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OPTIMAL CONTROL OF NON WELL-POSED DISTRIBUTED SYSTEMS

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1. INTRODUCTION

1.1. In the usual theory of optimal control of distributed systems one is given a partial differential equation

$$\mathscr{A}y = \mathscr{B}v, \tag{1.1}$$

where (1.1) is a *formal* writing; $\mathscr{A}$ denotes a partial differential operator, v denotes the *control function*, $\mathscr{B}$ is an operator which can be thought of as introducing *boundary conditions*, since to (1.1) one has to add boundary conditions and, in case $\mathscr{A}$ is an evolution operator, one has also to add *initial conditions*.

The assumption classically made is that (1.1) (subject to appropriate boundary and initial conditions) *defines a well-set problem*, i.e., that, given v in a suitable space, one can find a *unique solution* $y(v)$ of (1.1), and that the mapping $v \to y(v)$ is *continuous for suitable topologies*. We refer for instance to J.-L. Lions [2], [3] and to the bibliography therein.

The *problem of optimal control* then consists in finding

$$\inf J(v), \tag{1.2}$$

where v is subject to some constraints and where the *cost function* $J(v)$ is given by

$$J(v) = \Phi\big(y(v)\big) + \Psi(\|v\|), \tag{1.3}$$

where $\|v\|$ denotes a norm on a suitable space $\mathscr{V}$, $\lambda \to \Psi(\lambda)$ is a continuous function for $\lambda \geqslant 0$ such that $\Psi(\lambda) \to +\infty$ as $\lambda \to +\infty$, and where $\Phi(y)$ is a continuous functional on a space $\mathscr{Y}$.

The "*standard*" *theory* (J.-L. Lions [2], [3]) refers to the case where $v \to y(v)$ maps $\mathscr{V}$ continuously into $\mathscr{Y}$. If this is *not* the case, one is led (J.-L. Lions [4]) to introduce *new functional spaces* (*or sets*) defined by

$$\mathscr{U} = \{v \mid v \in \mathscr{V}, y(v) \in \mathscr{Y}\}. \tag{1.4}$$

Examples of these situations have been studied in J.-L. Lions [4].

1.2. There is another point of view, which is more general and *enables one to consider situations where* (1.1) *is not well-posed.* We adopt a different notation in order to avoid confusion. Let us present the idea on a simple example. We consider in a cylinder

$$Q = \Omega \times]0, T[\tag{1.5}$$

the *backward heat equation*:

$$\frac{\partial z}{\partial t} + \Delta z = v \quad \text{in} \quad Q, \tag{1.6}$$

where $v \in L^2(Q)$ and where $z \in L^2(Q)$. It is then possible to verify (see Section 3 below) that one can define the traces of z on $\Sigma = \Gamma \times]0, T[$, $\Gamma = \partial\Omega$ and on $t = 0$. We add to (1.6) the initial and boundary conditions

$$z = 0 \quad \text{on} \quad \Omega \quad \text{for} \quad t = 0, \tag{1.7}$$

$$z = 0 \quad \text{on} \quad \Sigma. \tag{1.8}$$

Of course, given v, (1.6), (1.7), (1.8) *do not* define z; we consider instead *the couples* $\{v, z\}$ *which are related by* (1.6), (1.7), (1.8).

We then consider the *functional* (the *cost function*)

$$J(v, z) = \int_Q (z - z_d)^2 \, dx\, dt + N \int_Q v^2 \, dx\, dt, \tag{1.9}$$

where z_d is given in $L^2(Q)$ and where N is given > 0, and we look for

$$\begin{cases} \inf J(v, z), \\ v, z \in L^2(Q) \text{ subject to (1.6), (1.7), (1.8).} \end{cases} \tag{1.10}$$

We easily show that this problem admits a unique solution and we show in Section 3 that one *can derive an optimality system.*

1.3. Actually there is, so to speak, an "intermediate case" between the situation of 1.1 and the situation of 1.2; this is when (in (1.1)) $\mathscr{A}$ is actually a "well-posed" operator but some of the boundary (or initial) conditions are missing.

Examples are presented in Section 2.

1.4. It is obvious that along the lines of what has been said in 1.2 and 1.3 an almost unlimited number of questions arise. Some of them are indicated in Section 3.

