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Introduction

In the present paper we deal with the functional inequalities

$$(0.1) \quad \psi[f(x)] \leq g[x, \psi(x)]$$

and

$$(0.2) \quad \psi[f(x)] \geq g[x, \psi(x)],$$

where f and g are given functions and ψ is an unknown function. We shall assume throughout this paper that there exists a solution of the functional equation

$$(0.3) \quad \varphi[f(x)] = g[x, \varphi(x)],$$

where φ is an unknown function. Thus, the two inequalities (0.1) and (0.2) always have a solution, because the solution of equation (0.3) satisfies (0.1) as well as (0.2).

The greater part of this paper will concern a comparison between solutions of an inequality and solutions of an equation. The approach to the problem of functional inequalities presented in this paper has been based on that of the theory of differential inequalities in Szarski's monograph [7].

A special case of functional inequalities (0.1) and (0.2) has been considered by Rumak [6]. Some theorems concerning a system of difference inequalities have been proved. In a special case of one difference inequality, the results given in [6] turn out to be special cases of the theorems which will be proved in Section 2 of this paper. So far we have not met any other results, except T. Rumak's paper, concerning functional inequalities in a single variable of first order.

In the present paper we shall deal, in general, with the continuous solutions of inequalities (0.1) and (0.2). It is only in Section 2 that the solutions will be considered without any assumptions about their continuity.

1. Preliminaries

The given functions f and g will be subjected to certain conditions.

HYPOTHESIS H_1 . *The function f is defined and strictly increasing in an interval $I = [\xi, b)$ and fulfils the condition*

$$(1.1) \quad f(\xi) = \xi, \quad f(x) < x \quad \text{for} \quad x \in (\xi, b).$$

HYPOTHESIS H₂. *The function f is continuous in I .*

Remark 1. The number ξ , as well as b , may be infinite. We say that a function ψ is *continuous at infinity* if there exists a finite limit $\lim_{x \rightarrow \xi} \psi(x)$. This limit will be considered as the value of function ψ at infinity. If $\xi = -\infty$, then $I = (-\infty, b)$.

Remark 2. Hypotheses H₁ and H₂ imply that ξ is an attractive fixed point of the function f in I , i.e.,

$$(1.2) \quad \lim_{n \rightarrow \infty} f^n(x) = \xi \quad \text{for } x \in I$$

(see [4]). Here $f^n(x)$ denotes the n -th iterate of the function f , i.e., $f^0(x) = x$, $f^{n+1}(x) = f[f^n(x)]$ for $n = 0, 1, \dots$. Hypotheses H₁ and H₂ also imply that ξ is the only fixed point of f in I .

Remark 3. The results obtained in this paper can also be applied to the case where the fixed point ξ is the right endpoint of the interval I , after a slight modification. Namely, in hypothesis H₁ we have to assume that it is the inequality $f(x) > x$ holds in I instead of the inequality occurring in condition (1.1). In particular, this is a case where one deals with the difference inequalities.

HYPOTHESIS H₃. *The function g is a real-valued function defined in a set $\Omega \subset I \times E$ and taking values in E , where E is a subset of the real line. Moreover, for every $x \in I$, the function g is strictly increasing with respect to the second variable and, for every $x \in I$, the following condition is fulfilled:*

$$\Gamma_x \subset \Omega_{f(x)},$$

where

$$(1.3) \quad \Omega_x = \{y: x, y \in \Omega\}$$

and Γ_x is the set of values of the function g for $y \in \Omega_x$.

Since, as has been mentioned in the introduction, we require that a solution φ of equation (0.3) should exist in I , the value η_0 of φ at the point ξ has to satisfy the condition

$$(1.4) \quad \eta_0 = g(\xi, \eta_0)$$

(see [4]).

When we consider continuous solutions of inequalities (0.1) and (0.2), we have to assume the following

HYPOTHESIS H₄.

- (i) *The function g is continuous in Ω .*
- (ii) *For every $x \in I$, the x -section of Ω , defined by formula (1.3), is an open interval.*
- (iii) *$\Gamma_x = \Omega_{f(x)}$ for every $x \in I$.*
- (iv) *$(\xi, \eta_0) \in \Omega$, where η_0 satisfies equality (1.4).*

We shall consider the solutions ψ of inequalities (0.1) and (0.2) and solutions φ of equation (0.3) such that their graphs lie in Ω , i.e.,

$$(1.5) \quad \psi(x), \varphi(x) \in \Omega_x \quad \text{for } x \in I.$$

2. Non-linear inequalities

In this section we are going to prove some general properties of solutions of functional inequalities (0.1) and (0.2).

THEOREM 2.1. *Let hypotheses H_1 and H_3 be fulfilled, let ψ be a solution of inequality (0.1) in I and let φ be a solution of equation (0.3) in I .*

(i) *If $x_0 \in (\xi, b)$ and*

$$(2.1) \quad \psi(x_0) < \varphi(x_0),$$

then

$$(2.2) \quad \psi[f^n(x_0)] < \varphi[f^n(x_0)] \quad \text{for } n = 0, 1, \dots$$

(ii) *If $x_0 \in I$ and*

$$(2.3) \quad \psi(x_0) \leq \varphi(x_0),$$

then

$$(2.4) \quad \psi[f^n(x_0)] \leq \varphi[f^n(x_0)] \quad \text{for } n = 0, 1, \dots$$

Proof. Since the proofs of both parts of the theorem are very similar, we are going to prove only the first part by induction. By virtue of (0.1), (2.1), hypothesis H_3 and (0.3), we obtain

$$\psi[f(x_0)] \leq g[x_0, \varphi(x_0)] < g[x_0, \varphi(x_0)] = \varphi[f(x_0)],$$

thus inequality (2.2) holds for $n = 1$.

Let us suppose that inequality (2.2) holds for a positive integer k . Then, putting $f^k(x_0)$ in place of x_0 in (2.1), we find in a similar way that inequality (2.2) holds for $n = k + 1$, which completes the inductive proof.

An immediate consequence of Theorem 2.1 and (1.2) is the following

COROLLARY 2.1. *Let hypotheses H_1 – H_3 be fulfilled, let ψ be a solution of inequality (0.1) in I and let φ be a solution of equation (0.3) in I . Moreover, let the functions ψ and φ be continuous at the point ξ . If there exists such a point $x_0 \in (\xi, b)$ that inequality (2.3) holds, then $\psi(\xi) \leq \varphi(\xi)$.*

Another consequence of Theorem 2.1 is the following

COROLLARY 2.2. *Let hypotheses H_1 and H_3 be fulfilled, let ψ be a solution of inequality (0.1) in I and let φ be a solution of equation (0.3) in I .*

(i) *If $x_0 \in (\xi, b)$ and*

$$(2.5) \quad \psi(x_0) > \varphi(x_0),$$

then

$$(2.6) \quad \psi[f^{-n}(x_0)] > \varphi[f^{-n}(x_0)]$$

for every positive integer n satisfying the condition

$$(2.7) \quad f^{-n}(x_0) \in I.$$

(ii) If $x_0 \in I$ and $\psi(x_0) \geq \varphi(x_0)$, then

$$\psi[f^{-n}(x_0)] \geq \varphi[f^{-n}(x_0)]$$

for every positive integer n satisfying condition (2.7).

Proof. Since both parts of the corollary can be proved in the same way, we are going to prove the first part only.

Let inequality (2.5) hold and let us suppose that there exists such a positive integer n satisfying condition (2.7) that

$$\psi[f^{-n}(x_0)] \leq \varphi[f^{-n}(x_0)].$$

Putting

$$(2.8) \quad f^{-n}(x_0) = t,$$

we have $\psi(t) \leq \varphi(t)$, whence, by virtue of (2.8) and Theorem 2.1, we obtain

$$\psi(x_0) = \psi[f^n(t)] \leq \varphi[f^n(t)] = \varphi(x_0),$$

contrary to (2.5). Thus inequality (2.6) is proved.

Let x_0 be an inner point of the interval I . Putting

$$I_k = [f^{k+1}(x_0), f^k(x_0)] \quad \text{for } k = 0, \pm 1, \pm 2, \dots,$$

we have

$$(\xi, x_0] = \bigcup_{n=0}^{\infty} I_n, \quad [f(x_0), b) = \bigcup_{n=0}^{K+2} I_{-n} \cup [f^{-(K+1)}(x_0), b),$$

where K is the greatest non-negative integer such that $f^{-(K+1)}(x_0) \in I$.

If such an integer K does not exist, then b has to be repulsive fixed point of the function f . If this is the case, we put $K = \infty$ (see [4]). As an immediate consequence of Theorem 2.1 and Corollary 2.2, we obtain the following

COROLLARY 2.3. *Let hypotheses H_1 – H_3 be fulfilled, let ψ be a solution of inequality (0.1) in I and let φ be a solution of equation (0.3) in I . Moreover, let $x_0 \in (\xi, b)$.*

(i) If

$$\psi(x) < \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) < \varphi(x) \quad \text{for } x \in (\xi, x_0].$$

(ii) *If*

$$\psi(x) \leq \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) \leq \varphi(x) \quad \text{for } x \in (\xi, x_0].$$

(iii) *If*

$$\psi(x) > \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) > \varphi(x) \quad \text{for } x \in [f(x_0), b].$$

(iv) *If*

$$\psi(x) \geq \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) \geq \varphi(x) \quad \text{for } x \in [f(x_0), b].$$

Similar results can be obtained in a similar way for the functional inequality (0.2). They are put together in the following

THEOREM 2.2. *Let hypotheses H_1 and H_3 be fulfilled, let ψ be a solution of inequality (0.2) in I and let φ be a solution of equation (0.3) in I . Moreover, let x_0 be an inner point of the interval I .*

(i) *If $\psi(x_0) > \varphi(x_0)$, then*

$$\psi[f^n(x_0)] > \varphi[f^n(x_0)] \quad \text{for } n = 0, 1, \dots$$

(ii) *If $\psi(x_0) \geq \varphi(x_0)$, then*

$$\psi[f^n(x_0)] \geq \varphi[f^n(x_0)] \quad \text{for } n = 0, 1, \dots$$

(iii) *If $\psi(x_0) < \varphi(x_0)$, then*

$$\psi[f^{-n}(x_0)] < \varphi[f^{-n}(x_0)]$$

for every positive integer n satisfying condition (2.7).

(iv) *If $\psi(x_0) \leq \varphi(x_0)$, then*

$$\psi[f^{-n}(x_0)] \leq \varphi[f^{-n}(x_0)]$$

for every positive integer n satisfying condition (2.7).

(v) *If hypothesis H_2 is fulfilled and*

$$\psi(x) > \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) > \varphi(x) \quad \text{for } x \in (\xi, b).$$

(vi) *If hypothesis H_2 is fulfilled and*

$$\psi(x) \geq \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) \geq \varphi(x) \quad \text{for } x \in (\xi, x_0].$$

(vii) If hypothesis H_2 is fulfilled and

$$\psi(x) < \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) < \varphi(x) \quad \text{for } x \in [f(x_0), b).$$

(viii) If hypothesis H_2 is fulfilled and

$$\psi(x) \leq \varphi(x) \quad \text{for } x \in [f(x_0), x_0],$$

then

$$\psi(x) \leq \varphi(x) \quad \text{for } x \in [f(x_0), b).$$

It is also very easy to prove an analogue of Corollary 2.1:

COROLLARY 2.4. *Let hypotheses H_1 – H_4 be fulfilled, let ψ be a solution of inequality (0.2) in I and let φ be a solution of equation (0.3) in I . Moreover, let the functions ψ and φ be continuous at the point ξ . If there exists such a point $x_0 \in (\xi, b)$ that $\psi(x_0) \geq \varphi(x_0)$, then*

$$\psi(\xi) \geq \varphi(\xi).$$

Let the functions ψ_1 and ψ_2 be defined in the interval I . In the sequel we are going to denote

$$(2.9) \quad \psi_1 \cup \psi_2(x) = \max[\psi_1(x), \psi_2(x)] \quad \text{for } x \in I,$$

$$(2.10) \quad \psi_1 \cap \psi_2(x) = \min[\psi_1(x), \psi_2(x)] \quad \text{for } x \in I.$$

THEOREM 2.3. *Let hypotheses H_1 and H_3 be fulfilled.*

(i) *If ψ_1 and ψ_2 satisfy inequality (0.1) in I , then $\psi_1 \cup \psi_2$ and $\psi_1 \cap \psi_2$ also satisfy (0.1) in I .*

(ii) *If ψ_1 and ψ_2 satisfy inequality (0.2) in I , then $\psi_1 \cup \psi_2$ and $\psi_1 \cap \psi_2$ also satisfy (0.2) in I .*

(iii) *If φ_1 and φ_2 satisfy equation (0.3) in I , then $\varphi_1 \cup \varphi_2$ and $\varphi_1 \cap \varphi_2$ also satisfy (0.3) in I .*

Proof. Let us assume that ψ_1 and ψ_2 satisfy inequality (0.1) in I and let $x \in I$. If

$$(2.11) \quad \psi_1(x) \leq \psi_2(x) \quad \text{and} \quad \psi_1[f(x)] \leq \psi_2[f(x)],$$

then, by virtue of (2.9), we have

$$\psi_1 \cup \psi_2(x) = \psi_2(x) \quad \text{and} \quad \psi_1 \cup \psi_2[f(x)] = \psi_2[f(x)],$$

whence $\psi_1 \cup \psi_2$ satisfies inequality (0.1) at x , because ψ_2 satisfies inequality (0.1) at x . Inequalities (2.11) imply, by virtue of (2.10), that

$$\psi_1 \cap \psi_2(x) = \psi_1(x) \quad \text{and} \quad \psi_1 \cap \psi_2[f(x)] = \psi_1[f(x)],$$

whence $\psi_1 \cap \psi_2$ satisfies inequality (0.1) at x , because ψ_1 satisfies inequality (0.1) at x .

