beta|l}(\sigma(t)) + \Gamma^a_{l\gamma}(\sigma(t))\Gamma^\gamma_{k\beta}(\sigma(t)) - R^a_{lk\beta}(\sigma(t))\big],$$

where $a, \beta = 1, \ldots, m$. Then (110) has the form

$$\ddot{X}(t)^a \underset{t}{\equiv} G^a_\beta(t)\dot{X}(t)^\beta + H^a_\beta(t)X(t)^\beta.$$

Substituting this and (111) into the identity

$$\dot{X}(t)^a = \dot{X}(0)^a + \int_0^t \ddot{X}(\tau)^a d\tau,$$

and also taking into account that

$$X(\tau)^\beta = \delta^{\beta m} + \int_0^\tau \dot{X}(\tau')^\beta d\tau',$$

we obtain

(112) $$\dot{X}(t)^\alpha = \int_0^t G^\alpha_\beta(\tau)\dot{X}^\beta(\tau)d\tau + \int_0^t H^\alpha_\beta(\tau)[\delta^{\beta m} + \int_0^\tau \dot{X}(\tau')^\beta d\tau']d\tau.$$

This enables us to estimate $|\dot{X}(t)^\alpha|$. Indeed, if

$$W := \sup\{|\dot{X}(t)^\alpha| : 0 \leqslant t \leqslant |a|,\ \alpha = 1, \ldots, m\},$$

$$G := \sup\{|G^\alpha_\beta(t)| : 0 \leqslant t \leqslant |a|,\ \alpha, \beta = 1, \ldots, m\},$$

$$H := \sup\{|H^\alpha_\beta(t)| : 0 \leqslant t \leqslant |a|,\ \alpha, \beta = 1, \ldots, m\},$$

then it follows from (112) that $W \leqslant mGW + H(1+mW)$, i.e.

(113) $$W[1 - m(G+H)] \leqslant H.$$

Since $R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\delta\beta|\gamma} - \Gamma^\alpha_{\gamma\beta|\delta} + \Gamma^\alpha_{\gamma\varrho}\Gamma^\varrho_{\delta\beta} - \Gamma^\alpha_{\delta\varrho}\Gamma^\varrho_{\gamma\beta}$, it is easily seen that

$$G \leqslant \sqrt{2(m-1)}\,\Gamma_0 |a|,$$

$$H \leqslant 3(m-1)(m\Gamma_0^2 + \Gamma_1)|a|^2.$$

Let us show that by (92)

(114) $$G + H \leqslant \frac{1}{2m}.$$

Obviously,

(115) $$G + H - \frac{1}{2m} \leqslant 3(m-1)(m\Gamma_0^2 + \Gamma_1)|a|^2 + \sqrt{2(m-1)}\,\Gamma_0|a| - \frac{1}{2m}.$$

If $\Gamma_0 = 0$, then $\Gamma_1 = 0$ and (114) holds. Thus we assume that $\Gamma_0 > 0$. The right-hand side of (115) is a polynomial of second degree of the variable $|a|$. This polynomial has one root negative and the other positive. Hence, if $|a|$ is not less than the positive root equal to

$$\frac{\sqrt{4\Gamma_0^2 + \dfrac{3}{m}\Gamma_1} - \Gamma_0}{3\sqrt{2(m-1)}(m\Gamma_0^2 + \Gamma_1)} = \frac{1}{m\sqrt{2(m-1)}\left[\left(4\Gamma_0 + \dfrac{3}{m}\Gamma_1\right)^{1/2} + \Gamma_0\right]},$$

then (114) holds. But fulfilment of this condition is ensured in (92). Now, by (114) and (113)

$$W \leqslant 2H \leqslant 6(m-1)(m^2\Gamma_0^2 + \Gamma_1)|a|^2.$$

Therefore, for every $0 \leqslant t \leqslant |a|$ we have

$$\left|\frac{d}{d\tau} p_t(\tau)^\alpha\Big|_{\tau=0}\right| = |X(t)^\alpha|$$
$$\leqslant |X(0)^\alpha| + \left|\int_0^t \dot{X}(t')^\alpha dt'\right| \leqslant 1 + tW$$
$$\leqslant 1 + 6(m-1)(m\Gamma_0^2 - \Gamma_1)|a|^3.$$

This and (104) applied to (101) give us

$$|E^\alpha(a)| \leqslant m^3\{\Gamma_1[1+6(m-1)(m\Gamma_0^2+\Gamma_1)|a|^3] + 2m\Gamma_0^2\}|a|^4. \tag{116}$$

Now (92) and (93) complete the proof of Lemma 9.

V. It follows from (94) and (102) that the matrix of the linear mapping $f'_a\colon \boldsymbol{R}^m \to T_{f(a)}(M)$, in the ordered basis $(e_\alpha)_1^m$ and $\left(\frac{\partial}{\partial x^\alpha}(f(a))\right)_1^m$, equals

$$\left[\begin{array}{cccc|c} 1 & & & & E^1(a) \\ & \ddots & 0 & & \vdots \\ & 0 & \ddots & & \vdots \\ & & & 1 & E^{m-1}(a) \\ \hline 0 & \cdots & & 0 & 1+E^m(a) \end{array}\right].$$

Thus (103) ensures that rank $f'_a = m$. ∎

For the sake of convenience we make a digression. If $\varkappa$ is a normal chart at a point $x \in M$, then we define two constants,

$$(117)\qquad \begin{aligned} A_4 &:= A_4(x, \varkappa) := \frac{1}{4\mu m^2 \gamma_1}, \\ A_5 &:= A_5(x, \varkappa) := \frac{A_3}{8\mu(m+2)}, \end{aligned}$$

where

$$\mu := 2^m m! \tag{118}$$

and $\gamma_0 = \gamma_0(x, \varkappa)$, $A_3 = A_3(x, \varkappa)$; cf. (41) and (90). The proof of Proposition 8 gives us all the arguments neccessary to prove the following theorem, which we shall use in the next section.

THEOREM 10. *We assume the hypothesis of Proposition* 8. *For the mapping* $f'_a\colon \mathbf{R}^m \to T_{f(a)}(M)$ *we have*:

1° *If*

$$|a|^2 \leqslant A_4 = A_4\big(q(s_m), \varkappa\big), \tag{119}$$

then

$$\|f'_a\| \leqslant 2m; \tag{120}$$

2° *If, in addition to* (119),

$$|a|^4 \leqslant A_5 = A_5\big(q(s_m), \varkappa\big), \tag{121}$$

then

$$(\mathrm{vol} f'_a)^2 := \det\big((f'_a e_\alpha | f'_a e_\beta)\big) \geqslant \tfrac{1}{2}, \tag{122}$$

where $\{e_\alpha\}_1^m$ *is the canonical orthonormal basis in* $\mathbf{R}^m$.

Proof. From (94) and (102) we know that

$$\|f'_a e_a\|_1^2 = \begin{cases} g_{aa}\big(f(a)\big), & a < m, \\ g_{mm}\big(f(a)\big) + g_{\alpha\beta}\big(f(a)\big)E^\alpha(a)E^\beta(a), & a = m. \end{cases} \tag{123}$$

As was done in the proof of Lemma 5 (cf. (52)), we find—using the Taylor Formula—that

$$\big|g_{\alpha\beta}\big(f(a)\big) - \delta_{\alpha\beta}\big| \leqslant m^2\gamma_1\big(q(s_m), \varkappa\big)\, l\big(q(s_m), f(a)\big)^2 = m^2\gamma_1|a|^2. \tag{124}$$

Applying (124), (103) and (119) to (123), we get

$$\begin{aligned} 0 &\leqslant g_{\alpha\beta}\big(f(a)\big)E^\alpha(a)E^\beta(a) \\ &\leqslant [m(1+m^2\gamma_1|a|^2) + (m^2-m)m^2\gamma_1|a|^2]\frac{|a|^8}{A_3^2} \\ &\leqslant m(1+m^3\gamma_1|a|^2)A_3^2|a|^8 \leqslant 2m\frac{|a|^8}{A_3^2} \leqslant 2m. \end{aligned} \tag{125}$$

If we take a unit vector $w = \lambda^\alpha e_\alpha$, i.e. $\sum_{\alpha=1}^m (\lambda^\alpha)^2 = 1$, then

$$\begin{aligned} \|f'_a w\|^2 &\leqslant m(\lambda^a)^2\|f'_a e_a\|^2 \\ &\leqslant m\Big[\sum_{a=1}^m g_{aa}\big(f(a)\big) + g_{\alpha\beta}\big(f(a)\big)E^\alpha(a)E^\beta(a)\Big] \\ &\leqslant m[m(1+m^2\gamma_1|a|^2) + 2m] \leqslant 4m^2, \end{aligned}$$

i.e. (120) holds.

