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BRANCHING OF ASYMPTOTICS FOR ELLIPTIC OPERATORS ON MANIFOLDS WITH EDGES

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The paper gives a summary of the authors' results on the asymptotic behaviour of solutions of elliptic equations on manifolds with edges. It is part of a program to deal with operators on manifolds with more general singularities or operators that degenerate at certain points or manifolds. Our theory contains a complete symbolic calculus which enables us to solve the problem of solutions with branching asymptotics, i.e. with asymptotic expansion of the form

$$u(y, t, x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j(y)} \zeta_{jk}(y, x) t^{p_j(y)} \log^k t \qquad \text{as } t \to 0,$$

where y is the point on the edge Y, t the distance to Y, $(p_j)_{j \in Z_+}$ a sequence of complex numbers and x the remaining coordinates. Both $p_j(y)$ and $m_j(y)$ may depend on y with a chaotic branching and jumping behaviour in y.

The pattern of couples (p_j, m_j) depends on the operator, and hence we have a rather individual sort of regularity of solutions. Roughly speaking, elliptic regularity for manifolds with edges (being the classical one in C^∞ or Sobolev spaces away the edge) is expressed here by the asymptotics, and it is really an urgent problem to give a precise description of this kind of regularity.

As far as the authors know the problem of establishing the branching of asymptotics of solutions has never been treated before, even in special cases. Our results give general answers. In addition we are convinced that the asymptotic regularity in the sense of Theorem 2 is of principal interest and that the continuous conormal asymptotics is a natural notion in this context.

Boundary problems in nonsmooth domains are an interesting and important subject in analysis (cf. [1], [2], [3], [6], [7], [8]). It is not our aim here to comment on the extensive literature. Let us refer to Kondrat'ev–Oleinik [2]. Our present work is based to a large extent on a precise algebra for the cone [8] and on the treatment of mixed boundary value problems [5] where the branching of asymptotics already occurs.

Let X be a Riemannian manifold with edge Y. Denote by $\check{X}$ the manifold with boundary which is obtained from X by resolving the singularity at Y, $\check{X} \to X$. Locally near $\partial\check{X}$ the manifold $\check{X}$ has the form $\Omega \times \bar{\boldsymbol{R}}_+ \times M$, $\Omega \subset \boldsymbol{R}^m$ open, M a smooth closed compact manifold, $n = \dim M$. It is convenient to consider operators lifted to $\check{X}$.

Let x be local coordinates on M, $t \in \boldsymbol{R}_+$, $y \in \Omega \subset \boldsymbol{R}^m$, and denote by (η, τ, ξ) the dual variables to (y, t, x). Differential operators on $\check{X}$ of order μ are assumed to have near $\partial\check{X}$ the form

$$A = \sum_{j=0}^{\mu} t^{-\mu+j} A_j,$$

$$A_j(y, D_y, t, D_t, x, D_x) = \sum_{|\alpha|+k \leqslant \mu-j} a_{\alpha k,j}(y, t, x, D_x)(tD_y)^\alpha (tD_t)^k,$$

where $a_{\alpha k,j}$ are differential operators on M, ord $a_{\alpha k,j} \leqslant \mu - j - |\alpha| - k$, smoothly depending on the parameters $y \in \Omega$, $t \in \boldsymbol{R}_+$. This form is well motivated because it appears in simplest cases, where the singularity is resolved by introducing polar coordinates. Using the Taylor expansion of $a_{\alpha k,j}$ at $t = 0$ we get

$$a_{\alpha k,j}(y, t, x, D_x) = \sum_{l=0}^{\infty} t^l \omega(c_l t) b_{\alpha k j l}(y, x, D_x) + \tilde{b}_{\alpha k,j}(y, t, x, D_x)$$

where $\tilde{b}_{\alpha k,j}$ is rapidly decreasing for $t \to 0$, $\omega \in C_0^\infty(\bar{\boldsymbol{R}}_+)$, $\omega \equiv 1$ near $t = 0$, and $\{c_l\}_{l \in \boldsymbol{Z}_+}$ is a sufficiently fast increasing sequence of positive reals. We are reduced to sums of strongly degenerate operators and converging sums of operators polynomially depending on t (near $t = 0$). Then

$$\begin{aligned} &A(y, D_y, t, D_t, x, D_x) \\ &\quad = t^{-\mu} \sum_{j=0}^{\mu} \Big\{ \sum_{|\alpha|+k \leqslant \mu-j} \Big[\sum_{l=0}^{\infty} t^{j+l} \omega(c_l t) b_{\alpha k j l}(y, x, D_x)(tD_y)^\alpha (tD_t)^k \\ &\qquad + \tilde{b}_{\alpha k,j}(y, t, x, D_x)(tD_y)^\alpha (tD_t)^k \Big] \Big\}. \end{aligned}$$

