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Vector-valued means and their applications in some vector-valued function spaces

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1. Introduction

The study of scalar-valued means can trace its origins back to the work of Lebesgue (1904) about the existence of a universal invariant measure on $\mathbb{R}$, the real line. The theory of scalar-valued means is still an active area of mathematical research. We refer to [6, 7, 14, 20, 22] for its development. However, little has been done on vector-valued means. This paper will continue our work in [26] to develop the theory and the applications of such means.

Let us recall some facts about scalar-valued means. Let S be a set and let $\mathcal{B}(S)$ be the space of bounded functions on S with the supremum norm. Given a subspace $\mathcal{F}$ of $\mathcal{B}(S)$ containing the constant functions, a scalar-valued mean is simply a positive linear functional on $\mathcal{F}$ with norm 1. By Alaoglu's theorem [10, V.4.2], the set $\mathrm{M}(\mathcal{F})$ of all means on $\mathcal{F}$ is weak* compact. Let ε be the evaluation map form S to $\mathcal{F}^*$, the dual space of $\mathcal{F}$. The convex hull of $\varepsilon(S)$ is weak* dense in $\mathrm{M}(\mathcal{F})$. If $\mathcal{F}$ is a C^*-algebra, then the set $\mathrm{MM}(\mathcal{F})$ of all multiplicative means on $\mathcal{F}$ is also weak* compact. Let $X = \mathrm{MM}(\mathcal{F})$. One can set up an isometry between $\mathcal{F}$ and $\mathcal{C}(X)$, the space of bounded continuous functions on X. By the Riesz representation theorem [10, IV.6.3], a scalar-valued mean on $\mathcal{F}$ can be identified with a probability measure on X.

In Sections 3 and 4, we will show that similar facts and others hold for vector-valued means. To start, we generalize the Alaoglu theorem about linear functionals to linear operators in the next section. In Section 3, we set up a general theory of vector-valued means. As the first applications of this theory, we show some operator extension theorems in Corollaries 3.6 and 3.7 and a (generalized) Riesz representation theorem of operators in (Theorem 4.2) Theorem 4.3. In the last part of Section 4, we identify a vector-valued mean with a probability measure on a compact set. In Section 5 we set up a compact semigroup structure for the set of vector-valued means. We discuss the invariant vector-valued means in Section 6.

Finally, we apply the results established in Section 2 through Section 6 to investigate two spaces of vector-valued functions on a semitopological semigroup S. Section 7 is about the space of vector-valued almost periodic functions and Section 8 is about the space of vector-valued weakly almost periodic functions. Our treatment is quite different from [8] in that, unless otherwise mentioned, S need not have an identity.

2. The Alaoglu theorem for operators

Let $\mathcal{X}, \mathcal{Y}$ be two normed linear spaces and let $\mathcal{Y}^*$ be the dual space of $\mathcal{Y}$. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. With the norm topology, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space. We can also furnish $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ with the following two topologies, both of them make $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ a locally convex topological space [10, VI.1.2, VI.1.3].

(1) The strong operator topology τ_s, which is the weakest topology of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ relative to which the mapping $U \to Ux : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Y}$ is continuous for each $x \in \mathcal{X}$.

(2) The weak operator topology τ_w, which is the weakest topology of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ relative to which the mapping $U \to y^*Ux : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \to \mathbb{C}$ is continuous for each $x \in \mathcal{X}$ and $y^* \in \mathcal{Y}^*$.

For $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$, we have one more topology which also makes $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ a locally convex topological space [10, p. 476]:

(3) The weak* operator topology τ_{w^*}, which is the weakest topology of $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ relative to which the mapping $U \to [Ux](y) : \mathcal{L}(\mathcal{X}, \mathcal{Y}^*) \to \mathbb{C}$ is continuous for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

If a subset B of $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ is τ_{w^*}-compact, then it is τ_{w^*}-closed since $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ is Hausdorff in τ_{w^*}. Conversely, we have the following Alaoglu theorem for operators.

THEOREM 2.1. *Let $\mathcal{X}, \mathcal{Y}$ be two normed linear spaces and let $\mathcal{Y}^*$ be the dual space of $\mathcal{Y}$. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ be the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}^*$. Then every norm bounded, τ_{w^*}-closed subset B of $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ is τ_{w^*}-compact. In the case that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, the τ_{w^*}-compactness of B implies its norm boundedness.*

Proof. Without losing the generality, we assume that $B = \{\mu \in \mathcal{L}(\mathcal{X}, \mathcal{Y}^*) : \|\mu\| \leq 1\}$.

For each $x \in \mathcal{X}$, set $K_x = \{y^* \in \mathcal{Y}^* : \|y^*\| \leq \|x\|\}$. By the Alaoglu theorem [10, V.4.2], $K_x \subset \mathcal{Y}^*$ is compact for the weak* topology $\sigma(\mathcal{Y}^*, \mathcal{Y})$. Then the product space

$$\prod := \prod \{K_x : x \in \mathcal{X}\}$$

is compact in the product topology.

Note that $\mu(x) \in K_x$ for $\mu \in B$ and $x \in \mathcal{X}$ since $\|\mu(x)\| \leq \|x\|$. The mapping $\mu \to \varphi_\mu = \{\mu(x) : x \in \mathcal{X}\} : B \to \prod$ is 1-1 continuous from the topology τ_{w^*} to the product topology. To show that B is τ_{w^*}-compact, we need to show that the image of B is closed in $\prod$.

Let $\varphi = \{\varphi(x) : x \in \mathcal{X}\} \in \prod$ and let the image $\{\varphi_{\mu_\alpha}\}$ of $\{\mu_\alpha\}$ converge to φ in $\prod$. We show that there is a $\mu \in B$ such that φ is the image of μ and $\mu_\alpha \to \mu$ in τ_{w^*}.

For this purpose, define $\mu : \mathcal{X} \to \mathcal{Y}^*$ by $\mu(x) = \varphi(x)$ for $x \in \mathcal{X}$. Then $\mu \in B$. For, let $x_i \in \mathcal{X}$, $\beta_i \in \mathbb{C}, i = 1, 2$, and let $y \in \mathcal{Y}$. We have

$$\begin{aligned}[\mu(\beta_1 x_1 + \beta_2 x_2)](y) &= [\varphi(\beta_1 x_1 + \beta_2 x_2)](y) \\ &= \lim_\alpha [\varphi_{\mu_\alpha}(\beta_1 x_1 + \beta_2 x_2)](y) = \lim_\alpha [\mu_\alpha(\beta_1 x_1 + \beta_2 x_2)](y) \\ &= \beta_1 \lim_\alpha [\mu_\alpha(x_1)](y) + \beta_2 \lim_\alpha [\mu_\alpha(x_2)](y) \\ &= \beta_1 \lim_\alpha [\varphi_{\mu_\alpha}(x_1)](y) + \beta_2 \lim_\alpha [\varphi_{\mu_\alpha}(x_2)](y) \\ &= \beta_1 [\varphi(x_1)](y) + \beta_2 [\varphi(x_2)](y) = [\beta_1 \mu(x_1) + \beta_2 \mu(x_2)](y).\end{aligned}$$

Therefore

$$\mu(\beta_1 x_1 + \beta_2 x_2) = \beta_1 \mu(x_1) + \beta_2 \mu(x_2).$$

This means that μ is linear from $\mathcal{X}$ to $\mathcal{Y}^*$. Since $\varphi \in \prod$, $\|\mu(x)\| = \|\varphi(x)\| \le \|x\|$ for each $x \in \mathcal{X}$. Therefore $\mu \in B$ and $\mu_\alpha \to \mu$ in τ_{w^*}. We have finished the proof of the first statement. For the second statement, note that the τ_{w^*}-compactness of B implies that the set $B_x = \{\mu(x) : \mu \in B\}$ is $\sigma(\mathcal{Y}^*, \mathcal{Y})$-compact in $\mathcal{Y}^*$ for every $x \in \mathcal{X}$. By the uniform boundedness theorem, B_x is norm bounded in $\mathcal{Y}^*$. Again by the uniform boundedness theorem, B is norm bounded.

The proof is complete.

The theorem reduces to Alaoglu's theorem when $\mathcal{Y} = \mathbb{C}$.

Remark 2.2. After having proved Theorem 2.1, we found that a result similar to the first statement in Theorem 2.1 appeared in [17, 18].

3. Vector-valued means

In this section, S denotes any set. Starting with this section, $\mathcal{X}$ always denotes a Banach space and $\mathcal{B}(S, \mathcal{X})$ denotes the space of bounded functions f from S to $\mathcal{X}$ with the supremum norm. When $\mathcal{X} = \mathbb{C}$, we will simply write $\mathcal{B}(S)$ for $\mathcal{B}(S, \mathbb{C})$. If B is a subset of a locally convex topological space (X, τ) then $\overline{\text{co}}^\tau B$ and $\overline{\text{cco}}^\tau B$ denote closed convex hull and closed convex circled hull respectively of B for τ.

Definition 3.1 [16, 19]. Let $\mathcal{A}$ be a subspace of $\mathcal{B}(S, \mathcal{X})$. A linear map $\mu : \mathcal{A} \to \mathcal{X}$ is called a *mean* on $\mathcal{A}$ provided $\mu(f) \in \overline{\text{co}} f(S)$, for all $f \in \mathcal{A}$. Denote by $\mathrm{M}(\mathcal{A})$ the set of all means on $\mathcal{A}$.

We embed $\mathcal{X}$ in its double dual space $\mathcal{X}^{**}$ canonically and let $\iota(\mathcal{X})$ denote its canonical image in $\mathcal{X}^{**}$; similarly we embed $f(S)$ in $\mathcal{X}^{**}$ for every $f \in \mathcal{A}$ and get a subspace $\iota(\mathcal{A})$ of $\mathcal{B}(S, \iota(\mathcal{X}))$. A function of $\mathcal{A}$ may be regarded as a function of $\iota(\mathcal{A})$, and vice versa. Replacing $\mathcal{A}$ and $\mathcal{X}$ by $\iota(\mathcal{A})$ and $\mathcal{X}^{**}$ respectively in Definition 3.1, one gets $\mathrm{M}(\iota(\mathcal{A}))$. A mean of $\mathrm{M}(\mathcal{A})$ may be regarded as a mean of $\mathrm{M}(\iota(\mathcal{A}))$, and vice versa. This leads to the following more general definition of means.

Definition 3.2. Let $\mathcal{A}$ be a subspace of $\mathcal{B}(S,\mathcal{X}^{**})$. A linear map $\mu:\mathcal{A}\to\mathcal{X}^{**}$ is called a w^* *mean* on $\mathcal{A}$ provided $\mu(f)\in\overline{\mathrm{co}}^{w^*}f(S)$, for all $f\in\mathcal{A}$, where w^* stands for the weak* topology $\sigma(\mathcal{X}^{**},\mathcal{X}^*)$. Denote by $w^*\mathrm{M}(\mathcal{A})$ the set of all w^* means on $\mathcal{A}$. In the case that $\mathcal{A}\subset\mathcal{B}(S,\mathcal{X})$, we define $w^*\mathrm{M}(\mathcal{A})$ to be $w^*\mathrm{M}(\iota(\mathcal{A}))$.

Both Definitions 3.1 and 3.2 will reduce to the definition of a scalar-valued mean when $\mathcal{X}=\mathbb{C}$ [1, 2.1.2].

A definition similar to Definition 3.2 appeared in [17, 23].

For a subspace $\mathcal{A}$ of $\mathcal{B}(S,\mathcal{X})$ ($\mathcal{B}(S,\mathcal{X}^{**})$), by a *constant function* f of $\mathcal{A}$ we mean that there is an $x\in\mathcal{X}$ ($x^{**}\in\mathcal{X}^{**}$) such that $f(s)=x$ (x^{**}) for all $s\in S$; by $\mathcal{A}$ containing the constant functions we mean that $\mathcal{A}$ contains all the constant functions. Let $\mu\in w^*\mathrm{M}(\mathcal{A})$. If $f\in\mathcal{A}$ is a constant function, then $\mu(f)$ is the constant.

The following proposition is obvious.

Proposition 3.3. *If $\mathcal{A}$ is a linear subspace of $\mathcal{B}(S,\mathcal{X}^{**})$ containing the constant functions, then each $\mu\in w^*\mathrm{M}(\mathcal{A})$ is in $\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$ with $\|\mu\|=1$.*

The question whether or not a $\mu\in w^*\mathrm{M}(\mathcal{A})$ is in $\mathrm{M}(\mathcal{A})$, in particular when μ is invariant (we will deal with the invariance in Section 6), is one to which we shall devote considerable attention. Therefore we will carefully distinguish the following cases: $\mathcal{A}\subset\mathcal{B}(S,\mathcal{X})$, $\iota(\mathcal{A})\subset\mathcal{B}(S,\iota(\mathcal{X}))$ and $\mathcal{A}\subset\mathcal{B}(S,\mathcal{X}^{**})$, as well as $\mathrm{M}(\mathcal{A})\subset\mathcal{L}(\mathcal{A},\mathcal{X})$, $\mathrm{M}(\iota(\mathcal{A}))\subset\mathcal{L}(\iota(\mathcal{A}),\iota(\mathcal{X}))$, $w^*\mathrm{M}(\mathcal{A})=w^*\mathrm{M}(\iota(\mathcal{A}))\subset\mathcal{L}(\iota(\mathcal{A}),\mathcal{X}^{**})$, $\mathrm{M}(\mathcal{A})\subset\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$ and $w^*\mathrm{M}(\mathcal{A})\subset\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$. However, individually we will not distinguish: an $f\in\mathcal{A}\subset\mathcal{B}(S,\mathcal{X})$ and an $f\in\iota(\mathcal{A})\subset\mathcal{B}(S,\iota(\mathcal{X}))$, as well as a $\mu\in\mathrm{M}(\mathcal{A})\subset\mathcal{L}(\mathcal{A},\mathcal{X})$ and a $\mu\in\mathrm{M}(\iota(\mathcal{A}))\subset\mathcal{L}(\iota(\mathcal{A}),\iota(\mathcal{X}))$; therefore, for $x^*\in\mathcal{X}^*$, the numerical functions $x^*[f(\cdot)]$ for $f\in\mathcal{A}$ and $f(\cdot)(x^*)$ for $f\in\iota(\mathcal{A})$ are the same. Since the most general case is $\mathcal{A}\subset\mathcal{B}(S,\mathcal{X}^{**})$ and $w^*\mathrm{M}(\mathcal{A})\subset\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$, we will mainly discuss this case. For instance, we stated Proposition 3.3 only for this case; the statements for other cases were simple corollaries and we did not state them. We will state some other cases when we discuss the applications.

For $\mathcal{A}\subset\mathcal{B}(S,\mathcal{X})$, it follows from Definitions 3.1 and 3.2 that $\mathrm{M}(\iota(\mathcal{A}))\subset w^*\mathrm{M}(\iota(\mathcal{A}))$; $\mu\in w^*\mathrm{M}(\iota(\mathcal{A}))\setminus\mathrm{M}(\iota(\mathcal{A}))$ if and only if there is some $f\in\iota(\mathcal{A})$ such that $\mu(f)\in\mathcal{X}^{**}\setminus\iota(\mathcal{X})$. In Section 6, after we give the definition of an invariant mean we will use this criterion to show that, in some circumstances, if $\mu\in w^*\mathrm{M}(\iota(\mathcal{A}))$ is invariant then μ is in $\mathrm{M}(\iota(\mathcal{A}))$, i.e., $\mu\in\mathrm{M}(\mathcal{A})$.

In general, we have the following proposition.

Proposition 3.4. *Let $\mathcal{A}$ be a subspace of $\mathcal{B}(S,\mathcal{X})$. If the range $f(S)$ is relatively weakly compact in $\mathcal{X}$ for every $f\in\mathcal{A}$, then $\mathrm{M}(\iota(\mathcal{A}))=w^*\mathrm{M}(\iota(\mathcal{A}))$.*

Proof. Let $\mu\in w^*\mathrm{M}(\iota(\mathcal{A}))$. Since $f(S)$ is relatively weakly compact in $\mathcal{X}$, by the Krein–Smulian theorem [10, V.6.4], $\overline{\mathrm{co}}f(S)$ is weakly compact in $\mathcal{X}$. We embed $\overline{\mathrm{co}}f(S)$ in its double dual space $\mathcal{X}^{**}$ canonically. Let $\iota[\overline{\mathrm{co}}f(S)]$ denote its canonical image in $\mathcal{X}^{**}$. Then $\iota[\overline{\mathrm{co}}f(S)]$ is $\sigma(\mathcal{X}^{**},\mathcal{X}^*)$-compact in $\mathcal{X}^{**}$ and is

$\sigma(\mathcal{X}^{**}, \mathcal{X}^*)$-closed. By Definition 3.2, $\mu(f) \in \iota[\overline{\text{co}} f(S)]$. That is, $\mu \in \mathrm{M}(\iota(\mathcal{A}))$. The proof is finished.

