MATHEMATICAL MODELS AND METHODS
IN MECHANICS
BANACH CENTER PUBLICATIONS, VOLUME 15
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

INTRODUCTORY REMARKS TO THE DYNAMICS OF CONTINUA WITH MICROSTRUCTURE

G. CAPRIZ

Istituto di Elaborazione della Informazione, Consiglio Nazionale delle Ricerche, Pisa, Italia

1. Introduction

These lectures were addressed to students having as a common background only a basic course in particle and continuum mechanics. Thus elementary topics were rivisited at first though with the slant of a scholar in continuum mechanics, emphasising constitutive laws, requirements of objectivity, etc.; in fact, Sections 1 to 4 could be intended as the sketch of a modest pedagogical proposal. The second part of the paper contains a succinct report on balance laws for continua with microstructure (as proposed in ([1])), the conciseness being in part justified by the lengthy introduction.

Section 2 contains a brief reference to the simplest representation theorems; Section 3 is devoted to some comments on the action-reaction principle and on the law of moment of momentum; systems of particles subject to perfect constraints are considered in Section 4, so as to lead in the following Sections 5 and 6 to a discussion on the allowable rigid velocity distributions. Thus a necessary, central condition on the Lagrangian components of internal forces is derived, a condition which is of the essence in the developments of the later sections. Section 5 contains also some examples which should render the reader familiar with the type of implications to be expected from the condition. Geometry, kinematics and dynamics of continua with general, finite-dimensional structure are discussed briefly in Section 7. In Section 8 it is finally shown how some special theories can be set within the general framework.

([1]) G. Capriz and P. Podio-Guidugli, *Materials with finite-dimensional structure*, in: A. P. S. Selvadurai (ed.), *Mechanics of Structured Media*, Elsevier, Amsterdam 1981, 255–268. The identifications mentioned at the end of Section 8 use notation of a paper by the same authors published in Ann. Mat. Pura. Appl. (4), 111 (1976), 195–217.

The scheme of notation is as follows: Greek letters (φ, ψ, ...) stand for scalar quantities (and are used also as indices of Lagrangian coordinates and quantities). Small latin letters (u, v, ...) stand for vectors in three-dimensional Euclidean space $\mathscr{E}$ (and are used also as indices, in particular as indices of components on an orthogonal system of reference $\mathfrak{x}$ in $\mathscr{E}$: e.g., u_i, v_i, $i = 1, 2, 3$). Capital latin letters (A, B, ...) stand for second-order tensors, with components A_{ij}, etc. on $\mathfrak{x}$; in particular 1 is the unit tensor; notice also the tensor product of vectors $a \otimes b$, such that $(a \otimes b)_{ij} = a_i b_j$. $\boldsymbol{e}$ is Ricci's third order permutation tensor; other letters from the same font ($\boldsymbol{a}$, $\boldsymbol{b}$, ...) are used for other third-order tensors. Fourth-order tensors are indicated with capital letters ($\boldsymbol{A}$, $\boldsymbol{B}$, ...). Repeated latin indices are summed from 1 to 3.

2. Scalar and vector functions of vectors

Let the scalar φ be a function of a vector u. If a cartesian reference $\mathfrak{x}$ is assigned, we can think of φ as a function of components u_1, u_2, u_3; it is not a generic function of those variables, however. In fact, if we change the reference from $\mathfrak{x}$ to $\mathfrak{x}'$ (and suppose that the matrix of direction cosines be Q_{rs}: $x'_r = Q_{rs} x_s$, $r = 1, 2, 3$), then u_1, u_2, u_3 change into $Q_{1r} u_r$, $Q_{2r} u_r$, $Q_{3r} u_r$ whereas φ must not change. The point is that, in constructing the scalar φ, direction and orientation of u have no role to play: *φ is a function of $|u|$ alone*:

$$\varphi = \hat{\varphi}(|u|). \tag{2.1}$$

Notice that the essential point in the proof is the negation of any preferred direction; only with that understanding the invariants associated with a vector are its modulus or functions of its modulus alone.

In this, as in other *representation theorems* which follow, when we say that some variable is a function of certain independent variables of vectorial or tensorial character, we mean that it is a function of those variables *alone*; it does *not* depend on other, non-expressed, vectorial or tensorial variables (more precisely, other, not explicitly declared, variables must be either scalars or *constant* tensors i.e. tensors whose components are the same in all Cartesian references). On the other hand, the function is meant to be defined over the whole space of vectors or tensors. Notice also that, when we speak of scalars or tensors, we mean *absolute* scalars and tensors (i.e., objects whose invariance or rule of variance apply also for changes from right-handed to left-handed systems) so that $\boldsymbol{e}$ is *not* a constant tensor.

A representation theorem of Cauchy shows how things stand when φ depends on $N (> 1)$ vectors: *if φ is a scalar function of N vectors $u^{(1)}, \ldots, u^{(N)}$ then it can be represented as a function of* (or it depends on those vectors through) *the scalar products $u^{(a)} \cdot u^{(b)}$ $(a, b = 1, \ldots, N)$ alone*

$$\varphi = \hat{\varphi}(|u^{(1)}|^2, u^{(1)} \cdot u^{(2)}, \ldots, |u^{(N)}|^2). \tag{2.2}$$

The proof follows from this remark: think of the geometric figure formed by the vectors $u^{(a)}$, $a = 1, \ldots, N$ all issuing from a common origin 0. If a rigid displacement is imposed upon the figure, lengths and angles do not change, nor do lengths and angles change after reflection. Hence any function of the type (2.2) is a scalar.

Vice versa, suppose that you have two systems of vectors $(0, u^{(a)})$, $a = 1, \ldots, N$, and $(0, v^{(a)})$, $a = 1, \ldots, N$, such that

$$u^{(a)} \cdot u^{(b)} = v^{(a)} \cdot v^{(b)}, \qquad \forall a, b; \tag{2.3}$$

then the two systems can be superposed through a rigid rotation, followed perhaps by a reflection. In fact, assume that $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ do not belong to the same plane (if all $u^{(a)}$, $a = 1, \ldots, N$, belong to the same plane the proof becomes simpler). By a rotation the plane of $v^{(1)}$, $v^{(2)}$ can be brought to coincide with the plane of $u^{(1)}$, $u^{(2)}$ and $v^{(1)}$ with $u^{(1)}$ (notice that, by (2.3), the length of any $v^{(a)}$ coincides with the length of the corresponding $u^{(a)}$); then $v^{(2)}$ either coincides with $u^{(2)}$ or can be brought to coincide through a rotation of π around $u^{(1)}$ (in fact $\cos(\widehat{u^{(1)}, u^{(2)}}) = \cos(\widehat{v^{(1)}, v^{(2)}})$; hence, if angles are taken as usual in $[0, \pi]$, $(\widehat{u^{(1)}, u^{(2)}}) = (\widehat{v^{(1)}, v^{(2)}})$). Finally, bringing to bear equality of angles again, one checks that either $v^{(3)}$ already coincides with $u^{(3)}$ or can be brought to coincide through a reflection on the plane of $u^{(1)}$, $u^{(2)}$. Once $v^{(1)}$, $v^{(2)}$, $v^{(3)}$ are superposed on $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ respectively, any other vector $v^{(a)}$, $a = 4, \ldots, N$ coincides with the corresponding vector $u^{(a)}$ because, by (2.3), its components on $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ are equal to those of $v^{(a)}$.

Consider now a vector f which is a function of a vector u: we can think of the components f_1, f_2, f_3 of f on a cartesian reference $\mathfrak{x}$ as functions of the components u_1, u_2, u_3 of u; but again the function must be a very particular one; in fact the relation

$$f_i = \hat{f}_i(u_1, u_2, u_3), \qquad i = 1, 2, 3, \tag{2.4}$$

must apply also if we change reference, when f_i goes into $Q_{ij} f_j$ and u_r into $Q_{rs} u_s$; i.e., the functional relation must apply

$$Q_{ij} \hat{f}_j(u_1, u_2, u_3) = \hat{f}_i(Q_{1r} u_r, Q_{2r} u_r, Q_{3r} u_r). \tag{2.5}$$

For instance, take the independent vector u parallel to the first axis of $\mathfrak{x}$ and calculate the corresponding values of f_1, f_2, f_3 through (2.4)

$$f_1 = \hat{f}_1(u_1, 0, 0), \qquad f_2 = \hat{f}_2(u_1, 0, 0), \qquad f_3 = \hat{f}_3(u_1, 0, 0).$$

Assume now that $\mathfrak{x}'$ differs from $\mathfrak{x}$ only because the orientations of the second and third axis are opposite (i.e., take, in the orthogonal matrix of reference change, $Q_{11} = 1$, $Q_{22} = -1$, $Q_{33} = -1$; all other Q_{rs} zero); apply (2.5) to obtain

$$\hat{f}_2(u_1, 0, 0) = -\hat{f}_2(u_1, 0, 0), \qquad \hat{f}_3(u_1, 0, 0) = -\hat{f}_3(u_1, 0, 0).$$

Hence f must be parallel to u:

$$f = \psi(u)u.$$

Here ψ must be a scalar function of u, as can easily be checked. In conclusion,

If the vector f is a function of a vector u, then it must have the form

$$f = \psi(|u|)u, \tag{2.6}$$

where ψ is any function of its argument.