The Fourier transform with respect to y yields the complete *edge symbol* $A(y, \eta, t, D_t, x, D_x)$, which is, of course, operator-valued. The Mellin transform M turns tD_t into multiplication by iz and we can write $A(y, \eta, t, D_t, x, D_x)$ as an infinite sum of Mellin operators of decreasing conormal orders with operator-valued symbols and a strongly degenerate operator:

$$\begin{aligned} &A(y, \eta, t, D_t, x, D_x) \\ &\quad = t^{-\mu} \sum_{j=0}^{\mu} \Big\{ \sum_{|\alpha|+k \leqslant \mu-j} \Big[\sum_{l=0}^{\infty} t^{j+l} \omega(c_l t)(t\eta)^\alpha \operatorname{op}_M(b_{\alpha k j l}(y, x, D_x))(iz)^k \\ &\qquad + \tilde{b}_{\alpha k,j}(y, t, x, D_x)(t\eta)^\alpha (tD_t)^k \Big] \Big\}. \end{aligned}$$

This means that the complete edge symbol takes values in the algebra $\mathfrak{N}^{\infty}(\boldsymbol{R}_+ \times M)$ (modulo the unessential factor $t^{-\mu}$), which was introduced in [8] for the treatment of conical points. The major advantage of $\mathfrak{N}^{\infty}(\boldsymbol{R}_+ \times M)$ is that it contains the precise parametrices of elliptic elements. It allows in the present situation to obtain a strong regularity theory.

Denote by $\mathrm{Diff}^{\mu}(\check{X})$ the space of all differential operators on $\check{X}$ of order $\leqslant \mu$ of the described form. It turns out to be contained, up to a weight factor, in a larger class $\mathfrak{N}^{\mu}(\check{X})$ of operators which are described near the edge by operator-valued edge symbols with values in the algebra $\mathfrak{N}^{\infty}(\boldsymbol{R}_+ \times M)$. One can prove that $\mathfrak{N}^{\infty}(\check{X})$ is a $*$-algebra. To each $\mathscr{A} \in \mathfrak{N}^{\mu}(\check{X})$ there is related a complete interior symbol $\sigma_{\psi}(\mathscr{A})$ and a complete edge symbol $\sigma_{\partial}(\mathscr{A})$ which induces a sequence of operator-valued Mellin symbols $\sigma_M^{-j}(A)$. For simplicity we assume the existence of homogeneous principal symbols.

The formulation of the results demands some function spaces on $\check{X}$. Recall from [8] the definition of the scale of Hilbert spaces $\mathscr{H}^s(\boldsymbol{R}_+ \times M)$, $s \in \boldsymbol{R}$. $\mathscr{H}^s(\boldsymbol{R}_+ \times M)$ consists of all distributions $u(t, x)$ having in local coordinates finite seminorms

$$\|\varphi u(t, x)\|^2 = \int_{\Gamma_{1/2}} \int_{\boldsymbol{R}^n} (1+|z|^2+|\xi|^2)^s |M_t F_x(\varphi u)(z, \xi)|^2 \, d\xi \, |dz|,$$

where $\Gamma_\varrho = \{z \in \boldsymbol{C}\colon \operatorname{Re} z = \varrho\}$, F_x is the Fourier transform in x and M_t the Mellin transform in t, and φ is a smooth function supported in the coordinate neighbourhood on M.