1.5. For other situations and further details, cf. J.-L. Lions [5]. The idea given in 1.2 has been introduced for *Navier Stokes equations* by A. V. Foursikov [1].

1.6. The plan of this paper is as follows:

2. SYSTEMS WITH INSUFFICIENT DATA

2.1. Partial information on the initial condition

Let Ω be a bounded open set of $\boldsymbol{R}^n$ with boundary Γ. In the cylinder $Q = \Omega \times]0, T[$, we consider the *heat equation*

$$\frac{\partial z}{\partial t} - \Delta z = v \quad \text{in} \quad Q, \tag{2.1}$$

where

$$v \in L^2(Q), \qquad z \in L^2(Q). \tag{2.2}$$

It follows from (2.1), (2.2) that(1)

$$\frac{\partial z}{\partial t} \in L^2\big(0, T; H^{-2}(\Omega)\big), \tag{2.3}$$

so that z is a.e. equal to a continuous function from $[0, T] \to H^{-1}(\Omega)$. Let K_0 be a set such that:

$$K_0 = \text{closed convex non-empty subset of } H^{-1}(\Omega). \tag{2.4}$$

We suppose that

$$z(0) \in K_0 \tag{2.5}$$

(where $z(0) = z|_{t=0}$).

We verify that one can also define the trace of z on $\Sigma = \Gamma \times]0, T[$, $\Gamma = \partial\Omega$ (cf. for instance J.-L. Lions [5], Chapter 7), so that it makes

(1) H^{-2} = dual of $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is the closure in $H^2(\Omega)$ (Sobolev space of order 2 for $L^2(\Omega)$) of smooth functions with compact support in Ω. Analogously $H^{-1}(\Omega)$ = dual of $H_0^1(\Omega)$.

sense to impose

$$z|_{\Sigma} = 0. \tag{2.6}$$

Remark 2.1. If $K_0 = \{0\}$ (or any given point in $H^{-1}(\Omega)$) then (2.1), (2.5), (2.6) uniquely define $z = z(v) = y(v)$ in the usual notation of Section 1.1 of the Introduction and one is led to a standard situation in Optimal Control. ∎

The *cost function* is defined by

$$J(v, z) = |z - z_d|_Q^2 + N|v|_Q^2, \tag{2.7}$$

where we set

$$|f|_Q^2 = \int_Q f^2\, dx\, dt,$$

where z_d is given in $L^2(Q)$ and N is given > 0.

Let us consider

$$\mathcal{U}_{ad} = \text{closed convex non-empty subset of } L^2(Q). \tag{2.8}$$

The *problem of optimal control* is now to find

$$\begin{cases} \inf J(v, z), \\ v \in \mathcal{U}_{ad}, \quad v, z \text{ subject to (2.1), (2.2), (2.5), (2.6).} \end{cases} \tag{2.9}$$

It is easily seen that (2.9) admits a *unique solution* $\{u, y\}$, which is called the *optimal couple*. We have

THEOREM 2.1. *The optimal couple $\{u, y\}$ which is the solution of* (2.9) *is characterized by the solution $\{u, y, p\}$ of*

$$\frac{\partial y}{\partial t} - \Delta y = u, \qquad -\frac{\partial p}{\partial t} - p = y - z_d \quad \text{in} \quad Q, \tag{2.10}$$

$$y = 0, \quad p = 0 \quad \text{on} \quad \Sigma, \tag{2.11}$$

$$y, p \in L^2(Q), \tag{2.12}$$

$$\begin{cases} \big(p(0), k - y(0)\big) \geqslant 0 \quad \forall k \in K_0, \ y(0) \in K_0, \\ p(T) = 0 \end{cases} \tag{2.13}$$

and

$$(p + Nu, v - u)_Q \geqslant 0 \quad \forall v \in \mathcal{U}_{ad}, \ u \in \mathcal{U}_{ad}. \tag{2.14}$$

Remark 2.2. In (2.14) $(p, v)_Q = \int_Q pv\, dx\, dt$. In $(2.13)_1$, $p(0) \in H_0^1(\Omega)$ so that $\big(p(0), k - y(0)\big)$ means the scalar product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. ∎

