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Homological methods in fixed-point theory of multi-valued maps

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WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

INTRODUCTION

One of the most remarkable theorems of topology is the Lefschetz Fixed-Point Theorem. In its most elementary form it may be formulated as follows: Let X denote a finite polyhedron and let $f\colon X \to X$ be a continuous, single-valued map. Then, by the use of the field of rationals Q as coefficients, f induces linear maps $f_{*n}\colon H_n(X) \to H_n(X)$. The number $\lambda(f) = \sum_n (-1)^n \mathrm{Trace}(f_{*n})$ is called the *Lefschetz number* of f. Then *a sufficient condition for f to have at least one fixed point is that $\lambda(f) \neq 0$.* This theorem was first proved by S. Lefschetz for manifolds, and by H. Hopf for polyhedra, and generalized by S. Lefschetz to all compact metric ANR-spaces. In 1967, A. Granas [41], using the generalized trace theory given by J. Leray [64], showed that the Lefschetz Fixed-Point Theorem remains true for compact (single-valued) self maps of arbitrary metric ANR-s. In some cases the theory of fixed points for ANR-s implies fixed-point theorems for some special classes of spaces which are not ANR-s, but which are intimately related to them: see [10], [23], [42], [61], [63], [85].

In 1946, S. Eilenberg and D. Montgomery [21] made the important observation that, by using the Vietoris Mapping Theorem [89] as a tool, the Lefschetz Fixed-Point Theorem could be carried over to the case of multi-valued acyclic maps of compact metric ANR-s, i.e., maps for which the image of every point is an acyclic compact set. Later, similar generalizations of the Lefschetz Fixed-Point Theorem and of the other topological theorems for acyclic maps were given by E. G. Begle [3], D. G. Bourgin [8], [9]. L. Górniewicz [29], [31]–[34], [38], A. Granas [38], [47], B. Halpern [49], C. Himmelberg [52], J. W. Jaworowski [54], [55], B. O'Neill [72], M. Powers [73]–[75], C. J. Rhee [76], H. Schirmer [78], W. L. Strother [82], [83], and S. A. Williams [91]. An important type of acyclic maps are those which are convex-valued. Various fixed-point theorems for compact operators were extended to this special type of maps: see [39], [40], [57]. As in the single-valued case, fixed-point theorems for multi-valued maps are very useful in many branches of mathematics; they have been applied, for instance, in the theory of games and more recently also in ordinary differential equations and optimal control theory.

In this paper we consider a new class of multi-valued maps which we have called *admissible maps*. The class of admissible maps contains acyclic maps and it is essentialy larger. The class of admissible maps and its properties are studied in Chapter III. The Lefschetz Fixed-Point Theorem for admissible maps and its consequences are studied in Chapters V and VI. In Chapter VII we consider some geometrical properties of admissible maps. Chapter I, II, IV are of auxiliary character and are devoted to homology theory, theory of trace and theory of ANR-spaces, respectively. The principal results of the paper are the following:

1. a modern proof of the Eilenberg–Montgomery Theorem,

2. the Lefschetz Fixed-Point Theorem for admissible maps of AANR-s, particularly, metric ANR-s,

3. the Lefschetz Fixed-Point Theorem for admissible maps of compact w-AANR-s,

4. the neighbourhood fixed-point property of compact acyclic subsets of w-AANR-s within the class of admissible maps,

5. the fixed-point property of the Tychonoff cube within the class of admissible maps,

6. Theorem on Antipodes for admissible maps in the Euclidean space R^n,

7. Theorem on Invariance of Domain for strongly admissible maps in the Euclidean space R^n.

All definitions, propositions, theorems and formal remarks are numbered consequtively by chapter and section. Thus Theorem III.2.7, for example, refers to item 7 in Chapter III, Section 2. The chapter number will be omitted for items in the same chapter.

I am very much indebted to Professor A. Granas for having inspired my interest in these problems.

I wish to express my gratitude to Dr. C. Bowszyc for valuable remarks concerning this paper.

I. HOMOLOGY

In this section we consider the Čech homology functor H with compact carriers and those of its properties which are of importance in the fixed-point theory of multi-valued maps. Therefore all facts concerning H are formulated only in the form necessary in the material which follows. The Čech homology and cohomology are of auxiliary importance.

1. Preliminaries. By a *pair* of spaces (X, X_0) we understand a pair consisting of a Hausdorff topological space X and of one its subsets X_0. A pair of the form $(X, \emptyset)$ will be identified with the space X. Let (X, X_0), (Y, Y_0) be two pairs; if $X \subset Y$ and $X_0 \subset Y_0$, then we say that (X, X_0) is a *subpair* of (Y, Y_0) and indicate this by writing $(X, X_0) \subset (Y, Y_0)$. A pair (X, X_0) is called *compact* if X is a compact space and X_0 is closed subset of X. By a *map* $f\colon (X, X_0)\to(Y, Y_0)$ we understand a continuous (single-valued) map $f\colon X\to Y$ satisfying the condition $f(X_0) \subset Y_0$. The category of all pairs and maps will be denoted by C. By $\tilde{C}$ will be denoted the subcategory of C consisting of all compact pairs and maps of such pairs. Two maps $f, g\colon (X, X_0)\to(Y, Y_0)$ are said to be *homotopic* (written $f \sim g$) provided that there is a map $h\colon (X\times\langle 0, 1\rangle, X_0\times\langle 0, 1\rangle)\to(Y, Y_0)$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for each $x\in X$. We observe that if (X, X_0) is a pair in $\tilde{C}$, then $(X\times\langle 0, 1\rangle, X_0\times\langle 0, 1\rangle)$ is also in $\tilde{C}$.

By H_* (H^*) we denote the *Čech homology* (*cohomology*) *functor* with the coefficients in the field of rational numbers Q from the category $\tilde{C}$ to the category $\mathscr{A}$ of graded vector spaces over Q and linear maps of degree zero. Thus, for a pair (X, X_0),

$$H_*(X, X_0) = \{H_q(X, X_0)\}, \quad (H^*(X, X_0) = \{H^q(X, X_0)\}),$$

is a graded vector space and, for a map $f\colon (X, X_0)\to(Y, Y_0)$, $H_*(f)$ ($H^*(f)$) is the induced linear map

$$f_* = \{f_{*q}\}\colon H_*(X, X_0)\to H_*(Y, Y_0)$$
$$(f^* = \{f^{*q}\}\colon H^*(Y, Y_0)\to H^*(X, X_0)),$$

where

$$f_{*q}\colon H_q(X, X_0) \to H_q(Y, Y_0), \qquad (f^{*q}\colon H^q(Y, Y_0) \to H^q(X, X_0)).$$

We assume as known that the functor H_* (H^*) satisfies all of the Eilenberg–Steenrod axioms for homology (cohomology). Recall ([22], Chapter IX) that a Čech homology (cohomology) theory can be defined on the category C. Then the Čech cohomology satisfies all of the Eilenberg–Steenrod axioms; however, the Čech homology satisfies all of the Eilenberg–Steenrod axioms except that of exactness.

By $\mathrm{Hom}_Q\colon \mathscr{A} \to \mathscr{A}$ we denote the contravariant functor which to a graded vector space $E = \{E_q\}$ assigns the conjugate graded space $\mathrm{Hom}_Q(E) = \{\mathrm{Hom}(E_q, Q)\}$ and to a linear map $l\colon E_1 \to E_2$ between graded spaces assigns the conjugate map $\mathrm{Hom}_Q(l)\colon \mathrm{Hom}_Q(E_2) \to \mathrm{Hom}_Q(E_1)$ given by the formula

$$\mathrm{Hom}_Q(l)(u) = u \circ l \quad \text{for every } u \in \mathrm{Hom}_Q(E_2).$$

We now formulate the Duality Theorem between the Čech homology and cohomology.

(1.1) THEOREM (see [34] or [51]). *On the category $\tilde{C}$ the functors H_* and $\mathrm{Hom}_Q \circ H^*$ are naturally isomorphic; in other words, for every $f\colon (X, X_0) \to (Y, Y_0)$ in $\tilde{C}$ we have the commutative diagram*

$$\begin{array}{ccc} H_*(X, X_0) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_Q\big(H^*(X, X_0)\big) \\ \Big\downarrow {\scriptstyle f_*} & & \Big\downarrow {\scriptstyle \mathrm{Hom}_Q(f^*)} \\ H_*(Y, Y_0) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_Q\big(H^*(Y, Y_0)\big) \end{array}$$

A graded vector space $E = \{E_q\}$ in $\mathscr{A}$ is said to be of *finite type* provided: (i) $\dim E_q < \infty$ for all q and (ii) $E_q = 0$ for almost all q.

The following fact is well known:

(1.2) *If E is a graded vector space of a finite type, then the graded vector space $\mathrm{Hom}_Q(E)$ is isomorphic to E; in particular, $\mathrm{Hom}_Q(E)$ is also of a finite type.*

A pair (X, X_0) in $\tilde{C}$ is said to be of *finite type with respect to H_* (H^*)* provided the graded vector space $H_*(X, X_0)$ ($H^*(X, X_0)$) is of finite type.

From (1.1) and (1.2) we instantly obtain:

(1.3) *A pair (X, X_0) in $\tilde{C}$ is of finite type with respect to H_* if and only if (X, X_0) is of a finite type with respect to H^*.*

For pairs (X, X_0), (Y, Y_0) in C we define the Cartesian product as the pair given by $(X, X_0) \times (Y, Y_0) = (X \times Y, X \times Y_0 \cup X_0 \times Y)$, where in $X \times Y$ the Cartesian product topology is given.

Given maps $f\colon (X, X_0)\to(Y, Y_0)$ and $g\colon (X', X_0')\to(Y', Y_0')$, we can define the product map $f\times g\colon (X, X_0)\times(X', X_0')\to(Y, Y_0)\times(Y', Y_0')$ by letting

$$(f\times g)(x, x') = \big(f(x), g(x')\big) \quad \text{for every } x\in X \text{ and } x'\in X'.$$

(1.4) KÜNNETH THEOREM ([81], p. 405). *For every two pairs* (X, X_0), (X', X_0') *in* $\tilde{C}$, *there is a linear isomorphism*

$$\mathscr{L}\colon H^*\big((X, X_0)\times(X', X_0')\big)\to H^*(X, X_0)\otimes H^*(X', X_0')$$

such that if

$$f\colon (X, X_0)\to(Y, Y_0) \quad \text{and} \quad g\colon (X', X_0')\to(Y', Y_0') \text{ in } \tilde{C},$$

then the following diagram commutes:

$$\begin{array}{ccc} H^*\big((X,X_0)\times(X', X_0')\big) & \xleftarrow{(f\times g)^*} & H^*\big((Y, Y_0)\times(Y', Y_0')\big) \\ \downarrow{\scriptstyle\mathscr{L}} & & \downarrow{\scriptstyle\mathscr{L}} \\ H^*(X, X_0)\otimes H^*(X', X_0') & \xleftarrow{f^*\otimes g^*} & H^*(Y, Y_0)\otimes H^*(Y', Y_0'). \end{array}$$

From (1.1), (1.4) and the commutativity of functors $\otimes$ and Hom_Q for graded vector spaces of finite type we have:

(1.5) THEOREM. *For every two pairs of finite type* (X, X_0), (X', X_0') *in* $\tilde{C}$, *there is a linear isomorphism*

$$\overline{\mathscr{L}}\colon H_*\big((X, X_0)\times(X', X_0')\big)\to H_*(X, X_0)\otimes H_*(X', X_0')$$

such that if $f\colon (X, X_0)\to(Y, Y_0)$ *and* $g\colon (X', X_0')\to(Y', Y_0')$ *are two maps of pairs of finite type, then the following diagram commutes:*

$$\begin{array}{ccc} H_*\big((X, X_0)\times(X', X_0')\big) & \xrightarrow{(f\times g)_*} & H_*\big((Y, Y_0)\times(Y', Y_0')\big) \\ \downarrow{\scriptstyle\overline{\mathscr{L}}} & & \downarrow{\scriptstyle\overline{\mathscr{L}}} \\ H_*(X, X_0)\otimes H_*(X', X_0') & \xrightarrow{f_*\otimes g_*} & H_*(Y, Y_0)\otimes H_*(Y', Y_0'). \end{array}$$

2. Maps in spaces of finite type. In this section we prove the following theorem:

(2.1) THEOREM. *Let* (X, d) *be a compact metric space of finite type with respect to* H^*. *Then there exists an* $\varepsilon > 0$ *such that for every two maps* $f, g\colon Y\to X$, *where* Y *is a compact space, the condition*

$$d\big(f(y), g(y)\big) < \varepsilon \quad \text{for each } y\in Y,$$

implies $f^* = g^*$.

First we prove the following lemma:

(2.2) LEMMA. *Let X be a normal topological space and $\alpha = \{U_1, \ldots, U_n\}$ a finite covering of X by open sets. Then there exists a covering $\beta = \{V_1, \ldots, V_n\}$ of X by open sets, such that for each $i = 1, \ldots, n$, $\bar{V}_i \subset U_i$ ($\bar{V}_i$ denotes the closure of V_i in X).*

Proof. Consider the following two closed subsets of X: $F = X \setminus U_i$, $F' = X \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{n} U_j$, where $i = 1, \ldots, n$ is an arbitrary but fixed number. Since $F \cap F' = \emptyset$, by the normality of X we find open subsets U and V_i of X such that: (i) $F \subset U$, (ii) $F' \subset V_i$ and (iii) $U \cap V_i = \emptyset$. Since $X \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{n} U_j \subset V_i$, we infer that the family $\{U_1, \ldots, U_{i-1}, V_i, U_{i+1}, \ldots, U_n\}$ is a covering of X by open subsets and $\bar{V}_i \subset U_i$.

Applying the above construction successively for each $i = 1, \ldots, n$, we obtain a covering $\beta = \{V_1, \ldots, V_n\}$ of X by open sets such that $\bar{V}_i \subset U_i$ for each $i = 1, \ldots, n$, and the proof of (2.2) is completed.

In the proof of (2.1) we will establish the following conventions. By a *covering* of X we understand a finite covering of X by open sets. If α, β are two coverings of X, then the symbol $\alpha \geqslant \beta$ means that α *refines* β. If α is a covering of X, then $N(\alpha)$ will stand for the finite simplicial complex which is the *nerve* of α and $H^*(N(\alpha))$ is the simplicial cohomology of $N(\alpha)$ with coefficientes in Q. If α, β are two coverings of X and $\alpha \geqslant \beta$, then by $i_{\alpha\beta}\colon N(\alpha) \to N(\beta)$ we denote a *simplicial map* given by a vertex transformation from $N(\alpha)$ to $N(\beta)$ taking a set V in α to a set U in β such that $V \subset U$. It is well known that $i^*_{\alpha\beta}\colon H^*(N(\beta)) \to H^*(N(\alpha))$ is independent of the choice of vertex transformations used to define $i_{\alpha\beta}$. Finally, for a map $f\colon Y \to X$ and a covering $\alpha = \{U_1, \ldots, U_n\}$ of X, we denote by $f^{-1}(\alpha)$ the covering of Y of the form

$$f^{-1}(\alpha) = \{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$$

and by $f_\alpha\colon N(f^{-1}(\alpha)) \to N(\alpha)$ a simplicial map given by the following vertex transformation:

$$f_\alpha(f^{-1}(U_i)) = U_i \quad \text{for each } i = 1, \ldots, n.$$

Proof of (2.1). Let $[u_{\alpha_1}], \ldots, [u_{\alpha_k}]$ be a basis of $H^*(X)$, where $u_{\alpha_i} \in H^*(N(\alpha_i))$ for each $i = 1, \ldots, k$. We choose a covering $\alpha = \{U_1, \ldots, U_n\}$ of X such that $\alpha \geqslant \alpha_i$ for all $i = 1, \ldots, k$. Consider simplicial maps $i_{\alpha\alpha_i}\colon N(\alpha) \to N(\alpha_i)$ for each $i = 1, \ldots, k$. Then

$$v^i_\alpha = i^*_{\alpha\alpha_i}(u_{\alpha_i}) \in [u_{\alpha_i}] \quad \text{for each } i.$$

Applying Lemma (2.2) to the covering α, we obtain a covering $\beta = \{V_1, \ldots, V_n\}$ such that $\bar{V}_i \subset U_i$ for each $i = 1, \ldots, n$. Let $i_{\beta\alpha}\colon N(\beta)$

$\to N(\alpha)$ be a simplicial map given by the vertex transformation $i_{\beta\alpha}(V_i) = U_i$ for each i. Then

$$w^i_\beta = i^*_{\beta\alpha}(v^i_\alpha) \in [u_{\alpha_i}] \quad \text{for each } i = 1, \ldots, k.$$

Let $\varepsilon = \min_i \operatorname{dist}(\overline{V}_i, X \setminus U_i)$. We may assume without loss of generality that $U_i \neq X$ for each i. Since $\overline{V}_i \cap X \setminus U_i = \emptyset$ and $\overline{V}_i$, $X \setminus U_i$ are compact, non-empty sets, we deduce that ε is a positive real number.

Let Y be a compact space and let $f, g\colon Y \to X$ be two maps such that $d(f(y), g(y)) < \varepsilon$ for each $y \in Y$. We assert that $f^* = g^*$. Consider the coverings $\gamma = f^{-1}(\alpha)$ and $\delta = g^{-1}(\beta)$. It is easy to see that

$$g^{-1}(V_i) \subset f^{-1}(U_i) \quad \text{for each } i = 1, \ldots, n \text{ and } \delta \geqslant \gamma.$$

Let $i_{\delta\gamma}\colon N(\delta) \to N(\gamma)$ be a simplicial map given by the vertex transformation $i_{\delta\gamma}(g^{-1}(V_i)) = f^{-1}(U_i)$ for each $i = 1, \ldots, n$. We have the following commutative diagram:

$$\begin{array}{ccc} N(\gamma) & \xrightarrow{f_\alpha} & N(\alpha) \\ \uparrow{\scriptstyle i_{\delta\gamma}} & & \uparrow{\scriptstyle i_{\beta\alpha}} \\ N(\delta) & \xrightarrow{g_\beta} & N(\beta) \end{array} .$$

This implies that $i^*_{\delta\gamma} f^*_\alpha(v^i_\alpha) = g^*_\beta(w^i_\beta)$ for each $i = 1, \ldots, k$ and hence we obtain $[f^*_\alpha(v^i_\alpha)] = [g^*_\beta(w^i_\beta)]$. Since $g^*([u_{\alpha_i}]) = [g^*_\beta(w^i_\beta)]$ and $f^*([u_{\alpha_i}]) = [f^*_\alpha(v^i_\alpha)]$, we find that the maps f^*, g^* are equal on a basis of $H^*(X)$. Finally, from this we deduce that $f^* = g^*$ and the proof of (2.1) is completed.

Using (1.1) we deduce that (2.1) is equivalent to the following:

(2.3) Theorem. *Let (X, d) be a compact metric space of finite type with respect to H_*. Then there exists an $\varepsilon > 0$ such that for every two maps $f, g\colon Y \to X$, where Y is a compact space, the condition:*

$$d(f(y), g(y)) < \varepsilon \quad \textit{for each } y \in Y,$$

implies $f_* = g_*$.

Remark. Note that Theorem (2.1) remains true in the case where Y is an arbitrary Hausdorff space.