If, instead of inequalities (2.11), the opposite inequalities hold, then the proof is identical.

Let us assume that

$$(2.12) \quad \psi_1(x) \leq \psi_2(x)$$

and

$$(2.13) \quad \psi_1[f(x)] \geq \psi_2[f(x)],$$

It follows from (2.9), (2.12) and (2.13) that

$$\psi_1 \cup \psi_2[f(x)] = \psi_1[f(x)] \quad \text{and} \quad \psi_1 \cup \psi_2(x) = \psi_2(x).$$

Hence, by virtue of (2.9) and (0.1),

$$\psi_1 \cup \psi_2[f(x)] = \psi_1[f(x)] \leq g[x, \psi_1(x)] \leq g[x, \psi_2(x)] = g[x, \psi_1 \cup \psi_2(x)],$$

because the function g is strictly increasing with respect to the second variable by virtue of hypothesis H_3 . Thus the function $\psi_1 \cup \psi_2$ satisfies inequality (0.1). It also follows from (2.10), (2.12), (2.13) and hypothesis H_3 that

$$\psi_1 \cap \psi_2[f(x)] = \psi_2[f(x)] \quad \text{and} \quad \psi_1 \cap \psi_2(x) = \psi_1(x).$$

Hence, by virtue of (2.10) and (0.1),

$$\psi_1 \cap \psi_2[f(x)] = \psi_2[f(x)] \leq \psi_1[f(x)] \leq g[x, \psi_1(x)] = g[x, \psi_1 \cap \psi_2(x)],$$

thus the function $\psi_1 \cap \psi_2$ also satisfies inequality (0.1). If we replace inequalities (2.12) and (2.13) by the opposite inequalities the proof is similar. The first part of the theorem is proved. The proof of the second part is quite similar.

Now, we are going to prove the last part of the theorem. Let φ_1 and φ_2 satisfy equation (0.3) in I and let $x \in I$. Let us also assume that the inequality

$$(2.14) \quad \varphi_1(x) \leq \varphi_2(x)$$

holds. Since φ_1 satisfies equation (0.3), it also satisfies inequality (0.1). Thus we may apply Theorem 2.1, which implies that

$$\varphi_1[f(x)] \leq \varphi_2[f(x)],$$

because φ_2 is a solution of equation (0.3) in I . Hence, by virtue of (2.9), (0.3) and (2.14), we obtain

$$\varphi_2[f(x)] = \varphi_2[f(x)] = g[x, \varphi_2(x)] = g[x, \varphi_1 \cup \varphi_2(x)].$$

The last equalities prove that the function $\varphi_1 \cup \varphi_2$ satisfies equation (0.3) at x . The proof for $\varphi_1 \cap \varphi_2$ is similar.

It is well known that every family of functions defined in an common set is a distributive lattice with respect to the operations \cup and \cap (see [8]). Thus, as an immediate consequence of Theorem 2.3, we obtain the following

THEOREM 2.4. *Let hypotheses H_1 and H_3 be fulfilled. Then each of the families of functions defined in I , namely:*

the family of all solutions of inequality (0.1) in I ,

the family of all solutions of inequality (0.2) in I , and

the family of all solutions of equation (0.3) in I ,

is a distributive lattice with respect to the operations \cup and \cap .

Since in next sections of this paper we shall deal with the continuous solutions of inequalities (0.1) and (0.2), it will be useful to adopt the following

DEFINITION 2.1. Let hypotheses H_1 – H_4 be fulfilled. We denote by L the family of all continuous solutions of inequality (0.1) in I , by L' the family of all continuous solutions of inequality (0.2) in I and by L_0 the family of all continuous solutions of equation (0.3) in I .

Since the operations \cup and \cap preserve the continuity of functions, we obtain, as an immediate consequence of Theorem 2.4, the following

THEOREM 2.5. *If hypotheses H_1 – H_4 are fulfilled, then each of the families L , L' and L_0 is a distributive lattice with respect to the operations \cup and \cap .*

3. Continuous solutions of homogeneous inequalities

In this section we shall deal with the homogeneous inequalities

$$(3.1) \quad \psi[f(x)] \leq g(x)\psi(x),$$

$$(3.2) \quad \psi[f(x)] \geq g(x)\psi(x),$$

where f and g are given functions and ψ is an unknown function. The function f will be subjected to the same conditions as in preceding sections. As regards the function g we shall assume the following

HYPOTHESIS H_5 . *The function g is defined and continuous in I and $g(x) > 0$ for $x \in I$.*

Under hypothesis H_5 we are allowed to use all the results of the preceding sections by virtue of the following

LEMMA 3.1. *If hypothesis H_5 is fulfilled, then the function $g(x, y) = g(x)y$ fulfils hypotheses H_3 and H_4 in $\Omega = I \times \mathbf{R}$.*

This lemma is very easy to verify.

Inequalities (3.1) and (3.2) are related to the homogeneous functional equation

$$(3.3) \quad \varphi[f(x)] = g(x)\varphi(x),$$

where φ is an unknown function. The results concerning the continuous solutions of the homogeneous inequalities, as well as the solutions of the homogeneous equation, depend on the behaviour of the following sequence

$$(3.4) \quad G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \quad x \in I, \quad n = 1, 2, \dots$$

It is easy to verify the following formula

$$(3.5) \quad G_{n+1}(x) = g(x)G_n[f(x)], \quad x \in I, \quad n = 1, 2, \dots$$

In the sequel, we shall consider the homogeneous inequalities in the following three cases:

(i) The limit

$$(3.6) \quad G(x) = \lim_{n \rightarrow \infty} G_n(x)$$

exists in I and G is continuous in I . Moreover, $G(x) \neq 0$ in I .

(ii) $\lim_{n \rightarrow \infty} G_n(x) = \infty$ for $x \in I$.

(iii) There exists an interval $J \subset I$ such that

$$(3.7) \quad \lim_{n \rightarrow \infty} G_n(x) = 0$$

uniformly in J .

We shall discuss the problem of homogeneous inequalities on the basis of the theory of the homogeneous equation (3.3). The results of this theory one can be found in [4]. Since we shall use those results many times in the present paper, it will be useful to quote them here as the following

LEMMA 3.2. *Let hypotheses H_1, H_2 and H_5 be fulfilled.*

In case (i) equation (3.3) has a one-parameter family of continuous solutions in I : for every real number c there exists exactly one continuous function φ satisfying (3.3) in I and fulfilling the condition

$$(3.8) \quad \varphi(\xi) = c.$$

This solution is given by

$$(3.9) \quad \varphi(x) = \frac{c}{G(x)} \quad \text{for } x \in I.$$

In case (ii) the function $\varphi(x) = 0$ for $x \in I$ is the only continuous solution of equation (3.3) in I .

In case (iii) equation (3.3) has in I a continuous solution depending on an arbitrary function. In this case every continuous solution φ of equation (3.3) in I fulfils the condition

$$(3.10) \quad \varphi(\xi) = 0.$$

There is a possibility that none of three cases (i), (ii), (iii), occurs. Then the function which is identically equal to zero in I is still the only continuous solution of equation (3.3) in I . We are not going to deal with such a case in this paper.

Let us notice that hypothesis H_5 implies that the function G , given by (3.6), has to be positive in I , as well as the functions G_n , given by (3.4). This enables us to prove the following

LEMMA 3.3. *Let hypotheses H_1 , H_2 and H_5 be fulfilled. If the function ψ satisfies inequality (3.1) in I , then*

$$(3.11) \quad \psi[f^n(x)] \leq G_n(x)\psi(x) \quad \text{for } x \in I, n = 1, 2, \dots$$

If the function ψ satisfies inequality (3.2) in I , then

$$(3.12) \quad \psi[f^n(x)] \geq G_n(x)\psi(x) \quad \text{for } x \in I, n = 1, 2, \dots$$

If the function φ satisfies equation (3.3) in I , then

$$(3.13) \quad \varphi[f^n(x)] = G_n(x)\varphi(x) \quad \text{for } x \in I, n = 1, 2, \dots$$

We omit the simple inductive proof of Lemma 3.3.

THEOREM 3.1. *Let hypotheses H_1 , H_2 and H_5 be fulfilled and let case (i) occur. If ψ is a continuous solution of inequality (3.1) in I , then*

$$(3.14) \quad \psi(x) \geq \varphi_0(x) \quad \text{for } x \in I,$$

where φ_0 is the continuous solution of equation (3.3) in I given by the formula

$$(3.15) \quad \varphi_0(x) = \frac{\psi(\xi)}{G(x)} \quad \text{for } x \in I.$$

The solution φ_0 satisfies the following conditions:

$$(3.16) \quad \varphi_0(\xi) = \psi(\xi)$$

and if φ is such a solution of equation (3.3) continuous in I that

$$(3.17) \quad \varphi(x_0) > \varphi_0(x_0)$$

for an $x_0 \in I$, then

$$(3.18) \quad \varphi(\xi) > \psi(\xi).$$

Proof. Condition (3.16) follows from Lemma 3.2. Now we are going to prove inequality (3.14). Let φ be a continuous solution of inequality (3.1) in I . It follows from Lemma 3.3 that

$$\frac{\psi[f^n(x)]}{G_n(x)} \leq \psi(x) \quad \text{for } x \in I, n = 1, 2, \dots,$$

whence, by virtue of (1.2) and (3.6), we obtain

$$\frac{\psi(\xi)}{G(x)} \leq \psi(x) \quad \text{for } x \in I,$$

because of the continuity of ψ . Thus inequality (3.14) has been proved.

Let us assume that φ is such a continuous solution of equation (3.3) that there exists such an $x_0 \in I$ that the inequality holds. Since the family of all continuous solutions of equation (3.3) is a one-parameter family in case (i) by virtue of Lemma 3.2, then inequality (3.17) implies the inequality $\varphi(x) > \varphi_0(x)$ for $x \in I$. The last inequality together with condition (3.16), which has already been proved, implies (3.18), which completes the proof of the theorem.

By virtue of Theorem 3.1, φ_0 is the greatest continuous solution of equation (3.3), which satisfies inequality (3.14) for a given ψ .

Theorem 3.1 given an analogue of the comparison theorems of the theory of differential inequalities. Namely, as an obvious consequence of Theorem 3.1, we obtain the following

THEOREM 3.2. *Let hypotheses H_1 , H_2 and H_5 be fulfilled and let case (i) occur. Moreover, let ψ be a continuous solution of inequality (3.1) in I and let φ be a continuous solution of equation (3.3) in I . If $\psi(\xi) \geq \varphi(\xi)$, then $\psi(x) \geq \varphi(x)$ for $x \in I$.*

In a similar way one can obtain analogous results for the functional inequality (3.2). We give them here without proof:

THEOREM 3.3. *Let hypotheses H_1 , H_2 and H_5 be fulfilled and let case (i) occur. If ψ is a continuous solution of inequality (3.2) in I , then*

$$(3.19) \quad \psi(x) \leq \varphi_0(x) \quad \text{for } x \in I,$$

where φ_0 is the continuous solution of equation (3.3) in I given by formula (3.15) and satisfying (3.16). Moreover, if φ is such a continuous solution of equation (3.3) that $\varphi(x_0) < \varphi_0(x_0)$ for an $x_0 \in I$, then $\varphi(\xi) < \psi(\xi)$.

Similar to Theorem 3.1, Theorem 3.3 means that φ_0 is the least continuous solution of equation (3.3) in I which satisfies inequality (3.19) for a given ψ .