This conclusion can be reached also by a direct, naive reasoning: to specify the direction of f we do not have at our disposal anything beside the direction of u, so, by Ockham's razor, the two directions must coincide.

The theorem just quoted is a special case of another general representation theorem of Cauchy, which we state below without proof:

If the vector f is a function of vectors $u^{(a)}$, $a = 1, \ldots, N$, then it has the form

$$f = \sum_{a=1}^{N} \psi^{(a)} u^{(a)} \tag{2.7}$$

where the coefficients $\psi^{(a)}$ are scalar functions of the vectors $u^{(b)}$ and hence of the form (2.2).

Sometimes one has to consider a scalar or a vector, which is a function of places (for instance of the places occupied by several particles $\mathscr{P}^{(a)}$). Places can be specified by vectors $p^{(a)}$ issuing from an origin 0, which, however, can be chosen arbitrarily. Then the relevant variables are not the vectors $p^{(a)}$, but rather the differences

$$d^{(a,b)} = p^{(a)} - p^{(b)},$$

which remain indifferent to the choice of 0.

Example 1. If f depends only on the places occupied by two particles, then it must be of the form

$$f = \psi(|d|)d, \qquad d = p^{(2)} - p^{(1)}. \tag{2.8}$$

This is, for instance, the general law of force in binary interaction.

Example 2. If φ is a scalar depending on the places occupied by two particles, then it must depend on their distance $\delta = |d|$ only

$$\varphi = \hat{\varphi}(\delta). \tag{2.9}$$

This is the case for potentials in binary interactions.

Example 3. If φ depends on the places occupied by three particles, then it can be given the form suggested directly by formula (2.2) or, alternatively, by

$$\varphi = \hat{\varphi}(\delta^{(1)}, \delta^{(2)}, \vartheta), \tag{2.10}$$

where $\delta^{(1)}$ and $\delta^{(2)}$ are distances of, say, the third particle $\mathscr{P}^{(3)}$ from $\mathscr{P}^{(1)}$ and $\mathscr{P}^{(2)}$ respectively and ϑ is the angle between the vectors $p^{(1)}-p^{(3)}$ and $p^{(2)}-p^{(3)}$. A more symmetric form can be also given to φ,

$$\varphi = \tilde{\varphi}(\delta^{(1)}, \delta^{(2)}, \delta^{(3)}), \tag{2.11}$$

with $\delta^{(3)}$ distance of $\mathscr{P}^{(1)}$ from $\mathscr{P}^{(2)}$, at the cost of restricting the region of definition of $\tilde{\varphi}$ to the range

$$0 \leqslant \delta^{(1)} \leqslant \delta^{(2)}+\delta^{(3)}, \qquad 0 \leqslant \delta^{(2)} \leqslant \delta^{(3)}+\delta^{(1)}, \qquad 0 \leqslant \delta^{(3)} \leqslant \delta^{(1)}+\delta^{(2)}.$$

Formula (2.11) gives the form of a potential for multiple interactions. Notice that the potential for multiple interaction need not be the sum of potentials for binary interactions.

EXAMPLE 4. If f depends on the place of three particles, then it must belong to the plane of the particles and can be given the form

$$f = \psi^{(1)} d^{(1)} + \psi^{(2)} d^{(2)}, \tag{2.12}$$

where

$$d^{(1)} = p^{(2)} - p^{(3)}, \qquad d^{(2)} = p^{(3)} - p^{(1)},$$

and $\psi^{(1)}$, $\psi^{(2)}$ are functions of $\delta^{(1)}$, $\delta^{(2)}$, $\delta^{(3)}$.

f in (2.12) could be the force on $\mathscr{P}^{(3)}$ due to the presence of $\mathscr{P}^{(1)}$, $\mathscr{P}^{(2)}$. Notice again that f need not be the sum of forces due to binary interactions. Consider, for instance, as an exercise, the case when f derives from a potential of the type (2.11), e.g.,

$$\tilde{\varphi} = \tfrac{1}{2} \sum_{r,s=1}^{3} \alpha_{rs} \delta^{(r)} \delta^{(s)}.$$

with the constants α_{rs} all different from zero.

In dynamics, cases are considered where a scalar or a vector depends on places and speeds of particles; but then deeper conditions of invariance are exploited. Consider, as an example, a vector f which depends on the place and speeds of two particles, $p^{(1)}$, $p^{(2)}$ and $\dot{p}^{(1)}$, $\dot{p}^{(2)}$ respectively. Seek the conditions under which the dependence is not influenced if the motion is seen instantaneously by an observer in arbitrary rigid motion. Then, first of all, only the difference of speeds $d = \dot{p}^{(1)} - \dot{p}^{(2)}$ becomes relevant (the origin of the observer frame can be put in the place occupied by $\mathscr{P}^{(2)}$, a speed of translation equal to $\dot{p}^{(2)}$ can be attributed to it, so that the speed of $\mathscr{P}^{(2)}$

vanishes and that of $\mathscr{P}^{(1)}$ becomes $\dot{d}$). By Cauchy's theorem (formula (2.7)) it must be

$$f = \alpha d + \beta \dot{d},$$

where α and β depend on d^2, $\dot{d}^2$, $d \cdot \dot{d}$; on the other hand the component of $\dot{d}$ transverse of d can be made arbitrary by giving the observer frame an appropriate speed of rotation around an axis through $\mathscr{P}^{(2)}$. In conclusion only the component of $\dot{d}$ along d counts (hence β must vanish) and we have

$$f = \alpha(\delta, \dot{\delta})d, \qquad \delta = |d|. \tag{2.13}$$

3. Comments on Newton's Lex Tertia

In the dynamics of a system formed by many particles *constitutive hypotheses* are needed to specify the laws of interaction; in most general terms the force acting on any one particle may depend on the placement of the system (the place of each particle) and its state of motion (the speed of each particle); but the dependence must be such as to satisfy a deep condition of invariance:

RULE OF OBJECTIVITY. *The law of force is invariant to any change of observer.*

We do not pursue here an analysis of various constitutive assumptions, but make use of results of Section 2 and of the rule just stated to restrict the range of choices.

Consider a binary system $\{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$. Then the forces $f^{(1)}$ and $f^{(2)}$ acting on $\mathscr{P}^{(1)}$ and $\mathscr{P}^{(2)}$ respectively must obey the restriction (2.13)

$$f^{(i)} = \alpha^{(i)}(\delta, \dot{\delta})d. \tag{3.1}$$

If the two particles are identical, then requirements of invariance alone lead to the condition

$$\alpha^{(1)}(\delta, \dot{\delta}) = -\alpha^{(2)}(\delta, \dot{\delta}). \tag{3.2}$$

In general, however, there does not seem to be a compelling logical argument in favour of (3.2). An explicit assumption is needed:

LEX TERTIA. *Relation* (3.2) *is valid for all binary systems.*

In other words, the resultant force of the system vanishes

$$f^{(1)} + f^{(2)} = 0. \tag{3.3}$$

Once account is taken of (3.1), eqn (3.3) is equivalent to

$$p^{(1)} \times f^{(1)} + p^{(2)} \times f^{(2)} = 0, \tag{3.4}$$

i.e., to the condition that the resultant moment vanishes. An alternative version of (3.4), trivial here but of some relevance later, is the condition

$$p^{(1)} \otimes f^{(1)} + p^{(2)} \otimes f^{(2)} \in \mathrm{Sym}. \tag{3.5}$$

Let us consider now, within $\mathscr{E}$, a system formed by N ($N > 2$) particles $\mathscr{P}^{(i)}$. Cauchy's theorem (formula (2.7)) helps little in general; one can say that, if all particles are in one plane, then the forces must be parallel to that plane, but not much more. A deeper hypothesis intervenes:

COROLLARIUM. *The actions exchanged by the particles of any isolated system* (i.e., a system which is all we think exists in space) *are equilibrated*; the resultant force and the resultant moment both vanish:

$$\sum_{i=1}^{N} f^{(i)} = 0, \tag{3.6}$$

$$\sum_{i=1}^{N} p^{(i)} \times f^{(i)} = 0, \quad \text{or} \quad \sum_{i=1}^{N} p^{(i)} \otimes f^{(i)} \in \mathrm{Sym}. \tag{3.7}$$

Naturally if all interactions among particles in a system were superpositions of binary interactions obeying (3.3), (3.4), conditions (3.6) and (3.7) would be trivially satisfied. On the other hand not all interactions obeying (3.6) and (3.7) need be in principle superpositions of binary interactions.

