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ALGEBRAS OF POLYNOMIAL GROWTH

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The present notes are a more or less faithful reproduction of lectures given at the Workshop on the present trends in Representation Theory of Algebras, Banach Center, Warsaw, April 1988. The aim of these notes is to give an introduction to the theory of representation-infinite finite-dimensional algebras, over an algebraically closed field, of polynomial growth, that is, algebras with infinitely many nonisomorphic indecomposable modules and for which there is a natural number m such that the indecomposable finite-dimensional modules occur, in each dimension $d \geqslant 2$, in a finite number of discrete and at most d^m one-parameter families. We will present below several results which exhibit important, for the general theory, classes of representation-infinite algebras of polynomial growth as well as some methods for the study of their indecomposable modules. In our opinion, the representation-infinite algebras of polynomial growth form the most interesting class of tame algebras but the theory of such algebras is still far from being complete. We pose a number of open problems which seem to be worth studying. In addition, we report on some recent investigations which are contained in the papers [ANS], [NS] and [S3].

The presented strategy for the classification of indecomposable modules over an arbitrary representation-infinite algebra of polynomial growth consists of two steps:

(1) a reduction, with the help of Galois converings, to the simply connected case,

(2) a classification of the indecomposable modules over the corresponding simply connected algebras using the known invariants and the extension

This paper is in final form and no version of it will be submitted for publication elsewhere.

process of known (minimal) representation-infinite simply connected algebras of polynomial growth.

The notes are divided into five sections. In Section 1 we recall some definitions needed in the paper. In Section 2 we introduce a hierarchy of tame algebras as well as we exhibit some known examples of such algebras. Section 3 is devoted to the Euclidean and Ringel algebras playing a crucial rôle in the study of simply connected algebras of polynomial growth. In Section 4 we present Galois covering techniques which permit the study of arbitrary algebras of polynomial growth to be reduced to the simply connected ones. Finally, using the concepts developed here we describe in Section 5 all standard representation-infinite selfinjective algebras of polynomial growth.

1. Preliminaries

1.1. Let k be a fixed algebraically closed field. A *locally bounded category* [BoG] is a k-category R satisfying the following conditions: (a) distinct objects are nonisomorphic; (b) the algebras $R(x, x)$ are local; (c) for each $x \in R$, $\sum_{y \in R}(\dim_k R(x, y) + \dim_k R(y, x)) < \infty$. Locally bounded categories can be constructed in the following way. Let Q be a locally finite quiver (the number of arrows starting or ending at any vertex is finite) with a set of vertices Q_0 and a set of arrows Q_1. The path category kQ of Q has objects the vertices of Q and, for $x, y \in Q_0$, the morphism space $kQ(x, y)$ consists of the formal linear combinations of paths from x to y. Let I be an ideal in kQ satisfying the following conditions: (d) for each pair of objects x, y of kQ, $I(x, y) = I \cap kQ(x, y)$ is contained in the subspace of $kQ(x, y)$ spanned by the paths of length $\geqslant 2$; (e) for each vertex $x \in Q_0$, there is a natural number N_x such that I contains each path of length $\geqslant N_x$ which starts or ends at x. Then (Q, I) is called a *bound quiver*. The residue category kQ/I, called the *bound quiver category*, is a locally bounded category. Conversely, each locally bounded category R is isomorphic to such a kQ/I, and Q is uniquely determined by R [BoG]. A locally bounded category $R = kQ/I$ is called *triangular* whenever Q has no oriented cycles. If R is a locally bounded category, an *R-module* is a k-linear covariant functor from R to the category of k-vector spaces. We denote by $\operatorname{mod} R$ the category of all finite-dimensional R-modules, that is, R-modules V with $\dim V = \sum_{x \in R} \dim_k V(x) < \infty$. If R is *bounded*, that is, $\sum_{x,y \in R} \dim_k R(x, y) < \infty$, the category $\operatorname{mod} R$ is equivalent to the category $\operatorname{mod} A$ of finite-dimensional left modules over the algebra $A = \bigoplus R$ formed by the quadratic matrices $a = (a_{yx})_{x,y \in R}$ such that $a_{yx} \in R(x, y)$. Conversely, to each finite-dimensional basic ($A/\operatorname{rad} A \simeq k \times \ldots \times k$) and connected (there are no central idempotents different from 0 and 1) algebra A we can attach the bounded category $R = R_A$ with $A = \bigoplus R$ whose objects are formed by a complete set E of primitive orthogonal idempotents e of A, $1 = \sum_{e \in E} e$, $R(e, f) = fAe$ and the composition is induced by the multiplication in A. *In the* *paper by an algebra is meant an associative, basic, connected, finite-dimensional k-algebra with identity.* We shall also identify an algebra A with the bounded category R_A.

1.2. For a locally bounded category R we denote by Γ_R its *Auslander–Reiten quiver*, that is, a valued translation quiver whose vertices are the isomorphism classes of indecomposable finite-dimensional R-modules, arrows are the irreducible maps valued by their multiplicities, and whose translation τ is the Auslander–Reiten translate $D\mathrm{Tr}$ [HPR]. A translation quiver T is called a *tube* [Ri3] if it contains a cyclic path and if its topological realization $|T| = S^1 \times \mathbf{R}_0^+$ (where S^1 is the unit circle and $\mathbf{R}_0^+$ the set of nonnegative real numbers). Further, T is a *stable tube* of rank r if it is of the form $\mathbf{Z}A_\infty/(\tau^r)$. A vertex x of T is called *projective-injective* provided $\tau x = 0$ and $\tau^{-1}x = 0$. Finally, T is said to be a *quasi-tube* if its full translation subquiver formed by all vertices which are not projective-injective is a tube.

1.3. We recall from [Gr], [MP] the notion of the fundamental group of a bound quiver. Let (Q, I) be a connected bound quiver. A relation $\varrho = \sum_{j=1}^{m} \lambda_j w_j \in I(x, y)$ is said to be *minimal* if $m \geqslant 2$ and for each nonempty proper subset J of $\{1, \ldots, m\}$, $\sum_{j\in J} \lambda_j w_j \notin I$. Denote by $m(I)$ the set of minimal relations of the ideal I, and by $\Pi_1(Q, x_0)$ the fundamental group of Q at the vertex x_0. Let $N(Q, m(I), x_0)$ be the normal subgroup of $\Pi_1(Q, x_0)$ generated by all elements of the form $[w^{-1}u^{-1}vw]$ where w is a walk from x_0 to x and u, v are paths from x to y such that there is an element $\sum_{j=1}^{m} \lambda_j w_j \in m(I)$ with $u = w_j$, $v = w_r$ for some $1 \leqslant j, r \leqslant m$. The *fundamental group* $\Pi_1(Q, I)$ of (Q, I) is defined to be the group

$$\Pi_1(Q, I) = \Pi_1(Q, x_0)/N(Q, m(I), x_0).$$

Following [AS1] a triangular locally bounded category R is called *simply connected* if, for any presentation $R \simeq kQ/I$ of R as a bound quiver category, the fundamental group $\Pi_1(Q, I)$ is trivial (see also Lemma 4.2). It follows from [BrG], [MP] that, for representation-finite algebras (categories), this definition coincides with that of [BoG]. An example of a representation-infinite simply connected algebra (bounded category) is provided by the bound quiver algebra (category) kQ/I, where Q is the quiver of Fig. 1 and I is generated by $\xi\eta$ and

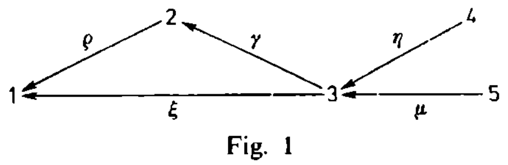


Fig. 1

$\xi\mu - \varrho\gamma\mu$. On the other hand, the algebra (category) given by the bound quiver (Q', I'), where Q' is illustrated in Fig. 2 and I' is generated by $\xi\mu - \varrho\gamma\mu$, is not simply connected. Indeed, it is isomorphic to kQ'/I'', where I'' is generated by $\xi\mu$, and $\Pi_1(Q', I'') \simeq \mathbf{Z}$.

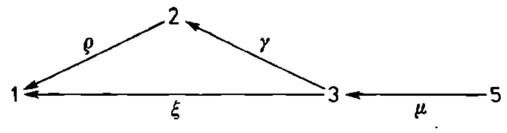


Fig. 2

1.4. Let A be a triangular algebra kQ/I. By $P(x)$ (resp. $I(x)$, $S(x)$) we denote the indecomposable projective (resp. injective, simple) A-module associated with $x \in Q_0$. If $M \in \operatorname{mod} A$, we set $\underline{\dim}\, M = (\dim_k \operatorname{Hom}_A(P(x), M))_{x \in Q_0}$. The *Cartan matrix* C_A of A is an $n \times n$-matrix (where $n = |Q_0|$) whose i-j-entry is $\dim_k \operatorname{Hom}_A(P(i), P(j))$. Since $\operatorname{gl\,dim} A < \infty$, C_A is $\mathbf{Z}$-invertible [Ri3] and we can consider the (nonsymmetric) bilinear form $\langle x, y \rangle_A = x C_A^{-T} y^T$, for x, y from the Grothendieck group $K_0(A) = \mathbf{Z}^n$ of A. For X, $Y \in \operatorname{mod} A$, we have

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_A^i(X, Y) \quad \text{(cf. [Ri3])}.$$

The associated quadratic form $\lambda_A(z) = \langle z, z \rangle_A$ is called the *Euler characteristic* of A. Let R be a minimal set of relations which generate I and denote by $r(x, y)$ the cardinality of $R \cap I(x, y)$ for each couple x, $y \in Q_0$. Then following [Bo1] the quadratic form q_A: $\mathbf{Z}^n \to \mathbf{Z}^n$ defined by

$$q_A(z) = \sum_{x \in Q_0} z(x)^2 - \sum_{(x \to y) \in Q_1} z(x)z(y) + \sum_{x, y \in Q_0} r(x, y) z(x) z(y),$$

$z = (z(x))_{x \in Q_0}$, is called the *Tits form* of A. If $\operatorname{gl\,dim} A \leqslant 2$, we have $q_A = \lambda_A$ [Bo1].

2. A hierarchy of tame algebras

In this section we introduce a hierarchy of tame algebras and show some known classes of such algebras.

2.1. Following Drozd [D] an algebra A is called *wild* if there is an A-$k\langle x, y\rangle$-bimodule M, where $k\langle x, y\rangle$ is the free k-algebra in two noncommuting variables x and y, such that M is a finitely generated free right $k\langle x, y\rangle$-module and the functor $M \otimes_{k\langle x, y\rangle} -$: $\operatorname{mod} k\langle x, y\rangle \to \operatorname{mod} A$ preserves isomorphism classes and indecomposability. An algebra A is called *tame* if, for any dimension d, there exists a finite family of (parametrizing) functors F_i: $\operatorname{mod} k[x] \to \operatorname{mod} A$, $1 \leqslant i \leqslant n_d$, where $k[x]$ is the polynomial algebra in one variable, satisfying the following conditions:

(i) For each i, $F_i = M_i \otimes_{k[x]} -$, where M_i is an A-$k[x]$-bimodule and a finitely generated free right $k[x]$-module.

(ii) All but a finite number (up to isomorphism) indecomposable

A-modules of dimension d are of the form $F_i(S)$ for some i and some simple $k[x]$-module S.

For equivalent definitions of tame algebras we refer to [DS1]. We have the following remarkable theorem of Drozd [D] (see also [CB]).

THEOREM [D]. *Every finite-dimensional algebra A is either tame or wild, and not both.*

Remarks. (1) Let F: $\operatorname{mod} k[x] \to \operatorname{mod} A$ be a functor satisfying the above condition (i). Then

$$\mathscr{X}_F = \{\lambda \in k \mid F(k[x]/(x-\lambda)) \text{ is indecomposable}\}$$

is a constructible subset of k in the Zariski topology, and hence finite or cofinite in k [DS1].

(2) Let F_1, F_2: $\operatorname{mod} k[x] \to \operatorname{mod} A$ be two functors satisfying (i). Then the set

$$\{\lambda \in k \mid F_1(k[x]/(x-\lambda)) \simeq F_2(k[x]/(x-\xi)) \text{ for some } \xi \in k\}$$

is finite or cofinite in k (again as a constructible subset of k).

For an algebra A and $d \geqslant 1$, we denote by $\mu_A(d)$ the least number of parametrizing functors satisfying (i) and (ii). An algebra A is called of *polynomial growth* [S2] if there is a natural number m such that, for any dimension $d \geqslant 2$, $\mu_A(d) \leqslant d^m$. Futher, A is of *linear growth* if there is a natural number n such that $\mu_A(d) \leqslant nd$ for all $d \geqslant 1$. Moreover, A is called *domestic* [Ri2] if there is a finite family of functors F_i: $\operatorname{mod} k[x] \to \operatorname{mod} A$, $1 \leqslant i \leqslant n$, satisfying (i) and (ii′) below:

(ii′) For each dimension d, all but a finite number of indecomposable A-modules of dimension d are of the form $F_i(V)$ for some i and some indecomposable $k[x]$-module V.

A is *n-parametric* if the minimal number of such functors is n. Observe that, if A is domestic, then there is a natural number n such that $\mu_A(d) \leqslant n$ for all $d \geqslant 1$. Indeed, if $F = M \otimes_{k[x]} -$: $\operatorname{mod} k[x] \to \operatorname{mod} A$ is a functor satisfying (i) and such that $\dim_k F(k[x]/(x-\lambda)^m) = mr$, for $\lambda \in k$, then the functor $F' = M' \otimes_{k[y]} -$: $\operatorname{mod} k[y] \to \operatorname{mod} A$, where $M' = M \otimes_{k[x]} k[x, y]/(x-y)^m$, also satisfies (i) and we have

$$F'(k[y]/(y-\lambda)) \simeq F(k[x]/(x-\lambda)^m) \quad \text{for all } \lambda \in k.$$

The second Brauer–Thrall conjecture (for a proof we refer to [F]) can be written as follows: an algebra A is representation-finite (only finitely many isoclasses of indecomposable modules) if and only if $\mu_A(d) = 0$ for all $d \geqslant 1$.

PROBLEM 1. *Let A be an algebra such that $\mu_A(d) \leqslant n$ for some n and all $d \geqslant 1$. Is A domestic?*

PROBLEM 2. *Let A be an algebra of polynomial growth. Is A of linear growth?*

A locally bounded category R is called tame (resp. of polynomial growth, linear growth, domestic) if so is every full finite subcategory of R (see [DS1]).