It follows from (103), (124) and (119) that

$$\left|g_{\alpha\gamma}(f(a))E^{\gamma}(a)\right| \leqslant [1+m^2\gamma_1|a|^2+(m-1)m^2\gamma_1|a|^2]\frac{|a|^4}{A_3} \leqslant 2\frac{|a|^4}{A_3}.$$

If we use this and (125) together with (94) and (102), we see that

$$(126)\qquad \left|(f'_a e_\alpha|f'_a e_\beta)-g_{\alpha\beta}(f(a))\right| \leqslant 4\frac{|a|^4}{A_3}+2m\frac{|a|^8}{A_3^2}$$

$$= 2\left(2+m\frac{|a|^4}{A_3}\right)\frac{|a|^4}{A_3} \leqslant 2(2+m)\frac{|a|^4}{A_3}.$$

Hence, by (124), (119) and (121),

$$|(f'_a e_\alpha|f'_a e_\beta)-\delta_{\alpha\beta}| \leqslant 2(2+m)\frac{|a|^4}{A_3}+m^2\gamma_1|a|^2$$

$$= \frac{1}{4\mu m}+\frac{1}{4\mu m} = \frac{1}{2\mu m} =: D,$$

and obviously

$$|(f'_a e_\alpha|f'_a e_\beta) \leqslant 1+D < 2.$$

Finally, using formula (56), we have

$$\left|\det\left((f'_a e_\alpha|f'_a e_\beta)\right)-1\right| \leqslant m!\sum_{j=0}^{m-1} 2^j D 1^{m-j-1} = m!\,D(2^m-1) < \tfrac{1}{2}. \ \blacksquare$$

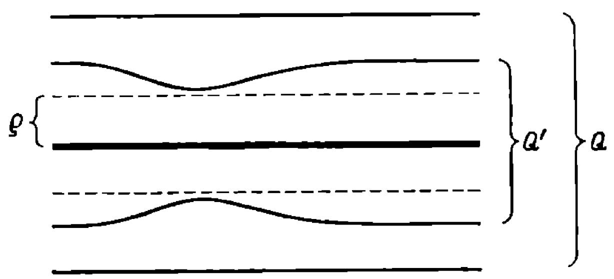
Let us now return to the main subject of the present section, i.e. to an investigation for what subset of Q the mapping f defined in (88) is a normal chart on a neighbourhood of the geodesic (84). We define the set Q' consisting of all $a = (s_1, \ldots, s_m) \in Q$ which satisfy (92) and (93). Then Proposition 8 tells us that—after restriction to Q'—the mapping

$$(127)\qquad f\colon Q' \to M$$

is a local diffeomorphism "into". The question arises whether it is also a (global) diffeomorphism "into"—in other words, whether it is injective. In order to answer this question we prove the following

PROPOSITION 11. *Let $\varrho > 0$ be such that*

$$(128)\qquad K^{m-1}(0, \varrho)\times[0, L] \subset Q'.$$



If $a^i = (s_1^i, \ldots, s_m^i) \in Q'$, $i = 1, 2$, *are such that* $s_m^1 \neq s_m^2$,

(129) $$|a^i| \leqslant \tfrac{1}{4}\varrho, \quad i = 1, 2$$

and

(130) $$f(a^1) = f(a^2),$$

then

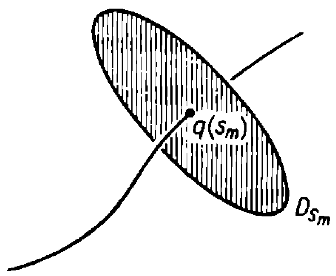
(131) $$|a^1| + |a^2| < |s_m^1 - s_m^2|.$$

Proof (a.a.). If (131) does not occur, then, by (129), we have

(132) $$|s_m^1 - s_m^2| \leqslant \tfrac{1}{2}\varrho.$$

Let us define:

1° $D_{s_m} := f(K^{m-1}(0, \varrho) \times \{s_m\})$,



where $s_m \in [0, L]$; D_{s_m} is an $(m-1)$-dimensional geodesic disc of radius ϱ and centre $q(s_m)$, which is normal to the geodesic q;

2° if $p \in D_{s_m}$, i.e. there exists a unique point $b \in K^{m-1}(0, \varrho)$ such. that $f(b, s_m) = p$, then

$$[q(s_m), p] := \{f(tb, s_m) \in D_{s_m}\colon\ 0 \leqslant t \leqslant 1\};$$

$[q(s_m), p]$ is the minimal geodesic interval in D_{s_m}, connecting the centre of D_{s_m} with p.

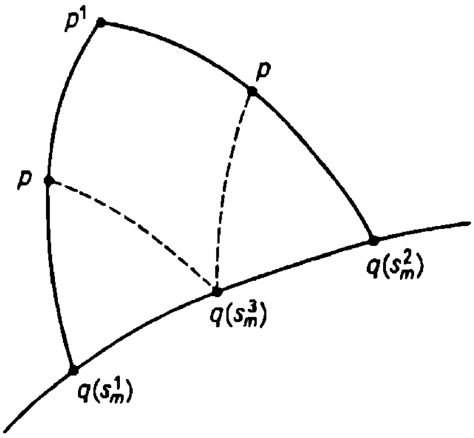
We are going to show that for every (small) $\varepsilon > 0$ there exist non-equal $s'_m, s''_m \in [s_m^1, s_m^2]$ such that $|s'_m - s''_m| < \varepsilon$ and $D_{s'_m} \cap D_{s''_m} \neq \emptyset$. This obviously contradicts the fact that $f\colon Q' \to M$ is a local diffeomorphism "into".

Let $p^1 := f(a_1) = f(a_2)$ and

$$s_m^3 := \tfrac{1}{2}(s_m^1 + s_m^2).$$

If we take a point $p \in [q(s_m^i), p^1]$, $i = 1$ or 2, then

$$\begin{aligned} l(q(s_m^3), p) &\leqslant l(q(s_m^3), q(s_m^i)) + l(q(s_m^i), p) \\ &\leqslant \tfrac{1}{2}|s_m^1 - s_m^2| + |a^i| \leqslant \tfrac{1}{2}\varrho < \varrho; \end{aligned}$$



cf. (132) and (129). Thus, there exists a point

$$p^2 \in D_{s_m^3} \cap \left([q(s_m^1), p^1] \cup [q(s_m^2), p^2]\right)$$

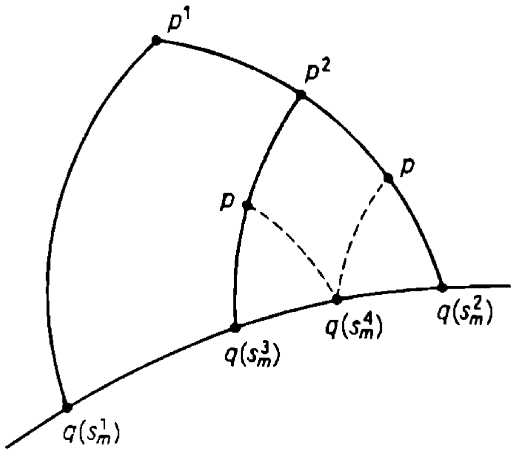
and

$$l\left(q(s_m^3), p^2\right) < \tfrac{1}{2}\varrho .$$

Without loss of generality we may assume that $p^2 \in [q(s_m^2), p^1]$. Then we define

$$s_m^4 := \tfrac{1}{2}(s_m^3 + s_m^2),$$

and, as before, we find that, for every point $p \in [q(s_m^i), p^2]$, $i = 2$ or 3,



$$\begin{aligned} l\left(q(s_m^4), p\right) &\leqslant l\left(q(s_m^4), q(s_m^i)\right) + l\left(q(s_m^i), p\right) \\ &\leqslant \tfrac{1}{2}|s_m^3 - s_m^2| + \begin{cases} l\left(q(s_m^2), p^1\right), & \text{if} \quad i = 2 \\ l\left(q(s_m^3), p^2\right), & \text{if} \quad i = 3 \end{cases} \\ &\leqslant \tfrac{1}{8}\varrho + \left.\begin{cases} \tfrac{1}{4}\varrho, & \text{if} \quad i = 2 \\ \tfrac{1}{2}\varrho, & \text{if} \quad i = 3 \end{cases}\right\} \leqslant \tfrac{5}{8}\varrho < \varrho . \end{aligned}$$

Thus, there exists a point

$$p^3 \in D_{\substack{4\\ sm}} \cap \big([q(s_m^2), p^2] \cup [q(s_m^3), p^2]\big)$$

and

$$l\big(q(s_m^4), p^3\big) \leqslant \tfrac{5}{8}\varrho.$$

Choosing $i = 2$ or $i = 3$ such that $p^3 \in [q(s_m^i), p^2]$, we define

$$s_m^5 := \tfrac{1}{2}(s_m^4 + s_m^i)$$

and we proceed as before.