THEOREM 3.4. *Let hypotheses H_1 , H_2 and H_5 be fulfilled and let case (i) occur. Moreover, let ψ be a continuous solution of inequality (3.2) in I and let φ be a continuous solution of equation (3.3) in I . If $\psi(\xi) \leq \varphi(\xi)$, then $\psi(x) \leq \varphi(x)$ for $x \in I$.*

Since equation (3.3) has, in case (i), a one-parameter family of continuous solutions in I (see Lemma 3.2), then for every point $(x, y) \in \Omega$, there exists exactly one continuous solution of (3.3) passing through that point. This justifies the following

DEFINITION 3.1. Let a function ψ be defined in I . If $x \in I$, we denote by $\eta(x)$ the value at ξ of continuous solution φ of equation (3.3), which satisfies the condition

$$(3.20) \quad \varphi(x) = \psi(x).$$

It is obvious that

$$(3.21) \quad \psi(x) = \frac{\eta(x)}{G(x)} \quad \text{for } x \in I,$$

by virtue of Lemma 3.2. Formula (3.21) and Lemma 3.2 immediately imply the following

LEMMA 3.4. *If hypotheses H_1 , H_2 and H_5 are fulfilled and case (i) occurs, then the function ψ is continuous in I , if and only if the function η is continuous in I .*

Here we give the definition of $\{f\}$ -monotonic (see [4]), which will be needed in the sequel:

DEFINITION 3.2. The function η defined in I will be called $\{f\}$ -increasing in I if the inequality

$$(3.22) \quad \eta[f(x)] \geq \eta(x) \quad \text{for } x \in I$$

holds.

The function η defined in I will be called $\{f\}$ -decreasing in I if the inequality

$$(3.23) \quad \eta[f(x)] \leq \eta(x) \quad \text{for } x \in I$$

holds.

We are going to show that there is a one-to-one correspondence between the family of $\{f\}$ -decreasing functions continuous in I and the family of continuous solutions of functional inequality (3.1), as well as between the family of $\{f\}$ -increasing functions continuous in I and the family of continuous solutions of functional inequality (3.2) in I . Namely, we are going to prove the following

THEOREM 3.5. *Let hypotheses H_1 , H_2 and H_5 be fulfilled and let case (i) occur. If ψ is a function defined and continuous in I , then*

1° *the necessary and sufficient condition for ψ to be a continuous solution of inequality (3.1) in I is that the function η , defined by Definition 3.1, be an $\{f\}$ -decreasing continuous function in I ;*

2° *the necessary and sufficient condition for ψ to be a continuous solution of inequality (3.2) in I is that the function η , defined by Definition 3.1, be an $\{f\}$ -increasing continuous function in I .*

Proof. We are going to prove the first part of the theorem. The proof of the second part is similar.

Let ψ be a continuous solution of inequality (3.1) in I . It follows from (3.21) and (3.1) that

$$\frac{\eta[f(x)]}{G[f(x)]} = \psi[f(x)] \leq g(x)\psi(x) = g(x) \frac{\eta(x)}{G(x)} = g(x) \frac{\eta(x)}{g(x)G[f(x)]}$$

for $x \in I$, by virtue of (3.5), (3.6) and because the function G is positive in I . Hence inequality (3.23) follows.

Conversely, let us assume that (3.23) holds. It follows from (3.21), (3.23), (3.5) and (3.6) that

$$\psi[f(x)] = \frac{\eta[f(x)]}{G[f(x)]} \leq \frac{\eta(x)}{G[f(x)]} = g(x) \frac{\eta(x)}{G(x)} = g(x)\psi(x) \quad \text{for } x \in I,$$

whence inequality (3.1) follows. Since the function ψ is continuous if and only if the function η is continuous, by virtue of (3.21) and the continuity of the function G , the proof is completed.

Since the function η defined by Definition 3.1, is determined uniquely by formula (3.21), we obtain from Theorem 3.5 the one-to-one correspondence between the family of continuous solutions of inequality (3.1) in I and the family of all continuous $\{f\}$ -decreasing functions. More precisely, Theorem 3.5 immediately implies the following

COROLLARY 3.1. *Let hypotheses H_1, H_2 and H_3 be fulfilled and let case (i) occur. The one-to-one correspondence between the family of continuous solutions ψ of inequality (3.1) in I taking values in E (the family of continuous solutions of inequality (3.2) in I taking values in E) and the family of continuous functions η , $\{f\}$ -decreasing in I ($\{f\}$ -increasing in I) and taking values in E , is given by the formula*

$$(3.24) \quad \eta(x) = G(x)\psi(x) \quad \text{for } x \in I.$$

Let us notice that if ψ is a continuous solution of inequality (3.1) in I , then Theorem 3.1 implies that if

$$(3.25) \quad \psi(\xi) > 0,$$

then

$$(3.26) \quad \psi(x) > 0 \quad \text{for } x \in I.$$

Thus, if η is an $\{f\}$ -decreasing ($\{f\}$ -increasing) function in I and (3.25) holds, then the function

$$(3.27) \quad \bar{\eta}(x) = \frac{\eta(x)}{\psi(\xi)} \quad \text{for } x \in I$$

is also an $\{f\}$ -decreasing ($\{f\}$ -increasing) function in I . It is very easy to see, in view of (3.25), that

$$(3.28) \quad \bar{\eta}(\xi) = 1.$$

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Similarly, Theorem 3.3 implies that if ψ is a continuous solution of inequality (3.2) in I satisfying the condition

$$(3.29) \quad \psi(\xi) < 0,$$

then

$$(3.30) \quad \psi(x) < 0 \quad \text{for} \quad x \in I.$$

Thus Corollary 3.1 implies immediately the following

COROLLARY 3.2. *Let hypotheses H_1 , H_2 and H_3 be fulfilled and let case (i) occur. The one-to-one correspondence between the family of continuous solutions ψ of inequality (3.1) in I , satisfying condition (3.25) (the family of continuous solutions of inequality (3.2), satisfying condition (3.29)) and taking values in E , and the family of continuous functions $\bar{\eta}$, $\{f\}$ -decreasing in I , satisfying condition (3.28) and taking values in E , is given by the formula*

$$(3.31) \quad \bar{\eta}(x) = \frac{\psi(x)}{\psi(\xi)} G(x) \quad \text{for} \quad x \in I.$$

The one-to-one correspondence between the family of continuous solutions ψ of inequality (3.1) in I , satisfying condition (3.30) (the family of continuous solutions of inequality (3.2) in I , satisfying condition (3.26)) taking values in E , and the family of continuous functions $\bar{\eta}$, $\{f\}$ -increasing in I , satisfying condition (3.28) taking values in E , is given by formula (3.31).

Since inequality (3.23) is a special case of inequality (3.1), then Theorem 3.2 implies that if $\eta(\xi) \geq 0$, then $\eta(x) \geq 0$ for $x \in I$. It follows from condition (3.21) that $\eta(\xi) = 0$, if and only if $\psi(\xi) = 0$. Thus Theorem 3.5 and Corollaries 3.1 and 3.2 enables us to reduce the investigation of solutions of inequality (3.1) to a special case of inequality (3.23) in the general case. In case where we restrict ourselves to the investigation of positive solutions of (3.1) only, this investigation can be reduced to the investigation of solutions of (3.23) such that either $\eta(\xi) = 0$, or $\eta(\xi) = 1$. In case of negative solutions of inequality (3.1) we can restrict ourselves to the investigation of solutions of inequalities (3.22) and (3.23) satisfying condition (3.28).

Inequalities (3.22) and (3.23) are related to the functional equation

$$(3.32) \quad \varphi[f(x)] = \varphi(x), \quad x \in I.$$

It is easy to see that this equation has a coefficient $g(x)$ equal to 1 identically in I , thus case (i) occurs. It follows from Lemma 3.2 that equation (3.32) has a one-parameter family of continuous solutions. This is the family of constant functions in I . Thus, by virtue of Theorem 3.2, we obtain

$$(3.33) \quad \eta(x) \geq \eta(\xi) \quad \text{for} \quad x \in I.$$

Thus Theorem 3.1, Corollary 3.1, and (3.33) imply the following

COROLLARY 3.3. *Let hypotheses H_1, H_2 and H_5 be fulfilled and let case (i) occur. Moreover, let us assume that ψ is a continuous solution of inequality (3.1) in I and*

$$(3.34) \quad \lim_{x \rightarrow b} \frac{1}{G(x)} = \infty,$$

where G is defined by formula (3.6).

If ψ satisfies condition (3.25), then

$$\lim_{x \rightarrow b} \psi(x) = \infty.$$

If ψ is a positive function bounded from above in I , then $\psi(\xi) = 0$.

COROLLARY 3.4. *Let hypotheses H_1, H_2 and H_5 be fulfilled and let case (i) occur. Moreover, let b be a repulsive fixed point of the function f in I and let (3.34) hold. If ψ is a continuous solution of inequality in I , satisfying (3.25) and bounded from above in I , then $\psi(x) = 0$ for $x \in I$.*

Proof. Let us suppose that ψ is a continuous solution of (3.1) in I , satisfying (3.25). Let x_0 be such an inner point of the interval I that

$$(3.35) \quad \psi(x_0) > 0.$$

Moreover, let φ be a continuous solution of equation (3.3) in I , passing through the point $(x_0, \psi(x_0))$, i.e.,

$$(3.36) \quad \varphi(x_0) = \psi(x_0).$$

Since equation (3.3) has the one-parameter family of continuous solutions in I on account of Lemma 3.2, then (3.35) implies that $\varphi(\xi) > 0$, whence

$$(3.37) \quad \lim_{x \rightarrow b} \varphi(x) = \infty,$$

by virtue of Lemma 3.2 and (3.34). It follows from Lemma 3.1, (3.36), Lemma 2.2 and (3.37) that

$$\lim_{n \rightarrow \infty} \psi[f^{-n}(x_0)] = \infty,$$

because for every positive integer n , $f^{-n}(x_0) \in I$ and the sequence $f^{-n}(x_0)$ converges to b for every $x_0 \in I$, $x_0 \neq \xi$ (see [4]). The last condition implies that if (3.35) holds, then the function ψ has to be unbounded from above in I . Therefore any solution ψ bounded from above has to satisfy the condition

$$\psi(x) < 0 \quad \text{for} \quad x \in I.$$

Hence, on account of (3.25) and Theorem 3.2, $\psi(x) = 0$ in I , which completes the proof.

In a similar way one can obtain results for inequality (3.2) analogous to the last two corollaries. These results are presented here as the following

COROLLARY 3.5. *Let hypotheses H_1, H_2 and H_3 be fulfilled and let case (i) occur. Moreover, let ψ be a continuous solution of inequality (3.2) in I and let (3.34) hold.*

If ψ satisfies (3.29), then $\lim_{x \rightarrow b} \psi(x) = -\infty$.

If ψ is negative function bounded from below in I , then (3.32) holds.

If ψ is bounded from below in I and satisfies (3.29) and if b is a repulsive fixed point of the function f in I , then $\psi(x) = 0$ for $x \in I$.

We are going to illustrate the foregoing results by examples.

EXAMPLE 3.1. The inequality

$$(3.38) \quad \psi(x/2) \leq e^{-x/2} \psi(x) \quad \text{for } x \in [0, b),$$

where b is an arbitrary positive number, is related to the equation

$$(3.39) \quad \varphi(x/2) = e^{-x/2} \varphi(x) \quad \text{for } x \in [0, b).$$

The continuous solutions of (3.39) are given by

$$\varphi(x) = ce^x \quad \text{for } x \in [0, b),$$

by virtue of Lemma 3.2, because in this case we have from (3.4) and (3.6)

$$\lim_{n \rightarrow \infty} G_n(x) = \lim_{n \rightarrow \infty} \prod_{k=0}^{n-1} e^{-x/2^k} = e^{-x} \quad \text{for } x \in [0, b),$$

thus case (i) occurs. In this example, the continuous solutions of (3.38) are given by

$$\psi(x) = \eta(x)e^x \quad \text{for } x \in [0, b),$$

where η is an arbitrary continuous function $\{f\}$ -decreasing in I , on account of Corollary 3.1. The continuous solutions of inequality in $[0, b)$ satisfying condition (3.25), can be written in the form

$$\psi(x) = c\bar{\eta}(x)e^x \quad \text{for } x \in [0, b),$$

where $\bar{\eta}$ is an arbitrary continuous function $\{f\}$ -decreasing in $[0, b)$ satisfying condition (3.28), and c is an arbitrary real number different from zero (see Corollary 3.2). In the special case where $b = \infty$ the assumptions of Corollaries 3.3 and 3.4 are fulfilled. Therefore, the only positive solution of (3.38) in $[0, \infty)$ continuous and bounded from above is the function $\psi(x) = 0$ for $x \in [0, \infty)$.

EXAMPLE 3.2. The inequality

$$(3.40) \quad \psi(\tfrac{1}{2}x) \leq \cos \tfrac{1}{2}x \psi(x) \quad \text{for } x \in [0, \pi)$$

is related with the equation

$$(3.41) \quad \psi\left(\frac{1}{2}x\right) = \cos\frac{1}{2}x\psi(x) \quad \text{for } x \in [0, \pi).$$

This is also case (i), because the function

$$G(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \in (0, \pi), \\ 1 & \text{for } x = 0, \end{cases}$$

is a continuous limit different from zero of the sequence $\prod_{i=0}^{n-1} \cos x/2^i$ (see [3]).

