POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CXXXVIII

SAM B. NADLER, Jr.

Arcwise accessibility in hyperspaces

WARSZAWA 1976

PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

1. Introduction. Several years ago Joe Stiles, then a graduate student at Tulane University, asked me the following question (see below for definitions):

(1.1) Is each member of $C(X)$ segmentwise accessible from $2^X - C(X)$?

Stiles' question was the initial motivation for the work in this paper.

In Section 2 we give answers to (1.1). In view of these answers and the fact that a segment is a special type of arc, the following question arose naturally:

(1.2) When is a singleton arcwise accessible from $2^X - C(X)$?

Of course, (1.1) and (1.2) are each part of the following question:

(1.3) Which members of $C(X)$ are arcwise accessible from $2^X - C(X)$?

This paper is concerned with (1.3). Obtaining answers to (1.2) comprises the bulk of the paper (Sections 3 and 4). Section 4 gives a detailed and fairly complete study of which points of 2^X arcwise disconnect 2^X (see Section 4 for definition). This topic is directly related to the study of arcwise accessibility of singletons (see especially Theorem 4.13), and has not been investigated before.

In our results which give sufficient conditions for arcwise accessibility from $2^X - C(X)$, we obtain a stronger conclusion; namely, arcwise accessibility from $C_2(X) - C(X)$ beginning with a two-point set. This means our accessibility takes place in a very small part of 2^X, one which people have investigated before (for example, [7], p. 29, and [15], p. 191).

In Section 5 we determine characterizations of hereditary indecomposability in terms of what sets arcs, which begin in $2^X - C(X)$ and terminate in $C(X)$, can begin and end with.

In Section 6 we pose some unsettled questions and show that any $A \in C(X)$ is continuumwise accessible from $C_2(X) - C(X)$ (see (6.8)).

We will use the following notation and terminology. The adjective *non-degenerate* refers to having more than one point. By a *continuum* we will mean a non-empty compact connected metric space. The point $p \in X$ is said to be a *cut point* of X if and only if $X - \{p\}$ is not connected. A continuum is said to be *rational* if and only if each point belongs to arbitrarily small open sets whose boundaries are countable. The symbol H denotes the Hausdorff metric as defined in [7], p. 23. We consider

$C(X)$ as canonically embedded in 2^X by the inclusion map

$$i\colon C(X)\to 2^X, \quad i(A) = A \quad \text{for all} \quad A\in C(X).$$

The symbol $C_2(X)$ denotes $\{A\in 2^X\colon A$ has at most two components$\}$, canonically embedded in 2^X by inclusion. Thus, in particular, $C(X) \subset C_2(X)$. The symbol μ denotes any continuous function from 2^X to $[0,1]$ satisfying

(a) if $A, B\in 2^X$ such that $A\subset B$ and $A\neq B$, then $\mu(A)<\mu(B)$;

(b) $\mu(X)=1$ and $\mu(\{x\})=0$ for all $x\in X$.

In [12], Whitney showed that such a function exists; we will call any continuous function satisfying (a) and (b) a *Whitney map for* 2^X. Some recent results on Whitney maps restricted to $C(X)$ appear in [4], [8] and [11]. By a *segment* we mean a segment in the sense of Kelley [7], p. 24 (see below, especially (1.4), for more discussion). An *order arc* in 2^X is an arc $a\subset 2^X$ such that if $A, B\in a$, then $A\subset B$ or $B\subset A$. Let $\Sigma\subset 2^X$; a member A of $C(X)$ is said to be $\left\{\begin{matrix}\textit{arcwise}\\ \textit{segmentwise}\end{matrix}\right\}$ *accessible from* $\Sigma - C(X)$ *beginning with* K if and only if there is a $\left\{\begin{matrix}\text{homeomorphism}\\ \text{segment}\end{matrix}\right\}$ $\sigma\colon [0,1]\to\Sigma$ such that $\sigma(0)=K$, $\sigma(1)=A$, and $\sigma(t)\in[\Sigma - C(X)]$ for all $t<1$. In this paper, Σ will be 2^X or $C_2(X)$.

The symbol $\overline{U}$ will denote the closure of U. Other terminology we use in this paper is either defined in the body of the paper or can be found in one or more of the references.

We give a brief discussion leading to the formulation of (1.4), a result we shall use several times. Though the concept of an order arc was not formulated in [2], the proof given there was, for the most part, devoted to producing what we call order arcs. Thus, Borsuk and Mazurkiewicz were the first to discover order arcs in 2^X. In 1942, Kelley defined the notion of a segment using the function discovered by Whitney [12] in 1932. Kelley does not relate his notion of segment to the work in [2]. The relationship between these is the following:

(1.4) THEOREM. *A subset of* 2^X *is the range of a segment if and only if it is an order arc or it consists of only one member of* 2^X.