The systems $\mathscr{S}$ of particles for which the statements above apply, are *isolated* systems. Usually one has to deal with subsystems, however. For the study of the evolution of a subsystem $\mathscr{A}$ (formed, say by the particles $\mathscr{P}^{(i)}$, $i = 1, 2, \ldots, K$; $K < N$) an appropriate classification of forces acting on the particles of $\mathscr{A}$ is of the essence: the force $f^{(i)}$ acting on the ith particle is split into the sum of the force $f^{\mathrm{I},(i)}$ (internal force) that would act upon it if $\mathscr{A}$ were isolated (i.e., if $\mathscr{S} - \mathscr{A}$ were removed) and of the force $f^{\mathrm{E},(i)} = f^{(i)} - f^{\mathrm{I},(i)}$ (external force). Notice that the vectors $f^{\mathrm{E},(i)}$ must account also for the alterations, if any, in the interactions among particles of $\mathscr{A}$ due to the presence of the particles of $\mathscr{S} - \mathscr{A}$. Newton's Lex Secunda requires for each particle the validity of the relation

$$\mu^{(i)} a^{(i)} = f^{\mathrm{E},(i)} + f^{\mathrm{I},(i)}, \tag{3.8}$$

where $a^{(i)}$ is the acceleration of the ith particle in a Galilean frame and $\mu^{(i)}$ its mass.

But, by the Corollarium, the system of forces $f^{\mathrm{I},(i)}$ ($i = 1, 2, \ldots, K$) is equilibrated; thus, if one introduces the resultant force of inertia

$$f^{\mathrm{M}} = -\sum_{i=1}^{K} \mu^{(i)} a^{(i)} \tag{3.9}$$

and the resultant moment

$$m_0^{\mathrm{M}} = -\sum_{i=1}^{K} p^{(i)} \times \mu^{(i)} a^{(i)}, \tag{3.10}$$

one obtains from (3.8) Euler's equations of balance

$$f^{M} + f^{E} = 0, \qquad m_0^{M} + m_0^{E} = 0, \tag{3.11}$$

where

$$f^{E} = \sum_{i=1}^{K} f^{E,(i)}, \qquad \text{and} \qquad m_0^{E} = \sum_{i=1}^{K} p^{(i)} \times f^{E,(i)}. \tag{3.12}$$

If we let p be the position vector of the centre of gravity $\mu p = \sum_{i=1}^{K} \mu^{(i)} p^{(i)}$ $(\mu = \sum_{i=1}^{K} \mu^{(i)})$ and we introduce the momentum and moment of momentum of $\mathscr{A}$, respectively,

$$\mu q = \sum_{i=1}^{K} \mu^{(i)} \dot{p}^{(i)}, \qquad \mu k_0 = \sum_{i=1}^{K} p^{(i)} \times \mu^{(i)} \dot{p}^{(i)}, \tag{3.13}$$

eqns (3.11) yield

$$\mu \dot{q} = f^{E}, \qquad \mu \dot{k}_0 = m_0^{E}, \tag{3.14}$$

whenever 0 is fixed or coincides with the centre of gravity.

Other global equations for $\mathscr{A}$ can be deduced, for instance:

(i) The kinetic energy theorem:

$$\mu \dot{\varkappa} = \pi \tag{3.15}$$

where

$$\mu \varkappa = \tfrac{1}{2} \sum_{i=1}^{K} \mu^{(i)} (\dot{p}^{(i)})^2 \tag{3.16}$$

is the kinetic energy and

$$\pi = \sum_{i=1}^{K} f^{(i)} \cdot \dot{p}^{(i)} \tag{3.17}$$

is the power of all forces (external *and* internal) acting on the particles of $\mathscr{A}$.

(ii) The virial theorem

$$\mu(\dot{\alpha} - 2\varkappa) = \beta \tag{3.18}$$

where

$$\mu \alpha = \sum_{i=1}^{K} p^{(i)} \cdot \mu^{(i)} \dot{p}^{(i)} \tag{3.19}$$

is the virial of momentum and

$$\beta = \sum_{i=1}^{K} p^{(i)} \cdot f^{(i)} \tag{3.20}$$

is the virial of all forces acting on the particles of $\mathscr{A}$.

To obtain (3.18) multiply both sides of (3.8) scalarly by $p^{(i)}$ and sum over i.

(iii) The equation of balance of generalized (or tensorial) moment of momentum

$$\mu(\dot{K}_0 - T) = M_0^{\mathrm{E}} + M_0^{\mathrm{I}} \tag{3.21}$$

where

$$\mu K_0 = \sum_{i=1}^{K} p^{(i)} \otimes \mu^{(i)} \dot{p}^{(i)} \tag{3.22}$$

is generalized (or tensorial) moment of momentum,

$$\mu T = \sum_{i=1}^{K} \mu^{(i)} \dot{p}^{(i)} \otimes \dot{p}^{(i)} \tag{3.23}$$

is the kinetic tensor and

$$M_0^{\mathrm{E}} = \sum_{i=1}^{K} p^{(i)} \otimes f^{\mathrm{E},(i)}, \qquad M_0^{\mathrm{I}} = \sum_{i=1}^{K} p^{(i)} \otimes f^{\mathrm{I},(i)} \tag{3.24}$$

are generalized (or tensorial) moments of external and internal forces respectively.

It is important to remark that, by the Corollarium, M_0^{I} is a symmetric tensor

$$M_0^{\mathrm{I}} \in \mathrm{Sym}. \tag{3.25}$$

Notice that, taking the trace of both sides of (3.21), one obtains (3.18) again, whereas, equating the vectors associated with the skew parts of right- and left-hand side of (3.21), one obtains $(3.14)_2$.

A final remark: if the kinematic state is rigid, there exist two vectors v (speed of translation) and r (rotational speed) such that

$$\dot{p}^{(i)} = v + r \times p^{(i)}. \tag{3.26}$$

The power of forces for such a kinematic state involves the resultant force and moment only. In fact

$$\pi^{\mathrm{R}} = \sum_{i=1}^{K} f^{(i)} \cdot (v + r \times p^{(i)}) = \Big(\sum_{i=1}^{K} f^{(i)}\Big) \cdot v + \Big(\sum_{i=1}^{K} p^{(i)} \times f^{(i)}\Big) \cdot r$$

and, because internal forces are balanced,

$$\pi^{\mathrm{R}} = f^{\mathrm{E}} \cdot v + m^{\mathrm{E}} \cdot r. \tag{3.27}$$

Actually *the property of internal forces to give null power for all rigid kinematic states* (3.26) *could be used instead of the conditions of balance*

$$f^{\mathrm{I}} \cdot v + m^{\mathrm{I}} \cdot r = 0, \qquad \forall v, r. \tag{3.28}$$

4. Constraints

In concrete problems a complete analysis of the behaviour of the subsystem $\mathscr{A}$ can be avoided through the acceptance of special hypotheses inspired by the physical situation. Drastic simplifications derive, for instance, from the introduction of geometrical constraints; a typical example is the internal constraint of rigidity.

One assumes that the vectors $p^{(i)}$ are completely specified once the values of a limited number n of parameters v_σ are assigned

$$p^{(i)} = \hat{p}^{(i)}(v_1, v_2, \ldots, v_n), \qquad i = 1, 2, \ldots, K. \tag{4.1}$$

The number n is always less and usually much less than $3K$; the parameters v_σ (the *Lagrangian variables*) are taken here to be non-dimensional. The constraint functions in the right-hand side of (4.1) are assumed to be defined over a domain of $\boldsymbol{R}^n$ and to be there sufficiently smooth so as to give meaning to the developments which follow. The parameters v_σ are assumed to be independent; a way to make this condition locally precise is to assume that the matrix

$$\pi_{A\sigma} = \frac{\partial \hat{p}_j^{(i)}}{\partial v_\sigma}, \qquad A = 3(i-1)+j, \qquad A = 1, 2, \ldots, 3K, \qquad \sigma = 1, 2, \ldots, n, \tag{4.2}$$

has everywhere characteristic n, so that in particular the vectors

$$\dot{p}^{(i)} = \sum_{\sigma=1}^{n} \frac{\partial p^{(i)}}{\partial v_\sigma} \dot{v}_\sigma$$

vanish if and only if all $\dot{v}_\sigma$ vanish. More generally, the relations (4.1) are required to define a differential manifold of dimension n imbedded in $\mathscr{E}^K$.