This construction leads to a sequence s_m^n, $n = 1, 2, \ldots$, such that $|s_m^n - s_m^{n+1}| = 2^{-n}\varrho$, and moreover for every n we have $D_{s_m^n} \cap (D_{s_m^{n-1}} \cup D_{s_m^{n-2}}) \neq \emptyset$, which completes the proof. ■

For our geodesic q introduced in (84) we define the constant

$$\text{(132)} \qquad r_q := \tfrac{1}{4} \inf_{s_m \in [0, L]} \big\{A_2\big(q(s_m), \varkappa\big), A_3\big(q(s_m), \varkappa\big)^{1/4}, r_0\big\},$$

where A_2 and A_3 are defined as in (90) and $\varkappa$, occurring next to $q(s_m)$, is the orthonormal chart at $q(s_m)$, spanned by the basis $\mathscr{B}_{s_m}$—cf. the beginning of the present section. (Thus r_q depends not only on q but also on the choice of $\mathscr{B}_0$.)

As a simple corollary to Propositions 8 and 11 we have

THEOREM 12. *If* $[0, L] \ni s \to q(s) \in M$ *is a minimal geodesic (connecting* $q(0)$ *and* $q(L)$*), and* $\mathscr{B}_0 = \big(u_k(0)\big)_1^m$ *is an orthonormal ordered basis in* $T_{q(0)}(M)$ *such that* (85) *holds, then the mapping*

$$\text{(133)} \qquad f\colon K^{m-1}(0, r_q) \times [0, L] \to M$$

defined by formula (88) *is a diffeomorphism "into", i.e. it is an (ortho-)normal chart on a neighbourhood of the geodesic* q; *here* r_q *is a constant given by* (132).

Proof. $K^{m-1}(0, r_q) \times [0, L] \subset Q'$, where Q' was defined before Proposition 11. Thus, by Proposition 8 the mapping (133) is a local diffeomorphism "into". Let us take $a^i = (s_1^i, \ldots, s_m^i) \in K^{m-1}(0, r_q) \times [0, L]$, $i = 1, 2$, such that $s_m^1 \neq s_m^2$. Then a^1 and a^2 satisfy the hypothesis of Proposition 11. If we assume $f(a^1) = f(a^2)$, then by the minimality of the geodesic q we have

$$\begin{aligned}|s_m^1 - s_m^2| &= l\big(q(s_m^1), q(s_m^2)\big)\\ &\leqslant l\big(q(s_m^1), f(a^1)\big) + l\big(f(a^2), q(s_m^2)\big)\\ &= |a^1| + |a^2| \quad \text{(contradiction)}.\end{aligned}$$

Thus (133) is an injection. ■

4. Minorization of the first positive eigenvalue of the laplacian

From now on we assume that our Riemannian manifold M is compact. One of the main foundations of this section are Theorems 7 and 12. They contain estimations of a local character, i.e. the estimating constants depend on the point (Theorem 7) or on the geodesic (Theorem 2) for which the estimation is made. To get rid of this dependence we define

$$\bar{\gamma}_i := \sup \gamma_i(x, \varkappa),$$

$$\underline{\gamma}_i := \inf \gamma_i(x, \varkappa); \quad \text{cf. (41)},$$

(134) $$\underline{\varrho}_0 := \inf \varrho_0(x, \varkappa) = \min\left\{r_0, \left(\frac{\sqrt[m]{\tfrac{3}{2}}-1}{m^2\bar{\gamma}_1}\right)^{1/2}\right\}; \quad \text{cf. (81)},$$

$$\bar{A}_i := \sup A_i(x, \varkappa),$$

$$\underline{A}_i := \inf A_i(x, \varkappa); \quad \text{cf. (83), (90), (117)};$$

here the bounds are taken over all $x \in M$ and all orthonormal charts $\varkappa$ at these points.

Now, just as a corollary to Theorem 7, we have

THEOREM 3. *If $\varphi \in C^\infty(M)$ is an eigenfunction of the laplacian, i.e.*

$$\Delta\varphi = \lambda\varphi,$$

then for every $x \in M$ and $u(x) \in T_x(M)$ we have

(135) $$|\langle u(x), d\varphi(x)\rangle| \leqslant B_1(\lambda)\|u(x)\| \sup_{y \in K(x, r_0)} |\varphi(y)|,$$

where

(136) $$B_1(\lambda) := 5m2^m(B_1 + \lambda r_0),$$
$$B_1 := 106\, m^{9+1/2}\left[(\bar{\gamma}_1^2 + \bar{\gamma}_2)r_0^2 + \bar{\gamma}_1 r_0 + \frac{1}{r_0} + m\left(\sqrt[m]{\tfrac{3}{2}}-1\right)^{-1/2}\sqrt{\bar{\gamma}_1}\right].$$

Proof. Since $(\min\{a, b\})^{-1} \leqslant \frac{1}{a} + \frac{1}{b}$ for $a, b > 0$, we see that $\frac{1}{\underline{\varrho}_0} \leqslant \frac{1}{r_0} + m\left(\sqrt[m]{\tfrac{3}{2}}-1\right)^{-1/2}\sqrt{\bar{\gamma}_1}$. Using this and the fact that $\underline{\varrho}_0 \leqslant r_0$, we obtain $\bar{A}_1 \leqslant B_1$. ∎

Let us write

(137) $$\varepsilon(\lambda) := \tfrac{1}{4}\min\left\{\frac{1}{B_1(\lambda)}, r_0, \underline{A}_2, \sqrt[4]{\underline{A}_3}, 4\sqrt{\underline{A}_4}, 4\sqrt[4]{\underline{A}_5}\right\}$$

(cf. (136) and (134)), and let

$$V := \int_M \overset{m}{\tau} \tag{138}$$

be the volume of the manifold M.

PROPOSITION 14. *If $\varphi \in C^\infty(M)$, $\Delta\varphi = \lambda\varphi$, $\lambda > 0$, $\|\varphi\| = 1$, then there exist two points $x, y \in M$ such that for every $x' \in K(x, \varepsilon(\lambda))$ and $y' \in K(y, \varepsilon(\lambda))$*

$$|\varphi(x') - \varphi(y')| > \frac{1}{2\sqrt{V}}. \tag{139}$$

Proof. Without loss of generality we may assume that $\sup|\varphi| = \sup\varphi =: \nu$. Since $1 = \|\varphi\| = \int \varphi^2 \overset{m}{\tau} \leqslant \nu^2 V$,

$$\nu \geqslant \frac{1}{\sqrt{V}}. \tag{140}$$

Let x be a point for which $\varphi(x) = \nu$. Since $\lambda > 0$, $(\varphi|1) = 0$. Thus, there exists such a $y \in M$ that $\varphi(y) < 0$, and therefore

$$\varphi(x) - \varphi(y) > \nu. \tag{141}$$

Let us take $x' \in K(x, \varepsilon(\lambda))$ and let C be the (unique) geodesic $[0, 1] \ni t \to x_t \in M$ connecting x' and x, i.e. $x_0 = x'$, $x_1 = x$. Then $\|\dot{x}_t\| \underset{t}{\equiv} l(x, x')$. Moreover, by Theorem 13 and (137) we have

$$\begin{aligned} |\varphi(x) - \varphi(x')| &= \Big|\int_C d\varphi\Big| \\ &= \Big|\int_0^1 \langle \dot{x}_t, d\varphi(x_t)\rangle dt\Big| \leqslant B_1(\lambda) l(x, x')\nu \\ &\leqslant B_1(\lambda)\varepsilon(\lambda)\nu \leqslant \nu/4. \end{aligned}$$

Analogously, $|\varphi(y) - \varphi(y')| \leqslant \nu/4$ for every $y' \in K(y, \varepsilon(\lambda))$. Combining this with (140) and (141), we obtain (139). ∎

Let $x, y \in M$ be as in the above proposition. We define the natural number

$$n := E\left(\frac{2\sqrt{m-1}\; l(x, y)}{\varepsilon(\lambda)}\right). \tag{142}$$

