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THOM'S LEMMA IN REAL GEOMETRY

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In his lecture notes of the Institute des Hautes Études Scientifiques (*Ensembles semianalitiques*, 1965), S. Łojasiewicz introduces in his study of stratifications of semianalytic sets the following "Lemme de Thom":

Let $P(x)$ be a polynomial of degree n in one variable with real coefficients. Then, every set of the kind

$$\Delta = \bigcap_{\nu=0}^{n} \{P^{(\nu)}(x) \in \theta_\nu\} \quad (\textit{where } \theta_\nu = \{(0, +\infty)\}, \{0\} \textit{ or } \{(-\infty, 0)\})$$

is connected; therefore it is either empty, a single point or an open interval.

In the same notes we have a proof-essentially using the "saucissonage à la Cohen" – of the semialgebricity of connected components of semialgebraic sets over $\boldsymbol{R}^n$. In particular, components of real algebraic sets are semialgebraic althought of a special type, as they can be described by zeroes of global Nash functions. The basic result of Risler [1975] showing the connectivity of zero sets of prime ideals in the ring of Nash functions on an open semialgebraic U uses, as a basic tool, the separation of disjoint closed semialgebraic sets in U by means of Nash functions (Mostowski [1976]). It turns out that this and other important geometric facts of real geometry (Efroymson [1976], Recio [1977], Bochnak and Efroymson [1980], Coste and Coste-Roy [1982], see also Lam [1984] for a survey of related results) are proved with the help of a generalization of Thom's lemma formulated here like in Coste [1982].

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DEFINITION. Let $P_1(x_1 \ldots x_n), \ldots, P_r(x_1 \ldots x_n)$ be a family of polynomials in $R[x_1 \ldots x_n]$. We say that it is a *separating family* if for each choice of signs θ_i, the set

$$\Delta = \bigcap_{i=1}^{r} \{P_i(x) \in \theta_i\}$$

is either empty or connected; if Δ is not empty it is further required that its closure (in the Euclidean topology) is described by relaxing the strict inequalities.

THOM'S LEMMA OF EFROYMSON. *Every finite family of polynomials in* $R[x_1 \ldots x_n]$ *can be enlarged to form a separating family.*

One particular remarkable consequence is the following:

COROLLARY (Łojasiewicz, loc. cit.). *Every open semialgebraic set* (resp., *closed semialgebraic set*) *can be written as a finite union of systems of the kind* $\{f_1(x) > 0, \ldots, f_e(x) > 0\}$ (resp. $\{f_1(x) \geqslant 0, \ldots, f_e(x) \geqslant 0\}$.

Extensions of Thom's Lemma to different rings have been obtained (see the case of Liouville extensions in Risler [1983] or for germs at a point of analytic functions in Fernandez et al. [1985]). Let us also mention the verification of Thom's Lemma – modifying suitably the notion of connected set – over real closed fields other that the reals (besides the work of Coste and Coste-Roy referred above, see also Delfs and Knebusch [1982] or Bradley [1983]).

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The case of linear inequality systems (cf. for details of the following in Miranda [1985])

$$\text{(S)} \qquad AX > A_0, \quad BX \geqslant B_0, \quad CX = C_0$$

(where $X = \begin{pmatrix} x_1 \\ x_n \end{pmatrix}$, A, B, C are $m \times n$ matrices, A_0, B_0, C_0 are $m \times 1$ matrices) presents some remarkable features in connection with Thom's Lemma. It is well known that the set of points in R^n – denoted by $\mathscr{R}(S)$ – satisfying all of the above equations and inequalities is convex and thus connected. It can be checked that if $\mathscr{R}(S)$ is non-empty, then its closure is described by

$$(\bar{\text{S}}) \qquad AX \geqslant A_0, \quad BX \geqslant B_0, \quad CX = C_0,$$

and therefore every family of linear polynomials in separating. Taking advantage of this remark it is easy to deduce the separation by linear polynomials of two disjoint and closed (with respect to (S)) solution sets of linear inequality systems.