It follows from Lemma 3.2 that the continuous solutions of (3.41) in $[0, \pi)$ are given by the formula $\varphi(x) = c\bar{\varphi}(x)$, where

$$\bar{\varphi}(x) = \begin{cases} \frac{x}{\sin x} & \text{for } x \in (0, \pi), \\ 1 & \text{for } x = 0, \end{cases}$$

and c is an arbitrary constant.

The continuous solutions of (3.40) are given by $\psi(x) = \eta(x)\varphi(x)$ for $x \in [0, \pi)$ and, if they satisfy condition (3.25), also by the formula $\psi(x) = \bar{\eta}(x)\varphi(x)$ for $x \in [0, \pi)$, where $\bar{\eta}$ is given by formula (3.31). The assumptions of Corollary 3.3 are fulfilled, but the assumptions of Corollary 3.4 are not. Thus every positive continuous solutions of (3.40) bounded in $[0, \pi)$ has to be equal to zero at the point zero, but, since π is not a fixed point of the function $x/2$ in $[0, \pi)$, there exist positive continuous solutions of (3.40) bounded from above which are not identically equal to zero in $[0, \pi)$. Such a solution is, for instance, the function

$$\psi(x) = \begin{cases} 0 & \text{for } x \in [0, \pi/2), \\ x - \frac{1}{2}\pi & \text{for } x \in [\pi/2, \pi). \end{cases}$$

In case (ii) we can prove the following

THEOREM 3.6. *Let hypotheses H_1 , H_2 and H_5 be fulfilled and let case (ii) occur. If ψ is a continuous solution of inequality (3.1) in I , then*

$$(3.42) \quad \psi(x) \geq 0 \quad \text{for } x \in I.$$

Proof. Let ψ be a continuous solution of (3.1) in I . It follows from Lemma 3.3 that inequality (3.11) holds. Hence

$$(3.43) \quad \psi(x) \geq \frac{\psi[f^n(x)]}{G_n(x)} \quad \text{for } x \in I,$$

because $G_n(x) > 0$ for $x \in I$, by virtue of (3.4) and hypothesis H_5 . Since case (ii) occurs, it follows from (1.2), the continuity of ψ and (3.43) that (3.42) holds.

Since the function $\varphi(x) = 0$ for $x \in I$ is the only continuous solution of equation (3.3) in I , by virtue of Lemma 3.2, Theorem 3.6 is, in case (ii), an analogue of Theorem 3.1. In a similar way one can obtain theorem analogous to Theorem 3.3:

THEOREM 3.7. *Let hypotheses H_1, H_2 and H_5 be fulfilled and let case (ii) occur. If ψ is a continuous solutions of inequality (3.2) in I , then*

$$(3.44) \quad \psi(x) \leq 0 \quad \text{for } x \in I.$$

Lack of continuous solutions of equation (3.3), other than the trivial one, makes it impossible to obtain any results analogous to Corollaries 3.1–3.4. However, inequality (3.1) may have quite a large family of continuous solutions in I . Let us consider the following

EXAMPLE 3.3. The inequality

$$(3.45) \quad \psi(x/2) \leq 2\psi(x) \quad \text{for } x \in [0, \infty)$$

is an example of such an inequality where case (ii) occurs. We do not know all the continuous solutions of this inequality, but we are going to give several examples: the functions

$$\varphi_1 = c, \quad \varphi_2(x) = cx, \quad \varphi_3(x) = \frac{c}{1+x}, \quad \varphi_4(x) = cx^2$$

are continuous solutions of (3.45) in $[0, \infty)$ for an arbitrary positive constant c , as well as their sums. One can verify this fact by a simple substitution.

In the sequel, we are going to consider the homogeneous functional inequality when case (iii) occurs. In this case the problem is much more complicated and we are able to obtain some results in a narrower class of solutions than the continuous ones. At the beginning, we are going to prove some lemmas.

LEMMA 3.5. *Let hypothesis H_1 be fulfilled and let g be a function defined in I and satisfying the condition*

$$(3.46) \quad g(x) > 0 \quad \text{for } x \in I.$$

If ψ satisfies inequality (3.1) in I , then the sequence

$$(3.47) \quad \psi_n(x) = \frac{\psi[f^n(x)]}{G_n(x)} \quad \text{for } x \in I,$$

is decreasing in I .

If ψ satisfies inequality (3.2) in I , then sequence (3.47) is increasing in I .

Proof. Since the proofs of both parts of the lemma are similar, we are going to prove only the first part.

Let ψ be a solution of inequality (3.1) in I . It follows from (3.47), (3.1) and (3.4) that

$$\psi_{n+1}(x) = \frac{\psi[f^{n+1}(x)]}{G_{n+1}(x)} \leq \frac{g[f^n(x)]\psi f^n(x)}{g[f^n(x)]G_n(x)} = \frac{\psi[f^n(x)]}{G_n(x)} = \psi_n(x)$$

for $x \in I$, thus the sequence ψ_n is decreasing in I .

LEMMA 3.6. *Let hypotheses H_1, H_2 and H_5 be fulfilled and let case (iii) occur. If ψ is a solution of inequality (3.1) in I , continuous at the point ξ , then*

$$(3.48) \quad \psi(\xi) \leq 0.$$

If ψ is a solution of inequality (3.2) in I continuous at the point ξ , then

$$(3.49) \quad \psi(\xi) \geq 0.$$

Proof. We are going to prove (3.48). The proof of (3.49) is similar. Let ψ be a solution of inequality (3.1) in I , at ξ . If case (iii) occurs, then the sequence G_n satisfies (3.7). Let $x \in J$. Since the function ψ is continuous at ξ , from (1.2), Lemma 3.3 and (3.7) we obtain condition (3.48).

DEFINITION 3.3. We denote by \mathcal{P}_1 the class of all continuous solutions of inequality (3.1) in I , satisfying the condition

$$(3.50) \quad \psi(\xi) = 0$$

and such that there exists, for each $\psi \in \mathcal{P}_1$, such a solution φ_1 of equation (3.3) in I that

$$(3.51) \quad \psi(x) \geq \varphi_1(x) \quad \text{for } x \in I.$$

We denote by \mathcal{P}_2 the class of all continuous solutions ψ of inequality (3.2) satisfying condition (3.50) and such that there exists, for each $\psi \in \mathcal{P}_2$, such a solution φ_2 of equation (3.3) in I that

$$\psi(x) \leq \varphi_2(x) \quad \text{for } x \in I$$

LEMMA 3.7. *Let hypotheses H_1, H_2 and H_5 be fulfilled and let case (iii) occur. If $\psi \in \mathcal{P}_1$, then the limit*

$$(3.52) \quad \varphi_0(x) = \lim_{n \rightarrow \infty} \frac{\psi[f^n(x)]}{G_n(x)}$$

exists in I and φ_0 is a solution of equation (3.3), continuous from above in I , continuous at the point ξ :

If $\psi \in \mathcal{P}_2$, then the limit (3.52) exists in I and φ_0

$$(3.53) \quad \psi(x) \geq \varphi_0(x) \quad \text{for } x \in I.$$

If $\psi \in \mathcal{P}_2$, then the limit (3.52) exists in I and φ_0 is a solution of equation (3.3), continuous from below in I , continuous at the point ξ and

$$\psi(x) \leq \varphi_0(x) \quad \text{for } x \in I.$$

Proof. Similarly to the precedings lemmas, we are going to prove the first part, because the proofs of both parts are similar. Let $\psi \in \mathcal{P}_1$. First, we are going to prove that the sequence ψ_n given by (3.47), is bounded from below at every $x \in I$. Indeed, since for every $x \in I$, the condition $f^n(x) \in I$ for $n = 1, 2, \dots$ is fulfilled on account of hypothesis H_1 , and there exists a solution φ_1 of (3.3) in I satisfying (3.51) on account of definition 3.3, we from (3.51) and (3.47) that

$$\psi_n(x) \geq \frac{\varphi_1[f^n(x)]}{G_n(x)} = \varphi_1(x) \quad \text{for } x \in I,$$

because φ_1 satisfies (3.3) and (3.13). Therefore the sequence ψ_n is a decreasing sequence bounded from below of continuous functions, where the limit (3.52) has to exist and is a function continuous from below as a limit of a decreasing sequence of continuous functions ψ_n . It follows from (3.52) and (3.47) that

$$\varphi_0(x) = \lim_{n \rightarrow \infty} \psi_n(x) = \lim_{n \rightarrow \infty} \frac{\psi[f^{n+1}(x)]}{G_n[f(x)]} = \lim_{n \rightarrow \infty} \frac{\psi[f^{n+1}(x)]}{G_{n+1}(x)} g(x)$$

for $x \in I$, by virtue of (3.5). Therefore φ_0 satisfies equation (3.3) in I . Condition (3.53) follows immediately from (3.1), from Lemma 3.3 and from (3.52). Since (3.52) and (3.51) imply that

$$\varphi_0(x) = \lim_{n \rightarrow \infty} \frac{\psi[f^n(x)]}{G_n(x)} \geq \lim_{n \rightarrow \infty} \frac{\varphi_1[f^n(x)]}{G_n(x)} = \varphi_1(x) \quad \text{for } x \in I$$

because φ_1 satisfies (3.3) in I , we obtain

$$0 = \varphi_1(\xi) \leq \liminf_{x \rightarrow \xi} \varphi_0(x) \leq \limsup_{x \rightarrow \xi} \varphi_0(x) \leq \varphi_0(\xi) = 0,$$

on account of continuity of φ_1 at ξ , Lemma 3.2, continuity from above of φ_0 and (3.50). It proves the continuity of φ_0 at ξ and, at the same time, completes the proof of the lemma.

LEMMA 3.8. *Let the assumptions of Lemma 3.7 be fulfilled and let φ be a solution of equation (3.3) in I . If $\psi \in \mathcal{P}_1$ and x_0 is such an inner point of the interval I that*

$$(3.54) \quad \varphi_0(x_0) < \varphi(x_0),$$

then there exists such a positive integer k that

$$(3.55) \quad \psi[f^k(x_0)] < \varphi[f^k(x_0)].$$

If $\varphi \in \mathcal{P}_2$ and x_0 is such an inner point of the interval I that

$$\varphi_0(x_0) > \varphi(x_0),$$

then there exists such a positive integer k that

$$\psi[f^k(x_0)] > \varphi[f^k(x_0)].$$

Proof. Let inequality (3.54) hold for an $x_0 \in (\xi, b)$ and let us suppose that inequality (3.55) does not hold for any positive integer k . This means that

$$(3.56) \quad \varphi[f^n(x_0)] \leq \psi[f^n(x_0)] \quad \text{for } n = 1, 2, \dots$$

Since the function g is positive in I , by virtue of hypothesis H_5 and φ satisfies equation (3.3) in I , we infer from (3.56) and (3.4) that

$$\varphi(x_0) = \frac{\varphi[f^n(x_0)]}{G_n(x_0)} \leq \frac{\psi[f^n(x_0)]}{G_n(x_0)} \quad \text{for } n = 1, 2, \dots$$

Hence, $\varphi(x_0) \leq \varphi_0(x_0)$, by virtue of (3.52). The last inequality contradicts inequality (3.54), which completes the proof of the first part of the lemma. The proof of the second part is similar.

If inequality (3.54) holds not only for one point x_0 , but also in the whole interval $[f(x_0), x_0]$, then for every x belonging to this interval there exists such a positive integer k that inequality (3.55) holds. However, k depends on x , and thus we cannot assert that there exists such a k that (3.55) holds in the whole interval $[f(x_0), x_0]$. Such an assertion requires some additional assumptions. Namely, we are going to prove the following

THEOREM 3.8. *Let hypotheses H_1 , H_2 and H_5 be fulfilled and let case (iii) occur. If $\psi \in \overline{\Psi}_1$ and φ is a continuous solution of equation (3.3) in I satisfying the inequality*

$$(3.57) \quad \varphi(x) > \varphi_0(x) \quad \text{for } x \in [f(x_0), x_0],$$

where x_0 is an inner point of I and φ_0 is given by formula (3.52), then there exists such a positive integer k that

$$(3.58) \quad \varphi[f^k(x)] > \psi[f^k(x)] \quad \text{for } x \in [f(x_0), x_0].$$

If $\psi \in \overline{\Psi}_2$ and φ is a continuous solution of equation (3.3) in I satisfying the inequality

$$\varphi(x) < \varphi_0(x) \quad \text{for } x \in [f(x_0), x_0],$$

where x_0 is an inner point of I and φ_0 is given by formula (3.52), then there exists such a positive integer k that

$$\varphi[f^k(x)] < \psi[f^k(x)] \quad \text{for } x \in [f(x_0), x_0].$$

Proof. Let us assume that the theorem is false. It means that there exist functions ψ and φ such that $\psi \in \overline{\Psi}_1$ and φ is a continuous solution of (3.3) in I , and there exists such an $x_0 \in (\xi, b)$ that (3.57) holds, but for every positive integer n there exists such a point $x_n \in [f(x_0), x_0]$ that

$$(3.59) \quad \varphi[f^n(x_n)] \leq \psi[f^n(x_n)].$$

Let us suppose that for a positive integer k the inequality

$$\varphi[f^k(x_n)] > \psi[f^k(x_n)]$$

holds. Then it follows from hypothesis H_1 and Theorem 2.1 that

$$\varphi[f^n(x_n)] > \psi[f^n(x_n)],$$

in spite of (3.59). Thus for every k the inequality

$$(3.60) \quad \varphi[f^k(x_n)] \leq \psi[f^n(x_n)]$$

holds. Since the interval $[f(x_0), x_0]$ is compact, there exists a subsequence x_{k_n} of the sequence x_n converging to a point $\bar{x} \in [f(x_0), x_0]$. It follows from the continuity of function φ and (3.60) that

$$\varphi[f^k(\bar{x})] \leq \psi[f^k(\bar{x})] \quad \text{for } k = 1, 2, \dots$$

The last inequality contradicts Lemma 3.8, which completes the proof of the first part of the theorem. The proof of the second part is similar.