3. The Čech homology functor with compact carriers. Let (X, X_0) be an arbitrary pair in C. We shall denote by $\mathscr{M} = \{(A_\alpha, A_{0\alpha})\}$ the directed set of all compact pairs such that $(A_\alpha, A_{0\alpha}) \subset (X, X_0)$ for each α, with the natural quasi-order relation $\leqslant$ defined by the condition

$$(A_\alpha, A_{0\alpha}) \leqslant (A_\beta, A_{0\beta}) \text{ if and only if } (A_\alpha, A_{0\alpha}) \subset (A_\beta, A_{0\beta}).$$

If $(A_\alpha, A_{0\alpha}) \leqslant (A_\beta, A_{0\beta})$, then we shall denote by $i_{\alpha\beta}\colon (A_\alpha, A_{0\alpha}) \to (A_\beta, A_{0\beta})$ the inclusion map. For each pair $(A_\alpha, A_{0\alpha})$ consider the graded vector space $H_*(A_\alpha, A_{0\alpha})$, together with the linear map $i_{\alpha\beta*}$ given for $(A_\alpha, A_{0\alpha}) \leqslant (A_\beta, A_{0\beta})$. Then the family $\{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta*}\}$ is a direct system in the category $\mathscr{A}$ over $\mathscr{M}$. We define a graded vector space

$$H(X, X_0) = \varinjlim_{\alpha}\{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta*}\}.$$

It is easy to see that

$$H(X, X_0) = \{H_q(X, X_0)\},$$

where

$$H_q(X, X_0) = \varinjlim_{\alpha}\{H_q(A_\alpha, A_{0\alpha}), i_{\alpha\beta*}\}, \quad \text{for each } q.$$

Let $f\colon (X, X_0)\to(Y, Y_0)$ be a map. Consider the directed sets $\mathscr{M} = \{(A_\alpha, A_{0\alpha})\}$ and $\mathscr{N} = \{(B_\gamma, B_{0\gamma})\}$ for (X, X_0) and (Y, Y_0) respectively. We define $F\colon \mathscr{M}\to\mathscr{N}$ by the formula

$$F\big((A_\alpha, A_{0\alpha})\big) = \big(f(A_\alpha), f(A_{0\alpha})\big) \quad \text{for each } (A_\alpha, A_{0\alpha})\in\mathscr{M}.$$

We observe that if $(A_\alpha, A_{0\alpha}) \leqslant (A_\beta, A_{0\beta})$ then

$$F\big((A_\alpha, A_{0\alpha})\big) \leqslant F\big((A_\beta, A_{0\beta})\big).$$

For each α, by $f_\alpha\colon (A_\alpha, A_{0\alpha})\to\big(f(A_\alpha), f(A_{0\alpha})\big)$ we denote a map given by $f_\alpha(x) = f(x)$ for each $x\in A$. Then the map F and the family $\{f_{\alpha*}\}$ is a map of directed systems $\{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta*}\}$ and $\{H_*(B_\gamma, B_{0\gamma}), i_{\delta\gamma*}\}$. We define the induced linear map for f, $H(f)$, by putting

$$H(f) = f_* = \varinjlim_{\alpha}\{f_{*\alpha}\}.$$

Then we have $f_{*q} = \varinjlim_{\alpha}\{f_{\alpha*q}\}$ for every q.

From the functoriality of $\varinjlim$ we deduce that $H\colon C\to\mathscr{A}$ is a covariant functor. The functor H is said to be the Čech homology functor with compact carriers.

We note that if (X, X_0) is a compact pair, then the family consisting of the single pair (X, X_0) is a cofinal subset of $\mathscr{M} = \{(A_\alpha, A_{0\alpha})\}$ for (X, X_0), and hence we obtain $H_*(X, X_0) = H(X, X_0)$. Similarly, if $f\colon (X, X_0)\to(Y, Y_0)$ is a map of compact pairs, then $H_*(f) = H(f)$.

The following properties of H clearly follow from the Eilenberg–Steenrod axioms for H_* and the simple properties of $\varinjlim$.

(3.1) *If $f, g\colon (X, X_0)\to(Y, Y_0)$ are homotopic maps, then the induced linear maps are equal, that is, $f_* = g_*$.*

(3.2) *Let (X, X_0) be a pair in C and let $i\colon X_0\to X$, $j\colon X\to(X, X_0)$ be inclusions. Then there exists a linear map*

$$\partial_q\colon H_q(X, X_0)\to H_{q-1}(X_0) \quad \textit{for each } q,$$

so that

$$\ldots \longrightarrow H_q(X_0) \xrightarrow{i_{*q}} H_q(X) \xrightarrow{j_{*q}} H_q(X, X_0) \xrightarrow{\partial_q} H_{q-1}(X_0) \longrightarrow \ldots$$

is exact.

The linear map ∂_q has the additional property of being natural in the following sense:

(3.3) *Given a map $f\colon (X, X_0) \to (Y, Y_0)$ in C, the diagram*

$$\begin{array}{ccc} H_q(X, X_0) & \xrightarrow{\partial_q} & H_{q-1}(X_0) \\ \downarrow{\scriptstyle f_{*q}} & & \downarrow{\scriptstyle (f_{X_0})_{*q-1}} \\ H_q(Y, Y_0) & \xrightarrow{\partial_q} & H_{q-1}(Y_0) \end{array}$$

commutes for all q, where $f_{X_0}\colon X_0 \to Y_0$ is given by the formula $f_{X_0}(x) = f(x)$ for each $x \in X_0$.

A pair (X, X_0) of finite type with respect to H is called *simply* of finite type.

We prove the following

(3.4) THEOREM. *Let (X, d) be a compact metric space of finite type. Then there exists an $\varepsilon > 0$ such that, for every two maps $f, g\colon Y \to X$, where Y is a Hausdorff space, the condition*

$$d\big(f(y), g(y)\big) < \varepsilon \quad \textit{for each } y \in Y$$

implies $f_* = g_*$.

Proof. Let ε be as in (2.3). Consider two maps f, g from a Hausdorff space Y to X. Let A be a compact subset of Y and let $f_A, g_A\colon A \to X$ be given by $f_A(y) = f(y)$, $g_A(y) = g(y)$ for each $y \in A$. We observe that f_A, g_A satisfies the assumptions of (2.3). So we have $(f_A)_* = (g_A)_*$. Since $f_* = \varinjlim_A \{(f_A)_*\}$ and $g_* = \varinjlim_A \{(g_A)_*\}$, we infer that $f_* = g_*$ and the proof of (3.4) is completed.

4. Vietoris maps. A space X is *acyclic* provided: (i) X is non-empty, (ii) $H_q(X) = 0$ for all $q \geqslant 1$ and (iii) $H_0(X) \approx Q$. A map $f\colon (X, X_0) \to (Y, Y_0)$ is *proper* provided for any compact B the counter image $f^{-1}(B)$ is also compact. A map $f\colon (X, X_0) \to (Y, Y_0)$ is said to be a *Vietoris map* provided the following conditions are satisfied:

(i) f is proper,

(ii) $f^{-1}(Y_0) = X_0$,

(iii) the set $f^{-1}(y)$ is acyclic for every $y \in Y$.

The following evident remark is of importance:

(4.1) *If* $f\colon (X, X_0) \to (Y, Y_0)$ *is a Vietoris map and* $(B, B_0) \subset (Y, Y_0)$, *then the map* $\tilde{f}\colon \big(f^{-1}(B), f^{-1}(B_0)\big) \to (B, B_0)$ *is also a Vietoris map, where* $\tilde{f}(x) = f(x)$ *for each* $x \in f^{-1}(B)$.

We shall require the following classical result:

(4.2) VIETORIS–BEGLE MAPPING THEOREM. *Let* X, Y *be compact spaces. If* $f\colon X \to Y$ *is a Vietoris map, then the induced map* $f_*\colon H_*(X) \underset{\sim}{\to} H_*(Y)$ *is a linear isomorphism.*

The Vietoris–Begle Mapping Theorem and the five lemma gives:

(4.3) THEOREM. *Let* (X, X_0), (Y, Y_0) *be compact pairs. If* $f\colon (X, X_0) \to (Y, Y_0)$ *is a Vietoris map, then* $f_*\colon H_*(X, X_0) \to H_*(Y, Y_0)$ *is a linear isomorphism.*

Now, from (4.3) we deduce the following theorem for non-compact pairs.

(4.4) THEOREM. *If* $f\colon (X, X_0) \to (Y, Y_0)$ *is a Vietoris map, then the induced map* $f_*\colon H(X, X_0) \to H(Y, Y_0)$ *is a linear isomorphism.*

Proof. Consider $\mathscr{M} = \{(A_\alpha, A_{0\alpha})\}$ and $\mathscr{N} = \{(B_\gamma, B_{0\gamma})\}$ for (X, X_0) and (Y, Y_0), respectively. Let $\mathscr{M}_0 = \big\{\big(f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})\big);\ (B_\gamma, B_{0\gamma}) \in \mathscr{N}\big\}$. Since f is a proper map, we have $\mathscr{M}_0 \subset \mathscr{M}$. It is easy to see that $\mathscr{M}_0$ is a cofinal subset of $\mathscr{M}$. Therefore we may assume without loss of generality that

$$H(X, X_0) = \varinjlim_{\alpha \in \mathscr{M}_0} \{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}.$$

Then for each $\gamma \in \mathscr{N}$ the map

$$f_\gamma\colon \big(f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})\big) \to (B_\gamma, B_{0\gamma})$$

is a Vietoris map of compact pairs. Using (4.3) we infer that

$$f_{\gamma *}\colon H_*\big(f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})\big) \underset{\sim}{\to} H_*(B_\gamma, B_{0\gamma})$$

is a linear isomorphism. Consequently, the linear map $f_* = \varinjlim_{\gamma \in \mathscr{N}} \{f_{\gamma *}\}$ is an isomorphism. The proof of (4.4) is completed.

5. Homology of open subsets of Euclidean spaces. Consider the subcategory $C_1 \subset C$ consisting of all pairs (U, V) such that U and V are open subsets in the Euclidean space R^n for some n, or U is a finite polyhedron and V is an open subset of U, and all maps of such pairs.

Since the family of all pairs of finite polyhedra $\{(K, K_0)\}$ is cofinal in the family of all compact pairs $\{(A, A_0)\}$ contained in (U, V), we obtain the following:

(5.1) *On the category* C_1 *the functors* H *and* $\bar{H}$ *are naturally isomorphic* ($\bar{H}$ denotes the singular homology functor with coefficients in Q).

Let $A \subset U \subset R^n$, where A is compact and U is open in R^n. We identify the nth sphere $S^n = \{x \in R^{n+1}, \|x\| = 1\}$ and $R^n \cup \{\infty\}$. Then from the excision axiom for singular homology and (5.1) we deduce:

(5.2) *The inclusion* $j\colon (U, U \setminus A) \to (S^n, S^n \setminus A)$ *induces an isomorphism*

$$j_*\colon H(U, U\setminus A) \to H(S^n, S^n \setminus A).$$

Let K be a finite polyhedron and U an open subset of R^n where $K \subset U$. Consider a Vietoris map $p\colon Y \to U$ and a map $q\colon Y \to K$ from a Hausdorff space Y to K. We prove the following:

(5.3) *There are isomorphisms* a_1, a_2, a_3 *such that the following diagram commutes*:

$$\begin{array}{ccccc}
H(U, U\setminus K)\otimes H(U) & \xleftarrow{\mathrm{Id}\otimes p_*} & H(U, U\setminus K)\otimes H(Y) & \xrightarrow{\mathrm{Id}\otimes q_*} & H(U, U\setminus K)\otimes H(K) \\
a_1 \Big\uparrow \wr & & \wr \Big\uparrow a_2 & & \wr \Big\uparrow a_3 \\
H\big((U, U\setminus K)\times U\big) & \xleftarrow{(\mathrm{Id}\times p)_*} & H\big((U, U\setminus K)\times Y\big) & \xrightarrow{(\mathrm{Id}\times q)_*} & H\big((U, U\setminus K)\times K\big)
\end{array}$$

Proof. It is easy to see that the families

$$\{(M, M_0)\times L\}, \quad \{(M, M_0)\times p^{-1}(L)\}, \quad \{(M, M_0)\times K\},$$

where M, M_0, L are finite polyhedra, are cofinal in families of all compact pairs contained in $(U, U\setminus K)\times U$, $(U, U\setminus K)\times Y$ and $(U, U\setminus K)\times K$, respectively. We observe that for every L the space $p^{-1}(L)$ is of finite type (p is a Vietoris map), so we may apply (1.5) and have the commutative diagram

$$\begin{array}{ccccc}
H_*(M, M_0)\otimes H_*(L) & \xleftarrow{\mathrm{Id}\otimes (p_L)_*} & H_*(M, M_0)\otimes H_*\big(p^{-1}(L)\big) & \xrightarrow{\mathrm{Id}\otimes (q_{p^{-1}(L)})_*} & H_*(M, M_0)\otimes H_*(K) \\
\wr \Big\uparrow & & \Big\uparrow \wr & & \Big\uparrow \wr \\
H_*\big((M, M_0)\times L\big) & \xleftarrow{(\mathrm{Id}\times p_L)_*} & H_*\big((M, M_0)\times p^{-1}(L)\big) & \xrightarrow{(\mathrm{Id}\times q_{p^{-1}(L)})_*} & H_*\big((M, M_0)\times K\big).
\end{array}$$

From the commutativity of the above diagram and the commutativity of $\varinjlim$ and $\otimes$ we simply deduce (5.3).

Consider the diagram

$$U \xleftarrow{p} Y \xrightarrow{q} K,$$

where p and q are as in (5.3). With the above diagram we associate the following:

$$(U, U\setminus K) \xleftarrow{\bar{p}} \big(Y, Y\setminus p^{-1}(K)\big) \xrightarrow{\bar{q}} (R^n, R^n\setminus\{0\}),$$

where $\bar{p}(y) = p(y)$ and $\bar{q}(y) = p(y) - q(y)$ for each $y \in Y$. We observe that $\bar{p}$ is a Vietoris map. Let $\Delta\colon (U, U\setminus K) \to (U, U\setminus K) \times U$ be a map given by $\Delta(x) = (x, x)$ and let $d\colon (U, U\setminus K) \times K \to (R^n, R^n\setminus\{0\})$ be given by $d(x, x') = x - x'$, for each $x \in U$ and $x' \in K$.

(5.4) Lemma. *The following diagram commutes*

$$\begin{array}{ccccc}
H(U, U\setminus K) & \xrightarrow{\Delta_*} & H(U, U\setminus K)\otimes H(U) & \xrightarrow{\mathrm{Id}\otimes q_* p_*^{-1}} & H(U, U\setminus K)\otimes H(K) \\
& {\searrow}\,{\scriptstyle \bar{q}_*\bar{p}_*^{-1}} & & {\scriptstyle d_*}\,{\swarrow} & \\
& & H(R^n, R^n\setminus\{0\}) & &
\end{array}$$

$$\bar{q}_*\bar{p}_*^{-1}\colon H(U, U\setminus K) \to H(R^n, R^n\setminus\{0\}), \qquad d_*\colon H(U, U\setminus K)\otimes H(K) \to H(R^n, R^n\setminus\{0\})$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
(U, U\setminus K)\times U & \xleftarrow{\mathrm{Id}\times p} & (U, U\setminus K)\times Y & \xrightarrow{\mathrm{Id}\times q} & (U, U\setminus K)\times K \\
\uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle f} & & \downarrow{\scriptstyle d} \\
(U, U\setminus K) & \xleftarrow{\bar{p}} & \big(Y, Y\setminus p^{-1}(K)\big) & \xrightarrow{\bar{q}} & (R^n, R^n\setminus\{0\}),
\end{array}$$

where the map f is given by $f(y) = \big(p(y), y\big)$ for each $y \in Y$. From the commutativity of the above diagram and (5.3) we obtain (5.4).

Let us fix for each n an orientation $1 \in H_n(S^n) \approx Q$ of the nth sphere $S^n = R^n \cup \{\infty\}$. Consider the diagram

$$S^n \xrightarrow{i} (S^n, S^n\setminus A) \xleftarrow{j} (U, U\setminus A)$$

in which A is a compact subset of U and U is open in R^n; i, j are inclusions. From (5.2) we infer that j_* is an isomorphism. We define the fundamental class O_A of the pair (U, A) by the equality

$$O_A = j_{*n}^{-1} i_{*n}(1).$$

(5.5) *Let $A \subset A_1 \subset V \subset U \subset R^n$, where A, A_1 are compact, V, U are open subsets of R^n and let $k\colon (V, V\setminus A_1) \to (U, U\setminus A)$ be the inclusion map. Then we have: $k_{*n}(O_{A_1}) = O_A$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
S^n & \xrightarrow{i} & (S^n, S^n\setminus A) & \xleftarrow{j} & (U, U\setminus A) \\
& {\searrow}\,{\scriptstyle i_1} & \uparrow{\scriptstyle k_1} & & \uparrow{\scriptstyle k} \\
& & (S^n, S^n\setminus A_1) & \xleftarrow{j_1} & (V, V\setminus A_1),
\end{array}$$

in which j_1, i_1, k_1 are inclusion maps. Applying H_n to the above diagram, we obtain (5.5).

Finally, we formulate Dold's Lemma in terms of Čech homology with compact carriers. Let $K \subset U \subset R^n$, where K is a finite polyhedron and U an open subset of R^n. We define the following maps:

$$t\colon U\times K \to K\times U, \quad t(x, x') = (x', x) \text{ for each } x\in U \text{ and } x'\in K,$$

$$O_K^\times\colon H(K)\to H(U, U\setminus K)\otimes H(K), \quad O_K^\times(u) = O_K\otimes u \text{ for each } u\in H(K),$$

$$\times\colon Q\otimes H(U)\to H(U), \quad \times(q\otimes u) = q\cdot u \text{ for each } u\in H(U),\ q\in Q.$$

(5.6) Lemma. *The composite*

$$l = l(K, U)\colon H(K) \xrightarrow{O_k^\times} H(U, U\setminus K)\otimes H(K) \xrightarrow{\Delta_*\otimes \mathrm{Id}} H(U, U\setminus K)\otimes H(U)\otimes H(K) \xrightarrow{\mathrm{Id}\otimes t_*} H(U, U\setminus K)\otimes H(K)\otimes H(U) \xrightarrow{d_*\otimes \mathrm{Id}} Q\otimes H(U) \xrightarrow{\times} H(U)$$

coincides with the linear map

$$i_*\colon H(K)\to H(U).$$

Remark. Dold's Lemma was given in terms of singular homology in [20] (comp. also [14], p. 153). Lemma (5.6), in view of (5.1), clearly follows from the original statement of Dold's Lemma.

References

§ 1. Eilenberg, Steenrod [22], Górniewicz [34], Hurewicz, Wallman [51], Spanier [81].

§ 2. Granas [42], Skordev [80].

§ 3. Eilenberg, Steenrod [22], Spanier [81], Teleman [84].

§ 4. Begle [3], Skljarenko [79], Spanier [81], Vietoris [89].

§ 5. Dold [20], Brown [14], Górniewicz [37], Spanier [81], Teleman [84].

II. THE LEFSCHETZ NUMBER

In what follows all the vector spaces are taken over Q.

1. The (ordinary) Lefschetz number. Let $f\colon E \to E$ be an endomorphism of a finite-dimensional vector space E. If $v_1, \ldots, v_n$ is a basis for E, then we can write

$$f(v_i) = \sum_{j=1}^{n} a_{ij} v_j, \qquad \text{for all } i = 1, \ldots, n.$$

The matrix $[a_{ij}]$ is called the matrix of f (with respect to the basis $v_1, \ldots$ $\ldots, v_n$). Let $A = [a_{ij}]$ be an $(n \times n)$-matrix; then the trace of A is defined as $\sum_{i=1}^{n} a_{ii}$. If $f\colon E \to E$ is an endomorphism of a finite-dimensional vector space E, then the trace of f, written $\operatorname{tr}(f)$, is the trace of the matrix of f with respect to some basis for E. If E is a trivial vector space then, by definition, $\operatorname{tr}(f) = 0$. It is a standard result that the definition of the trace of an endomorphism is independent of the choice of the basis for E.

We recall the following two basic properties of the trace:

(1.1) *Assume that in the category of finite-dimensional vector spaces the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{\;f\;} & E'' \\ \uparrow{\scriptstyle f'} & \nwarrow{\scriptstyle g} & \uparrow{\scriptstyle f''} \\ E' & \xrightarrow{\;f\;} & E'' \end{array}$$

Then $\operatorname{tr}(f') = \operatorname{tr}(f'')$; *in other words* $\operatorname{tr}(gf) = \operatorname{tr}(fg)$.

(1.2) *Given a commutative diagram of finite-dimensional vector spaces with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \end{array}$$

we have $\operatorname{tr}(f) = \operatorname{tr}(f') + \operatorname{tr}(f'')$.

Let $E = \{E_q\}$ be a graded vector space in $\mathscr{A}$ of finite type. If $f = \{f_q\}$ is an endomorphism of degree zero of such a graded vector space, then the (*ordinary*) *Lefschetz number* $\lambda(f)$ of f is defined by

$$\lambda(f) = \sum_q (-1)^q \operatorname{tr}(f_q).$$

Let E be a finite-dimensional vector space and $v_1, \ldots, v_n$ a basis for E. We define a basis $v^1, \ldots, v^n$ for $\operatorname{Hom}_Q(E)$ by putting

$$v^i(v_j) = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

The basis $v^1, \ldots, v^n$ is called the *conjugate basis* to $v_1, \ldots, v_n$. For a vector space E and any integer q, define a linear map

$$\Theta_q\colon \operatorname{Hom}_Q(E)\otimes E \to \operatorname{Hom}(E, E)$$

by letting

$$\Theta_q(u\otimes v)(v') = (-1)^q u(v')\cdot v, \quad \text{for} \quad u \in \operatorname{Hom}_Q(E), \quad v, v' \in E,$$

and extend Θ_q to all $\operatorname{Hom}_Q(E)\otimes E$.