Proof. We show that an order arc is the range of a segment, the rest of (1.4) being a simple consequence of definitions. Let a be an order arc in 2^X. Let $A_0=\bigcap a$ and let $A_1=\bigcup a$. It is not difficult to show that $A_0\in a$ and $A_1\in a$. It is then easy to show that μ restricted to a, denoted by $\underline{\mu}$, is a homeomorphism of a onto the interval $[\mu(A_0), \mu(A_1)]$. Define

$$\varrho\colon [0,1]\to[\mu(A_0), \mu(A_1)]$$

by the formula

$$\varrho(t) = (1-t)\cdot\mu(A_0)+t\cdot\mu(A_1)$$

for all $t\in[0,1]$. Now, let $\sigma\colon [0,1]\to\alpha$ be given by $\sigma = \mu^{-1}\cdot\varrho$. Clearly, σ is a continuous function from $[0,1]$ onto α such that $\sigma(0) = A_0$, $\sigma(1) = A_1$, and $\mu(\sigma(t)) = \varrho(t)$ for all $t\in[0,1]$. Now, let $t', t''\in[0,1]$ such that $t' < t''$. Then,

$$\mu(\sigma(t')) = \varrho(t') < \varrho(t'') = \mu(\sigma(t'')).$$

Since α is an order arc, $\sigma(t')\subset\sigma(t'')$ or $\sigma(t'')\subset\sigma(t')$. Hence, by (a) above, $\sigma(t')\subset\sigma(t'')$. Thus, σ is a segment in the sense of [7] whose range is α. This completes our proof of (1.4).

We express our appreciation to B. J. Ball for looking over parts of the manuscript and for some helpful suggestions.

2. Segmentwise accessibility. The following two results answer Stiles' question (1.1).

(2.1) THEOREM. *For any $x\in X$, $\{x\}$ is not segmentwise accessible from $2^X - C(X)$.*

Proof. An easy consequence of 2.2 of [7].

(2.2) THEOREM. *Every non-degenerate member of $C(X)$ is segmentwise accessible from $C_2(X)-C(X)$ beginning with a two-point set.*

Proof. Let M be a non-degenerate subcontinuum of X and let $q\in M$. Taking R to be $\{q\}$ in Theorem 5 of [1], p. 500, we see that there is a point $p\in[M-\{q\}]$ such that the union E of all subcontinua of $M-\{p\}$ which contain q is dense in M. Let $D=\{e_1, e_2, \ldots, e_n, \ldots\}$ be a countable dense subset of E. For each $i = 1, 2, \ldots$, let Z_i be a non-degenerate continuum in E such that $q, e_i\in Z_i$. For each $n = 1, 2, \ldots$, let $M_n = \bigcup_{i=1}^{n} Z_i$. It is easy to see that

$$\{M_n\}_{n=1}^{\infty} \quad \text{converges to } M; \tag{1}$$

$$M_1\subset M_2\subset\ldots\subset M_n\subset\ldots; \tag{2}$$

$$p\in\big[M-\bigcup_{n=1}^{\infty} M_n\big]. \tag{3}$$

Furthermore, we may assume, by passing to a subsequence if necessary, that

$$M_n \neq M_{n+1} \quad \text{for} \quad n = 1, 2, \ldots \tag{4}$$

Now, by (2) above and 2.3 of [7], there is a segment σ_1 from $\{q\}$ to M_1 and, for each $n = 1, 2, \ldots$, there is a segment σ_{n+1} from M_n to M_{n+1}.

For each $n = 1, 2, \ldots$, let f_n: $\left[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}\right] \to 2^M$ be given by

$$f_n(t) = \sigma_n(2^n \cdot t + 2 - 2^n) \cup \{p\}$$

(thus, f_n is σ_n "speeded up" with the point p adjoined). Define f: $[0, 1] \to 2^M$ by

$$f(t) = \begin{cases} f_n(t), & \frac{2^{n-1}-1}{2^{n-1}} \leqslant t \leqslant \frac{2^n-1}{2^n}, \\ M, & t = 1. \end{cases}$$

Note that since $\sigma_n(1) = \sigma_{n+1}(0)$ for each $n = 1, 2, \ldots, f$ is a function. From (4) it follows that f is one-to-one (the one-to-oneness of f on $[0, \frac{1}{2}]$, i.e. of f_1, follows from the fact that M_1 is non-degenerate). It is routine to verify, using (1), that f is continuous and, using (3), that $f(t) \in C(X)$ if and only if $t = 1$. Since $0 \leqslant s \leqslant t \leqslant 1$ implies $f(s) \subset f(t)$, our result now follows from Theorem 1.4 above and the fact that $f(0) = f_1(0) = \sigma_1(0) \cup \{p\} = \{q, p\}$.