2.2. Hereditary algebras. Let H be a hereditary algebra, that is, $\operatorname{gl}\dim H \leqslant 1$. Then $H = k\Delta$ for some connected quiver Δ without oriented cycles, and we have the following theorem:

THEOREM [DF], [N]. *The following conditions are equivalent:*

(i) *H is representation-infinite and tame.*
(ii) *H is one-parametric.*
(iii) *The underlying graph $\bar{\Delta}$ of Δ is one of the Euclidean quivers of Fig. 3.*

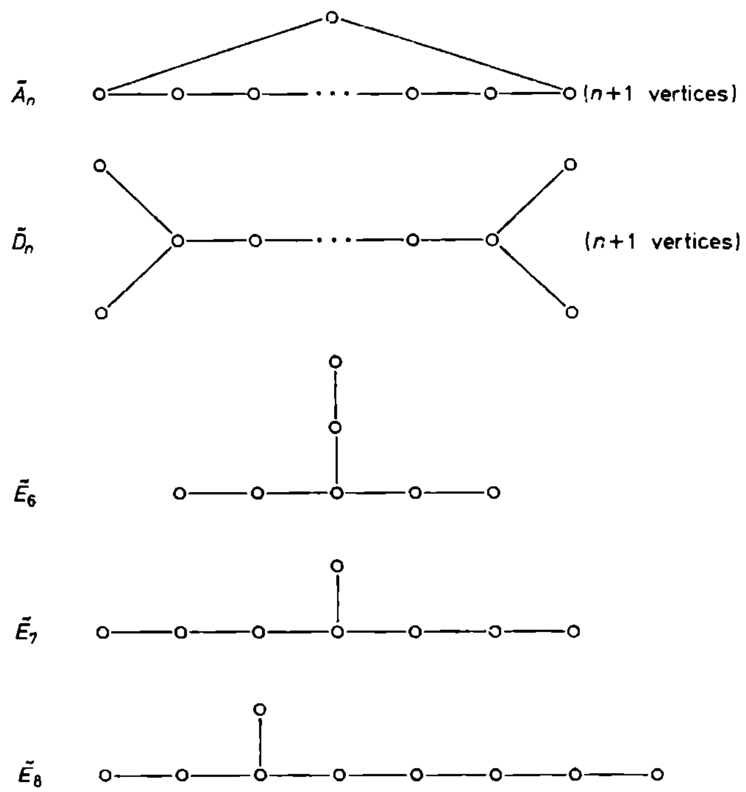


Fig. 3

2.3. Tilted algebras of Euclidean type. Let H be a hereditary algebra of Euclidean type. A module $U \in \operatorname{mod} H$ is called *preprojective* (resp. *preinjective*) if there are only finitely many isoclasses of indecomposable H-modules V such that $\operatorname{Hom}_H(V, U) \neq 0$ (resp. $\operatorname{Hom}_H(U, V) \neq 0$). Let T be a multiplicity-free tilting module, that is, an H-module satisfying the conditions: (1) $\operatorname{Ext}^1(T, T) = 0$ and (2) T is a direct sum of n (n = the rank of $K_0(H)$) pairwise nonisomorphic indecomposable modules. Then $B = \mathrm{End}_E(T)$ is called a *tilted algebra of Euclidean type.*

THEOREM [HR]. *The following conditions are equivalent:*

(i) *B is representation-infinite.*
(ii) *B is one-parametric.*
(iii) *T has no simultaneously preprojective and preinjective direct summand.*

2.4. Trivial extension algebras. The *trivial extension* $T(A)$ of an algebra A by its minimal injective cogenerator bimodule $DA = \mathrm{Hom}_k(A, k)$ is the algebra whose additive structure is that of the group $A \oplus DA$, and whose multiplication is defined by

$$(a, f)(b, g) = (ab, ag + fb)$$

for $a, b \in A$ and $f, g \in DA$. Then $T(A)$ is selfinjective and, in fact, symmetric.

THEOREM [T]. *Let H be a hereditary algebra. The following conditions are equivalent:*

(i) *$T(H)$ is representation-infinite and tame.*
(ii) *$T(H)$ is 2-parametric.*
(iii) *H is of Euclidean type.*

For example, if H is given by the quiver of type $\tilde{D}_4$ of Fig. 4, then $T(H)$ is the bound quiver algebra kQ/I, where Q is the quiver of Fig. 5 and I is generated by $\alpha_i\beta_i - \alpha_j\beta_j$, $\beta_j\alpha_i$, $i \neq j$, $1 \leqslant i, j \leqslant 4$.

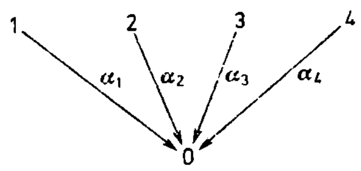


Fig. 4

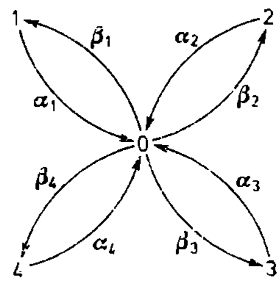


Fig. 5

For a classification of all representation-infinite trivial extension algebras of polynomial growth we refer to [ANS], [NS], [Ne].

2.5. Triangular matrix algebras. Denote by $T_2(A)$ the algebra

$$\begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$$

of 2×2 upper-triangular matrices over an algebra A.

THEOREM [L]. *Let $H = k\Delta$ be a hereditary algebra. The following conditions are equivalent*:

(i) *$T_2(H)$ is representation-infinite and tame.*
(ii) *$T_2(H)$ is representation-infinite of polynomial growth.*
(iii) *$\bar{\Delta}$ is one of the Dynkin graphs*

$$A_5:\ \circ - \circ - \circ - \circ - \circ \quad \text{or} \quad D_4:\ \begin{tikzcd}[column sep=small, row sep=small] & \circ \arrow[d, no head] & \\ \circ \arrow[r, no head] & \circ \arrow[r, no head] & \circ \end{tikzcd}.$$

Moreover, for Δ different from $\circ \to \circ \to \circ \to \circ \to \circ$, $T_2(H)$ is nondomestic.

For example, if $\Delta = \circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ \xleftarrow{\gamma} \circ \xrightarrow{\sigma} \circ$, $H = k\Delta$, then $T_2(H)$ is the bound quiver algebra KQ/I, where Q is the quiver

$$\begin{tikzcd} \circ \arrow[r, "\alpha"] \arrow[d, "\xi"'] & \circ \arrow[r, "\beta"] \arrow[d, "\eta"'] & \circ \arrow[d, "\varrho"'] & \circ \arrow[l, "\gamma"'] \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \circ \arrow[d, "\psi"'] \\ \circ \arrow[r, "\alpha'"'] & \circ \arrow[r, "\beta'"'] & \circ & \circ \arrow[l, "\gamma'"] \arrow[r, "\sigma'"'] & \circ \end{tikzcd}$$

and I is generated by $\alpha'\xi - \eta\alpha$, $\beta'\eta - \varrho\beta$, $\gamma'\varphi - \varrho\gamma$, $\sigma'\varphi - \psi\sigma$.

THEOREM [S1]. *Let A^n be the truncated polynomial algebra $k[x]/(x^n)$. The following conditions are equivalent*:

(i) *$T_2(A^n)$ is representation-infinite and tame.*
(ii) *$T_2(A^n)$ is nondomestic of polynomial growth.*
(iii) *$n = 4$.*

$T_2(A^4)$ is the bound quiver algebra kQ/I given by the quiver

$$Q:\ \begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\gamma"'] & \circ \arrow[loop right, "\beta"] \end{tikzcd}$$

and the ideal I generated by α^4, β^4 and $\gamma\alpha - \beta\gamma$.

The tame triangular matrix algebras $T_2(A)$ over arbitrary Nakayama algebras are classified in [S1] and those over arbitrary selfinjective algebras in [HM]. For a classification of all tame algebras of the form $T_2(A)$ we refer to [LS].

2.6. Group algebras. Denote by D_m, $m \geqslant 1$, S_m, $m \geqslant 3$, Q_m, $m \geqslant 2$, the following 2-groups:

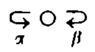
the dihedral groups $D_m = \langle g, h \mid g^2 = h^{2^m} = 1,\ hg = gh^{-1} \rangle$,

the semidihedral groups $S_m = \langle g, h \mid g^2 = h^{2^m} = 1,\ hg = gh^{2^{m-1}-1} \rangle$,

the quaternion groups $Q_m = \langle g, h \mid g^2 = h^{2^{m-1}},\ g^4 = 1,\ hg = gh^{-1} \rangle$.

THEOREM [BD]. *Let G be a finite group, let* char $k = p > 0$ *and suppose p divides the order of G. The group algebra kG is representation-infinite and tame if and only if $p = 2$ and any Sylow 2-subgroup of G is one of the groups D_m, S_m or Q_m.*

The group algebras kD_m were investigated in [B], [Ri1]. Moreover, the group algebra kG is representation-infinite domestic (resp. of polynomial growth) if and only if $p = 2$, 2 divides the order of G and any Sylow 2-subgroup of G is isomorphic to the Klein four-group D_1. This follows from the fact that, if H is one of the groups D_m, $m \geqslant 2$, S_m, $m \geqslant 3$, or Q_m, $m \geqslant 2$, then the group algebra kH has a quotient isomorphic to the bound quiver algebra kQ/I where Q is the quiver



and I is generated by α^2, β^2, $\beta\alpha\beta$, $\alpha\beta\alpha$, which is by [S2, Lemma 1] tame but not of polynomial growth. The representation-infinite polynomial growth group algebras AG of finite groups G over arbitrary finite-dimensional algebras A are classified in [S2].

2.7. Gelfand–Ponomarev algebras. The algebras $k[x, y]/(xy, x^n, y^n)$, $n \geqslant 3$, investigated by Gelfand and Ponomarev [GP] (see also [DS3]) are tame but not of polynomial growth.

3. Euclidean and Ringel algebras

In this section we describe the module categories $\operatorname{mod} R$ for two important classes of algebras of polynomial growth: Euclidean and Ringel algebras. We start with some basic definitions.

3.1. Let H be a hereditary algebra of Euclidean type. Of special interest are the tilted algebras $C = \operatorname{End}_H(T)$, where T is a preprojective tilting H-module, called *tame concealed algebras* [Ri3]. They have the following characterization:

THEOREM [HV]. *The class of tame concealed algebras coincides with the class of tame algebras A with a preprojective component in the Auslander–Reiten quiver Γ_A and such that all factor algebras A/AeA, where e is a primitive orthogonal idempotent of A, are representation-finite.*

The tame concealed algebras have been classified by Happel and Vossieck [HV]. The only tame concealed algebras of type $\tilde{A}_n$ are the hereditary algebras. There are four kinds of tame concealed algebras of type $\tilde{D}_n$ (see Fig. 6, where the unoriented edges may be oriented arbitrarily). Moreover, there are $4302 = 56 + 437 + 3809$ tame concealed algebras of types $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ given by the bound quivers with 7, 8, 9 vertices respectively, which are very clearly arranged in 144 families by Happel and Vossieck [HV]. In a parallel work [Bo2] Bongartz obtained the same list with another characterization.

THEOREM [Bo2]. *The class of tame concealed algebras of types $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ coincides with the class of minimal representation-infinite tame simply*

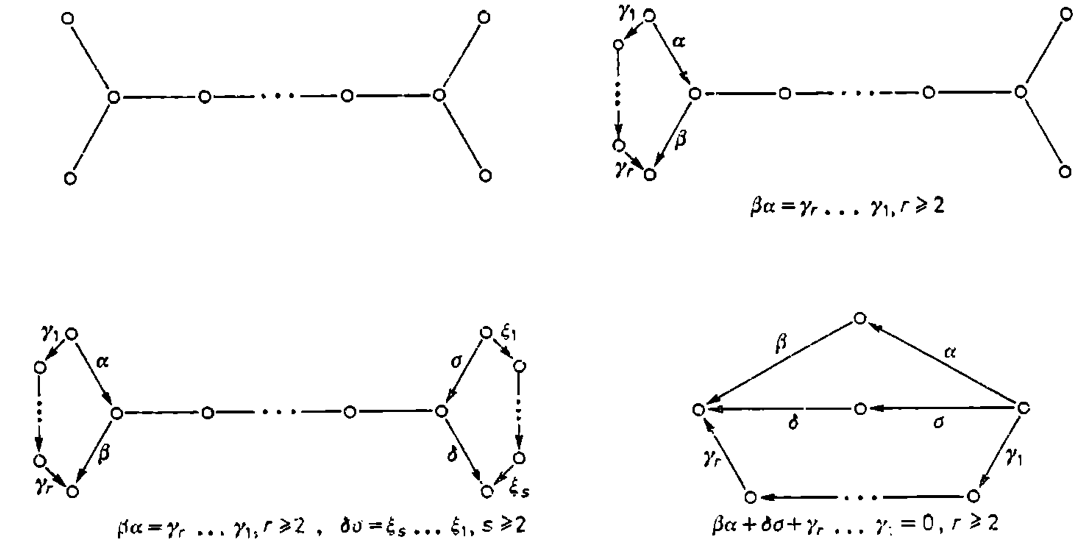


Fig. 6

connected algebras A, where minimal means that every full convex proper subcategory of the category R_A of A is representation-finite.

Recall now the structure of the Auslander-Reiten quiver of a tame concealed algebra C. From [Ri3], Γ_C is the disjoint union

$$\mathscr{P}^C \vee \mathscr{T}^C \vee \mathscr{Q}^C,$$

where $\mathscr{P}^C$ (resp. $\mathscr{Q}^C$) is a preprojective (resp. preinjective) component of Γ_C consisting of the isoclasses of all indecomposable preprojective (resp. preinjective) C-modules and $\mathscr{T}^C$ is a $P_1(k)$-family of stable tubes T_λ^C, $\lambda \in P_1(k) = k \cup \{\infty\}$. Moreover, the ordering from the left to right indicates that there are nonzero maps only from any of these components to itself and to the components to its right.

3.2. We shall now define the tubular extension of a tame concealed algebra. Let P be a finite connected subquiver of the quiver of Fig. 7 containing the root b. Let I be the ideal in kP generated by all compositions $\beta\alpha$. Then the quotient category (algebra) kP/I is called a *branch* in b (in the sense of [Ri3]).