Through (4.1) a partial specification of the physical properties of $\mathscr{A}$ is given; it is implied that the system evolves in accordance with (4.1) because appropriate forces $f^{C,(i)}$ (*constraint forces*) are evoked. To avoid indetermination, specific hypotheses are needed with regard to $f^{C,(i)}$; a very convenient assumption requires $f^{C,(i)}$ to have null power for any kinematic state $\dot{p}^{(i)}$ compatible with (4.1), precisely that

$$\sum_{i=1}^{K} f^{C,(i)} \cdot \left(\sum_{\sigma=1}^{n} \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma} \dot{v}_\sigma \right) = 0, \qquad \forall \dot{v}_\sigma;$$

then the constraints are called *perfect* (or frictionless). This condition implies

$$\sum_{i=1}^{K} f^{C,(i)} \cdot \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma} = 0, \qquad \sigma = 1, 2, \ldots, n. \tag{4.3}$$

A new classification of forces is therefore in order: the forces due to the constraints, already quoted, and the active forces $f^{A,(i)}$:

$$f^{A,(i)} = f^{(i)} - f^{C,(i)}. \tag{4.4}$$

If we multiply both members of (3.8) scalarly by $\frac{\partial \hat{p}^{(i)}}{\partial v_\sigma}$ and sum with respect to i, we obtain, in view of (4.3),

$$\sum_{i=1}^{K} \mu^{(i)} a^{(i)} \cdot \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma} = \sum_{i=1}^{K} f^{A,(i)} \cdot \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma}. \tag{4.5}$$

If we notice also that

$$\frac{\partial \hat{\dot{p}}^{(i)}}{\partial \dot{v}_\sigma} = \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma}$$

and recall the definition (3.16) of kinetic energy we can derive from (4.5) the *equations of Lagrange*

$$\mu\left(\frac{d}{d\tau}\left(\frac{\partial \varkappa}{\partial \dot{v}_\sigma}\right) - \frac{\partial \varkappa}{\partial v_\sigma}\right) = \lambda_\sigma, \tag{4.6}$$

where the Lagrangian components λ_σ of active forces are defined by

$$\lambda_\sigma = \sum_{i=1}^{K} f^{A,(i)} \cdot \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma}. \tag{4.7}$$

These forces could depend on placement and kinematic state of the whole system $\mathscr{S}$, and not only of $\mathscr{A}$. A final simplifying hypothesis is now accepted: *the placement and kinematic state of $\mathscr{S} - \mathscr{A}$ influences λ_σ at most through an explicit dependence on the time τ*; each λ_σ is taken to be an assigned function of $v_1, \ldots, v_n, \dot{v}_1, \ldots, \dot{v}_n, \tau$:

$$\lambda_\sigma = \hat{\lambda}_\sigma(v_1, \ldots, v_n; \dot{v}_1, \ldots, \dot{v}_n; \tau). \tag{4.8}$$

If we put

$$\mu_{\varrho\sigma} = \frac{1}{\mu} \sum_{i=1}^{K} \mu^{(i)} \frac{\partial \hat{p}^{(i)}}{\partial v_\varrho} \cdot \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma}, \tag{4.9}$$

so that

$$\varkappa = \tfrac{1}{2} \sum_{\varrho,\sigma=1}^{n} \mu_{\varrho\sigma} \dot{v}_\varrho \dot{v}_\sigma, \tag{4.10}$$

eqns (4.6) can be put in a more explicit form

$$\sum_{\beta=1}^{n} \mu_{\alpha\beta} \ddot{v}_\beta + \sum_{\beta,\gamma=1}^{n} \left(\frac{\partial \mu_{\beta\alpha}}{\partial v_\gamma} - \tfrac{1}{2} \frac{\partial \mu_{\beta\gamma}}{\partial v_\alpha}\right) \dot{v}_\beta \dot{v}_\gamma = \frac{1}{\mu} \lambda_\alpha. \tag{4.11}$$

It is important to notice that the quantities $\mu_{\alpha\beta}$ form a positive-definite symmetric matrix. In fact $\varkappa$ (compare (4.10) with (3.16)) vanishes if and only if all $\dot{q}^{(i)}$ vanish, or, in other words, if and only if all $\dot{v}_\sigma$ vanish, in view of the property of the matrix (4.2).

In particular

$$\det \mu_{\alpha\beta} > 0;$$

hence an inverse matrix $\mu_{\alpha\beta}^{-1}$ exists, so that we can deduce from (4.11) an equivalent system

$$\ddot{v}_\alpha = \sum_{\beta,\gamma,\delta=1}^{n} \mu_{\alpha\delta}^{-1}\left(\frac{1}{2}\frac{\partial\mu_{\beta\gamma}}{\partial v_\delta} - \frac{\partial\mu_{\beta\delta}}{\partial v_\gamma}\right)\dot{v}_\beta \dot{v}_\gamma + \\ + \mu^{-1}\sum_{\delta=1}^{n} \mu_{\alpha\delta}^{-1}\hat{\lambda}_\delta(v_1, \ldots, v_n; \dot{v}_1, \ldots, \dot{v}_n; t). \tag{4.12}$$

In conclusion we have given eqns (4.6) the form of a system of n ordinary differential equations in the n unknown functions $v_\alpha(\tau)$ and the system is in normal form. Under known conditions of regularity of the right-hand sides, theorems of existence and uniqueness of solutions can be applied; our scheme is *predictive*: once the initial conditions (i.e., placement and kinematic state of $\mathscr{A}$) are given, we can determine the evolution of the system.

Remark. In the following analysis both classifications of forces encountered so far (internal-external and active-constraint) are required; for the set of forces belonging to a class Z we set

$$\lambda_\sigma^Z = \sum_{i=1}^{K} f^{Z,(i)} \cdot \frac{\partial p^{(i)}}{\partial v_\sigma}.$$

With this notation, λ_σ above (and in the following pages) should really be written λ_σ^A where A stands for active, and condition (4.3) could be spelled $\lambda_\sigma^C = 0$, or

$$\lambda_\sigma^{IC} + \lambda_\sigma^{EC} = 0. \tag{4.13}$$

Having in mind our final goal (which is a theory of continua) we are interested below primarily in cases where λ_σ^I coincides with λ_σ^{IA} and λ_σ^E with λ_σ^{EA}; this is the case, for instance, when the constraints are purely internal ($f^{EC,(i)} = 0$) or purely external ($f^{IC,(i)} = 0$); however, we leave the matter open for the moment.

5. Virtual rigid velocity distributions

In the developments of Section 4, leading to (4.6) or (4.12) no use was made of the fundamental property (3.28) of internal forces: the Lex Tertia did not have a role. Let us make up for this deficiency now.

We need to define the subset $\mathscr{C}^{(R)} \equiv \{\dot{v}_\sigma^R\}$ of $\mathbf{R}^n$ which contains the values $\dot{v}_\sigma^R$ of v_σ which give rise to rigid velocity distributions for $\mathscr{A}$. $\mathscr{C}^{(R)}$ is a linear subspace of $\boldsymbol{R}^n$ of dimension not greater than 6; in fact, for all i, any one of its elements satisfies

$$\sum_{\sigma=1}^{n} \frac{\partial \hat{p}^{(i)}}{\partial v_\sigma} \dot{v}_\sigma^R = v + r \times p^{(i)}, \tag{5.1}$$

for some choice of the vectors v and r.

In terms of components, system (5.1) can be written as follows

$$\sum_{\sigma=1}^{n} \Pi_{A\sigma} \dot{v}_\sigma^R = v_{A-3[(A-1)/3]} + e_{A-3[(A-1)/3],m,n} r_m p_n^{(1+[(A-1)/3])}, \tag{5.2}$$

where $A = 1, 2, \ldots, 3K$ and $[(A-1)/3]$ is the integer part of the fraction between brackets.

System (5.2) can be considered as a system of $3K$ linear equations in the n unknowns $\dot{v}_\sigma^R$ and we have assumed already that the $(3K \times n)$-matrix $\Pi_{A\sigma}$ has maximal characteristic n. For the compatibility of system (5.2) the complete $(3K \times (n+1))$-matrix

$$(\Pi_{A\sigma} | v_{A-3[(A-1)/3]} + e_{A-3[(A-1)/3],m,n} r_m p_n^{(1+[(A-1)/3])}) \tag{5.3}$$

must also have characteristic n. Hence all its minors of order $n+1$ must vanish; these conditions can be interpreted as a set of linear homogeneous conditions on v and r. Introducing a characteristic length α to avoid a mismatch of dimension, we can say that the 6-vector $(\alpha^{-1} v | r)$ must belong to a linear subspace $\mathscr{R}$ of $\boldsymbol{R}^6$

$$(\alpha^{-1} v | r) \in \mathscr{R} \subset \boldsymbol{R}^6. \tag{5.4}$$

The dimension of $\mathscr{R}$ may range from 0 to 6; either extreme cases $\mathscr{R} \equiv \{0\}$ of $\mathscr{R} \equiv \boldsymbol{R}^6$ may occur.