(2.3) **Remark.** The real use of Bing's theorem in the proof of (2.2) was to show we could begin with a two-point set (properties of segments preclude a segment which begins in $C_2(X)$ from going outside $C_2(X)$). It is somewhat easier to show segmentwise accessibility beginning with a countable set. Let $a \in M$, let σ be a segment from $\{a\}$ to M, and let $a_n \in \left[M - \sigma\left(\frac{n-1}{n}\right)\right]$ for each $n = 1, 2, \ldots$ (see 2.1, 2.2, and 2.3 of [7]). The sequence $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{a_{n_i}\}_{i=1}^{\infty}$, converging to a point $a_0 \in M$. Define f: $[0, 1] \to 2^X$ by $f(t) = \sigma(t) \cup \{a_{n_i}: i = 1, 2, \ldots\} \cup \{a_0\}$ and use Theorem 1.4 above. This argument shows that M is segmentwise accessible from $2^X - C(X)$ beginning with the countable set $\{a, a_0, a_{n_1}, a_{n_2}, \ldots\}$.

3. Arcwise accessibility of singletons. In the previous section we established in (2.2) that each non-degenerate subcontinuum of X is arcwise (in fact, segmentwise) accessible from $2^X - C(X)$. Thus, we reduced our study of (1.3) to a study of (1.2). Also, (2.1) ruled out segmentwise accessibility in relation to (1.2).

We devote this section to giving some answers to (1.2). In (3.3) we obtain a necessary condition in order that a singleton be arcwise accessible from $2^X - C(X)$ (also, see (4.13) and (4.14)). This condition enables us to conclude that no singleton is arcwise accessible from $2^X - C(X)$ for certain continua X ((3.4) and (3.5)). We then obtain a number of sufficient conditions in order that a singleton be arcwise accessible from $2^X - C(X)$ ((3.7) through (3.9)). We use one of these results, namely (3.9), to show that every rational continuum Q has a dense set of points $p \in Q$ such that $\{p\}$ is arcwise accessible from $2^Q - C(Q)$ (for a precise statement, see (3.11)). In (3.12) we give an example of a rational continuum Q_0 with a dense set of points $q \in Q_0$ such that $\{q\}$ is not arcwise accessible from $2^{Q_0} - C(Q_0)$.

The following lemma extends 8.1 of [7] in two directions: first, to 2^X instead of $C(X)$ and, second, to locally connected continua instead of arcs.

(3.1) Lemma. *Let Y be an indecomposable continuum and let $\Lambda \dot{\subset} 2^Y$ be a locally connected continuum. If $\bigcup \Lambda = Y$ and if $\Lambda \cap C(Y) \neq \emptyset$, then $Y \in \Lambda$.*

Proof. Let $k\colon [0,1] \to \Lambda$ be a continuous function with $k([0,1]) = \Lambda$ (see Theorem 3–30 of [5], p. 129). Since $\Lambda \cap C(Y) \neq \emptyset$, there is a $t_0 \in [0,1]$ such that $k(t_0) \in C(Y)$. Let

$$a = \text{l.u.b.}\ \big(\{t \leqslant t_0\colon\ \bigcup k([t,1]) = Y\}\big)$$

and let

$$b = \text{g.l.b.}\ \big(\{t \geqslant t_0\colon\ \bigcup k([a,t]) = Y\}\big).$$

Notice that, by the continuity of k and of union [7], p. 23, it follows that $\bigcup k([a,b]) = Y$. Furthermore, $[a,b]$ is minimal with respect to the property of

(*) being a closed interval containing t_0 and having the union of its images be all of Y.

Assume first that $a < t_0$. Let s_0 be such that $a < s_0 \leqslant t_0$. Since $k([s_0,b])$ is a subcontinuum of 2^Y and since $k([s_0,b]) \cap C(Y) \neq \emptyset$ (note: $k(t_0) \in [k([s_0,b]) \cap C(Y)]$), we have from 1.2 of [7] that $\bigcup k([s_0,b])$ is a subcontinuum of Y. Also, by the minimality of $[a,b]$ with respect to (*), $\bigcup k([s_0,b]) \neq Y$ and, thus, is completely contained in the composant of Y determined by $k(t_0)$. It now follows that

$$\bigcup \{\bigcup k([s,b])\colon\ a < s \leqslant t_0\} = \bigcup k\big((a,b]\big)$$

is completely contained in the composant of Y determined by $k(t_0)$. Now, Y has uncountably many mutually disjoint composants, each dense in Y [5], p. 140. Thus, it follows from $\bigcup k([a,b]) = Y$ and compactness of $k(a)$ that $k(a) = Y$. A similar argument shows that if we assume $t_0 < b$, then we obtain $k(b) = Y$. Finally, if $a = t_0 = b$, $k(t_0) = Y$. This completes the proof of the lemma.