(1.3) LEMMA. *If the vector space E is finite-dimensional, then Θ_q is an isomorphism.*

Proof. Let $v_1, \ldots, v_n$ be a basis for E and $v^1, \ldots, v^n$ the conjugate basis to $v_1, \ldots, v_n$. Then every element a in $\operatorname{Hom}_Q(E)\otimes E$ has the following form:

$$a = \sum_{i,j=1}^{n} a_{ij} v^i \otimes v_j.$$

If $\Theta_q(a) = 0$, then

$$\Theta_q(a)(v_k) = (-1)^q \sum_{j=1}^{n} a_{kj} v^k(v_k)\cdot v_j = (-1)^q \sum_{j=1}^{n} a_{kj}\cdot v_j = 0$$

so $a_{kj} = 0$ for all k, j, which implies that $a = 0$. To prove Θ_q is onto, let $\in \operatorname{Hom}(E, E)$. Then we can write

$$f(v_j) = a_{j1}v_1 + \ldots + a_{jn}v_n \quad \text{for} \quad j = 1, \ldots, n.$$

Let $a = (-1)^q \sum_{m,k=1}^{n} a_{mk} v^m \otimes v_k$. For each $j = 1, \ldots, n$ we see that

$$\Theta_q(a)(v_j) = (-1)^{2q} \sum_{k=1}^{n} a_{jk}\cdot v_k = f(v_j).$$

So f and $\Theta_q(a)$ agree on a basis for E, which implies that Θ_q is onto. The proof of (1.3) is completed.

Define $e\colon \operatorname{Hom}_Q(E)\otimes E \to Q$ as the evaluation map

$$e(u\otimes v) = u(v) \quad \text{for} \quad u \in \operatorname{Hom}_Q(E),\ v \in E.$$

(1.4) LEMMA. *If E is a finite-dimensional vector space and $f: E \to E$ is a linear map, then*

$$e(\Theta_q^{-1}(f)) = (-1)^q \operatorname{tr}(f).$$

Proof. Take a basis $v_1, \ldots, v_n$ for E and write

$$f(v_j) = \sum_{k=1}^{n} a_{jk} v_k \quad \text{for} \quad j = 1, \ldots, n.$$

From the proof of (1.3) we know that

$$\Theta_q^{-1}(f) = (-1)^q \sum_{m,k=1}^{n} a_{mk}(v^m \otimes v_k),$$

so

$$e(\Theta_q^{-1}(f)) = (-1)^q \sum_{k,m=1}^{n} a_{mk} \cdot v^m(v_k) = (-1)^q \sum_k a_{kk} = (-1)^q \operatorname{tr}(f)$$

and the proof of (1.4) is completed.

Let $E = \{E_q\}$ be a graded vector space of finite type. Define the following graded vector spaces:

(1) $E^* = \{E_q^*\}$, where $E_q^* = \operatorname{Hom}_Q(E_{-q})$,

(2) $\operatorname{Hom}(E, E) = \{(\operatorname{Hom}(E, E))_k\}$, where

$$(\operatorname{Hom}(E, E))_k = \bigoplus_{-q+i=k} \operatorname{Hom}(E_q, E_i),$$

(3) $E^* \otimes E = \{(E^* \otimes E)_k\}$, where $(E^* \otimes E)_k = \bigoplus_{q+i=k} E_q^* \otimes E_i$.

Define $\Theta: (E^* \otimes E)_0 \to (\operatorname{Hom}(E, E))_0$ by letting

$$\Theta(u_q \otimes v_i) = \Theta_q(u_q \otimes v_i), \quad \text{for} \quad u_q \in \operatorname{Hom}_Q(E_q),\ v_i \in E_i,\ q = i$$

and extend Θ_q to all $(E^* \otimes E)_0$; and $e: (E^* \otimes E)_0 \to Q$ by letting

$$e(u_q \otimes v_i) = u_q(v_i), \quad \text{for} \quad u_q \in \operatorname{Hom}_Q(E_q),\ v_i \in E_i,\ q = i$$

and extend e to all $(E^* \otimes E)_0$.

It is immediate from Lemma (1.4) that

(1.5) THEOREM. *If $f: E \to E$ is a linear map of degree zero on a graded vector space of finite type E, then $e(\Theta^{-1}(f)) = \lambda(f)$.*

2. The generalized Lefschetz number. Let $f: E \to E$ be an endomorphism of an arbitrary vector space E. Denote by $f^{(n)}: E \to E$ the nth iterate of f and observe that the kernels

$$\operatorname{Ker} f \subset \operatorname{Ker} f^{(2)} \subset \ldots \subset \operatorname{Ker} f^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of E. Let us now put

$$N(f) = \bigcup_n \operatorname{Ker} f^{(n)} \quad \text{and} \quad \tilde{E} = E/N(f).$$

Clearly, f maps $N(f)$ into itself and therefore induces the endomorphism

$$\tilde{f}\colon \tilde{E}\to\tilde{E}$$

on the factor space $\tilde{E} = E/N(f)$.

(2.1) *We have* $f^{-1}(N(f)) = N(f)$; *consequently, the kernel of the induced map* $\tilde{f}\colon \tilde{E}\to\tilde{E}$ *is trivial, i.e.,* $\tilde{f}$ *is a monomorphism.*

Proof. If $v\in f^{-1}(N(f))$, then $f(v)\in N(f)$. This implies that for some n we have $f^{(n)}(f(v)) = 0 = f^{(n+1)}(v)$ and $v\in N(f)$. Conversely, if $v\in N(f)$, then $f^{(n)}(v) = 0$ for some n; then $f^{(n)}(f(v)) = 0$ and hence $f(v)\in N(f)$, i.e., $v\in f^{-1}(N(f))$.

Let $f\colon E\to E$ be an endomorphism of a vector space E. Assume that $\dim\tilde{E} < +\infty$; in this case we define the generalized trace $\mathrm{Tr}(f)$ of f by putting $\mathrm{Tr}(f) = \mathrm{tr}(\tilde{f})$.

(2.2) *Let* $f\colon E\to E$ *be an endomorphism. If* $\dim E < +\infty$, *then* $\mathrm{Tr}(f) = \mathrm{tr}(f)$.

Proof. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N(f) & \longrightarrow & E & \longrightarrow & E/N(f) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} & & \\
0 & \longrightarrow & N(f) & \longrightarrow & E & \longrightarrow & E/N(f) & \longrightarrow & 0
\end{array}$$

in which $\bar{f}$ is induced by f.

Applying (1.2), to the above diagram, we obtain

(1) $$\mathrm{tr}(f) = \mathrm{tr}(\bar{f})+\mathrm{tr}(\tilde{f}),\quad \text{where}\quad \mathrm{tr}(\tilde{f}) = \mathrm{Tr}(f).$$

We prove that $\mathrm{tr}(\bar{f}) = 0$. Since $\dim E < +\infty$, we may assume that $N(f) = \mathrm{Ker} f^{(n)}$ for some $n\geqslant 1$. Now consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ker}(f) & \longrightarrow & \mathrm{Ker}(f^{(2)}) & \longrightarrow \cdots \longrightarrow & \mathrm{Ker}(f^{(n-1)}) & \longrightarrow & \mathrm{Ker}(f^{(n)})\\
\downarrow{\scriptstyle 0=\bar{f}_1} & \swarrow{\scriptstyle f_1} & \downarrow{\scriptstyle \bar{f}_2} & & \downarrow{\scriptstyle \bar{f}_{n-1}} & \swarrow{\scriptstyle f_n} & \downarrow{\scriptstyle \bar{f}_n=\bar{f}}\\
\mathrm{Ker}(f) & \longrightarrow & \mathrm{Ker}(f^{(2)}) & \longrightarrow \cdots \longrightarrow & \mathrm{Ker}(f^{(n-1)}) & \longrightarrow & \mathrm{Ker}(f^{(n)})
\end{array}$$

where the maps $\bar{f}_i$, f_i, $i = 1,\ldots,n$, are given by f (observe that if $v\in\mathrm{Ker}(f^{(i)})$, then $f(v)\in\mathrm{Ker}(f^{(i-1)})$, for every $i > 1$).

Then, from (1.1) we infer

$$\mathrm{tr}(\bar{f}) = \mathrm{tr}(\bar{f}_{n-1}) = \ldots = \mathrm{tr}(\bar{f}_2) = \mathrm{tr}(\bar{f}_1) = 0.$$

Finally, from (1) we obtain $\mathrm{Tr}(f) = \mathrm{tr}(\tilde{f}) = \mathrm{tr}(f)$ and the proof is completed.

Let $f = \{f_q\}$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We say that f is a *Leray endomorphism* provided the graded vector space $\tilde{E} = \{\tilde{E}_q\}$ is of finite type. For such an f we define the (*generalized*) *Lefschetz number* $\Lambda(f)$ of f by putting

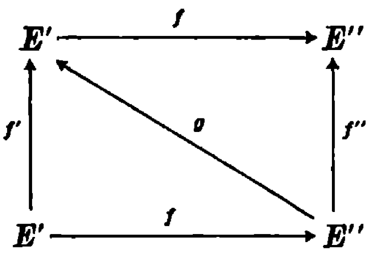
$$\Lambda(f) = \sum_q (-1)^q \operatorname{Tr}(f_q).$$

It is immediate from (2.2) that

(2.3) *Let $f\colon E \to E$ be an endomorphism of degree zero. If E is a graded vector space of finite type, then $\Lambda(f) = \lambda(f)$.*

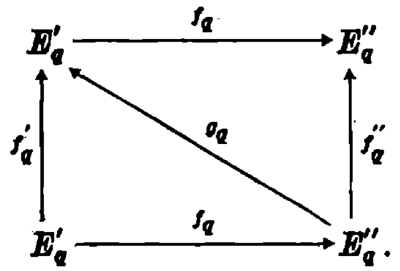
The following property of the Leray endomorphism is of importance:

(2.4) *Assume that in the category $\mathscr{A}$ the following diagram commutes:*

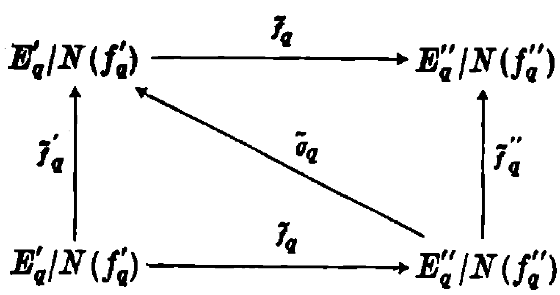


Then if either f' or f'' is a Leray endomorphism, then so is the other Leray endomorphism, and in that case $\Lambda(f') = \Lambda(f'')$.

Proof. By assumption we have, for each q, the following commutative diagram in the category of vector spaces:



For the proof it is sufficient to show that if either $\operatorname{Tr}(f'_q)$ or $\operatorname{Tr}(f''_q)$ is defined, then so is the other trace, and in that case $\operatorname{Tr}(f'_q) = \operatorname{Tr}(f''_q)$. We observe that the commutativity of the above diagram implies that the following diagram commutes:



Since $\tilde{f}_q$ and $\tilde{g}_q$ are monomorphisms, the commutativity of the above diagram implies that $\dim(E'_q/N(f'_q)) < +\infty$ if and only if $\dim(E''_q/N(f''_q)) < +\infty$, and hence we conclude that $\mathrm{Tr}(f'_q)$ is defined if and only if $\mathrm{Tr}(f''_q)$ is defined. Moreover, from (1.1) we deduce that $\mathrm{Tr}(f'_q) = \mathrm{Tr}(f''_q)$, if $\mathrm{Tr}(f'_q)$ or $\mathrm{Tr}(f''_q)$ is defined. The proof of (2.4) is completed.

References

§ 1. Bowszyc [7], Brown [14], Dold [20], Lefschetz [63], van der Walt [90].

§ 2. Bowszyc [7], Granas [41], [44], Leray [64], [65].

III. MULTI-VALUED MAPS

This chapter is devoted to the definitions and basic properties of multi-valued maps. The notion of admissible and s-admissible maps and their properties establish the main result of this chapter. By a space we shall always understand a Hausdorff topological space.

1. Upper semi-continuous and compact multi-valued maps. Let X and Y be two spaces and assume that for every point $x \in X$ a non-empty subset $\varphi(x)$ of Y is given; in this case, we say that φ is a *multi-valued map* from X to Y and we write $\varphi\colon X \to Y$. In what follows the symbols φ, ψ, χ will be reserved for multi-valued maps; single-valued maps will be denoted by $f, g, p, q, \ldots$

Let $\varphi\colon X \to Y$ be a multi-valued map. We associate with φ the diagram of continuous maps

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y$$

in which

$$\Gamma_\varphi = \{(x, y) \in X \times Y;\ y \in \varphi(x)\}$$

is the graph of φ and the natural projections p_φ and q_φ are given by

$$p_\varphi(x, y) = x \quad \text{and} \quad q_\varphi(x, y) = y.$$

The image of a subset $A \subset X$ under φ is $\varphi(A) = \bigcup_{x \in A} \varphi(x)$. The counter-image of a subset $B \subset Y$ under φ is $\varphi^{-1}(B) = \{x \in X;\ \varphi(x) \subset B\}$.

A multi-valued map $\varphi\colon X \to Y$ is called *upper semi-continuous* (u.s.c.) provided (i) $\varphi(x)$ is compact for each $x \in X$ and (ii) for each open set $V \subset Y$ the counter-image $\varphi^{-1}(V)$ is an open subset of X.

(1.1) PROPOSITION. *If $\varphi\colon X \to Y$ is a u.s.c. multi-valued map, then the graph Γ_φ of φ is a closed subset of $X \times Y$.*

Proof. We prove that $X \times Y \setminus \Gamma_\varphi$ is an open subset of $X \times Y$. Let $(x, y) \in X \times Y \setminus \Gamma_\varphi$, i.e., $y \notin \varphi(x)$. Since Y is a Hausdorff space and $\varphi(x)$ is compact, we may choose an open neighbourhood V_y of y in Y and $V_{\varphi(x)}$ of $\varphi(x)$ in Y such that $V_y \cap V_{\varphi(x)} = \emptyset$. By assumption, we infer that $U_x = \varphi^{-1}(V_{\varphi(x)})$ is an open neighbourhood of x in X. Consequently, the set $U_x \times V_y$ is an open neighbourhood of (x, y) in $X \times Y$. We observe that if

$(x', y') \in U_x \times V_y$, then $\varphi(x') \in V_{\varphi(x)}$. This implies that $U_x \times V_y \subset X \times Y \setminus \Gamma_\varphi$ and the proof of (1.1) is completed.

(1.2) PROPOSITION. *If $\varphi\colon X \to Y$ is a u.s.c. map and A a compact subset of X, then the image $\varphi(A)$ of A under φ is compact.*

Proof. Let $\{V_t\}$ be an open covering of $\varphi(A)$. Since $\varphi(x)$ is compact, we infer that for each $x \in X$ there exist a finite number of sets V_t such that $\varphi(x) \subset W_x$, where W_x is the union of the sets V_t. This implies that the family $\{W_x\}_{x \in A}$ is open covering of $\varphi(A)$. Let $U_x = \varphi^{-1}(W_x)$ for each $x \in A$. From the u.s.c. of φ we infer that $\{U_x\}_{x \in A}$ is an open covering of A. Since A is compact, there exist a finite number of sets $U_{x_1}, \ldots, U_{x_n}$ in U_x such that $\{U_{x_1}, \ldots, U_{x_n}\}$ is a covering of A. Then the family $\{W_{x_1}, \ldots, W_{x_n}\}$ is a covering of $\varphi(A)$ and, since for each $i = 1, \ldots, n$ the set W_{x_i} is the union of a finite number of sets V_t, we deduce that there exist a finite number of sets in $\{V_t\}$ which is a covering of $\varphi(A)$. The proof of (1.2) is completed.

Let $\varphi\colon X \to Y$ be a multi-valued map, A a subset of X and B a subset of Y. If $\varphi(A) \subset B$, then the contraction of φ to the pair (A, B) is a multi-valued map $\varphi'\colon A \to B$ defined by $\varphi'(a) = \varphi(a)$ for each $a \in A$. A contraction of φ to the pair (A, Y) is simply a restriction $\varphi|_A$ of φ to A. Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be two multi-valued maps. Then the composition of φ and ψ is a map $\psi \circ \varphi\colon X \to Z$ defined by $\psi\circ\varphi(x) = \psi(\varphi(x))$ for each $x \in X$.

The following fact simply results from (1.2):

(1.3) PROPOSITION. *If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are two u.s.c. maps, then the composition $\psi\circ\varphi$ of φ and ψ is u.s.c.*

A multi-valued map $\varphi\colon X \to Y$ is called *compact* provided the image $\varphi(X)$ of X under φ is contained in a compact subset of Y.

From (1.2) and (1.3) we obtain

(1.4) PROPOSITION. *Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be two u.s.c. maps. If φ or ψ is a compact map, then the composition $\psi\circ\varphi$ of φ and ψ is compact, u.s.c.*

For a multi-valued map $\varphi\colon X \to X$ by φ^m, $m \geqslant 1$, we denote the mth iteration of φ (i.e., $\varphi^m = \underbrace{\varphi\circ\varphi\circ\ldots\circ\varphi}_{m\text{th}}$).

(1.5) PROPOSITION. *Let $\varphi\colon X \to X$ be a u.s.c. map. If φ is a compact map, then the set $C_\infty = \bigcap_{m \geqslant 1} \varphi^m(X)$ is compact and non-empty.*

Proof. By assumption the set $X_0 = \overline{\varphi(X)}$ is compact and non-empty. Applying (1.2) and (1.3), we infer that $\varphi^m(X_0)$ is compact and non-empty for each $m \geqslant 1$. Since $X_0 \subset X$, we infer that $\varphi^m(X_0) \subset \varphi^m(X)$ for each $m \geqslant 1$. Since $\varphi(X) \subset X_0$, we have $\varphi^{m+1}(X) \subset \varphi^m(X_0)$ for each $m \geqslant 1$. So

$$C_\infty = \bigcap_{m \geqslant 1} \varphi^m(X) = \bigcap_{m \geqslant 1} \varphi^m(X_0)$$

and, as a decreasing family of compact non-empty sets, it is compact and non-empty. The proof of (1.5) is completed.

Let $\varphi\colon X \to X$ be a multi-valued map and A a subset of X. A point $x \in X$ is called a fixed point for φ whenever $x \in \varphi(x)$; moreover, if $x \in A$ in this case, we say that φ has a fixed point in A. Let $\varphi, \psi\colon X \to Y$ be two multi-valued maps. If $\varphi(x) \subset \psi(x)$ for each $x \in X$, then we say that φ is a selector of ψ and indicate this by $\varphi \subset \psi$.

Let $p\colon X \to Y$ be a single-valued map from the space X onto Y. We associate with such a p the multi-valued map $\varphi_p\colon Y \to X$ given by

$$\varphi_p(y) = p^{-1}(y) \quad \text{for each } y \in Y.$$

(1.6) Lemma. *If $p\colon X \to Y$ is a closed map from X onto Y, then for every open subset U of X the counter-image of U under φ_p is an open subset of Y.*

Proof. Let U be an open subset of X. We prove that the set $Y \setminus \varphi_p^{-1}(U)$ is closed. We have

$$\begin{aligned} Y \setminus \varphi_p^{-1}(U) &= \{y \in Y;\ \varphi_p(y) \cap (X \setminus U) \neq \emptyset\} \\ &= \{y \in Y;\ p^{-1}(y) \cap (X \setminus U) \neq \emptyset\} = p(X \setminus U). \end{aligned}$$

Since $X \setminus U$ is a closed subset of X and p is a closed map, the above equality implies (1.6).

(1.7) Lemma. *Let $p\colon X \to Y$ be a proper (single-valued) map from X onto the metric space Y. Then p is a closed map.*

Proof. Let A be a closed subset of X. We prove that $p(A)$ is closed subset of Y. Consider a sequence $\{y_n\}$ of points in $p(A)$ and assume that $\lim_n y_n = y$. For the proof it is sufficient to show that $y \in p(A)$. The set $B = \{y_n\} \cup \{y\}$ is a compact subset of Y. By assumption, the set $A_0 = p^{-1}(B)$ is compact and hence the set $A \cap A_0$ is compact. The continuity of p implies that the set $p(A \cap A_0)$ is compact. We observe that, for each n, $y_n \in p(A \cap A_0) \subset p(A)$. Finally, the condition $\lim_n y_n = y$ implies that $y \in p(A \cap A_0) \subset p(A)$ and the proof of (1.7) is completed.

Remark. For an arbitrary Hausdorff space Y Lemma (1.7) is not true (comp. [6], p. 156 and 164).

It is immediate from (1.6) and (1.7) that

(1.8) Proposition. *Let $p\colon X \to Y$ be a proper map from X onto the metric space Y. Then the map $\varphi_p\colon Y \to X$ is u.s.c.*

2. Admissible maps. A u.s.c. map $\varphi\colon X \to Y$ is said to be acyclic provided the set $\varphi(x)$ is acyclic for every point $x \in X$.