We choose a minimal geodesic $[x, y]$ connecting the points x and y. Let $y_1 \in [x, y]$ be the (unique) point for which

$$l(x, y_1) = n\frac{\varepsilon(\lambda)}{2\sqrt{m-1}}. \tag{143}$$

Then

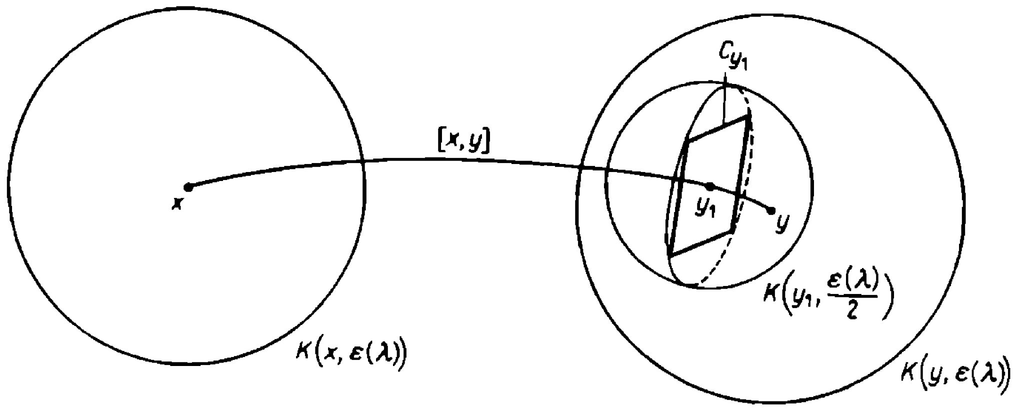
$$(144)\quad l(y_1, y) = l(x, y) - l(x, y_1) = \frac{\varepsilon(\lambda)}{2\sqrt{m-1}}\left[\frac{2\sqrt{m-1}\, l(x, y)}{\varepsilon(\lambda)} - E\left(\frac{2\sqrt{m-1}\, l(x, y)}{\varepsilon(\lambda)}\right)\right] < \frac{\varepsilon(\lambda)}{2\sqrt{m-1}}.$$

Let us fix an ordered orthonormal basis $\mathscr{B} = (u_i)_1^m$ in $T_{y_1}(M)$, such that u_m is tangent to $[x, y]$. Then the set

$$(145)\qquad C_{y_1} := \exp_{y_1} \frac{\varepsilon(\lambda)}{2\sqrt{m-1}} \left\{\sum_{i=1}^{m-1} \lambda_i u_i \colon \ |\lambda_i| \leqslant \tfrac{1}{2}\right\}$$

is an $(m-1)$-dimensional (geodesic) cube in M, with the centre at y_1 and the edge $\varepsilon(\lambda)/2\sqrt{m-1}$, orthogonal to the geodesic $[x, y]$. It is easily seen that

$$(146)\qquad C_{y_1} \subset K(y, \varepsilon(\lambda)).$$



Indeed, if $y' \in C_{y_1}$, then $l(y', y_1) \leqslant \sqrt{m-1}\,\frac{\varepsilon(\lambda)}{2\sqrt{m-1}} = \frac{\varepsilon(\lambda)}{2}$. Thus, by (144), $l(y', y) < \varepsilon(\lambda)$.

We are going to apply the results of Section 3 to that part of $[x, y]$ which lays between y_1 and x. According to the notation of that section we have

$$(147)\qquad L := l(y_1, x) = n\frac{\varepsilon(\lambda)}{2\sqrt{m-1}},$$

and $[0, L] \ni s \to q(s) \in M$ is the natural parametrization of the part of

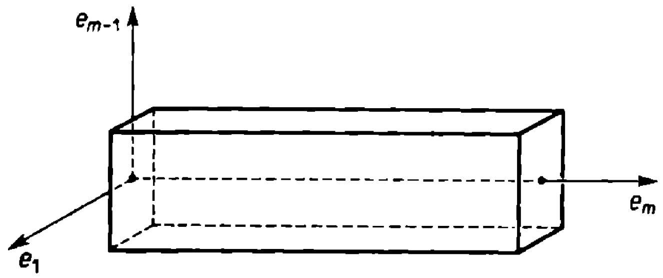
$[x, y]$, connecting y_1 and x, i.e. $q(0) = y_1$, $q(L) = x$. As the orthonormal ordered basis $\mathscr{B}_0$ in $T_{q(0)}(M)$ we take the basis $\mathscr{B}$.

Let

(148) $$\mathring{P} := \{(\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m \colon 0 \leqslant \lambda_m \leqslant n;\ |\lambda_i| \leqslant \tfrac{1}{2},\ i = 1, \ldots, m-1\}$$

and

(149) $$P := \frac{\varepsilon(\lambda)}{2\sqrt{m-1}} \mathring{P}.$$



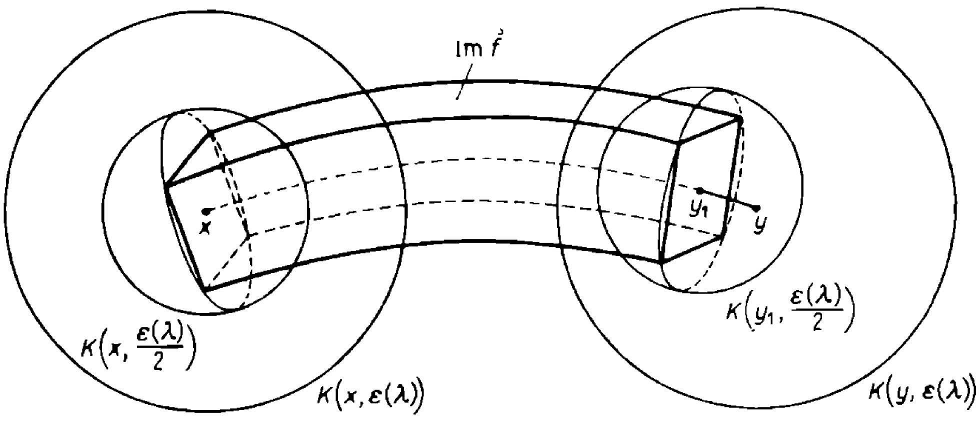
Then formula (81) defines a mapping

(150) $$f\colon P \to M,$$

i.e. $P \subset Q$; cf. (87) and (137). The definition (137) of $\varepsilon(\lambda)$ ensures that we may use Theorem 12. It tells us that the mapping (150) is a diffeomorphism "into". Thus also the mapping

(151) $$\mathring{P} \ni a \to \mathring{f}(a) := f\left(\frac{\varepsilon(\lambda)}{2\sqrt{m-1}} a\right) \in M$$

is a diffeomorphism "into".



Defining $\varepsilon(\lambda)$, we were careful to make it possible to use **Theorem 10**, which gives estimations of $\|f_a'\|$ and $\mathrm{vol} f_a'$ for each $a \in P$. In this way we obtain

(152) $$\|\overset{\circ}{f}{}'_a\| \leqslant \frac{m\varepsilon(\lambda)}{\sqrt{m-1}}$$

and

(153) $$(\mathrm{vol}\overset{\circ}{f}{}'_a)^2 := \det\left((\overset{\circ}{f}{}'_a e_i | \overset{\circ}{f}{}'_a e_j)\right) \geqslant \tfrac{1}{2}\left[\frac{\varepsilon(\lambda)}{2\sqrt{m-1}}\right]^{2m},$$

where $\{e_i\}_1^m$ is the canonical orthonormal basis in $\boldsymbol{R}^m$.

Now, let us introduce several notions that appear in paper [10] (Sections 1 and 2), the results of which we are going to use later.

Obviously, $\overset{\circ}{P}$ is the sum of n copies of unit cubes in $\boldsymbol{R}^m$. Let us triangulate the unit cube I^m in the canonical way. The triangulation is defined inductively with respect to m; if $a_0, \ldots, a_{m-1}$ are the vertices of an $(m-1)$-simplex lying in the base I^{m-1} of the cube I^m, then for every $k = 0, \ldots$ $\ldots, m-1$ the vertices

$$a_k, \ldots, a_{m-1}, a_0+e_m, \ldots, a_k+e_m$$

span an m-simplex, in I^m, over our $(m-1)$-simplex. By translations we get a triangulation of the whole $\overset{\circ}{P}$. This triangulation will be called *of 0-th generation*. Dividing all the edges of our cubes into $2, 4, 8, \ldots, 2^k, \ldots$ parts, we get partitions of these cubes into $2^m, 2^{2m}, 2^{3m}, \ldots, 2^{km}, \ldots$ smaller cubes (respectively). The triangulation of the cubes we started with determines—by dilations and translations—a triangulation of all smaller cubes. The triangulations of $\overset{\circ}{P}$ obtained in this way will be called *of 1-th, 2-nd, 3-rd, ..., k-th, ... generation* (respectively).