Of course, the evaluation mapping $\varepsilon : S \to \mathcal{L}(\mathcal{A}, \mathcal{X})$ defined by

$$\varepsilon(s)(f) = f(s) \qquad (s \in S, f \in \mathcal{A})$$

is in $\mathrm{M}(\iota(\mathcal{A}))$.

Before stating the following theorem, let us recall some facts about convexity of a set whose proofs can be found in [10, V.2.4]. Let X be a linear topological space and let $A, B \subset X$, $\alpha \in \mathbb{C}$. Then we have the following facts:

(i) $\overline{\text{co}}(\alpha A) = \alpha \overline{\text{co}} A$;
(ii) $\overline{\text{co}}(A + B) \supset \overline{\text{co}} A + \overline{\text{co}} B$.
(iii) if $\overline{\text{co}} A$ is compact, then $\overline{\text{co}}(A + B) = \overline{\text{co}} A + \overline{\text{co}} B$.

THEOREM 3.5. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, \mathcal{X}^{**})$. Then, for τ_{w^*},*

(1) $w^*\mathrm{M}(\mathcal{A})$ *is convex and compact*;

(2) *if ε is the evaluation map $S \to \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$, then* $\text{co}(\varepsilon(S))$ *is dense in* $w^*\mathrm{M}(\mathcal{A})$;

(3) *if S is a topological space and $\mathcal{A} \subset w^*\mathcal{C}(S, \mathcal{X}^{**})$, the subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ whose members f are such that $f(\cdot)(x^*) \in \mathcal{C}(S)$ for every $x^* \in \mathcal{X}^*$, then $\varepsilon : S \to \mathrm{M}(\mathcal{A})$ is continuous.*

Proof. (1) The convexity of $w^*\mathrm{M}(\mathcal{A})$ follows directly from Definition 3.2 and the facts (i) and (ii) above. To show that $w^*\mathrm{M}(\mathcal{A})$ is compact, by Theorem 2.1 we need to show that it is τ_{w^*}-closed. Let $\{\mu_\alpha\} \subset w^*\mathrm{M}(\mathcal{A})$ converge to $\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ for τ_{w^*}. Then $\mu_\alpha(f) \to \mu(f)$ for each $f \in \mathcal{A}$ in $\sigma(\mathcal{X}^{**}, \mathcal{X}^*)$, and since $\mu_\alpha(f) \in \overline{\text{co}}^{w^*} f(S)$ for all α, $\mu(f) \in \overline{\text{co}}^{w^*} f(S)$. Therefore, $\mu \in w^*\mathrm{M}(\mathcal{A})$.

(2) Clearly, $\text{co}(\varepsilon(S)) \subset \mathrm{M}(\mathcal{A}) \subset w^*\mathrm{M}(\mathcal{A})$. Therefore $\overline{\text{co}}^{\tau_{w^*}}(\varepsilon(S)) \subset w^*\mathrm{M}(\mathcal{A})$. To show that $\overline{\text{co}}^{\tau_{w^*}}(\varepsilon(S)) \supset w^*\mathrm{M}(\mathcal{A})$, let $\mu \in w^*\mathrm{M}(\mathcal{A})$ and set $E = (\mathcal{L}(\mathcal{A}, \mathcal{X}^{**}), \tau_{w^*})$. By Definition 3.2, $\mu(f) \in \overline{\text{co}}^{w^*} f(S)$ for every $f \in \mathcal{A}$, i.e., $\mu(f)(x^*) \in \overline{\text{co}}\{f(S)(x^*)\}$ for every $f \in \mathcal{A}$ and $x^* \in \mathcal{X}^*$. Thus, for $m \in \mathbb{N}$, $f_i \in \mathcal{A}$, and $x_i^* \in \mathcal{X}^*$, $i = 1, \ldots, m$, by the fact (iii),

$$\sum_{i=1}^{m} \mu(f_i)(x_i^*) \in \sum_{i=1}^{m} \overline{\text{co}}\{f_i(S)(x_i^*)\} = \overline{\text{co}}\Big\{ \sum_{i=1}^{m} f_i(S)(x_i^*) \Big\}. \tag{3.1}$$

We claim that $\mu \in \overline{\text{co}}^{\tau_{w^*}}(\varepsilon(S))$. Otherwise, by the separation theorem [10, V.2.10], there is a $\varphi \in E^*$, the dual space of E, such that

$$\operatorname{Re} \varphi(\mu) > \sup\{\operatorname{Re} \overline{\text{co}}[\varphi(\varepsilon(S))]\}. \tag{3.2}$$

For $f \in \mathcal{A}$ and $x^* \in \mathcal{X}^*$, define $f \circ x^* : E \to \mathbb{C}$ by $f \circ x^*(\nu) = \nu(f)(x^*)$. Then $f \circ x^* \in E^*$. By [10, V.3.9], $E^* = \text{sp}\{f \circ x^* : f \in \mathcal{A},\ x^* \in \mathcal{X}^*\}$. So there are $f_i \in \mathcal{A}$, and $x_i^* \in \mathcal{X}^*$, $i = 1, \ldots, n$, such that $\varphi = \sum_{i=1}^{n} f_i \circ x_i^*$. Here we omit the coefficients before $f_i \circ x_i^*$'s because $\mathcal{A}$ is a linear space. It follows from (3.2) that

$$\operatorname{Re}\Big(\sum_{i=1}^{n}\mu(f_i)(x_i^*)\Big) > \sup\Big\{\operatorname{Re}\overline{\mathrm{co}}\Big(\sum_{i=1}^{n} f_i(S)(x_i^*)\Big)\Big\},$$

which contradicts (3.1).

(3) is obvious.

Set

$$\mathcal{IL}(\mathcal{A},\mathcal{X}^{**}) = \bigcup_{n=1}^{\infty}\{n\overline{\mathrm{cco}}^{\tau_{w^*}}[w^*\mathrm{M}(\mathcal{A})]\}. \tag{3.3}$$

It is obvious that $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ is a linear subspace of $\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$.

The following two corollaries are extension theorems for operators on function spaces. The first comes from Theorem 3.5(1),(2).

Corollary 3.6. *Every w^* mean on $\mathcal{A}$ extends to a w^* mean on $\mathcal{B}(S,\mathcal{X}^{**})$.*

Corollary 3.7. *Every member of $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ extends to a member of $\mathcal{IL}(\mathcal{B}(S,\mathcal{X}^{**}),\mathcal{X}^{**})$.*

Proof. Set $B = \mathcal{B}(S,\mathcal{X}^{**})$. By Theorem 2.1, for all $n \in \mathbb{N}$, $n\overline{\mathrm{cco}}^{\tau_{w^*}}[w^*\mathrm{M}(\mathcal{A})]$ is τ_{w^*}-compact in $\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$ and $n\overline{\mathrm{cco}}^{\tau_{w^*}}[w^*\mathrm{M}(B)]$ is τ_{w^*}-compact in $\mathcal{L}(B,\mathcal{X}^{**})$. Let $\mu \in n\overline{\mathrm{cco}}^{\tau_{w^*}}[w^*\mathrm{M}(\mathcal{A})]$. There is a net $\{\mu_\alpha\} \subset n\mathrm{cco}[w^*\mathrm{M}(\mathcal{A})]$ such that $\mu_\alpha \to \mu$ in τ_{w^*}. It follows from Corollary 3.6 that each μ_α extends to a member μ'_α of $n\mathrm{cco}[w^*\mathrm{M}(B)]$. There are a subset $\{\mu'_\beta\}$ of $\{\mu'_\alpha\}$ and a $\mu' \in n\overline{\mathrm{cco}}^{\tau_{w^*}}[w^*\mathrm{M}(B)]$ such that $\mu'_\beta \to \mu'$ in τ_{w^*}. Thus, μ' is an extension in $\mathcal{L}(B,\mathcal{X}^{**})$ of μ. The proof is complete.

Let $\mathcal{A} \subset \mathcal{B}(S,\mathcal{X}^{**})$ and set

$$\mathcal{F} = \mathrm{sp}\{f(\cdot)(x^*) : f \in \mathcal{A}, x^* \in \mathcal{X}^*\}. \tag{3.4}$$

$\mathrm{M}(\mathcal{F})$ denotes the set of all scalar-valued means on $\mathcal{F}$. Now we have the following isomorphism theorem between $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ and $\mathcal{F}^*$.

Theorem 3.8. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S,\mathcal{X}^{**})$ containing the constant functions. Then there is an isometric* (*isomorphic*) *τ_{w^*}-$\sigma(\mathcal{F}^*,\mathcal{F})$ homeomorphism $\mu \to \varphi_\mu : w^*\mathrm{M}(\mathcal{A}) \to \mathrm{M}(\mathcal{F})$ $(\mathcal{IL}(\mathcal{A},\mathcal{X}^{**}) \to \mathcal{F}^*)$ such that*

$$\mu(f)(x^*) = \varphi_\mu[f(\cdot)(x^*)] \quad (f \in \mathcal{A}, x^* \in \mathcal{X}^*). \tag{3.5}$$

Proof. First, we show that $w^*\mathrm{M}(\mathcal{A})$ is isometrically homeomorphic to $\mathrm{M}(\mathcal{F})$. Denote by ε and ε' the evaluation maps from S to $\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$ and S to $\mathcal{F}^*$ respectively. By Theorem 3.5, $w^*\mathrm{M}(\mathcal{A})$ is τ_{w^*}-compact and $\mathrm{co}(\varepsilon(S))$ is τ_{w^*}-dense in $w^*\mathrm{M}(\mathcal{A})$. Analogously, $\mathrm{M}(\mathcal{F})$ is $\sigma(\mathcal{F}^*,\mathcal{F})$-compact and $\mathrm{co}(\varepsilon'(S))$ is $\sigma(\mathcal{F}^*,\mathcal{F})$-dense in $\mathrm{M}(\mathcal{F})$. Note the isometry $\varepsilon(s) \in \mathrm{M}(\mathcal{A}) \to \varepsilon'(s) \in \mathrm{M}(\mathcal{F})$ for $s \in S$. (3.5) holds for all $s \in S$, $f \in \mathcal{A}$ and $x^* \in \mathcal{X}^*$. The τ_{w^*}-compactness of $w^*\mathrm{M}(\mathcal{A})$ and $\sigma(\mathcal{F}^*,\mathcal{F})$-compactness of $\mathrm{M}(\mathcal{F})$ imply that for a $\mu \in w^*\mathrm{M}(\mathcal{A})$ there exists a unique

$\varphi_\mu \in \mathrm{M}(\mathcal{F})$ such that (3.5) holds, and vice versa. $\mu \to \varphi_\mu$ is a τ_{w^*}-$\sigma(\mathcal{F}^*, \mathcal{F})$ homeomorphism between $w^*\mathrm{M}(\mathcal{A})$ and $\mathrm{M}(\mathcal{F})$. Since $\mu \in w^*\mathrm{M}(\mathcal{A})$ and $\varphi_\mu \in \mathrm{M}(\mathcal{F})$, it follows from Proposition 3.3 that $\|\mu\| = \|\varphi\| = 1$.

Similarly, for every $n \in \mathbb{N}$, $n\overline{\mathrm{cco}}^{\tau_{w^*}}[w^*\mathrm{M}(\mathcal{A})]$ is homeomorphic to $n\overline{\mathrm{cco}}^{w^*}\mathrm{M}(\mathcal{F})$.

To show that the map is an isomorphism, we need to show that it is onto $\mathcal{F}^*$. For this purpose, we show that $\overline{\mathrm{cco}}^{w^*}\varepsilon'(S)$ is the norm closed unit ball of $\mathcal{F}^*$, where w^* stands for the weak* topology $\sigma(\mathcal{F}^*, \mathcal{F})$. Suppose the statement is false, i.e., there is a $\varphi \in \mathcal{F}^*$ with $\|\varphi\| = 1$ but $\varphi \notin \overline{\mathrm{cco}}^{w^*}\varepsilon'(S)$. By the separation theorem, there is a function $F \in \mathcal{F}$ such that

$$\mathrm{Re}\,\varphi(F) > \sup\{\mathrm{Re}\,\overline{\mathrm{cco}}(F(S))\} = \|F\|.$$

But $\mathrm{Re}\,\varphi(F) \le \|\varphi(F)\| \le \|F\|$. We get a contradiction.

Since $\overline{\mathrm{cco}}^{w^*}\varepsilon'(S) \subset \overline{\mathrm{cco}}^{w^*}(\mathrm{M}(\mathcal{F}))$ and $\overline{\mathrm{cco}}^{w^*}(\mathrm{M}(\mathcal{F}))$ is contained in the norm closed unit ball of $\mathcal{F}^*$, $\overline{\mathrm{cco}}^{w^*}(\mathrm{M}(\mathcal{F}))$ is the norm closed unit ball of $\mathcal{F}^*$. Similarly one can show that $n\overline{\mathrm{cco}}^{w^*}(\mathrm{M}(\mathcal{F}))$ is the norm closed ball of $\mathcal{F}^*$ with radius n. Therefore the map is onto $\mathcal{F}^*$. The proof is complete.

COROLLARY 3.9. *Let* $\mu \in \mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$. *Then there exist* $\mu_j \in w^*\mathrm{M}(\mathcal{A})$ *and* $\alpha_j \ge 0$, $1 \le j \le 4$, *such that*

$$\mu = \alpha_1\mu_1 - \alpha_2\mu_2 + i(\alpha_3\mu_3 - \alpha_4\mu_4).$$

Proof. Since each $\mu \in \mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$ extends to a member of $\mathcal{IL}(\mathcal{B}(S, \mathcal{X}^{**}), \mathcal{X}^{**})$ (Corollary 3.7), we show the corollary in the case that $\mathcal{A} = \mathcal{B}(S, \mathcal{X}^{**})$. By Theorem 3.8, there is a unique $\varphi_\mu \in \mathcal{F}^*$ for μ such that (3.5) holds. By [1, 2.1.4], there exist $\varphi_j \in \mathrm{M}(\mathcal{F})$ and $\alpha_j \ge 0$, $1 \le j \le 4$, such that

$$\varphi = \alpha_1\varphi_1 - \alpha_2\varphi_2 + i(\alpha_3\varphi_3 - \alpha_4\varphi_4).$$

Again by Theorem 3.8, there is a $\mu_i \in w^*\mathrm{M}(\mathcal{A})$ corresponding to φ_i such that (3.5) holds. Let $\mu' = \alpha_1\mu_1 - \alpha_2\mu_2 + i(\alpha_3\mu_3 - \alpha_4\mu_4)$. It is clear that (3.5) holds for μ' and φ_μ too. By uniqueness, $\mu = \mu'$. The proof is complete.

Remark 3.10. Three questions arise from [17, p. 722]. (i) Is $\mathrm{co}(\varepsilon(S))$ τ_{w^*}-dense in $w^*\mathrm{M}(\mathcal{A})$? (ii) Does every w^* mean on $\mathcal{A}$ extend to a w^* mean on $\mathcal{B}(S, \mathcal{X}^{**})$? Let ε' be as in the proof for Theorem 3.8. If $\mathcal{F}$ is a C^*-subalgebra of $\mathcal{B}(S)$ containing the constant functions, then it follows from [1, 2.1.16] (or [21]) that the set of extreme points of $\mathrm{M}(\mathcal{F})$ is precisely $\mathrm{MM}(\mathcal{F}) = \overline{\varepsilon'(S)}^{w^*}$. Let $w^*\mathrm{MM}(\mathcal{A}) = \overline{\varepsilon(S)}^{\tau_{w^*}}$. (iii) Is the set of extreme points of $w^*\mathrm{M}(\mathcal{A})$ precisely $w^*\mathrm{MM}(\mathcal{A})$ if $\mathcal{F}$ is a C^*-subalgebra of $\mathcal{B}(S)$ containing the constant functions? Theorem 3.5(2) and Corollary 3.6 have affirmatively answered the first two questions respectively. The third one can be answered affirmatively by simply applying Theorem 3.8.