Remark 2.3. If $K_0 = \{0\}$, (2.13) reduces to

$$y(0) = 0, \quad p(T) = 0 \quad \text{in} \quad \Omega, \tag{2.15}$$

and one obtains the usual optimality system; cf. J.-L. Lions [2]. ∎

Remark 2.4. If $K_0 = H^{-1}(\Omega)$ (no information at all on $z(0)$), then (2.13) reduces to

$$p(0) = 0, \quad p(T) = 0 \quad \text{in} \quad \Omega. \; ∎ \tag{2.16}$$

Remark 2.5. If one takes

$$K_0 = \{\varphi |\ \varphi \in H^{-1}(\Omega), \varphi \geqslant 0 \text{ in } \Omega\}, \tag{2.17}$$

then (2.13) becomes

$$p(0) \geqslant 0, \quad y(0) \geqslant 0, \quad p(0)y(0) = 0, \quad p(T) = 0 \quad \text{in} \quad \Omega. \; ∎ \tag{2.18}$$

Sketch of proof. One *defines* p by the second equation in (2.10) and in (2.11) and by $p(T) = 0$. Then one transforms by integration by parts the optimality condition

$$\begin{cases} (y - z_d, z - y)_Q + N(u, v - u)_Q \geqslant 0 & \forall v \in \mathcal{U}_{ad}, \\ v \text{ and } z \text{ subject to (2.1), (2.2), (2.5), (2.6).} \end{cases} \tag{2.19}$$

2.2. Partial information on boundary conditions

Let Ω be an open set of R^n with boundary (cf. Fig. 1)

$$\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1. \tag{2.20}$$

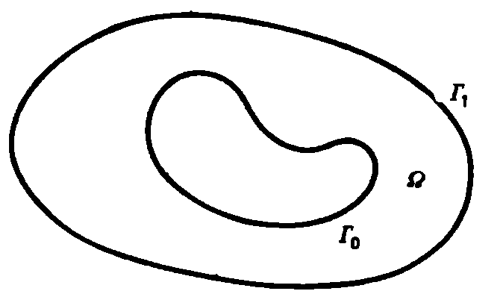

Fig. 1

Let us consider the equation

$$-\Delta z = v \quad \text{in} \quad \Omega, \tag{2.21}$$

where

$$v, z \in L^2(\Omega), \tag{2.22}$$

$$z = 0 \quad \text{on} \quad \Gamma_0. \tag{2.23}$$

If z and $\Delta z \in L^2(\Omega)$, then (cf. J.-L. Lions and E. Magenes [6]) one can define

$$\left.\frac{\partial z}{\partial \nu}\right|_{\Gamma_1} \text{ as an element of } H^{-3/2}(\Gamma_1). \tag{2.24}$$

Let us consider

$$K = \text{closed convex non-empty subset of } H^{-3/2}(\Gamma_1). \tag{2.25}$$

We shall suppose that

$$\left.\frac{\partial z}{\partial \nu}\right|_{\Gamma} \in K. \tag{2.26}$$

We are also given

$$\mathscr{U}_{ad} = \text{closed convex non-empty subset of } L^2(\Omega) \tag{2.27}$$

and we consider the *cost function*

$$J(v, z) = |z - z_d|^2 + N|v|^2, \tag{2.28}$$

where $|\varphi|^2 = \int_\Omega \varphi^2 dx$, z_d is given in $L^2(\Omega)$ and N is greater than 0. The problem of *optimal control* is now *to find*

$$\begin{cases} \inf J(v, z), \\ v \in \mathscr{U}_{ad}, \quad v \text{ and } z \text{ subject to (2.21), (2.22), (2.23), (2.26).} \end{cases} \tag{2.29}$$