For every $k \geqslant 0$ the diffeomorphism $\overset{\circ}{f}$ transfers the triangulation of kth generation, of $\overset{\circ}{P}$, giving us a triangulation of $\mathrm{Im}\overset{\circ}{f} \subset M$, which will also be called of kth generation. Let

(154) $$S_k := \Delta(a_0, \ldots, a_m) := \left\{a_0 + \sum_{i=1}^{m} \alpha_i(a_i - a_0):\ 0 \leqslant \alpha_i \leqslant 1,\ \sum_{i=1}^{m} \alpha_i \leqslant 1\right\}$$

denote an m-simplex, of kth generation, in $\overset{\circ}{P}$, with vertices $a_0, \ldots, a_m$; then

(155) $$s_k := \overset{\circ}{f}(S_k)$$

is an m-simplex of kth generation, in $\mathrm{Im}\overset{\circ}{f}$. We denote

(156) $$x_i := \overset{\circ}{f}(a_i), \quad i = 0, \ldots, m.$$

Then we define

$$r(s_k) := \max_{i,j} l(x_i, x_j),$$

(157)

$$r_k := \max_{s_k \subset \mathring{P}} r(s_k)$$

and

(158) $$g(s_k, x_0) := \det\big((\overline{x_0, x_i} \,|\, \overline{x_0, x_j})\big);$$

cf. footnote (2), p. 9. If we write

$$C_1 := \inf_{a \in \mathring{P}} (\mathrm{vol}\,\mathring{f}'_a)^2,$$

$$C_2 := \sup_{a \in \mathring{P}} \|\mathring{f}'_a\|,$$

then Theorem 2 of [10], which we are going to use, states that for sufficiently great k's—i.e. greater than a certain k_0—

$$\frac{g(s_k, x_0)}{r_k^{2m}} \geqslant \frac{C_1}{2(4mC_2)^{2m}},$$

which, together with (152) and (153), shows us that for sufficiently great k's

(159) $$\frac{g(s_k, x_0)}{r_k^{2m}} \geqslant 4^{-3m-1} m^{-m}.$$

It is easily seen that the points $a_0 := (0, \ldots, 0)$ and $a_1 := (0, \ldots \ldots, 0, 2^{-k})$ belonging to $\mathring{P}$ are neighbouring vertices of the triangulation of kth generation. Moreover, $\mathring{f}$ maps them into the geodesics q connecting y_1 and x. Thus, by (157), we have

(160) $$r_k \geqslant l(\mathring{f}(a_0), \mathring{f}(a_1)) = l\left(q(0), q\left(\frac{\varepsilon(\lambda)}{2\sqrt{m-1}} 2^{-k}\right)\right) = \frac{\varepsilon(\lambda)}{2\sqrt{m-1}} 2^{-k}.$$

Applying (160) to (159), we see that for sufficiently great k's

(161) $$g(s_k, x_0) \geqslant 4^{-(k+4)m-1} m^{-4m} \varepsilon(\lambda)^{2m};$$

here s_k is an arbitrary m-simplex of kth generation in $\mathrm{Im}\,\mathring{f}$.

If we take two vertices x_i, x_j of an m-simplex s_k (cf. (155), (156)), then the difference $l(x_i, x_j) - \|\mathring{f}'_a(a_j - a_i)\|$ decreases to zero more quickly than $\|a_j - a_i\|$, as $k \to \infty$. More precisely, there exists a sequence

(162) $$\alpha_k \underset{k\to\infty}{\to} 0$$

such that $|l(x_j, x_i) - \|\mathring{f}'_{a_i}(a_j - a_i)\|| \leqslant \alpha_k \|a_j - a_i\|$.

Thus

$$l(x_j, x_i) \leqslant \|\mathring{f}'_{a_i}(a_j - a_i)\| + \alpha_k \|a_j - a_i\|$$
$$\leqslant (C_2 + \alpha_k)\|a_j - a_i\| \leqslant (C_2 + \alpha_k) 2^{-k}\sqrt{m}.$$

Hence, by (157), we have

$$r_k \leqslant (C_2 + \alpha_k)\sqrt{m}\, 2^{-k}. \tag{163}$$

The estimations (161) and (163) will be used later.

Our triangulation of 0th generation can be extended from the set $\operatorname{Im}\mathring{f}$ to the entire manifold M. Every m-simplex of the extended triangulation is a diffeomorphic image of the canonical m-simplex

$$\Delta^m := \Delta(0, e_1, \ldots, e_m) \quad \text{(cf. (154))} \tag{164}$$

in $\boldsymbol{R}^m$, where $\{e_i\}_1^m$ is the canonical (orthonormal) basis in $\boldsymbol{R}^m$. Such triangulating diffeomorphism determines the family of finer and finer triangulations of the m-simplex; each member of the family is nothing but the preceding member (triangulation) divided "into halves". It is not difficult to ensure that two such families on neighbouring m-simplexes fit together on the common face; it suffices that the triangulating diffeomorphisms coincide on this face. Moreover, we may assume that if for every m-simplex which we started with we take the kth triangulation (i.e. the kth member of the respective family), then the resulting triangulation of M coincides on $\operatorname{Im}\mathring{f}$ with the triangulation of kth generation defined before—see p. 45. This triangulation of M will also be called of kth generation.

Let X_k be the set of vertices (0-simplexes) of the triangulation of kth generation of M. In X_k we have the following relation $\sim$: let $x_1, x_2 \in X_k$, then $x_1 \sim x_2$ iff x_1 and x_2 belong to the same edge (1-simplex). In [10] the set X_k with the relation $\sim$ was called a net. The notion of difference form on X_k was introduce (Chapter II) and some difference operators were defined. Here we shall use only some of the results of [10] and therefore we confine ourselves merely to recalling a few definitions.

Let X be a net, i.e. $X = X_0, X_1, X_2, \ldots$ For every $k = 0, 1, \ldots$ we define

$$\begin{aligned} A_k(X) &:= \{(x_0, \ldots, x_k) \in X^k\colon\ x_i \sim x_j \text{ and } x_i \neq x_j \text{ for } i \neq j\}, \\ \mathscr{A}_k(X) &:= \big\{\{x_0, \ldots, x_k\} \subset X\colon\ (x_0, \ldots, x_k) \in A_k(X)\big\}. \end{aligned} \tag{165}$$

A *difference k-form* on X is a mapping

$$\overset{k}{\omega}\colon\ A_k(X) \to \boldsymbol{R} \tag{166}$$

such that for every $(x_0, \ldots, x_k) \in A_k(X)$ and every permutation $\pi \in \Pi(k)$ of k elements

(167) $$\overset{k}{\omega}(x_0, x_{\pi(1)}, \ldots, x_{\pi(k)}) = (\operatorname{sgn}\pi)\overset{k}{\omega}(x_0, \ldots, x_k).$$

We define the (exterior) difference operator

$$d\colon T^k(X) \to T^{k+1}(X),$$

where $T^k(X)$ is the space of difference k-forms. Here we shall use d only in the case $k = 0$ and

(168) $$(d\overset{0}{\omega})(x_0, x_1) := \overset{0}{\omega}(x_1) - \overset{0}{\omega}(x_0)$$

for every $(x_0, x_1) \in A_1(X)$.

The elements of the set $\mathscr{A}_m(X)$ will be denoted by s, which should not be confused with (155). If $s = \{x_0, \ldots, x_m\} \in \mathscr{A}_m(X)$, then we define

(169) $$g_{ij}(s, x_0) := \langle \overline{x_0, x_i} \otimes \overline{x_0, x_j}, g(x_0) \rangle,$$

where $i, j = 1, \ldots, m$, and

$$g(s, x_0) := \det\big(g_{ij}(s, x_0)\big).$$

(Thus, if $X = X_k$ and s is the set of vertices of an m-simplex $s_k = \overset{\circ}{f}(S_k)$, then $g(s, x_0) = g(s_k, x_0)$; cf. (155), (158).) Since the triangulating mappings—mapping Δ^m into M—are diffeomorphisms, for sufficiently great k's we have $g(s, x_0) > 0$.