Let g^{μ} be a strictly positive function on $\boldsymbol{R}_+ \times M$ which equals t^{μ} near $t = 0$ and 1 outside some other neighbourhood of $t = 0$. Define $\mathscr{H}^{s,\varrho} = g^{\varrho} \mathscr{H}^s$, $\varrho \in \boldsymbol{R}$, so that $\mathscr{H}^s = \mathscr{H}^{s,0}$. Moreover, define the space $H^s(\check{X})$, $s \in \boldsymbol{R}$, as the subspace of those $u \in H^s_{\mathrm{loc}}(\operatorname{int} \check{X})$ for which near Y

$$u|_{\Omega \times \boldsymbol{R}_+ \times M} \in C^{\infty}(\Omega, \mathscr{H}^s(\boldsymbol{R}_+ \times M)),$$

$\Omega \subset Y$ an arbitrary coordinate neighbourhood. Set

$$H^{s,\varrho}(\check{X}) = g^{\varrho} H^s, \qquad \varrho \in \boldsymbol{R}.$$

For fixed $(y, \eta) \in T^* Y \setminus 0$ the operator $A(y, \eta, t, D_t, x, D_x)$ defines a continuous mapping

$$A(y, \eta, t, D_t, x, D_x)\colon \mathscr{H}^{s,\varrho}(\boldsymbol{R}_+ \times M) \to \mathscr{H}^{s-\mu,\varrho-\mu}(\boldsymbol{R}_+ \times M)$$

for each $\varrho \in \boldsymbol{R}$. Therefore near Y we get a continuous mapping

$$A(y, D_y, t, D_t, x, D_x)\colon C^{\infty}(\Omega, \mathscr{H}^{s,\varrho}(\boldsymbol{R}_+ \times M)) \to C^{\infty}(\Omega, \mathscr{H}^{s-\mu,\varrho-\mu}(\boldsymbol{R}_+ \times M))$$

and globally

$$A\colon H^{s,\varrho}(\check{X}) \to H^{s-\mu,\varrho-\mu}(\check{X}).$$

The solvability theory for the differential operators considered above is formulated in terms of function spaces with asymptotics. Recall some notions of [8]. We start from the description of singularity types occurring in the so-called discrete conormal asymptotics.

A sequence $P = \{(p_j, m_j, L_j)\}_{j\in \mathbf{Z}_+}$, $p_j \in \mathbf{C}$, Re $p_j < \frac{1}{2} - \varrho$, $\varrho \in \mathbf{R}$ fixed, Re $p_j \to -\infty$ as $j \to \infty$, $m_j \in \mathbf{Z}_+$, $L_j \subset C^\infty(M)$ finite-dimensional, is called a *singularity type* with respect to ϱ. Denote by $\underline{P}^\varrho$ the set of all such P, and $\underline{P}^0 =: \underline{P}$.

Then $\mathscr{H}_P^{s;\varrho}(\mathbf{R}_+ \times M)$ denotes the subspace of $\mathscr{H}^{s,\varrho}(\mathbf{R}_+ \times M)$ of all functions $v(t, x)$ possessing for $t \to 0$ an asymptotics of the form

$$v(t, x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} \zeta_{jk}(x) t^{-p_j} \log^k t$$

where $\zeta_{jk} \in L_j$. The precise meaning of this asymptotics can be expressed in terms of a countable number of norms making $\mathscr{H}_P^{s;\varrho}(\mathbf{R}_+ \times M)$ a Fréchet space (cf. [8]). We shall also need the inductive limit $\mathscr{H}_{\underline{P}}^{s;\varrho} := \varinjlim_{P \in \underline{P}} \mathscr{H}_P^{s;\varrho}$.