(3.2) Theorem. *Let $\Lambda \subset 2^X$ be a locally connected continuum such that $\Lambda \cap C(X) \neq \emptyset$. Then $\bigcup \Lambda$ is a subcontinuum of X such that $[\bigcup \Lambda] \in \Lambda$ or $\bigcup \Lambda$ is decomposable.*

Proof. Let $Y = \bigcup \Lambda$. By 1.2 of [7] Y is a continuum. Assume Y is indecomposable. Then, by Lemma 3.1, $Y \in \Lambda$.

In particular, then, we have the following theorem about arcwise accessibility of singletons (we refer the reader to (4.14), where we obtain an additional necessary condition in order that a singleton be arcwise accessible).

(3.3) THEOREM. *Let $x_0 \in X$. If $\{x_0\}$ is arcwise accessible from $2^X - C(X)$, then x_0 belongs to arbitrarily small decomposable subcontinua of X. In fact, if $f\colon [0,1]\to 2^X$ is continuous such that $f(t) \in [2^X - C(X)]$ for all $t < 1$ and $f(1) = \{x_0\}$, then $\bigcup f([t,1])$ is a decomposable subcontinuum of X, with x_0 in it, for all $t < 1$.*

Proof. Let $t_0 < 1$. Then $f([t_0, 1])$ satisfies the conditions on $\mathcal{A}$ in Theorem 3.2. Hence, $\bigcup f([t_0, 1])$ is a subcontinuum of X such that $\bigcup f([t_0,1]) \in f([t_0,1])$ or $\bigcup f([t_0,1])$ is decomposable. Since $f(t) \in [2^X - C(X)]$ for all $t < 1$ and $f(1) = \{x_0\}$, $\bigcup f([t_0, 1]) \notin f([t_0, 1])$. Therefore, $\bigcup f([t_0, 1])$ is decomposable. This completes the proof of Theorem 3.3.

As an immediate consequence of (3.3), we have the following result which the reader should compare with (6.8.1).

(3.4) COROLLARY. *If Y is an hereditarily indecomposable continuum, then no singleton is arcwise accessible from $2^Y - C(Y)$.*

There are decomposable continua D such that no singleton is arcwise accessible from $2^D - C(D)$. The following example illustrates this.

(3.5) EXAMPLE. Let $D = J \cup K$, where J and K are hereditarily indecomposable continua and $J \cap K$ is a *non-degenerate* proper subcontinuum of each. Clearly, D is decomposable. To see that no singleton is arcwise accessible from $2^D - C(D)$, we first show that any decomposable subcontinuum of D must contain $J \cap K$. To see this let M be a decomposable subcontinuum of D. Since K is hereditarily indecomposable, there is a point $p \in U = M - K$ and, since J is hereditarily indecomposable, $U \neq M$. Hence, by (10.1) of [14], p. 16, the component L of $\overline{U}$ containing p intersects $\overline{U} - U$. Since $U \subset J$, $\overline{U} \subset J$. Also $\overline{U} - U = [(\overline{M-K}) - (M-K)] \subset K$, the containment being valid by virtue of the compactness of M. Thus, $[\overline{U} - U] \subset [J \cap K]$ and we now have that $L \cap [J \cap K] \neq \emptyset$. Thus, since $L \subset \overline{U} \subset J$, L and $J \cap K$, are intersecting subcontinua of J. From the hereditary indecomposability of J we conclude that

$$L \supset [J \cap K] \quad \text{or} \quad [J \cap K] \supset L,$$

the latter containment being false because $p \in [L - (J \cap K)]$. Hence,

$$M \supset [J \cap K].$$

This completes the proof that any decomposable subcontinuum of D contains $J \cap K$. It follows from this that no point of D belongs to arbitrarily small decomposable subcontinua of D. Therefore, by Theorem 3.3, no singleton is arcwise accessible from $2^D - C(D)$.

It was crucial for Example 3.5 that $J\cap K$ was non-degenerate, as the next example shows.

(3.6) EXAMPLE. If D, J, and K are as in Example 3.5 *except* that $J\cap K$ consists of only a single point w, then $\{w\}$ is arcwise accessible from $2^D - C(D)$. This is a consequence (see Corollary 3.9 below) of the following general theorem which we will use later.