Now, in every space $T^k(X)$ we can introduce a unitary structure. We shall need it only for $k = 1$; if $\overset{1}{\omega}, \overset{1}{\varrho} \in T^1(X)$, then

$$(\overset{1}{\omega}|\overset{1}{\varrho}) = \sum_{\substack{s \in \mathscr{A}_m(X)\\ s = \{x_0, \ldots, x_m\}}} \frac{1}{(m+1)!} \sum_{r=0}^{m} \Bigg[\sum_{\substack{i,j=0\\ i,j \neq r}}^{m} g^{ij}(s, x_r)\overset{1}{\omega}(x_r, x_i)\overset{1}{\varrho}(x_r, x_j)\Bigg]\sqrt{g(s, x_r)}.$$

Here $g^{ij}(s, x_r)$ is the matrix inverse to $g_{ij}(s, x_r)$, and both matrices are positive defined. So we have

(170) $$\|\overset{1}{\omega}\|^2 = \sum_{s \in \mathscr{A}_m(X)} \|\overset{1}{\omega}\|_s^2,$$

where, for any $s = \{x_0, \ldots, x_m\} \in \mathscr{A}_m(X)$,

(171) $$\|\overset{1}{\omega}\|_s^2 := \frac{1}{(m+1)!} \sum_{r=0}^{m} \Bigg[\sum_{\substack{i,j=0\\ i,j \neq r}}^{m} g^{ij}(s, x_r)\overset{1}{\omega}(x_r, x_i)\overset{1}{\omega}(x_r, x_j)\Bigg]\sqrt{g(s, x_r)}.$$

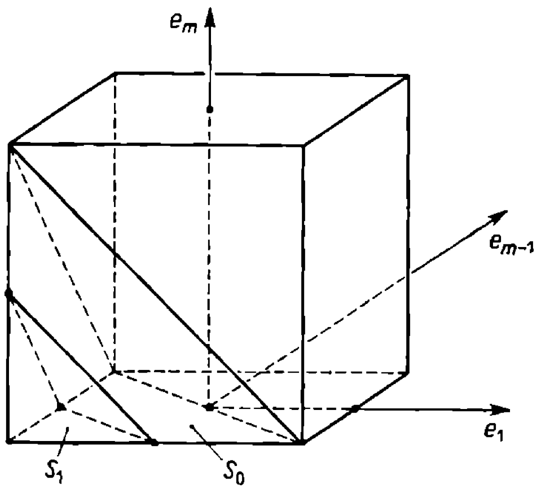
Since the matrices $\big(g^{ij}(s, x_r)\big)$ are positive defined, each summand of (171) contained in brackets is non-negative.

On page 43 we have written about the canonical triangulation of the unit cube I^m in R^m. Let us select one m-simplex of the triangulation, say Δ^m defined in (164). By definition, the triangulation of 0th generation on $\mathring{P}$ contains the m-simplex

$$S_0 := -\tfrac{1}{2}(1, \ldots, 1, 0) + \Delta^m.$$

Thus the triangulation of kth generation on $\mathring{P}$ contains the m-simplex

$$S_k := -\tfrac{1}{2}(1, \ldots, 1, 0) + 2^{-k}\Delta^m.$$

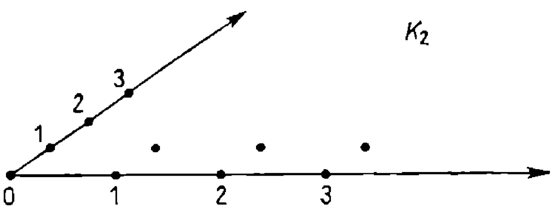


Hence on the base of $\mathring{P}$ we have 2^k m-simplexes

$$S_k^\mu := 2^{-k}\mu + S_k,$$

where $\mu = (\mu_1, \ldots, \mu_m)$ runs over the set

(172) $\quad K_k := \{\mu \in Z^m\colon\ \mu_m = 0;\ 0 \leqslant \mu_i \leqslant 2^k - 1,\ i = 1, \ldots, m\}.$



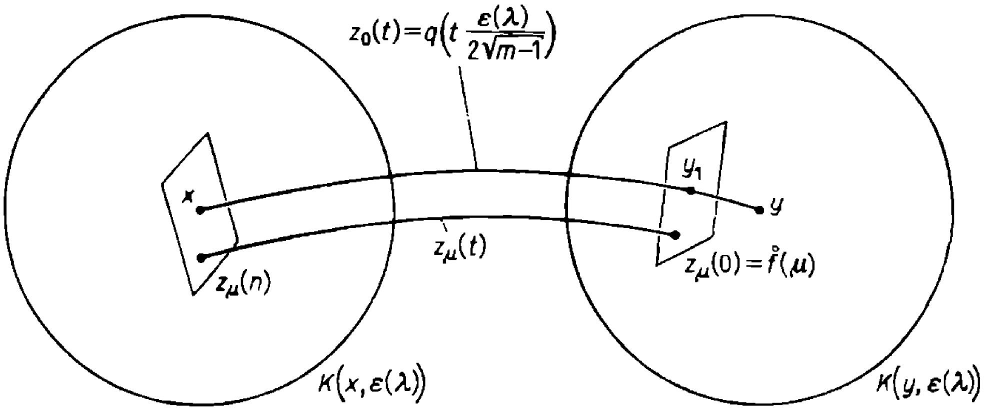
All these S_k^μ belong to the triangulation of kth generation on $\mathring{P}$. Thus also the m-simplexes

(173) $\quad S_k^{\mu,\nu} := (0, \ldots, 0, 2^{-k}\nu) + S_k^\mu, \qquad \nu = 0, 1, 2, \ldots, n2^k - 1$

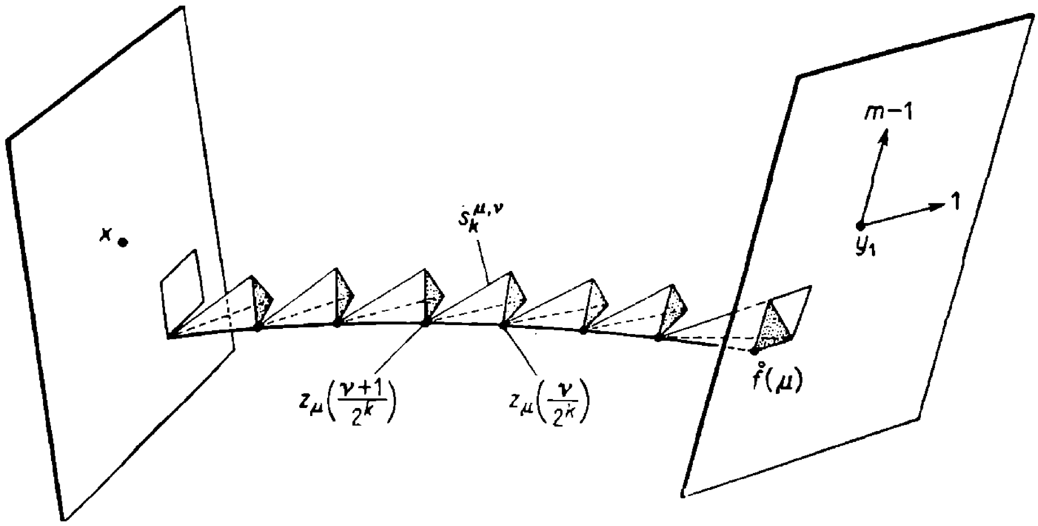
(cf. (148)) belong to the triangulation of kth generation, of $\mathring{P}$.

Let us fix a number k labelling the triangulations on M, and hence on $\dot{P}$. If $\mu \in K_k$, then we define

$$z_\mu(t) := \mathring{f}(\mu + te_m), \qquad 0 \leqslant t \leqslant n. \tag{174}$$



Moreover, we denote by $s_k^{\mu,\nu}$ the set of vertices of the m-simplex $\mathring{f}(S_k^{\mu,\nu})$. Thus $s_k^{\mu,\nu} \in \mathscr{A}_m(X_k)$.



It follows from Proposition 14 that for the eigenfunction φ

$$|\varphi(z_\mu(0)) - \varphi(z_\mu(n))| \geqslant \frac{1}{2\sqrt{V}}.$$

But since

$$\varphi(z_\mu(0)) - \varphi(z_\mu(n)) \quad \sum_{\nu=0}^{n2^k-1} \left[\varphi\left(z_\mu\left(\frac{\nu}{2^k}\right)\right) - \varphi\left(z_\mu\left(\frac{\nu+1}{2^k}\right)\right)\right],$$

we have

(175) $$\frac{1}{4V} \leqslant n2^k \sum_{\nu=0}^{n2^k-1} \left[\varphi\left(z_\mu\left(\frac{\nu}{2^k}\right)\right) - \varphi\left(z_\mu\left(\frac{\nu+1}{2^k}\right)\right)\right]^2.$$

Let us consider one of the summands of the above sum. We denote by $x_0, \ldots, x_m$ the vertices of the m-simplex $\mathring{f}(S_k^{\mu,\nu})$, i.e. $s_k^{\mu,\nu} = \{x_0, \ldots, x_m\}$. Obviously, $z_\mu\left(\frac{\nu}{2^k}\right)$, $z_\mu\left(\frac{\nu+1}{2^k}\right) \in s_k^{\mu,\nu}$. So, we may assume that $x_i = z_\mu\left(\frac{\nu+i}{2^k}\right)$, $i = 0, 1$. Thus

(176) $$\left[\varphi\left(z_\mu\left(\frac{\nu}{2^k}\right)\right) - \varphi\left(z_\mu\left(\frac{\nu+1}{2^k}\right)\right)\right]^2 \leqslant \sum_{i=1}^{m} [\varphi(x_i) - \varphi(x_0)]^2.$$

Using (163), we have

(177) $$|g_{ij}(s_k^{\mu,\nu}, x_0)| = |(\overline{x_0, x_i}|\overline{x_0, x_j})| \leqslant \|\overline{x_0 x_i}\|\,\|\overline{x_0, x_j}\| \leqslant r_k^2 \leqslant (C_2 + a_k)^2 m\, 4^{-k}.$$

This estimation permits us to make use of the following

LEMMA 15. *Let (a_{ij}) be a symmetric, positive defined, real $m \times m$-matrix such that $|a_{ij}| \leqslant a$; then for every $(w_1, \ldots, w_m) \in \boldsymbol{R}^m$*

$$\sum_{i=1}^{m} w_i^2 \leqslant m^{3/2} a \sum_{i,j=1}^{m} a^{ij} w_i w_j,$$

where $(a^{ij}) := (a_{ij})^{-1}$.