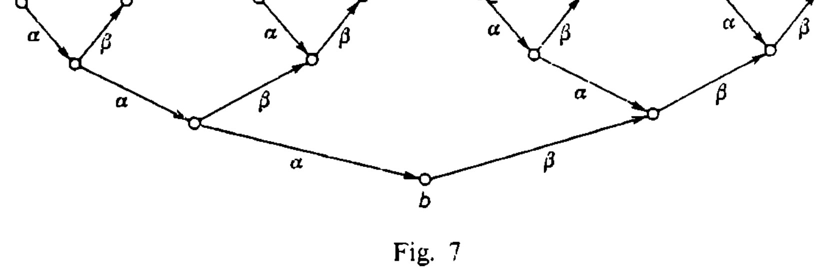


Fig. 7

Let C be a tame concealed category with a $P_1(k)$-tubular family T_λ^C, $\lambda \in P_1(k)$, and let r_λ be the rank of T_λ^C. Let $B = (B_1, \ldots, B_s)$ be a finite family of branches and $E = (E_1, \ldots, E_s)$ a finite sequence of pairwise nonisomorphic simple regular C-modules, that is, modules from the tubes which are not starting points of irreducible epimorphisms. The *tubular extension* $R = C[E, B]$ of C by E and B is the category whose objects are formed by the disjoint union of all objects of $C, B_1, \ldots, B_s$ and the morphism spaces are defined as follows: $R(x, y) = C(x, y)$ if $x, y \in C$, $R(x, y) = B_i(x, y)$ if $x, y \in B_i$, $R(x, y) = 0$ if $y \in B_j$, $x \in C \cup B_i$, $i \neq j$, and for $x \in B_i$, $y \in C$, we put $R(x, y) = E_i(y) \otimes_k B_i(x, b)$. The composition is the unique possible one. The *tubular type* $n_R = (n_\lambda)_{\lambda \in P_1(k)}$ of R is defined by

$$n_\lambda = r_\lambda + \sum_{E_i \in T_\lambda^C} |B_i|,$$

where $|B_i|$ denotes the cardinality of B_i. Since almost all n_λ are equal to 1, we shall write, instead of (n_λ), the finite sequence containing at least two n_λ, including all those which are larger than 1, arranged in nondecreasing order. The tubular extension $R = [E, B]$ is said to be a *Euclidean* (resp. *Ringel*) *category* if its tubular type n_R is one of the following: (p, q), $1 \leqslant p \leqslant q$, $(2, 2, m)$, $m \geqslant 2$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$ (resp. $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$, or $(2, 2, 2, 2)$). The associated algebra $\bigoplus R$ is then called a *Euclidean* (resp. *Ringel*) *algebra*. It follows from [Ri3, 4.9] that the class of Euclidean algebras coincides with the class of representation-infinite tilted algebras of Euclidean types having a complete slice in their preinjective component.

3.3. Let C be the hereditary algebra given by the quiver of Fig. 8. Then the

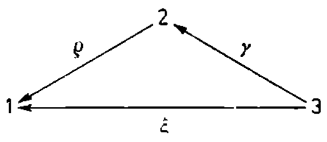


Fig. 8

tubular $P_1(k)$-family $\mathscr{T}^C$ consists of the tubes depicted in Fig. 9, where the vertical lines have to be identified in order to obtain a tube. Let $B = (B_1, B_2, B_3)$ be the family of branches

$$B_1 = \{b\}, \quad B_2 = \{b\}, \quad B_3 = \{c\},$$

where c is shown in Fig. 10, and $E = (E_1, E_2, E_3)$ (see Fig. 11) a sequence of simple regular C-modules lying respectively in the tubes T_∞^C, T_1^C, T_0^C.

Then $D = \bigoplus C[E, B]$ is a Euclidean algebra of type $n_D = (2, 3, 4)$ isomorphic to the bound quiver algebra kQ_D/I_D, where Q_D is the quiver of Fig. 12 and I_D is generated by $\beta\alpha$, $\gamma\sigma$, $\xi\eta$, $\xi\mu - \varrho\gamma\mu$.

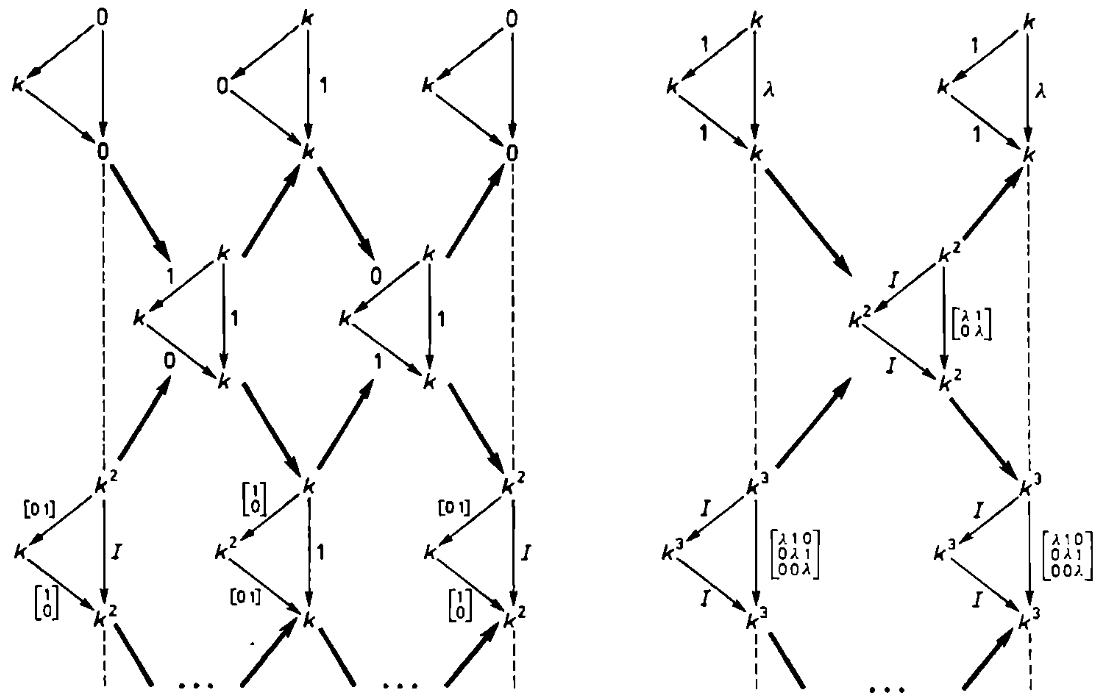


Fig. 9. T_∞^C and T_λ^C, $\lambda \in k$

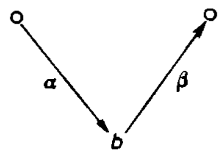


Fig. 10

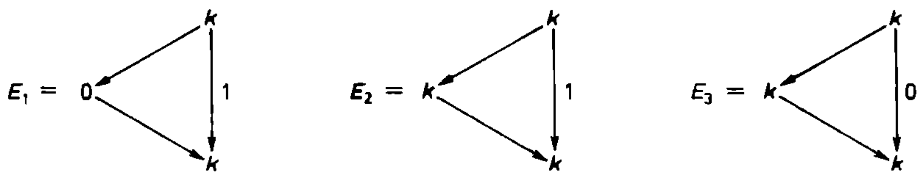


Fig. 11

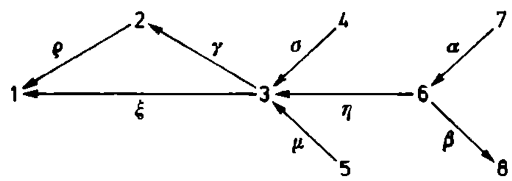


Fig. 12

Similarly, if $\bar{B} = (\bar{B}_1, \bar{B}_2, \bar{B}_3)$, where

$$\bar{B}_1 = \{\circ \to b\}, \quad \bar{B}_2 = \{b\}, \quad \bar{B}_3 = \{c\},$$

where c is shown in Fig. 10, and $E = (E_1, E_2, E_3)$ as above, then $S = \bigoplus C[E, \bar{B}]$ is a Ringel algebra of type (2, 4, 4). Moreover, S is isomorphic to the bound quiver algebra kQ_S/I_S, where Q_S is the quiver of Fig. 13 and I_S is generated by $\beta\alpha$, $\gamma\sigma$, $\xi\eta$, $\xi\mu - \varrho\gamma\mu$.

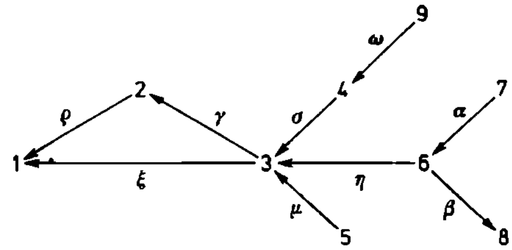


Fig. 13

3.4. Let A be a Euclidean or Ringel algebra. It follows from [Ri3] that $\operatorname{gl\,dim} A \leqslant 2$ and consequently the Tits form q_A coincides with the Euler characteristic λ_A. Moreover, by [Ri3], q_A is positive-semidefinite. The elements $x \in K_0(A) = \mathbf{Z}^n$ satisfying $q_A(x) = 0$, called *radical vectors*, form a subgroup $\operatorname{rad} q_A$ of $K_0(A)$, the radical of q_A. The rank of $\operatorname{rad} q_A$ is called the *radical rank* of q_A. An element $x \in K_0(A)$ such that $q_A(x) = 1$ is called a *root* of q_A. Finally, an element $x \in K_0(A) = \mathbf{Z}^n$ is positive provided $x \neq 0$ and $x_i \geqslant 0$ for all i. The following theorem shows that positive roots and positive radical vectors are convenient invariants in classifying the indecomposable (finite-dimensional) A-modules.

THEOREM [Ri3]. *Let A be a Euclidean or Ringel algebra. Then the indecomposable finite-dimensional A-modules are controlled by q_A, that is, the following conditions are satisfied*:

(α) *For any indecomposable finite-dimensional A-module X, the dimension vector $\underline{\dim} X$ of X is either a connected positive root or a connected positive radical vector of q_A.*

(β) *For any connected positive root x of q_A in $K_0(A)$, there is precisely one isomorphism class of indecomposable A-modules X satisfying $\underline{\dim} X = x$.*

(γ) *For any connected positive radical vector x of q_A in $K_0(A)$, there is an infinite, one-parameter, family of isomorphism classes of indecomposable A-modules X satisfying $\underline{\dim} X = x$.*

3.5. Let $A = \bigoplus C[E, B]$ be a Euclidean algebra. Then from [Ri3, 4.9] the Auslander–Reiten quiver Γ_A of A is the disjoint union

$$\Gamma_A = \mathscr{P}^A \vee \mathscr{T}^A \vee \mathscr{Q}^A,$$

where $\mathscr{P}^A = \mathscr{P}^C$ is a preprojective component, $\mathscr{Q}^A$ is a preinjective component with a complete Euclidean slice, and $\mathscr{T}^A = (T_\lambda^A)_{\lambda \in P_1(k)}$ is a tubular $P_1(k)$-family obtained from the tubular family $\mathscr{T}^C = (T_\lambda^C)_{\lambda \in P_1(k)}$ by ray insertions in the tubes containing the simple regular modules E_i from $E = (E_1, \ldots, E_s)$. Moreover, q_A is positive-semidefinite with radical rank 1 and hence A is one-parametric. Finally, the indecomposable modules X with $\underline{\dim}\, X \in \operatorname{rad} q_A$ are C-modules.

For the Euclidean algebra D (3.3), the preprojective component $\mathscr{P}^D = \mathscr{P}^C$, of type $\tilde{A}_3$, is shown in Fig. 14, and the preinjective component $\mathscr{Q}^D$, with

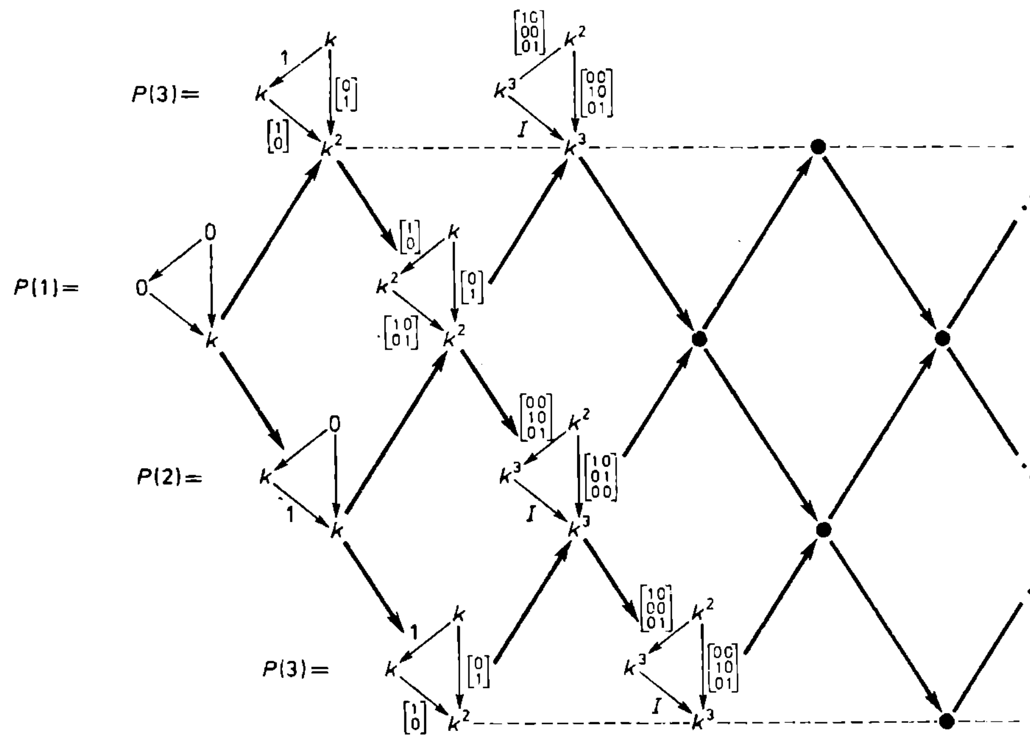


Fig. 14

a complete slice of type $\tilde{E}_7$, is illustrated in Fig. 15, where we replace the indecomposable modules by the dimension vectors. Finally, $T_\lambda^D = T_\lambda^C$ for $\lambda \neq 0, 1, \infty$, and T_∞^D (see Fig. 16) is obtained from T_∞^C by one ray insertion containing the module $P(4)$, T_0^D (Fig. 17) is obtained from T_0^C by one ray insertion containing $P(5)$, and T_1^D (Fig. 18) is obtained from T_1^C by three ray insertions containing the modules $P(8)$, $P(6)/P(8)$ and $P(7)$.