The topology of this space is so strong that each continuous family of functions in $\mathscr{H}_P^{s;\varrho}(\mathbf{R}_+ \times M)$ automatically has exponents in the asymptotics independent of the parameter. This was the main reason for introducing in [5] the notion of continuous asymptotics which will allow a general branching of the exponents with varying parameters. Instead of discrete sets, multiplicities and finite-dimensional spaces, a *continuous singularity type* consists just of a closed simply connected set $\Delta \subset \{z \in \mathbf{C}\colon \text{Re } z \leqslant \frac{1}{2} - \varrho\}$ with the property that $\Delta \cap \{z \in \mathbf{C}\colon \varrho_1 \leqslant \text{Re } z \leqslant \varrho_2\}$ is compact for all $\varrho_1, \varrho_2 \in \mathbf{R}$. Then $\mathscr{H}_\Delta^{s;\varrho}(\mathbf{R}_+ \times M)$ is defined as the subspace of $\mathscr{H}^{s,\varrho}(\mathbf{R}_+ \times M)$ of those functions whose Mellin image admits a holomorphic extension to $\mathbf{C} \backslash \Delta$ with certain growth conditions defining the norms (this is a generalization of the notion in [5]). Denote by $C_\Delta^\infty(\check{X})$ the subspace of $C^\infty(\text{int } \check{X})$ of all functions u which are near Y, in local coordinates on $\Omega \subset Y$, of the form $u \in C^\infty(\Omega, \mathscr{H}_\Delta^\infty(\mathbf{R}_+ \times M))$. There is a natural countable system of norms making $C_\Delta^\infty(\check{X})$ a nuclear Fréchet space. Denote by $C_{\underline{\Delta}}^\infty(\check{X})$ the inductive limit of all $C_\Delta^\infty(\check{X})$ for Δ as described.

Now we give a slight modification of the "dotted spaces" with asymptotics from [5], here in the vector-valued case. Let $\Omega \ni y \mapsto P(y) \in \underline{P}$ be a family of discrete singularity types which are continuous in the sense that, roughly speaking, for each neighbourhood K of the set $\{p_j(y_0)\}_{j\in \mathbf{Z}_+} \subset \mathbf{C}$ there is a neighbourhood $U \subset \Omega$ of y_0 such that $p_j(y)$ belongs to K for all $j \in \mathbf{Z}_+$ and $y \in U$. Then we define $C^\infty(\Omega, C_{\dot{P}}^\infty(\mathbf{R}_+ \times M))$ as the space of all $u(y, t, x)$ which belong to $C_{P(y)}^\infty(\mathbf{R}_+ \times M)$ for each fixed y and, when restricted to the above-mentioned small neighbourhood U, belong to $C^\infty(U, C_K^\infty(\mathbf{R}_+ \times M))$. This space is equipped with the projective limit topology.

Denote by $C^\infty(\Omega, C_{\dot{\Delta}}^\infty(\mathbf{R}_+ \times M))$ the inductive limit of all spaces $C^\infty(\Omega, C_{\dot{P}}^\infty(\mathbf{R}_+ \times M))$ such that all $\{p_j(y)\}$ belong to Δ. If we define $C_{\dot{\Delta}}^\infty(\check{X})$ in the same way as $C_\Delta^\infty(\check{X})$ we get a continuous embedding $C_{\dot{\Delta}}^\infty(\check{X}) \hookrightarrow C_\Delta^\infty(\check{X})$. We shall also use the space $\mathscr{H}^{-\infty}(\check{X})$ of all extendible distributions u on $\check{X}$

which belong in local coordinates near the boundary to $H^{-\infty}(\Omega, \mathscr{H}^{-\infty}(\mathbf{R}_+ \times M))$.

Now we can state the first regularity theorem.

1. THEOREM. *Let $\mathscr{A} \in \mathfrak{N}^{\mu}(\check{X})$, let the principal interior symbol on the compressed cotangent bundle be invertible up to the boundary and assume that the principal Mellin symbol is elliptic in the sense of $\mathfrak{N}^{\mu}(\mathbf{R}_+ \times M)$ for a given weight $\varrho \in \mathbf{R}$. Then*

$$\mathscr{A}u \in C^{\infty}_{\Delta}(\check{X}) \text{ and } u \in H^{-\infty,\varrho}(\check{X}) \quad \text{imply} \quad u \in C^{\infty}_{\Delta}(\check{X}).$$

This regularity result shows that the branching of asymptotics of the right-hand side implies the same property of the solution. Of course, in general, there is a "loss of tangent regularity".

Note that the operator $\mathscr{A}$ does not define, in general, a Fredholm operator. Both the kernel and the cokernel of $\mathscr{A}\colon C^{\infty}_{\Delta}(\check{X}) \to C^{\infty}_{\Delta}(\check{X})$ may be infinite-dimensional.

In order to get also tangent regularity and the Fredholm property we have to add, to the assumptions of Theorem 1, the bijectivity of the edge symbol. This is not a consequence of the assumptions of Theorem 1. However, upon imposing extra trace and potential conditions with respect to Y, the bijectivity of the edge symbol can be achieved provided a certain topological obstruction vanishes.