Let us notice that if $\psi \in \overline{\mathcal{P}}_1$ ($\psi \in \overline{\mathcal{P}}_2$) and if the function φ_0 is continuous in I , then φ_0 is the greatest (the smallest) continuous solution of equation (3.3) in I satisfying inequality (3.53) (the inequality opposite to (3.53)), in view of Theorem 3.8.

In the sequel we shall consider narrower classes of functions:

DEFINITION 3.4. We denote by Φ the family of all continuous solutions of equation (3.3) in I satisfying the condition

$$(3.61) \quad \varphi(x) > 0 \quad \text{for } (x, b).$$

We denote by \mathcal{P}_1 the family of all continuous solutions of inequality (3.1) in I satisfying condition (3.42).

We denote by \mathcal{P}_2 the family of all continuous functions $\psi \in \overline{\mathcal{P}}_2$ for which $\varphi_2 \in \Phi$ (see Definition 3.3), satisfying condition (3.42).

Let us notice that if $\psi \in \mathcal{P}_1$, then condition (3.42) implies condition (3.50), on account of Lemma 3.6. Thus, $\mathcal{P}_1 \subset \overline{\mathcal{P}}_1$ with $\varphi_1(x) = 0$ for $x \in I$ (see Definition 3.3). But condition (3.42) implies neither (3.50) nor $\psi \in \overline{\mathcal{P}}_2$ for any continuous solution ψ of inequality (3.2) in I . Since we want to deal with the positive continuous solutions of (3.2), we have to make an additional assumption that $\mathcal{P}_2 \subset \overline{\mathcal{P}}_2$. Here we have to renounce the duality which one could observe so far between (3.1) and (3.2).

It is necessary for further considerations in this section to have the condition $\Phi \neq \emptyset$ fulfilled. For this reason we shall assume the following

HYPOTHESIS H_6 . *There exists such an inner point x_0 of the interval I that $\lim_{n \rightarrow \infty} G_n(x) = 0$ uniformly in the interval $[f(x_0), x_0]$, where the sequence G_n is given by formula (3.4).*

It is easy to see that hypothesis H_6 implies that case (iii) occurs, but in case (iii) hypothesis H_6 does not have to be fulfilled (see [1]). The condition $\Phi \neq \emptyset$ is fulfilled if and only if hypothesis H_6 is fulfilled (see [9]).

THEOREM 3.9. *Let hypotheses H_1, H_2, H_5 , and H_6 be fulfilled and let ψ be a continuous solution of inequality (3.1) in I . If $\varphi_0 \in \Phi$, where φ_0 is given by formula (3.52), and then satisfies inequality $\psi(x) > 0$ for $x \in (\xi, b)$ and*

$$(3.62) \quad \lim_{x \rightarrow \xi} \frac{\psi(x)}{\varphi_0(x)} = 1.$$

Proof. It follows immediately from (3.53) and (3.61) that $\psi(x) > 0$ for $x \in (\xi, b)$ on account of Lemma 3.7 and definition 3.4. Let ε be an arbitrary positive number. Put

$$(3.63) \quad c = 1 + \varepsilon.$$

The function $c\varphi_0(x)$ is a continuous function in I , by virtue of definition 3.4 and satisfies equation (3.3) in I by virtue of Lemma 3.7 and Since $\varepsilon > 0$, it follows from (3.63) and (3.61) that

$$(3.64) \quad \varphi(x) = c\varphi_0(x) \geq \varphi_0(x) \quad \text{for } x \in I.$$

Let x_0 be an inner point of the interval I . Condition (3.64) implies that inequality (3.57) holds; thus there exists such a positive integer k that (3.58) holds on account of Theorem 3.8. Let us denote $d = f^{k-1}(x_0)$. Inequality (3.58) implies, in view of Corollary 2.3, that

$$(3.65) \quad \psi(x) \leq \varphi(x) \quad \text{for } x \in (\xi, d).$$

Inequalities (3.53) and (3.65) imply that

$$1 \leq \frac{\psi(x)}{\varphi_0(x)} \leq c = 1 + \varepsilon \quad \text{for } x \in (\xi, d),$$

by virtue of (3.63) and (3.64). The last condition proves (3.62), which completes the proof.

One can prove in a similar way the following

THEOREM 3.10. *Let hypotheses H_1, H_2, H_5 and H_6 be fulfilled and let $\psi \in \Psi_2$ be a function satisfying the condition $\psi(x) \neq 0$ for $x \in (\xi, b)$. If $\varphi_0 \in \Phi$, where φ_0 is given by formula (3.52), then condition (3.62) is fulfilled.*

In case (iii) one can obtain theorems similar to Theorems 3.2 and 3.4 but in a narrower class of functions than the class of continuous functions. Let us define this class.

DEFINITION 3.5. Let $\bar{\varphi} \in \Phi$. We denote by $\Phi(\bar{\varphi})$ the family of all functions $\varphi \in \Phi$ such that the limit

$$(3.66) \quad \lim_{x \rightarrow \xi} \frac{\varphi(x)}{\bar{\varphi}(x)} = a$$

exists.

We denote by Φ_1 (Φ_2) the family of all functions $\psi \in \Psi_1$ ($\psi \in \Psi_2$) such that the limit

$$(3.67) \quad \lim_{x \rightarrow \xi} \frac{\psi(x)}{\bar{\varphi}(x)} = d$$

exists for a function $\bar{\varphi} \in \Phi$.

We are going to prove some properties of the families of functions defined above.

LEMMA 3.9. *If $\bar{\varphi} \in \Phi$ and $\varphi \in \Phi(\bar{\varphi})$, then*

$$(3.68) \quad \lim_{x \rightarrow \xi} \frac{\varphi(x)}{\bar{\varphi}(x)} \neq 0.$$

Proof. Let $\bar{\varphi} \in \Phi$ and $\varphi \in \Phi(\bar{\varphi})$. Then the functions $\bar{\varphi}$ and φ satisfy equation (3.3) on account of Definitions 3.4 and 3.5. It follows from (3.3) and (3.61) that the function

$$(3.69) \quad \hat{\varphi}(x) = \begin{cases} \frac{\varphi(x)}{\bar{\varphi}(x)} & \text{for } x \in (\xi, b), \\ \lim_{x \rightarrow \xi} \frac{\varphi(x)}{\bar{\varphi}(x)} & \text{for } x = \xi, \end{cases}$$

satisfies the functional equation

$$(3.70) \quad \hat{\varphi}[f(x)] = \hat{\varphi}(x) \quad \text{for } x \in I$$

and $\hat{\varphi}$ is continuous in I .

It follows from Lemma 3.2 that equation (3.70) has a one-parameter family of continuous solutions in I given by the formula

$$(3.71) \quad \hat{\varphi}(x) = c \quad \text{for } x \in I,$$

where c is an arbitrary real number. It follows from (3.71), (3.69) and (3.66) that $a = c$, whence $a \neq 0$, because the condition $a = 0$ implies $\varphi(x) = 0$ for $x \in I$ by virtue of (3.71) and (3.69), in spite of the assumption that $\varphi \in \Phi(\bar{\varphi})$, Definition 3.5 and (3.61). Thus condition (3.68) has been proved.

LEMMA 3.10. *If $\bar{\varphi} \in \Phi$ and $\varphi_1, \varphi_2 \in \Phi(\bar{\varphi})$, then the limit*

$$(3.72) \quad \lim_{x \rightarrow \xi} \frac{\varphi_2(x)}{\varphi_1(x)}$$

exists and it is different from zero.

Proof. Let us denote

$$a_1 = \lim_{x \rightarrow \xi} \frac{\varphi_1(x)}{\bar{\varphi}(x)}, \quad a_2 = \lim_{x \rightarrow \xi} \frac{\varphi_2(x)}{\bar{\varphi}(x)}.$$

It follows from Lemma 3.9 that $a_1 a_2 \neq 0$, whence the limit (3.72) exists and it is equal to $a_2/a_1 \neq 0$.

LEMMA 3.11. *If $\bar{\varphi} \in \Phi$, then the family $\Phi(\bar{\varphi})$ is a one-parameter family of functions*

$$(3.73) \quad \varphi(x) = a\bar{\varphi}(x) \quad \text{for } x \in I,$$

where a is given by formula (3.66).

Proof. Let $\bar{\varphi} \in \Phi$ and $\varphi \in \Phi(\bar{\varphi})$. It follows from Definitions 3.4 and 3.5 that $\bar{\varphi}$ and φ are continuous solutions of (3.3) in I , whence the function $\hat{\varphi}$ given by (3.69) satisfies (3.70) in I . This implies on account of (3.69) and (3.10) that (3.73) holds.

THEOREM 3.11. *Let hypotheses H_1, H_2, H_5 and H_6 be fulfilled. If $\psi \in \Phi_1$ ($\psi \in \Phi_2$) and $\bar{\varphi} \in \Phi$ is such a function that (3.67) holds, then*

$$(3.74) \quad \psi(x) \geq d\bar{\varphi}(x) \quad \text{for } x \in I \quad (\psi(x) \leq d\bar{\varphi}(x) \quad \text{for } x \in I),$$

where c is defined by formula (3.67).

Proof. We are going to prove the theorem in the case where $\psi \in \Phi_1$, because the proof in the other case is similar. Since ψ satisfies inequality (3.1) in view of Definition 3.5, then the function

$$(3.75) \quad \eta(x) = \begin{cases} \frac{\psi(x)}{\bar{\varphi}(x)} & \text{for } x \in (\xi, b), \\ \lim_{x \rightarrow \xi} \frac{\psi(x)}{\bar{\varphi}(x)} & \text{for } x = \xi, \end{cases}$$

is a continuous solution of inequality (3.23) in I by virtue of (3.75). Therefore inequality (3.74) follows from Lemma 3.2 and from Theorem 3.1 on account of (3.73), (3.67) and (3.61).

COROLLARY 3.6. *Let the assumptions of Theorem 3.11 be fulfilled. If the limit (3.67) is different from zero, then there exists, for every $\varphi \in \Phi(\bar{\varphi})$, a limit $\lim_{x \rightarrow \xi} \frac{\psi(x)}{\varphi(x)}$.*

Proof. If $\varphi \in \Phi(\bar{\varphi})$, then Lemma 3.11 implies that

$$\lim_{x \rightarrow \xi} \frac{\psi(x)}{\varphi(x)} = \lim_{x \rightarrow \xi} \frac{\psi(x)}{a\bar{\varphi}(x)}$$

by virtue of (3.73) and (3.67).

As an immediate consequence of the above corollary and Lemma 3.9, we obtain the following

COROLLARY 3.7. *Let the assumptions of Theorem 3.11 be fulfilled.*

If $\varphi \in \Phi \setminus \Phi(\bar{\varphi})$ and the limit $\bar{c} = \lim_{x \rightarrow \xi} \frac{\psi(x)}{\varphi(x)}$ exists, then $\bar{c} = 0$.

THEOREM 3.12. *Let hypotheses H_1, H_2, H_5 and H_6 be fulfilled. If $\psi \in \Phi_1$ ($\psi \in \Phi_2$) and limit (3.67) is different from zero, then there exists a uni-*

quely determined function η , continuous and $\{f\}$ -decreasing ($\{f\}$ -increasing) in I and such that

$$(3.76) \quad \psi(x) = \varphi_0(x)\eta(x) \quad \text{for } x \in I$$

and

$$(3.77) \quad \eta(\xi) = 1,$$

where the function φ_0 is given by formula (3.52) and $\varphi_0 \in \Phi$.