Proposition (1.2) implies

(2.1) *If* $\varphi\colon X \to Y$ *is an acyclic map, then the natural projection* $p_\varphi\colon \Gamma_\varphi \to X$ *is a Vietoris map.*

Using Theorem I.4.4 for an acyclic map $\varphi\colon X \to Y$, we define the linear map

$$\varphi_*\colon H(X) \to H(Y)$$

by putting

$$\varphi_* = (q_\varphi)_* \circ [(p_\varphi)_*]^{-1}.$$

φ_* is said to be induced by the multi-valued map φ. It is easy to see that if $\varphi = f$ (i.e., φ is a single-valued continuous map), then $\varphi_* = f_*$.

Let $\varphi\colon X \to Y$ be a multi-valued map. A pair (p, q) of single-valued, continuous maps of the form $X \xleftarrow{p} Z \xrightarrow{q} Y$ is called a *selected pair* of φ (written $(p, q) \subset \varphi$) if the following two conditions are satisfied:

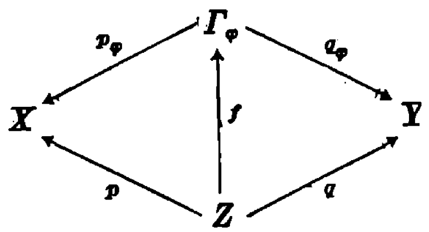
(i) p is a Vietoris map,

(ii) $q(p^{-1}(x)) \subset \varphi(x)$ for each $x \in X$.

(2.2) Remark. We observe that if φ is a compact map and $(p, q) \subset \varphi$, then q is a compact map.

(2.3) Proposition. *If* $\varphi\colon X \to Y$ *is an acyclic map and* $(p, q) \subset \varphi$, *then* $q_* p_*^{-1} = \varphi_*$.

Proof. Let (p, q) be a selected pair of φ of the form $X \xleftarrow{p} Z \xrightarrow{q} Y$. Consider the commutative diagram



in which $f(z) = (p(z), q(z))$ for every $z \in Z$.

The condition $q(p^{-1}(x)) \subset \varphi(x)$ implies that $(p(z), q(z)) \in \Gamma_\varphi$. Applying to the above diagram the functor H, we obtain $q_* p_*^{-1} = (q_\varphi)_* \circ [(p_\varphi)_*]^{-1}$, and the proof is completed.

From (1.8) and (2.3) we simply deduce

(2.4) Proposition. *If* $p\colon Z \to X$ *is a Vietoris map from* Z *onto the metric space* X, *then the map* $\varphi_p\colon X \to Z$ *is acyclic and* $(\varphi_p)_* = p_*^{-1}$.

(2.5) Definition. A multi-valued map $\varphi\colon X \to Y$ is called *admissible* provided there exists a selected pair (p, q) of φ.

We observe that if φ has an acyclic selector or, in particular, a continuous single-valued selector, then φ is an *admissible map*.

(2.6) Definition. An admissible map $\varphi\colon X \to Y$ is called *strongly admissible* (*s-admissible*) provided there exists a selected pair (p, q) of φ such that $(p, q) = \varphi$, i.e., $q(p^{-1}(x)) = \varphi(x)$ for each $x \in X$.

EXAMPLES. 1. Every acyclic map is not only admissible but also s-admissible. For example, the pair (p_φ, q_φ) is a selected pair of acyclic map φ such that $(p_\varphi, q_\varphi) = \varphi$.

2. We observe that if $\varphi\colon X\to Y$ is an s-admissible map, then $\varphi(x)$ is a connected set for each $x\in X$. The map $\varphi\colon [0,1]\to[0,1]$ given by

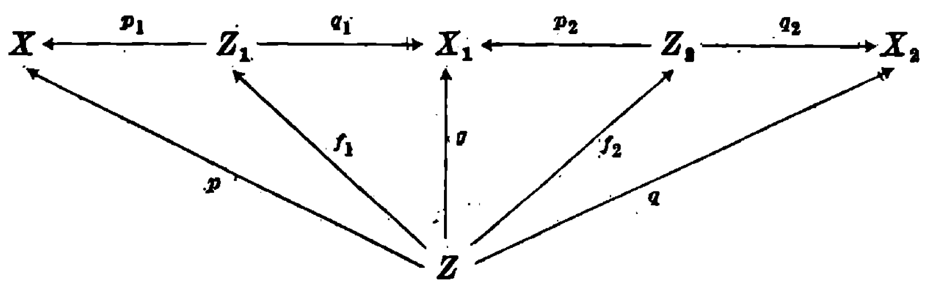
$$\varphi(t) = \begin{cases} t & \text{for} \quad t\neq 0, \\ \{0,1\} & \text{for} \quad t = 0 \end{cases}$$

is an admissible map but φ is not an s-admissible map.

(2.7) THEOREM. *Let $\varphi\colon X\to X_1$ and $\psi\colon X_1\to X_2$ be two admissible maps. Then the composition $\psi\circ\varphi\colon X\to X_2$ is an admissible map, and for every selected pair $(p_1, q_1)\subset\varphi$ and $(p_2, q_2)\subset\psi$ there exists a selected pair (p, q) of $\psi\circ\varphi$ such that*

$$q_{2*}\circ(p_{2*})^{-1}\circ q_{1*}\circ(p_{1*})^{-1} = q_*\circ p_*^{-1}.$$

Proof. Let $(p_1, q_1)\subset\varphi$ and $(p_2, q_2)\subset\psi$. Consider the commutative diagram



in which

$$Z = \{(z_1, z_2)\in Z_1\times Z_2;\ q_1(z_1) = p_2(z_2);\ p(z_1, z_2) = p_1(z_1)\},$$

$$q(z_1, z_2) = q_2(z_2), \quad f_1(z_1, z_2) = z_1, \quad f_2(z_1, z_2) = z_2, \quad g(z_1, z_2) = q_1(z_1)$$

o r each $(z_1, z_2)\in Z$.

Since $f_1^{-1}(z_1)$ is homeomorphic to $p_2^{-1}(q_1(z_1))$ and p_2 is a Vietoris map, we deduce that f_1 is a Vietoris map. Hence p, as the composite $p\circ f_1$, is a Vietoris map. Moreover, we have $q(p^{-1}(x))\subset\psi(\varphi(x))$ for each $x\in X$. Applying to the above diagram the functor H, we obtain

$$q_{2*}(p_{2*})^{-1}\circ q_{1*}(p_{1*})^{-1} = q_*p_*^{-1}$$

and the proof of (2.7) is completed.

(2.8) THEOREM. *If $\varphi\colon X\to X_1$ and $\psi\colon X_1\to X_2$ are two s-admissible maps, then the composition $\psi\circ\varphi\colon X\to X_2$ is an s-admissible map and for every $(p_1, q_1) = \varphi$ and $(p_2, q_2) = \psi$ there exists a $(p, q) = \psi\circ\varphi$ such tha*

$$q_{2*}(p_{2*})^{-1}\circ q_{1*}(p_{1*})^{-1} = q_*p_*^{-1}.$$

The proof of (2.8) is analogous to the proof of (2.7).

Theorem (2.8) implies that the composition of two acyclic maps is an s-admissible map.

3. Homotopy and selectors. Let $\varphi\colon X\to Y$ be an admissible map. Define the set $\{\varphi\}_*$ of linear maps from $H(X)$ to $H(Y)$ by putting

$$\{\varphi\}_* = \{q_* p_*^{-1};\ (p, q) \subset \varphi\};$$

$\{\varphi\}_*$ is said to be an induced set of linear maps by the map φ. From (2.3) we infer that if φ is an acyclic map then $\{\varphi\}_* = \{\varphi_*\}$.

(3.1) THEOREM. *Let $\varphi, \psi\colon X\to Y$ be two admissible maps. If $\varphi\subset\psi$, then $\{\varphi\}_*\subset\{\psi\}_*$.*

For the proof of (3.1) we observe that if $(p, q)\subset\varphi$, then $(p, q)\subset\psi$.

From (3.1) and (2.3) we obtain

(3.2) COROLLARY. *Let $\psi\colon X\to Y$ be an acyclic map and $\varphi\colon X\to Y$ an admissible map. If $\varphi\subset\psi$, then $\{\varphi\}_* = \{\psi_*\}$.*

(3.3) EXAMPLE. Let S^n denote the unit n-sphere in the Euclidean space R^{n+1}. Define the map $\varphi\colon S^n\to S^n$ by $\varphi(x) = S^n$ for each $x\in S^n$. It is easy to see that φ is an admissible map and hence every continuous (single-valued) map $f\colon S^n\to S^n$ is a selector of φ. Therefore Theorem (3.1) implies that $\{\varphi\}_*$ is an infinite set. Moreover, we assert that φ is an s-admissible map and in this case, if the dimension of S^n is even, there exist two selected pairs, $(p, q) = \varphi$ and $(p', q') = \varphi$, such that $q_*p_*^{-1}\neq q'_*(p'_*)^{-1}$. In order to show this, we define the maps $\psi_1, \psi_2\colon S^n\to S^n$ by

$$\psi_1(x) = \{y\in S^n;\ \|x-y\|\leqslant \tfrac{3}{2}\}\quad\text{and}\quad \psi_2(x) = \psi_1(-x)\quad\text{for each } x\in S^n.$$

We have

$$\varphi(x) = \psi_1(\psi_1(x)) = \psi_2(\psi_1(x))\quad\text{for each } x\in S^n$$

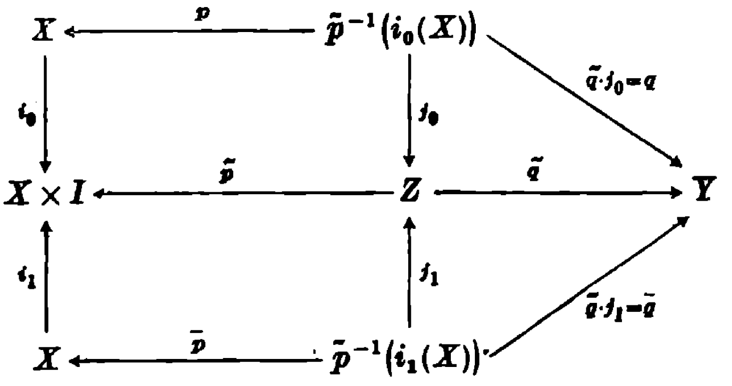
and (2.8) implies that φ is an s-admissible map. Since $\mathrm{Id}_{S^n}\subset\psi_1$ and $(-\mathrm{Id}_{S^n})\subset\psi_2$, from (3.2) we infer that $\psi_{1*} = \mathrm{Id}_{H(S^n)}$ and $(-\mathrm{Id}_{S^n})_* = \psi_{2*}$. Applying Theorem (2.8) again, we deduce that there exist two selected pairs, $(p, q) = \varphi$ and $(p', q') = \varphi$, such that $q_*p_*^{-1} = \psi_{1*}\circ\psi_{1*}$ and $q'_*(p'_*)^{-1} = \psi_{2*}\circ\psi_{1*}$. Finally, this implies that $q_*p_*^{-1}\neq q'_*(p'_*)^{-1}$ for $\varphi\colon S^{2k}\to S^{2k}$.

(3.4) DEFINITION. Two admissible maps $\varphi, \psi\colon X\to Y$ are called *homotopic* (written $\varphi\sim\psi$) provided there exists an admissible map $\chi\colon X\times I\to Y$, where $I = [0, 1]$, such that

$$\chi(x, 0)\subset\varphi(x)\quad\text{and}\quad\chi(x, 1)\subset\psi(x)\quad\text{for each } x\in X.$$

(3.5) THEOREM. *Let $\varphi, \psi\colon X\to Y$ be two admissible maps. Then $\varphi\sim\psi$ implies that there exist selected pairs $(p, q)\subset\varphi$ and $(\bar{p}, \bar{q})\subset\psi$ such that $q_*\circ p_*^{-1} = \bar{q}_*\circ\bar{p}_*^{-1}$.*

Proof. Let $(\tilde{p}, \tilde{q}) \subset \chi$. Consider the commutative diagram



in which $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$ for each $x \in X$, j_0, j_1 are inclusions and p, $\bar{p}$ are given as the first coordinates of $p(z)$ for every $z \in \tilde{p}^{-1}(i_0(X))$ and $z \in \tilde{p}^{-1}(i_1(X))$, respectively. Then p, $\bar{p}$ are Vietoris maps and we have $(p, q) \subset \varphi$, $(\bar{p}, \bar{q}) \subset \psi$. We observe that $i_{0*} = i_{1*}$ is a linear isomorphism. This and the commutativity of the above diagram imply $q_* \circ p_*^{-1} = \bar{q}_* \circ \bar{p}_*^{-1}$. This proves Theorem (3.5).

(3.6) Corollary. *Let φ, ψ: $X \to Y$ be two admissible maps. Then $\varphi \sim \psi$ implies $\{\varphi\}_* \cap \{\psi\}_* \neq \emptyset$.*

(3.7) Corollary. *Let φ, ψ: $X \to Y$ be two acyclic maps. Then $\varphi \sim \psi$ implies $\varphi_* = \psi_*$.*

Example. Let φ, ψ_1: $S^n \to S^n$ be as in (3.3). Define the map χ: $S^n \times I \to S^n$ by $\chi(x, t) = \psi_1(x)$. Then χ is a homotopy joining φ with ψ_1 but $\{\psi_1\}_* = \{\psi_{1*}\}$ is a set consisting of one element; however, $\{\varphi\}_*$ is an infinite set.

4. Lefschetz maps. An admissible map φ: $X \to X$ is called a *Lefschetz map* provided for each selected pair $(p, q) \subset \varphi$ the linear map $q_* p_*^{-1}$: $H(X) \to H(X)$ is a Leray endomorphism.

For every Lefschetz map φ: $X \to X$ we may define the Lefschetz set

$$\Lambda(\varphi) = \{\Lambda(q_* p_*^{-1});\ (p, q) \subset \varphi\}.$$

The following facts are simple consequences of (3.1), (3.6) and (3.7), respectively:

(4.1) Proposition. *Let φ, ψ: $X \to X$ be two Lefschetz maps. Then $\varphi \subset \psi$ implies $\Lambda(\varphi) \subset \Lambda(\psi)$.*

(4.2) Proposition. *Let φ, ψ: $X \to X$ be two Lefschetz maps. Then $\varphi \sim \psi$ implies $\Lambda(\varphi) \cap \Lambda(\psi) \neq \emptyset$.*

(4.3) Proposition. *Let φ, ψ: $X \to X$ be two acyclic maps. If $\varphi \subset \psi$ or $\varphi \sim \psi$, then φ is a Lefschetz map if and only if ψ is a Lefschetz map and in this case $\Lambda(\varphi) = \Lambda(\psi)$.*

EXAMPLE. Let X be a space which is not of finite type. Define the maps $f, \varphi: X \to X$ by $\varphi(x) = X$, $f(x) = x_0$ for each $x \in X$. Then φ is an admissible map. We have $f \subset \varphi$ and $\mathrm{Id}_X \subset \varphi$ but f_* is a Leray endomorphism and $\mathrm{Id}_{H(X)}$ is not a Leray endomorphism.

References

§ 1. Bourbaki [6], Kuratowski [62].

§ 2–§ 4. Begle [3], Bowszyc [7], Bryszewski, Górniewicz [16], Eilenberg, Montgomery [21], Górniewicz [29], [32], [34], [36], [37], Granas [41], [44], [46], Granas, Górniewicz [38], Jaworowski, Powers [56], Leray [64], [65], O'Neill [72], Powers [73], [75], Rhee [76], Strother [82].

IV. ANR-s, AANR-s and w-AANR-s

In this chapter we recall the notions and basic properties which are essential in the fixed-point theory of multi-valued maps, of ANR-s, AANR-s and w-AANR-s.

1. ANR-s. A single-valued continuous map $f: X \to Y$ is said te be an *r-map* if there is a continuous single-valued map $g: Y \to X$ which is a right inverse of f, that is such that the composition $f \circ g: Y \to Y$ is the identity map Id_Y. If there exists an r-map $f: X \to Y$, then the space Y is called an *r-image* of the space X. The maps called retractions are a special kind of r-maps. Suppose that Y is a subset of X. Then map $f: X \to Y$ is said to be a *retraction* if the inclusion $i: Y \to X$ is a right inverse of f, i.e., $f(x) = x$ for all points $x \in X$. A subset X_0 of a space X is said to be a *retract* of X if there is a retraction of X onto X_0. A closed subset X_0 of a space X is said to be a *neighbourhood* retract in the space X provided X_0 is the retract of an open subset of X which contains X_0.

We denote by ANR the class of metrizable absolute neighbourhood retracts. A metrizable space X belongs to ANR provided, for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, the set $h(X)$ is a neighbourhood retract in Y.

In what follows we shall make use of the following facts from general topology:

(1.1) (Kuratowski Theorem). *Every metrizable space is embeddable into a Banach space; in particular, any topologically complete metrizable space can be embedded as a closed subset of a Banach space.*

(1.2) (Arens–Eells Theorem). *Every metrizable space can be embedded as a closed subset of a normed space.*

(1.3) (Aleksandrov Theorem). *Every open subset of a topologically complete metrizable space can be topologically complete metrizable.*

We prove the following

(1.4) THEOREM. *In order that $X \in$ ANR it is necessary and sufficient that X be an r-image of an open subset of a normed space.*

Proof. Let $X \in$ ANR. By Theorem (1.2) there exists an embedding $h: X \to E$ of X into a normed space E such that $h(X)$ is closed in E. Then there is a retraction $r: U \to h(X)$ of an open subset U of E which contains $h(X)$. Then $h^{-1} \circ r: U \to X$ is clearly an r-map. Now suppose that X is an r-image of a set U which is open in a normed space E. Let $f\colon U\to X$ be an r-map and $g\colon X\to U$ a right inverse for f. Consider a homeomorphism h mapping X onto a closed subset of a metric space Y. Then $g_1 = g\circ h^{-1}$ maps $h(X)$ into $U\subset E$ and so, by the generalized theorem of Tietze, there is a continuous extension $\tilde{g}_1$ of g_1 mapping Y into E. Let U' be the counter-image of U under $\tilde{g}_1$. Then U' is a neighbourhood of $h(X)$ in Y. Setting $r(y) = h\circ f\circ\tilde{g}_1(y)$ for $y\in U'$, we obtain a retraction map r and the proof is completed.

By applying (1.1), (1.3), instead of (1.2), we obtain analogously

(1.5) THEOREM. *A metrizable space is a topologically complete* ANR *if and only if it is an r-image of an open set in a Banach space.*

From (1.4) clearly follows

(1.6) *Every open subset of an* ANR *is* ANR.

Similarly, (1.3) and (1.5) implies

(1.7) *Every open subset of a topologically complete* ANR *is a topologically complete* ANR.

The following facts are well known:

(1.8) (a) *Every (finite) polyhedron is a compact* ANR;

(b) *Every compact* ANR *is a space of finite type*;

(c) *Every convex subset of a normed space is an* ANR;

(d) *Suppose that the metrizable space X is the union of two closed subsets X_1 and X_2 and that $X_0 = X_1\cap X_2$. If $X_0, X_1, X_2\in$* ANR, *then $X\in$* ANR.

Now we prove the following geometrical fact:

(1.9) LEMMA. *If U is open in a Banach space E and $X\subset U$ is compact, then there exists a compact $C\in$* ANR *such that $X\subset C\subset U$.*

Proof. Cover X by a finite number of closed balls $W_1,\ldots,W_n\subset U$ and denote by C_i the convex closure of the compact set $X\cap W_i$ for each $i=1,\ldots,n$. By the Mazur Lemma, every C_i is compact. From the inclusions $C_i\subset W_i\subset U$ we conclude that X is contained in the compact set $C=\bigcup_{i=1}^{n} C_i\subset U$.

Now we show by induction on n that the union of n compact, convex sets is an ANR. The statement is true if $n=1$ (comp. (1.8) (c)). Assume that the result is true for any integer less than n. By hypothesis $Y=\bigcup_{i=1}^{n-1} C_i$ and C_n are ANR-s. Further, $Y\cap C_n = \big(\bigcup_{i=1}^{n-1} C_i\big)\cap C_n = \bigcup_{i=1}^{n-1}(C_i\cap C_n)$, which by the induction hypothesis is an ANR. Thus $C = Y\cup C_n$ is the union of two ANR-s whose intersection is an ANR and (1.8) (d) implies that C is an ANR. This completes the induction and shows (1.9).

2. Approximation Theorem. In this section all maps are continuous and single-valued. We recall that a map $f\colon X\to Y$ is compact if it maps

X into a compact subset of Y. By homotopy we understand ordinary single-valued homotopy.

(2.1) Approximation Theorem. *Let U be an open subset of a normed space E and let $f\colon X \to U$ be a compact map. Then for every $\varepsilon > 0$ there exist a (finite) polyhedron $K_\varepsilon \subset U$ and a map $f_\varepsilon\colon X \to U$, called an ε-approximation of f, such that*

(i) $\|f_\varepsilon(x) - f(x)\| < \varepsilon$ *for all* $x \in X$,

(ii) $f_\varepsilon(X) \subset K_\varepsilon$,

(iii) f_ε *is homotopic to* f.

