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SOME ALGEBRAIC ASPECTS OF MULTIGRID METHODS

WILGARD LANG

Technische Universität, Karl-Marx-Stadt, DDR

In this paper we present a combination of iteration methods with multigrid steps and discuss some of its properties. It is our goal to solve a nonsymmetric system of linear equations

$$Ay = f, \quad A \in \mathbf{R}^{N\times N}, \quad y, f \in \mathbf{R}^N. \tag{1}$$

We interpret the indices $1, \ldots, N$ of the unknowns as the numbers of points of a fine grid. We perform a permutation of $(1 \ldots N)$ to $(k_1 \ldots k_n\, k_{n+1} \ldots k_N)$ so that $k_1, \ldots, k_n$ are the numbers of points of a coarser grid. This is only done for description purposes. We then replace (1) by the equation

$$Kx = b \tag{2}$$

with

$$K = \mathfrak{C} A \mathfrak{C}^T = \begin{pmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{pmatrix},$$

$$x = \mathfrak{C} y, \quad b = \mathfrak{C} f.$$

First we consider a two-grid step; it can be described as follows. Let x be an actual approximation of the solution $K^{-1}b$ and let

$$\tilde{x} = x + Br \quad \text{with } r = b - Kx,$$

$$\tilde{e} = x^* - \tilde{x} = (I - BK)e \quad \text{with } x^* = K^{-1}b,$$

$$\tilde{r} = (I - KB)r = K(I - BK)e.$$

There $B = PCR$ is a two-grid operator with $P \in \mathbf{R}^{N\times n}$, $R \in \mathbf{R}^{n\times N}$ and $\operatorname{rank} P = \operatorname{rank} R = n$. P is a prolongation matrix and R is a restriction matrix. The matrix C in the two-grid operator is an element of $\mathbf{R}^{n\times n}$ and has the form

$$C = (RKP)^{-1}.$$

Following the ideas of Greenbaum [1], it can be shown that

(3) (a) the nullspace $N(I-BK)$ of the matrix $I-BK$ is equal to the range $\boldsymbol{R}(P)$ of the prolongation matrix P,
(b) $\boldsymbol{R}(K(I-BK)) \subset N(R)$.
(c) the matrices BK and $I-BK$ are projectors,
(d) the direct sum of $N(I-BK)$ and $\boldsymbol{R}(I-BK)$ is the full space $\boldsymbol{R}^N$: $N(I-BK) \dot{+} \boldsymbol{R}(I-BK) = \boldsymbol{R}^N$.

In the case where K is a symmetric positive-definite matrix, the choice of the prolongation matrix $P = R^T$ is subject to the following sharper requirements:
(a) $B^T = B$ is positive-semidefinite,
(b) $I-BK$ is selfadjoint in the K-inner product,
(c) $\boldsymbol{R}(P) \oplus N(R) = \boldsymbol{R}^N$ ($\oplus$ denoting the orthogonal sum),
(d) $\boldsymbol{R}(I-BK) \overset{K}{\oplus} N(I-BK) = \boldsymbol{R}^N$ ($\overset{K}{\oplus}$ denoting the K-orthogonal sum),
(e) $\boldsymbol{R}(K(I-BK)) = N(R)$.

In the sequel we want to combine a two-grid step with a step of an iteration method

$$\hat{x} = x + Qr$$

with a suitable matrix Q. This is possible in two ways:

$$\text{“pre-smoothing”:} \quad x \to \hat{x} \to \tilde{\hat{x}}, \quad \text{or}$$
$$\text{“post-smoothing”:} \quad x \to \tilde{x} \to \hat{\tilde{x}}.$$

We obtain accordingly

$$\tilde{\hat{e}} = (I-BK)(I-QK)e,$$
$$\hat{\tilde{e}} = (I-QK)(I-BK)e,$$
$$\tilde{\hat{r}} = K(I-BK)(I-QK)e,$$
$$\hat{\tilde{r}} = (I-QK)K(I-BK)e.$$