Proof. Let $\psi \in \Phi_1$ and let $\bar{\varphi} \in \Phi$ be such a function that there exists a limit (3.67) different from zero. It follows from Corollary 3.6 and from Lemma 3.11 that there exists such a $\varphi \in \Phi(\bar{\varphi})$ that

$$(3.78) \quad \lim_{x \rightarrow \xi} \frac{\psi(x)}{\varphi(x)} = 1.$$

The function φ is given by (3.73), where $a = d$ and d is given by formula (3.67). Let us denote

$$(3.79) \quad \eta(x) = \begin{cases} \frac{\psi(x)}{\varphi(x)} & \text{for } x \in (\xi, b), \\ 1 & \text{for } x = \xi. \end{cases}$$

Since the functions ψ and φ are continuous in I , (3.79) and (3.78) imply the continuous of η in I and (3.77) holds in view of (3.79). Since ψ satisfies (3.1) and φ satisfies (3.3) by virtue of Definitions 3.5 and 3.4 and $\varphi(x) > 0$ in I , (ξ, b) in view of Definition 3.4, we infer that η satisfies inequality (3.23). Hence

$$(3.80) \quad \psi(x) = \varphi(x)\eta(x) \quad \text{for } x \in I.$$

Therefore

$$\psi[f^n(x)] = \varphi[f^n(x)]\eta[f^n(x)] = G_n(x)\varphi(x)\eta[f^n(x)] \quad \text{for } x \in I, n = 1, 2, \dots,$$

where the sequence G_n is given by formula (3.4). Thus we obtain

$$\varphi(x) = \frac{\psi[f^n(x)]}{G_n(x)\eta[f^n(x)]} \quad \text{for } x \in I, n = 1, 2, \dots,$$

whence $\varphi(x) = \varphi_0(x)$ for $x \in I$, by virtue of (3.77) and (3.52). Therefore formula (3.76) has been proved.

Since $\bar{\varphi} \in \Phi$, we have $\psi(x) > 0$ in I , on account of Theorem 3.11 and (3.61). Therefore we infer from (3.76) that $\varphi_0(x) > 0$ for $x \in (\xi, b)$, because $\eta(x) > 0$ for $x \in I$ by virtue of (3.23), (3.77) and Theorem 3.1. The continuity of φ_0 follows from the continuity of ψ and η , in view of (3.76) proved earlier. Hence $\varphi_0 \in \Phi$, which completes the proof.

It is easy to verify the following

LEMMA 3.12. *If φ is a positive solution of equation (3.3) in I and η is a function $\{f\}$ -decreasing ($\{f\}$ -increasing) in I , then the function ψ given by formula (3.80) satisfies inequality (3.1) (inequality (3.2)) in I .*

We obtain from Theorem 3.12 and from Lemma 3.12 the following

THEOREM 3.13. *Let hypotheses H_1, H_2, H_5 , and H_6 be fulfilled. The following conditions are equivalent:*

- 1° $\psi \in \Phi_1$ ($\psi \in \Phi_2$),
- 2° $\varphi_0 \in \Phi$, where φ_0 is given by formula (3.52),
- 3° the function η given by formula (3.79) with $\varphi(x) = \varphi_0(x) \in \Phi$ for $x \in I$ is a continuous $\{f\}$ -decreasing ($\{f\}$ -increasing) function in I satisfying condition (3.77).

Since $\Phi \subset \Phi_1 \cap \Phi_2$ on account of Definitions 3.4 and 3.5, we infer that for every $\varphi \in \Phi$ the function φ_0 given by (3.52), equals $\varphi(x)$ and $\varphi_0 \in \Phi$. Thus Theorems 3.12 and 3.13 give the general solution of inequality (3.1) in I in the class Φ_1 . These two theorems give, as an immediate consequence, the following

COROLLARY 3.8. *If hypotheses H_1, H_2, H_5 and H_6 are fulfilled, then for every $\psi \in \Phi_1$ ($\psi \in \Phi_2$) there exists exactly one function $\varphi_0 \in \Phi$ and exactly one continuous function η $\{f\}$ -decreasing ($\{f\}$ -increasing) in I satisfying condition (3.77) and such that (3.76) holds.*

Let us notice that if $\varphi_0 \in \Phi$, then $\psi \in \Phi_1$, but if φ_0 is a continuous solution of (3.3) in I satisfying the inequality $\varphi_0(x) \geq 0$ in I , then ψ does not have to belong to Φ_1 . We are going to give a sufficient condition for φ_0 to be a trivial solution of (3.3) in I .

THEOREM 3.14. *Let hypotheses H_1, H_2, H_5 and H_6 be fulfilled and let ψ be a continuous solution of inequality (3.1) in I , satisfying the condition*

$$(3.81) \quad \psi(x) > 0 \quad \text{for} \quad x \in (\xi, b).$$

Moreover, let there exist limit

$$(3.82) \quad q = \lim_{x \rightarrow \xi} \frac{\psi[f(x)]}{\psi(x)}.$$

If $q < g(\xi)$, then

$$(3.83) \quad \varphi_0(x) = 0 \quad \text{for} \quad x \in I,$$

where φ_0 is given by formula (3.52).

Proof. Since the function ψ is continuous in I , then the function

$$(3.84) \quad \bar{g}(x) = \begin{cases} \frac{\psi[f(x)]}{g(x)\psi(x)} & \text{for } x \in (\xi, b), \\ \frac{q}{g(\xi)} & \text{for } x = \xi, \end{cases}$$

is a continuous function in I and $\bar{g}(x) > 0$ for $x \in I$. In view of the assumption that $q < g(\xi)$, we have $\bar{g}(\xi) < 1$, whence

$$\lim_{n \rightarrow \infty} \prod_{i=0}^{n-1} \bar{g} f^i(x) = \lim_{n \rightarrow \infty} \prod_{i=0}^{n-1} \frac{\psi[f^{i+1}(x)]}{g[f^i(x)]\psi[f^i(x)]} = \lim_{n \rightarrow \infty} \frac{f^n(x)}{G_n(x)\psi(x)} = 0$$

uniformly in $[\xi, x_0)$, where x_0 is an arbitrary inner point of I (see [4]). Hence we obtain (3.83) by virtue of (3.52).

It is easy to see that if $\varphi \in \Phi$, then $\lim_{x \rightarrow \xi} \frac{\varphi[f(x)]}{\varphi(x)} = g(\xi)$, because

φ satisfies (3.3) in I . Further, if η is a continuous function, $\{f\}$ decreasing in I and satisfying (3.77), then η is positive in I , on account of (3.22) and Theorem 3.1. Therefore we obtain, as an immediate consequence of Theorem 3.12, the following

COROLLARY 3.9. *Let hypotheses H_1, H_2, H_5 and H_6 be fulfilled. If $\varphi \in \Phi_1$, and inequality (3.81) holds, then limit (3.82) exists and $q = g(\xi)$.*

4. Continuous solutions of non-linear inequalities

In this section we shall deal with continuous solutions of inequalities (0.1) and (0.2). We shall assume the following

HYPOTHESIS H_7 . *There exists such a neighbourhood*

$$U: [\xi, \xi + c) \times (\eta_0 - d, \eta_0 + d), \quad c > 0, \quad d > 0, \quad U \subset \Omega,$$

where η_0 is a solution of equation (1.4), and such a positive function γ , continuous in the interval $[\xi, \xi + c)$, that

$$(4.1) \quad |g(x, y_1) - g(x, y_2)| \geq \gamma(x)|y_1 - y_2| \quad \text{for} \quad (x, y_1), (x, y_2) \in \Omega.$$

Let us denote

$$(4.2) \quad G_n(x) = \prod_{i=0}^{n-1} \gamma[f^i(x)] \quad \text{for} \quad x \in [\xi, \xi + c), \quad n = 1, 2, \dots$$

In this section sequence (4.2) will be subjected to the following

HYPOTHESIS H_8 . *The sequence G_n converges to a limit satisfying one of the following conditions:*

$$(4.3) \quad 1^\circ \quad \lim_{n \rightarrow \infty} G_n(x) = G(x) \quad \text{for} \quad x \in [\xi, \xi + c),$$

where G is a continuous function different from zero in $[\xi, \xi + c)$,

$$(4.4) \quad 2^\circ \quad \lim_{n \rightarrow \infty} G_n(x) = \infty \quad \text{for} \quad x \in [\xi, \xi + c).$$

It follows from hypothesis H_7 , (4.2) and condition 1° of hypothesis H_8 that

$$(4.5) \quad G(x) > 0 \quad \text{for} \quad x \in [\xi, \xi + c).$$

Let us notice that hypotheses H_1 - H_4 , H_7 and H_8 imply that there exists at most one continuous solution of equation (0.3) in I satisfying the condition $\varphi(\xi) = \eta_0$ (see [2]).

THEOREM 4.1. *Let hypotheses H_1 - H_4 , H_7 and H_8 be fulfilled, let ψ be a continuous solution of inequality (0.1) in I and let φ be a continuous solution of equation (0.3) in I satisfying condition*

$$(4.6) \quad \varphi(\xi) \in (\eta_0 - d, \eta_0 + d).$$

If

$$(4.7) \quad \eta_0 = \psi(\xi) \geq \varphi(\xi),$$

then

$$(4.8) \quad \psi(x) \geq \varphi(x) \quad \text{for } x \in I.$$

Proof. First, let us assume that ψ is a continuous solution of inequality (0.1) in I satisfying the condition

$$(4.9) \quad \psi(\xi) = \varphi(\xi).$$

Thus, there exists such a positive number $c_0 < c$ that $(x, \psi(x)) \in U$ for $x \in [\xi, \xi + c_0)$ and $(x, \varphi(x)) \in U$ for $x \in [\xi, \xi + c_0)$. Let us write

$$(4.10) \quad \bar{\psi}(x) = \min[\psi(x), \varphi(x)] \quad \text{for } x \in [\xi, \xi + c_0).$$

It follows from Theorem 2.3 that the function $\bar{\psi}$ is a continuous solution of (0.1) in $[\xi, \xi + c_0)$. The function

$$(4.11) \quad \psi_0(x) = \varphi(x) - \bar{\psi}(x) \quad \text{for } x \in [\xi, \xi + c_0)$$

is, in view of (4.10), a continuous function satisfying the inequality

$$(4.12) \quad \psi_0(x) \geq 0 \quad \text{for } x \in [\xi, \xi + c_0).$$

Thus we infer from (4.11), (4.12), (0.1) and (0.3) and also hypothesis H_8 , (1.1) and (4.1) that

$$\begin{aligned} \psi_0[f(x)] &= \varphi[f(x)] - \bar{\psi}[f(x)] \geq g[x, \varphi(x)] - g[x, \bar{\psi}(x)] \\ &\geq |g[x, \varphi(x)] - g[x, \bar{\psi}(x)]| \geq \gamma(x) |\varphi(x) - \bar{\psi}(x)| = \gamma(x) \psi_0(x). \end{aligned}$$

Hence the function ψ_0 is a continuous solution of inequality (3.2) in $[\xi, \xi + c_0)$ and satisfies condition

$$(4.13) \quad \psi_0(x) = 0,$$

by virtue of (4.9), (4.10), (4.11) and (4.12). Therefore hypothesis H_8 , Theorem 3.4, 3.7 and Lemma 3.2 imply, in view of (4.13) and (4.12), that

$$\psi_0(x) = 0 \quad \text{for } x \in [\xi, \xi + c_0),$$

whence

$$\bar{\psi}(x) = \varphi(x) \quad \text{for } x \in [\xi, \xi + c_0),$$

by virtue of (4.11). Thus, on account of (4.10), inequality (4.8) holds in the interval $[\xi, \xi + c_0)$. Now we are going to show that inequality (4.8) holds in the interval I . Indeed, if there exists such a point $x_0 \in I$ that

$$(4.14) \quad \psi(x_0) < \varphi(x_0),$$

then there exists such a positive integer n that $f^n(x_0) \in [\xi, \xi + c_0)$ by virtue of (1.2), and $\psi[f^n(x_0)] < \varphi[f^n(x_0)]$ by virtue of (4.14) and Theorem 2.1. The last inequality contradicts the fact that inequality (4.8) holds in the interval $[\xi, \xi + c_0)$, which has already been proved.

If $\psi(\xi) > \varphi(\xi)$, then the assumption of the existence of such a point $x_0 \in I$ that inequality (4.14) holds leads to contradiction in the same way. Thus the proof of Theorem 4.1 is complete.