To end this section, we make some observations. Let $\mu \in \mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$. Then $\mu \in n\overline{\mathrm{cco}}^{\tau_{w^*}}[w^*\mathrm{M}(\mathcal{A})]$ for some $n \in \mathbb{N}$. There is a net $\{\mu_\alpha\}$ of $n\mathrm{cco}[w^*\mathrm{M}(\mathcal{A})]$ with $\mu_\alpha = \sum_{i=1}^{n(\alpha)} c_{\alpha,i}\mu_{\alpha,i}$, where $\mu_{\alpha,i} \in w^*\mathrm{M}(\mathcal{A})$ and $\sum_{i=1}^{n(\alpha)} |c_{\alpha,i}| \le n$, such that

$\mu_\alpha \to \mu$ in τ_{w^*}. Since $\mu_\alpha(f) = \sum_{i=1}^{n(\alpha)} c_{\alpha,i}\mu_{\alpha,i}(f) \in \overline{\mathrm{co}}^{w^*}(\sum_{i=1}^{n(\alpha)} c_{\alpha,i}f(S))$ for every $f \in \mathcal{A}$, $\mu(f) \in \overline{\mathrm{sp}}^{w^*}\{f(s) : s \in S\} \subset \mathcal{X}^{**}$, where w^* stands for the weak* topology $\sigma(\mathcal{X}^{**}, \mathcal{X}^*)$. Here, we in fact have pointed out a necessary condition for a $\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ to be in $\mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$, that is, $\mu(f) \in \overline{\mathrm{sp}}^{w^*}\{f(s) : s \in S\}$ for all $f \in \mathcal{A}$. This condition will be further understandable when we discuss integration in the next section and express $\mu(f)$ an integral of f with respect to some measure μ. However, we do not know if the condition is also sufficient. The condition shows that $\mathcal{IL}(\mathcal{A}, \mathcal{X}^{**}) \neq \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ in general. For example, let $S = \{s\}$ be a single point set. Let H be a Hilbert space. Define $f_h(s) = h$, for $h \in H$. Let $\mathcal{A} = \mathcal{B}(S, H) = \{f_h : h \in H\}$. The linear maps from $\mathcal{A}$ to H may be regarded as maps from H to H. Then $\mathcal{IL}(\mathcal{A}, H)$ consists of identity and its multiples. In particular, if H has an infinite, countable, orthonormal basis $\{e_i\}$, then the shift operator which maps e_i to e_{i+1} is not in $\mathcal{IL}(\mathcal{A}, H)$.

4. The Riesz representation theorems for operators

We follow [12] to define integration of vector-valued functions.

Let (S, Σ, μ) be a measure space and let $\mathcal{X}$ be a Banach space. A bounded function $f : S \to \mathcal{X}^{**}$ is said to be *scalarwise integrable* if $f(\cdot)(x^*)$ is integrable for every $x^* \in \mathcal{X}^*$. In this case, it follows from [12, 8.14.11(a)] that there is a unique $x^{**} \in \mathcal{X}^{**}$ such that

$$x^{**}(x^*) = \int_S f(s)(x^*)\,d\mu \qquad (x^* \in \mathcal{X}^*). \tag{4.1}$$

DEFINITION 4.1. Let (S, Σ, μ) be a measure space and let $f \in \mathcal{B}(S, \mathcal{X}^{**})$ be scalarwise integrable. Then the *integral* $\int_S f\,d\mu$ is defined to be the unique $x^{**} \in \mathcal{X}^{**}$ satisfying (4.1). If $f \in \mathcal{B}(S, \iota(\mathcal{X}))$ and there is an $x \in \mathcal{X}$ such that f and $\iota(x) \in \mathcal{X}^{**}$ satisfy (4.1), then we will say that $f \in \mathcal{B}(S, \mathcal{X})$ is *integrable* and define $\int_S f d\mu = x$.

Let X be a compact, Hausdorff, topological space and let $\mu \in \mathrm{rca}(X)$, the linear space of regular countable additive measures on X. By Definition 4.1, $\int_X f\,d\mu$ exists for every f in $w^*\mathcal{C}(X, \mathcal{X}^{**})$ or $w\mathcal{C}(S, \mathcal{X})$, the space of weakly continuous functions from X to $\mathcal{X}$.

The following theorem is a generalized Riesz representation theorem for operators.

THEOREM 4.2. *Let X be a compact, Hausdorff, topological space. Set $\mathcal{A} = w^*\mathcal{C}(X, \mathcal{X}^{**})$. Then there is an isometric isomorphism between $\mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$ and* $\mathrm{rca}(X)$ *such that*

$$\mu(f) = \int_X f\,d\mu \qquad (f \in \mathcal{A}), \tag{4.2}$$

where μ denotes both a member of $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ and the corresponding measure on X. Furthermore, $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ is a τ_{w^*}-closed subspace of $\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$.

Proof. Since $\mathcal{A}=w^*\mathcal{C}(X,\mathcal{X}^{**})$, $\mathcal{F}=\mathcal{C}(X)$. By Theorem 3.8, there is an isomorphism $\mu\to\varphi_\mu:\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})\to\mathcal{F}^*$ such that (3.5) holds. It is obvious that $\|\mu\|\le\|\varphi_\mu\|$. To show that it is isometric, we need to show that $\|\mu\|\ge\|\varphi_\mu\|$.

Let $F\in\mathcal{C}(X)$ be such that $\|F\|=1$. Let $x_0^*\in\mathcal{X}^*$ and $x_0^{**}\in\mathcal{X}^{**}$ be such that $\|x_0^*\|=\|x_0^{**}\|=1$ and $x_0^{**}(x_0^*)=1$. Set $f=F(\cdot)x_0^{**}$. Then $f\in w^*\mathcal{C}(X,\mathcal{X}^{**})$ and $\|f\|=1$. By (3.5), $\varphi_\mu(F)=\varphi_\mu[f(\cdot)(x_0^*)]=\mu(f)(x_0^*)$. Therefore $|\varphi_\mu(F)|\le\|\mu\|$. Since F is arbitrary in $\mathcal{C}(X)$, $\|\varphi_\mu\|\le\|\mu\|$.

By the Riesz representation theorem [10, IV.6.3], there is an isometric isomorphism $\varphi_\mu\to\mu:\mathcal{F}^*\to\mathrm{rca}(X)$ such that

$$\varphi_\mu(F)=\int_X F(u)\,\mu(du)\qquad(F\in\mathcal{F}).\tag{4.3}$$

It follows from (3.5) and (4.3) that

$$\mu(f)(x^*)=\int_X f(u)(x^*)\,\mu(du)\qquad(f\in\mathcal{A},x^*\in\mathcal{X}^*).\tag{4.4}$$

It follows from Definition 4.1 and (4.4) that (4.2) holds.

To show that $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ is τ_{w^*}-closed in $\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$, let $\{\mu_\alpha\}\subset\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ and $\mu\in\mathcal{L}(\mathcal{A},\mathcal{X}^{**})$ such that $\mu_\alpha\to\mu$ in τ_{w^*}. We want to show that $\mu\in\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$. By Theorem 3.8, there is $\varphi_{\mu_\alpha}\in\mathcal{F}^*$ for μ_α such that (3.5) holds. We claim that $\{\varphi_{\mu_\alpha}(F)\}$ converges for every $F\in\mathcal{F}$. In fact, let x_0^* and x_0^{**} be as above. Then $f=F(\cdot)x_0^{**}\in\mathcal{A}$ and $F=f(\cdot)(x_0^*)$. We have

$$\lim_\alpha\varphi_{\mu_\alpha}(F)=\lim_\alpha\varphi_{\mu_\alpha}[f(\cdot)(x_0^*)]=\lim_\alpha\mu_\alpha(f)(x_0^*)=\mu(f)(x_0^*).$$

By the uniform boundedness theorem, $\{\|\varphi_{\mu_\alpha}\|\}$ is bounded. Define $\varphi_\mu:\mathcal{F}\to\mathbb{C}$ by

$$\varphi_\mu(F)=\lim_\alpha\varphi_{\mu_\alpha}(F)\qquad(F\in\mathcal{F}).$$

It follows that $\varphi_\mu\in\mathcal{F}^*$ and

$$\mu(f)(x^*)=\varphi_\mu[f(\cdot)(x^*)]\qquad(f\in\mathcal{A},x^*\in\mathcal{X}^*).$$

Therefore $\mu\in\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$. The proof is complete.

Let $\mathcal{A}\subset\mathcal{B}(S,\mathcal{X})$. In (3.3), replace $\mathcal{X}^{**}$, $w^*\mathrm{M}(\mathcal{A})$, and τ_{w^*} by $\mathcal{X}$, $\mathrm{M}(\mathcal{A})$, and τ_w respectively. We have

$$\mathcal{IL}(\mathcal{A},\mathcal{X})=\bigcup_{n=1}^\infty\{n\overline{\mathrm{co}}^{\tau_w}\mathrm{M}(\mathcal{A})\}.\tag{4.5}$$

The following theorem is a Riesz representation theorem for operators.

THEOREM 4.3. *Let X be as in Theorem 4.2 and let $w\mathcal{C}(X,\mathcal{X})$ be the space of functions weakly continuous from X to $\mathcal{X}$. Set $\mathcal{A}=w\mathcal{C}(X,\mathcal{X})$. Then there is an* *isometric isomorphism between* $\mathcal{IL}(\mathcal{A},\mathcal{X})$ *and* $\mathrm{rca}(X)$ *such that*

$$\mu(f)=\int\limits_X f\,d\mu \qquad (f\in\mathcal{A}), \tag{4.6}$$

where μ *denotes both a member of* $\mathcal{IL}(\mathcal{A},\mathcal{X})$ *and the corresponding measure on* X*. Furthermore,* $\mathcal{IL}(\mathcal{A},\mathcal{X})$ *is a* τ_w*-closed subspace of* $\mathcal{L}(\mathcal{A},\mathcal{X})$*.*

Proof. By Proposition 3.4, we have $\mathrm{M}(\iota(\mathcal{A}))=w^*\mathrm{M}(\iota(\mathcal{A}))$. Note that $\mathcal{F}=\mathrm{sp}\{f(\cdot)(x^*):f\in\iota(\mathcal{A}),x^*\in\mathcal{X}^*\}=\mathcal{C}(X)$. The rest of the proof is similar to the proof of the previous theorem.

We call Theorem 4.2 the *generalized Riesz representation theorem* for operators and call Theorem 4.3 the *Riesz representation theorem for operators* because the integral of (4.2) is in $\mathcal{X}^{**}$ and that of (4.6) is in $\mathcal{X}$. The two theorems reduce to the Riesz representation theorem when $\mathcal{X}=\mathbb{C}$.

Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S,\mathcal{X}^{**})$ containing the constant functions and let ε be the evaluation map. Set $w^*\mathrm{MM}(\mathcal{A})=\overline{\varepsilon(S)}^{\tau_{w^*}}$. Let $X=w^*\mathrm{M}(\mathcal{A})$ (respectively $X=w^*\mathrm{MM}(\mathcal{A})$). By Theorem 3.5, X is τ_{w^*}-compact. For each $f\in\mathcal{A}$ the function $\widehat{f}$ defined by

$$\widehat{f}(\mu)=\mu(f)\qquad(\mu\in X)$$

is in $w^*\mathcal{C}(X,\mathcal{X}^{**})$. Further, we define

$$\widehat{\mathcal{A}}=\{\widehat{f}:f\in\mathcal{A}\}.$$

$\widehat{\mathcal{A}}$ is a linear subspace of $w^*\mathcal{C}(X,\mathcal{X}^{**})$. The mapping $f\to\widehat{f}:\mathcal{A}\to w^*\mathcal{C}(X,\mathcal{X}^{**})$ is an isometry since

$$\begin{aligned}\|\widehat{f}\|&=\sup\{\|\mu(f)\|:\mu\in X\}\le\|f\|=\sup\{\|f(s)\|:s\in S\}\\&=\sup\{\|\widehat{f}(\varepsilon(s))\|:s\in S\}\le\|\widehat{f}\|.\end{aligned}$$

Note that the evaluation map ε has the dual map $\varepsilon^*:w^*\mathcal{C}(X,\mathcal{X}^{**})\to\mathcal{B}(S,\mathcal{X}^{**})$ such that

$$\varepsilon^*(\widehat{f})=f\qquad(f\in\mathcal{A}).$$

Summarizing the discussion above, we have the following proposition.

PROPOSITION 4.4. *Let* $\mathcal{A}$ *be linear subspace of* $\mathcal{B}(S,\mathcal{X}^{**})$ *containing the constant functions and let* X *be the space* $w^*\mathrm{M}(\mathcal{A})$ (*respectively*, $w^*\mathrm{MM}(\mathcal{A})$) *with topology* τ_{w^*}*. Then the mapping* $f\to\widehat{f}$ *is an isometric isomorphism of* $\mathcal{A}$ *with* $\widehat{\mathcal{A}}$*. The inverse of this mapping is* $\varepsilon^*:\widehat{\mathcal{A}}\to\mathcal{A}$*, where* $\varepsilon:S\to X$ *is the evaluation mapping.*

The assumptions in the following theorem are weaker than those used in Theorems 4.2 and 4.3 to get the desired isomorphism.

THEOREM 4.5. *Let* $\mathcal{A}$ *and* $\widehat{\mathcal{A}}$ *be as in Proposition* 4.4 *and let* $X=w^*\mathrm{MM}(\mathcal{A})$*. Suppose that* $\mathcal{F}=\mathrm{sp}\{f(\cdot)(x^*):f\in\mathcal{A},x^*\in\mathcal{X}^*\}$ *is a* C^**-subalgebra of* $\mathcal{B}(S)$*.*

Then there is an isomorphism between $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ *and* $\mathrm{rca}(X)$ *such that*

$$\mu(f)=\int\limits_X \widehat{f}\,d\mu \qquad (f\in\mathcal{A}), \tag{4.7}$$

where μ *denotes both a member of* $\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})$ *and the corresponding measure on* X.

Proof. For simplicity, we use $\cong$ to denote the isometric isomorphism of two Banach spaces. Let ε and ε' be as in the proof of Theorem 3.8. Set $X'=\overline{\varepsilon'(S)}^{w^*}$. By the Gelfand–Naimark representation theorem [1, 2.1.9], $\mathcal{F}\cong\mathcal{C}(X')$ with inverse map ε'^*. As in the proof for the isometric homeomorphism between $w^*\mathrm{M}(\mathcal{A})$ and $\mathrm{M}(\mathcal{F})$ in Theorem 3.8, one shows that X is isometrically homeomorphic to X'. Therefore, $\mathcal{C}(X)\cong\mathcal{C}(X')$.

Let $\widehat{\mathcal{F}}=\mathrm{sp}\{\widehat{f}(\cdot)(x^*):\widehat{f}\in\widehat{\mathcal{A}},x^*\in\mathcal{X}^*\}$. We claim that $\widehat{\mathcal{F}}=\mathcal{C}(X)$. It is obvious that $\widehat{\mathcal{F}}\subset\mathcal{C}(X)$. To show the inverse inclusion, let $\widehat{F}\in\mathcal{C}(X)$. Since $\mathcal{C}(X)\cong\mathcal{C}(X')$, there is an $\widehat{F}'\in\mathcal{C}(X')$ such that

$$\widehat{F}(\varepsilon(s))=\widehat{F}'(\varepsilon'(s)) \qquad (s\in S).$$

Then $\varepsilon'^*(\widehat{F}')=F\in\mathcal{F}$ such that

$$\varepsilon'^*(\widehat{F}')(s)=\widehat{F}'(\varepsilon'(s))=F(s) \qquad (s\in S).$$

Therefore

$$\widehat{F}(\varepsilon(s))=F(s) \qquad (s\in S).$$

Since $F\in\mathcal{F}$, there are $f_i\in\mathcal{A}$ and $x_i^*\in\mathcal{X}^*$, $i=1,\ldots,n$, such that $F=\sum_{i=1}^n f_i(\cdot)(x^*)$. Note that $\widehat{f}_i\in\widehat{\mathcal{A}}$, $i=1,\ldots,n$, and $F(s)=\sum_{i=1}^n f_i(s)(x^*)=\sum_{i=1}^n \widehat{f}_i(\varepsilon(s))(x^*)$. Hence

$$\widehat{F}(\varepsilon(s))=\sum_{i=1}^n \widehat{f}_i(\varepsilon(s))(x^*) \qquad (s\in S).$$

Thus, $\widehat{F}$ and $\sum_{i=1}^n \widehat{f}_i(\cdot)(x^*)$ are identical on $\varepsilon(S)$. Since $\widehat{F}$, $\sum_{i=1}^n \widehat{f}_i(\cdot)(x^*)\in\mathcal{C}(X)$ and $\varepsilon(S)$ is dense in X, $\widehat{F}=\sum_{i=1}^n \widehat{f}_i(\cdot)(x^*)$, i.e., $\widehat{F}\in\widehat{\mathcal{F}}$.