Proof. Given $\varepsilon > 0$ (which we may assume to be sufficiently small), $f(X)$ is contained in the union of a finite number of open balls $V(y_i, \varepsilon)$ with $V(y_i, 2\varepsilon) \subset U$, $i = 1, \ldots, k$. Putting for $x \in X$

$$f(x) = \frac{\sum_{i=1}^{k} \lambda_i(x) \cdot y_i}{\sum_{i=1}^{k} \lambda_i(x)},$$

where $\lambda_i(x) = \max\{0, \varepsilon - \|f(x) - y_i\|\}$, we obtain the map f_ε satisfying (i) and (ii). Clearly, the values of f_ε are in a polyhedron $K_\varepsilon \subset U$ with vertices $y_1, \ldots, y_k$. To prove assertion (iii), note that for each $x \in X$ the points $f(x)$ and $f_\varepsilon(x)$ belong to some ball $V \subset U$. Therefore the family g_t defined by the formula

$$g_t(x) = t \cdot f(x) + (1-t) f_\varepsilon(x), \quad \text{for} \quad (x, t) \in X \times \langle 0, 1 \rangle$$

has values in U. Thus $g_t\colon X \to U$ is a homotopy joining f with f_ε and the proof is complete.

3. AANR-s. The class of AANR-s was first studied by H. Noguchi.

(3.1) Definition. Let (X, A) be a pair of metric spaces and let ε be a positive real number. A continuous (single-valued) map $r_\varepsilon\colon X \to A$ is called an *ε-retraction* provided $d(r_\varepsilon(a), a) < \varepsilon$ for all $a \in A$.

A subspace A of a metric space X is said to be an *approximative retract* of X provided for each $\varepsilon > 0$ there exists an ε-retraction $r_\varepsilon\colon X \to A$.

(3.2) Definition. A metrizable space X is said to be an *approximative* ANR (AANR) provided for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metric space Y, the set $h(X)$ is an approximative retract of some open set U in Y.

Although not necessarily locally connected, the AANR-s enjoy many familiar properties of ANR spaces. In particular:

(3.3) *Every compact* AANR X *is of finite type.*

(3.4) Definition. An AANR X is said to be *admissible* provided there exist a homeomorphism h mapping X onto a closed subset $h(X)$

of a normed space E and an open neighbourhood U of $h(X)$ in E such that the following two conditions are satisfied:

(i) $h(X)$ is an approximative retract of U,

(ii) the inclusion $i\colon h(X)\to U$ induces a monomorphism $i_*\colon H(h(X))\to H(U)$.

(3.5) PROPOSITION. *Every* ANR *in an admissible* AANR.

Proof. Let $X\in$ ANR. Using the Arens–Eells embedding theorem, we obtain a homeomorphism h mapping X into a normed space E such that (i) $h(X)$ is closed of E, (ii) there exists a retraction $r\colon U\to h(X)$, where U is an open neighbourhood of $h(X)$ in E. Then the inclusion $i\colon h(X)\to U$ is the right inverse of r and we have $ri = \mathrm{Id}_{h(X)}$. Hence we infer that $r_* i_* = \mathrm{Id}_{H(h(X))}$ and this implies that i_* is a monomorphism.

(3.6) PROPOSITION. *Every compact* AANR *is an admissible* AANR.

Proof. Using the Arens–Eells embedding theorem (or the Kuratowski embedding theorem), we may assume without loss of generality that X is an approximative retract of some open neighbourhood U of X in a normed space E. Since X is of finite type, from Theorem I.3.4 we deduce that there exists an $\varepsilon_0 > 0$ such that for every two maps $f, g\colon X\to X$, the condition $\|f(x)-g(x)\| < \varepsilon_0$ implies $f_* = g_*$.

Choose an $\varepsilon > 0$ such that $\varepsilon < \varepsilon_0$ and consider the two maps Id, $r_\varepsilon \circ i\colon X\to X$, where $r\colon U\to X$ is an ε-retraction and $i\colon X\to U$ is an inclusion map. By Theorem I.3.4 we infer that $\mathrm{Id}_{H(X)} = (r_\varepsilon)_*\cdot i_*$, and this implies that $i_*\colon H(X)\to H(U)$ is a monomorphism.

(3.7) PROPOSITION. *Every acyclic* AANR *is an admissible* AANR.

For the proof of (3.7) observe that if X is an acyclic space and $X\subset Y$, then the inclusion $i\colon X\to Y$ induces a monomorphism $i_*\colon H(X)\to H(Y)$.

The following lemma is of importance:

(3.8) LEMMA. *Let X be an* AANR. *Assume that X is an approximative retract of an open subset U in a normed space E and $i\colon X\to U$ induces a monomorphism $i_*\colon H(X)\to H(U)$. Then for every compact subset $K\subset X$ there exists a positive real number $\varepsilon(K)$ such that for every $\varepsilon < \varepsilon(K)$ and for every ε-retraction $r_\varepsilon\colon U\to X$ we have*

$$(r_\varepsilon)_* i_* j_* = j_*, \quad \textit{where} \quad j\colon K\to X \textit{ is the inclusion map}.$$

Proof. Let $\varepsilon(K) > 0$ be a number smaller than the distance $\mathrm{dist}(K, \partial U)$ from the compact set K to the boundary ∂U of U in E. From the definition of $\varepsilon(K)$ we infer that for each $x\in X$ and $\varepsilon < \varepsilon(K)$ the interval $t\cdot ir_\varepsilon ij(x) + (1-t)\cdot ij(x)$, where $0\leqslant t\leqslant 1$, is entirely contained in U. This implies that $ir_\varepsilon ij$ and ij are homotopic for every $\varepsilon < \varepsilon(K)$. Since i_* is a monomorphism, we get $(r_\varepsilon)_* i_* j_* = j_*$ for each $\varepsilon < \varepsilon(K)$ and the proof is completed.

4. w-AANR-s. A closed subspace X of a metric space Y is called a *weak approximative neighbourhood retract* in Y provided for every $\varepsilon > 0$ there exist an open neighbourhood U_ε of X in Y and an ε-retraction $r_\varepsilon\colon U_\varepsilon \to X$.

(4.1) DEFINITION. A metrizable space X is said to be *weakly* AANR (w-AANR) provided for each emedding $h\colon X \to Y$, Y being a metric space and $h(X)$ being closed in Y, the space $h(X)$ is a weak approximative neighbourhood retract in Y.

It is easy to see that there exists a compact w-AANR which is not of finite type.

We prove the following simple geometrical fact:

(4.2) LEMMA. *Let X be a weak approximative neighbourhood retract in a normed space E and let K be a compact subset of X. Then for each open neighbourhood W of K in X there exists a positive real number $\delta(W)$ such that $K \subset r_\varepsilon^{-1}(W)$ for each $0 < \varepsilon < \delta(W)$, where r_ε denotes any ε-retraction related to X.*

Proof. Let W be an open neighbourhood of K in X and let $\partial(W)$ denote the boundary of W in X. Then $K \cap \partial(W) = \emptyset$. We define $f(x) = \inf_{y \in \partial(W)} \|x - y\|$. Since $\partial(W)$ as a closed subset of X is closed in E, then $K \cap \partial(W) = \emptyset$ implies $f(x) > 0$ for every $x \in K$ and thus $f\colon K \to (0, +\infty)$. Since K is compact, we deduce that $\delta(W) = \inf_{x \in K}\{f(x)\}$ is a positive real number. Then for every $0 < \varepsilon < \delta(W)$ we have $K \subset r_\varepsilon^{-1}(W)$ and the proof is completed.

References

§ 1. Borsuk [5], Bourbaki [6], Kuratowski [62], Arens and Eells [1].
§ 2. Granas [41], [44], [46].
§ 3. Gmurczyk [28], Jaworowski [53], Noguchi [70].
§ 4. Clapp [18].

V. THE LEFSCHETZ FIXED-POINT THEOREM

We shall now propose the application of the Čech homology with compact carriers and the theory of Lefschetz number by establishing a general fixed-point theorem for admissible maps, which contains the classical Lefschetz Fixed-Point Theorem (for single-valued maps) and the well-known Eilenberg–Montgomery Fixed-Point Theorem for acyclic maps. The principal results of this chapter are Theorems (3.2) and (5.1). Moreover, Sections 1 and 2 contain a modern proof of the Eilenberg–Montgomery Theorem, and Sections 4 and 6 present some applications of those results to the fixed-point theory of admissible maps.

1. The index of coincidence. Let $p, q\colon Y \to X$ be two single-valued maps. A *coincidence* of the pair (p, q) is a point $y \in Y$ with $p(y) = q(y)$.

(1.1) LEMMA. *Let $p, q\colon Y \to U$ be two single-valued, continuous maps from a space Y to an open subset U of the Euclidean space R^n. Assume that p is a Vietoris map and q is a compact map. Then the set*

$$\varkappa_{p,q} = \{x \in U;\ x \in q(p^{-1}(x))\}$$

is compact.

Proof. Consider the multi-valued map $\psi = q \circ \varphi_p\colon U \to U$ (comp. Section 1 in III). Then, in view of III.1.8 we deduce that ψ is a compact u.s.c. map. Let

$$\varkappa_\psi = \{x \in U;\ x \in \psi(x)\}.$$

Then $\varkappa_\psi = \varkappa_{p,q}$ and hence from III.1.1 we deduce that $\varkappa_{p,q}$ is a compact set. The proof of (1.1) is completed.

Consider the diagram

$$U \xleftarrow{p} Y \xrightarrow{q} U,$$

in which U, Y, p, q are as in (1.1).

With the above diagram we associate diagram

$$(1)\qquad (U, U \setminus \varkappa_{p,q}) \xleftarrow{\bar{p}} (Y, Y \setminus p^{-1}(\varkappa_{p,q})) \xrightarrow{\bar{q}} (R^n, R^n \setminus \{0\}),$$

where $\bar{p}(y) = p(y)$ and $\bar{q}(y) = p(y) - q(y)$ for each $y \in Y$.

We define the index of coincidence $I(p, q)$ of the pair (p, q) by putting (comp. Section I.5)

$$I(p, q) = \bar{q}_*(\bar{p}_*)^{-1}(O_{\varkappa_{p,q}}) \in H_n(R^n, R^n \setminus \{0\}) \approx Q.$$

(1.2) Proposition. *If $I(p, q) \neq 0$, then the pair (p, q) has a coincidence.*

Proof. Indeed, if $p(y) \neq q(y)$ for each $y \in Y$, then $\varkappa_{p,q} = \emptyset$. Then we have

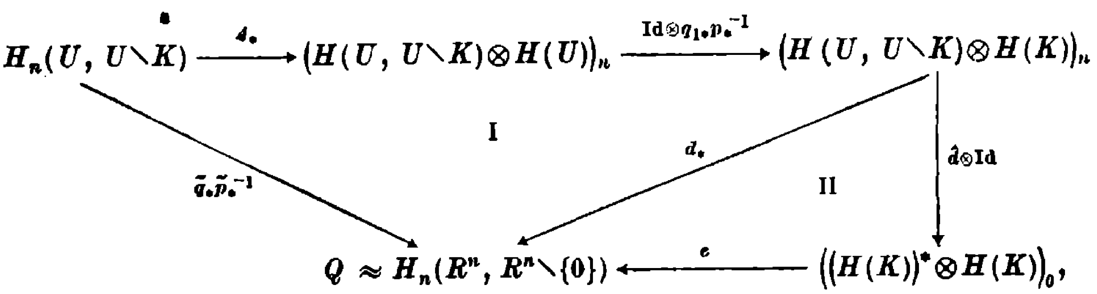
$$I(p, q) = \bar{q}_*(\bar{p}_*)^{-1}(O_{\varkappa_{p,q}}) = \bar{q}_*\bar{p}_*^{-1}(0) = 0.$$

From I.5.5 clearly follows

(1.3) Proposition. *If A is a compact set such that $\varkappa_{p,q} \subset A \subset U$, then $I(p, q) = \tilde{q}_*(\tilde{p}_*)^{-1}(O_A)$, where $\tilde{p}, \tilde{q}$ are given by the same formula as $\bar{p}, \bar{q}$ in* (1).

(1.4) Proposition. *Let K be a finite polyhedron such that $q(Y) \subset K$; then there exists an element $a \in (H(K))^* \otimes H(K)$ such that $I(p, q) = e(a)$.*

Proof. Consider the diagram



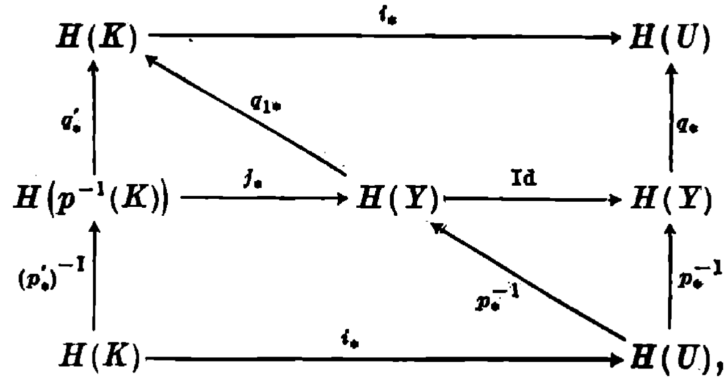
in which $q_1\colon Y \to K$ is the contraction of q to the pair (Y, K) and $\hat{d}\colon H(U, U \setminus K) \to (H(K))^*$ is a linear map of degree $(-n)$ given by

$$\hat{d}(u)(v) = d_*(u \otimes v) \quad \text{for} \quad u \in H(U, U \setminus K) \text{ and } v \in H(K).$$

The subdiagram I commutes (see I.5.4). The commutativity of II follows by easy computation. We define $a = (\hat{d} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes q_{1*}p_*^{-1}) \Delta_*(O_K)$. From the commutativity of the above diagram and (1.3) we deduce $I(p, q) = e(a)$ and the proof of (1.4) is completed.

(1.5) Coincidence Theorem. *Let $p, q\colon Y \to U$ be two continuous (single-valued) maps from a space Y to an open subset $U \subset R^n$. If p is a Vietoris map and q is a compact map, then $q_*p_*^{-1}$ is a Leray endomorphism, and if $\Lambda(q_*p_*^{-1}) \neq 0$, then the pair (p, q) has a coincidence.*

Proof. Since q is a compact map, there exists a finite polyhedron K such that $q(Y) \subset K \subset U$. We have the commutative diagram



in which i_*, j_* are induced linear maps by the inclusions $i\colon K \to U$ and $j\colon p^{-1}(K) \to Y$, respectively, and q'_*, q_{1*}, p'_* are induced linear maps by the contractions of q and p, respectively.

The commutativity of the above diagram and II.2.3 imply

$$\Lambda(q_* p_*^{-1}) = \lambda\big(q'_*(p'_*)^{-1}\big),$$

and hence $q_* p_*^{-1}$ is a Leray endomorphism.

Assume that $\Lambda(q_* p_*^{-1}) \neq 0$. For the proof it is sufficient to show that

$$\lambda\big(q'_*(p'_*)^{-1}\big) = I(p, q) \tag{1}$$

(comp. also Section 5 in I).

Consider the following diagram:

$$\begin{array}{ccccc}
H(U, U\setminus K)\otimes H(U)\otimes H(K) & \xrightarrow{\mathrm{Id}\otimes t_*} & H(U, U\setminus K)\otimes H(K)\otimes H(U) & \xrightarrow{d_*\otimes \mathrm{Id}} & Q\otimes H(U) \approx H(U) \\
\Big\downarrow {\scriptstyle \hat{d}\otimes q_{1*}p_*^{-1}\otimes \mathrm{Id}} & & \Big\downarrow {\scriptstyle \hat{d}\otimes \mathrm{Id}\otimes q_{1*}p_*^{-1}} & & \Big\downarrow {\scriptstyle q_{1*}p_*^{-1}} \\
\big(H(K)\big)^*\otimes H(K)\otimes H(K) & \xrightarrow{\mathrm{Id}\otimes t_*} & \big(H(K)\big)^*\otimes H(K)\otimes H(K) & \xrightarrow{e\otimes \mathrm{Id}} & Q\otimes H(K) \approx H(K).
\end{array}$$

The commutativity of the above diagram is obtained by simple calculation.

Let

$$a = (\hat{d}\otimes \mathrm{Id})(\mathrm{Id}\otimes q_{1*}p_*^{-1})\,\Delta_*(O_K) \in \mathrm{Hom}_Q\big(H(K)\big)\otimes H(K).$$

Since $e(a) = I(p, q)$ (see (1.4)), for the proof of (1) it is sufficient to show that

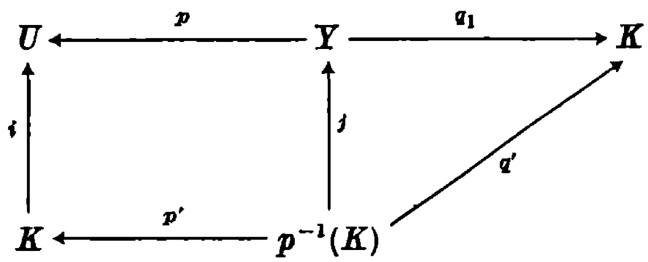
$$\Theta(a) = q'_*(p'_*)^{-1} \tag{2}$$

(comp. Section 1 in II).

If we follow $\Delta_*(O_K)\otimes u \in H(U, U\setminus K)\otimes H(U)\otimes H(K)$ along $\downarrow \to\to$, we get $(\Theta(a))(u)$. If we follow it along $\to\to$, by Dold's Lemma (I.5.6) we get $i_*(u)$. Therefore, for the proof of (2) it is sufficient to show that

$$(3) \qquad q_{1*}p_*^{-1}i_* = q'_*(p'_*)^{-1}.$$

Consider the following commutative diagram:



Applying to the above diagram the functor H, we obtain (3) and the proof of the Coincidence Theorem is completed.

2. The Lefschetz Fixed-Point Theorem for open subsets in R^n. The Coincidence Theorem implies the following Lefschetz Fixed-Point Theorem:

(2.1) THEOREM. *Let U be an open subset in R^n and $\varphi\colon U\to U$ an admissible compact map. Then* (i) *φ is a Lefschetz map, and* (ii) *$\Lambda(\varphi) \neq \{0\}$ implies that φ has a fixed point.*

Proof. Let (p, q) be a selected pair of φ. Since φ is compact, the pair (p, q) satisfies all the assumptions of the Coincidence Theorem. Then (1.5) implies that φ is a Lefschetz map. If $\Lambda(\varphi) \neq \{0\}$, then there exists a $(p, q) \subset \varphi$ such that $\Lambda(q_*p_*^{-1}) \neq 0$ and by (1.5) we obtain a point $y \in Y$ such that $p(y) = q(y)$. Let $y \in p^{-1}(x)$; then $x \in q(p^{-1}(x)) \subset \varphi(x)$ and the proof of (2.1) is completed.

(2.2) THEOREM. *Let X be a retract of an open subset $U \subset R^n$ and $\varphi\colon X\to X$ an admissible compact map. Then* (i) *φ is a Lefschetz map, and* (ii) *if $\Lambda(\varphi) \neq \{0\}$, then φ has a fixed point.*

Proof. Let $r\colon U\to X$ be a retraction map, $i\colon X\to U$ the inclusion map and $\psi\colon U\to U$ a multi-valued map given by $\psi = i\varphi r$. From III.2.7 we deduce that ψ is an admissible map. The compactness of ψ is evident. Let (p, q) be a selected pair of φ. Since r, i, as single-valued maps, are acyclic, then for every selected pair $(p_1, q_1) \subset r$ and $(p_2, q_2) \subset i$, by III.2.3 we have

$$q_{1*}p_{1*}^{-1} = r_* \quad \text{and} \quad q_{2*}p_{2*}^{-1} = i_*.$$

Applying again III.2.7, we deduce that there exists a selected pair $(\bar{p}, \bar{q}) \subset \psi$ such that

$$(1) \qquad \bar{q}_* \bar{p}_*^{-1} = i_* q_* p_*^{-1} r_*.$$

Consider the following diagram:

$$\begin{array}{ccc} H(X) & \xrightarrow{\quad i_* \quad} & H(U) \\ {\scriptstyle q_*p_*^{-1}} \Big\uparrow & {\scriptstyle q_*p_*^{-1}r_*} & \Big\uparrow {\scriptstyle i_*q_*p_*^{-1}r_*} \\ H(X) & \xrightarrow{\quad i_* \quad} & H(U) \end{array}$$

Since $r_* i_* = \mathrm{Id}_{H(X)}$, this diagram commutes. Using (1), (1.5) and II.2.4, we infer that $q_* p_*^{-1}$ is a Leray endomorphism.