The radical vectors of q_D are as shown in Fig. 19, with $m \in \mathbf{Z}$.

3.6. Let now A be a Ringel algebra. Then there is exactly one full convex tame concealed subcategory $C = C_0$ of the category R_A such that $A = \bigoplus C_0[E, B]$ for a sequence $E = (E_1, \ldots, E_s)$ of pairwise nonisomorphic simple regular C_0-modules and a sequence $B = (B_1, \ldots, B_s)$ of branches. Let h_0 be the smallest positive (connected) radical vector of q_{C_0}. Further, there is, by

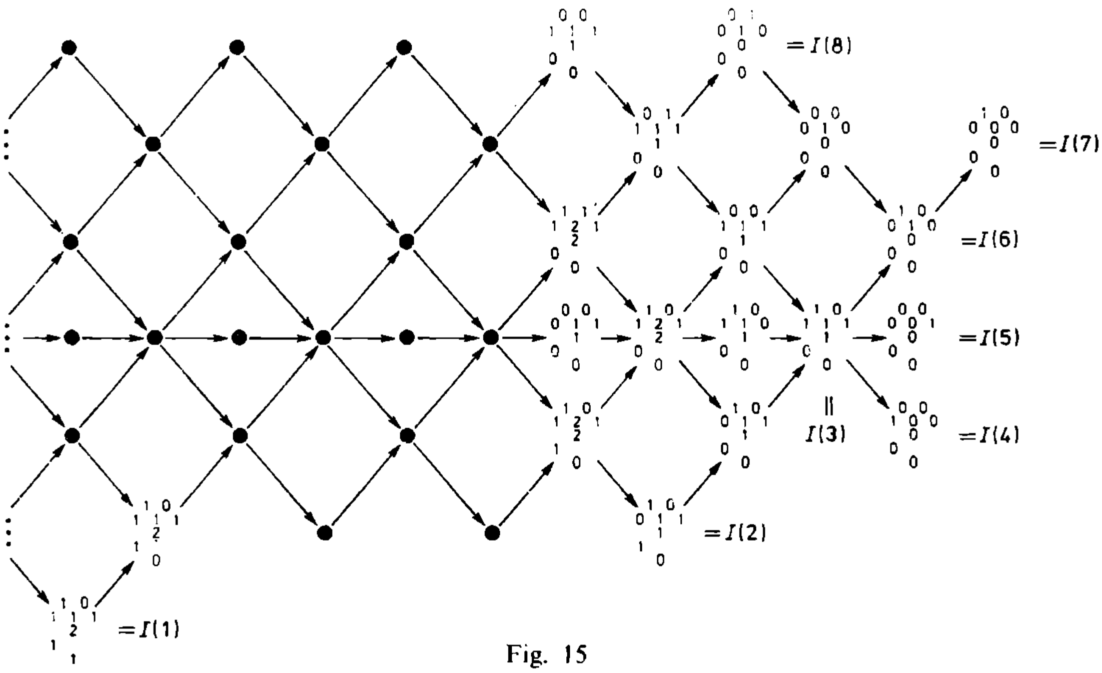


Fig. 15

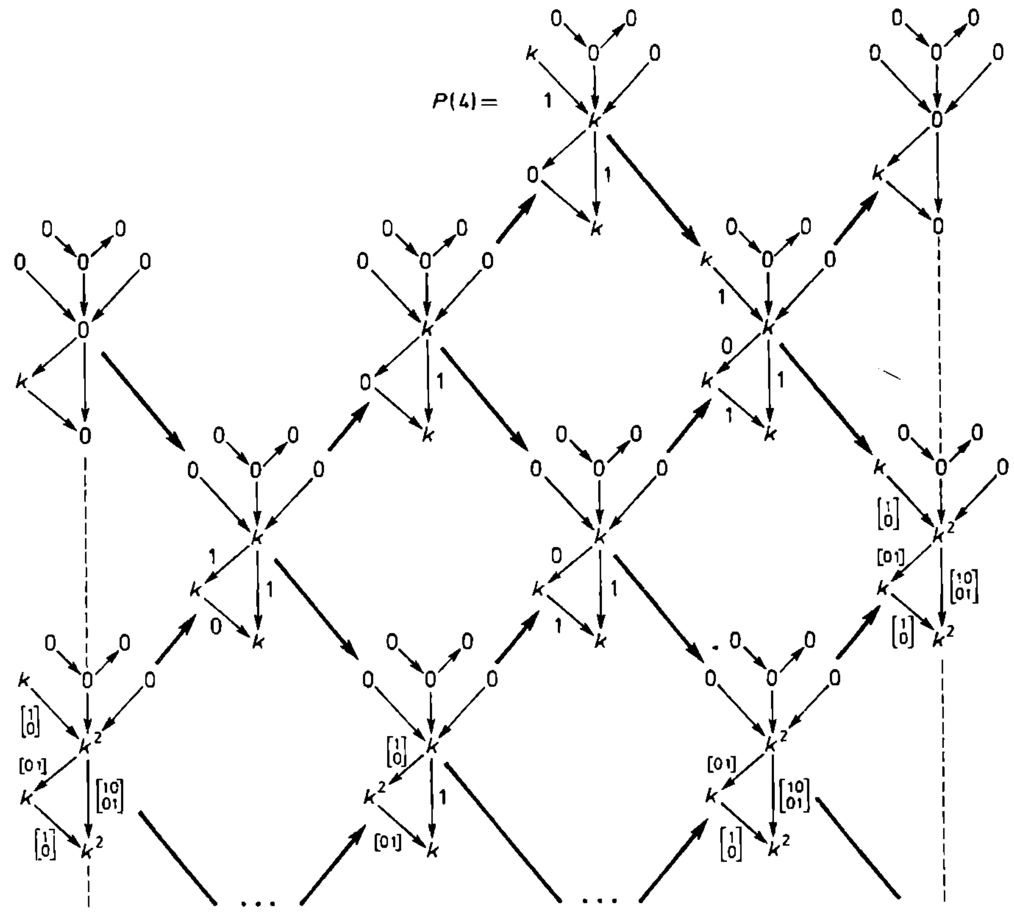


Fig. 16. T^D_∞

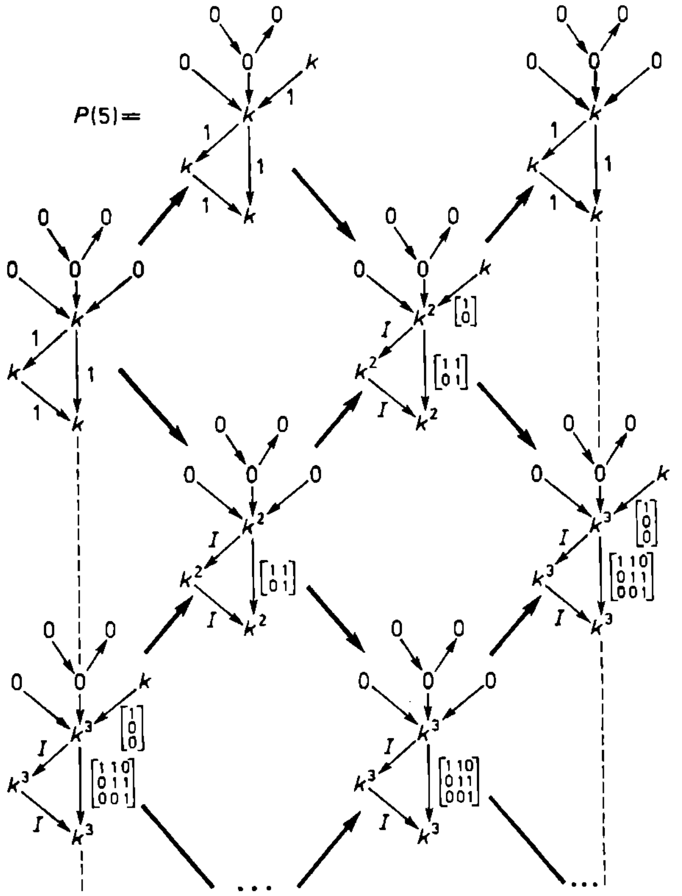


Fig. 17. T_0^D

[Ri3; 5.2], a full convex tame concealed subcategory C_∞ of R_A (uniquely determined by A) such that $A = \bigoplus[B', E']C_\infty$ is a tubular coextension (the dual procedure to the tubular extension) of C_∞ for some sequence $E' = (E'_1, \ldots, E'_r)$ of pairwise nonisomorphic simple regular C_∞-modules and a sequence of branches $B' = (B'_1, \ldots, B'_r)$. Let h_∞ be the smallest positive radical vector of q_C. Then from [Ri3, 5.2], Γ_A is the disjoint union

$$\mathscr{P}^A \vee \mathscr{T}_0^A \vee \bigvee_{\gamma \in Q_0^+} \mathscr{T}_\gamma^A \vee \mathscr{T}_\infty^A \vee \mathscr{Q}^A,$$

where $\mathscr{P}^A = \mathscr{P}^{C_0}$ is a preprojective component, $\mathscr{Q}^A = \mathscr{Q}^{C_\infty}$ is a preinjective component, $\mathscr{T}_0^A$ is a tubular $P_1(k)$-family obtained from $\mathscr{T}^{C_0}$ by a finite number of ray insertions in the tubes containing the modules $E_1, \ldots, E_s$, $\mathscr{T}_\infty^A$ is a tubular $P_1(k)$-family obtained from $\mathscr{T}^{C_\infty}$ by a finite number of coray insertions in the tubes containing the modules $E'_1, \ldots, E'_r$, and each $\mathscr{T}_\gamma^A$, $\gamma \in \mathbf{Q}_0^+$, where $\mathbf{Q}_0^+$ is the set of all positive rationals, is a stable tubular $P_1(k)$-family of tubular type n_A. The form q_A is positive-semidefinite of radical rank 2, $\mathbf{Z}h_0 + \mathbf{Z}h_\infty$ is a subgroup of $\operatorname{rad} q_A$ of finite index, $h_0 + h_\infty$ is a sincere positive

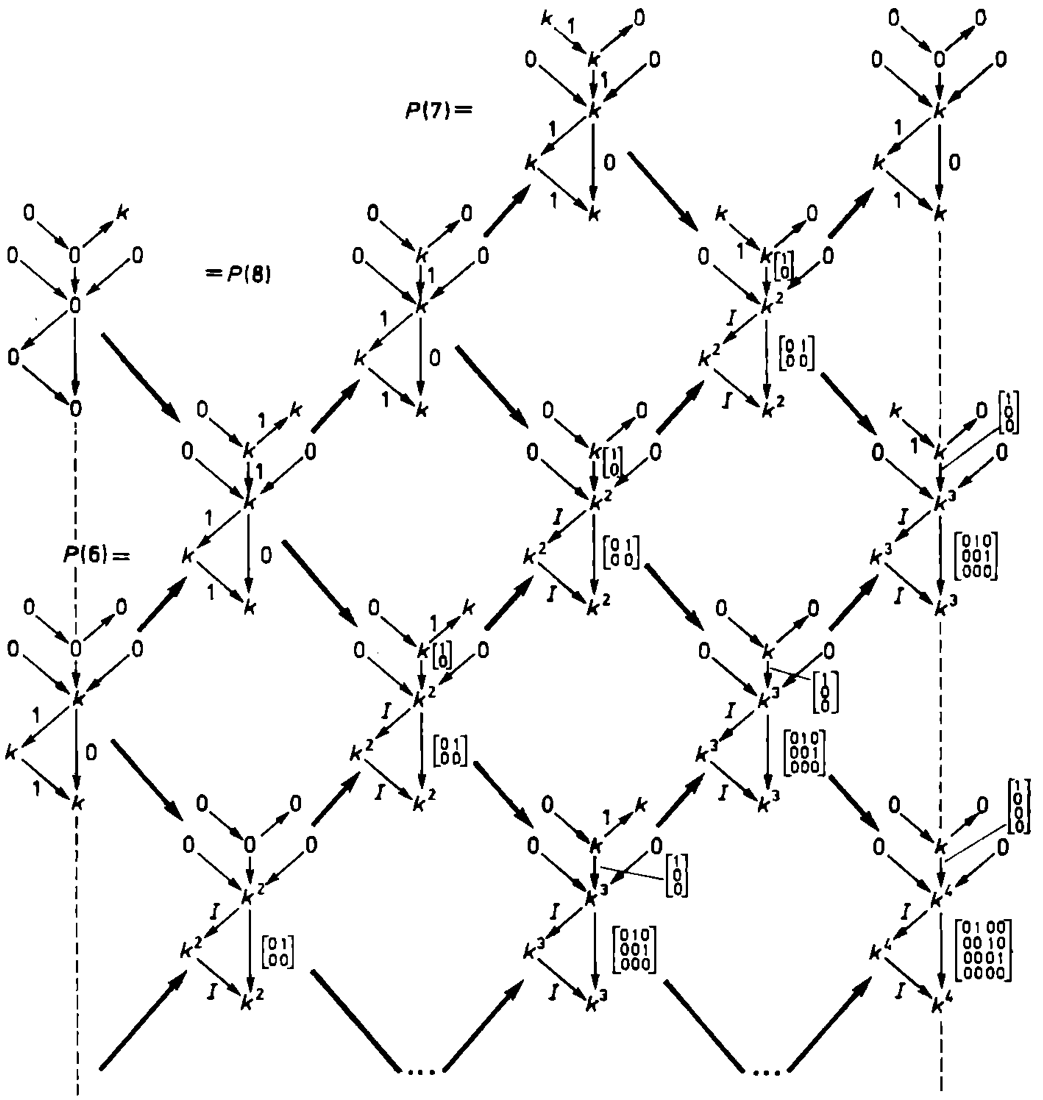


Fig. 18. T_1^D

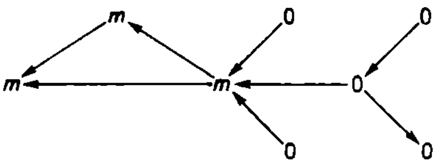


Fig. 19

radical vector and the nonsincere positive radical vectors are positive multiples of h_0 and h_∞. Every connected positive radical vector x of q_A is of the form $x = \alpha_0 h_0 + \alpha_\infty h_\infty$, where α_0, α_∞ are nonnegative rationals, and, if $\alpha_0 = p_0/r_0$, $(p_0, r_0) = 1$, $\alpha_\infty = p_\infty/r_\infty$, $(p_\infty, r_\infty) = 1$, then $r_0 \leqslant$ the maximal coordinate of h_0 and $r_\infty \leqslant$ the maximal coordinate of h_∞. For each $\gamma \in \mathbf{Q}_0^+ \cup \{0, \infty\}$, the tubes of rank 1 in $\mathcal{T}_\gamma^A$ consist of indecomposable modules X with $\underline{\dim}\, X = \alpha_0 h_0 + \alpha_\infty h_\infty$ and $\gamma = \alpha_\infty/\alpha_0$ (the index of X). Then $\mu_A(d)$ is the cardinality of the set consisting of all $\gamma \in \mathbf{Q}_0^+ \cup \{0, \infty\}$ such that the tubes of rank 1 in $\mathcal{T}_\gamma^A$ contain indecomposable modules of dimension d. Hence $\mu_A(d) \leqslant m_A d$, where m_A denotes the maximum of the coordinates of h_0 and h_∞. Therefore $\mu_A(d) \leqslant 2d$, $3d$, $4d$, $6d$ whenever $n_A = (2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ respectively. Thus we have proved the second part of the lemma below.