It follows from Theorem 3.8 and $\mathcal{F}^*\cong\widehat{\mathcal{F}}^*$ that the maps $\mu\to\varphi_\mu\to\widehat{\varphi}_\mu:\mathcal{IL}(\mathcal{A},\mathcal{X}^{**})\to\mathcal{F}^*\to\widehat{\mathcal{F}}^*$ are isomorphisms such that

$$\mu(f)(x^*)=\varphi_\mu[f(\cdot)(x^*)]=\widehat{\varphi}_\mu[\widehat{f}(\cdot)(x^*)] \qquad (f\in\mathcal{A},\ x^*\in\mathcal{X}^*).$$

By the Riesz representation theorem, $\mathcal{C}(X)^*\cong\mathrm{rca}(X)$. Therefore

$$\widehat{\varphi}_\mu[\widehat{f}(\cdot)(x^*)]=\int\limits_X \widehat{f}(u)(x^*)\,\mu(du) \qquad (f\in\mathcal{A},\ x^*\in\mathcal{X}^*),$$

where μ also denotes the corresponding measure on X. Thus

$$\mu(f)(x^*)=\int\limits_X \widehat{f}(u)(x^*)\,\mu(du) \qquad (f\in\mathcal{A},\ x^*\in\mathcal{X}^*).$$

It follows from Definition 4.1 that

$$\mu(f) = \int_X \widehat{f} d\mu \qquad (f \in \mathcal{A}).$$

The proof is complete.

Definition 4.6. Let X be a compact, Hausdorff, topological space. A nonnegative regular Borel measure μ on X is called a *probability measure* if $\mu(X) = 1$. The set of all such probability measures on X will be denoted by $P(X)$. The *support* $\operatorname{supp}\mu$ of a probability measure μ on X is defined by

$$\operatorname{supp}\mu = \bigcap\{C \subset X : C \text{ is closed and } \mu(C) = 1\}.$$

Equivalently, $\operatorname{supp}\mu$ is the unique closed subset of X with the properties $\mu(\operatorname{supp}\mu) = 1$, and $\mu(W \cap \operatorname{supp}\mu) > 0$ for any open $W \subset X$ that meets $\operatorname{supp}\mu$.

Remark 4.7. Under the identification in Theorem 4.5, one can see that the vector-valued w^* means on $\mathcal{A}$ are precisely the probability measures on X, and members of $w^*\mathrm{MM}(\mathcal{A})$ are the probability measures whose supports consist of a single point.

5. Introversion and semigroups of vector-valued means

The main result of the section is Theorem 5.9. To lead to the theorem, we need a sequence of definitions and statements.

In this section, S denotes a semigroup. Let $f \in \mathcal{B}(S, \mathcal{X}^{**})$. Then the *right* (respectively, *left*) *translate* $R_s f$ of f by $s \in S$ is the map $R_s f(t) = f(ts)$ (respectively, $L_s f(t) = f(st)$) for all $t \in S$.

$\mathcal{A} \subset \mathcal{B}(S, \mathcal{X}^{**})$ is said to be *right* (respectively, *left*) *translation invariant* if $R_S\mathcal{A} = \{R_s f : s \in S, f \in \mathcal{A}\} \subset \mathcal{A}$ (respectively, $L_S\mathcal{A} = \{L_s f : s \in S, f \in \mathcal{A}\} \subset \mathcal{A}$). $\mathcal{A}$ is said to be *translation invariant* if it is both right and left translation invariant.

We will say S is *semitopological* if the multiplication $(s, t) \to st : S \times S \to S$ is separately continuous.

Definition 5.1. (1) Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S, \mathcal{X}^{**})$. For a linear map μ from $\mathcal{A}$ to $\mathcal{X}^{**}$, define the *left introversion operator* $T_\mu : \mathcal{A} \to \mathcal{B}(S, \mathcal{X}^{**})$ by

$$T_\mu f(s) = \mu(L_s f) \qquad (f \in \mathcal{A}, s \in S)$$

and analogously define the *right introversion operator* $U_\mu : \mathcal{A} \to \mathcal{B}(S, \mathcal{X}^{**})$ by

$$U_\mu f(s) = \mu(R_s f) \qquad (f \in \mathcal{A}, s \in S).$$

(2) If $T_\mu\mathcal{A} \subset \mathcal{A}$ for all $\mu \in \mathrm{M}(\mathcal{A})$ (resp., $\mu \in w^*\mathrm{M}(\mathcal{A})$), we will say that $\mathcal{A}$ is *left* (w^*) *introverted*; we will say that A is *right* (w^*) *introverted* if $U_\mu\mathcal{A} \subset \mathcal{A}$. $\mathcal{A}$ is (w^*) *introverted* if it is both left and right (w^*) introverted.

(3) Set $w^*\mathrm{MM}(\mathcal{A}) = \overline{\varepsilon(S)}^{\tau_{w^*}}$. If $T_\mu\mathcal{A} \subset \mathcal{A}$ for all $\mu \in w^*\mathrm{MM}(\mathcal{A})$, we will say that $\mathcal{A}$ is left *w^*m-introverted*; we will say that A is *right w^*m-introverted* if $U_\mu\mathcal{A} \subset \mathcal{A}$. $\mathcal{A}$ is *w^*m-introverted* if it is both left and right w^*m-introverted.

(4) Let $\mathcal{A} \subset \mathcal{B}(S, \mathcal{X})$. We will say that $\mathcal{A}$ is *left* (w^*) [w^*m-] *introverted* if $\iota(\mathcal{A})$ is left (w^*) [w^*m-] introverted. The right introvertedness can be defined analogously.

It is obvious that if $\mathcal{A}$ is left (right, both) w^* introverted then $\mathcal{A}$ is left (right, both) introverted. When we investigate some function spaces later, we will mainly check the w^* introvertedness of the spaces. Note that if $\mathcal{A}$ is left (right, both) w^* introverted then, by Corollary 3.9, $T_\mu f$ ($U_\mu f$, both $T_\mu f$ and $U_\mu f$) is in $\mathcal{A}$ for all $\mu \in \mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$ and $f \in \mathcal{A}$.

PROPOSITION 5.2. *Let $\mathcal{A}$ be as in Definition* 5.1(1) *and let μ and φ_μ be as in* (3.5). *Then for $f \in \mathcal{A}$ and $x^* \in \mathcal{X}^*$ we have*

$$[(T_\mu f)(\cdot)](x^*) = T_{\varphi_\mu}[f(\cdot)(x^*)], \tag{5.1}$$

$$[(U_\mu f)(\cdot)](x^*) = U_{\varphi_\mu}[f(\cdot)(x^*)]. \tag{5.2}$$

Proof. Set $G = [(T_\mu f)(\cdot)](x^*)$ and $g = f(\cdot)(x^*)$. To show (5.1), we need to show that

$$G(s) = (T_{\varphi_\mu} g)(s) \qquad (s \in S).$$

It follows from Definition 5.1 (1) and (3.5) that

$$\begin{aligned} G(s) &= [(T_\mu f)(s)](x^*) = [\mu(L_s f)](x^*) = \varphi_\mu[L_s f(\cdot)(x^*)] \\ &= \varphi_\mu(L_s g) = (T_{\varphi_\mu} g)(s). \end{aligned}$$

Similarly we can show (5.2). The proof is complete.

Set $\mathcal{B} = \mathcal{B}(S, \mathcal{X}^{**})$ and $\mathcal{B}x^* = \{f(\cdot)(x^*) : f \in \mathcal{B}\}$ for $x^* \in \mathcal{X}^*$. Define $\pi : \mathcal{B} \to \prod\{\mathcal{B}x^* : x^* \in \mathcal{X}^*\}$ by $\pi(f)(x^*) = f(\cdot)(x^*)$. Let $\mathcal{A} \subset \mathcal{B}$. Since $f \in \mathcal{B} \setminus \mathcal{A}$ implies that $\pi(f) \in \pi(\mathcal{B}) \setminus \pi(\mathcal{A})$, we have the following proposition.

PROPOSITION 5.3. *A function f is in $\mathcal{A}$ if and only if $\pi(f)$ is in $\pi(\mathcal{A})$.*

Recall that $w^*\mathcal{C}(S, \mathcal{X}^{**})$ [$w\mathcal{C}(S, \mathcal{X})$] is a subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ [$\mathcal{B}(S, \mathcal{X})$] such that $f \in w^*\mathcal{C}(S, \mathcal{X}^{**})$ [$f \in w\mathcal{C}(S, \mathcal{X})$] if and only if $f(\cdot)(x^*) \in \mathcal{C}(S)$ [$x^*f(\cdot) \in \mathcal{C}(S)$] for every $x^* \in \mathcal{X}^*$. It is easy to check that $w^*\mathcal{C}(S, \mathcal{X}^{**})$ [$w\mathcal{C}(S, \mathcal{X})$] is a closed subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ [$\mathcal{B}(S, \mathcal{X})$]. Let $\mathcal{A} = w\mathcal{C}(S, \mathcal{X})$. It follows from Proposition 3.4 that $\mathrm{M}(\iota(\mathcal{A})) = w^*\mathrm{M}(\iota(\mathcal{A}))$. Now we use the previous proposition in the following example.

EXAMPLE 5.4. $w\mathcal{C}(S, \mathcal{X})$ (resp., $w^*\mathcal{C}(S, \mathcal{X}^{**})$) is ($w^*$) introverted if S is a compact semitopological semigroup.

Proof. We give the proof only for the case $\mathcal{A} = w^*\mathcal{C}(S, \mathcal{X}^{**})$. For $\mu \in w^*\mathrm{M}(\mathcal{A})$ and $f \in \mathcal{A}$, we must show that both $T_\mu f$ and $U_\mu f$ are in $\mathcal{A}$. Note that $\{f(\cdot)(x^*) : f \in \mathcal{A}\} = \mathcal{C}(S)$ for all $0 \neq x^* \in \mathcal{X}^*$. By the previous proposition, we need to show that $\pi(T_\mu f), \pi(U_\mu f) \in \pi(\mathcal{A})$, i.e., $[(T_\mu f)(\cdot)](x^*), [(U_\mu f)(\cdot)](x^*) \in \mathcal{C}(S)$ for all $x^* \in \mathcal{X}^*$.

By (5.1), $[(T_\mu f)(\cdot)](x^*) = T_{\varphi_\mu}[f(\cdot)(x^*)]$. Since $f(\cdot)(x^*) \in \mathcal{C}(S)$ and $\mathcal{C}(S)$ is introverted because S is a compact semitopological semigroup [1, 2.2.5], we see that $T_{\varphi_\mu}[f(\cdot)(x^*)] \in \mathcal{C}(S)$. That is, $[(T_\mu f)(\cdot)](x^*) \in \mathcal{C}(S)$ for every $x^* \in \mathcal{X}^*$.

Similarly we can show that $[U_\mu f(\cdot)](x^*) \in \mathcal{C}(S)$. The proof is complete.

Let $\{f_\alpha\} \subset \mathcal{A} \subset \mathcal{B}(S, \mathcal{X}^{**})$ be a net and $f \in \mathcal{A}$. By $f_\alpha \to f$ w^* pointwise we mean that $f_\alpha(s)(x^*) \to f(s)(x^*)$ for all $s \in S$ and $x^* \in \mathcal{X}^*$.

Proposition 5.5. *Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ containing the constant functions and let $\varepsilon : S \to \mathrm{M}(\mathcal{A})$ be the evaluation mapping. Then*

(1) *for each $\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$, $T_\mu : \mathcal{A} \to \mathcal{B}(S, \mathcal{X}^{**})$ is a bounded linear transformation with $\|T_\mu\| \le \|\mu\|$;*

(2) *the mapping $\mu \to T_\mu : \mathcal{L}(\mathcal{A}, \mathcal{X}^{**}) \to \mathcal{L}(\mathcal{A}, \mathcal{B}(S, \mathcal{X}^{**}))$ is a bounded transformation;*

(3) *if $\mu \in w^*\mathrm{M}(\mathcal{A})$, then $T_\mu(x^{**}) = x^{**}$, $x^{**} \in \mathcal{X}^{**}$;*

(4) *for all $s \in S$ and $\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$*

$$T_\mu L_s = L_s T_\mu \qquad T_\mu R_s = T_{R_s^* \mu} \qquad T_{\varepsilon(s)} = R_s,$$

where $R_s^* : \mathcal{L}(\mathcal{A}, \mathcal{X}^{**}) \to \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ is the adjoint of R_s;

(5) *if $f \in \mathcal{A}$, then $\{T_\mu f : \mu \in \mathrm{M}(\mathcal{A})\}$ (resp., $\{T_\mu f : \mu \in w^*\mathrm{M}(\mathcal{A})\}$) is the closure in $\mathcal{B}(S, \mathcal{X}^{**})$ of $\mathrm{co}(R_S f)$ in the topology of (w^*) pointwise convergence on S.*

The proof of the proposition above is like that for [1, 2.2.3]; so we omit it.

Definition 5.6. Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ containing the constant functions, and define

$$Z_T = \{\nu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**}) : T_\nu \mathcal{A} \subset \mathcal{A}\} \quad \text{and} \quad Z_U = \{\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**}) : U_\mu \mathcal{A} \subset \mathcal{A}\}.$$

If $\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ and $\nu \in Z_T$, define $\mu\nu : \mathcal{A} \to X$ by

$$\mu\nu(f) = \mu(T_\nu f) \qquad (f \in \mathcal{A}).$$

If $\mu \in Z_U$ and $\nu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$, define $\mu * \nu : \mathcal{A} \to \mathcal{X}^{**}$ by

$$\mu * \nu(f) = \nu(U_\mu f) \qquad (f \in \mathcal{A}).$$

Definition 5.7. A (w^*) [w^*m-] *admissible subspace* $\mathcal{A}$ of $\mathcal{B}(S, \mathcal{X}^{**})$ is a norm closed, translation invariant, left (w^*) [w^*m-] introverted subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ containing the constant functions. In the case that $\mathcal{A} \subset \mathcal{B}(S, \mathcal{X})$, we will say $\mathcal{A}$ is (w^*) [w^*m-] *admissible* if $\iota(\mathcal{A})$ is (w^*) [w^*m-] admissible. In the case that $\mathcal{X} = \mathbb{C}$, an admissible subspace $\mathcal{A} \subset \mathcal{B}(S)$ is also required to be conjugate closed.

Let S be a semigroup. Define $\varrho_t : S \to S$ and $\lambda_t : S \to S$ by

$$\varrho_t = st, \quad \lambda_t = ts \quad (s \in S).$$

S is called a *right topological semigroup* if it is a topological space and ϱ_t is continuous for all $t \in S$. Set

$$\Lambda(S) = \{s \in S : \lambda_s \text{ is continuous}\}.$$

An *affine semigroup* S is a semigroup and a convex subset of a vector space in such a way that ϱ_t and λ_t are affine mappings for each $t \in S$. The requirement that ϱ_t and λ_t be affine means that if $r, s \in S$ and $a, b \in [0,1]$ with $a+b=1$ then

$$(ar+bs)t = art + bst \quad \text{and} \quad t(ar+bs) = atr + bts,$$

where $(+)$ denotes vector addition.

The following lemma summarizes the properties of the operation $(\mu,\nu) \to \mu\nu$. The proof is similar to that of [1, 2.2.9]. We omit the statements of the corresponding properties of the operation $(\mu,\nu) \to \mu * \nu$.

LEMMA 5.8. *Let $\mathcal{A}$ be as in Definition* 5.7 *and let $\varepsilon : \mathcal{A} \to \mathcal{X}^{**}$ be the evaluation mapping. Then*

(1) *Z_T is a linear subspace of $\mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ containing $\varepsilon(S)$*;

(2) *$\mu\nu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ for all $\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ and $\nu \in Z_T$*;

(3) *if $\mu \in \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$, $\nu \in Z_T$ and $s \in S$, we have*

$$T_{\mu\nu} = T_\mu \circ T_\nu, \quad \varepsilon(s)\nu = L_s^*\nu, \quad \mu\varepsilon(s) = R_s^*\mu, \quad \|\mu\nu\| \le \|\mu\|\|\nu\|,$$

where $L_s^* : \mathcal{L}(\mathcal{A}, \mathcal{X}^{**}) \to \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ is the adjoint of L_s;

(4) *Z_T is a right topological semigroup.*

The following theorem, which describes the compact semigroup structure for the set of w^* means, is the main result of the section. The theorem is essentially a consequence of the preceding lemma and Theorem 3.5.