To prove (ii) assume that $\Lambda(\varphi) \neq \{0\}$. Then there exists a $(p, q) \subset \varphi$ such that $\Lambda(q_* p_*^{-1}) \neq 0$ and, as in the proof of the first part of this theorem, we obtain $(\bar{p}, \bar{q}) \subset \psi$ such that

$$\Lambda(\bar{q}_* \bar{p}_*^{-1}) = \Lambda(q_* p_*^{-1}) \neq 0.$$

From Theorem (2.1) we obtain a point x such that $x \in \psi(x)$. We have $x \in i\varphi r(x) \subset X$ and, since r is a retraction map, we deduce that $r(x) = x$. Finally, we obtain $x \in \varphi(x)$ and the proof is completed.

Theorem (2.2) and IV.1.8(a) give

(2.3) COROLLARY. *Let K be a finite polyhedron and $\varphi: K \to K$ an admissible map. Then $\Lambda(\varphi) \neq \{0\}$ implies that φ has a fixed point.*

3. The Lefschetz Fixed-Point Theorem for AANR-s. Before stating the main result of this section in full generality, we shall first consider the following special case:

(3.1) THEOREM. *Let U be an open subset in a normed space E and let $\varphi: U \to U$ be an admissible compact map. Then* (a) *φ is a Lefschetz map, and* (b) *$\Lambda(\varphi) \neq \{0\}$ implies that φ has a fixed point.*

Proof. Let $p, q: Y \to U$ be a pair of maps such that $(p, q) \subset \varphi$. By applying to q the Approximation Theorem (see IV.2.1), we get a sequence $\{K_n\}$ of finite polyhedra $K_n \subset U$ and a sequence of maps $q_n: Y \to U$ such that the following conditions are satisfied:

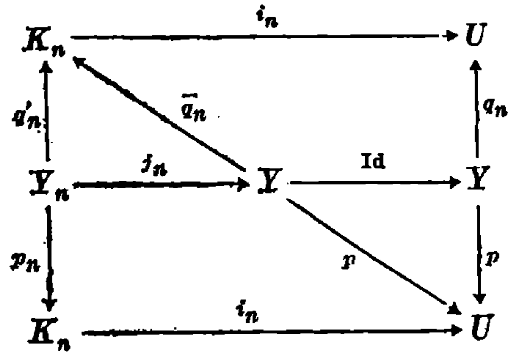
(i) $\|q_n(y) - q(y)\| < 1/n$ for all $y \in Y$ and for every n,

(ii) $q_n(Y) \subset K_n$ for each n,

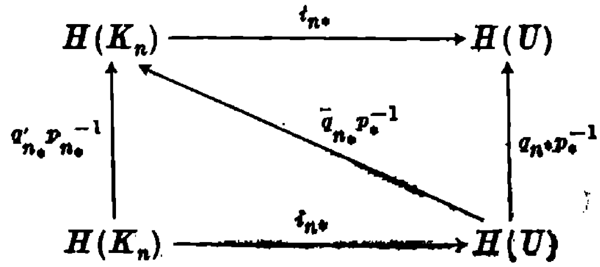
(iii) q_n is homotopic to q for each n.

Let $q_n': Y_n = p^{-1}(K_n) \to K_n$, $\bar{q}_n: Y \to K_n$, $p_n: Y_n \to K_n$ be contractions of q_n and p, respectively, and let $i_n: K_n \to U$, $j_n: Y_n \to Y$ be inclusions.

Now, for every n, we consider the following commutative diagram:



Consequently its image under H also commutes and the diagram



commutes. Since every K_n is of finite type, $q'_{n*}p_{n*}^{-1}$ is a Leray endomorphism. Then, by II.2.4, $q_{n*}p_*^{-1}$ is also a Leray endomorphism and $\lambda(q'_{n*}p_{n*}^{-1}) = \Lambda(q_{n*}p_*^{-1})$. Finally, condition (iii) implies that $q_*p_*^{-1}$ is a Leray endomorphism, and moreover we have

(iv) $\lambda(q'_{n*}p_{n*}^{-1}) = \Lambda(q_{n*}p_*^{-1}) = \Lambda(q_*p_*^{-1})$.

To prove (b) assume that $\Lambda(\varphi) \neq \{0\}$. Then there exists a $(p, q) \subset \varphi$ such that $\Lambda(q_*p_*^{-1}) \neq 0$. As in the first part of this proof, we obtain a sequence $\{K_n\}$ of polyhedra and sequences of $\{q_n\}$, $\{q'_n\}$, $\{p_n\}$, associated with q and p, respectively, such that the conditions (i)–(iv) are satisfied. Consider, for each n, the map ψ_n: $K_n \to K_n$ given by $\psi_n = q'_n \circ \varphi_{p_n}$. Then ψ_n is an admissible map and hence $(p_n, q'_n) \subset \psi_n$ for each n.

Using (2.3) and (iv), we obtain a sequence $\{x_n\}$ of points in U such that $x_n \in \psi_n(x_n)$ for each n. For each n we denote by y_n a point in $p_n^{-1}(x_n)$ such that $q'_n(y_n) = x_n$. Then we have

$$x_n = p_n(y_n) = q'_n(y_n) = p(y_n) = q_n(y_n) \quad \text{for each } n.$$

We put $q(y_n) = \bar{x}_n$ for each n. Since q is a compact map, we may assume that there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ such that $\lim_k \bar{x}_{n_k} = x$. Then from (i) we deduce that $\lim_k x_{n_k} = x$ and hence we have

(v) $\bar{x}_{n_k} \in qp^{-1}(x_{n_k})$, $\{\bar{x}_{n_k}\} \to x$ and $\{x_{n_k}\} \to x$.

Since the map $\psi = q \circ \varphi_p$ is u.s.c. (comp. III.1.8), then, in view of (v) and III.1.1, we have $x \in \psi(x) = q(p^{-1}(x)) \subset \varphi(x)$, and the proof of (3.1) is completed.

Now we are able to state the principal result of this paper.

(3.2) THEOREM. *Let X be an admissible* AANR *and let φ: $X \to X$ be an admissible compact map. Then*: (a) *φ is a Lefschetz map, and* (b) $\Lambda(\varphi) \neq \{0\}$ *implies that φ has a fixed point.*

Proof. Since X is an admissible AANR, we may assume that there exists an open subset of a normed space E such that the following two conditions are satisfied:

(i) X is an approximative retract of U,

(ii) the inclusion i: $X \to U$ induces a monomorphism i_*: $H(X) \to H(U)$.

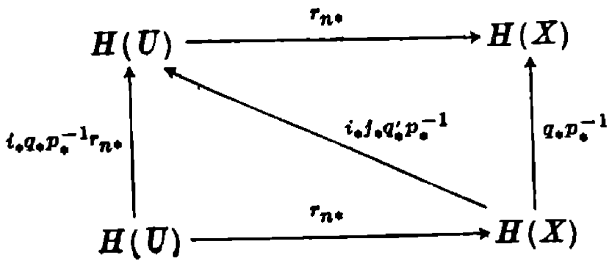
Let r_n: $U \to X$ be a $(1/n)$-retraction. We have

(iii) $\|r_n(x) - x\| < 1/n$ for each $x \in X$ and for every n.

Let p, q: $Y \to X$ be a pair of maps such that $(p, q) \subset \varphi$. Consider for each n an admissible compact map ψ_n: $U \to U$ given by $\psi_n = i_* q_* \varphi_{*p} r_n$. Using III.2.7 and III.2.3, we choose a selected pair $(p_n, q_n) \subset \psi_n$ such that

(iv) $q_{n*} p_{n*}^{-1} = i_* q_* p_*^{-1} r_{n*}$, for each n.

Since q is a compact map, we infer that the set $A = \overline{q(Y)}$ is compact. Consider for each n the diagram



where q': $Y \to A$ is given by $q'(y) = q(y)$ for each $y \in Y$ and j: $A \to X$ is an inclusion. From Lemma IV.3.8 we obtain $r_{n*} i_* j_* = j_*$ for all $n > n_0$. Since $j_* q'_* = (j \circ q')_* = q_*$, we deduce that the above diagram commutes for each $n > n_0$. Consequently, from II.2.4, (iv) and (3.1) we conclude that $q_* p_*^{-1}$ is a Leray endomorphism. Thus the assertion (a) is proved.

To prove (b) assume that $\Lambda(\varphi) \neq \{0\}$. Then there exists a selected pair $(p, q) \subset \varphi$ such that $\Lambda(q_* p_*^{-1}) \neq 0$. Let $(p_n, q_n) \subset \psi_n$ where p_n, q_n and ψ_n are obtained as in first part of the proof. Then, from II.2.4 and (iv) we have

$$\Lambda(q_{n*} p_{n*}^{-1}) = \Lambda(i_* q_* p_*^{-1} r_{n*}) = \Lambda(q_* p_*^{-1}) \neq 0, \quad \text{for each } n > n_0.$$

This, in view of Theorem (3.1), implies that ψ_n has a fixed point for each $n > n_0$. We find a sequence $\{x_n\}$ in the compact set A such that:

(v) $x_n \in \psi_n(x_n)$ for each $n > n_0$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

(vi) $\lim\limits_k x_{n_k} = x$.

Then from (iii) we obtain

(vii) $\lim\limits_k r_{n_k}(x_{n_k}) = x$.

Conditions (v), (vi), (vii) give

(viii) $\{r_{n_k}(x_{n_k})\}\to x$, $x_{n_k}\in q\varphi_p r_{n_k}(x_{n_k})$ and $\{x_{n_k}\}\to x$.

Finally, the u.s.c. of $\psi = q\circ\varphi_p$ (III.1.8), in view of (viii) and III.1.1, implies $x\in\psi(x) = q\circ\varphi_p(x) = qp^{-1}(x)\subset\varphi(x)$ and the proof of Theorem (3.2) is completed.

We now draw a few immediate consequences of Theorem (3.2).

(3.3) COROLLARY. *Let X be an* ANR *or a compact* AANR *and let $\varphi\colon X\to X$ be an admissible compact map. Then* (i) *φ is a Lefschetz map, and* (ii) *$\Lambda(\varphi)\neq\{0\}$ implies that φ has a fixed point.*

For acyclic maps we obtain the following

(3.4) COROLLARY. *Let X be an admissible* AANR *or, in particular, either of the following:*

(a) *an* ANR,

(b) *a compact* AANR.

If $\varphi\colon X\to X$ is a compact acyclic map, then (i) *φ is a Lefschetz map, and* (ii) *$\Lambda(\varphi)\neq 0$ implies that φ has a fixed point.*

From (3.4) and III.4.3 we deduce

(3.5) COROLLARY. *Let X be an admissible* AANR *and let $\varphi,\psi\colon X\to X$ be two compact acyclic maps which satisfy one of the following conditions:*

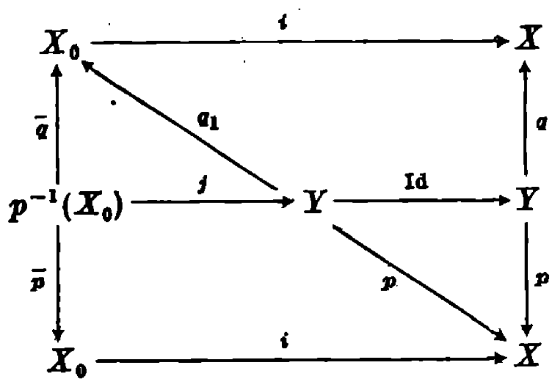
(i) *φ is a selector of ψ,*

(ii) *φ is homotopic to ψ.*

Then both φ and ψ are Lefschetz maps, $\Lambda(\varphi)=\Lambda(\psi)$, and $\Lambda(\psi)\neq 0$ implies that φ has a fixed point.

(3.6) COROLLARY. *Let X be an admissible* AANR *and $\varphi\colon X\to X$ an admissible compact map. Assume further that $\varphi(X)$ is contained in an acyclic subset X_0 of X. Then $\Lambda(\varphi)=\{1\}$ and φ has a fixed point.*

Proof. Let $p,q\colon Y\to X$ be a pair of maps such that $(p,q)\subset\varphi$. Write the diagram



in which p, q, q_1 are contractions of p and q, respectively and i, j are inclusions. Then its image under H also commutes. Since $\Lambda(\bar q_*\bar p_*^{-1})=1$ from

II.2.4, we have $\Lambda(q_* p_*^{-1}) = 1$ for every $(p, q) \subset \varphi$, and from Theorem (3.2) we obtain (3.6).

A space X has the fixed-point property within the class of admissible compact maps provided any admissible compact map $\varphi\colon X \to X$ has a fixed point.

(3.7) Corollary. *Let X be an acyclic* AANR *or, in particular, either of the following*:

(i) *an acyclic* ANR,

(ii) *a contractible open set in a normed space.*

Then X has the fixed-point property within the class of admissible compact maps.

This simply follows from (3.6) and IV.3.7. Similarly, from (3.6) and IV.1.8(c), we have

(3.8) Corollary (The Schauder Fixed-Point Theorem). *Let X be a convex subset of a normed space. Then X has the fixed-point property within the class of admissible compact maps.*

Finally, we prove the following proposition, well-known for single-valued maps:

(3.9) Proposition. *Assume that a space X has the fixed-point property within the class of admissible, compact maps. Then every retract of X has the fixed-point property within the class of admissible compact maps.*

Proof. Assume that X has the fixed-point property within the class of admissible compact maps. Let $A \subset X$ be a retract of X and let $r\colon X \to A$ be the corresponding retraction. Let $\varphi\colon A \to A$ be an admissible compact map. Define the map $\psi\colon X \to X$ by putting $\psi = i\varphi r$, where $i\colon A \to X$ is the inclusion map. From III.2.7 we deduce that ψ is an admissible compact map. By assumption, there exists a point x such that $x \in \psi(x)$, but $\psi(X) \subset A$, and therefore $x \in A$. Since r is a retraction map, we have $r(x) = x$ and hence $x \in \varphi(x)$. This completes the proof.

4. Neighbourhood fixed-point property. It was shown by K. Borsuk that there exist in the Euclidean space R^3 acyclic continua without the fixed-point property within the class of single-valued maps. This shows that the Lefschetz Fixed-Point Theorem cannot be extended to arbitrary compacta.

Let A be an acyclic compact subset of the Euclidean space R^n. Then the Lefschetz Fixed-Point Theorem implies that for every open neighbourhood U of A in R^n and for every admissible map $\varphi\colon U \to A$ there exists a fixed point of φ. This allows us to introduce the following notion:

We say that a compact acyclic subset A of a space X has the *neighbourhood fixed-point property* in X if for every open neighbourhood U of A in X and for every admissible map $\varphi\colon U \to A$ there exists a fixed point of φ.

(4.1) THEOREM. *If X is an* w-AANR, *then every compact acyclic subset of X has the neighbourhood fixed-point property in X.*

Proof. Let A be a compact acyclic subset of X, U an open neighbourhood of A in X and $\varphi\colon U \to A$ an admissible map. We prove that φ has a fixed point. By the Arens–Eells Theorem we may assume without loss of generality that X is a weak approximative neighbourhood retract of a normed space E. For each $n = 1, 2, \ldots$, let $r_n\colon V_n \to X$ be the corresponding $(1/n)$-retraction of some neighbourhood V_n of X in E. Applying IV.4.2 to the pair (U, A), we infer that for $n > n_0 = n_0(U)$ the set A is contained in $r_n^{-1}(U)$. Let (p, q) be a selected pair of φ and let $\psi\colon U \to A$ be a map given by $\psi = q \circ \varphi_p$. For each $n > n_0$ define the map $\psi_n\colon W_n = r_n^{-1}(U) \to W_n$ by putting $\psi_n = i_n \psi r_n$, where $i_n\colon A \to W_n$ is the inclusion map. Then ψ_n is an admissible map such that $\psi_n(W_n)$ is contained in a compact acyclic space A. Since W_n, as an open subset of E, is ANR space from (3.6), we deduce that ψ_n has a fixed point for each $n > n_0$. Then, as in the proof of (3.2) (see p. 43), we obtain a fixed point of φ. This proves (4.1).

Since each compact metric space may be regarded as a compact subset of the Hilbert cube, we infer in particular from (4.1), that every compact acyclic and metric space has the neighbourhood fixed-point property in the Hilbert cube.

5. The Lefschetz Fixed-Point Theorem for w-AANR-s. In this section we prove the following

(5.1) THEOREM. *Let X be a compact* w-AANR *of finite type and let $\varphi\colon X \to X$ be an admissible map. Then $\Lambda(\varphi) \neq \{0\}$ implies that φ has a fixed point.*

Proof. We may assume without loss of generality that X is a weak approximative neighbourhood retract in a Banach space E. For each $n = 1, 2, \ldots$ let $r_n\colon U_n \to X$ be a $(1/n)$-retraction from an open neighbourhood of X in E to X. We have

$$\|x - r_n(x)\| < 1/n, \quad \text{for all } x \in X. \tag{1}$$

For each n let $i_n\colon X \to U_n$ be the inclusion map. By assumption we infer that there exists a selected pair $(p, q) \subset \varphi$ such that $\lambda(q_* p_*^{-1}) \neq 0$. Let $\psi\colon X \to X$ be a map given by $\psi = q \cdot \varphi_p$. Then ψ is an admissible map and hence $(p, q) = \psi$. Define for each n a map $\psi_n\colon U_n \to U_n$ by putting

$$\psi_n = i_n \psi r_n.$$

From III.2.7 and III.2.3 we deduce that for each n there exists a selected pair $(p_n, q_n) \subset \psi_n$ such that

$$q_{n*} p_{n*}^{-1} = i_{n*} q_* p_*^{-1} r_{n*}. \tag{2}$$

Consider for each n the diagram

$$\begin{array}{ccc} H(X) & \xrightarrow{\quad i_{n*} \quad} & H(U_n) \\ {\scriptstyle q_*p_*^{-1}}\Big\uparrow & \nwarrow {\scriptstyle q_*p_*^{-1}r_{n*}} & \Big\uparrow{\scriptstyle i_{n*}q_*p_*^{-1}r_{n*}} \\ H(X) & \xrightarrow{\quad i_{n*} \quad} & H(U_n) \end{array}$$

Since X is a compact space of finite type, we deduce from I.3.4 that

$$r_{n*}i_{n*} = \mathrm{Id}_{H(X)} \qquad \text{for all } n > n_0.$$

This implies that for each $n > n_0$ the above diagram commutes and hence (2) and II.2.4 gives

$$\lambda(q_*p_*^{-1}) = \Lambda(q_{n*}p_{n*}^{-1}) \neq 0 \qquad \text{for all } n > n_0.$$

Thus Theorem (3.1) implies that ψ_n has a fixed point for each $n > n_0$. Using the procedure followed in the proof of (3.2) (see p. 43), we obtain a fixed point of φ, and the proof is completed.

(5.2) Corollary. *If X is an acyclic compact* w-AANR, *then X has the fixed-point property within the class of admissible maps.*

In particular, for acyclic maps Theorem (5.1) and III.4.3 give

(5.3) Corollary. *Let X be a compact* w-AANR *and let φ, ψ: $X \to X$ be two acyclic maps which satisfy one of the following conditions:*

(i) *φ is a selector of ψ,*

(ii) *φ is homotopic to ψ.*

Then $\lambda(\varphi) = \lambda(\psi)$ and $\lambda(\psi) \neq 0$ implies that φ has a fixed point.

6. Two consequences of the Lefschetz Fixed-Point Theorem. Let A be a non-empty subset of a space X and let i: $A \to X$ be the inclusion map; call A a *homologically trivial subset of X* provided (i) $\dim \operatorname{Im} i_{*0} = 1$, and (ii) $i_{*k} = 0$ for all $k \geq 1$.