(3.7) THEOREM. *Let $p\in X$. Assume that there exist subcontinua K_1, $L_1, K_2, L_2, \ldots, K_n, L_n, \ldots$ such that $\bigcap_{n=1}^{\infty} K_n = \{p\} = \bigcap_{n=1}^{\infty} L_n$ and such that, for any $i = 1, 2, \ldots$, $K_i \not\subset \bigcup_{n=1}^{\infty} L_n$ and $L_i \not\subset \bigcup_{n=1}^{\infty} K_n$. Then $\{p\}$ is arcwise accessible from $C_2(X) - C(X)$ beginning with a two-point set.*

Proof. For each $i = 1, 2, \ldots$, let $x_i \in [K_i - \bigcup_{n=1}^{\infty} L_n]$ and let $y_i \in [L_i - \bigcup_{n=1}^{\infty} K_n]$ such that the points x_i are all distinct and the points y_i are all distinct (this is possible because $\bigcap_{n=1}^{\infty} K_n = \{p\} = \bigcap_{n=1}^{\infty} L_n$). Now, since $C(K_n\cup K_{n+1})$ and $C(L_n\cup L_{n+1})$ are arcwise connected for all $n = 1, 2, \ldots$ ([2] or 2.7 of [7]), there exist arcs $\alpha_n \subset C(K_n\cup K_{n+1})$ and $\beta_n \subset C(L_n\cup L_{n+1})$, α_n having non-cut points $\{x_n\}$ and $\{x_{n+1}\}$ and β_n having non-cut points $\{y_n\}$ and $\{y_{n+1}\}$. For each $n = 1, 2, \ldots$, let

$$I_n = \left[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}\right],$$

let $f_n\colon I_{2\cdot n-1}\to\alpha_n$ be a homeomorphism onto α_n such that

$$f_n\left(\frac{2^{2\cdot n-2}-1}{2^{n-1}}\right) = \{x_n\} \quad\text{and}\quad f_n\left(\frac{2^{2\cdot n-1}-1}{2^{2\cdot n-1}}\right) = \{x_{n+1}\},$$

and let $g_n\colon I_{2\cdot n}\to\beta_n$ be a homeomorphism onto β_n such that

$$g_n\left(\frac{2^{2\cdot n-1}-1}{2^{2\cdot n-1}}\right) = \{y_n\} \quad\text{and}\quad g_n\left(\frac{2^{2\cdot n}-1}{2^{2\cdot n}}\right) = \{y_{n+1}\}.$$

Now define $k\colon [0,1]\to 2^X$ by

$$k(t) = \begin{cases} f_n(t)\cup\{y_n\}, & t\in I_{2\cdot n-1},\\ g_n(t)\cup\{x_{n+1}\}, & t\in I_{2\cdot n},\\ \{p\}, & t=1.\end{cases}$$

It is easy to verify that k is a function. Using that the points x_i and y_i $(i = 1, 2, \ldots)$ are all distinct, it is easy to see that k is one-to-one. The continuity of k follows routinely using that $\bigcap K_n = \{p\} = \bigcap_{n=1}^{\infty} L_n$. Since $f_n(t) \subset [K_n \cup K_{n+1}]$ for each $t \in I_{2 \cdot n-1}$ and $y_n \notin [K_n \cup K_{n+1}]$, $f_n(t) \cup \{y_n\}$ is not connected for any $n = 1, 2, \ldots$ Similarly, $g_n(t) \cup \{x_{n+1}\}$ is not connected for any $n = 1, 2, \ldots$ It follows that $k(t)$ is not connected unless $t = 1$, when $k(t) = \{p\}$. Finally, we note that $k(0) = f_1(0) \cup \{y_1\} = \{x_1\} \cup \{y_1\} = \{x_1, y_1\}$; thus, $\{p\}$ is arcwise accessible from $C_2(X) - C(X)$ beginning with the two-point set $\{x_1, y_1\}$.

(3.8) COROLLARY. *If p is a point of a continuum Y for which there exist non-degenerate subcontinua A and B of Y such that $\{p\}$ is a component of $A \cap B$, then $\{p\}$ is arcwise accessible from $C_2(Y) - C(Y)$ beginning with a two-point set.*

Proof. By 2.3 of [7] there exist segments $\sigma_1\colon [0, 1] \to C(A)$ and $\sigma_2\colon [0, 1] \to C(B)$ such that $\sigma_1(0) = \{p\} = \sigma_2(0)$, $\sigma_1(1) = A$, and $\sigma_2(1) = B$. For each $n = 1, 2, \ldots$, let $K_n = \sigma_1(1/n)$ and let $L_n = \sigma_2(1/n)$. Clearly, $\bigcap_{n=1}^{\infty} K_n = \{p\} = \bigcap_{n=1}^{\infty} L_n$. Also, since $\{p\}$ is a component of $A \cap B$ and $\sigma_j(t)$ is a non-degenerate continuum with p in it for each $j = 1$ and 2 and all $t > 0$, $K_i \not\subset B = L_1 = \bigcup_{n=1}^{\infty} L_n$ and $L_i \not\subset A = K_1 = \bigcup_{n=1}^{\infty} K_n$ for any $i = 1, 2, \ldots$ Thus, Corollary 3.8 follows from Theorem 3.7.