THEOREM 5.9. (1) *If $\mathcal{A}$ is a w^* admissible subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ $[\mathcal{B}(S, \mathcal{X})]$ then, for τ_{w^*} and multiplication $(\mu,\nu) \to \mu\nu$, $w^*\mathrm{M}(\mathcal{A})$ is a compact right topological affine subsemigroup of $\mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$, $\mathrm{co}(\varepsilon(S)) \subset \Lambda(w^*\mathrm{M}(\mathcal{A}))$, $\mathrm{M}(\mathcal{A})$ is a subsemigroup of $w^*\mathrm{M}(\mathcal{A})$, and $\varepsilon : S \to w^*\mathrm{M}(\mathcal{A})$ is a homomorphism.*

(2) *Set $w^*\mathrm{MM}(\mathcal{A}) = \overline{\varepsilon(S)}^{\tau_{w^*}}$. If $\mathcal{A}$ is a w^*m-admissible subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ $[\mathcal{B}(S, \mathcal{X})]$, then for τ_{w^*}, and multiplication $(\mu,\nu) \to \mu\nu$, $w^*\mathrm{MM}(\mathcal{A})$ is a compact right topological subsemigroup of $w^*\mathrm{M}(\mathcal{A})$, $\varepsilon(S) \subset \Lambda(w^*\mathrm{MM}(\mathcal{A}))$ and $\varepsilon : S \to w^*\mathrm{MM}(\mathcal{A})$ is a homomorphism.*

Suppose that S is a compact semitopological semigroup. By Example 5.4, $w\mathcal{C}(S, \mathcal{X})$ (resp., $w^*\mathcal{C}(S, \mathcal{X}^{**})$) is (w^*) introverted. Hence $\mu\nu$, $\mu * \nu \in \mathrm{M}(w\mathcal{C}(S, \mathcal{X}))$ (resp., $w^*\mathrm{M}(w^*\mathcal{C}(S, \mathcal{X}^{**}))$); indeed, they are equal.

PROPOSITION 5.10. *Suppose that S is a compact, semitopological semigroup and let $\mathcal{A} = w^*\mathcal{C}(S, \mathcal{X}^{**})$. Then*

(1) *$\mu\nu = \mu * \nu$ for all $\mu, \nu \in w^*\mathrm{M}(\mathcal{A})$*;

(2) *for τ_{w^*} and multiplication $(\mu,\nu) \to \mu\nu$, $w^*\mathrm{M}(\mathcal{A})$ is a compact semitopological affine semigroup*;

(3) *if* S *is also a topological semigroup*, *so is* $w^*\mathrm{M}(\mathcal{A})$ *in* τ_{w^*}.

The results hold if we replace $w^*\mathcal{C}(S, \mathcal{X}^{**})$, $w^*\mathrm{M}(\mathcal{A})$ *and* τ_{w^*} *by* $w\mathcal{C}(S, \mathcal{X})$, $\mathrm{M}(\mathcal{A})$ *and* τ_w *respectively.*

Proof. (1) For $f \in \mathcal{A}$ and $x^* \in \mathcal{X}^*$, it follows from Definition 5.6, (3.5) and (5.1) that

$$\mu\nu(f)(x^*) = \mu[T_\nu f](x^*) = \varphi_\mu\{[(T_\nu f)(\cdot)](x^*)\}$$
$$= \varphi_\mu\{T_{\varphi_\nu}[f(\cdot)(x^*)]\} = \varphi_\mu\varphi_\nu[f(\cdot)(x^*)].$$

Similarly we can show that

$$\mu * \nu(f)(x^*) = \varphi_\mu * \varphi_\nu[f(\cdot)(x^*)].$$

By [1, 2.2.12], $\varphi_\mu\varphi_\nu[f(\cdot)(x^*)] = \varphi_\mu * \varphi_\nu[f(\cdot)(x^*)]$. Therefore

$$\mu\nu(f) = \mu * \nu(f) \qquad (f \in w^*\mathcal{C}(S, \mathcal{X}^{**})),$$

i.e., $\mu\nu = \mu * \nu$.

(2) is a consequence of (1) and Theorem 5.9(1).

To verify (3), we need to show that if $\mu_\alpha \to \mu$ and $\nu_\alpha \to \nu$ for τ_{w^*} then $\mu_\alpha\nu_\alpha \to \mu\nu$ for τ_{w^*}. By [1, 2.2.12(c)], $\varphi_{\mu_\alpha}\varphi_{\nu_\alpha}[f(\cdot)(x^*)] \to \varphi_\mu\varphi_\nu[f(\cdot)(x^*)]$ for all $f \in \mathcal{A}$ and $x^* \in \mathcal{X}^*$. We know from the proof of (1) that $\mu_\alpha\nu_\alpha(f)(x^*) = \varphi_{\mu_\alpha}\varphi_{\nu_\alpha}[f(\cdot)(x^*)]$ and $\mu\nu(f)(x^*) = \varphi_\mu\varphi_\nu[f(\cdot)(x^*)]$. Therefore $\mu_\alpha\nu_\alpha \to \mu\nu$ for τ_{w^*}.

The proof is similar when we replace $w^*\mathcal{C}(S, \mathcal{X}^{**}), w^*\mathrm{M}(\mathcal{A})$ and τ_{w^*} by $w\mathcal{C}(S, \mathcal{X})$, $\mathrm{M}(\mathcal{A})$ and τ_w respectively. The proof is finished.

6. Invariant vector-valued means

In this section, S denotes a semigroup.

Definition 6.1. Let $\mathcal{A}$ be left (right) translation invariant subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ $[\mathcal{B}(S, \mathcal{X})]$ containing the constant functions. Then a $\mu \in w^*\mathrm{M}(\mathcal{A})$ is said to be *left* (*right*) *invariant* if $\mu(L_s f) = \mu(f)[\mu(R_s f) = \mu(f)]$ for all $f \in \mathcal{A}$ and $s \in S$. $\mathrm{LIM}(\mathcal{A})$ $[\mathrm{RIM}(\mathcal{A})]$ denotes the set of all left (right) invariant means on $\mathcal{A}$. $\mathcal{A}$ is said to be *left* (*right*) *amenable* if $\mathrm{LIM}(\mathcal{A}) \neq \emptyset$ $[\mathrm{RIM}(\mathcal{A}) \neq \emptyset]$. If $\mathcal{A}$ is translation invariant, we set

$$\mathrm{IM}(\mathcal{A}) = \mathrm{LIM}(\mathcal{A}) \cap \mathrm{RIM}(\mathcal{A})$$

and call members of $\mathrm{IM}(\mathcal{A})$ *invariant means*. $\mathcal{A}$ is said to be *amenable* if $\mathrm{IM}(\mathcal{A}) \neq \emptyset$.

For a subspace $\mathcal{A} \subset \mathcal{B}(S, \mathcal{X})$, we know that $\mathrm{M}(\iota(\mathcal{A})) \subset w^*\mathrm{M}(\iota(\mathcal{A}))$. Therefore, if $\mu \in \mathrm{LIM}(\mathcal{A})$ $[\mathrm{LIM}(\mathcal{A}), \mathrm{IM}(\mathcal{A})]$, μ may not be in $\mathrm{M}(\iota(\mathcal{A}))$. However, the following theorem tells us that, in some circumstances, μ is in $\mathrm{M}(\iota(\mathcal{A}))$, i.e., $\mu \in \mathrm{M}(\mathcal{A})$.

Theorem 6.2. *Let* $\mathcal{A} \subset \mathcal{B}(S, \mathcal{X})$ *and let* $\mathcal{A}$ *be left* (*right*, *both*) w^* *introverted. Then every* $\mu \in \mathrm{LIM}(\mathcal{A})$ $[\mathrm{LIM}(\mathcal{A}), \mathrm{IM}(\mathcal{A})]$ *is in* $\mathrm{M}(\mathcal{A})$.

P r o o f. We only show the left introverted case. By Definition 5.1(4), $\iota(\mathcal{A})$ is left introverted, i.e., $T_\mu f \in \iota(\mathcal{A})$ for all $f \in \iota(\mathcal{A})$. Therefore $T_\mu f(S) \subset \iota(\mathcal{X})$. Let $s \in S$. Since $\mu \in \mathrm{LIM}(\mathcal{A})$, $\mu(f) = \mu(L_s f) = (T_\mu f)(s) \in \iota(\mathcal{X})$ for all $f \in \iota(\mathcal{A})$. Therefore $\mu \in \mathrm{M}(\mathcal{A})$. The proof is complete.

The following theorem comes from Theorem 3.8.

THEOREM 6.3. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ containing the constant functions and let $\mathcal{F} = \mathrm{sp}\{f(\cdot)(x^*) : f \in \mathcal{A}, x^* \in \mathcal{X}^*\}$. Let μ and φ_μ be as in* (3.5). *Suppose $\mathcal{A}$ is left* (*right*, *both*) *translation invariant. Then $\mu \in w^*\mathrm{M}(\mathcal{A})$ is left* (*right*, *both*) *invariant if and only if $\varphi_\mu \in \mathrm{M}(\mathcal{F})$ is left* (*right*, *both*) *invariant.*

The following theorem is an extension of [15, 18.10] about the space of numerical almost periodic functions to $w^*\mathcal{C}(S, \mathcal{X}^{**})$.

THEOREM 6.4. *Let S be a subsemigroup of a locally compact*, *Hausdorff*, *topological group G with left Haar measure λ, and suppose there exists a net $\{H_\alpha\}$ of Borel subsets of G contained in S such that $0 < \lambda(H_\alpha) < \infty$ for all α and such that*

$$\lim_\alpha \lambda(sH_\alpha \triangle H_\alpha)/\lambda(H_\alpha) = 0 \qquad (s \in S), \tag{6.1}$$

where $sH_\alpha \triangle H_\alpha = (sH_\alpha \setminus H_\alpha) \cup (H_\alpha \setminus sH_\alpha)$. Then $w^*\mathcal{C}(S, \mathcal{X}^{**})$ has a left invariant mean.

P r o o f. Set $\mathcal{A} = w^*\mathcal{C}(S, \mathcal{X}^{**})$. By Definition 4.1, for every α, the integral

$$\lambda(H_\alpha)^{-1} \int\limits_{H_\alpha} f(t)\,\lambda(dt) \qquad (f \in \mathcal{A}) \tag{6.2}$$

is in $\mathcal{X}^{**}$. The map $\mu_\alpha : \mathcal{A} \to \mathcal{X}^{**}$ defined by (6.2) is a bounded linear operator. Since $\mu_\alpha(f) \in \overline{\mathrm{co}}^{w^*} f(S)$ for every $f \in \mathcal{A}$, $\mu_\alpha \in w^*\mathrm{M}(\mathcal{A})$.

Let $x^* \in \mathcal{X}^*$ with $\|x^*\| \le 1$. If $s \in S$ then, by the left invariance of λ,

$$\begin{aligned}|\mu_\alpha(L_s f - f)(x^*)| &\le \lambda(H_\alpha)^{-1} \int\limits_G |1_{sH_\alpha} - 1_{H_\alpha}| \|f\| \|x^*\|\, d\lambda \\ &\le \|f\| \lambda(sH_\alpha \triangle H_\alpha)/\lambda(H_\alpha).\end{aligned}$$

It follows from (6.1) that any τ_{w^*}-limit point of the net $\{\mu_\alpha\}$ is a left invariant mean on $w^*\mathcal{C}(S, \mathcal{X}^{**})$. The proof is complete.

As in the scalar case, we have the following proposition, whose proof is similar to that of [1, 2.3.5]; so we omit it.

PROPOSITION 6.5. *Let $\mathcal{A}$ be a w^* admissible subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ and let $\varepsilon : S \to \mathcal{L}(\mathcal{A}, \mathcal{X}^{**})$ be the evaluation mapping.*

(1) $\mathrm{LIM}(\mathcal{A})$ *is the set of right zeros of* $w^*\mathrm{M}(\mathcal{A})$; *hence if $\mathcal{A}$ is left amenable*, *then* $\mathrm{LIM}(\mathcal{A})$ *is a closed ideal of* $w^*\mathrm{M}(\mathcal{A})$ *contained in every right ideal.*

(2) *If $\mathcal{A}$ is right amenable*, *then* $\mathrm{RIM}(\mathcal{A})$ *is a closed left ideal of* $w^*\mathrm{M}(\mathcal{A})$.

Corollary 6.6. *Let $\mathcal{A}$ be a w^* admissible subspace of $\mathcal{B}(S,\mathcal{X}^{**})$. If $\mathcal{A}$ is left and right amenable, then it is amenable.*

Proof. If $\mu\in \mathrm{LIM}(\mathcal{A})$ and $\nu\in\mathrm{RIM}(\mathcal{A})$, then $\mu\nu\in\mathrm{IM}(\mathcal{A})$.

Corollary 6.7. *Let $\mathcal{A}$ be a w^* admissible right introverted subspace of $\mathcal{B}(S,\mathcal{X}^{**})$ such that $\mu\nu=\mu*\nu$ for all $\mu,\nu\in w^*\mathrm{M}(\mathcal{A})$. Then $\mathcal{A}$ has at most one invariant mean.*

Proof. By the proposition and its right introverted analog, if $\mu,\nu\in\mathrm{IM}(\mathcal{A})$, then $\nu=\mu\nu=\mu*\nu=\mu$.

Theorem 6.8. *Let S be an abelian group. Then $\mathcal{B}(S,\mathcal{X}^{**})$ is amenable.*

Proof. Let $\mathcal{A}=\mathcal{B}(S,\mathcal{X}^{**})$ and set $\mathcal{F}=\mathrm{sp}\{f(\cdot)(x^*) : f\in\mathcal{A}, x^*\in\mathcal{X}^*\}$. Then $\mathcal{F}=\mathcal{B}(S)$. By [1, 2.3.8], $\mathcal{B}(S)$ is amenable. By Theorem 6.3, $\mathcal{B}(S,\mathcal{X}^{**})$ is amenable.

Theorem 6.9. *Let G be a compact, Hausdorff, topological group. Then $w\mathcal{C}(G,\mathcal{X})$ (resp., $w^*\mathcal{C}(G,\mathcal{X}^{**})$) has a unique invariant mean μ.*

Proof. The mean μ can be expressed as

$$\mu(f)=\int_G f\,d\nu \qquad (f\in w\mathcal{C}(G,\mathcal{X}),\ \text{resp. } f\in w^*\mathcal{C}(G,\mathcal{X}^{**})),$$

where ν is normalized Haar measure on G; the uniqueness follows from Proposition 5.10 and Corollary 6.7.

The scalar version of the next theorem is [1, 2.3.14]; a small modification of the proof of [1, 2.3.14] yields a proof of the present theorem.

Theorem 6.10. *Let S be a compact, Hausdorff, semitopological semigroup. Then the following assertions hold*:

(1) *$w^*\mathcal{C}(S,\mathcal{X}^{**})$ is left (right) amenable if and only if S has a unique minimal right (left) ideal*;
(2) *$w^*\mathcal{C}(S,\mathcal{X}^{**})$ is amenable if and only if the minimal ideal of S is a compact topological group.*

(1) *and* (2) *hold if we replace $w^*\mathcal{C}(G,\mathcal{X}^{**})$ by $w\mathcal{C}(S,\mathcal{X})$.*

To state the following theorems, we need a definition and some results from [1], which will also be used in Section 8.

Definition 6.11. An element e of a semigroup S is said to be an *idempotent* if $e^2=e$. The set of all idempotents of S is denoted by $E(S)$. An $e\in E(S)$ is said to be *minimal* if Se is a minimal left ideal of S.

Proposition 6.12 [1, 1.2.8]. *Let e be an idempotent of a semigroup S. Then the following statements are equivalent.*

(1) *e is minimal*;
(2) *eS is a minimal right ideal of S*;

(3) *eSe $(= eS \cap Se)$ is a group (and hence is the maximal subgroup of S containing e).*

By [1, 1.3.1], S has a minimal idempotent if S is a compact, Hausdorff, semitopological semigroup.

Suppose S is a compact, Hausdorff, semitopological semigroup. Let $\mathcal{A} = w^*\mathcal{C}(S, \mathcal{X}^{**})$. As in the proof of Example 5.4 and Proposition 5.10, one can show that $\mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$ is a semitopological semigroup with respect to the product $\mu\nu$ in Definition 5.6, and $\mu\nu = \mu * \nu$. By Theorem 4.2, the generalized Riesz representation theorem of operators, for a μ of $\mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$ there is a unique regular, complex, Borel measure μ on S such that

$$\mu(f) = \int_S f\, d\mu \qquad (f \in \mathcal{A}).$$

Furthermore, the product of a pair of elements μ, ν in $\mathcal{IL}(\mathcal{A}, \mathcal{X}^{**})$ may be viewed as convolution of measures on S. In fact, for $f \in \mathcal{A}$ we have

$$\begin{aligned}\mu\nu(f) = \mu(T_\nu f) &= \int_S \int_S f(st)\, \nu(dt)\, \mu(ds)\\ &= \mu * \nu(f) = \nu(U_\mu f) = \int_S \int_S f(st)\, \mu(ds)\, \nu(dt).\end{aligned}$$

THEOREM 6.13. *Suppose S is a compact, Hausdorff, semitopological semigroup. Let $\mathcal{A} = w^*\mathcal{C}(S, \mathcal{X}^{**})$ be left amenable and $\lambda \in \mathrm{LIM}(\mathcal{A})$. Then, for each minimal idempotent $e \in S$, λ is uniquely expressible as a product $\mu\nu$, where μ is normalized Haar measure on the compact topological group Se and $\operatorname{supp}\nu \subset E(eS)$. Moreover, $\operatorname{supp}\lambda = S \cdot \operatorname{supp}\nu$, hence $\operatorname{supp}\lambda$ is a left ideal contained in $K(S)$, the minimal ideal of S.*

Proof. For λ, let φ_λ be as in (3.5) of Theorem 3.8. By Theorem 6.3, $\varphi_\lambda \in \mathrm{LIM}(\mathcal{C}(S))$. So $\mathcal{C}(S)$ is left amenable. By [1, 2.5.1], we have a unique expression $\varphi_\lambda = \varphi_\mu\varphi_\nu$, where φ_μ is the normalized Haar measure on the compact topological group Se and $\operatorname{supp}\varphi_\mu \subset E(eS)$. Again by Theorem 3.8, there are μ and ν corresponding to φ_μ and φ_ν respectively such that (3.5) hold. Therefore $\lambda = \mu\nu$. The properties of μ an ν follow from those of φ_μ and φ_ν respectively. The proof is finished.