LEMMA. *A is nondomestic of linear growth.*

Proof. In the above notation, for each prime p, there are infinitely many pairwise nonisomorphic indecomposable A-modules X with $\underline{\dim}\, X = h_0 + p h_\infty$. Suppose that A is domestic and let $F_1, \ldots, F_n$: $\operatorname{mod} k[x] \to \operatorname{mod} A$ be a family of functors satisfying the corresponding conditions (i) and (ii′). Let $\underline{\dim}\, F_i(k[x]/(x-\lambda)) = r_i$, $\lambda \in k$, $1 \leqslant i \leqslant n$. Thus, $\underline{\dim}\, F_i(k[x]/(x-\lambda)^m) = m r_i$. From our assumption, there are prime numbers $p \neq q$, $1 \leqslant i \leqslant n$, and $a, b \in \mathbf{N}$ such that $h_0 + p h_\infty = a r_i$ and $h_0 + q h_\infty = b r_i$. Since h_0 and h_∞ are not sincere, this implies that $a r_i = b r_i$ and consequently $p = q$, a contradiction. Therefore, A is nondomestic.

Let S be the Ringel algebra of type $(2, 4, 4)$ considered in 3.3. Then $S = \bigoplus C_0[E, \bar{B}] = \bigoplus [E', B']C_\infty$, where $C_0 = C$, E, B are as in 3.3, C_∞ is the tame concealed category kQ'/I' of type $(2, 3, 4)$ given by the quiver Q' of Fig. 20

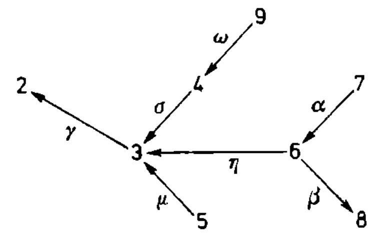


Fig. 20

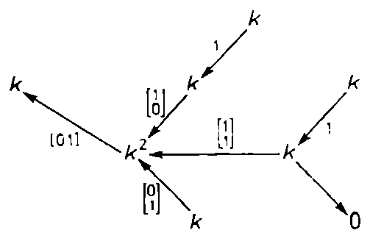


Fig. 21

and the ideal I' generated by $\gamma\sigma$ and $\beta\alpha$, B' consists of one branch $\{b\}$, and E' consists of one simple regular C_∞-module depicted in Fig. 21, lying in the tube of rank 3 in $\mathcal{T}^{C_\infty}$ shown in Fig. 22, where we replace any module by its dimension vector, and the vertical lines have to be identified in order to obtain

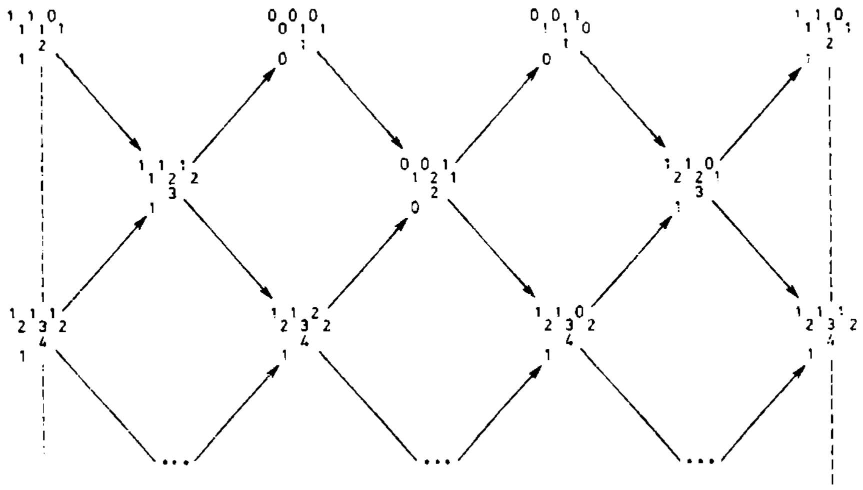


Fig. 22

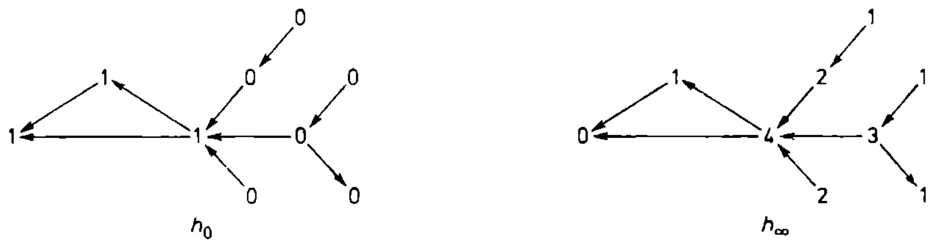


Fig. 23

a tube. Hence, in our example, h_0 and h_∞ are as shown in Fig. 23. Moreover, $\mathscr{P}^S = \mathscr{P}^D = \mathscr{P}^C$, $\mathscr{Q}^S = \mathscr{Q}^{C_\infty}$, $\mathscr{T}_0^S$ is obtained from $\mathscr{T}^D$ by one ray insertion in the tube T_∞^D, this changed tube is shown in Fig. 24, and $\mathscr{T}_\infty^S$ is obtained from $\mathscr{T}^{C_\infty}$ by one coray insertion in the described tube of rank 3, and this changed tube is illustrated in Fig. 25.

Remark. The Euclidean algebras of type $\neq (p, q)$ and all Ringel algebras are simply connected. It is expected that every simply connected algebra of polynomial growth can be obtained from such algebras by extensions and glueings.

3.7. There is a well-known conjecture that a simply connected algebra A is tame if and only if the Tits form q_A is weakly nonnegative, that is, $q_A(x) \geq 0$ for all positive vectors $x \in K_0(A)$. We now pose the corresponding problems concerning simply connected algebras of polynomial growth.

Let A be a simply connected algebra and $q = q_A$. Denote by $\mathrm{rad}^+ q$ the set of all nonnegative vectors x from $K_0(A) \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}^n$ such that $q(x) = 0$. A subset V of $\mathrm{rad}^+ q$ is said to be a *half space* provided $ax + by \in V$ for all $x, y \in V$ and $a, b \in \mathbf{Q}^+$ (the set of all nonnegative rationals). The *dimension* $\dim V$ of a half space V is the maximal number of linearly independent (over $\mathbf{Q}$)

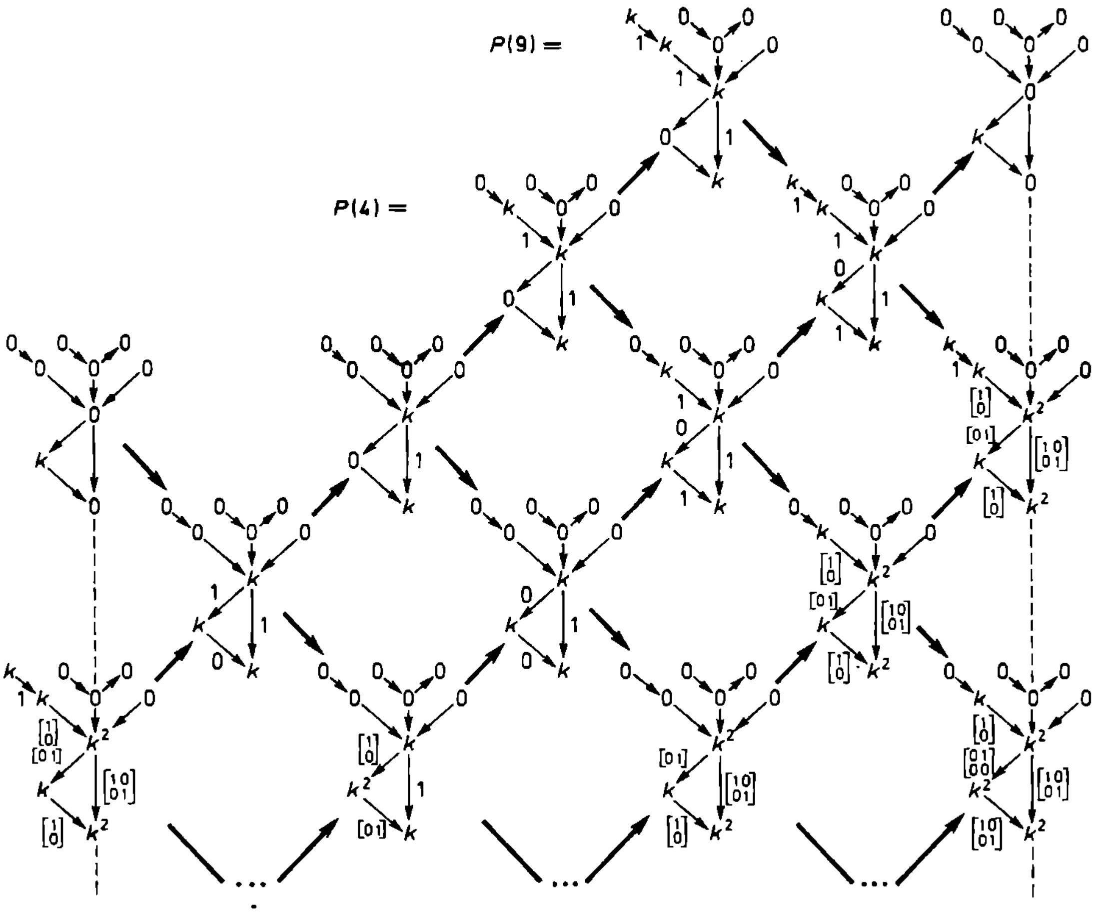


Fig. 24

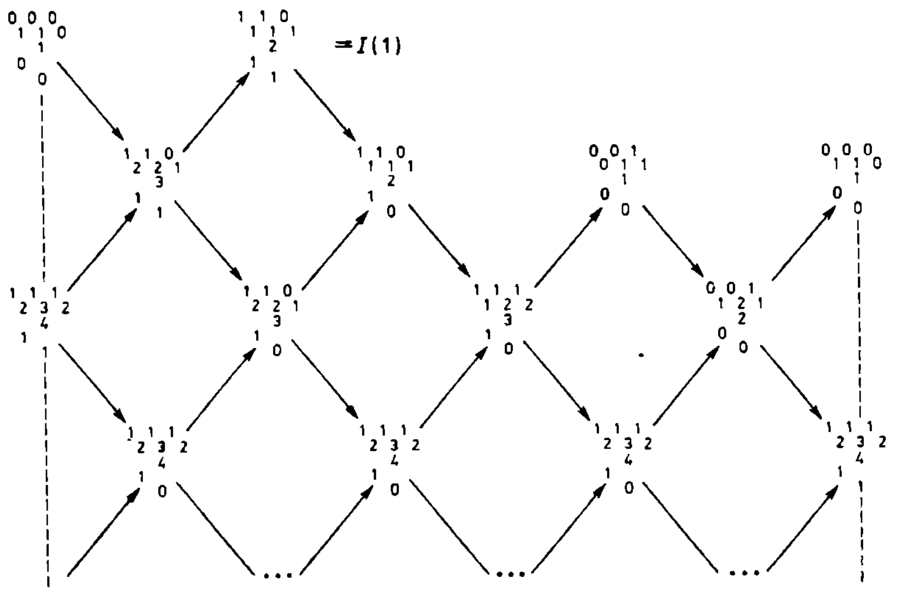


Fig. 25

vectors of V. Finally, the *rank* r_A^+ of $\operatorname{rad}^+ q_A$ is the maximum of the dimensions $\dim V$ of all half spaces V of $\operatorname{rad}^+ q_A$.

PROBLEM 3. *Prove that a simply connected algebra A is domestic if and only if $r_A^+ \leqslant 1$.*

The corresponding problem for partially ordered sets is solved in [NZ].

PROBLEM 4. *Find a characterization of simply connected algebras of polynomial growth in terms of the Tits form.*

4. Galois coverings of tame algebras

In this section we describe Galois covering techniques playing an important rôle in the study of algebras of polynomial growth.