Once condition (5.4) is satisfied, the system (5.2) can be uniquely solved in terms of $\dot{v}_\sigma^R$, which become necessarily linear functions of $\alpha^{-1} v$ and r, so that a representation of the following form is possible

$$\dot{v}_\sigma^R = \alpha^{-1} v^{(\sigma)} \cdot v + r^{(\sigma)} \cdot r, \tag{5.5}$$

with each $(v^{(\sigma)} | r^{(\sigma)}) \in \mathscr{R}$. Actually one may ask if $\mathscr{R}$ can be larger that span $(v^{(\sigma)} | r^{(\sigma)})$. If that were the case non-null 6-vectors $(\alpha^{-1} v^\perp | r^\perp) \in \mathscr{R}$ would exist orthogonal to all $(v^{(\sigma)} | r^{(\sigma)})$ and such that all the corresponding $\dot{v}_\sigma^R$ would be zero; hence $v^\perp + r^\perp \times p^{(i)}$ must vanish for all $p^{(i)}$. In conclusion all points must belong to a straight line and $r^\perp$ have the direction of that line. Thus, generally, (i.e., if the exceptional circumstance is excluded) $\mathscr{R} \equiv$ span $(v^{(\sigma)} | r^{(\sigma)})$; if special mention to the contrary is not made, this identity is presumed valid from now on.

Going back to (3.28), one is led to require

$$f^{\mathrm{I}} \cdot v + m^{\mathrm{I}} \cdot r = 0, \qquad \forall (\alpha^{-1} v | r) \in \mathscr{R}, \tag{5.6}$$

which is the only condition relevant now in view of the admissible rigid velocity distributions. In fact one can introduce, in general, a reduced resultant force $\tilde{f}^{\text{I}}$ and reduced resultant moment $\tilde{m}^{\text{I}}$ of internal forces

$$\tilde{f}^{\text{I}} = \alpha^{-1} \sum_{\sigma=1}^{n} \lambda_\sigma^{\text{I}} v^{(\sigma)}, \qquad \tilde{m}^{\text{I}} = \sum_{\sigma=1}^{n} \lambda_\sigma^{\text{I}} r^{(\sigma)}; \tag{5.7}$$

observe that, by (4.7) and (5.5), eqn (5.6) is equivalent to

$$\tilde{f}^{\text{I}} \cdot v + \tilde{m}^{\text{I}} \cdot r = 0, \qquad \forall (\alpha^{-1} v | r) \in \mathscr{R},$$

which in its turn applies if and only if

$$\tilde{f}^{\text{I}} = 0, \qquad \tilde{m}^{\text{I}} = 0, \tag{5.8}$$

because the 6-vector $(\alpha \tilde{f}^{\text{I}} | \tilde{m}^{\text{I}})$ belongs to $\mathscr{R}$. One can easily check that the result is valid also in the circumstance which we have called exceptional above.

In conclusion, for the complete direct specification of a constrained system of the type introduced in Section 4, it is appropriate to choose also the subspace $\mathscr{R}$ (of dimension not greater than 6) of $\boldsymbol{R}^n$ and a set of n 6-vectors $(v^{(\sigma)}|r^{(\sigma)})$ (such that span $(v^{(\sigma)}|r^{(\sigma)}) \equiv \mathscr{R}$, except for one special circumstance) so as to represent rigid velocity distributions, and require that the Lagrangian components of internal forces satisfy (5.8). The choice of $\mathscr{R}$ and $(v^{(\sigma)}|r^{(\sigma)})$ reflects the specific kinematic behaviour of the system.

EXAMPLE 1. Let $\mathscr{P}^{(1)}$, $\mathscr{P}^{(2)}$ be two points constrained to move along an assigned plane curve γ (a case of purely external constraint); choose as Lagrangian coordinates v_1, v_2 the arc lengths measured from a fixed origin along γ to $\mathscr{P}^{(1)}$, $\mathscr{P}^{(2)}$ respectively, divided by the factor α. Suppose that the only active force derives from a binary interaction. λ_1 and $-\lambda_2$ become the components along the tangents to γ at $\mathscr{P}^{(1)}$, $\mathscr{P}^{(2)}$ respectively of one vector representing the interaction multiplied by α.

Any virtual rigid velocity distribution, being plane, must have either translatory or rotatory character. The second instance occurs if the tangents to γ at $\mathscr{P}^{(1)}$, $\mathscr{P}^{(2)}$ are not parallel; then the virtual rotation is around an axis (of unit vector c) normal to the plane through the point $\mathscr{C}$ of intersection of the normals to γ at $\mathscr{P}^{(1)}$, $\mathscr{P}^{(2)}$ and

$$v = 0, \qquad r = \beta c \qquad (\beta \text{ arbitrary scalar}).$$

Correspondingly, if we call $\alpha\varrho_1$, $\alpha\varrho_2$ the distances of $\mathscr{P}^{(1)}$, $\mathscr{P}^{(2)}$ from $\mathscr{C}$, we have

$$v^{(\sigma)} = 0, \qquad r^{(\sigma)} = \varrho_\sigma c;$$

then, condition (5.8) requires

$$-\lambda_1/\lambda_2 = \varrho_2/\varrho_1. \tag{5.9}$$

A simple geometric construction proves that this equation is satisfied if and only if λ_1 and $-\lambda_2$ are, as implied in the premises, the components of the same vector along the tangents to γ at $\mathscr{p}^{(1)}$, $\mathscr{p}^{(2)}$ respectively. The "advantage" of (5.9) is in its intrinsic character, not requiring explicit reference to the properties of binary interactions.

EXAMPLE 2. Let $e^{(i)}$ be K fixed vectors not all belonging to one plane and G a non-singular tensor which is a function of one lagrangian variable v. Consider the system with one degree of freedom constrained as follows $p^{(i)} = G(v)e^{(i)}$.

Then the kinetic energy and the Lagrangian component of force are respectively

$$\mu\varkappa = \tfrac{1}{2}\mu I\cdot(H^T H)\dot{v}^2, \qquad \lambda = H^T\cdot M$$

where

$$I = \frac{1}{\mu}\sum_{i=1}^{K}\mu^{(i)} p^{(i)}\otimes p^{(i)}, \qquad H = \frac{dG}{dv}G^{-1}, \qquad M = \sum_{i=1}^{K} p^{(i)}\otimes f^{A,(i)}.$$

Rigid virtual velocity distributions are possible only with

$$v = 0, \qquad r = -\tfrac{1}{2}\dot{v}^R e(H).$$

There is only one 6-vector in the set $\{(v^{(\sigma)}|r^{(\sigma)})\}$ and it is $(0|0)$ if $e(H) = 0$ and otherwise $\left(0\middle| -\frac{1}{2}\frac{e(H)}{(e(H))^2}\right)$. The reduced resultant and resultant moment are automatically zero in the first case and otherwise

$$\tilde{f}^{\mathrm{I}} = 0, \qquad \tilde{m}^{\mathrm{I}} = -\tfrac{1}{2}\lambda^{\mathrm{I}}\frac{e(H)}{(e(H))^2};$$

hence, conditions (5.8) do not restrict λ^{I} at all if skw $H = 0$; if this relation is not satisfied, then λ^{I} must vanish.

This example shows that two dynamic systems with one degree of freedom may have a totally different kinetic behaviour which reflects on the allowable choice of the lagrangian component of internal forces: two extreme cases are $G\in\mathrm{Orth}^+$ (when $H\in\mathrm{Skw}$ and hence λ^{I} is necessarily zero) and $G\in\mathrm{Sph}$ (when λ^{I} is not constrained at all).

EXAMPLE 3. Let $\xi^{(i)}$ be K constants and $c^{(\sigma)}$, $\sigma = 1, 2, 3$, three orthogonal unit vectors. Consider the system with three degrees of freedom subject to the constraint

$$p^{(i)} = \xi^{(i)}\sum_{\sigma=1}^{3} v_\sigma c^{(\sigma)}.$$

We can write

$$p^{(i)} = \xi^{(i)} d, \qquad d = \sum_{\sigma=1}^{3} v_\sigma c^{(\delta)}.$$

Kinetic energy and Lagrangian components of force are respectively

$$\mu \varkappa = \tfrac{1}{2} \mu \iota \dot{d}^2, \qquad \lambda_\sigma = g^{\mathrm{A}} \cdot c^{(\sigma)},$$

where

$$\iota = \frac{1}{\mu} \sum_{i=1}^{K} \mu^{(i)} (\xi^{(i)})^2, \qquad g^{\mathrm{A}} = \sum_{i=1}^{K} \xi^{(i)} f^{\mathrm{A},(i)}.$$

In a rigid velocity distribution $v = 0$ and r is arbitrary, whereas

$$\dot{d}^R = \sum_{\sigma=1}^{3} \dot{v}_\sigma^R c^{(\sigma)} = r \times d \qquad \text{or} \qquad \dot{v}_\sigma^R = (d \times c^{(\sigma)}) \cdot r$$

hence

$$v^{(\sigma)} = 0, \qquad r^{(\sigma)} = d \times c^{(\sigma)}.$$

This example brings forth one of the circumstances which we have called exceptional in Section 5. Here $\mathscr{R}$ is larger than span $(v^{(\sigma)} | r^{(\sigma)})$: all $r^{(\sigma)}$ are orthogonal to d, whereas r is arbitrary.