Setting $x^k = x$ and $x^{k+1} = \tilde{\hat{x}}$ or $x^{k+1} = \hat{\tilde{x}}$, we get for $k = 0, 1, \ldots$ a two-level iteration process. For

$$Q = \begin{pmatrix} K_1 & 0 \\ K_{21} & K_2 \end{pmatrix}^{-1}$$

this becomes a combination of Block–Gauß–Seidel iteration with a two-grid method. For the Block–Gauß–Seidel iteration we have

$$Q = \begin{pmatrix} K_1^{-1} & 0 \\ -K_2^{-1} K_{21} K_1^{-1} & K_2^{-1} \end{pmatrix},$$

$$I-QK = \begin{pmatrix} 0 & -K_1^{-1} K_{12} \\ 0 & K_2^{-1} \quad K_{21} K_1^{-1} K_{12} \end{pmatrix}.$$

First we consider the pre-smoothing. If the relation

$$\boldsymbol{R}(I-QK) \subset N(I-BK)$$

is valid, then $\tilde{e}=0$. We have

$$\boldsymbol{R}(I-QK) = \operatorname{span}\begin{pmatrix} -K_1^{-1}K_{12} \\ K_2^{-1}K_{21}K_1^{-1}K_{12} \end{pmatrix} \subset \operatorname{span}\begin{pmatrix} I_n \\ -K_2^{-1}K_{21} \end{pmatrix}.$$

Consequently, if

$$P = \begin{pmatrix} I_n \\ -K_2^{-1}K_{21} \end{pmatrix} \tag{4}$$

then

$$\boldsymbol{R}(I-QK) \subset \operatorname{span}\begin{pmatrix} I_n \\ -K_2^{-1}K_{21} \end{pmatrix} = \operatorname{span} P = \boldsymbol{R}(P) = N(I-BK),$$

and therefore $\tilde{e}=0$, so that $\tilde{x}$ is the exact solution of equation (2).

Following Ruge, Stüben [3,4], in the algorithm AMG a good choice of prolongation matrix $\tilde{P}$ for a symmetric positive-definite system in a multi-grid-iteration process is achieved by

$$P = \begin{pmatrix} I_n \\ -D_2K_{21} \end{pmatrix}, \quad D_2 = \operatorname{Diag}(d_{n+1}, \ldots, d_N), \quad d_i = k_{ii}^{-1};$$

this is an approximation of the matrix P from (4).

In a similar way we can consider the post-smoothing. If the relation $\boldsymbol{R}(K(I-BK)) \subset N(I-KQ)$ is valid, then $\hat{r}=0$ and, for a regular K, also $\hat{e}=0$. We have

$$I-KQ = \begin{pmatrix} K_{12}K_2^{-1}K_{21}K_1^{-1} & -K_{12}K_2^{-1} \\ 0 & 0 \end{pmatrix},$$

$$N(I-KQ) = \{x \in \boldsymbol{R}^N : K_{12}K_2^{-1}(-K_{21}K_1^{-1} \quad I_{N-n})x = 0\}.$$

Hence, for

$$R = (-K_{21}K_1^{-1} \quad I_{N-n}),$$

we get

$$N(R) \subset N(I-KQ)$$

and therefore, in view of (3 b),

$$\boldsymbol{R}(K(I-BK)) \subset N(R) \subset N(I-KQ)$$

and so $\hat{e}=0$.

In each of the two-grid steps one has to solve a linear system with the matrix $\tilde{K} = RKP$. In the case of pre-smoothing,

$$\tilde{K}_V = (I_n X)K\begin{pmatrix} I_n \\ -K_2^{-1}K_{21}\end{pmatrix} = K_1 - K_{12}K_2^{-1}K_{21}$$

for any $X \in \boldsymbol{R}^{n\times(N-n)}$; therefore we choose $X = 0$.

Similarly, in the case of post-smoothing

$$\tilde{K}_n = (-K_{21}K_1^{-1} \quad I_{N-n})K\begin{pmatrix} Y \\ I_{N-n}\end{pmatrix} = K_2 - K_{21}K_1^{-1}K_{12}$$

for any $Y \in \boldsymbol{R}^{n\times(N-n)}$; therefore, we set $Y = 0$.