The unique solution $\{u, y\}$ of (2.29) is characterized by

THEOREM 2.2. *The solution* $\{u, y\}$ *of* (2.29) *is defined by the unique solution* $\{u, y, p\}$ *of the optimality system*

$$-\Delta y = u, \quad -\Delta p = y - z_d \quad \text{in} \quad \Omega, \tag{2.30}$$

$$y = p = 0 \quad \text{on} \quad \Gamma_0, \tag{2.31}$$

$$\begin{cases} \dfrac{\partial y}{\partial \nu} \in K, \quad \left(p, k - \dfrac{\partial y}{\partial \nu}\right)_{\Gamma_1} \geqslant 0 \quad \forall k \in K, \\ \dfrac{\partial p}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \end{cases} \tag{2.32}$$

and

$$(p + Nu, v - u) \geqslant 0 \quad \forall v \in \mathscr{U}_{ad}, \quad u \in \mathscr{U}_{ad}. \tag{2.33}$$

3. NON-WELL-POSED SYSTEMS

3.1. Backward heat equation

We return to (1.6) in the Introduction. We consider v and z such that

$$v, z \in L^2(Q) \tag{3.1}$$

and

$$\frac{\partial z}{\partial t} + \Delta z = v \quad \text{in} \quad Q. \tag{3.2}$$

It follows (as in Section 2) that the following conditions make sense:

$$z(0) = 0 \quad \text{in} \quad \Omega, \tag{3.3}$$

$$z = 0 \quad \text{on} \quad \Sigma. \tag{3.4}$$

We define

$$J(v, z) = |z - z_d|_Q^2 + N|v|_Q^2 \tag{3.5}$$

and we consider the problem

$$\begin{cases} \inf J(v, z), \\ v \text{ and } z \text{ subject to (3.1)–(3.4).} \end{cases} \tag{3.6}$$

We have

THEOREM 3.1. *The unique solution $\{u, y\}$ of* (3.6) *is characterized by the unique solution $\{u, y, p\}$ of*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \Delta y = u, \quad p + Nu = 0, \\ -\dfrac{\partial p}{\partial t} + \Delta p = y - z_d \quad \textit{in} \quad Q, \end{cases} \tag{3.7}$$

$$y = p = 0 \quad \textit{on} \quad \Sigma, \tag{3.8}$$

$$y(0) = 0, \quad p(T) = 0 \quad \textit{in} \quad \Omega, \tag{3.9}$$

$$y, p \in L^2(Q). \tag{3.10}$$

Remark 3.1. If one replaces Δ by $-\Delta$, one is led to the usual optimal system in such problems; cf. J.-L. Lions [2]. Cf. also Section 3.2 below. ■

Remark 3.2. The solutions y, p of the optimality system (3.7)–(3.10) are *weak* solutions; they become *stronger* if Δ is replaced by $-\Delta$. ■

Remark 3.3. We refer to J.-L. Lions [5] for the proof of Theorem 3.1. We use an approximation by a penalty method. It is to be noticed that p *cannot be defined* by $(3.7)_2$ and the conditions on p which appear in (3.8), (3.9), since the *backward* problem is *not* well-posed for $\partial/\partial t - \Delta$.

Remark 3.4. A *direct* solution of (3.7)–(3.10) (i.e., without reference to optimal control) can be given as follows: We define

$$\mathscr{V} = \left\{\varphi \,\middle|\, \varphi, \frac{\partial \varphi}{\partial t} - \Delta\varphi \in L^2(Q),\ \varphi(T) = 0,\ \varphi|_\Sigma = 0\right\}, \tag{3.11}$$

which is a *Hilbert space* for the norm

$$\|\varphi\|_{\mathscr{V}} = \left(|\varphi|_Q^2 + \left|\frac{\partial\varphi}{\partial t} - \Delta\varphi\right|_Q^2\right)^{1/2}.$$

We multiply $(3.7)_2$ by $-\dfrac{\partial\varphi}{\partial t} + \Delta\varphi$. We obtain

$$\begin{aligned}\left(\frac{\partial p}{\partial t} - \Delta p, \frac{\partial\varphi}{\partial t} - \Delta\varphi\right)_Q &= \left(\frac{\partial y}{\partial t} + \Delta y, \varphi\right)_Q + \left(z_d, \frac{\partial\varphi}{\partial t} - \Delta\varphi\right)_Q \\ &= \left(\text{using } (3.7)_1\right) = -\frac{1}{N}(p, \varphi)_Q + \left(z_d, \frac{\partial\varphi}{\partial t} - \Delta\varphi\right)_Q.\end{aligned} \tag{3.12}$$