THEOREM 6.14. *Let S, $\mathcal{A}$ and λ be as in the previous theorem. Then λ is in $\mathrm{ex}(\mathrm{LIM}(\mathcal{A}))$ if and only if $\operatorname{supp}\lambda$ is a minimal left ideal of S. In this case $\operatorname{supp}\lambda$ is a compact topological group and λ is normalized Haar measure on $\operatorname{supp}\lambda$.*

Proof. As in the proof for the previous theorem, the properties of λ follow from those of φ_λ. The theorem is a consequence of [1, 2.5.2].

Let S be a compact, Hausdorff, semitopological semigroup. Let $\mathcal{A} = w\mathcal{C}(S, \mathcal{X})$. As above, by Theorem 4.3, the Riesz representation theorem of operators, we have the following theorems.

THEOREM 6.15. *Suppose S is a compact, Hausdorff, semitopological semigroup. Let $\mathcal{A} = w\mathcal{C}(S, \mathcal{X})$ be left amenable and $\lambda \in \mathrm{LIM}(\mathcal{A})$. Then, for each minimal idempotent $e \in S$, λ is uniquely expressible as a product $\mu\nu$, where μ is normalized Haar measure on the compact topological group Se and $\operatorname{supp}\nu \subset E(eS)$. Moreover, $\operatorname{supp}\lambda = S \cdot \operatorname{supp}\nu$, hence $\operatorname{supp}\lambda$ is a left ideal contained in $K(S)$, the minimal ideal of S.*

THEOREM 6.16. *Let S, $\mathcal{A}$ and λ be as in the previous theorem. Then λ is in $\mathrm{ex}(\mathrm{LIM}(\mathcal{A}))$ if and only if $\operatorname{supp}\lambda$ is a minimal left ideal of S. In this case $\operatorname{supp}\lambda$ is a compact topological group and λ is normalized Haar measure on $\operatorname{supp}\lambda$.*

7. Vector-valued almost periodic functions

For the theory of numerical almost periodic functions and their generalizations, we refer to [1–5, 8, 11]. For some developments for vector-valued functions, we refer to [1, 5, 13, 24, 25]. In previous work on vector-valued almost periodic functions, people usually considered continuous vector-valued functions on a semitopological semigroup S with an identity. However, we assume that the functions are only weakly continuous and S need not have an identity.

DEFINITION 7.1. Let S be a semitopological semigroup. A function $f \in w\mathcal{C}(S, \mathcal{X})$ is said to be *almost periodic* if the set $\{R_S f\}$ of right translates of f is norm relatively compact in $w\mathcal{C}(S, \mathcal{X})$. The set of all almost periodic functions on S is denoted by $\mathcal{AP}(S, \mathcal{X})$.

One can verify that $\mathcal{AP}(S, \mathcal{X})$ is a translation invariant, norm closed subspace of $w\mathcal{C}(S, \mathcal{X})$ containing the constant functions. Note that an $f \in w\mathcal{C}(S, \mathcal{X})$ may be regarded as an element of $w^*\mathcal{C}(S, \iota(\mathcal{X}))$, and vice versa. We have the following theorem.

THEOREM 7.2. *Let $\mathcal{A} = \mathcal{AP}(S, \mathcal{X})$ and let $f \in \mathcal{A}$. Then $T_\mu f \in \mathcal{A}$ for every $\mu \in w^*\mathrm{M}(\mathcal{A})$. The map $(\mu, \nu) \to \mu\nu(f) : w^*\mathrm{M}(\mathcal{A}) \times w^*\mathrm{M}(\mathcal{A}) \to \mathcal{X}$ is $\tau_{w^*} \times \tau_{w^*}$-$\sigma(\mathcal{X}, \mathcal{X}^*)$ continuous.*

Proof. By Proposition 5.5(5), the set $\{T_\mu f : \mu \in w^*\mathrm{M}(\mathcal{A})\}$ is the closure in $\mathcal{B}(S, \mathcal{X}^{**})$ of $\mathrm{co}\{R_S f\}$ in the topology of w^* pointwise convergence on S. Since $f \in \mathcal{AP}(S, \iota(\mathcal{X}))$, $\{R_S f\}$ is norm relatively compact in $w^*\mathcal{C}(S, \iota(\mathcal{X}))$. By the Mazur theorem [1, A.1], $\overline{\mathrm{co}} R_S f$, the norm closure in $w^*\mathcal{C}(S, \iota(\mathcal{X}))$ of $\mathrm{co}\{R_S f\}$, is norm compact. Therefore the w^* pointwise and norm closures of $\mathrm{co}\{R_S f\}$ coincide. Since $\mathcal{AP}(S, \iota(\mathcal{X}))$ is norm closed and translation invariant, $\{T_\mu f : \mu \in w^*\mathrm{M}(\mathcal{A})\} \subset \mathcal{AP}(S, \iota(\mathcal{X}))$.

It is clear that the map $\mu \to T_\mu f : w^*\mathrm{M}(\mathcal{A}) \to \mathcal{AP}(S, \iota(\mathcal{X}))$ is τ_{w^*}-w^* pointwise continuous. By the previous paragraph, the w^* pointwise topology and norm topology on $\{T_\mu f : \mu \in w^*\mathrm{M}(\mathcal{A})\}$ are the same. Therefore, the map $\mu \to T_\mu f$ is

τ_{w^*}-norm continuous. Now, for every $x^* \in \mathcal{X}^*$,

$$\begin{aligned}|\mu(T_\nu f)(x^*) - \mu_0(T_{\nu_0} f)(x^*)| &\le |\mu(T_\nu f)(x^*) - \mu(T_{\nu_0} f)(x^*)| + |\mu(T_{\nu_0} f)(x^*) - \mu_0(T_{\nu_0} f)(x^*)| \\ &\le \|T_\nu f - T_{\nu_0} f\|\|x^*\| + |(\mu-\mu_0)(T_{\nu_0} f)(x^*)|.\end{aligned}$$

Therefore when $\mu \to \mu_0$, $\nu \to \nu_0$ in τ_{w^*}, $\mu\nu(f) \to \mu_0\nu_0(f)$ in $\sigma(\mathcal{X}, \mathcal{X}^*)$. The proof is complete.

DEFINITION 7.3. A *semigroup compactification* of a semitopological semigroup S is a pair (ψ, X), where X is a compact, Hausdorff, right topological semigroup and $\psi : S \to X$ is a continuous homomorphism such that $\overline{\psi(S)} = X$ and

$$\psi(S) \subset \Lambda = \{t \in X : \text{the function } s \to ts : X \to X \text{ is continuous}\}.$$

DEFINITION 7.4. Let S be a semitopological semigroup and let P be a property of compactifications (ψ, X) of S. A *P-compactification* of S is a compactification of S that has the given property P.

Set $\mathcal{A} = \mathcal{AP}(S, \mathcal{X})$. Then $\mathcal{F} = \operatorname{sp}\{f(\cdot)(x^*) : f \in \iota(\mathcal{A}), x^* \in \mathcal{X}^*\} = \mathcal{AP}(S)$, the space of numerical almost periodic functions on S. Let ε and ε' be the evaluation maps from S to $\mathcal{L}(\iota(\mathcal{A}), \iota(\mathcal{X}))$ and from S to $\mathcal{F}^*$ respectively. Set $w^*\mathrm{MM}(\iota(\mathcal{A})) = \overline{\varepsilon(S)}^{\tau_{w^*}}$ and $\mathrm{MM}(\mathcal{F}) = \overline{\varepsilon'(S)}^{w^*}$. We have the following theorem.

THEOREM 7.5. *$\mathcal{AP}(S, \mathcal{X})$ is w^* admissible. $(\varepsilon, w^*\mathrm{MM}(\mathcal{A}))$ is a universal topological semigroup compactification of S.*

Proof. It follows from the previous theorem and the discussion in the paragraph before Theorem 7.2 that $\mathcal{AP}(S, \iota(\mathcal{X}))$ is w^* admissible. By Definition 5.7, $\mathcal{AP}(S, \mathcal{X})$ is w^* admissible. It follows from Theorem 5.9(2) and Definition 7.3 that $(\varepsilon, w^*\mathrm{MM}(\mathcal{A}))$ is a semigroup compactification of S. As in the proof for Theorem 3.8, $w^*\mathrm{MM}(\mathcal{A})$ is isometrically homeomorphic to $\mathrm{MM}(\mathcal{F})$. Since $\mu\nu(f)(x^*) = \varphi_\mu\varphi_\nu[f(\cdot)(x^*)]$, $w^*\mathrm{MM}(\mathcal{A})$ is homomorphic to $\mathrm{MM}(\mathcal{F})$. Since $\mathrm{MM}(\mathcal{F})$ is a universal topological semigroup compactification of S [1, 4.1.15], so is $w^*\mathrm{MM}(\mathcal{A})$.

COROLLARY 7.6. *If S is a semitopological group, then $\mathcal{AP}(S, \mathcal{X})$ has a unique invariant mean μ. Let φ_μ be as in* (3.5) *for μ. If $0 \ne f \in \mathcal{AP}(S, \iota(\mathcal{X}))$, $x^* \in \mathcal{X}^*$ such that $f(\cdot)(x^*) \ne 0$ then $\varphi_\mu(|f(\cdot)(x^*)|) > 0$.*

Proof. By the proof of Theorem 7.5, $w^*\mathrm{MM}(\mathcal{A})$ is homeomorphically homomorphic to $\mathrm{MM}(\mathcal{F})$. Since $\mathrm{MM}(\mathcal{F})$ is a topological group when S is a semitopological group [1, 4.1.2], $w^*\mathrm{MM}(\mathcal{A})$ is a compact, Hausdorff, topological group. Set $G = w^*\mathrm{MM}(\mathcal{A})$. By Theorem 6.9, $w^*\mathcal{C}(G, \iota(\mathcal{X}))$ has a unique invariant mean μ. Let $\widehat{\mathcal{A}}$ be as in Proposition 4.4. The restriction to $\widehat{\mathcal{A}}$ of μ is invariant. This finishes the proof of the existence and uniqueness of the first statement. The second statement follows from [1, 4.1.14]. The proof is complete.

THEOREM 7.7. *If S is a locally compact, noncompact, topological group then $\mathcal{AP}(S,\mathcal{X})\cap\mathcal{C}_0(S,\mathcal{X})=\{0\}$, where $\mathcal{C}_0(S,\mathcal{X})$ is the space of functions that vanish at infinity.*

Proof. If $f\in\mathcal{AP}(S,\iota(\mathcal{X}))\cap\mathcal{C}_0(S,\iota(\mathcal{X}))$ and $f\neq 0$, then there is some $x^*\in\mathcal{X}^*$ such that the function $f(\cdot)(x^*)\neq 0$ and is in $\mathcal{AP}(S)\cap\mathcal{C}_0(S)$, a contradiction to [1, 1.4.15].

In the case that the functions are scalar-valued, it follows from [1, 4.1.4] that for a function $f\in\mathcal{C}(S)$, the relative compactness of $\{R_S f\}$ is equivalent to the relative compactness of $\{L_S f\}$. The next example shows that this is not true for vector-valued functions.

EXAMPLE 7.8. Let $\mathbb{N}=\{1,2,\ldots\}$. Define a product on $\mathbb{N}$ by $nm=n$, for $n,m\in\mathbb{N}$. Define a function $f\in\mathcal{C}(\mathbb{N},l_2)$ by $f(a)=(0,\ldots,1,0,\ldots)$, the nth component is 1 and others are 0. Since $R_n f(m)=f(mn)=f(m)$, $\{R_{\mathbb{N}}f\}=\{f\}$. So $f\in\mathcal{AP}(\mathbb{N},l_2)$. However, $L_n f(m)=f(n)$. Therefore the set $\{L_{\mathbb{N}}f\}$ is not relatively compact in $\mathcal{C}(\mathbb{N},l_2)$.

This will not happen if S has an identity.

THEOREM 7.9. *Let S be a semitopological semigroup with identity e. If $f\in\mathcal{AP}(S,\mathcal{X})$, then the following statements hold*:

(1) $f\in\mathcal{C}(S,\mathcal{X})$;
(2) *the range $f(S)$ of f is norm relatively compact in $\mathcal{X}$*;
(3) *the map $s\to L_s f: S\to\mathcal{C}(S,\mathcal{X})$ is norm continuous*;
(4) *$\{L_S f\}$ is relatively compact in $\mathcal{C}(S,\mathcal{X})$.*

Proof. (1) Let $s_\alpha\to s$ in S. We need to show that $\|f(s_\alpha)-f(s)\|\to 0$. Since $f\in w^*\mathcal{C}(S,\iota(\mathcal{X}))$, $f(s_\alpha)\to f(s)$ in $\sigma(\iota(\mathcal{X}),\mathcal{X}^*)$. Note that $f(s_\alpha)=R_{s_\alpha}f(e)$. Since $\{R_{s_\alpha}f\}$ is norm relatively compact in $w^*\mathcal{C}(S,\iota(\mathcal{X}))$ and $\mathcal{AP}(S,\iota(\mathcal{X}))$ is norm closed in $w^*\mathcal{C}(S,\iota(\mathcal{X}))$, there are a $g\in\mathcal{AP}(S,\iota(\mathcal{X}))$ and subnet $\{s_\beta\}$ of $\{s_\alpha\}$ such that $\|R_{s_\beta}f-g\|\to 0$. It follows that $\|R_{s_\beta}f(e)-g(e)\|\to 0$. So $f(s)=g(e)$ and $\|f(s_\alpha)-f(s)\|\to 0$.

(2) A subset of a Banach space is norm relatively compact if and only if it is totally bounded. Therefore, for $\varepsilon>0$ there exists a finite subset K of S such that

$$\min\{\|R_s f-R_t f\|: t\in K\}<\varepsilon\qquad (s\in S).$$

Note that $R_s f(e)=f(s)$, for every $s\in S$. Hence

$$\min\{\|f(s)-f(t)\|: t\in K\}\le\min\{\|R_s f-R_t f\|: t\in K\}<\varepsilon\qquad (s\in S).$$

Thus $f(S)$ is relatively compact in $\mathcal{X}$.

(3) Since

$$\|L_t f-L_s f\|=\sup_{r\in S}\|L_t f(r)-L_s f(r)\|=\sup_{r\in S}\|R_r(f(t)-f(s))\|\le\|f(t)-f(s)\|,$$

(3) is a consequence of (1).

(4) By the Mazur theorem, the norm closure $\overline{\mathrm{co}}f(S)$ is norm compact. By [26, Proposition 1.5], $\mathrm{M}(\mathcal{A})$ is compact in the strong operator topology τ_s. Define a map $\mu \to g(\mu) : \mathrm{M}(\mathcal{A}) \to \mathcal{C}(\mathrm{M}(\mathcal{A}), \mathcal{X})$ by $[g(\mu)](\nu) = \mu(T_\nu f)$. We claim that the map is τ_s-norm continuous, i.e., $\mu_\alpha \to \mu$ for τ_s implies

$$\|g(\mu_\alpha) - g(\mu)\| = \sup_{\nu \in \mathrm{M}(\mathcal{A})} \|\mu_\alpha(T_\nu f) - \mu(T_\nu f)\| \to 0.$$

Note that $w^*\mathrm{M}(\mathcal{A}) = \mathrm{M}(\mathcal{A})$ because of (2) and Proposition 3.4. It follows from the proof of Theorem 7.2 that $\{T_u f : \mu \in \mathrm{M}(\mathcal{A})\}$ is norm compact in $\mathcal{C}(S, \mathcal{X})$. For $\delta > 0$, there are ν_i, $1 \le i \le n$, such that

$$\min\{\|T_\nu f - T_{\nu_i} f\| : 1 \le i \le n\} < \delta/3 \qquad (\nu \in \mathrm{M}(\mathcal{A})).$$

Since $\mu_\alpha \to \mu$ in τ_s, there is an α_0 such that

$$\|(\mu_\alpha - \mu)T_{\nu_i} f\| < \delta/3 \qquad (\alpha > \alpha_0, 1 \le i \le n).$$

Now, for $\alpha > \alpha_0$ and $\nu \in \mathrm{M}(\mathcal{A})$,

$$\begin{aligned}\|[g(\mu_\alpha) - g(\mu)](\nu)\| &= \|\mu_\alpha(T_\nu f) - \mu(T_\nu f)\| \\ &\le \min_i\{\|\mu_\alpha(T_\nu f) - \mu_\alpha(T_{\nu_i} f)\| \\ &\quad + \|\mu_\alpha(T_{\nu_i} f) - \mu(T_{\nu_i} f)\| + \|\mu(T_{\nu_i} f) - \mu(T_\nu f)\|\} \\ &\le \min_i\{2\|T_\nu f - T_{\nu_i} f\|\} + \delta/3 < \delta.\end{aligned}$$

Therefore, when $\alpha > \alpha_0$, $\|g(\mu_\alpha) - g(\mu)\| < \delta$. Since $L_s f(t) = [g(\varepsilon(s))](\varepsilon(t))$ and $\mathrm{M}(\mathcal{A})$ is τ_s-compact, it follows that $L_S f$ is norm relatively compact. The proof is complete.