(An easy proof is given in [10], Lemma 50. ■) Thus, by (176), (177) and then (161) we obtain

(178) $$\left[\varphi\left(z_\mu\left(\frac{\nu}{2^k}\right)\right) - \varphi\left(z_\mu\left(\frac{\nu+1}{2^k}\right)\right)\right]^2$$
$$\leqslant (C_2 + a_k)^2 m^{5/2} 4^{-k} \sum_{i,j=1}^{m} g^{ij}(s_k^{\mu,\nu}, x_0)[\varphi(x_i) - \varphi(x_0)][\varphi(x_j) - \varphi(x_0)]$$
$$\leqslant (C_2 + a_k)^2 m^{2m+5/2} \varepsilon(\lambda)^{-m} 2^{k(m-2)+4m-1} \times$$
$$\times \sum_{i,j=1}^{m} g^{ij}(s_k^{\mu,\nu}, x_0)[\varphi(x_i) - \varphi(x_0)][\varphi(x_j) - \varphi(x_0)]\sqrt{g(s_k^{\mu,\nu}, x_0)}.$$

In order to simplify this long expression we introduce the following notation: let $\omega_k \in T^1(X_k)$ be defined as

(179) $$\omega_k(x', x'') := \varphi(x'') - \varphi(x'),$$

where $(x', x'') \in A_1(X_k)$. Then, according to (171),

$$\|\omega_k\|^2_{s_k^{\mu,\nu}}$$
$$= \frac{1}{(m+1)!} \sum_{r=0}^{m} \sum_{\substack{i,j=0 \\ i,j\neq r}}^{m} g^{ij}(s_k^{\mu,\nu}, x_r)[\varphi(x_i)-\varphi(x_r)][\varphi(x_j)-\varphi(x_r)]\sqrt{g(s_k^{\mu,\nu}, x_r)}$$
$$\geqslant \frac{1}{(m+1)!} \sum_{i,j=1}^{m} g^{ij}(s_k^{\mu,\nu}, x_0)[\varphi(x_i)-\varphi(x_0)][\varphi(x_j)-\varphi(x_0)]\sqrt{g(s_k^{\mu,\nu}, x_0)}.$$

This, combined with (178), gives us

(180) $$\left[\varphi\left(z_\mu\left(\frac{\nu}{2^k}\right)\right) - \varphi\left(z_\mu\left(\frac{\nu+1}{2^k}\right)\right)\right]^2 \leqslant 2^{k(m-2)} B_{2,k}(\lambda)\|\omega_k\|^2_{s_k^{\mu,\nu}},$$

where

(181) $$B_{2,k}(\lambda) := (C_2+a_k)^2(m+1)!\,m^{2m+5/2}2^{4m-1}\varepsilon(\lambda)^{-m}.$$

Now, if we apply (180) to (175), then we obtain

(182) $$\frac{1}{4V} \leqslant n2^{k(m-1)}B_{2,k}(\lambda)\sum_{\nu=0}^{n2^k-1}\|\omega_k\|^2_{s_k^{\mu,\nu}}.$$

Let us notice that the set K_k consists of $2^{k(m-1)}$ elements. Then, adding inequalities (182) for all $\mu \in K_k$, we have

(183) $$\frac{1}{4V} \leqslant nB_{2,k}(\lambda)\sum_{\mu\in K_k}\sum_{\nu=0}^{n2^k-1}\|\omega_k\|^2_{s_k^{\mu,\nu}}.$$

All the $s_k^{\mu,\nu}$ appearing in the above inequality form a family of $2^{k(m-1)}$ elements of long chains of m-simplexes belonging to $\mathscr{A}_m(X_k)$. One such chain (for a fixed $\mu \in K_k$) is drawn on page 50. The others are "parallel" to it. This family (of chains of m-simplexes) was determined by our choice of Δ^m as an m-simplex of the canonical triangulation of the unit cube I^m; see p. 49. Of course, that choice was arbitrary. Thus, instead of Δ^m, we may select any other of the $m!$ m-simplexes of the canonical triangulation of I^m. Every such new choice leads to another family of "parallel" long chains of m-simplexes belonging to $\mathscr{A}_m(X_k)$ and gives us an inequality analogous to (183). Let $\mathscr{F}$ be the set of all m-simplexes belonging to the long chains of those $m!$ families. Now, if we add all our $m!$ inequalities analogous to (183), then, by (170), we obtain

(184) $$\frac{1}{4V} \leqslant \frac{n}{m!}B_{2,k}(\lambda)\sum_{s\in\mathscr{F}}\|\omega_k\|^2_s \leqslant \frac{n}{m!}B_{2,k}(\lambda)\|\omega_k\|^2.$$

In [10], Section 11, there were defined restriction operators

$$R_k\colon \bigwedge^r(M)\to T^r(X_k),$$

where $\bigwedge^r(M):=\Gamma(\bigwedge^r T^*(M))$. For $r=0,1$ they are as follows: if $\varphi\in C^\infty(M)$, then

$$(R_k\varphi)(x_0):=\varphi(x_0)\qquad\text{for every } x_0\in X_k=A_0(X_k);$$

if $\omega\in\bigwedge^1(M)$, then

$$(R_k\omega)(x_0,x_1):=\langle\overline{x_0,x_1},\,\omega(x_0)\rangle.$$

Thus, looking at (179) and (168), we have

$$\omega_k=dR_k\varphi,$$

where φ is our eigenfunction of the laplacian. Theorem 56 and formula (68) in [10] state that

$$\|\omega_k-R_k d\varphi\|\xrightarrow[k\to\infty]{}0$$

and

$$\|R_k d\varphi\|-\|d\varphi\|\xrightarrow[k\to\infty]{}0.$$

Therefore $\lim\limits_{k\to\infty}\|\omega_k\|^2=\|d\varphi\|^2=(d^*d\varphi|\varphi)=\lambda$, because $\|\varphi\|=1$. This fact combined with (184) and then with (181), (162) and (152) gives

$$\text{(185)}\qquad \frac{1}{4V}\leqslant\frac{n}{m!}\,\lambda\lim_{k\to\infty}B_{2,k}(\lambda)\leqslant n\lambda\,\frac{(m+1)m^{2(m+2)+1/2}}{m-1}\,2^{4m-1}\varepsilon(\lambda)^{2-m}.$$

If δ is the diameter of M, i.e.

$$\text{(186)}\qquad \delta:=\sup_{x,y\in M}l(x,y),$$

then, by (143),

$$n=\frac{2\sqrt{m-1}}{\varepsilon(\lambda)}\,l(x,y_1)\leqslant\frac{2\sqrt{m-1}}{\varepsilon(\lambda)}\,\delta.$$

This and (185) give us

$$\lambda\geqslant B_3\frac{\varepsilon(\lambda)^{m-1}}{\delta V},$$

where

$$\text{(187)}\qquad B_3:=\frac{2^{-2(2m+1)}\sqrt{m-1}}{(m+1)m^{2(m+2)+\frac12}}.$$

The above consideration, i.e. those following Proposition 14, can be summarized in

THEOREM 16. *If $\lambda > 0$ is an eigenvalue of the laplacian on a compact, Riemannian, m-dimensional manifold M, then it satisfies the inequality*

$$\lambda \geqslant B_3 \frac{\varepsilon(\lambda)^{m-1}}{\delta V}, \tag{188}$$

where δ is the diameter of M (see (186)), V is the volume of M (see (138)), B_3 is the constant (187) and $\varepsilon(\lambda)$ is defined in (137).