The first condition (5.8) is automatically satisfied, whereas the second requires that $g^{\mathrm{I}} = \sum_{i=1}^{K} \xi^{(i)} f^{\mathrm{I},(i)}$ be parallel to d.

EXAMPLE 4. We consider here a system with nine degrees of freedom and interpret the nine Lagrangian coordinates as the components of a second order tensor G with respect to the usual frame. In fact we substitute indices from 1 to 9 with a couple of indices running from 1 to 3

$$G = \sum_{\alpha,\beta=1}^{3} v_{\alpha\beta} c^{(\alpha)} \otimes c^{(\beta)}.$$

Borrowing notation from Example 2, we consider a system constrained as follows

$$p^{(i)} = G e^{(i)}$$

and assume for simplicity

$$\sum_{i=1}^{K} \mu^{(i)} e^{(i)} = 0.$$

The kinetic energy becomes

$$\mu \varkappa = \tfrac{1}{2} \mu \operatorname{tr} (\dot{G} T \dot{G}^T) \qquad \text{or} \qquad \mu \varkappa = \tfrac{1}{2} \mu \operatorname{tr} (W I W^T),$$

where

$$\mu T = \sum_{i=1}^{K} \mu^{(i)} e^{(i)} \otimes e^{(i)}, \qquad W = \dot{G} G^{-1}, \qquad I = GTG^T.$$

The Lagrangian components of forces are related with the generalized moment of forces as follows

$$(G^{-1} M)^T = \sum_{i=1}^{K} f^{\Lambda,(i)} \otimes e^{(i)} = \sum_{\alpha,\beta=1}^{n} \lambda_{\alpha\beta}\, c^{(\alpha)} \otimes c^{(\beta)}$$

and the equations of Lagrange become the equations of generalized moment of momentum for affine bodies

$$\mu G T \ddot{G} = M.$$

Rigid velocity distributions are possible with $v = 0$ and r arbitrary; the vectors $v^{(\varrho,\sigma)}$ vanish, whereas the τth component of $r^{(\varrho,\sigma)}$ coincides with $-e_{\varrho\sigma\tau}$, so that conditions (5.8) require

$$M^{\mathrm{I}} \in \mathrm{Sym}.$$

6. Systems with unconstrained centre of gravity

For our purposes it is appropriate to consider explicitly the special case of a constrained system, for which the centre of gravity is allowed arbitrary placements; in that case the analysis of Section 5 becomes simpler: the reduced formulae are quoted below.

Let us begin with the general remark that the position vector p of the centre of gravity is also a function of $v_1, \ldots, v_n$ and if we put

$$q^{(i)} = p^{(i)} - p,$$

we have also

$$\dot{p}^{(i)} = \sum_{\sigma=1}^{n} \frac{\partial \hat{p}}{\partial v_\sigma} \dot{v}_\sigma + \sum_{\sigma=1}^{n} \frac{\partial \hat{q}^{(i)}}{\partial v_\sigma} \dot{v}_\sigma.$$

Now the matrix $\Delta_{3(i-1)+j,\sigma} = \dfrac{\partial q_j^{(i)}}{\partial v_\sigma}$, contrary to the matrix $\Pi_{3(i-1)+j,\sigma}$, need not have maximal characteristic. Then a set of non-null values $\tilde{\dot{v}}_\sigma$ of $\dot{v}_\sigma$ may exist, such that

$$\sum_{\sigma=1}^{n} \frac{\partial q^{(i)}}{\partial v_\sigma} \tilde{\dot{v}}_\sigma = 0;$$

as a consequence the corresponding vectors $\dot{p}^{(i)}$ are independent of the index i and a translatory speed distribution becomes possible. The interest lies here with the case when any choice of v is always possible, a case when the matrix $\Delta_{A\sigma}$ has characteristic exactly equal to $n-3$, and it is generally possible to express $q^{(i)}$ in terms of $m=n-3$ lagrangian coordinates only, say $v_1, v_2, \ldots, v_m$. The simple example given below shows that the requirement that the centre of gravity may take up any placement does not imply necessarily that any rigid translation speed is admissible.

However, outside that example, we will restrict our attention to the case when *the vectors $q^{(i)}$ can be expressed in terms of the first m Lagrangian variables only; and besides the last three variables can be taken to be proportional* (by a factor α^{-1}) *to the coordinates of the centre of gravity.*

The assumption does not seem to be too heavy in view of our goal and, on the other hand, allows one to simplify the formal developments considerably.

EXAMPLE. With the usual notation let the constraint relation be

$$p^{(i)} = p + \gamma |p|^2 e^{(i)},$$

where again $e^{(i)}$ are fixed vectors not all in the same direction but with

$$\sum_{i=1}^{K} \mu^{(i)} e^{(i)} = 0;$$

γ is a constant scalar. The system has three degrees of freedom; the Lagrangian variables are the components of p (i.e., the coordinates of the centre of gravity) divided by α. The kinetic energy is given by

$$\mu\varkappa = \tfrac{1}{2}\mu(\dot{p}^2 + 4\iota\gamma^2 (p\cdot\dot{p})^2),$$

where now

$$\mu\iota = \sum_{i=1}^{K} \mu^{(i)} (e^{(i)})^2.$$

The Lagrangian components of active forces are the components of the vector

$$\alpha(f^{\mathrm{A}} + 2\gamma\beta^{\mathrm{A}} p)$$

on the usual frame; here f^{A} is the resultant active force $\sum_{i=1}^{K} f^{\mathrm{A},(i)}$ and β^{A} is the virial $\sum_{i=1}^{K} e^{(i)} \cdot f^{\mathrm{A},(i)}$.

The set $\mathscr{R}$ is given by all 6-vectors of the type $(e\times p|0)$, with e an arbitrary vector; the quantities v_σ^R coincide with the components of the vector $\alpha^{-1} v$. The vectors $v^{(\sigma)}$ are the vector components normal to p of the unit vectors of the frame of reference:

$$v^{(\sigma)} = \left(1 - \frac{p\otimes p}{|p|^2}\right) c^{(\sigma)}.$$

The reduced resultant moment $\tilde{m}^{\mathrm{I}}$ vanishes identically, whereas the first condition (5.8) requires

$$\left(1-\frac{p\otimes p}{|p|^2}\right)f^{\mathrm{I}}=0,$$

with an obvious notation.

We conclude with the remark that the equations of Lagrange can be written compactly as follows

$$\mu(1+4\eta\gamma^2\,p\otimes p)\,\ddot{p}=f^{\mathrm{E}}+2\gamma\beta^{\mathrm{E}}\,p,$$

a vector equation which has similarities with the first Euler equation but involves modifiers both in the right- and left-hand side.

Let us return to the main topic of this section. It is obvious that a translatory speed distribution is possible, because now as a consequence of the assumption

$$p^{(i)}=q^{(i)}(v_1,\ldots,v_m)+\alpha(v_{n-2}c^{(1)}+v_{n-1}c^{(2)}+v_n c^{(3)}),$$

we have

$$\dot{p}^{(i)}=\sum_{\sigma=1}^{m}\frac{\partial q^{(i)}}{\partial v_\sigma}\dot{v}_\sigma+\alpha(\dot{v}_{n-2}c^{(1)}+\dot{v}_{n-1}c^{(2)}+\dot{v}_n c^{(3)}),$$

and we need only take $\dot{v}_\sigma=0$ for $\sigma=1,2,\ldots,m$ to obtain values of $\dot{p}^{(i)}$ independent of the index i. We can think of $\mathscr{R}$ as the product $\mathscr{R}=\boldsymbol{R}^3\times\tilde{\mathscr{R}}$, where $\tilde{\mathscr{R}}$ is a linear subspace of $\boldsymbol{R}^3$, where the vectors lie which represent allowable rotational speeds. The vectors $v^{(\sigma)}$ vanish for $\sigma=1,\ldots,m$, and $v^{(n-2)}=c^{(1)}$, $v^{(n-1)}=c^{(2)}$, $v^{(n)}=c^{(3)}$; vice versa $r^{(n-2)}$, $r^{(n-1)}$, $r^{(n)}$ vanish.

$\tilde{f}^{\mathrm{I}}$ coincides with the complete resultant of internal forces and the first eqn (5.8) requires it to be zero. All interest centres on $\tilde{\mathscr{R}}$, the set $\{r^{(\sigma)}|\sigma=1,\ldots,m\}$ and $\tilde{m}^{\mathrm{I}}=\sum_{\sigma=1}^{m}\lambda_\sigma r^{(\sigma)}$.