4.1. Let R be a locally bounded category and let G be a group of k-linear automorphisms of R. We assume that the action of G on R is free, that is, $gx \neq x$ for each object x of R and $g \neq 1$ in G. Following Gabriel [G] consider the quotient category R/G. The objects of R/G are the G-orbits of G in the set of objects of R. A morphism f: $a \to b$ of R/G is a family $f = ({}_yf_x) \in \prod R(x, y)$, where x, y range over a, b respectively and f satisfies the relation $g({}_yf_x) = {}_{gy}f_{gx}$ for all g and all x, y. The composition ef of f: $a \to b$ and e: $b \to c$ in R/G is defined by ${}_zef_x = \sum_{y \in b} {}_ze_y\, {}_yf_x$; this sum makes sense since R is locally bounded. The canonical projection F: $R \to R/G$ which assigns to each object x of R its G-orbit Gx and to each morphism $\xi \in R(x, y)$ the family $F\xi$ such that ${}_{hy}F\xi_{gx} = g\xi$ or 0 according as $h = g$ or $h \neq g$, is called the *Galois covering* of $\Lambda = R/G$ with group G. We have the following isomorphisms induced by F:

$$\bigoplus_{Fy=a} R(x, y) \xrightarrow{\sim} \Lambda(Fx, a), \qquad \bigoplus_{Fy=a} R(y, x) \xrightarrow{\sim} \Lambda(a, Fx).$$

EXAMPLES. (1) Let R_1 be the bound quiver category kQ'/I', where Q' is the quiver

$$\cdots \leftleftarrows -2 \underset{\beta_{-2}}{\overset{\alpha_{-2}}{\leftleftarrows}} -1 \underset{\beta_{-1}}{\overset{\alpha_{-1}}{\leftleftarrows}} 0 \underset{\beta_{0}}{\overset{\alpha_{0}}{\leftleftarrows}} 1 \underset{\beta_{1}}{\overset{\alpha_{1}}{\leftleftarrows}} 2 \leftleftarrows \cdots$$

and I' is generated by $\alpha_{i-1}\alpha_i$, $\beta_{i-1}\beta_i$, $\alpha_{i-1}\beta_i - \beta_{i-1}\alpha_i$, $i \in \mathbf{Z}$. Let G be the infinite cyclic group of k-linear automorphisms of R_1 generated by the shift g: $g(i) = i+1$, $g(\alpha_i) = \alpha_{i+1}$, $g(\beta_i) = \beta_{i+1}$. Then $\Lambda_1 = R_1/G$ is the bound quiver category kQ/I, where Q is the quiver

$$\underset{\alpha}{\circlearrowleft} \circ \underset{\beta}{\circlearrowright}$$

and I is generated by α^2, β^2 and $\alpha\beta - \beta\alpha$. Observe that $\bigoplus \Lambda_1$ is isomorphic to $k[x, y]/(x^2, y^2)$.

(2) Let $R_2 = k\tilde{Q}/\tilde{I}$, where $\tilde{Q}$ is the quiver

$$\begin{tikzcd}
\cdots & \underline{-2} \arrow[l] \arrow[d, "\gamma_{-2}"] & \underline{-1} \arrow[l, "\alpha_{-2}"'] \arrow[d, "\gamma_{-1}"] & \underline{0} \arrow[l, "\alpha_{-1}"'] \arrow[d, "\gamma_{0}"] & \underline{1} \arrow[l, "\alpha_{0}"'] \arrow[d, "\gamma_{1}"] & \underline{2} \arrow[l, "\alpha_{1}"'] \arrow[d, "\gamma_{2}"] & \cdots \arrow[l] \\
\cdots & -2 \arrow[l] & -1 \arrow[l, "\beta_{-2}"] & 0 \arrow[l, "\beta_{-1}"] & 1 \arrow[l, "\beta_{0}"] & 2 \arrow[l, "\beta_{1}"] & \cdots \arrow[l]
\end{tikzcd}$$

and $\tilde{I}$ is generated by $\alpha_i\alpha_{i+1}\alpha_{i+2}\alpha_{i+3}$, $\beta_i\beta_{i+1}\beta_{i+2}\beta_{i+3}$, $\gamma_i\alpha_i-\beta_i\gamma_{i+1}$, $i\in\mathbf{Z}$. Let G be the infinite cyclic group generated by the shift g: $g(\underline{i}) = \underline{i+1}$, $g(i) = i+1$, $g(\alpha_i) = \alpha_{i+1}$, $g(\beta_i) = \beta_{i+1}$, $g(\gamma_i) = \gamma_{i+1}$. The quotient category $\Lambda_2 = R_2/G$ is the bound quiver category kQ/I given by the quiver Q

$$\begin{tikzcd}
\circ \arrow[loop left, "\alpha"] \arrow[r, "\gamma"] & \circ \arrow[loop right, "\beta"]
\end{tikzcd}$$

and the ideal $I = \langle\alpha^4, \beta^4, \gamma\alpha-\beta\gamma\rangle$, and $\bigoplus \Lambda_2 \simeq T_2(k[x]/(x^4))$ (cf. 2.5).

(3) Let $R_3 = k\tilde{Q}/\tilde{I}$, where $\tilde{Q}$ is the quiver of Fig. 26 and $\tilde{I}$ is generated by

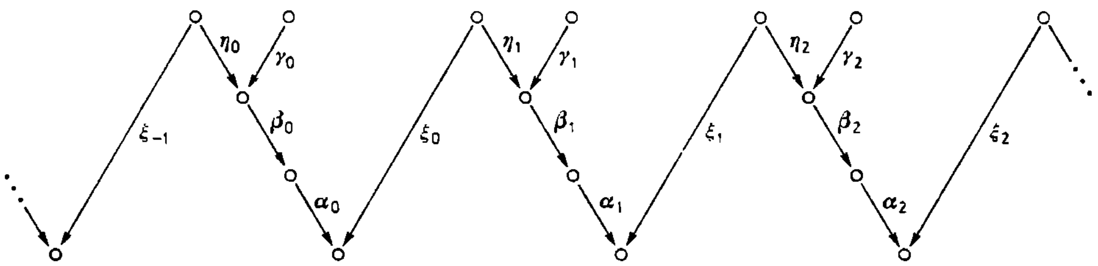


Fig. 26

$\alpha_i\beta_i\gamma_i$, $i\in\mathbf{Z}$, and let G be the infinite cyclic group generated by the shift g: $g(\alpha_i) = \alpha_{i+1}$, $g(\beta_i) = \beta_{i+1}$, $g(\gamma_i) = \gamma_{i+1}$, $g(\eta_i) = \eta_{i+1}$ and $g(\xi_i) = \xi_{i+1}$. Then $\Lambda_3 = R_3/G$ is the bound quiver category kQ/I, where Q is the quiver of Fig. 27 and I is generated by $\alpha\beta\gamma$.

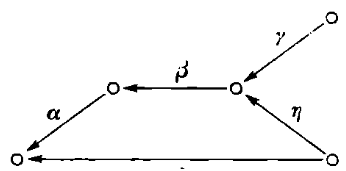


Fig. 27

(4) Let $R_4 = k\tilde{Q}/\tilde{I}$, where $\tilde{Q}$ is the quiver of Fig. 28 and $\tilde{I}$ is generated by all paths α^2, β^2, $\alpha\beta\alpha$, $\beta\alpha\beta$. Let G be the free (nonabelian) group of k-linear automorphisms of R_4 generated by the α-shift and β-shift. Then $\Lambda_4 = R_4/G$ is the bound quiver category kQ/I, where Q is the quiver

$$\begin{tikzcd}
\circ \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"]
\end{tikzcd}$$

and I is generated by α^2, β^2, $\alpha\beta\alpha$, $\beta\alpha\beta$ (cf. 2.6).

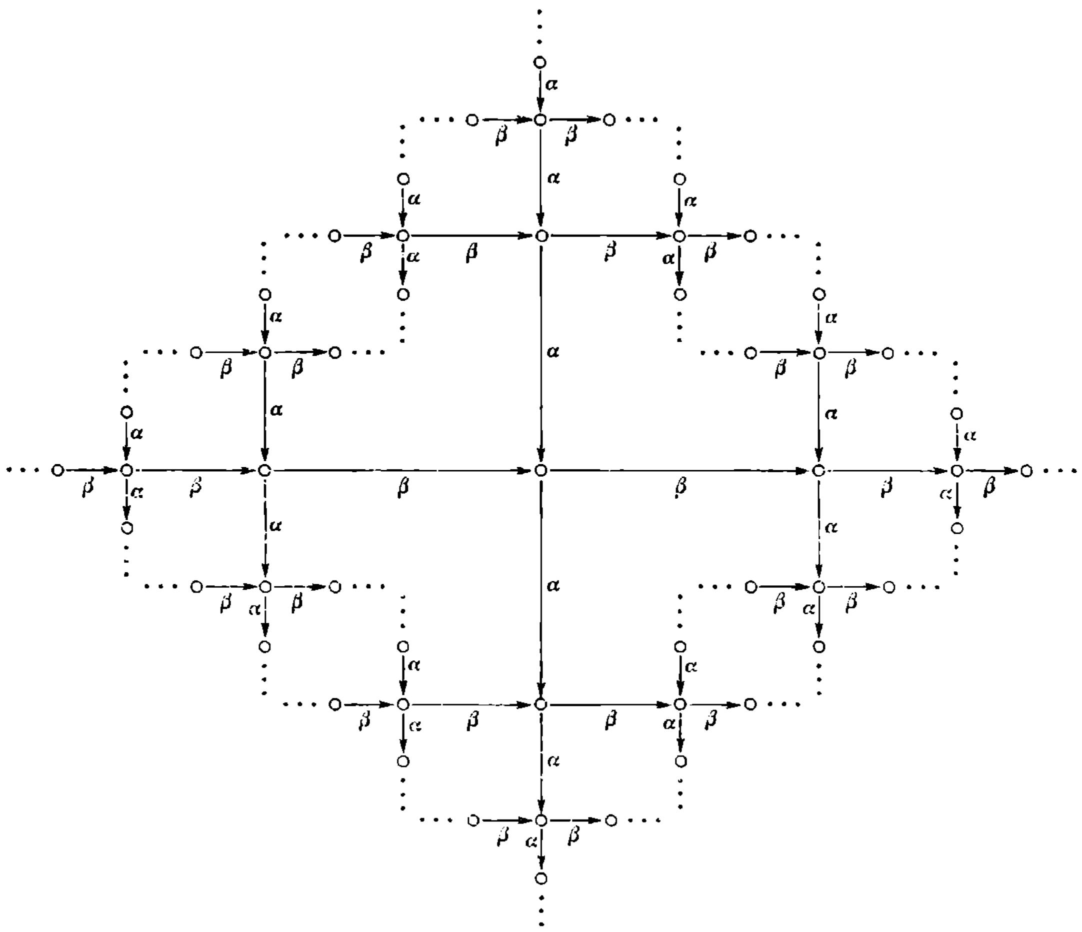


Fig. 28

4.2. Let $\Lambda = kQ/I$ be a locally bounded category and assume that the fundamental group (1.3) $\Pi_1(Q, I) = \Pi_1(Q, x_0)/N(Q, m(I), x_0)$ of (Q, I) is nontrivial. We now construct a Galois covering

$$F\colon R = k\tilde{Q}/\tilde{I} \to \Lambda = kQ/I$$

with group $G = \Pi_1(Q, I)$ called the *universal Galois covering* of kQ/I (see [Gr], [MP]).

Let W be the topological universal cover of Q with base point at x_0, that is, the vertices of W are the homotopy classes of walks starting at x_0 and for two such vertices y_1, y_2 we set $y_1 \to y_2$ in W if and only if there exist representatives w_1, w_2 of the classes y_1, y_2 respectively and an arrow α in Q with $w_2 = \alpha w_1$. There is a natural (left) action of $\Pi_1(Q, x_0)$ on W. Then the normal subgroup $N(Q, m(I), x_0)$ acts on W and we denote by $\tilde{Q}$ the orbit quiver $W/N(Q, m(I), x_0)$. Moreover, the induced action of $\Pi_1(Q, I)$ on $\tilde{Q}$ gives a map $p\colon \tilde{Q} \to Q$. Thus we have a functor $p\colon k\tilde{Q} \to kQ$. Finally, we denote by $\tilde{I}$ the ideal in $k\tilde{Q}$ generated by all liftings, through p, of generators of I. Consequently, we obtain the required Galois covering

$$F\colon R = k\tilde{Q}/\tilde{I} \to \Lambda = kQ/I$$

with group $\Pi_1(Q, I)$.

In examples (2), (3), (4) of 4.1, $R = k\tilde{Q}/\tilde{I} = R_i$ for $\Lambda = kQ/I = \Lambda_i$, $i = 2, 3, 4$. On the other hand, for $\Lambda = \Lambda_1 = kQ/I$ in (1), $R = k\tilde{Q}/\tilde{I}$ is different from R_1. Indeed, $\tilde{Q}$ is the quiver of Fig. 29 and $\tilde{I}$ is generated by all paths α^2, β^2

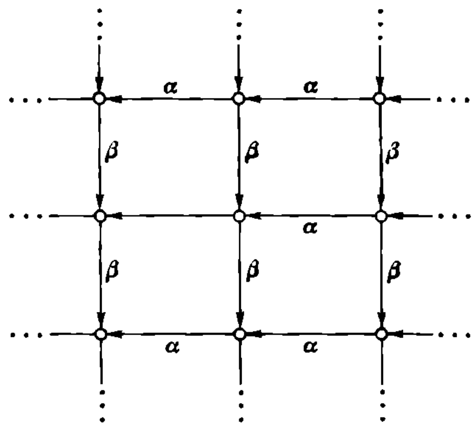


Fig. 29

and $\alpha\beta - \beta\alpha$. Here $\Lambda_1 = R/H$, where $H = \Pi_1(Q, I)$ is the free abelian group generated by the horizontal α-shift g and the vertical β-shift h. Moreover, $R_1 = R/(gh^{-1})$ is not simply connected but R is simply connected. Observe also that the categories R_2, R_3 and R_4 are simply connected.

LEMMA. *Let Λ be a triangular locally bounded category. Then Λ is simply connected if and only if Λ does not admit a proper Galois covering.*

Proof. Suppose that Λ is not simply connected. Then there is a presentation $\Lambda \simeq kQ/I$ of Λ with $\Pi_1(Q, I)$ nontrivial. Hence the above-constructed Galois covering $R = k\tilde{Q}/\tilde{I} \to kQ/I = \Lambda$ with group $\Pi_1(Q, I)$ is proper. Conversely, let $F\colon R \to \Lambda$ be a Galois covering with nontrivial group G. Then, by [Gr], Λ is a G-graded category. Take a presentation $f\colon kQ/I \xrightarrow{\sim} \Lambda$ of Λ such that $f(\alpha)$ is G-homogeneous for any arrow α in Q. Then there is a presentation $k\bar{Q}/\bar{I} \simeq R$ of R such that F is induced by a morphism $(\bar{Q}, \bar{I}) \to (Q, I)$ of bound quivers [Gr]. Further, by [MP], the Galois covering $k\tilde{Q}/\tilde{I} \to kQ/I$ factors through F and hence $\Pi_1(Q, I)$ is nontrivial. Therefore Λ is not simply connected and we are done.

We now propose the following definition (see [S3]). A locally bounded category Λ is said to be *standard* provided Λ admits a Galois covering $R \to \Lambda$ with R simply connected. If Λ is (locally) representation-finite, this notion coincides with that introduced in [BoG].