We note the following evident facts:

(6.1) (a) *If $A \subset X \subset Y$ and A is a homologically trivial subset of X, then A is a homologically trivial subset of Y.*

(b) *If $A_0 \subset A \subset X$ and A is a homologically trivial subset of X, then A_0 is a homologically trivial subset of X.*

(c) *If $A \subset X$ and A or X is an acyclic space, then A is a homologically trivial subset of X.*

(6.2) Theorem. *Let X be a metric space and assume that the Lefschetz Fixed-Point Theorem for X, within the class of admissible compact maps, holds. If φ: $X \to X$ is an admissible compact map and for some $m \geq 1$ the set $\varphi^m(X)$ is a homologically trivial subset of X, then* (i) $\Lambda(\varphi) = \{1\}$, *and* (ii) *φ has a fixed point.*

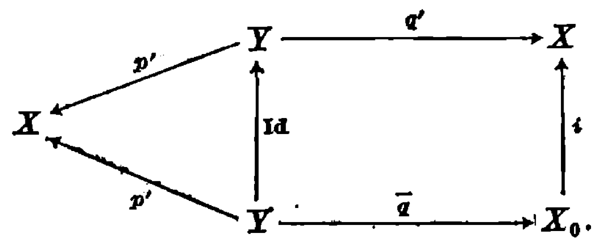
Proof. Assume that $\varphi^m(X)$, $m \geqslant 1$, is a homologically trivial subset of X. Let $\varphi^m(X) = X_0$ and let $i\colon X_0 \to X$ denote the inclusion map. First we observe that $\varphi^m\colon X \to X$, as the composition of admissible maps, is also admissible. Let $\tilde{\varphi}^m\colon X \to X_0$ be the contraction map of φ^m to the pair (X, X_0). It is easy to see that $\tilde{\varphi}^m$ is an admissible map. We have $\varphi^m = i \circ \tilde{\varphi}^m$. Let (p, q) be a selected pair of φ. Then, in view of III.2.7, there exists a selected pair $p', q'\colon Y \to X$ of φ^m such that

$$(1) \qquad q'_* p'^{-1}_* = \underbrace{q_* p_*^{-1} \cdots q_* p_*^{-1}}_{m\text{th}}.$$

Observe that the pair $(p', \bar{q}) \subset \tilde{\varphi}^m$, where $\bar{q}\colon Y \to X_0$ is the contraction map of q' to the pair (Y, X_0), is a selected pair of $\tilde{\varphi}^m$. We assert that

$$(2) \qquad q'_* p'^{-1}_* = i_* \bar{q}'_* p'^{-1}_*.$$

In this order, consider the following commutative diagram:



Applying to the above diagram the functor H, we obtain (2). For $n \geqslant 1$, $i_{*n} = 0$ and hence we have

$$i_{*n} q_{*n} (p'_{*n})^{-1} = q'_{*n} (p'_{*n})^{-1} = q_{*n} p_{*n}^{-1} \cdots q_{*n} p_{n*}^{-1} = 0.$$

Since $q_{*n} p_{*n}^{-1}$ is nilpotent for $n \geqslant 1$, it follows that

$$\operatorname{Tr}(q_{*n} p_{*n}^{-1}) = 0 \quad \text{for} \quad n \geqslant 1.$$

For $n = 1$, it follows that since the rank of i_{*n} is 1, the rank of $q_{*0} p_{*0}^{-1}$ must be 1. Hence

$$\Lambda(q_* p_*^{-1}) = \operatorname{tr}(q_{*0} p_{*0}^{-1}) = 1$$

and the proof of (i) is completed; (ii) simply follows from (i).

(6.3) THEOREM. *Let X be a topologically complete* ANR *and $\varphi\colon X \to X$ an admissible compact map. Let K be a compact subset of X which is invariant under φ. Suppose also that $C_\infty = \bigcap\limits_{m \geqslant 1} \varphi^m(X)$ is contained in K and that each compact subset of C_∞ is a homologically trivial subset of K. Then φ has a fixed point.*

Let X be a compact space, A a closed subset of X and $i\colon A \to X$ the inclusion map. Then from Theorem I.1.1 we obtain

(6.4) *A is a homologically trivial subset of X if and only if A is a cohomologically trivial subset of X, i.e.,* $\dim \operatorname{Im} i^{*0} = 1$ *and $i^{*n} = 0$ for each $n \geqslant 1$.*

(6.5) LEMMA. *Let X be a compact space of finite type and let $\{A_n\}_{n\geq 1}$ be a sequence of closed, non-empty subsets of X such that $A_{n+1} \subset A_n$ for each $n \geq 1$. Assume further that the set $A = \bigcap_{n\geq 1} A_n$ is a cohomologically trivial. subset of X. Then there exists a number $n \geq 1$ such that A_n is a cohomologically trivial subset of X.*

Proof. By the continuity of the Čech cohomology we have

$$H^m(A) = \varinjlim\{H^m(A_n)\}.$$

By assumption, each reduced cohomology class of $H^m(X)$ is annihilated by the maps induced by the respective inclusions. Hence, there exists for each such class v an integer $n(v)$ such that v is annihilated by the map $i^{*m}_{n(v)}$, where $i_{n(v)}\colon A_{n(v)}\to X$ denote the inclusion map. Since X is of finite type, we infer that $H^*(X)$ has a finite basis. Thus there must exist an integer $n = n(v)$ for all reduced cohomology classes v in such a finite basis. For this n, however, A_n is a cohomologically trivial subset of X and the proof of (6.5) is completed.

Proof of (6.3). By IV.1.1 we may assume without loss of generality that X is a retract of an open subset U in a Banach space E. Let $r\colon U\to X$ be a retraction and $i\colon X\to U$ the inclusion map. Define an admissible map (see III.2.7) $\psi_1\colon U\to U$ by putting $\psi_1 = i\circ\varphi\circ r$. Then K is an invariant subset under ψ_1 and moreover

$$\bigcap_{m\geq 1}\psi_1^m(U) = \bigcap_{m\geq 1}\varphi^m(X).$$

Let (p, q) be a selected pair of ψ_1. Define a map $\psi\colon U\to U$ by putting $\psi = q\circ\varphi_p$. Then ψ is a u.s.c., compact, admissible map (comp. III.1.8). We have

$$\bigcap_{m\geq 1}\psi^m(U) \subset \bigcap_{m\geq 1}\psi_1^m(U).$$

It is easy to see that K is an invariant subset under ψ. Since ψ is a compact map, we infer that the set $A = K\cup\left(\overline{\psi(U)}\right)$ is a compact subset of U. Applying Lemma IV.1.9 to the pair (U, A), we obtain a compact ANR C such that $A \subset C \subset U$. Then the contraction $\tilde{\psi}$ of ψ to the pair (C, C) is an admissible map. From III.1.4 we infer that $C'_\infty = \bigcap_{m\geq 1}\tilde{\psi}^m(C)$ is a compact and non-empty subset of C. Since $C'_\infty \subset K$, from the assumption and (6.1) we conclude that C'_∞ is a homologically trivial subset of C. Hence we infer from (6.4) that C'_∞ is a cohomologically trivial subset of C. Applying Lemma (6.5) to the pair (C, C'_∞), we infer that there exists an integer $m \geq 1$ such that $\tilde{\psi}^m(C)$ is a cohomologically trivial subset of C. Then, in view of (6.4), we deduce that $\tilde{\psi}^m(C)$ is a homologically trivial subset of C

for some $m \geqslant 1$. Hence Theorem (6.2) implies that $\tilde{\varphi}$ has a fixed point and therefore φ has a fixed point. The proof of Theorem (6.3) is completed.

(6.6) COROLLARY. *Let X be a compact* ANR *and let $\varphi: X \to X$ be an admissible map. If the set $C_\infty = \bigcap_{m \geqslant 1} \varphi^m(X)$ is a homologically trivial subset of X, then φ has a fixed point.*

References

§ 1–§ 2. Begle [3], Dold [20], Eilenberg, Montgomery [21], Górniewicz [37] O'Neill [72], Strother [83].

§ 3. Calvert [17], Górniewicz [29], [32], [34], [36]–[38], Granas [38], Jaworowski [55], Kakutani [57], Powers [73]–[75].

§ 4. Borsuk [4], Górniewicz [31], Kinoshita [58], Lopez [66].

§ 5. Górniewicz [37].

§ 6. Górniewicz [33].

In connection with the material in this chapter see also: Bowszyc [7], Browder [10]–[12], Brown [13]–[15], Fadell [23]–[25], Fuller [26], [27], Granas [41]–[44], [46], Hajek [48], Halpern [50], Jaworowski [54], [56], Knill [59]–[61], Lefschetz [63], Leray [64], [65], Nakaoka [69], O'Neill [71], Schauder [77], Skordev [80], Thompson [85]–[88], van der Walt [90].

VI. FIXED-POINT PROPERTY OF THE TYCHONOFF CUBE

In this chapter all spaces will be assumed to be compact Hausdorff.

1. Almost fixed points. For a space X we denote by $\mathrm{Cov}(X)$ the directed set of all finite open coverings of X. Let $\varphi\colon X \to X$ be a multi-valued map and $a \in \mathrm{Cov}(X)$. A point $x \in X$ is said to be an *a-fixed point* for φ provided there exists a member $U \in a$ such that (i) $x \in U$ and (ii) $\varphi(x) \cap U \neq \emptyset$. Clearly, if $a, \beta \in \mathrm{Cov}(X)$ and a refines β, then every a-fixed point for φ is also a β-fixed point for φ.

(1.1) LEMMA. *Let $\varphi\colon X \to X$ be a u.s.c. map. Assume that there exists a cofinal family of coverings $\mathscr{D} = \{a\} \subset \mathrm{Cov}(X)$ such that φ has an a-fixed point for every $a \in \mathscr{D}$. Then φ has a fixed point.*

Proof. Suppose that φ has no fixed points. Then for each $x \in X$ there are open neighbourhoods V_x and $U_{\varphi(x)}$ of x and $\varphi(x)$, respectively, such that $V_x \cap U_{\varphi(x)} = \emptyset$. From the u.s.c. of φ we deduce that the set $V = \varphi^{-1}(U_{\varphi(x)})$ is an open neighbourhood of x in X. Let $W_x = V_x \cap V$; then we have (i) $\varphi(W_x) \subset U_{\varphi(x)}$ and (ii) $W_x \cap U_{\varphi(x)} = \emptyset$. Since X is a compact space, we infer that there exists a finite number of sets $W_{x_1}, \ldots, W_{x_n}$ such that $X = \bigcup_{i=1}^{n} W_{x_i}$. Putting $\beta = \{W_{x_1}, \ldots, W_{x_n}\}$, we get a covering of X such that φ has no β-fixed point. If a is a member of $\mathscr{D}$ that refines β, then φ has no a-fixed point, and thus we obtain a contradiction.

2. Fixed-point property for infinite products. Let $\{X_i\}_{i \in I}$ be a family of compact spaces indexed by an infinite set I and let $X = \prod_{i \in I} X_i$ be their topological product. Denote by $\mathscr{J} = \{J\}$ the family of all finite subsets of I; given $J \in \mathscr{J}$, we put $X_J = \prod_{i \in J} X_i$.

(2.1) THEOREM. *The infinite product $X = \prod_{i \in I} X_i$ of compact spaces has the fixed-point property within the class of admissible maps if and only if every finite product $X_J = \prod_{i \in J} X_i$ $(J \in \mathscr{J})$ has the fixed-point property within the class of admissible maps.*

Proof. Choose in each X_i a point x_i^0 and define $\tilde{X}_J \subset X$ as follows:

$$\{x_i\} \in \tilde{X}_J \Leftrightarrow \begin{cases} x_i \in X_i & \text{for} \quad i \in J, \\ x_i = x_i^0 & \text{for} \quad i \notin J. \end{cases}$$

Clearly we may identify $\tilde{X}_J$ with X_J. Next we define a subset $\mathscr{D} = \{a\} \subset \mathrm{Cov}\ (X)$ as follows: $a \in \mathscr{D}$ provided a is a finite covering consisting of open sets of the form $U_J = \prod_{i \in J} U_i$ with U_i open in X_i and $U_i = X_i$ for all $i \notin J$. By the Theorem of Tychonoff and taking into account the definition of the product topology, we conclude that $\mathscr{D}$ is cofinal in $\mathrm{Cov}(X)$. Let $a \in \mathscr{D}$; it follows from the definition of the set $\mathscr{D}$ that a determines a finite set of essential indices $J(a)$. Take r_a: $X \to \tilde{X}_{J(a)}$ to be the projection and s_a: $\tilde{X}_{J(a)} \to X$ the inclusion.

Assume that every finite product $X_J = \prod_{i \in J} X_i$ has the fixed-point property within the class of admissible maps. Let φ: $X \to X$ be an admissible map. We prove that φ has a fixed point. Let p, q: $Y \to X$ be a selected pair of φ. Consider the map ψ: $X \to X$ given by $\psi = q \circ \varphi_p$. Then from III.1.6, III.1.3 and III.2.7 we deduce that ψ is a u.s.c., admissible maps For each $a \in \mathscr{D}$, consider the map ψ_a: $\tilde{X}_{J(a)} \to \tilde{X}_{J(a)}$ given by $\psi_a = r_a \psi s_a$. Then III.1.3 and III.2.7 imply that ψ_a is a u.s.c., admissible map for each $a \in \mathscr{D}$. By assumption, there exists a point $x^a \in \tilde{X}_{J(a)}$ such that

$$(1) \qquad x^a \in \psi_a(x^a) = r_a \psi s_a(x^a) = r_a \psi(x^a), \qquad \text{for each } a \in \mathscr{D}.$$

Let U be a member of a such that $x^a \in U$. Then from (1) we deduce that $\psi(x^a) \cap U \neq \emptyset$. This implies that x^a is an a-fixed point of ψ, and hence from (1.1) we infer that ψ has a fixed point. Finally, since $\psi(x) \subset \varphi(x)$ for each $x \in X$, we conclude that φ has a fixed point.

Conversely, assume that X has the fixed-point property within the class of admissible maps and that there exists a finite set $J \in \mathscr{I}$ such that X_J has no fixed-point property within the class of admissible maps. We may assume without loss of generality that there is an admissible map ψ: $\tilde{X}_J \to \tilde{X}_J$ such that $x \notin \psi(x)$, for each $x \in \tilde{X}_J$. Let r_J: $X \to \tilde{X}_J$ be the projection and s_J: $\tilde{X}_J \to X$ the inclusion. Then we have the admissible map φ: $X \to X$ given by $\varphi = s_J \psi r_J$. By assumption there exists a point $x \in X$ such that

$$x \in \varphi(x) = s_J \psi r_J(x).$$

This implies that $r_J(x) \in r_J s_J \psi(r_J(x))$ and thus we obtain a contradiction. The proof of (2.1) is completed.

From V.3.7 and (2.1) we obtain

(2.2) COROLLARY. *An arbitrary Tychonoff cube has the fixed-point property within the class of admissible maps.*

Corollary (2.2) and Proposition V.3.9 give

(2.3) COROLLARY. *Every retract of a Tychonoff cube has the fixed-point property within the class of admissible maps.*

References

§ 1–§ 2. Granas [45].

VII. ADMISSIBLE MAPS OF SUBSETS OF EUCLIDEAN SPACES

In this chapter we consider some geometrical properties of admissible and s-admissible maps. In what follows we shall denote by S^n the unit sphere in the Euclidean $(n+1)$-space R^{n+1}, by K^{n+1} the unit closed ball in R^{n+1} and by P^{n+1} the space R^{n+1} without the point 0. The main results of this chapter are the following: (i) Brouwer's type fixed-point theorem for admissible maps from K^{n+1} to R^{n+1}, (ii) theorem on antipodes for admissible maps and (iii) theorem on invariance of domain for s-admissible maps. The proofs of these results depend on the concept of degree of admissible maps.

1. Degree of admissible maps. In this section we define the degree of an admissible map $\varphi\colon S_1^n \to S_2^n$, where S_i^n $(i = 1, 2)$ are two spaces which have the homology of an n-sphere S^n.

Note that, in particular, the spaces P^{n+1} or $R^{n+1}\setminus\{x_0\}$, where x_0 is a point in R^{n+1}, have the homology as S^n. We orient S_i^n by choosing generators $\beta_i \in H_n(S_i^n)$, $i = 1, 2$.

Consider the diagram

$$S_1^n \xleftarrow{p} Y \xrightarrow{q} S_2^n,$$

in which p is a Vietoris map and q is a continuous (single-valued) map.

In this case we define the degree $\deg(p, q)$ of the pair (p, q) as the unique rational number which satisfies the condition

$$q_{*n} p_{*n}^{-1}(\beta_1) = \deg(p, q)\cdot\beta_2.$$

(1.1) DEFINITION. Let $\varphi\colon S_1^n \to S_2^n$ be an admissible map. By the *degree* $\mathrm{Deg}(\varphi)$ of φ we shall understand the following set of rational numbers:

$$\mathrm{Deg}(\varphi) = \{\deg(p, q);\ (p, q) \subset \varphi\}.$$

From III.3.1 and III.3.5 we obtain

(1.2) *Let* $\varphi, \psi\colon S_1^n \to S_2^n$ *be two admissible maps. Then*

(i) $\varphi \sim \psi$ *implies that* $\mathrm{Deg}(\varphi) \cap \mathrm{Deg}(\psi) \neq \emptyset$,

(ii) $\varphi \subset \psi$ *implies that* $\mathrm{Deg}(\varphi) \subset \mathrm{Deg}(\psi)$.

From III.2.3 we simply deduce that if φ is an acyclic map, then the set $\mathrm{Deg}(\varphi)$ consists of exactly one rational number. Applying III.2.3, III.3.1 and III.3.5, we have

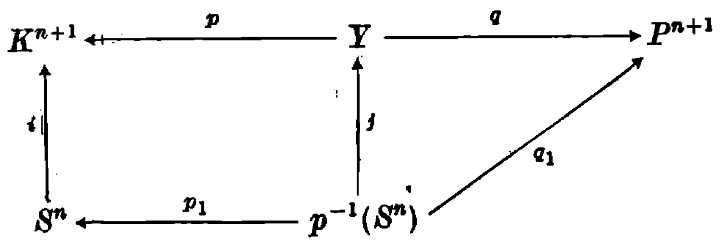
(1.3) *Let $\varphi, \psi\colon S_1^n \to S_2^n$ be two acyclic maps. If $\varphi \sim \psi$ or $\varphi \subset \psi$, then* $\mathrm{Deg}(\varphi) = \mathrm{Deg}(\psi)$.

Let $\varphi\colon K^{n+1} \to R^{n+1}$ be an admissible map and assume that $\varphi(S^n) \subset R^{n+1} \setminus \{x_0\}$ for some $x_0 \in R^{n+1}$. By $\varphi|_{S^n}\colon S^n \to R^{n+1} \setminus \{x_0\}$ we denote the contraction of φ to the pair $(S^n, R^{n+1} \setminus \{x_0\})$. In this case with every selected pair $(p, q) \subset \varphi$ we associate the selected pair $(p_1, q_1) \subset \varphi|_{S^n}$ as follows: let $p\colon Y \to K^{n+1}$, $q\colon Y \to R^{n+1}$; then $p_1\colon p^{-1}(S^n) \to S^n$ and $q_1\colon p^{-1}(S^n) \to R^{n+1} \setminus \{x_0\}$ are given as contractions of p and q, respectively. We define $\mathrm{Deg}(\varphi, x_0)$ of φ by putting

$$\mathrm{Deg}(\varphi, x_0) = \{\deg(p_1, q_1);\ (p, q) \subset \varphi\}.$$

Clearly, $\mathrm{Deg}(\varphi, x_0) \subset \mathrm{Deg}(\varphi|_{S^n})$, and the following example shows that $\mathrm{Deg}(\varphi, x_0) \neq \mathrm{Deg}(\varphi|_{S^n})$.

Example. Let $\varphi\colon K^{n+1} \to R^{n+1}$ be a map given by $\varphi(x) = S^n$ for each $x \in K^{n+1}$. We assert that $\mathrm{Deg}(\varphi, 0) = \{0\}$. Indeed, let (p, q) be a selected pair of φ, where p maps a space Y onto K^{n+1} and q maps Y into R^{n+1}. Since $(p, q) \subset \varphi$, we have $q(Y) \subset P^{n+1}$ and consequently we obtain the commutative diagram



in which i, j are inclusions. Since $H_n(K^{n+1}) = 0$, from the commutativity of the above diagram we obtain $\deg(p_1, q_1) = 0$. This implies that $\mathrm{Deg}(\varphi, 0) = \{0\}$. It is easy to see (comp. Example III.3.3) that $\mathrm{Deg}(\varphi|_{S^n}) \neq 0$. Finally, we note that if $\|x_0\| > 1$, $x_0 \in R^{n+1}$, then

$$\mathrm{Deg}(\varphi, x_0) = \mathrm{Deg}(\varphi|_{S^n}) = \{0\},$$

where $\varphi|_{S^n}$ is regarded as the map from S^n to $R^{n+1} \setminus \{x_0\}$.

(1.4) Lemma. *Let $\varphi\colon K^{n+1} \to R^{n+1}$ be an admissible map such that $\varphi(S^n) \subset R^{n+1} \setminus \{x_0\}$ for some x_0 in R^{n+1}. If* $\mathrm{Deg}(\varphi, x_0) \neq \{0\}$, *then there exists a point $x \in K^{n+1}$ such that $x_0 \in \varphi(x)$.*

Proof. For the proof we may assume without loss of generality that $x_0 = 0$. By assumption there exists a selected pair $(p, q) \subset \varphi$ such that $\deg(p_1, q_1) \neq 0$. Suppose that $0 \notin \varphi(K^{n+1})$. Thus we may regard the map q as a map to P^{n+1}, and we obtain the diagram as in the above example.

Therefore we deduce that $\deg(p_1, q_1) = 0$ and this is a contradiction. The proof of (1.4) is completed.

Next, from (1.4) we deduce a fixed-point theorem for admissible maps.