The equations of Lagrange become

$$\mu\ddot{p}=f^{\mathrm{E}},$$

$$\mu\left(\frac{d}{d\tau}\left(\frac{\partial\tilde{\varkappa}}{\partial\dot{v}_\sigma}\right)-\frac{\partial\tilde{\varkappa}}{\partial v_\sigma}\right)=\lambda_\sigma,\qquad\sigma=1,2,\ldots,m, \tag{6.1}$$

where $\mu\tilde{\varkappa}$ is the reduced kinetic energy

$$\mu\tilde{\varkappa}=\tfrac{1}{2}\sum_{\varrho,\sigma=1}^{m}\mu_{\varrho\sigma}\dot{v}_\varrho\dot{v}_\sigma, \tag{6.2}$$

which is again a positive definite quadratic form in $\dot{v}_\sigma$ for $\sigma=1,2,\ldots,m$.

Equations (6.1), (6.2) are the point of departure of the developments of the following sections.

In conclusion the systems $\tilde{\mathscr{S}}$ that interest us are defined by the number m of "extra" degrees of freedom, by the positive constant μ and by the functions $\mu_{\varrho\sigma}(v_\tau)$, which form a positive definite matrix. From a kinematic point of view the specification of the system is complete when the subspace $\tilde{\mathscr{R}}$ of $\boldsymbol{R}^3$ is given together with the set $\{r^{(\sigma)}\}$ of vectors belonging to $\tilde{\mathscr{R}}$. The dynamic behaviour of $\tilde{\mathscr{S}}$ is the consequence of the choice of the vector function $f^{\mathrm{E}}(p;\, \dot{p};\, v_\sigma;\, \dot{v}_\sigma;\, \tau)$ and the m scalar functions λ_σ of the same variables. Besides, $\lambda_\sigma^{\mathrm{I}}$ must satisfy the condition

$$\sum_{\sigma=1}^{m} \lambda_\sigma^{\mathrm{I}} r^{(\sigma)} = 0. \tag{6.3}$$

The most interesting special case is when both $\lambda_\sigma^{\mathrm{CI}}$ and $\lambda_\sigma^{\mathrm{CE}}$ vanish, not only their sum (because the constraints are, for instance, purely internal; see Remark in Sect. 4); then $\lambda_\sigma^{\mathrm{I}}$ are explicit functions of the variables p, $\dot{p}$, v_σ, $\dot{v}_\sigma$, τ and condition (6.3) must apply for all their values and thus restricts the allowable choices of $\lambda_\sigma^{\mathrm{I}}$.

Such special case is the basis for the developments of the last two sections.

7. Continuous bodies with finite-dimensional microstructure

We wish to model here a continuous body $\mathfrak{B}$ with microstructure. $\mathfrak{B}$ is a set of particles $\mathscr{X}$, which is equipped with a class of one-to-one mappings $p(\mathscr{X})$ from $\mathfrak{B}$ into open subsets $\mathscr{B}$ of the three-dimensional Euclidean space. Each mapping $p(\mathscr{X})$ specifies, however, only an *apparent placement* of $\mathfrak{B}$. The microstate of each particle must still be made precise: it is assumed that the microstate is determined when a set of m non-dimensional Lagrangian variables $v_\sigma(\mathscr{X})$ is given. Thus a *complete placement* of $\mathfrak{B}$ is specified by a choice of both $p(\mathscr{X})$ and $v_\sigma(\mathscr{X})$. Correspondingly a complete motion of $\mathfrak{B}$ is a mapping of a time interval $[0, \tau)$ into an appropriate set of complete placements. At the outset there is no need to declare the mechanical meaning of the variables v_σ, although it must be presumed that they represent a greater kinematic freedom tempered perhaps by conditionings of purely internal character (as in the situations envisaged at the end of Section 6). To respect this requirement a first necessary step is to assign appropriately the set $\tilde{\mathscr{R}}(\mathscr{X})$ of the rigid rotational speeds r allowed in the microstructure; more precisely if (v, s) are the vectors which specify a rigid velocity distribution for $\mathfrak{B}$

$$\dot{p} = v + s \times p,$$

then the projector $\mathscr{T}$ from $\boldsymbol{R}^3$ into $\tilde{\mathscr{R}}$ must be assigned such that

$$r = \mathscr{T}s, \qquad \{r\} \equiv \tilde{\mathscr{R}}, \qquad \forall s, \tag{7.1}$$

and this projector must be objective.

Besides a set $r^{(\sigma)}$ ($\sigma = 1, 2, \ldots, m$) of vectors must be given within $\tilde{\mathscr{R}}$ which determines the kinematic consequences on the Lagrangian variables of a global rigid rotation of $\mathfrak{B}$:

$$\dot{v}_\sigma^R = r^{(\sigma)} \cdot r = r^{(\sigma)} \cdot \mathscr{T}s = r^{(\sigma)} \cdot s. \tag{7.2}$$

Both $\mathscr{T}$ and the set $\{r^{(\sigma)}\}$ may depend on the particle, and on the values of the variables v_σ, as occurs already in the examples of the two previous sections.

Relations (7.1) and (7.2) connect in a definite manner the microstructure to the whole body and are thus constitutive in character.

The next step in the definition of $\mathfrak{B}$ is the specification of the kinetic energy $\varkappa_{\mathscr{M}}$ of $\mathfrak{B}$ associated with a given kinetic state ($\dot{p}$; $\dot{v}_\sigma$). It is assumed that a smooth density function ϱ of $\mathscr{X}$ (or of p) exists which represents the mass per unit volume and which is subject to the condition of conservation of mass

$$\int_{\mathscr{M}} \varrho = \text{const}. \tag{7.3}$$

Besides we need a set of functions $\mu_{\varrho\sigma}$ of $\mathscr{X}$ and v_σ, which are subject to the condition to be the coefficients of a definite positive form, such that

$$\varkappa_{\mathscr{M}} = \int_{\mathscr{M}} \varrho(\tfrac{1}{2}\dot{p}^2 + \tilde{\varkappa}), \qquad \tilde{\varkappa} = \tfrac{1}{2} \sum_{\varrho,\sigma=1}^{m} \mu_{\varrho\sigma} \dot{v}_\varrho \dot{v}_\sigma. \tag{7.4}$$

We have here an obvious strict analogy with (6.2).

The restriction implied by (7.4) is heavy; much more complex situations could be envisaged. We are led here by a criterium of simplicity.

The equation of balance of momentum is carried over from the theory of classical continua. It is assumed that external forces are of two types: body force of density b per unit mass and surface traction of density t per unit surface so that the analogue of $(6.1)_1$ becomes

$$\int_{\mathscr{M}} \varrho\ddot{p} = \int_{\mathscr{M}} \varrho b + \int_{\partial\mathscr{M}} t. \tag{7.5}$$

The Lagrangian components of external forces are assumed to be expressed by similar totals of external body force with density β_σ per unit mass, and surface traction of density τ_σ per unit surface. Besides one must recall that the Lagrangian internal force need not be zero; its density per unit volume is called here $-\zeta_\sigma$. In all, the Lagrangian equations $(6.1)_2$ are adapted for the body $\mathscr{B}$ as follows

$$\int_{\mathscr{B}} \varrho\left(\frac{d}{dt}\left(\frac{\partial \tilde{\varkappa}}{\partial \dot{v}_\sigma}\right) - \frac{\partial \tilde{\varkappa}}{\partial v_\sigma}\right) = \int_{\mathscr{B}} (\varrho \beta_\sigma - \zeta_\sigma) + \int_{\partial \mathscr{B}} \tau_\sigma. \tag{7.6}$$

Equations (7.5), (7.6) are assumed to apply also when they are referred to an arbitrary subbody of $\mathfrak{B}$. Then, under conditions of sufficient smoothness of the fields involved, it is possible to follow a traditional argument and prove that a tensor field T exists such that

$$t = Tn, \tag{7.7}$$

where n is the exterior normal to $\partial\mathscr{B}$. Similarly one arrives at the proof of the existence of m vector fields $t^{(\sigma)}$ such that

$$\tau_\sigma = n \cdot t^{(\sigma)}. \tag{7.8}$$

Finally it is possible to state local consequences of (7.5), (7.6) in the form

$$\begin{gathered} \varrho \ddot{p} = \varrho b + \operatorname{div} T, \\ \varrho\left(\frac{d}{d\tau}\left(\frac{\partial \tilde{\varkappa}}{\partial \dot{v}_\sigma}\right) - \frac{\partial \tilde{\varkappa}}{\partial v_\sigma}\right) = \varrho \beta_\sigma - \zeta_\sigma + \operatorname{div} t^{(\sigma)}. \end{gathered} \tag{7.9}$$