Let us write

$$\begin{aligned} &H_1 := 5m2^{m+2}B_1; \quad \text{cf. } (136),\\ &H_2 := 5m2^{m+2}r_0, \\ &H_3 := \tfrac{1}{4}\min\{r_0, \underline{A}_2, \sqrt[4]{\underline{A}_3}, 4\sqrt{\underline{A}_4}, 4\sqrt[4]{\underline{A}_5}\}; \quad \text{cf. } (134).\end{aligned} \tag{189}$$

COROLLARY 17. *For every $c > 0$, at least one of the inequalities*

$$\lambda \geqslant \frac{B_3 H_3^{m-1}}{\delta V}, \quad \lambda > c, \quad \lambda \geqslant \frac{B_3(H_1 + cH_2)^{1-m}}{\delta V}, \tag{190}$$

holds. In particular, if we take $c = B_3 H_3^{m-1}/V$, then we arrive at (⁂) on page 6.

Proof. Let c be fixed. There are two possibilities: $\lambda > c$ and $\lambda \leqslant c$. Let us analyse the second one. By definition (137), $\varepsilon(\lambda) = \min\{(H_1 + \lambda H_2)^{-1}, H_3\}$. If $\varepsilon(\lambda) = H_3$, then by (188) we get the first inequality of (190). If $\varepsilon(\lambda) = (H_1 + \lambda H_2)^{-1}$, then, using the fact that $\lambda \leqslant c$, we obtain the third inequality of (190). ∎

Appendix

When formulas (152) and (153) are obtained, a shorter way leading to an even better minorization of λ_1 than (188) is possible. The difference between these two minorizations is not essential; they differ on a factor depending only on the dimension m of the manifold M. However, that new estimate we are going to present now, is obtained more directly, i.e. without dealing with difference approximations. But before stating the theorem I would like to express my gratefulness to Tadeusz Bałaban who substancially contributed to this simplification.

Thus returning back to page 44 we are going to prove the following

THEOREM 17. *If $\lambda > 0$ is an eigenvalue of the laplacian on a compact Riemannian m-dimensional manifold M, $m \geqslant 3$, then it satisfies the in-*

equality

(189′) $$\lambda > \bar{B}\frac{\varepsilon(\lambda)^{m-1}}{\delta V},$$

where

(190′) $$\bar{B} := [2^{m+11/2}m^{7/2}(m-1)^{(m-1)/2}]^{-1},$$

δ is the diameter of M (see (186)), *V is the volume of M (see* (138)), *and $\varepsilon(\lambda)$ is defined in* (137).

Proof. Let $\varphi \in C^\infty(M)$, $\|\varphi\| = 1$, $\Delta\varphi = \lambda\varphi$. We shall use the local chart defined by the mapping

(191) $$f\colon p \to M; \quad \text{see (151)}.$$

The domain of this chart, i.e. Imf, is a domain connecting the balls $K(x, \varepsilon(\lambda))$ and $K(y, \varepsilon(\lambda))$; cf. the figure on page 44. Let us take an interval

$$[0, n] \ni t \to (\lambda_1, \ldots, \lambda_{m-1}, t) \in P, \qquad |\lambda_i| \leqslant 1/2,$$

running along P. Then its f-image connects the points

$$x' := f(\lambda_1, \ldots, \lambda_{m-1}, 0),$$
$$y' := f(\lambda_1, \ldots, \lambda_{m-1}, n),$$

laying in the balls $K(x, \varepsilon(\lambda))$ and $K(y, \varepsilon(\lambda))$, respectively. If we denote this f-image by $[x', y']$, then, using Proposition 14, we have

$$\frac{1}{2\sqrt{V}} < \varphi(x') - \varphi(y')$$
$$= \int\limits_{[x',y']} d\varphi = \int\limits_0^n \frac{\partial\varphi\circ f}{\partial\lambda_m}(\lambda_1, \ldots, \lambda_{m-1}, \lambda_m)\,d\lambda_m.$$

The integration of both sides over the unit $(m-1)$-cube: $|\lambda_i| \leqslant 1/2$, $i = 1, \ldots, m-1$, and the Schwarz inequality give

$$\frac{1}{2\sqrt{V}} < \int\limits_P \frac{\partial\varphi\circ f}{\partial\lambda_m}(\lambda)\,d\lambda_1 \wedge \ldots \wedge d\lambda_m$$
$$\leqslant \Big[\int\limits_P d\lambda_1 \wedge \ldots \wedge d\lambda_m\Big]^{1/2}\Big[\int\limits_P \Big|\frac{\partial\varphi\circ f}{\partial\lambda_m}(\lambda)\Big|^2 d\lambda_1 \wedge \ldots \wedge d\lambda_m\Big]^{1/2}$$
$$= \sqrt{n}\Big[\int\limits_P \Big|\frac{\partial\varphi\circ f}{\partial\lambda_m}(\lambda)\Big|^2 d\lambda_1 \wedge \ldots \wedge d\lambda_m\Big]^{1/2}$$
$$\leqslant \sqrt{n}\Big[\int\limits_P \sum_{i=1}^m \Big|\frac{\partial\varphi\circ f}{\partial\lambda_i}(\lambda)\Big|^2 d\lambda_1 \wedge \ldots \wedge d\lambda_m\Big]^{1/2}$$

We shall need the following algebraic

LEMMA. *If (a_{ij}) is a symmetric positive defined real $m\times m$-matrix such that $|a_{ij}|\leqslant a$, then for every $(w_1,\ldots,w_m)\in \boldsymbol{R}^m$*

$$\sum_{i=1}^{m} w_i^2 \leqslant m^{3/2} a \sum_{i,j=1}^{m} a^{ij} w_i w_j,$$

where (a^{ij}) is the matrix $(a_{ij})^{-1}$. (An easy proof is given in [5], Lemma 50.)

We are going to apply this lemma to the matrix of the coordinates

$$g_{ij}(f(\lambda)) = (f'_\lambda e_i | f'_\lambda e_j)$$

of the metric tensor, with respect to the local chart $f^{-1}\colon \operatorname{Im} f \to \boldsymbol{R}^m$ which we have mentioned at the beginning of the proof. By (152) we have

$$|g_{ij}(f(\lambda))| \leqslant \|f' e_i\| \, \|f' e_j\| \leqslant \frac{m^2 \varepsilon(\lambda)^2}{m-1}.$$

Thus

$$\sum_{i=1}^{m} \left| \frac{\partial \varphi\circ f}{\partial \lambda_i}(\lambda)\right|^2 \leqslant \frac{m^{7/2}\varepsilon(\lambda)^2}{m-1} \sum_{i,j=1}^{m} g^{ij}(f(\lambda)) \frac{\partial\varphi\circ f}{\partial\lambda_i}(\lambda)\frac{\partial\varphi\circ f}{\partial\lambda_j}(\lambda).$$

Since $\lambda = (\Delta\varphi|\varphi) = \|d\varphi\|^2$, we are going to estimate $\|d\varphi\|^2$ from below.

$$\lambda = \|d\varphi\|^2 \geqslant \int_{\operatorname{Im} f} \|d\varphi(x)\|^2 \overset{m}{\tau}(x)$$

$$= \int_P \sum_{i,j=1}^{m} g^{ij}(f(\lambda)) \frac{\partial\varphi\circ f}{\partial\lambda_i}(\lambda)\frac{\partial\varphi\circ f}{\partial\lambda_j}(\lambda)\left[\det\left(g_{ij}(f(\lambda))\right)\right]^{1/2} d\lambda_1\wedge\ldots\wedge d\lambda_m$$

$$\geqslant \frac{m-1}{m^{7/2}\varepsilon(\lambda)^2}\int_P \sum_{i=1}^{m}\left|\frac{\partial\varphi\circ f}{\partial\lambda_i}(\lambda)\right|^2 \left[\det\left(g_{ij}(f(\lambda))\right)\right]^{1/2} d\lambda_1\wedge\ldots\wedge d\lambda_m.$$

By (153) we have

$$\left[\det\left(g_{ij}(f(\lambda))\right)\right]^{1/2} \geqslant \frac{\varepsilon(\lambda)^m}{2^{m+1/2}(m-1)^{m/2}}$$

and therefore, using also (192), we obtain

$$\lambda \geqslant \frac{\varepsilon(\lambda)^{m-2}}{2^{m+1/2} m^{7/2}(m-1)^{(m-2)/2}} \int_P \sum_{i=1}^{m} \left|\frac{\partial\varphi\circ f}{\partial\lambda_i}(\lambda)\right|^2 d\lambda_1\wedge\ldots\wedge d\lambda_m$$

$$> \frac{\varepsilon(\lambda)^{m-2}}{2^{m+1/2} m^{7/2}(m-1)^{(m-2)/2}}\cdot\frac{1}{4\sqrt{n}}.$$

To complete the proof it is enough to notice that by (142)

$$n \leqslant \frac{2\sqrt{m-1}\, l(x,y)}{\varepsilon(\lambda)} \leqslant \frac{2\sqrt{m-1}\,\delta}{\varepsilon(\lambda)}. \ \blacksquare$$

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