PROBLEM 5. *Let R be a simply connected tame locally bounded category and let $R \to R/G = \Lambda$ be a Galois covering with Λ bounded. Is G torsion-free?*

Recall that, in case R is locally representation-finite and simply connected, G is a free group [BoG], [BrG], [MP].

4.3. Let R be a locally bounded category. An R-module M is called *locally finite-dimensional* if $\dim_k M(x)$ is finite for all objects x of R. We denote by $\operatorname{MOD} R$ the category of all R-modules, by $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) the category of all locally finite-dimensional (resp. finite-dimensional) R-modules, by $\operatorname{Ind} R$ (resp. $\operatorname{ind} R$) the full subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) formed by all indecomposable objects, and by $\operatorname{Ind} R/\simeq$ (resp. $\operatorname{ind} R/\simeq$) the set of isoclasses of objects in $\operatorname{Ind} R$ (resp. $\operatorname{ind} R$).

Let G be a group of k-linear automorphisms of R. The group G acts on $\operatorname{MOD} R$ by the translations ${}^g(-)$ which assign to each $M \in \operatorname{MOD} R$ the R-module ${}^gM = M \circ g^{-1}$. For each $M \in \operatorname{MOD} R$ we denote by G_M the stabilizer $\{g \in G \mid {}^gM \simeq M\}$. *Throughout this section we assume that G acts freely on* $\operatorname{ind} R/\simeq$. Observe that this is true in case G is torsion-free.

Let F: $R \to R/G$ be a Galois covering and F_λ: $\operatorname{MOD} R \to \operatorname{MOD} R/G$ the *push-down functor* [BoG] which assigns to each $M \in \operatorname{MOD} R$ the module $F_\lambda M \in \operatorname{MOD} R/G$ defined as follows: For each object $a \in R/G$, we set

$$(F_\lambda M)(a) = \bigoplus_{x \in a} M(x),$$

where x ranges over all objects x of R such that $F(x) = a$; if $a \xrightarrow{\alpha} b$ is a morphism of R/G, the map $(F_\lambda M)(\alpha)$: $(F_\lambda M)(a) \to (F_\lambda M)(b)$ assigns to $(\mu_x) \in \bigoplus_{x \in a} M(x)$ the family $\left(\sum_x M({}_x\alpha_y)(\mu_x)\right) \in \bigoplus_{y \in b} M(y)$, where ${}_x\alpha_y$ is determined by $\sum_{y \in b} F({}_x\alpha_y) = \alpha$.

THEOREM [G]. *The functor F_λ induces an injection from the set* $(\operatorname{ind} R/\simeq)/G$ *of G-orbits in* $\operatorname{ind} R/\simeq$ *into* $(\operatorname{ind} R/G)/\simeq$.

4.4. Let $\operatorname{ind}_1 R/G$ be the full subcategory of $\operatorname{ind} R/G$ consisting of all objects isomorphic to $F_\lambda M$ for some $M \in \operatorname{ind} R$, called indecomposable R/G-modules of the *first kind* [DS3], and $\operatorname{ind}_2 R/G$ the full subcategory of $\operatorname{ind} R/G$ formed by the remaining indecomposables, called indecomposable R/G-modules of the *second kind.* The category R is called *G-exhaustive* if $\operatorname{ind} R = \operatorname{ind}_1 R$. Denote by $F_.$: $\operatorname{MOD} R/G \to \operatorname{MOD} R$ the *pull-up functor* which assigns to each $N \in \operatorname{MOD} R/G$ the module $F_.N = N \circ F \in \operatorname{MOD} R$. The *support* $\operatorname{supp} M$ of $M \in \operatorname{MOD} R$ is the full subcategory of R formed by all objects $x \in R$ such that $M(x) \neq 0$. A module $Y \in \operatorname{Ind} R$ is called *weakly G-periodic* [DS3] if $\operatorname{supp} Y$ is infinite and $\operatorname{supp} Y/G_Y$ is finite. We have the following characterization of indecomposable R/G-modules of the first and second kind.

PROPOSITION [DS3]. *For $X \in \operatorname{ind} R/G$ the following conditions hold*:

(1) *$X \in \operatorname{ind}_1 R/G$ if and only if $F_.X \simeq \bigoplus_{i \in I} Z_i$, where all $Z_i \in \operatorname{ind} R$.*

(2) *$X \in \mathrm{ind}_2 R/G$ if and only if $F_{.}X \simeq \bigoplus_{i\in I} Y_i$, where all Y_i are weakly G-periodic.*

As a consequence we have the following corollaries.

COROLLARY 1 [DS3]. *If G acts freely on $\mathrm{Ind}\, R/\simeq$, then R is G-exhaustive.*

COROLLARY 2 [DS3]. *If R admits a weakly G-periodic module, then R is not G-exhaustive.*

Now we formulate a handy sufficient condition for R to be G-exhaustive. For each object x of R, denote by $R(x)$ the full subcategory of R consisting of the objects of all $\mathrm{supp}\, M$, where $M \in \mathrm{ind}\, R$ is such that $M(x) \neq 0$. Then, following [DS2], R is called *locally support-finite* if $R(x)$ is finite for all $x \in R$. Observe that, if R is locally representation-finite (for each $x \in R$, there are only finitely many isoclasses of $M \in \mathrm{ind}\, R$ with $M(x) \neq 0$), then R is locally support-finite.

THEOREM [DS3]. *Let R be a locally support-finite category. Then $\mathrm{Ind}\, R = \mathrm{ind}\, R$. In particular, R is G-exhaustive for any group G of k-linear automorphisms of R acting freely on $\mathrm{ind}\, R/\simeq$. Moreover, in this case:*

(1) *R/G is tame (resp. domestic, of polynomial growth) if and only if so is R.*

(2) *$\Gamma_R/G \simeq \Gamma_{R/G}$.*

The category R_1 (4.1) is locally support-finite and, for each $n \in \mathbf{Z}$, the category $R_1(n)$ is the full subcategory of R_1 consisting of the objects $n-1$, n, $n+1$. Hence the algebra Λ_1 is one-parametric. Similarly, the support of any indecomposable finite-dimensional R_2-module is contained in one of the categories B_m, $m \in \mathbf{Z}$, where B_m denotes the full subcategory of R_2 formed by the objects $m+i$ and $\underline{m+i}$, $-3 \leqslant i \leqslant 3$ (for details we refer to [S1]). Hence R_2 is locally support-finite. Moreover, R_2 is nondomestic of linear growth, and consequently so is Λ_2.

PROBLEM 6. *Let R be a locally bounded category and let G be a group of k-linear automorphisms of R acting freely on $\mathrm{ind}\, R/\simeq$. Assume that R is G-exhaustive. Is R locally support-finite?*

4.5. A *line* in a locally bounded category R is a full convex subcategory L of R which is isomorphic to the path category of a linear quiver (of type A_n, A_∞ or ${}_\infty A_\infty$). A line L is *G-periodic* if its stabilizer $G_L = \{g \in G \mid gL = L\}$ is nontrivial. Let L be a G-periodic line in R. We associate with L a canonical indecomposable (weakly G-periodic) R-module B_L by setting $B_L(x) = k$ for $x \in L$, $B_L(x) = 0$ for $x \notin L$ and $B_L(\alpha) = \mathrm{id}_k$ for each arrow α in L. Then $G_{B_L} \simeq G_L$ is an infinite cyclic group and the group algebra kG_{B_L} is isomorphic to the algebra $k[T, T^{-1}]$ of Laurent polynomials. Moreover, $F_\lambda B_L$ is an R/G-$k[T, T^{-1}]$-bimodule such that, for each $x \in R$, $F_\lambda B_L(Gx)$ is a free $k[T, T^{-1}]$-module of rank $\sum_{y \in W_x} \dim_k B_L(y)$, where W_x is a set of representatives of the G_{B_L}-orbits in Gx. We denote by Φ^L the functor

$$F_\lambda B_L \otimes_{k[T,T^{-1}]}(-)\colon \operatorname{mod} k[T, T^{-1}] \to \operatorname{mod} R/G.$$

Let $\mathscr{L}$ be the set of all subcategories supp Y, where Y ranges over all weakly G-periodic R-modules, and let $\mathscr{L}_0$ be a fixed set of representatives of the G-orbits in $\mathscr{L}$.

THEOREM [DS3]. *Let R be a locally bounded category and G a group of k-linear automophisms of R acting freely on* $\operatorname{ind} R/\simeq$. *Assume that $\mathscr{L}$ consists only of lines in R. Then:*

(i) *Every indecomposable module from* $\operatorname{ind}_2 R/G$ *is of the form $\Phi^L(V)$ for some $L \in \mathscr{L}_0$ and some indecomposable finite-dimensional $k[T, T^{-1}]$-module V.*

(ii) $$\Gamma_{R/G} = \Gamma_R/G \vee \bigvee_{L \in \mathscr{L}_0} \Gamma_{k[T,T^{-1}]},$$

where $\Gamma_{k[T,T^{-1}]}$ is the translation quiver of the category of indecomposable finite-dimensional $k[T, T^{-1}]$-modules, consisting of a k^*-family ($k^* = k\backslash\{0\}$) of stable tubes of rank one.

(iii) *R/G is tame if and only if so is R.*

In 4.1, every full convex finite subcategory of R_3 is representation-finite and $\mathscr{L}$ consists only of one line. Hence, Λ_3 is one-parametric. The category R_4 (4.1) is special biserial [DS3; 5.2] and hence the supports of indecomposable locally finite-dimensional R_4-modules are lines. Therefore, by the above theorem, Λ_4 is tame but, as shown in [S2], it is not of polynomial growth.

We expect that Theorems 4.4 and 4.5 form sufficient covering techniques for the study of indecomposable finite-dimensional modules over algebras of polynomial growth. In fact, we believe that the following conjecture is true.

PROBLEM 7. *Let R be a simply connected locally bounded category of polynomial growth and let G be a groupp of k-linear automorphisms of R acting freely on* $\operatorname{ind} R/\simeq$. *Prove that all weakly G-periodic R-modules are linear.*

5. Selfinjective algebras

In this section, with the help of Galois coverings, we describe the selfinjective standard representation-infinite algebras of polynomial growth. The coverings used here are constructed from the Euclidean and Ringel categories.

5.1. Let R be a locally bounded category. The *repetitive category* of R is the locally bounded selfinjective category $\hat{R}$ whose objects are pairs $(n, x) = x_n$, $x \in R$ and $n \in \mathbf{Z}$, and $\hat{R}(x_n, y_n) = \{n\} \times R(x, y)$, $\hat{R}(x_{n+1}, y_n) = \{n\} \times DR(y, x)$ and $\hat{R}(x_p, y_q) = 0$ if $p \neq q, q+1$, where DV denotes the dual vector space $\operatorname{Hom}_k(V, k)$. We shall identify R with the full subcategory of $\hat{R}$ consisting of all objects x_0, $x \in R$. For a sink x of R, we denote by $\sigma_x^+ R$ the full subcategory of

$\hat{R}$ consisting of the objects x_1 and y_0, for all $y \neq x$. Further, we denote by ν_R: $\hat{R} \to \hat{R}$ the *Nakayama automorphism* of $\hat{R}$ which assigns to each object x_n of $\hat{R}$ the object x_{n+1}. A group G of k-linear automorphisms of $\hat{R}$ is said to be *admissible* if it acts freely on the objects of $\hat{R}$ and $\hat{R}/G$ is bounded. Observe that, if R is bounded, then the infinite cyclic groups $(\nu_{\hat{R}}^m)$, $m \in \mathbf{Z}$, are admissible.

The repetitive categories are important objects in the study of derived categories. Recall that, by a result of Happel [H], if the global dimension of R is finite, then the stable module category $\underline{\mathrm{mod}}\,\hat{R}$ of $\mathrm{mod}\,\hat{R}$ (modulo projective-injectives) is equivalent as a triangulated category to the derived category $D^{\mathrm{b}}(R)$ of bounded complexes over $\mathrm{mod}\,R$. For a characterization of derived categories $D^{\mathrm{b}}(R)$ of polynomial growth we refer to [AS2] and [AS3].

Examples. (1) Let D be the Euclidean category defined in 3.3. Then $\hat{D}$ is the bound quiver category $k\hat{Q}_D/\hat{I}_D$, where $\hat{Q}_D$ is the quiver of Fig. 30 and $\hat{I}_D$ is

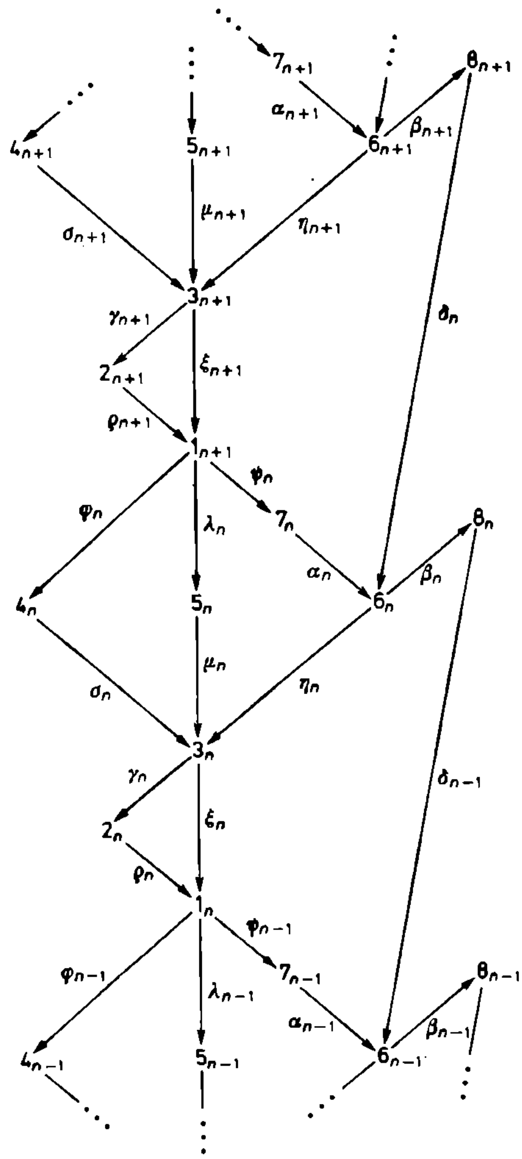


Fig. 30

generated by $\beta_n\alpha_n$, $\gamma_n\sigma_n$, $\xi_n\eta_n$, $\xi_n\mu_n - \varrho_n\gamma_n\mu_n$, $\eta_n\delta_n$, $\delta_n\beta_{n+1} - \alpha_n\psi_n\varrho_{n+1}\gamma_{n+1}\eta_{n+1}$, $\lambda_n\xi_{n+1} - \lambda_n\varrho_{n+1}\gamma_{n+1}$, $\psi_n\xi_{n+1}$, $\varphi_n\varrho_{n+1}$, $\sigma_n\varphi_n + \mu_n\lambda_n + \eta_n\alpha_n\psi_n$, $n \in \mathbf{Z}$.