(3.9) COROLLARY. *If p is a cut point of a continuum Y, then $\{p\}$ is arcwise accessible from $C_2(Y) - C(Y)$ beginning with a two-point set.*

Proof. Since $Y - \{p\}$ is not connected, $Y - \{p\} = U \cup V$, where U and V are disjoint non-empty open sets. By Theorem 4 of [10], p. 133, $U \cup \{p\}$ and $V \cup \{p\}$ are connected. Letting $A = U \cup \{p\}$ and $B = V \cup \{p\}$, the corollary now follows from Corollary 3.8.

(3.10) LEMMA. *Every non-degenerate rational continuum contains a subcontinuum with a cut point.*

Proof. Let Q be a non-degenerate rational continuum, let $x \in Q$, and let U be an open subset of Q such that $x \in U$, $\overline{U} \neq Q$, and $\overline{U} - U$ is countable. Let $V = Q - \overline{U}$ and let $W = Q - \overline{V}$. It is not difficult to verify that $[\overline{V} - V] \subset [\overline{U} - U]$ and that $[\overline{V} - V] = [\overline{W} - W]$. Thus, since every compact countable (metric) space has an isolated point, there is an isolated point p of $[\overline{V} - V] = [\overline{W} - W]$. By Theorem 8 of [10], p. 185, there is a continuum $A \subset [V \cup \{p\}]$ such that $p \in A \neq \{p\}$ and a continuum $B \subset [W \cup \{p\}]$ such that $p \in B \neq \{p\}$. Since $A \cap B = \{p\}$, $A \cup B$ is a subcontinuum of Q with cut point p.

(3.11) THEOREM. *If Q is a non-degenerate rational continuum, then there is a dense subset D of Q such that, for each $p \in D$, $\{p\}$ is arcwise accessible from $C_2(Q) - C(Q)$ beginning with a two-point set. In fact, such a subset D exists so that $Q - D$ is punctiform.*

Proof. Let $D = \{p \in Q: p$ is a cut point of some subcontinuum of $Q\}$. The accessibility property follows from Corollary 3.9. Since every subcontinuum of a rational continuum is rational, it follows from Lemma 3.10 that D intersects every subcontinuum of Q; hence, $Q - D$ is punctiform. This implies (using 10.1 of [14], p. 16) that D is a dense subset of Q.

One might suspect that each singleton is arcwise accessible from $2^Y - C(Y)$ when Y is hereditarily decomposable. However, this is not the case *even* for rational continua, as the next example shows.

(3.12) EXAMPLE. We construct a rational continuum Q_0 with a dense subset Z_0 such that, for any $z \in Z_0$, $\{z\}$ is not arcwise accessible from $2^{Q_0} - C(Q_0)$. The construction of Q_0 is similar to that of various continua in [0].

Our "basic building block" is

$$X_1 = \{(x, \sin[1/x]): 0 < |x| \leqslant 1\} \cup Y_1,$$

where

$$Y_1 = \{(0, y): |y| \leqslant 1\}.$$

Let $D_1 = \{x_1^1, x_2^1, \ldots, x_n^1, \ldots\}$ be a countable subset of X_1 which intersects every non-degenerate subcontinuum of X_1 (i.e., D_1 is *continuumwise-dense* in X_1). We also assume that the (four) end points of the arc components of X_1 are not in D_1 and that the enumeration of D_1 is one-to-one. We now let X_2 be the continuum obtained by "inserting a copy of X_1 in X_1 at x_1^1", as indicated below in Figure 3.13.