In a similar way one can prove the following

THEOREM 4.2. *Let hypotheses H_1 – H_4 , H_7 and H_8 be fulfilled, let ψ be a continuous solution of inequality (0.2) in I and let φ be a continuous solution of equation (0.3) in I satisfying condition (4.6).*

If

$$\eta_0 = \psi(\xi) \leq \varphi(\xi),$$

then

$$\psi(x) \leq \varphi(x) \quad \text{for } x \in I.$$

Theorems 4.1 and 4.2 together with Theorem 2.5 enables us to obtain some results concerning the families L , L' and L_0 (see Definition 2.1) in terms of lattice theory:

COROLLARY 4.1. *Let hypotheses H_1 – H_4 , H_7 and H_8 be fulfilled. The necessary condition for $\varphi \in L \cup L'$ to belong to L_0 is each of the following two conditions:*

1° *The principal filter of element $\varphi \in L$ in the lattice L is a prime filter, i.e., φ is an element of L irreducible for union⁽¹⁾.*

2° *The principal ideal of element $\varphi \in L'$ in the lattice L' is a prime ideal, i.e., φ is an element of L' irreducible for intersection⁽¹⁾.*

(1) The set $L^\varphi = \{\psi \in L: \psi > \varphi\}$ is called the *principal filter of element φ in L* . The filter is prime in L if for any $\psi_1, \psi_2 \in L$, the condition $\psi_1 \cup \psi_2 \in L^\varphi$ implies either $\psi_1 \in L^\varphi$ or $\psi_2 \in L^\varphi$.

An element $\varphi \in L$ called *irreducible for union* in L if L^φ is a prime filter.

The set $L'_\varphi = \{\psi \in L': \psi < \varphi\}$ is called the *principal ideal of element φ in L'* . The ideal L'_φ is prime in L' if for any $\psi_1, \psi_2 \in L'$ the condition $\psi_1 \cap \psi_2 \in L'_\varphi$ implies either $\psi_1 \in L'$ or $\psi_2 \in L'_\varphi$.

An element φ is called *irreducible for intersection* in L' if L'_φ is a prime ideal.

Proof. We are going to prove the first part of the corollary. The proof of the second part is similar.

Let $\psi_1, \psi_2 \in L$ and $\varphi \in L_0$. If $\varphi \leq \psi_1 \cup \psi_2 = \psi$, where \leq is defined as an ordering relation in the lattice, then $\psi(\xi) \geq \varphi(\xi)$ whence either $\psi_1(\xi) \geq \varphi(\xi)$ or $\psi_2(\xi) \geq \varphi(\xi)$, by virtue of (2.9). Therefore either ψ_1 or ψ_2 satisfies the conditions required in Theorem 4.1 and, consequently, at least one of these two functions satisfies inequality (4.8), which completes the proof of the corollary.

Conditions 1° and 2° of Corollary 4.1 are also sufficient for φ to belong to L_0 under the additional

HYPOTHESIS H_9 . *For every point $(x, y) \in \Omega$, there exists exactly one continuous solution of equation (0.3) in I passing through that point.*

First, we are going to prove some lemmas.

LEMMA 4.1. *Let hypotheses H_1 and H_3 be fulfilled and let ψ_1 and ψ_2 be solutions of inequality (0.1) in I such that*

$$(4.15) \quad \psi_1(x) \leq \psi_2(x) \quad \text{for} \quad x \in I.$$

Moreover, let x_0 be an inner point of I . If ψ_0 is an arbitrary function defined in the interval $[f(x_0), x_0]$ satisfying the conditions

$$(4.16) \quad \psi_1(x) \leq \psi_0(x) \leq \psi_2(x) \quad \text{for} \quad x \in [f(x_0), x_0],$$

$$(4.17) \quad \psi_1[f(x_0)] = \psi_0[f(x_0)], \quad \psi_2(x_0) = \psi_0(x_0),$$

then the function

$$(4.18) \quad \psi(x) = \begin{cases} \psi_1(x) & \text{for } x \in [\xi, f(x_0)], \\ \psi_0(x) & \text{for } x \in [f(x_0), x_0], \\ \psi_2(x) & \text{for } x \in [x_0, b], \end{cases}$$

is a solution of inequality (0.1) in I .

Proof. Let $x \in I$. There are four cases possible:

1° $x, f(x) \in [\xi, f(x_0)]$.

Since $\psi(x) = \psi_1(x)$ and $\psi[f(x)] = \psi_1[f(x)]$ and ψ_1 satisfies (0.1) we infer that ψ satisfies (0.1).

2° $x \in [f(x_0), x_0]$.

In this case $f(x) \in [\xi, f(x_0)]$ by virtue of hypothesis H_1 , and then

$$\psi[f(x)] = \psi_1[f(x)] \leq g[x, \psi_1(x)] \leq g[x, \psi_0(x)] = g[x, \psi(x)]$$

by virtue of (4.18), (0.1), hypothesis H_3 , (4.16) and (4.17).

3° $f(x) \in [f(x_0), x_0]$.

In this case $x \in [x_0, b)$ by virtue of hypothesis H_1 , and then

$$\psi[f(x)] = \psi_0[f(x)] \leq \psi_2[f(x)] \leq g[x, \psi_2(x)] = g[x, \psi(x)]$$

in view of (4.18), (4.16), (0.1) and (4.17).

4° $x, f(x) \in [x_0, b)$.

Since ψ_2 satisfies (0.1) in I , we can prove that ψ satisfies (0.1) in the same way as in case 1°.

LEMMA 4.2. *Let hypotheses H_1 – H_4 and H_9 be fulfilled and let ψ be a continuous solution of inequality (0.1) in I . If there exists such an inner point of I that*

$$(4.19) \quad [f(x_0)] \leq g[x_0, \psi(x_0)],$$

then

$$(4.20) \quad \varphi_0(x) \leq \varphi_1(x) < \varphi_2(x) \quad \text{for } x \in I,$$

where φ_1, φ_0 , and φ_2 are the continuous solutions of equation (0.3) in I determined by the conditions

$$(4.21) \quad \varphi_0(\xi) = \psi(\xi), \quad \varphi_1[f(x_0)] = \psi[f(x_0)], \quad \varphi_2(x_0) = \psi(x_0).$$

Moreover, there exists such an interval $(\alpha, \beta) \subset [f(x_0), x_0]$ that

$$(4.22) \quad \varphi_0(x) < \psi(x) < \varphi_2(x) \quad \text{for } x \in (\alpha, \beta).$$

Proof. Let ψ be a continuous solution of (0.1) in I and let x_0 be an inner point of I such that (4.19) holds. It follows from hypotheses H_1 – H_4 , H_7 and H_8 that the functions φ_1 and φ_2 are determined uniquely by (4.21), thus the continuity of these functions implies that two cases are possible: either

$$(4.23) \quad \varphi_1(x) < \varphi_2(x) \quad \text{for } x \in I$$

or

$$(4.24) \quad \varphi_1(x) \geq \varphi_2(x) \quad \text{for } x \in I.$$

The last inequality cannot hold. Indeed, if it did, then $\varphi_1(x_0) \geq \varphi_2(x_0)$. Therefore it follows from (4.21), (0.3), hypothesis H_3 , (4.20) and (4.23) that

$$\psi[f(x_0)] = \varphi_1[f(x_0)] = g[x_0, \varphi_1(x_0)] \geq g[x_0, \psi(x_0)],$$

which contradicts (4.19). Since (4.21) implies that

$$\varphi_0(\xi) = \psi(\xi) \leq \varphi_1(\xi),$$

in view of Corollary 2.4, then $\psi(x) \leq \varphi_1(x)$ for $x \in I$, which can be proved in the same way as (4.23). The last inequality and inequalities (4.22) being proved, we also have proved (4.20).

It follows from (4.21) that

$$(4.25) \quad \psi(x) \geq \varphi_0(x) \quad \text{for } x \in I,$$

by virtue of Theorem 4.1. The continuity of ψ , (4.19), (4.21), (4.20) and (4.25) imply that there exists such an interval $(\alpha, \beta) = [f(x_0), x_0]$ that (4.22) holds. This ends the proof of the lemma.

LEMMA 4.3. *Under the hypotheses of Lemma 4.2, there exists such an $x_1 \in [f(x_0), x_0]$ that $\psi(x) \leq \varphi_3(x)$ for $x \in [\xi, f(x_0)]$, where φ_3 is the continuous function satisfying (0.3) in I determined by the condition*

$$(4.26) \quad \varphi_3(x_1) = \psi(x_1).$$

Proof. Let x_1 be an arbitrary point of the interval (α, β) (see the preceding lemma) and let φ_3 be the continuous solution of (0.3) determined by (4.26). Thus conditions (4.26) and (4.21) imply, in view of Lemma 4.2, that

$$\varphi_0(\xi) = \psi(\xi) < \varphi_3(\xi) < \varphi_2(\xi),$$

whence

$$(4.27) \quad \varphi_0(x) < \varphi_3(x) < \varphi_2(x) \quad \text{for } x \in I.$$

The point x_1 can be chose so close to the point β that $\varphi_3(x) \geq \psi(x)$ for $x \in [\xi, f(x_0)]$, in view of continuity of functions φ_3 , φ_2 and ψ , which completes the proof.

THEOREM 4.3. *Let hypotheses H_1-H_4 and H_7-H_9 be fulfilled. The necessary and sufficient condition for a function $\varphi \in L$ to belong to L_0 (for a function $\varphi \in L'$) to belong to L_0 is that the principal filter (the principal ideal) of φ in the lattice L (in the lattice L') be a prime filter (a prime ideal).*

Proof. We are going to prove the theorem for L . The proof for L' is similar. The necessity of the condition follows directly from Corollary 4.1. To prove its sufficiency, let us assume that there exists such a continuous solution ψ of (0.1) in I that ψ does not satisfy (0.3) in I , i.e., $\psi \in L \setminus L_0$. If this is the case, then there exists a point $x_0 \in I$ such that inequality (4.19) holds. It follows from hypothesis H_1 that $x_0 \neq \xi$. Let x_1 be the point defined in Lemma 4.3. We define a function φ_3 in the following way: let φ be an arbitrary function defined and continuous in the interval $[f(x_0), x_1]$ and satisfying the condition

$$(4.28) \quad [\varphi f(x_0)] = \varphi_0[f(x_0)], \quad \varphi(x_1) = \psi(x_1),$$

where φ_0 is defined by (4.21), and

$$(4.29) \quad \varphi(x) < \psi(x) \quad \text{for } x \in [f(x_0), x_1].$$

Let us put

$$(4.30) \quad \psi_3(x) = \varphi(x) \quad \text{for } x \in [f(x_0), x_1],$$

$$(4.31) \quad \psi_3(x) = \varphi(x) \quad \text{for } x \in (x_1, x_0].$$

The continuity of ψ_3 follows from the continuity of ψ and φ from (4.28), (4.30) and (4.31).

Let $\bar{\psi}$ be an arbitrary function defined and continuous in the interval $[x_1, x_0]$ and satisfying the conditions

$$(4.32) \quad \bar{\psi}(x_0) = \psi(x_0),$$

$$(4.33) \quad \bar{\psi}(x_3) < \psi(x_3) \quad \text{for } x \in [x_1, x_3],$$

where x_3 is an arbitrary inner point of the interval $[x_1, x_0]$,

$$(4.34) \quad \varphi_3(x) \leq \psi(x) \leq \varphi_2(x) \quad \text{for } x \in [x_1, x_0],$$

where φ_3 is defined by (4.26) and φ_2 is defined by (4.21).

Let us define the function ψ_4 in the following way:

$$(4.35) \quad \psi_4(x) = \max[\psi(x), \varphi_3(x)] \quad \text{for } x \in [f(x_0), x_1],$$

$$(4.36) \quad \psi_4(x) = \bar{\psi}(x) \quad \text{for } x \in [x_1, x_0].$$

The continuity of ψ_4 follows from the continuity of φ_3 , ψ and $\bar{\psi}$ and from (4.32).

Now, we can define the following solutions of (0.1) in I :

$$(4.37) \quad \psi_1(x) = \begin{cases} \varphi_0(x) & \text{for } x \in [\xi, f(x_0)], \\ \psi_3(x) & \text{for } x \in (f(x_0), x_0], \\ \psi(x) & \text{for } x \in (x_0, b), \end{cases}$$

$$(4.38) \quad \psi_2(x) = \begin{cases} \psi(x) & \text{for } x \in [\xi, f(x_0)], \\ \psi_4(x) & \text{for } x \in [f(x_0), x_0], \\ \max[\psi(x), \varphi_2(x)] & \text{for } x \in [x_0, b]. \end{cases}$$

The continuity of ψ_1 follows from (4.37), from the continuity of the functions φ_0 , φ_3 and ψ and from (4.28), (4.30) and (4.31). The continuity of ψ_2 follows from (4.38), from the continuity of the functions ψ , ψ_4 , φ_3 and φ_2 and Lemma 4.3 and (4.34). The function ψ_1 satisfies inequality (0.1) by virtue of (4.25) and Lemma 4.1. The function ψ_2 satisfies inequality (0.1), by virtue of (4.27) and Lemma 4.1. We are going to show that

$$(4.39) \quad \psi_1 \cup \psi_2(x) \geq \psi(x) \quad \text{for } x \in I.$$

Inequality (4.39) holds in the interval $[\xi, f(x_0)]$, by virtue of (4.29), (4.37), (4.38) and (4.27). It follows from (4.37), (4.38), (4.35), (4.27), (4.31) and (4.25), then (4.39) also holds in the interval $[f(x_0), x_0]$. In the interval $[x_0, b)$, the inequality follows from (4.37), (4.38) and (4.25).