It can be also shown that if (S) is open and not empty, then (S) is given precisely by $AX > A_0$.

Slack inequalities are a relevant topic in linear programming; an inequality $f \geqslant 0$ is slack on a set A if $\{x \in A: f(x) > 0\} \neq \emptyset$; otherwise it is singular. Clearly if $B_j X = B_{j0}$ is a consequence of $CX = C_0$, then the inequality $B_j X \geqslant B_0$ is singular on $\mathcal{R}(S)$. Thus it is often natural to consider only systems (S) such that $\mathcal{R}(S) \neq \emptyset$ and for all $j = 1, \ldots, m$, $B_j X = B_{j0}$ is not a consequence of $CX = C_0$. Still some singular inequalities might arise; the following criterion is useful in this sense:

Every $B_j X \geqslant B_{j0}$ is slack on $\mathcal{R}(S)$ iff $CX = C_0$ is the affine hull of $\mathcal{R}(S)$ iff the closure of $\mathcal{R}(S)$ is equal to the closure of $\mathcal{R}(S')$, where

$$(\mathrm{S}') \qquad AX > A_0, \quad BX > B_0, \quad CX = C_0.$$

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If we consider systems (S) of inequalities of higher degree it is clear that we can find counter examples to every statement mentioned in the preceeding section, even if we change in the above criterion the affine hull of $\mathcal{R}(S)$ by the Zariski closure (or algebraic hull of $\mathcal{R}(S)$).

To fix some notation, let (S) be a system of polynomial equations and inequalities

$$\begin{aligned} &Q_i(x_1, \ldots, x_n) > 0, && i = 1, \ldots, r, \\ &P_j(x_1, \ldots, x_n) \geqslant 0, && j = 1, \ldots, s, \\ &H_k(x_1, \ldots, x_n) = 0, && k = 1, \ldots, m; \end{aligned}$$

we assume that the set $\mathcal{R}(S)$ of points in R^n defined as above is non-empty.

Further we suppose that the algebraic set given by $(H_1, \ldots, H_m)$ is irreducible (otherways we can decompose (S) as union of systems with this property). We omit the inequalities $P_i \geqslant 0$ such that $P_i = 0$ on $\{H_1 = 0, \ldots, H_m = 0\}$ (i.e., the P_i's that belong to the real radical of $(H_1, \ldots, H_m)$). Then we say ([6], [15]) that the (remaining) set $\{P_1 \geqslant 0, \ldots, P_l \geqslant 0\}$ is *locally* slack on $\mathcal{R}(S)$ if for every point $(x) \in \mathcal{R}(S)$ and every neighborhood $W(x)$, the Zariski closure of $W(x) \cap \mathcal{R}(S)$ has equations $H_1 = 0, \ldots, H_m = 0$. Locally slack systems enjoy some good properties of the linear case: first, the closure in $\mathcal{R}(S)$ of $\mathcal{R}(S')$, where

$$(\mathrm{S}') \quad Q_i > 0, P_j > 0, H_k = 0, \qquad i = 1, \ldots, r, \; j = 1, \ldots, l, \; k = 1, \ldots, m,$$

is equal to $\mathcal{R}(S)$; second, the algebraic set $\{H_1 = 0, \ldots, H_m = 0\}$ gives the Zariski closure of $\mathcal{R}(S)$; third, every $P_j \geqslant 0$ is slack on $\mathcal{R}(S)$. Now, using Thom's Lemma, it can be shown that *every semialgebraic set can be written as a finite union of sets of the form $\mathcal{R}(S)$, with the corresponding weak inequalities locally slack on $\mathcal{R}(S)$.*

For applications of this type of decomposition to the expression of non-negative functions over semialgebric sets as sums of squares, see Dubois and Recio [1984].

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