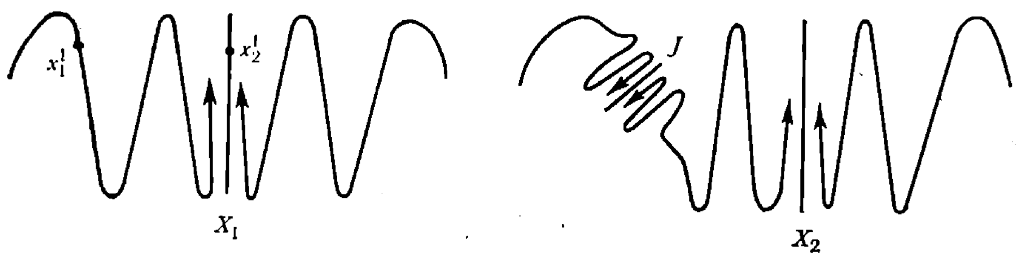

Figure 3.13

Letting the "newly inserted" arc be denoted by J, we let f_1 be a continuous function from X_2 onto X_1 such that (i) $f_1^{-1}(x_1^1) = J$ and (ii) f_1 is a "homeomorphism near the identity" on $X_2 - J$.

Next we indicate how to construct X_3 and f_2. First, let D_2 be a countable continuumwise-dense subset of X_2 such that the (six) end points of arc components of X_2 are not in D_2. Let $\{x_1^2, x_2^2, \ldots\}$ be a one-to-one enumeration of D_2. Now, let X_3 be the continuum obtained by "inserting copies of X_1 in X_2, one copy at $f_1^{-1}(x_2^1)$ and another at x_1^2, as in Figure 3.14 (if $f_1(x_1^2) = x_2^1$, then we "insert" only one copy of X_1 in X_2 to form X_3).

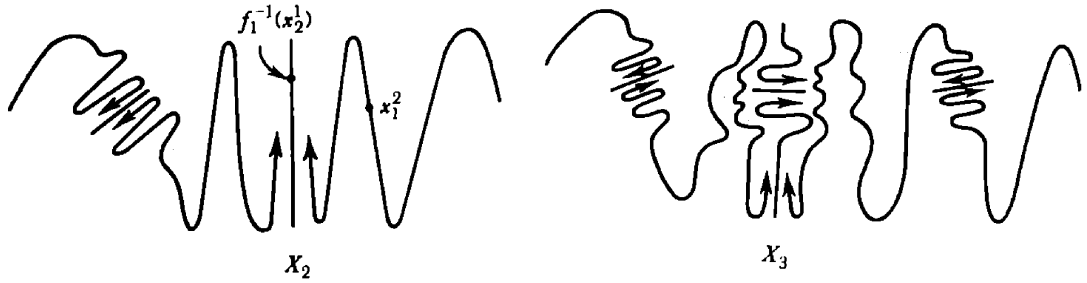

Figure 3.14

We let f_2 be a continuous function from X_3 onto X_2, defined in a manner similar to the way we defined f_1, so that f_2 shrinks the "newly inserted" arcs to $f_1^{-1}(x_2^1)$ and x_1^2, respectively.

Now, let D_3 be a countable continuumwise-dense subset of X_3 such that none of the end points of arc components of X_3 are in D_3. We obtain X_4 and f_3: $X_4 \to X_3$ by the process used to obtain the previous continua and maps, this time making sure that copies of X_1 are "inserted in X_3 at $(f_1 \circ f_2)^{-1}(x_3^1), f_2^{-1}(x_2^2)$, and at the first enumerated point of D_3.

Continuing this process we produce an inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$. Let Q_0 denote the inverse limit space of $\{X_n, f_n\}_{n=1}^{\infty}$.

First, let us note that Q_0 is rational. This is a consequence of the following lemma which is easy to prove using 2.2 of [3].

(3.15) LEMMA. *Let Y be the inverse limit of $\{Y_n, f_n\}_{n=1}^{\infty}$, where Y_n is a continuum and f_n continuously maps Y_{n+1} onto Y_n for each $n = 1, 2, \ldots$ Also assume that, for each $n = 1, 2, \ldots$,*

(3.15.1) *Y_n has a base $B_n = \{U_i^n: i = 1, 2, \ldots\}$ of open sets such that, for each $i = 1, 2, \ldots$, $\overline{U_i^n} - U_i^n$ is countable and $(f_n \circ f_{n+1} \circ \ldots \circ f_{n+k})^{-1}(z)$ consists of only one point for each $z \in [\overline{U_i^n} - U_i^n]$ and each $k = 0, 1, \ldots$*

Then, Y is a rational continuum.

Next, let $Z_0 = \{(p_1, p_2, \ldots) \in Q_0: p_n \in D_n \text{ for some } n\}$.

Since D_n is a dense subset of X_n for each $n = 1, 2, \ldots, Z_0$ is a dense subset of Q_0. Now we show that for any $p \in Z_0$, $\{p\}$ is not arcwise accessible from $2^{Q_0} - C(Q_0)$. To do this we will use some of the results in the next section. First let us observe that since Q_0 is hereditarily decomposable, it suffices by (4.13) to show that each point of Z_0 belongs to arbitrarily small subcontinua of Q_0 each of which arcwise disconnects 2^{Q_0}. Hence, by (4.4), it suffices to show that each point of Z_0 belongs to arbitrarily small non-degenerate subcontinua of Q_0 each of which satisfies (4.4.1).