The algorithmic realization of the combination of one step Block–Gauß–Seidel method with a two-grid step has the form

$$\hat{x} = x + Qr, \qquad \tilde{\hat{x}} = \hat{x} + B_V\hat{r}$$

with

$$B_V = P_V C_V R_V = P_V(R_V K P_V)^{-1} R_V,$$

$$P_V = \begin{pmatrix} I_n \\ -K_2^{-1}K_{21}\end{pmatrix}, \qquad R_V = (I_n \quad 0),$$

$$C_V = (K_1 - K_{12}K_2^{-1}K_{21})^{-1} = \tilde{K}_V^{-1}.$$

If $r = b - Kx = (r_1^T r_2^T)^T$, then the following algorithm is a good realization:

$$r_2 := K_2^{-1} r_2,$$

$$r_1 := r_1 - K_{12} r_2,$$

$$r_1 := C_V r_1,$$

$$r_2 := r_2 - K_2^{-1}K_{21} r_1,$$

$$x := x + r.$$

For the case where K is a symmetric positive-definite matrix, we can consider as the effect of the post-smoothing the operator

$$B = R^T C R \quad \text{with } R = (I_n - K_{12}K_2^{-1}),$$

$$C = (K_1 - K_{12}K_2^{-1}K_{21})^{-1}, \qquad K_{21} = K_{12}^T.$$

We are interested in the K-norm of the error $\hat{e} = (I - QK)(I - BK)e$

$$\|(I-QK)(I-BK)e\|_K \leqslant \|(I-QK)(I-BK)\|_K \|e\|_K.$$

The K-norm of matrix $(I-QK)(I-BK)$ is given by

$$\|(I-QK)(I-BK)\|_K = \max\left(\|(I-QK)y\|_K : \|y\|_K = 1,\ y \in \boldsymbol{R}(I-BK)\right).$$

To estimate it, we can use the singular values of the matrix $I-QK$; in the K-unitary space these are the roots of the eigenvalues of the matrix

$$(I-Q^T K)(I-QK) = \begin{pmatrix} 0 & -K_1^{-1} K_{12} K_2^{-1} K_{21} K_1^{-1} K_{12} \\ 0 & K_2^{-1} K_{21} K_1^{-1} K_{12} \end{pmatrix}.$$

If $\sigma_1, \ldots, \sigma_N$ are the singular values in question, then $\sigma_1 = \ldots = \sigma_n = 0$ and $\sigma_{n+1}^2, \ldots, \sigma_N^2$ are the eigenvalues of the matrix

$$K_2^{-1} K_{21} K_1^{-1} K_{12}.$$

If $v_1, \ldots, v_N$ are the corresponding singular vectors, i.e.,

$$\langle v_i, v_j \rangle_K = \delta_{ij}, \quad 1 \leqslant i, j = N,$$

then

$$y = \sum_{i=1}^{N} \langle y, v_i \rangle_K v_i, \quad y \in \boldsymbol{R}^N$$

and

$$\|(I-QK)y\|_K^2 = \sum_{i=n+1}^{N} \langle y, v_i \rangle_K^2 \sigma_i^2.$$

First, we see that

$$\|(I-QK)(I-BK)\|_K = \sigma_N.$$

Further we can show that for $y = (y_1^T y_2^T)^T$ and

$$v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \begin{pmatrix} -K_1^{-1} K_{12} \\ I_{N-n} \end{pmatrix} v_{i2} \quad \text{for } i = n+1, \ldots, N,$$

the properties $\|y\|_K = 1$ and $y \in \boldsymbol{R}(I-BK)$ imply

$$(Ky, v_i) = (1-\sigma_i^2)(K_2 y_2, v_{i2})$$

and

$$(K_2 y_2, y_2) = \sum_{i=1}^{N} (1-\sigma_i^2)(K_2 y_2, v_{i2})^2 = 1.$$

Therefore,

$$\begin{aligned} \|(I-QK)y\|_K^2 &= \sum_{i=n+1}^{N} (K_2 y_2, v_{i2})^2 (1-\sigma_i^2)^2 \sigma_i^2 \\ &\leqslant \sum_{i=n+1}^{N} (K_2 y_2, v_{i2})^2 (1-\sigma_i^2) \cdot \max_i \left((1-\sigma_i^2)\sigma_i^2\right) \leqslant 1/2 \end{aligned}$$

and

$$\|(I-QK)(I-BK)\|_K \leqslant \left(\max_i \left((1-\sigma_i^2)\sigma_i^2\right)\right)^{1/2} = (1/2)\sqrt{2}.$$

The resulting estimate for the error is

$$\|\hat{e}\|_K \leqslant \tilde{\sigma}_N \|e\|_K \quad \text{or} \quad \|\hat{e}\|_K = (1/2)\sqrt{2}\,\|e\|_K.$$

All these considerations have only theoretical meaning. But they can be useful in estimating an approximate choice of the prolongation or restriction matrices or of an approximate solution of similar linear equations with K_1, K_2, K_V or K_N.

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