Let $\mathcal{AP}'(S, \mathcal{X})$ be the space of all functions f of $w\mathcal{C}(S, \mathcal{X})$ such that $L_S f$ is norm relatively compact in $w\mathcal{C}(S, \mathcal{X})$. If S is a semitopological semigroup with identity e, then, by Theorem 7.9(4), $\mathcal{AP}(S, \mathcal{X}) \subset \mathcal{AP}'(S, \mathcal{X})$. Analogously we can show that $\mathcal{AP}(S, \mathcal{X}) \supset \mathcal{AP}'(S, \mathcal{X})$. Note that to verify Theorem 7.9(4) we use only Theorem 7.9(2). Therefore, if $f(S)$ is norm relatively compact in $\mathcal{X}$ for all $f \in \mathcal{AP}(S, \mathcal{X})$, then $\mathcal{AP}(S, \mathcal{X}) \subset \mathcal{AP}'(S, \mathcal{X})$. Analogously we have the reverse inclusion. Therefore we have the following theorem.

THEOREM 7.10. *Let S be a semitopological semigroup. If S has an identity, or $f(S)$ is norm relatively compact in $\mathcal{X}$ for all $f \in \mathcal{AP}(S, \mathcal{X}) \cup \mathcal{AP}'(S, \mathcal{X})$, then $\mathcal{AP}(S, \mathcal{X}) = \mathcal{AP}'(S, \mathcal{X})$.*

8. Vector-valued weakly almost periodic functions

DEFINITION 8.1. Let S be a semitopological semigroup. A function $f \in w\mathcal{C}(S, \mathcal{X})$ is said to be *weakly almost periodic* if $R_S f$ is weakly relatively compact in $w\mathcal{C}(S, \mathcal{X})$. The set of all weakly almost periodic functions is denoted by $\mathcal{WAP}(S, \mathcal{X})$.

The next theorem presents a characterization of $\mathcal{WAP}(S,\mathcal{X})$ that is analogous to Grothendieck's characterization of $\mathcal{WAP}(S)$, the C^*-algebra of weakly almost periodic functions on S. First we need some notation.

Let $w\mathcal{C}(S,\mathcal{X})^*$ be the dual space of $w\mathcal{C}(S,\mathcal{X})$, let E be a bounded subset of $w\mathcal{C}(S,\mathcal{X})^*$, and let $\mathcal{A}\subset w\mathcal{C}(S,\mathcal{X})$. We shall say that $\mathcal{A}$ has the *double limit property* with respect to E ($\mathcal{A}$ DLP E) if

$$\lim_m \lim_n \varphi_m(f_n)=\lim_n \lim_m \varphi_m(f_n),$$

whenever $\{f_n\}\subset\mathcal{A}$ and $\{\varphi_m\}\subset E$ are sequences such that all the limits exist.

The following lemma is [1, A.11].

LEMMA 8.2. *Let $\mathcal{A}$ be a bounded subset of a Banach space $\mathcal{Y}$ and let E be a subset of $\mathcal{Y}^*$ such that the norm closed, convex, circled hull of E is the norm closed unit ball of $\mathcal{Y}^*$. Then the weakly closed, convex, circled hull of $\mathcal{A}$ is weakly compact if and only if $\mathcal{A}$ DLP E.*

THEOREM 8.3. *Let S be a semitopological semigroup and let $f\in w\mathcal{C}(S,\mathcal{X})$. Then $f\in\mathcal{WAP}(S,\mathcal{X})$ if and only if*

$$\lim_m \lim_n x_m^*[f(s_m t_n)]=\lim_n \lim_m x_m^*[f(s_m t_n)],$$

whenever $\{x_m^*\}$ is a sequence in $\mathcal{X}^*$, $\|x_m^*\|=1$, $m=1,2,\ldots,$ and $\{s_m\}$, $\{t_n\}$ are sequences in S such that all the limits exist.

Proof. Let $D=\{x^*\in\mathcal{X}^*:\|x^*\|=1\}$. For $s\in S$ and $x^*\in D$, the functional

$$x^*\circ s: f\to x^*[f(s)] \qquad (f\in w\mathcal{C}(S,\mathcal{X}))$$

is in $w\mathcal{C}(S,\mathcal{X})^*$. Set

$$E=\{x^*\circ s: x^*\in D, s\in S\}.$$

We claim that $\overline{\mathrm{cco}}^{w^*}E$ is the norm closed unit ball B of $w\mathcal{C}(S,\mathcal{X})^*$. Suppose there is a $\varphi\in B\setminus\overline{\mathrm{cco}}^{w^*}E$. By the separation theorem, there is an $f\in w\mathcal{C}(S,\mathcal{X})$ such that

$$\mathrm{Re}\,\varphi(f)>\sup\{\mathrm{Re}\,\overline{\mathrm{cco}}^{w^*}E(f)\}=\sup\{\mathrm{Re}\,\overline{\mathrm{cco}}\{x^*f(s):x^*\in D, s\in S\}\}=\|f\|.$$

But $\mathrm{Re}\,\varphi(f)\le\|\varphi(f)\|\le\|f\|$. We get a contradiction. Now the necessity and sufficiency are consequences of Lemma 8.2. The proof is finished.

COROLLARY 8.4. *If S has an identity, then the range $f(S)$ is weakly relatively compact in $\mathcal{X}$ for every $f\in\mathcal{WAP}(S,\mathcal{X})$.*

Proof. If we take $s_m=e$, $m=1,2,\ldots,$ in the theorem, then

$$\lim_m \lim_n x_m^*[f(t_n)]=\lim_n \lim_m x_m^*[f(t_n)],$$

whenever $\{x_m^*\}\subset D$ and $\{t_n\}\subset S$ such that all the limits exist. By [1, A.5], the range $f(S)$ is weakly relatively compact in $\mathcal{X}$.

One can show that $\mathcal{WAP}(S,\mathcal{X})$ is a translation invariant, norm closed subspace of $w\mathcal{C}(S,\mathcal{X})$ containing the constant functions. Let $\mathcal{A}=\mathcal{WAP}(S,\mathcal{X})$.

As in the proof for Theorem 7.2, one can show that $T_\mu f \in \mathcal{A}$ for all $f \in \mathcal{A}$ and $\mu \in w^*\mathrm{M}(\mathcal{A})$. As in the proof for the second statement in Theorem 7.5, one shows that $(\varepsilon, w^*\mathrm{MM}(\mathcal{A}))$ is a universal semitopological semigroup compactification of S. Therefore we have the following theorem.

THEOREM 8.5. *Let S be a semitopological semigroup. Then $\mathcal{WAP}(S,\mathcal{X})$ is w^* admissible.* $(\varepsilon, w^*\mathrm{MM}(\mathcal{A}))$ *is a universal semitopological semigroup compactification of S.*

Let $\mathcal{WAP}'(S,\mathcal{X})$ be the space of all functions $f \in w\mathcal{C}(S,\mathcal{X})$ such that $L_S f$ is weakly relatively compact in $w\mathcal{C}(S,\mathcal{X})$. As in the case for almost periodic functions, we have the following theorem whose proof is similar to that for Theorem 7.10.

THEOREM 8.6. *Let S be a semitopological semigroup. If S admits an identity, or the range $f(S)$ is weakly relatively compact in $\mathcal{X}$ for every $f \in \mathcal{WAP}(S,\mathcal{X}) \cup \mathcal{WAP}'(S,\mathcal{X})$, then $\mathcal{WAP}(S,\mathcal{X}) = \mathcal{WAP}'(S,\mathcal{X})$.*

Recall from Section 6 that S has a minimal idempotent if S is a compact, Hausdorff, semitopological semigroup. Set $\mathcal{A} = \mathcal{WAP}(S,\mathcal{X})$. It follows from Theorem 8.5 that $w^*\mathrm{MM}(\mathcal{A})$ has a minimal idempotent. It follows from Proposition 6.12 that in $w^*\mathrm{MM}(\mathcal{A})$ there are a minimal left ideal, a minimal right ideal, and a group.

The following theorem extends [1, 4.2.14] about $\mathcal{WAP}(S)$ to $\mathcal{WAP}(S,\mathcal{X})$.

THEOREM 8.7. *Let S be a semitopological semigroup. Set $\mathcal{A} = \mathcal{WAP}(S,\mathcal{X})$ and let $X = w^*\mathrm{MM}(\mathcal{A})$. Set $K = K(X)$, the minimal ideal of X; and set $E = E(K)$, the set of all idempotents of K. Then*

(1) *$\mathcal{WAP}(S,\mathcal{X})$ is left amenable if and only if K is a minimal right ideal of X;*

(2) *$\mathcal{WAP}(S,\mathcal{X})$ is right amenable if and only if K is a minimal left ideal of X;*

(3) *$\mathcal{WAP}(S,\mathcal{X})$ is amenable if and only if K is a compact topological group.*

If $\mathcal{WAP}(S,\mathcal{X})$ is left amenable, then for any $\mu \in \mathrm{LIM}(\mathcal{A})$ and $e \in E$ there is a probability measure ν on E such that

$$\mu(f) = \int_E \int_{Xe} \widehat{f}(vu)\, \lambda_e(dv)\, \nu(du) \qquad (f \in \mathcal{A}), \tag{8.1}$$

where λ_e denotes normalized Haar measure on the compact topological group Xe. Furthermore, $\mu \in \mathrm{ex}\,\mathrm{LIM}(\mathcal{A})$ if and only if for some $e \in E$,

$$\mu(f) = \int_{Xe} \widehat{f}(v)\, \lambda_e(dv) \qquad (f \in \mathcal{A}). \tag{8.2}$$

The corresponding statements for the right amenable case also hold. (*Replace Xe by eX in* (8.1) *and* (8.2), *and replace vu by uv in* (8.1).)

If $\mathcal{A}$ is amenable, then the invariant mean μ on $\mathcal{A}$ is unique and is given by (8.2).

Proof. Note that $\mathcal{F} = \mathrm{sp}\{f(\cdot)(x^*) : x^* \in \mathcal{X}^*, f \in \iota(\mathcal{A})\} = \mathcal{WAP}(S)$, the C^*-algebra of weakly almost periodic functions on S. X is homeomorphically homomorphic to $\mathrm{MM}(\mathcal{F})$. By Theorem 6.3, $\mathcal{WAP}(S, \mathcal{X})$ is left amenable if and only if $\mathcal{WAP}(S)$ is left amenable. The latter condition is equivalent to K being a minimal right ideal of $\mathrm{MM}(\mathcal{F})$ [1, 4.2.14]. This shows (1). One can show (2) and (3) similarly.

Suppose $\mathcal{A}$ is left amenable and $\mu \in \mathrm{LIM}(\mathcal{A})$. Let $\widehat{\mathcal{A}}$ be as in Theorem 4.5 and let $\widehat{\mathcal{F}}$ be as in the proof of Theorem 4.5. Note that $\mu \in \mathrm{M}(\mathcal{A})$ (Theorem 6.2). By Theorem 4.5,

$$\mu(f) = \int_X \widehat{f}\, d\mu \qquad (f \in \mathcal{A}), \tag{8.3}$$

where μ denotes both a member of $\mathrm{M}(\mathcal{A})$ and the corresponding measure on X.

For μ, let φ_μ and $\widehat{\varphi}_\mu$ be as in the proof of Theorem 4.5. Then $\varphi_\mu \in LI\mathrm{M}(\mathcal{F})$ (Theorem 6.3). We claim that $\widehat{\varphi}_\mu \in \mathrm{LIM}(\widehat{\mathcal{F}})$. In fact, let $\widehat{F} \in \widehat{\mathcal{F}}$. There is an $F \in \mathcal{F}$ corresponding to $\widehat{F}$. Then

$$\widehat{\varphi}_\mu(L_{\varepsilon(s)}\widehat{F}) = \varphi_\mu(L_s F) = \varphi_\mu(F) = \widehat{\varphi}_\mu(\widehat{F}) \qquad (s \in S).$$

Now $\widehat{\varphi}_\mu \in \mathrm{LIM}(\widehat{\mathcal{F}})$ because X is a compact semitopological semigroup (Theorem 8.5) and the map $x \to L_x\widehat{F} : X \to \mathcal{C}(X)$ is weakly continuous [1, 4.2.8].

Note that $\widehat{F} = \mathcal{C}(X)$ and $\mathcal{C}(X) = \mathrm{sp}\{x^* f(\cdot) : f \in w\mathcal{C}(X, \mathcal{X}), x^* \in \mathcal{X}^*\}$. It follows from Theorem 6.3 that $\mu \in \mathrm{LIM}(w\mathcal{C}(X, \mathcal{X}))$. Now, (8.1) is a consequence of Theorem 6.15 and (8.3).

Similarly, (8.2) is a consequence of Theorem 6.16 and (8.3).

If $\mathcal{A}$ is amenable, then (8.2) obviously defines an invariant mean on $\mathcal{A}$. By Theorem 6.3, the space $\mathcal{WAP}(S)$ is amenable. Since $\mathcal{WAP}(S)$ has at most one invariant mean [1, 2.3.28], the invariant mean defined by (8.2) is unique on $\mathcal{A}$ (Theorem 6.3).

The proof is complete.

Before we state the next decomposition theorem we introduce some notation. Let $\mathcal{A} = \mathcal{WAP}(S, \mathcal{X})$, let $\mu \in \mathrm{LIM}(\mathcal{A})$, let φ_μ be as in (3.5) for μ, and let D be as in the proof of Theorem 8.3. Define

$$\mathcal{WAP}_0(S, \mathcal{X}) = \{f \in \mathcal{A} : 0 \in \overline{R_S f}\},$$

where the closure of $R_S f$ is taken in the weak topology on $\mathcal{A}$, and define

$$\mathcal{WAP}_\mu(S, \mathcal{X}) = \{f \in \mathcal{A} : \varphi_\mu(|x^* f(\cdot)|) = 0 \text{ for all } x^* \in D\}.$$

It is readily verified that both $\mathcal{WAP}_0(S, \mathcal{X})$ and $\mathcal{WAP}_\mu(S, \mathcal{X})$ are left translation invariant, norm closed subsets of $\mathcal{WAP}(S, \mathcal{X})$. Moreover, $\mathcal{WAP}_\mu(S, \mathcal{X})$ is also a subspace of $\mathcal{WAP}(S, \mathcal{X})$.

The following theorem generalizes [8, Theorem 4.11] to the extent that the semitopological semigroup S need not have an identity. We state the theorem only for the left amenable case. We omit its right amenable analog.

THEOREM 8.8. *Let S be a semitopological semigroup, let $\mathcal{A} = \mathcal{WAP}(S,\mathcal{X})$ and let $X = w^*\mathrm{MM}(\mathcal{A})$. If $\mathcal{A}$ is left amenable then the following statements hold*:

(1) *If e and e' are two minimal idempotents of X, then $T_e(\mathcal{A}) = T_{e'}(\mathcal{A})$.*

(2) *Define $\mathcal{SAP}(S,\mathcal{X}) = T_e(\mathcal{A})$ for some* (*hence any*) *minimal idempotent e of X and call the member of $\mathcal{SAP}(S,\mathcal{X})$ strongly almost periodic. Then $\mathcal{SAP}(S,\mathcal{X}) \subset \mathcal{AP}(S,\mathcal{X})$. Let $\mathcal{F} = \mathrm{sp}\{x^*[f(\cdot)] : x^* \in \mathcal{X}^*, f \in \mathcal{SAP}(S,\mathcal{X})\}$, then $\mathcal{F} = \mathcal{SAP}(S)$, the space of strongly almost periodic functions on S.* (*For the definition of $\mathcal{SAP}(S)$, see* [1, 4.3.2].)