(2) Let S be the Ringel category defined in 3.3. Then $\hat{S}$ is the bound quiver category $k\hat{Q}_S/\hat{I}_S$, where $\hat{Q}_S$ is the quiver of Fig. 31 and

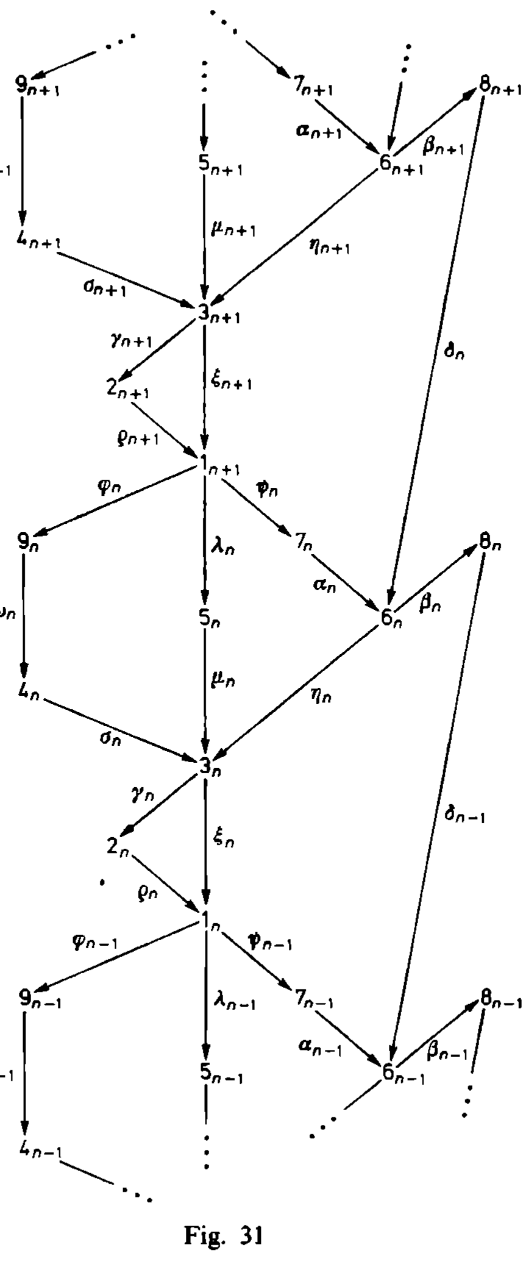


Fig. 31

$\hat{I}_S$ is generated by $\beta_n\alpha_n$, $\gamma_n\sigma_n$, $\xi_n\eta_n$, $\xi_n\mu_n - \varrho_n\gamma_n\mu_n$, $\eta_n\delta_n$, $\psi_n\xi_{n+1}$, $\varphi_n\varrho_{n+1}$, $\delta_n\beta_{n+1} - \alpha_n\psi_n\varrho_{n+1}\gamma_{n+1}\eta_{n+1}$, $\lambda_n\xi_{n+1} - \lambda_n\varrho_{n+1}\gamma_{n+1}$, $\sigma_n\omega_n\varphi_n + \mu_n\lambda_n + \eta_n\alpha_n\psi_n$, $\omega_n\varphi_n\xi_{n+1}\sigma_{n+1}\omega_{n+1}$, $n \in \mathbf{Z}$.

5.2. Let B be a Euclidean category, say of Euclidean type Δ, and let g_B be the rank of $K_0(B)$. It is shown in [ANS] that the support of any indecomposable nonprojective $\hat{B}$-module is contained in one of the following full subcategories of $\hat{B}$:

$$\nu_{\hat{B}}^{m}(\sigma_{x_t}^{+}\ldots\sigma_{x_1}^{+}B), \qquad m\in\mathbf{Z},\ 1\leqslant t\leqslant g_B,$$

where $x_1, \ldots, x_{g_B}$ is a suitably ordered sequence of all objects of B. The categories $\sigma_{x_t}^{+}\ldots\sigma_{x_1}^{+}B$ are, by [TW], [H], iterated tilted algebras of Euclidean type Δ, and hence domestic. Therefore, $\hat{B}$ is locally support-finite, domestic and the Auslander–Reiten quiver $\Gamma_{\hat{B}}$ of $\hat{B}$ is as shown in Fig. 32 (we refer to [ANS]

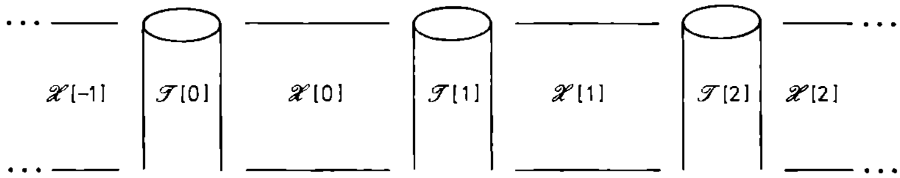


Fig. 32

for details), where $\mathscr{T}[m]$, $m\in\mathbf{Z}$, are $P_1(k)$-families of quasi-tubes with tubular type equal to the tubular type n_B of B, $\mathscr{X}[m]$, $m\in\mathbf{Z}$, are flat components with the stable parts of the form $\mathbf{Z}\Delta$, and $\nu_{\hat{B}}\mathscr{T}[m]=\mathscr{T}[m+2]$, $\nu_{\hat{B}}\mathscr{X}[m]=\mathscr{X}[m+2]$, for all $m\in\mathbf{Z}$. Moreover, there are nonzero maps (in mod $\hat{B}$) only from any of these components to itself and to the components to its right.

Let D be the Euclidean category defined in 3.3. Then the stable parts of $\mathscr{X}[m]$ are of the form $\mathbf{Z}\tilde{E}_7$, the $P_1(k)$-families $\mathscr{T}[m]$ are of type (2, 3, 4), and the projective-injective $\hat{B}$-modules are placed as follows: $P(1_m)$ is contained in $\mathscr{X}[2m]$, $P(3_m)$ in $\mathscr{X}[2m+1]$, $P(2_m)$ and $P(8_m)$ are respectively in tubes of rank 3 and 4 of $\mathscr{T}[2m+1]$, $P(5_m)$ in a tube of rank 2 of $\mathscr{T}[2m+2]$, $P(4_m)$ in a tube of rank 3 of $\mathscr{T}[2m+2]$, and $P(6_m)$, $P(7_m)$ in a quasi-tube of $\mathscr{T}[2m+2]$ of the form shown in Fig. 33, where the vertical lines have to be identified in order to obtain a quasi-tube.

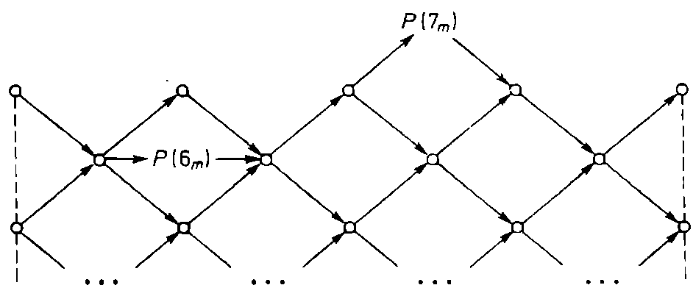


Fig. 33

Applying Theorem 4.4 we obtain the following theorem:

Theorem. *Let B be a Euclidean category and let G be an admissible infinite cyclic group of k-linear automorphisms of $\hat{B}$. Then $A=\bigoplus\hat{B}/G$ is a standard representation-infinite domestic selfinjective algebra and $\Gamma_A\simeq\Gamma_{\hat{B}}/G$ is obtained* *from $\Gamma_{\hat{B}}$ by identifying, via the push-down functor F_λ associated with the Galois covering $F: \hat{B} \to \hat{B}/G$, $\mathscr{T}[m]$ with $\mathscr{T}[m+r]$ and $\mathscr{X}[m]$ with $\mathscr{X}[m+r]$ for some $r \geqslant 1$ and all $m \in \mathbf{Z}$.*

5.3. Let R be a Ringel category. It is shown in [NS] that the support of any indecomposable nonprojective R-module is contained in one of the full subcategories of $\hat{R}$

$$\nu_R^m(\sigma_{x_t}^+ \cdots \sigma_{x_1}^+ R), \qquad m \in \mathbf{Z},\ 1 \leqslant t \leqslant g_R,$$

where g_R denotes the rank of $K_0(R)$ and $x_1, \ldots, x_{g_R}$ is a suitably ordered sequence of all objects of R. The categories $\sigma_{x_1}^+ \ldots \sigma_{x_t}^+ R$, $1 \leqslant t \leqslant g_R$, are tilting-cotilting [AS2] equivalent to R, and hence are of polynomial (even linear) growth. Therefore, $\hat{R}$ is locally support-finite, nondomestic of polynomial growth, and the Auslander–Reiten quiver $\Gamma_{\hat{R}}$ is as shown in Fig. 34, where,

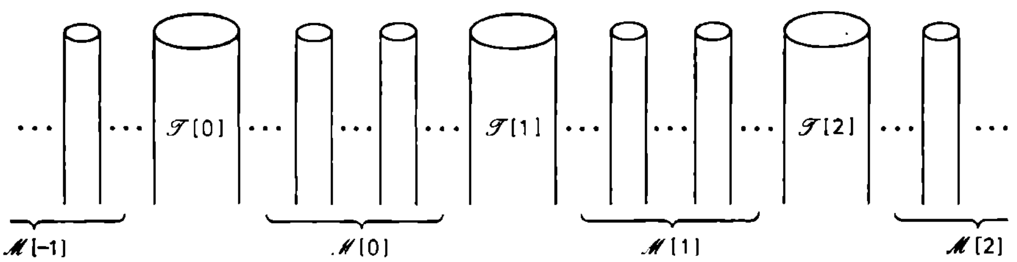


Fig. 34

for each $m \in \mathbf{Z}$, $\mathscr{T}[m]$ is a nonstable $P_1(k)$-family of quasi-tubes of tubular type equal to n_R, $\mathscr{M}[m] = \bigvee_{\gamma \in \mathbf{Q}_{m+1}^m} \mathscr{T}_\gamma^m$, $\mathbf{Q}_{m+1}^m = \mathbf{Q} \cap (m, m+1)$, and for each $\gamma \in \mathbf{Q}_{m+1}^m$, $\mathscr{T}_\gamma^m$ is a stable tubular $P_1(k)$-family of type n_R. Further, $\nu_R \mathscr{T}[m] = \mathscr{T}[m+s]$ and $\nu_R \mathscr{M}[m] = \mathscr{M}[m+s]$ for some s, $3 \leqslant s \leqslant g_R$. Moreover, there are nonzero maps (in mod $\hat{R}$) only from any of these components to itself and to the components to its right.

Let S be the Ringel category of type (2, 4, 4) defined in 3.3. Then the indecomposable projective-injective $\hat{S}$-modules are placed as follows: $P(1_m)$ in one of the tubes of rank 4 in $\mathscr{T}[m]$, $P(2_m)$ and $P(8_m)$ in two different tubes of rank 4 in $\mathscr{T}[m+1]$, $P(3_m)$ in one of the tubes of rank 4 in $\mathscr{T}[m+2]$, $P(5_m)$ in the tube of rank 2 in $\mathscr{T}[m+3]$, $P(4_m)$ and $P(9_m)$ in the same tube of rank 4 in $\mathscr{T}[m+3]$, and $P(6_m)$, $P(7_m)$ in the quasi-tube of rank 4 in $\mathscr{T}[m+3]$ described above. In particular, $\nu_S \mathscr{T}[m] = \mathscr{T}[m+4]$, $m \in \mathbf{Z}$.

Applying Theorem 4.4 we obtain the following result:

THEOREM. *Let R be a Ringel category and let G be an admissible infinite cyclic group of k-linear automorphisms of $\hat{R}$. Then $A = \bigoplus \hat{R}/G$ is a standard nondomestic selfinjective algebra of polynomial growth and $\Gamma_A \simeq \Gamma_{\hat{R}}/G$ is obtained from $\Gamma_{\hat{R}}$ by identifying, via the push-down functor F_λ associated with the*

Galois covering $F: \hat{R} \to \hat{R}/G$, $\mathscr{T}[m]$ *with* $\mathscr{T}[m+r]$ *and* $\mathscr{T}_\gamma^m$ *with* $\mathscr{T}_\gamma^{m+r}$, *for some* $r \geqslant 3$ *and all* $m \in \mathbf{Z}$, $\gamma \in \mathbf{Q}_{m+1}^m$.

5.4. It has recently been shown by the author [S3] that the following converse to Theorems 5.2 and 5.3 holds.

THEOREM [S3]. *Let* A *be a standard representation-infinite domestic (resp. nondomestic of polynomial growth) selfinjective algebra. Then* A *is isomorphic to* $\bigoplus \hat{B}/G$ *for some Euclidean (resp. Ringel) category* B *and an admissible infinite cyclic group* G *of* k*-linear automorphisms of* $\hat{B}$.

For more details concerning the selfinjective algebras of polynomial growth we refer to [ANS], [NS], [S3].

The presented classification of standard representation-infinite selfinjective algebras of polynomial growth raises the following problems.

PROBLEM 8. *Prove that a finite-dimensional algebra* A *is domestic if and only if all but a finite number of connected components of* Γ_A *are stable tubes of rank* 1.

The Auslander–Reiten quiver of all known classes of algebras of polynomial growth has the following property: all but a finite number of its connected components are stable tubes. On the other hand, the group algebras kQ_m of quaternion groups Q_m (2.6) in characteristic 2 are not of polynomial growth and their Auslander–Reiten quiver consists only of tubes of rank 1 or 2, and so it also has the above property. We pose the following problem.

PROBLEM 9. *Characterize finite-dimensional algebras of polynomial growth in terms of the Auslander–Reiten quiver.*

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