If we define

$$a(p, \varphi) = \left(\frac{\partial p}{\partial t} - \Delta p, \frac{\partial\varphi}{\partial t} - \Delta\varphi\right)_Q + \frac{1}{N}(p, \varphi)_Q, \tag{3.13}$$

we see that (3.12) is equivalent to

$$\begin{cases} a(p, \varphi) = \left(z_d, \dfrac{\partial\varphi}{\partial t} - \Delta\varphi\right)_Q & \forall\varphi \in \mathscr{V}, \\ p \in \mathscr{V}, \end{cases} \tag{3.14}$$

a *problem which admits a unique solution*. Once p is known, y is given by

$$y = z_d - \frac{\partial p}{\partial t} + \Delta p. \blacksquare$$

3.2. Completely non well-posed evolution problems

Let $m(t)$ be given as in Fig. 2. The main property we are going to use is that m is continuous and

$$\begin{aligned} m(t) &> 0 \quad \text{for} \quad t \in [0, t_0[, \\ m(t) &< 0 \quad \text{for} \quad t \in]t_0, T]. \end{aligned} \tag{3.15}$$

We now consider, instead of (3.2), the equation

$$\frac{\partial z}{\partial t} + m(t)\Delta z = v \quad \text{in} \quad Q, \tag{3.16}$$

conditions (3.1), (3.3), (3.4) being unchanged([2]).

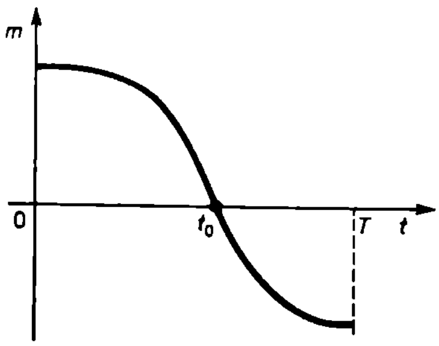

Fig. 2

Remark 3.5. The operator $\frac{\partial}{\partial t} + m(t)\Delta$ is "not well-posed" when a Cauchy data is given at $t = 0$ *and also* when Cauchy data are given at $t = T$; we express this fact by saying that the operator is *completely non well-posed*. In the case of Section 3.1, we can think of the problem as a *backward* problem with *no information* at $t = T$ and with a *state constraint* at $t = 0$. ■

Let us consider the cost function (3.5). The problem analogous to (3.6) *admits a unique solution* $\{u, y\}$ which is characterized by the solution $\{u, y, p\}$ of the following *optimality system*:

THEOREM 3.2. *The optimality system for* (3.16), (3.1), (3.3)–(3.6) *is given by*

$$\begin{cases} \dfrac{\partial y}{\partial t} + m(t)\Delta y = u, \quad p + Nu = 0, \\ -\dfrac{\partial p}{\partial t} + m(t)\Delta p = y - z_d \quad in \quad Q, \end{cases} \tag{3.17}$$

with conditions (3.8)–(3.10) *unchanged.* ■

([2]) One can verify that (3.4) still makes sense.

Remark 3.6. Let us mention some other ***completely non well-posed systems***:

$$\frac{\partial z}{\partial t} - \Delta z = v, \tag{3.18}$$

subject to

$$\frac{\partial z}{\partial t} - \frac{\partial z}{\partial \nu} = 0 \quad \text{on} \quad \Sigma, \tag{3.19}$$

and

$$z(0) = 0. \ \blacksquare$$

Another example is given by the system

$$\left\{\begin{array}{l} \dfrac{\partial z_1}{\partial t} + \Delta z_1 - z_2 = v_1, \\ \dfrac{\partial z_2}{\partial t} - \Delta z_2 + z_1 = v_2, \\ z_1 = z_2 = 0 \quad \text{on} \quad \Sigma, \\ z_i \in L^2(Q), \\ z_1(0) = z_2(0) = 0. \ \blacksquare \end{array}\right. \tag{3.20}$$

4. OTHER SITUATIONS

4.1. Higher order evolution problems

Let us consider

$$\frac{\partial^m z}{\partial t^m} - \Delta z = v \quad \text{in} \quad Q, \quad m \text{ integer} \geqslant 3, \tag{4.1}$$

$$z, v \in L^2(Q), \tag{4.2}$$

$$z = 0 \quad \text{on} \quad \Sigma(^3) \tag{4.3}$$

and without information for $t = 0$.