(3) *Let e be a minimal idempotent of X. Then $T_e : \mathcal{A} \to \mathcal{SAP}(S,\mathcal{X})$ is a projection. Therefore $\mathcal{WAP}(S,\mathcal{X}) = \mathcal{SAP}(S,\mathcal{X}) \oplus T_e^{-1}(0)$.*

(4) *For any extreme left invariant mean μ on $\mathcal{A}$,*

$$\mathcal{WAP}(S,\mathcal{X}) = \mathcal{SAP}(S,\mathcal{X}) \oplus \mathcal{WAP}_\mu(S,\mathcal{X}) \text{ and } \mathcal{WAP}_\mu(S,\mathcal{X}) \subset \mathcal{WAP}_0(S,\mathcal{X}).$$

Conversely, *if the conditions* (1)–(3) *are satisfied*, *then $\mathcal{WAP}(S,\mathcal{X})$ is left amenable.*

Proof. (1) Let K be as in Theorem 8.7. Then K is a minimal right ideal of X. By [1, 1.2.12], $e'X = eX$. So there is a $\mu \in X$ such that $e'\mu = e$. Let $f \in \mathcal{A}$. It follows from Lemma 5.8(3) that $T_e f = T_{e'\mu} f = T_{e'} T_\mu f$. Since $T_\mu f \in \mathcal{A}$ (Theorem 8.5), $T_e f \in T_{e'}(\mathcal{A})$, i.e., $T_e(\mathcal{A}) \subset T_{e'}(\mathcal{A})$. Similarly, one can show that $T_e(\mathcal{A}) \supset T_{e'}(\mathcal{A})$.

(2) Suppose that there is an $f \in \mathcal{WAP}(S,\mathcal{X})$ such that $T_e f \in \mathcal{WAP}(S,\mathcal{X}) \setminus \mathcal{AP}(S,\mathcal{X})$. As in the proof of Theorem 8.3, one can show that $\mathcal{A}^*$, the dual space of $\mathcal{A}$, is $\mathrm{sp}\{x^* \circ \mu : x^* \in D, \mu \in \mathcal{IL}(\mathcal{A},\mathcal{X})\}$. By the separation theorem, there is $\sum_{i=1}^n x_i^* \circ \mu_i \in \mathcal{A}^*$ such that

$$\mathrm{Re} \sum_{i=1}^n x_i^* \circ \mu_i(T_e f) > \sup\Big\{\mathrm{Re} \sum_{i=1}^n x_i^* \circ \mu_i(\mathcal{AP}(S,\mathcal{X}))\Big\},$$

i.e.,

$$\mathrm{Re} \sum_{i=1}^n \mu_i(T_e f)(x_i^*) > \sup\Big\{\mathrm{Re} \sum_{i=1}^n \mu_i(g)(x_i^*) : g \in \mathcal{AP}(S,\iota(\mathcal{X}))\Big\}.$$

It follows from (3.5) that

$$\mathrm{Re} \sum_{i=1}^n \varphi_{\mu_i}[T_e f(\cdot)(x_i^*)] > \sup\Big\{\mathrm{Re} \sum_{i=1}^n \varphi_{\mu_i}[g(\cdot)(x_i^*)] : g \in \mathcal{AP}(S,\iota(\mathcal{X}))\Big\}.$$

Since $[T_e f(\cdot)](x_i^*) = T_{\varphi_e}[f(\cdot)(x_i^*)]$ (Proposition 5.2) and $T_{\varphi_e}[f(\cdot)(x_i^*)] \in \mathcal{AP}(S)$ [1, 4.3.12], $[T_e f(\cdot)](x_i^*) \in \mathcal{AP}(S)$ for $1 \le i \le n$. Note that $\{g(\cdot)(x^*) : g \in \mathcal{AP}(S,\iota(\mathcal{X}))\} = \mathcal{AP}(S)$ for any $0 \ne x^* \in \mathcal{X}^*$. Therefore, $\sum_{i=1}^n \varphi_{\mu_i}[T_e f(\cdot)(x_i^*)] \in \{\sum_{i=1}^n \varphi_{\mu_i}[g(\cdot)(x_i^*)] : g \in \mathcal{AP}(S,\iota(\mathcal{X}))\}$. We get a contradiction.

To show that $\mathcal{F} = \mathcal{SAP}(S)$, note that for $0 \neq x^* \in \mathcal{X}^*$, $x^*[T_e(\mathcal{A})] = \{[T_e f(\cdot)](x^*) : f \in \iota(\mathcal{A})\}$. By (5.1), $x^*[T_e(\mathcal{A})] = \{T_{\varphi_e}[f(\cdot)(x^*)] : f \in \iota(\mathcal{A})\} = T_{\varphi_e}(\mathcal{WAP}(S))$. It follows from the proof for (i)⇒(ii)⇒(iii) of [1, 4.3.12] that $T_{\varphi_e}(\mathcal{WAP}(S)) = \mathcal{SAP}(S)$. Therefore $\mathcal{F} = \mathcal{SAP}(S)$.

(3) By Lemma 5.8(3), $T_e^2 = T_{e^2} = T_e$.

(4) Since K is a minimal right ideal of X, Xe is a group for any minimal idempotent e of X [1, 1.2.13]. It follows from Theorem 8.7 that, for $\mu \in \mathrm{LIM}(\mathcal{A})$, there is a minimal idempotent $e \in X$ such that

$$\mu(f) = \int_G \widehat{f}(u)\,du \qquad (f \in \mathcal{A}),$$

where G is the compact topological group $Xe = eXe$, du is normalized Haar measure on G, $\varepsilon^* \widehat{f} = f$. Therefore

$$\varphi_\mu(x^* f) = \int_G \widehat{f}(u)(x^*)\,du \qquad (f \in \mathcal{A}).$$

Note that $\int_G |\widehat{f}(u)(x^*)|\,du = 0$ if and only if $R_e \widehat{f}(\cdot)(x^*) = 0$. Since $R_e \widehat{f}(\cdot)(x^*) = 0$ for all $x^* \in D$ is equivalent to $R_e \widehat{f} = 0$, $f \in \mathcal{WAP}_\mu(S, \mathcal{X})$ if and only if $R_e \widehat{f} = 0$. Since $\varepsilon^*(R_e \widehat{f}) = T_e f$, we see that $\mathcal{WAP}_\mu(S, \mathcal{X}) = T_e^{-1}(0)$. By (3), $\mathcal{A} = \mathcal{SAP}(S, \mathcal{X}) \oplus \mathcal{WAP}_\mu(S, \mathcal{X})$.

Let $f \in T_e^{-1}(0)$. Since $R_S f$ is relatively weakly compact in $\mathcal{WAP}(S, \mathcal{X})$, one can show that $\{T_\mu f : \mu \in w^*\mathrm{MM}(\mathcal{A})\}$ is the weak closure in $\mathcal{A}$ of $R_S f$. Therefore $f \in \mathcal{WAP}_0(S, \mathcal{X})$.

Conversely, suppose the conditions (1)–(3) are satisfied. Since T_e is a projection from $\mathcal{A}$ onto $\mathcal{SAP}(S, \mathcal{X})$, T_{φ_e} is a projection from $\mathcal{WAP}(S)$ onto $\mathcal{SAP}(S)$. By Proposition 5.5(4), $T_e L_s = L_s T_e$ for all $s \in S$. Therefore $T_{\varphi_e} L_s = L_s T_{\varphi_e}$. Thus, T_{φ_e} satisfies the condition (iii) of [1, 4.3.12], which is equivalent to $\mathcal{WAP}(S)$ being left amenable. By Theorem 6.3, $\mathcal{WAP}(S, \mathcal{X})$ is left amenable.

The proof is complete.

THEOREM 8.9. *Let S be a semitopological semigroup and let $\mathcal{A} = \mathcal{WAP}(S, \mathcal{X})$. If $\mathcal{WAP}(S, \mathcal{X})$ is amenable then $\mathcal{WAP}(S, \mathcal{X}) = \mathcal{SAP}(S, \mathcal{X}) \oplus \mathcal{WAP}_0(S, \mathcal{X})$. In this case $\mathcal{WAP}_0(S, \mathcal{X}) = \mathcal{WAP}_\mu(S, \mathcal{X})$, where μ is the unique invariant mean on $\mathcal{A}$.*

Proof. By the previous theorem, it suffices to show that if $\mathcal{WAP}(S, \mathcal{X})$ is amenable then $\mathcal{WAP}_0(S, \mathcal{X}) \subset \mathcal{WAP}_\mu(S, \mathcal{X})$. Now if $f \in \mathcal{WAP}_0(S, \mathcal{X})$ then the set $E = \{\nu \in X : T_\nu f = 0\}$ is nonempty and is a closed left ideal of X. Hence E contains a minimal idempotent e [1, 1.3.1]. Since $K(X)$ is a group (Theorem 8.7(3)), it follows from the proof for Theorem 8.8(4) that $T_e^{-1}(0) = \mathcal{WAP}_\mu(S, \mathcal{X})$. Therefore, $f \in \mathcal{WAP}_\mu(S, \mathcal{X})$, as required. The proof is complete.

Since $\mathcal{WAP}(S, \mathcal{X})$ has at most one invariant mean, we apply Theorem 6.4 respectively to $\mathcal{WAP}(\mathbb{R}, \mathcal{X})$, $\mathcal{WAP}(\mathbb{R}^+, \mathcal{X})$, $\mathcal{WAP}(\mathbb{Z}, \mathcal{X})$, and $\mathcal{WAP}(\mathbb{N}, \mathcal{X})$ and get the unique invariant means:

$$\mu(f) = \lim_{t\to\infty} (2t)^{-1} \int_{-t}^{t} f(s)\,ds \qquad (f \in \mathcal{WAP}(\mathbb{R}, \mathcal{X}))$$

$$\mu(f) = \lim_{t\to\infty} t^{-1} \int_{0}^{t} f(s)\,ds \qquad (f \in \mathcal{WAP}(\mathbb{R}^+, \mathcal{X}))$$

$$\mu(f) = \lim_{n\to\infty} (2n+1)^{-1} \sum_{k=-n}^{n} f(k) \qquad (f \in \mathcal{WAP}(\mathbb{Z}, \mathcal{X}))$$

and

$$\mu(f) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} f(k) \qquad (f \in \mathcal{WAP}(\mathbb{N}, \mathcal{X})).$$

Since $\mathbb{R} = \mathbb{R}^+ \cup \mathbb{R}^-$ and $\mathbb{Z} = \mathbb{N} \cup \mathbb{N}^-$, we can also represent the unique invariant means on $\mathcal{WAP}(\mathbb{R}, \mathcal{X})$ and $\mathcal{WAP}(\mathbb{Z}), \mathcal{X}$, respectively, by

$$\mu(f) = \lim_{t\to\infty} t^{-1} \int_{0}^{t} f(s)\,ds \qquad (f \in \mathcal{WAP}(\mathbb{R}, \mathcal{X}))$$

and

$$\mu(f) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} f(k) \qquad (f \in \mathcal{WAP}(\mathbb{Z}, \mathcal{X})).$$

The next example generalizes the results in [24, Theorem 3.4] and [25].

EXAMPLE 8.10. If $S = (\mathbb{R}^+, +)$, then $\mathcal{AP}(S, \mathcal{X}) = \mathcal{AP}(\mathbb{R}, \mathcal{X})|_S \oplus C_0(S, \mathcal{X})$.

Proof. We use Theorem 8.9. Let $f \in \mathcal{AP}(S, \mathcal{X})$. By Theorem 8.9, $f = g + h$, where $g \in \mathcal{SAP}(S, \mathcal{X})$ and $h \in \mathcal{WAP}_0(S, \mathcal{X})$. By Theorem 8.8(2), $g \in \mathcal{AP}(S, \mathcal{X})$. So $h \in \mathcal{AP}(S, \mathcal{X})$. Therefore $\|R_{t_n} h\| \to 0$ for some sequence $\{t_n\} \subset S$. So, $h \in C_0(S, \mathcal{X})$. As in the proof for [1, 4.3.14], one can show that g can be extended to a function $g_1 \in \mathcal{AP}(\mathbb{R}, \mathcal{X})$. The proof is finished.

THEOREM 8.11. *Let S be a semitopological semigroup and suppose that S contains a dense subset A such that the closure $\overline{At}$ of At is equal to S for all $t \in A$. Then $\mathcal{WAP}(S, \mathcal{X})$ is left amenable if and only if the following conditions hold*:

(1) *S has a topological left identity*, *that is*, *there is a net $\{s_\alpha\}$ in S such that $\lim_\alpha s_\alpha s = s$ for all $s \in S$.*

(2) *S has nonempty center.*

(3) *$\overline{sS} = S$ for all $s \in S$.*

Proof. By [1, 4.2.15], $\mathcal{WAP}(S)$ is left amenable if and only if any one of the above conditions holds. The theorem is a consequence of Theorem 6.3.

COROLLARY 8.12. *Let S be a semitopological semigroup such that either S contains a dense subgroup or S is a dense subsemigroup of a semitopological group. Then $\mathcal{WAP}(S,\mathcal{X})$ is amenable.*

Proof. By Theorem 8.11 and its right amenable analog, $\mathcal{WAP}(S,\mathcal{X})$ is both left and right amenable, hence amenable [Corollary 6.6].

THEOREM 8.13. *Let S be a semitopological semigroup containing a dense topologically left (or right) simple subsemigroup S', i.e., $S't = S'$ ($tS' = S'$) for all $t \in S'$. Assume that S has a topological identity. Then $\mathcal{WAP}(S,\mathcal{X})$ is amenable.*

Proof. By [1, 4.2.17], $\mathcal{WAP}(S)$ is amenable if S satisfies the conditions in the theorem. Now the theorem is a consequence of Theorem 6.3.

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References

[1] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups: Function Spaces, Compactifications, Representations*, Wiley, New York, 1989.

[2] A. S. Besicovitch, *Almost Periodic Functions*, Dover, New York, 1954.

[3] H. A. Bohr, *Almost Periodic Functions*, Chelsea, New York, 1951.

[4] R. B. Burckel, *Weakly Almost Periodic Functions on Semigroups*, Gordon and Breach, New York, 1970.

[5] C. Corduneanu, *Almost Periodic Functions*, 2nd ed., Chelsea, New York, 1989.

[6] M. M. Day, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509–544.

[7] —, *Semigroups and amenability*, in: Semigroups, K. W. Folley (ed.), Academic Press, New York and London, 1969, 5–53.

[8] K. de Leeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. 105 (1961), 63–97.

[9] J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leurs applications*, Acta Sci. Math. (Szeged) 12 (1950), 213–227.

[10] N. Dunford and J. T. Schwartz, *Linear Operators I*, Wiley, New York, 1958.

[11] W. F. Eberlein, *Abstract ergodic theorems and weakly almost periodic functions*, Trans. Amer. Math. Soc. 67 (1949), 217–240.

[12] R. E. Edwards, *Functional Analysis: Theory and Applications*, Holt, Rinehart and Winston, New York, 1965.

[13] S. Goldberg and P. Irwin, *Weakly almost periodic vector valued functions*, Dissertationes Math. 157 (1979).

[14] F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand, New York, 1969.

[15] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer, New York, 1963.

[16] T. Husain, *Amenability of locally compact groups and vector-valued function spaces*, Sympos. Math. 16 (1975), 417–431.

[17] T. Husain and J. C. S. Wong, *Invariant means on vector valued functions I*, Ann. Scuola Norm. Sup. Pisa (3) 27 (1973), 717–727.

[18] R. V. Kadison, *The trace in finite operator algebras*, Proc. Amer. Math. Soc. 12 (1961), 973–977.
[19] P. Milnes, *On vector-valued weakly almost periodic functions*, J. London Math. Soc. (2) 22 (1980), 467–472.
[20] A. L. T. Paterson, *Amenability*, Amer. Math. Soc., Providence, R.I., 1988.
[21] R. R. Phelps, *Extreme positive operators and homomorphisms*, Trans. Amer. Math. Soc. 108 (1963), 265–274.
[22] J.-P. Pier, *Amenable Locally Compact Groups*, Wiley, New York, 1984.
[23] J. F. Price, *Means of vector valued functions and projections which commute with the action of a group*, Ann. Scuola Norm. Sup. Pisa (3) 26 (1972), 515–531.
[24] W. M. Ruess and W. H. Summers, *Compactness in spaces of vector valued continuous functions and asymptotic almost periodicity*, Math. Nachr. 135 (1988), 7–33.
[25] S. Zaidman, *Almost-Periodic Functions in Abstract Spaces*, Pitman, London, 1985.
[26] C. Zhang, *Vector-valued means and weakly almost periodic functions*, Internat. J. Math. and Math. Sci. 17 (1994), 227–238.