Let us introduce the class $\mathfrak{W}^{\infty}(\check{X})$ of operators

$$\mathscr{A} = \begin{pmatrix} \underline{A} & P \\ T & Q \end{pmatrix} \colon \begin{matrix} C^{\infty}_{\Delta}(\check{X}) \\ \oplus \\ C^{\infty}(Y, \mathbf{C}^{j_1}) \end{matrix} \to \begin{matrix} C^{\infty}_{\Delta}(\check{X}) \\ \oplus \\ C^{\infty}(Y, \mathbf{C}^{j_2}) \end{matrix}$$

where $\underline{A} = A + W + G \in \mathfrak{N}^{\infty}(\check{X})$, $Q \in L^{\infty}_{\mathrm{cl}}(Y)$, T is a trace operator and P a potential operator with respect to Y with some specific structure. Roughly speaking, $\begin{pmatrix} W+G & P \\ T & Q \end{pmatrix}$ is a pseudo-differential operator on Y with an operator-valued symbol $a(y, \eta)$, a special edge symbol. Each $\mathscr{A} \in \mathfrak{W}^{\infty}$ is uniquely determined by the complete symbol $\underline{\sigma}(\mathscr{A})$ consisting of the complete interior symbol, the complete Mellin symbol and the complete edge symbol modulo a small space of negligible operators. $\mathscr{A} \in \mathfrak{W}^{\infty}$ is called *elliptic* if $\underline{\sigma}(\mathscr{A})$ is invertible in $\mathrm{Symb}(\mathfrak{W}^{\infty})$. The ellipticity can be described on the principal symbol level.

2. THEOREM. *Let $\mathscr{A} \in \mathfrak{W}^{\infty}$ be elliptic. Then there exists a parametrix $\mathscr{B} \in \mathfrak{W}^{\infty}$ of $\mathscr{A}$ and $\mathscr{A}$ defines a Fredholm mapping*

$$\mathscr{A}\colon C^{\infty}_{\Delta}(\check{X}) \oplus C^{\infty}(Y, \mathbf{C}^{j_1}) \to C^{\infty}_{\Delta}(\check{X}) \oplus C^{\infty}(Y, \mathbf{C}^{j_2}).$$

Moreover, $\mathscr{A}w \in C^{\infty}_{\Delta}(\check{X}) \oplus C^{\infty}(Y, \mathbf{C}^{j_2})$ and $w \in \mathscr{H}^{-\infty}(\check{X}) \oplus H^{-\infty}(Y, \mathbf{C}^{j_1})$ imply $w \in C^{\infty}_{\Delta}(\check{X}) \oplus C^{\infty}_{\Delta}(Y, \mathbf{C}^{j_1})$.

Even stronger results can be obtained for the subalgebra $\mathfrak{W}^{\cdot\infty}$ of $\mathfrak{W}^{\infty}$, where in all Mellin symbols, Green potential symbols and trace symbols the dotted spaces are taken. Then we have

3. THEOREM. *Let $\mathcal{A} \in \mathfrak{W}^{\cdot\infty}$ be elliptic. Then there exists a parametrix $\mathcal{B} \in \mathfrak{W}^{\cdot\infty}$ of $\mathcal{A}$, and $\mathcal{A}w \in C^{\infty}_{\Delta^{\cdot}}(\check{X}) \oplus C^{\infty}(Y, C^{j_2})$, $w \in \mathcal{H}^{-\infty}(\check{X}) \oplus H^{-\infty}(Y, C^{j_1})$ imply $w \in C^{\infty}_{\Delta^{\cdot}}(\check{X}) \oplus C^{\infty}(Y, C^{j_1})$.*

Clearly the calculus applies to systems too.

Another variant of this theory gives analogous results for boundary value problems on manifolds where the boundary is of the form X as in the beginning. Here we have to apply the results of [6].

The results were formulated here in inductive limits of spaces. Of course, in any concrete case the precise singularity types of a solution may be expressed in terms of the associated principal Mellin symbol of $\mathcal{A}$. Here the situation is similar to that for the cone, cf. [6], [8]. The details will be published in a forthcoming paper.

Added in proof (June 1987). More details on the proofs may be found in [9], [10].

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