(3) This condition makes sense when z and $\dfrac{\partial^m z}{\partial t^m} - \Delta z \in L^2(Q)$.

If the *cost function* is given by

$$J(v, z) = |z - z_d|_Q^2 + N|v|_Q^2, \tag{4.4}$$

the *optimality system* is given by

$$\begin{cases} \dfrac{\partial^m y}{\partial t^m} - \Delta y = u, \quad p + Nu = 0, \\ (-1)^m \dfrac{\partial^m p}{\partial t^m} - \Delta p = y - z_d \quad \text{in} \quad Q, \\ \dfrac{\partial^j p}{\partial t^j}(0) = \dfrac{\partial^j p}{\partial t^j}(T) = 0, \quad 0 \leqslant j \leqslant m-1, \quad \text{in} \quad \Omega, \\ y = p = 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{4.5}$$

4.2. Unstable non-linear systems

One can consider systems governed by *unstable* non-linear models, such as

$$\frac{\partial z}{\partial t} - \Delta z - z^3 = v \quad \text{in} \quad Q \tag{4.6}$$

subject to

$$v \in L^2(Q), \quad z \in L^6(Q), \tag{4.7}$$

$$z(0) = 0, \tag{4.8}$$

$$z = 0 \quad \text{on} \quad \Sigma. \tag{4.9}$$

Again one thinks of the *set of couples* $\{v, z\}$ subject to (4.6)–(4.9) and *not* of the solution z of these equations, once v is given *since in general there is no global (in time) solution of this problem.*

One can again solve optimal control problems, for instance for the functional

$$J(v, z) = \frac{1}{6} \int_Q (z - z_d)^6 dx\, dt + \frac{N}{2} \int_Q v^2 dx\, dt. \tag{4.10}$$

If u, y is a solution, there exists a p such that

$$p, \frac{\partial p}{\partial x_i}, \frac{\partial^2 p}{\partial x_i \partial x_j}, \frac{\partial p}{\partial t} \in L^{6/5}(Q), \tag{4.11}$$

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y - y^3 = u, \\ -\dfrac{\partial p}{\partial t} - \Delta p - 3y^2 p = (y - z_d)^5, \quad p + Nu = 0, \\ y = p = 0 \quad \text{on} \quad \Sigma, \\ y(0) = 0, \quad p(T) = 0 \quad \text{in} \quad \Omega. \ \blacksquare \end{cases} \tag{4.12}$$

4.3. Cauchy systems

Let Ω be given as in Section 2.2 and let us consider

$$\begin{cases} z \in L^2(\Omega), \\ \Delta z = 0 \quad \text{in} \quad \Omega, \\ z|_{\Gamma_0} = v_0, \quad \left.\dfrac{\partial z}{\partial \nu}\right|_{\Gamma_0} = v_1 \quad \text{on} \quad \Gamma_0. \end{cases} \tag{4.13}$$

No information is given on Γ_1. One can consider

$$J(v_0, v_1, z) = |z - z_d|^2 + N_0 |v_0|^2_{\Gamma_0} + N_1 |v_1|^2_{\Gamma_0} \tag{4.14}$$

and minimize (4.14) with various constraints on v_0, v_1. Cf. J.-L. Lions [5], Chapter 7, for further details and also for applications in Game Theory.

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[1] A. V. Foursikov, *On some problems of control...*, Dokl. Akad. Nauk SSSR 252 (1980), 1066–1070.

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[6] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1, 2, Dunod, Paris 1968 [English translation by P. Kenneth, Springer-Verlag, 1971, 1972].

Added in proof (August 1984): Further results are presented in the following publications of the author:

1. *Contrôle des systèmes distribués singuliers*, Gauthiers-Villars, 1983, English translation, Russian translation 1985.

2. *Some remarks on the optimal control of singular distributed systems*, in: *Summer Institute on Nonlinear Analysis, Berkeley 1983*, A.M.S. Publications.

3. *Distributed Systems with Incomplete Data and Lagrange Multipliers*, Erice 1984.