However, it follows from (4.30), Lemma 4.2 and (4.37) that the inequality $\psi_1(x) \geq \psi(x)$ does not hold in the whole interval I . It also

follows from (4.34), (4.36), Lemma 4.2 and (4.38) that the inequality $\psi_2(x) \geq \psi(x)$ does not hold in the whole interval I . Since the function $\psi_1 \cup \psi_2$ belongs to L , in view of Theorem 3.3 and Definition 2.1. The existence of such functions ψ_1 and ψ_2 means that the principal filter of ψ is not a prime one in L , which contradicts the assumptions of the theorem. In this way the proof of Theorem 4.3 is complete.

Theorem 4.3 shows that the conditions of irreducibility for union characterizes the family L_0 within the family L , and the condition of irreducibility for intersection characterizes L_0 within the family L' .

We are going to study the relations between L and L' , and the families of $\{f\}$ -decreasing functions continuous in I and $\{f\}$ -increasing functions continuous in I , respectively.

DEFINITION 4.1. If hypotheses H_3 , H_4 and H_5 are fulfilled, we define the function $R: \Omega \rightarrow \Omega_\xi$, where Ω_x is defined by (1.3), by the formula

$$(4.40) \quad R(x, y) = \varphi(\xi) \quad \text{for} \quad (x, y) \in \Omega,$$

where φ is a solution of equation (0.3) in I determined by the condition

$$(4.41) \quad \varphi(x) = y.$$

As an immediate consequence of the above definition we obtain the following

COROLLARY 4.2. *If hypotheses H_1 - H_4 and H_5 are fulfilled, then the function R is invertible with respect to the second variable.*

LEMMA 4.4. *If hypotheses H_1 - H_4 and H_5 are fulfilled, then the function R is continuous in Ω .*

Proof. Let

$$P(x, y), P_n(x_n, y_n) \in \Omega \quad \text{for} \quad n = 1, 2, \dots$$

and

$$(4.42) \quad \lim_{n \rightarrow \infty} P_n = P.$$

Let us denote by φ_n the continuous solution of equation (0.3) in I satisfying the condition

$$(4.43) \quad \varphi_n(x_n) = y_n \quad \text{for} \quad n = 1, 2, \dots$$

Further, let us denote by φ the continuous solution of equation (0.3) in I satisfying condition (4.41). Hypothesis H_5 ensures that the functions φ_n and φ are determined uniquely. It follows from (4.43), (4.41) and (4.42) and from hypothesis H_5 that

$$(4.44) \quad \lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x)$$

uniformly in I (see [10]), whence $\lim_{n \rightarrow \infty} \varphi_n(\xi) = \varphi(\xi)$. Therefore the func-

tion R is continuous at the point P . Since P is an arbitrary point of Ω , the continuity of R has been proved.

DEFINITION 4.2. Let ψ be a function defined in I and let its graph lie in Ω . We denote

$$(4.45) \quad \eta(x) = \varphi(\xi) = R[x, \psi(x)] \quad \text{for } x \in I,$$

where φ satisfies (4.41) with $y = \psi(x)$.

LEMMA 4.5. *Let hypotheses H_1 - H_4 and H_9 be fulfilled, let ψ be a function defined in I with its graph in Ω and let η be defined by formula (4.45). The necessary and sufficient condition for η to be continuous in I is the continuity of ψ in I .*

Proof. Since the function R is continuous in Ω , by virtue of Lemma 4.4 then the continuity of ψ in I implies the continuity of η in I . Thus the condition is sufficient. In order to prove the necessity, let us take such $x, x_n \in I$ for $n = 1, 2, \dots$ that

$$(4.46) \quad \lim_{n \rightarrow \infty} x_n = x.$$

Definition 4.1 and the continuity of η in I imply that

$$\lim_{n \rightarrow \infty} \varphi_n(\xi) = \lim_{n \rightarrow \infty} \eta(x_n) = \eta(x) = \varphi(\xi),$$

where φ is determined by condition (4.43) with $y = \varphi(x) = \psi(x)$ and φ_n is determined by

$$(4.47) \quad \varphi_n(x_n) = \psi(x_n) \quad \text{for } n = 1, 2, \dots$$

Since the sequence φ_n converges to φ , uniformly in I (see [5]), we have

$$\lim_{n \rightarrow \infty} \varphi(x_n) = \lim_{n \rightarrow \infty} \varphi_n(x_n) = \varphi(x) = \psi(x),$$

by virtue of (4.47) and (4.46). This proves the continuity of ψ at x and then the continuity of ψ in the whole interval I , because x is an inner point of I .

Remark 4.1. Let us notice that the function R is a prime integral of equation (0.3) in I , i.e., R is constant on every solution of (0.3).

In the sequel we will be able to prove that there exists a one-to-one correspondence between the family of $\{f\}$ -decreasing ($\{f\}$ -increasing) functions continuous in I and the family of continuous solutions of inequality (0.1) (of inequality (0.2)) in I , i.e., we are going to prove the following

THEOREM 4.4. *Let hypothesis H_1 - H_4 and H_9 be fulfilled. The one-to-one correspondence between the family L (the family L') and the family of continuous functions η , $\{f\}$ -decreasing in I ($\{f\}$ -increasing in I) is given by formula (4.45), where $\psi \in L$ ($\psi \in L'$).*

Proof. Let $\psi \in L$ and $t \in I$. We denote by φ_1 the continuous solution of equation (0.1) in I satisfying the condition

$$(4.48) \quad \varphi_1[f(t)] = \psi[f(t)]$$

and by φ_2 the continuous solution of equation (0.3) in I satisfying the condition

$$(4.49) \quad \varphi_2(t) = \psi(t).$$

Hypothesis H_0 ensures that such solutions of (0.3) exist and are determined uniquely. It follows from (4.48), (0.1), (4.49) and (0.3) that

$$\varphi_1[f(t)] = \psi[f(t)] \leq g[t, \psi(t)] = g[t, \varphi_2(t)] = \varphi_2[f(t)].$$

Since the family of continuous solutions of equation (0.3) in I is a one-parameter one by virtue of hypothesis H_0 , the last inequality has to hold in the whole interval I , i.e.,

$$\varphi_1(x) \leq \varphi_2(x) \quad \text{for } x \in I.$$

Thus $\varphi_1(\xi) \leq \varphi_2(\xi)$, whence

$$(4.50) \quad \eta[f(t)] = \varphi_1(\xi) \leq \varphi_2(\xi) = \eta(t) = R[t, \psi(t)]$$

for any $t \in I$, in view of (4.45). It follows from (4.50) that η is an $\{f\}$ -decreasing function in I (see Definition 3.2) and it is a continuous function in I , by virtue of (4.50) and (4.45) (see Lemma 4.5).

Conversely, let us assume that ψ is a function defined in I and η , defined by formula (4.45), is an $\{f\}$ -decreasing in function continuous in I . Therefore it follows from (4.45) that

$$R[f(x), \psi(f(x))] = \eta[f(x)] \leq \eta(x) = R[x, \psi(x)]$$

and

$$(4.51) \quad \varphi_1(\xi) = \eta[f(x)] \leq \eta(x) = \varphi_2(\xi),$$

where φ_1 and φ_2 are defined by conditions (4.48) and (4.49), respectively. Consequently, conditions (4.48), (4.51), (0.3) and (4.49) imply that

$$\psi[f(x)] = \varphi_1[f(x)] \leq \varphi_2[f(x)] = g[x, \varphi_2(x)] = g[x, \psi(x)] \quad \text{for } x \in I.$$

Therefore ψ satisfies inequality (0.1) in I and is a continuous function in I , which completes the proof of the theorem.

Let us denote by K the family of all $\{f\}$ -decreasing functions continuous in I , by K' the family of all $\{f\}$ -increasing functions continuous in I and by K_0 the family of all functions constant in I . In each of these three cases the functions take values in the set E . Moreover, let us denote by T the map $T: L \cup L' \rightarrow K \cup K'$, defined by (4.45).

THEOREM 4.5. *Let hypotheses H_1 – H_4 and H_9 be fulfilled. Then the map T is an isomorphism between*

- 1° L and K ,
- 2° L' and K' ,
- 3° L_0 and K_0 .

Proof. We are going to prove point 1° of the theorem. The proofs of the other points are similar. It follows from Theorem 4.4 that T is an invertible map of L onto K . We are going to prove the continuity of T . Let ψ_n be a sequence of elements of L converging to an element $\psi \in L$. Therefore, by virtue of (4.45) and the continuity of function R , ensured by Lemma 4.4, we obtain

$$T(\psi_n)(x) = \psi_n(x) = R[x, \psi_n(x)] \rightarrow R[x, \psi(x)] \quad \text{for } x \in I,$$

where η_n is defined by formula (4.45) (with ψ_n instead of ψ). Thus T is a continuous map.

Now, let η_n be a sequence of elements of K converging to an element $\eta \in K$. Since T is an invertible map, we can denote by ψ_n the sequence defined by

$$\psi_n(x) = T^{-1}(\eta_n)(x) \quad \text{for } x \in I, n = 1, 2, \dots,$$

where T^{-1} is an inverse map to T . It follows from Definition 4.2 that

$$(4.52) \quad \eta_n(x) = R[x, \psi_n(x)] \quad \text{for } x \in I, n = 1, 2, \dots$$

and for every $x \in I$, there exists such a sequence $\varphi_n \in L_0$ (depending on x) that

$$\eta_n(x) = \varphi_n(\xi) \quad \text{for } n = 1, 2, \dots,$$

where

$$(4.53) \quad \varphi_n(x) = \psi_n(x) \quad \text{for } n = 1, 2, \dots$$

Since

$$(4.54) \quad \lim_{n \rightarrow \infty} \eta_n(x) = \eta(x) \quad \text{for } x \in I,$$

then

$$\lim_{n \rightarrow \infty} \varphi_n(\xi) = \varphi(\xi),$$

whence

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x)$$

uniformly in I (see [5]), where φ is an element of L_0 satisfying the condition $\varphi(\xi) = \eta(x)$. Thus there exists a limit

$$\psi(x) = \lim_{n \rightarrow \infty} \psi_n(x) = \lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x) \quad \text{for } x \in I,$$

by virtue of (4.53) and (4.54). It follows from (4.52) and (4.54) that (4.45) holds, because R is continuous in Ω in view of Lemma 4.4. Thus $\psi \in L$ by virtue of Theorem 4.4. This way we have proved that the map T is a homeomorphism. Now, we are going to show that T is a lattice isomorphism between the lattice L and the lattice K (K is a lattice in view of Theorem 2.5).

Let $\psi_1, \psi_2 \in L$ and let, for an $x \in I$, φ_1 and φ_2 be such elements of L_0 that

$$(4.55) \quad \varphi_1(x) = \psi_1(x), \quad \varphi_2(x) = \psi_2(x).$$

Further, let us denote by $\eta_1, \eta_2 \in K$ the functions defined by

$$\eta_1 = T(\psi_1), \quad \eta_2 = T(\psi_2).$$

It follows from (4.45) that

$$\eta_1(x) = R[x, \psi_1(x)], \quad \eta_2(x) = R[x, \psi_2(x)]$$

and

$$(4.56) \quad \eta_1(x) = \varphi_1(\xi), \quad \eta_2(x) = \varphi_2(\xi),$$

by virtue of (4.30). Let us assume that for an arbitrary fixed $x_0 \in I$ the inequality

$$(4.57) \quad \psi_1(x_0) \leq \psi_2(x_0)$$

holds. Then $\psi_1 \cap \psi_2(x_0) = \psi_1(x_0)$, by virtue of (2.10). Hence

$$R[x_0, \psi_1 \cap \psi_2(x_0)] = R[x_0, \psi_1(x_0)] = \eta_1(x_0) = \eta_1 \cap \eta_2(x_0),$$

because $\eta_1(x_0) \leq \eta_2(x_0)$, in view of (4.57), (4.55), hypothesis H_9 , (4.58) and (2.10). In the case where, instead of (4.57), the inequality $\psi_1(x_0) \geq \psi_2(x_0)$ holds, the proof is similar. Therefore we have proved that $T(\psi_1 \cap \psi_2) = T(\psi_1) \cap T(\psi_2)$. In a similar way one can prove that $T(\psi_1 \cup \psi_2) = T(\psi_1) \cup T(\psi_2)$. One can prove, in a similar way, the equality $T(\psi_1 \cup \psi_2) = T(\psi_1) \cup T(\psi_2)$. Thus T is an isomorphism between the lattices L and K . The proof of the first part of the theorem is completed. The other two parts can be proved in a similar way.

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