A theorem of kinetic energy is now easily derived from eqns (7.9) multiplying scalarly both sides of $(7.9)_1$ by $\dot{p}$, multiplying both members of $(7.9)_2$ by $\dot{v}_\sigma$ and summing over σ from 1 to m and integrating finally over $\mathscr{B}$. After integration by parts, one obtains

$$\begin{aligned} \frac{d}{d\tau} \int_{\mathscr{B}} \varrho(\tfrac{1}{2}\dot{p}^2 + \tilde{\varkappa}) & \\ = \int_{\mathscr{B}} \varrho\Big(b \cdot \dot{p} + \sum_{\sigma=1}^{m} \beta_\sigma \dot{v}_\sigma\Big) &+ \int_{\partial\mathscr{B}} \Big(t \cdot \dot{p} + \sum_{\sigma=1}^{m} \tau_\sigma \dot{v}_\sigma\Big) - \\ &- \int_{\mathscr{B}} \Big(T \cdot \operatorname{grad} \dot{p} + \sum_{\sigma=1}^{m} \zeta_\sigma \dot{v}_\sigma + \sum_{\sigma=1}^{m} t^{(\sigma)} \cdot \operatorname{grad} \dot{v}_\sigma\Big). \end{aligned} \tag{7.10}$$

The terms in this energy balance equation are easily interpreted. The term in the left-hand side is the time rate of kinetic energy; the first two integrals in the right-hand side give the power of external body force and surface traction; the last integral must thus be interpreted as an expression of the power of internal actions.

It remains for us to require that the power density of internal actions be zero for any rigid velocity distribution when

$$\operatorname{grad} \dot{p}^R = es, \qquad \dot{v}_\sigma^R = r^{(\sigma)} \cdot s,$$

where s is any constant vector. The condition is easily deduced by substitution in the relation

$$T \cdot \operatorname{grad} \dot{p}^R + \sum_{\sigma=1}^{m} \zeta_\sigma \dot{v}_\sigma^R + \sum_{\sigma=1}^{m} t^{(\sigma)} \cdot \operatorname{grad} \dot{v}_\sigma^R = 0,$$

leading to

$$eT + \sum_{\sigma=1}^{m} \left(\zeta_\sigma r^{(\sigma)} + (\operatorname{grad} r^{(\sigma)}) t^{(\sigma)}\right) = 0, \tag{7.11}$$

or alternatively to

$$\operatorname{skw} T + \tfrac{1}{2} e\Big(\sum_{\sigma=1}^{m} \left(\zeta_\sigma r^{(\sigma)} + (\operatorname{grad} r^{(\sigma)}) t^{(\sigma)}\right)\Big) = 0. \tag{7.12}$$

Needless to say this condition requires T to be symmetric when the microstructure is absent; in general it states a necessary link between macro- and microbehaviour.

8. Examples of microstructures

(i) *One degree of freedom.* The governing equations are Cauchy's balance equation $(7.9)_1$ and a scalar equation

$$\varrho\left(\mu_{11} \ddot{v}_1 + \tfrac{1}{2} \frac{d\mu_{11}}{dv_1} \dot{v}_1^2\right) = \varrho\beta_1 - \zeta_1 + \operatorname{div} t^{(1)}. \tag{8.1}$$

Two important simple subcases come to mind. In the first case one tries to picture rigid particles which can spin around a materially fixed axis. Then: (i) r is the vector component of s in the direction of Fc, where F is the macro position gradient and c an appropriate constant unit vector; (ii) hence $\mathscr{T}$ is given by $(c \cdot F^T Fc)^{-1} F(c \otimes c) F^T$; (iii) $r^{(1)}$ is given by $(c \cdot F^T Fc)^{-1/2} Fc$.

Finally μ_{11} is a positive constant. In the left-hand side of (8.1) the second addendum vanishes; besides (7.11) gives

$$eT = \zeta_1 (c \cdot F^T Fc)^{-1/2} Fc + \left(\operatorname{grad} \left((c \cdot F^T Fc)^{-1/2} Fc\right) t^{(1)}\right)$$

which is a restriction on the constitutive prescriptions of T, ζ_1 and $t^{(1)}$.

In the second case one tries to picture a continuum with expanding and contracting spherical microstructure: e.g., fluids with inert bubbles. Then $\mathscr{T}$ vanishes and $r^{(1)}$ is taken to be zero; as a consequence T is symmetric. If a suggestion deriving from a formula of Rayleigh (on expanding bubbles in an indefinite incompressible medium) is taken up, one can choose $\mu_{11} = \alpha v_1^3$ (α a constant) and the left-hand side of (8.1) becomes $\varrho\alpha v_1^{3/2}(v_1^{3/2}\dot{v}_1)^{\cdot}$.

(ii) *Three degrees of freedom.* Suppose that v_1, v_2, v_3 are the components with respect to a fixed cartesian reference of a vector d, as in Example 3 of Section 5.

Then it is convenient to think also ζ_σ and β_σ as the components of two vectors z and l respectively and to introduce also the tensors

$$(8.2)\qquad \begin{aligned} M &= \sum_{\varrho,\sigma=1}^{3} \mu_{\varrho\sigma} c^{(\varrho)} \otimes c^{(\sigma)}, \\ H &= \sum_{\sigma=1}^{3} c^{(\sigma)} \otimes t^{(\sigma)}. \end{aligned}$$

Then the balance equation $(7.9)_2$ can be given the form

$$(8.3)\qquad \varrho\left(M\ddot{d}+\left(\frac{\partial(M\dot{d})}{\partial d}-\tfrac{1}{2}\left(\frac{\partial(M\dot{d})}{\partial d}\right)^T\right)\dot{d}\right) = \varrho l - z + \operatorname{div} H,$$

which appears in many theories of structured materials.

As we have not given d any specific significance so far, we can proceed to many different choices of $\mathscr{T}$ and the vectors $r^{(\sigma)}$. If we are inspired by Example 3 of Section 5 we can choose

$$\mathscr{T} = 1 - \frac{d \otimes d}{|d|^2}, \qquad r^{(\sigma)} = d \times c^{(\sigma)};$$

introduction of these vectors in (7.12) and use of $(8.2)_2$ leads to the condition (already appearing in Ericksen's theory of liquid crystals)

$$\operatorname{skw}\left(T + d \otimes z + (\operatorname{grad} d) H^T\right) = 0.$$

As an alternative we may think of using the three Lagrangian variables to specify a rigid rotation

$$Q = e^{ed}, \qquad Q \in \mathrm{Orth}.$$

This assumption implies that $\dot{d}$ may be interpreted directly as the vector r assigning an arbitrary angular velocity:

$$\tilde{\mathscr{R}} = \boldsymbol{R}^3, \qquad \mathscr{T} = 1, \qquad \dot{d}^R = r, \qquad r^{(\sigma)} = c^{(\sigma)}.$$

Reference to (7.12) leads to the condition $eT - z = 0$.

(iii) *Nine degrees of freedom.* Assume that $m = 9$ and, as in the Example 4 of Section 5, that the Lagrangian coordinates are the components with respect to the usual orthogonal basis of a second-order tensor G. Then it is again convenient to think of ζ_σ and β_σ as the components of two tensors X and B respectively, the components of the vectors $t^{(\sigma)}$ as the components of a third-order tensor $\boldsymbol{t}$ and $\mu_{\varrho\sigma}$ as the components of a fourth-order tensor $\boldsymbol{M}$ so that

$$\tilde{\chi} = \tfrac{1}{2}\dot{G}\cdot(M\dot{G}).$$

Correspondingly eqn $(7.9)_2$ becomes

$$\varrho\left(M\ddot{G}+\left(\frac{\partial(M\dot{G})}{\partial G}-\frac{1}{2}\left(\frac{\partial(M\dot{G})}{\partial G}\right)^T\right)\dot{G}\right) = \varrho B - X + \operatorname{div} \boldsymbol{t}.$$

This equation can be brought to coincide with the balance equation of generalized moment of momentum of continua with affine microstructure: The identifications are

$$\boldsymbol{M} = 1 \boxtimes \mathscr{T} \quad (\text{i.e., } \boldsymbol{M}_{ijhk} = \delta_{ih}\mathscr{T}_{jk}), \qquad B = (G^{-1}L)^T, \qquad \boldsymbol{t}_{ijk} = G_{jl}^{-1}\boldsymbol{h}_{lik}.$$

There are many possible choices for the vectors $r^{(\sigma)}$, the components of which can be set as components of a third-order tensor. In the theory of affine bodies one takes

$$\dot{G}^R = SG, \qquad S \in \mathrm{skw}.$$

As a consequence one finds that condition (7.12) requires that the tensor

$$Z_{ij} = T_{ij} + X_{ih}G_{jh} + G_{jrs}\boldsymbol{t}_{irs}$$

be symmetric.

Presented to the semester
Mathematical Models and Methods
in Mechanics