(1.5) THEOREM (Brouwer's Fixed-Point Theorem). *Let* $\varphi\colon K^{n+1}\to R^{n+1}$ *be an admissible map such that* $\varphi(S^n)\subset K^{n+1}$. *Then* φ *has a fixed point.*

Proof. For the proof we may assume without loss of generality that $x\notin\varphi(x)$ for each $x\in S^n$. Let $(p, q)\subset\varphi$ be a selected pair of φ of the form $K^{n+1}\overset{p}{\leftarrow} Y\overset{q}{\rightarrow} R^{n+1}$. Define a continuous single-valued map $f\colon Y\to R^{n+1}$ by putting $f(y) = p(y)-q(y)$ for every $y\in Y$. From the above assumption we infer that $f(y)\in P^{n+1}$, for all $y\in p^{-1}(S^n)$. Let $\psi\colon K^{n+1}\to R^{n+1}$ be given as the composition $\psi = f\circ\varphi_p$. Observe that the pair (p, f) is a selected pair of ψ. It is easy to see that $\psi(S^n)\subset P^{n+1}$. We assert that

$$\text{(i)}\qquad \operatorname{Deg}(\psi, 0)\neq\{0\}.$$

To prove (i) we define the following two maps:

$g\colon p^{-1}(S^n)\to P^{n+1}$, $g(y) = p(y)$ for each $y\in p^{-1}(S^n)$,

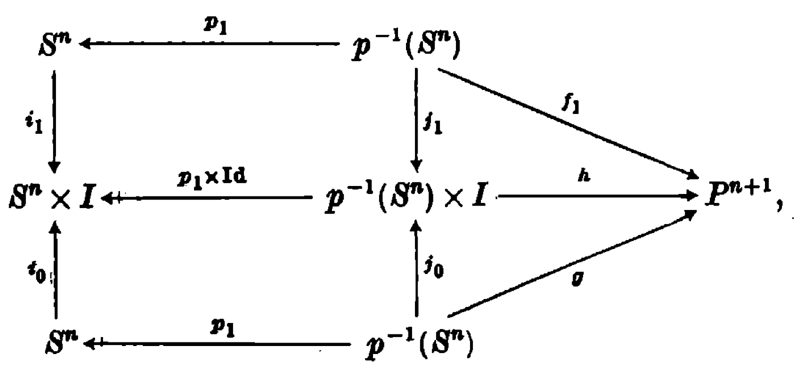
$h\colon p^{-1}(S^n)\times I\to P^{n+1}$, $h(y, t) = p(y)-t\cdot q(y)$ for each $y\in p^{-1}(S^n)$ and $t\in I$.

Observe that the map h is well defined. Indeed, for $t = 0$ and $t = 1$ it is evident. Assume that there exist a number $0 < t < 1$ and a point $y\in p^{-1}(S^n)$ such that $p(y)-t\cdot q(y) = 0$. Then we have

$$1 = \|p(y)\| = t\cdot\|q(y)\|\leqslant t < 1,$$

(since $q(y)\in K^{n+1}$ for all $y\in p^{-1}(S^n)$), and we obtain a contradiction. Let (p_1, f_1) be the associate pair to the selected pair $(p, f)\subset\psi$ (comp. the definition of degree for maps from K^{n+1} to R^{n+1}).

Consider the commutative diagram



in which $i_1(x) = (x, 1)$, $i_0(x) = (x, 0)$, for each $x\in S^n$ and $j_1(y) = (y, 1)$, $j_0(y) = (y, 0)$, for each $y\in p^{-1}(S^n)$. It is evident that $p\times\mathrm{Id}$ is a Vietoris map. Applying to the above diagram the functor H_n, we obtain: $f_{1*n}(p_{1*n})^{-1} = g_{*n}(p_{1*n})^{-1}$. Since $g_{*n}(p_{1*n})^{-1} \neq 0$, we have $\deg(p_1, f_1) \neq 0$. Finally, we observe that ψ satisfies all the assumptions of Lemma (1.4). This implies that there exists a point $x \in K^{n+1}$ such that $0 \in \psi(x)$. Thus $x \in \varphi(x)$ and the proof of (1.5) is completed.

2. Theorem on antipodes. In this section by M we denote a compact space which has the homology of the unit n-sphere S^n. An u.s.c. map Φ: $M \to M$ is called *involution* provided the condition $(x, y) \in \Gamma_\Phi$ implies that $(y, x) \in \Gamma_\Phi$ for every point (x, y) in $M \times M$.

In this section we generalize to admissible maps the following well-known version of the theorem on antipodes for single-valued maps:

(2.1) THEOREM. *Let g: $M \to M$ be a single-valued involution and let f: $M \to S^n$ be a continuous single-valued map such that $f(x) \neq f(g(x))$ for each $x \in M$. Then the induced homomorphism f_{*n}: $H_n(M) \to H_n(S^n)$ is a non-zero homomorphism.*

(2.2) THEOREM. *Let Φ: $M \to M$ be an acyclic involution and let φ: $M \to P^{n+1}$ be an admissible map such that the following condition is satisfied:*

every radius with origin at the zero point of R^{n+1} has an empty intersection with the set $\varphi(x)$ or $\varphi(\Phi(x))$ for each $x \in M$.

Then $0 \notin \mathrm{Deg}(\varphi)$.

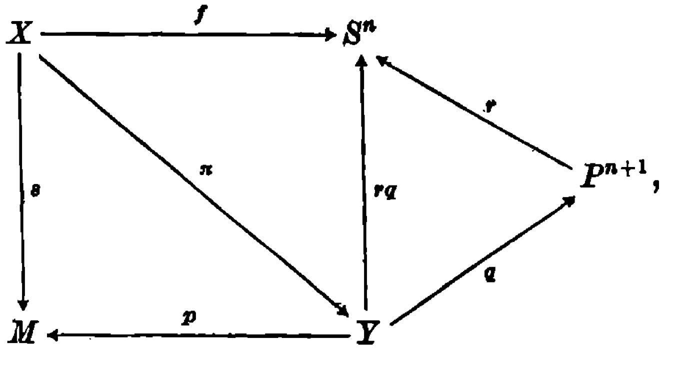
Proof. Let (p, q) be a selected pair of φ of the form

$$M \xleftarrow{p} Y \xrightarrow{q} P^{n+1}.$$

We prove that $\deg(p, q) \neq 0$. Define the set X by putting:

$$X = \{(x, x', y, y');\ x \in M,\ x' \in \Phi(x),\ y \in p^{-1}(x),\ y' \in p^{-1}(x')\}.$$

Since p, as the Vietoris map, is proper, M is compact and Φ is a u.s.c. map, we infer that X is a compact set. Consider the diagram



in which: $s(x, x', y, y') = x$; $f(x, x', y, y') = \dfrac{q(y)}{\|q(y)\|}$; $\pi(x, x', y, y') = y$ for each $(x, x', y, y') \in X$ and $r(z) = \dfrac{z}{\|z\|}$ for each $z \in P^{n+1}$.

It is easy to see that the above diagram commutes. The map s has the following decomposition:

$$(x, x', y, y') \mapsto (x, x', y) \mapsto (x, x') \mapsto x.$$

Since the maps given in the decomposition of s are determined by Vietoris maps p and p_Φ, respectively, we deduce that s is a Vietoris map. This implies that X has the homology of the n-sphere S^n. Define a single-valued involution $g\colon X \to X$ by putting

$$g(x, x', y, y') = (x', x, y', y) \quad \text{for every } (x, x', y, y') \in X.$$

We assert that

$$f(x, x', y, y') \neq f\big(g(x, x', y, y')\big) \quad \text{for all } (x, x', y, y') \in X.$$

Indeed, we have

$$f(x, x', y, y') = \frac{q(y)}{\|q(y)\|} \quad \text{and} \quad f\big(g(x, x', y, y')\big) = \frac{q(y')}{\|q(y')\|}.$$

Since $q\big(p^{-1}(x)\big) \subset \varphi(x)$ for each $x \in M$, from the assumption we deduce that $f(x, x', y, y') \neq f\big(g(x, x', y, y')\big)$. Therefore, from (2.1) we infer that f_{*n} is a non-zero homomorphism. Since p_*, s_*, r_* are isomorphisms, the commutativity of the above diagram implies that

$$q_{*n}(p_{*n})^{-1} = (r_{*n})^{-1} f_{*n} (s_{*n})^{-1} \neq 0$$

and $\deg(p, q) \neq 0$. The proof of (2.2) is completed.

We shall now draw a few consequences from Theorem (2.2).

(2.3) COROLLARY. *Let $\Phi\colon M \to M$ be an acyclic involution and let $\varphi\colon M \to S^n$ be an admissible map which satisfies the following condition:*

$$\varphi(x) \cap \varphi(y) = \emptyset, \quad \textit{for each } x \in M \textit{ and for each } y \in \Phi(x).$$

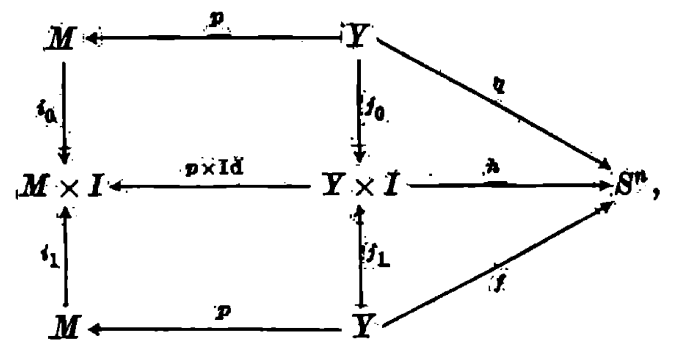
Then $0 \notin \mathrm{Deg}(\varphi)$.

(2.4) COROLLARY. *Let $\Phi\colon M \to M$ and $\varphi\colon M \to S^n$ be as in* (2.3). *Then* $\varphi(M) = S^n$.

Proof. Let (p, q) be a selected pair of φ of the form

$$M \xleftarrow{p} Y \xrightarrow{q} S^n.$$

From (2.3) we infer that $\deg(p, q) \neq 0$. Assume that there exists a point $u_0 \in S^n \setminus q(p^{-1}(M))$. We prove that this implies $\deg(p, q) = 0$. Indeed, we have the commutative diagram



in which $f(y) = -u_0$ for all $y \in Y$, $h(y, t) = \dfrac{t\cdot q(y)+(t-1)\cdot u_0}{\|t\cdot q(y)+(t-1)\cdot u_0\|}$ for each $y \in Y$ and $t \in I$ (observe that the map h is well defined). Since f is a constant map, we conclude that f_{*n} is zero homomorphism. The commutativity of the above diagram implies that

$$q_{*n}p_{*n}^{-1} = f_{*n}p_{*n}^{-1} = 0$$

and $\deg(p, q) = 0$. This contradicts the fact that $\deg(p, q) \neq 0$, and the proof of (2.4) is completed.

(2.5) Corollary. *Let Φ: $M \to M$ be an acyclic involution and φ: $M \to R^n$ be an admissible map. Then there exists a point $(x, y) \in \Gamma_\Phi$ such that $\varphi(x) \cap \varphi(y) \neq \emptyset$.*

Proof. We may regard the map φ as the map from M to $S^n \approx R^n \cup \{\infty\}$. Then (2.5) clearly follows from (2.4).

Putting $M = S^n$ and $\Phi = -\mathrm{Id}_{S^n}$ from (2.5), we obtain:

(2.6) Corollary (Borsuk–Ulam Theorem). *For every admissible map φ: $S^n \to R^n$ there exists a point $x \in S^n$ such that $\varphi(x) \cap \varphi(-x) \neq \emptyset$.*

3. Theorem on the invariance of domain. In this section we generalize the classical Brouwer Invariance of Domain Theorem to s-admissible maps.

(3.1) Lemma. *Let φ: $K^{n+1} \to R^{n+1}$ be an s-admissible map such that*

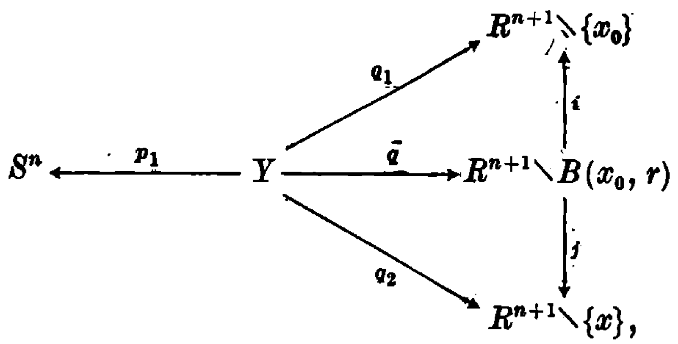
$$\varphi(S^n) \subset R^{n+1} \setminus \{x_0\} \quad \textit{for some } x_0 \in R^{n+1}.$$

If $0 \notin \mathrm{Deg}(\varphi, x_0)$, then there exists an open neighbourhood V_{x_0} of x_0 in R^{n+1} such that

$$0 \notin \mathrm{Deg}(\varphi, x) \quad \textit{for every } x \in V_{x_0}.$$

Proof. From III.2.6, III.1.8, III.1.3 and III.1.2 we deduce that $\varphi(S^n)$ is a compact subset of R^{n+1}. Since $x_0 \notin \varphi(S^n)$, we infer that there exists a positive number r such that $B(x_0, r) \cap \varphi(S^n) = \emptyset$, where $B(x_0, r)$ denotes the open ball in R^{n+1} with centre at x_0 and radius r. Consider a point $x \in B(x_0, r)$. Let (p, q) be a selected pair of φ and let (p_1, q_1), (p_1, q_2) be associated pairs to (p, q) for points x_0 and x, respectively.

Then we have the commutative diagram



in which $\bar{q}$ is a contraction of q and i, j are inclusions. Since i_*, j_* are isomorphisms, from the commutativity of the above diagram and the assumption $0 \notin \mathrm{Deg}(\varphi, x_0)$ we infer that $0 \notin \mathrm{Deg}(\varphi, x)$ and the proof is completed.

An s-admissible map $\varphi\colon X \to Z$ is called an *ε-map* if the condition $\varphi(x) \cap \varphi(x') \neq \emptyset$ implies $d(x, x') < \varepsilon$ for every $x, x' \in X$, where d denotes a metric on the space X.

(3.2) Lemma. *Let* $\varphi\colon K^{n+1} \to R^{n+1}$ *be a* 1-*map and* $z_0 \in \varphi(0)$. *Then* (i) $\varphi(S^n) \subset R^{n+1} \setminus \{z_0\}$ *and* (ii) $0 \notin \mathrm{Deg}(\varphi, z_0)$.

Proof. The assertion (i) simply follows from the fact that φ is a 1-map. Next, we prove (ii). Let $(p, q) = \varphi$ be a selected pair of φ of the form

$$K^{n+1} \xleftarrow{p} Y \xrightarrow{q} R^{n+1}.$$

Let $y_0 \in p^{-1}(0)$ be a point such that $q(y_0) = z_0$. Define the maps

$$p_1\colon p^{-1}(S^n) \to S^n, \qquad q_1\colon p^{-1}(S^n) \to R^{n+1} \setminus \{z_0\}.$$

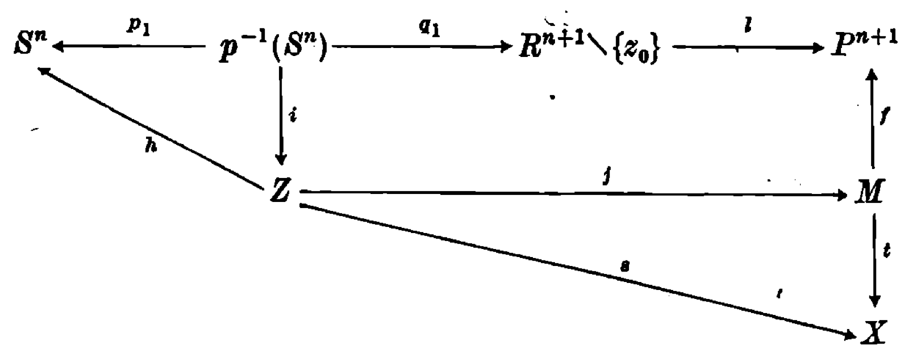
For the proof of (ii) it is sufficient to show that $\deg(p_1, q_1) \neq 0$. Define the following sets:

$$X = \{(x, x') \in K^{n+1} \times K^{n+1};\ \|x - x'\| = 1\},$$

$$M = \{(x, x', y, y');\ (x, x') \in X,\ y \in p^{-1}(x),\ y' \in p^{-1}(x')\},$$

$$Z = \{(x, x', y, y');\ (x, x', y, y') \in M,\ x' = 0\}.$$

It is easy to see that X, M, Z are compact sets. Consider the diagram



in which $i(y) = (p_1(y), 0, y, y_0)$, $h(x, 0, y, y') = x$, $l(z) = z - z_0$, $j(x, 0, y, y') = (x, 0, y, y')$, $t(x, x', y, y') = (x, x')$, $s(x, 0, y, y') = (x, 0)$, $f(x, x', y, y') = q(y) - q(y')$.

Since φ is a 1-map, we find that $f(x, x', y, y') \neq 0$ for each point in M. It is evident that the above diagram commutes. As in the proof of (2.2) (comp. pp. 57–58), we deduce that h, t are Vietoris maps and the map s is a Vietoris map onto $s(Z)$. Since $s(Z)$ is a deformation retract of X, we infer that s_* is an isomorphism. Therefore the commutativity of the above diagram implies that j_*, i_* are isomorphisms. This implies that M has the homology of S^n. Define the involution map $g\colon M \to M$ by putting $g(x, x', y, y') = (x', x, y', y)$ for every point in M. Then

$$f(g(x, x', y, y')) \neq f(x, x', y, y')$$

for each point in M and Theorem (2.2) implies that $f_{*n} \neq 0$. Consequently, from the above diagram we conclude that $l_{*n} q_{1*n} \neq 0$. Finally, we infer that $q_{1*n} \neq 0$ and this implies that $\deg(p_1, q_1) \neq 0$. The proof of (3.2) is completed.

(3.3) Remark. It is evident that Lemma (3.2) remains true when K^{n+1} is any closed ball in R^{n+1} with radius ε and φ is any ε-map.

Now we are able to state the principal results of this section.

(3.4) THEOREM. *Let ε be a positive real number. If $\varphi\colon R^{n+1} \to R^{n+1}$ s an ε-map, then $\varphi(R^{n+1})$ is an open subset of R^{n+1}.*

Proof. Let $y \in \varphi(R^{n+1})$. We prove that $y \in \operatorname{Int}\varphi(R^{n+1})$. Assume that $y \in \varphi(x)$ for some $x \in R^{n+1}$. Let K^{n+1} be a closed ball in R^{n+1} with the centre at x and radius ε, and let S^n_ε denote the boundary of K^{n+1}_ε in R^{n+1}. Let ψ be the restriction of φ to the ball K^{n+1}_ε. Then ψ is an ε-map and $\psi(S^n) \subset R^{n+1} \setminus \{y\}$. Therefore Lemma (3.2) (comp. also (3.3)) implies that $0 \notin \operatorname{Deg}(\psi, y)$. Applying Lemma (3.1) to the map ψ, we obtain an open neighbourhood V_y of y in R^{n+1} such that $0 \notin \operatorname{Deg}(\psi, y')$ for every $y' \in V_y$.

Consequently, Lemma (1.4) implies that $V_y \subset \varphi(R^{n+1})$ and the proof of (3.4) is completed.

(3.5) THEOREM. *Let U be an open subset of R^{n+1} and $\varphi\colon U \to R^{n+1}$ an s-admissible map. Assume further that for every point $x_1, x_2 \in U$ the condition $x_1 \neq x_2$ implies $\varphi(x_1) \cap \varphi(x_2) = \emptyset$. Then $\varphi(U)$ is an open subset of R^{n+1}.*

Proof. From the assumption we infer that φ is an ε-map for each $\varepsilon > 0$. Let $y \in \varphi(x)$ for some $x \in U$. We prove that $y \in \operatorname{Int}\varphi(U)$. Since U is an open subset of R^{n+1}, there exists an $\varepsilon > 0$ such that the closed ball K_ε^{n+1}, with centre x and radius ε, is contained in U. Let ψ denote the restriction of φ to the set K_ε^{n+1}. Since ψ is an ε-map, $y \notin \varphi(S_\varepsilon^n)$. Applying (3.2), (3.1) and (1.4), as in the proof of (3.4), we obtain $y \in \operatorname{Int}\varphi(U)$. The proof of Theorem (3.5) is completed.

References

§ 1. Bourgin [8], [9], Bryszewski, Górniewicz [16].

§ 2. Bryszewski, Górniewicz [16], Davies [19], Granas, Jaworowski [47].

§ 3. Bryszewski, Górniewicz [16], Granas, Jaworowski [47].

In connection with the material in this chapter see also: Górniewicz [30], Granas [39], [40], Himmelberg [52], Jaworowski [54], Kakutani [57], Nagumo [68], Rhee [76], Schirmer [78], Strother [83].

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