To do this fix $p = (p_1, p_2, \ldots) \in Z_0$ and fix i. Let E denote the inverse limit space of $\{E_n, f_n | E_{n+1}\}_{n=1}^{\infty}$, where

$$E_n = \begin{cases} \{p_n\}, & n \leqslant i, \\ (f_i \circ f_{i+1} \circ \ldots \circ f_{n-1})^{-1}(p_i), & n > i, \end{cases}$$

and the "vertical line" denotes the restriction of f_n to E_{n+1}. It is easy to see from the construction of $\{X_n, f_n\}_{n=1}^{\infty}$ that there exists m such that if $n \geqslant m$, then E_n is a non-degenerate proper subcontinuum of X_n which satisfies (4.4.1). Using this and some of the results in Section 2 of [3] it follows that E is a non-degenerate proper subcontinuum of Q_0 which satisfies (4.4.1). By using the base for Q_0 exhibited in 2.2 of [3], it follows by making appropriate choices for i that E can be "as small as we wish". This completes the verification of the properties of Q_0 promised above. We mention that Q_0 is a rational chainable continuum which contains no arc.

4. Compacta in X which arcwise disconnect 2^X or $C(X)$. Recall that 2^X and $C(X)$ are each arcwise connected. A point of an arcwise connected continuum is said to *arcwise disconnect* the continuum if and only if the continuum without the point is not arcwise connected. This section gives fairly thorough answers to the question of which points (i.e., compacta in X) arcwise disconnect 2^X or $C(X)$.

The following simple lemma will be used many times.

(4.1) LEMMA. *If E is a subcontinuum of X, then each of $C(X) - C(E)$ and $2^X - 2^E$ is arcwise connected.*

Proof. Let $A \in 2^X$. By 2.3 of [7], there is a segment σ: $[0, 1] \to 2^X$ such that $\sigma(0) = A$ and $\sigma(1) = X$. Furthermore, if $A \in C(X)$, then, by 2.6 of [7], $\sigma(t) \in C(X)$ for all $t \in [0, 1]$. Also, since $A = \sigma(0) \subset \sigma(t)$ for all $t \in [0, 1]$ (see 2.2 of [7]), we have that if $A \not\subset E$, then $\sigma(t) \not\subset E$ for $t \in [0, 1]$. The lemma now follows.

Note that, as a simple consequence of (4.1), $2^X - \{x\}$ is arcwise connected for any $x \in X$. This fact will be used without mentioning it in some of the proofs later on.

(4.2) THEOREM. *Let $A \in 2^X$. If $2^X - \{A\}$ is not arcwise connected, then $A \in C(X)$.*

Proof. Assume $A \notin C(X)$ and let K and L be non-empty disjoint compact sets such that $A = K \cup L$. Let $B \in [2^X - \{A\}]$. It suffices to show there is an arc from B to X missing A. If $B \not\subset A$, such an arc exists by 2.3 of [7]. Hence, we assume $B \subset A$. We also assume, without loss of generality, that $B \cap K \neq \emptyset$ and $B \cap L \neq L$. Now, by 2.3 of [7], there is a segment σ: $[0, 1] \to 2^X$ from $B \cap K$ to X. Let

$$t_0 = \text{l.u.b.}\ \big(\{t \in [0, 1]\colon\ \sigma(t) \subset K\}\big)$$

and let

$$s_0 = \text{g.l.b.}\ (\{t \in [0,1]\colon\ \sigma(t) \cap L \neq \emptyset\}).$$

By 2.2 of [7], $\sigma(t) \subset K$ for all $t \leqslant t_0$ and $\sigma(t) \cap L \neq \emptyset$ for all $t \geqslant s_0$. Hence, there is a number u such that $t_0 < u < s_0$. Now, $\sigma(u) \not\subset K$ and $\sigma(u) \cap L = \emptyset$. Thus, $\sigma(u) \cap [X - A] \neq \emptyset$. We now show

$$(*) \qquad \sigma(t) \cup B \neq A$$

for any $t \in [0, 1]$.

Suppose $\sigma(t') \cup B = A$ for some $t' \in [0, 1]$. Since $B \cap L$ is a proper subset of L, $t' \geqslant s_0$. Thus, $t' > u$ and we have, by 2.2 of [7], that $\sigma(t') \supset \sigma(u)$. Therefore, $\sigma(t') \cap [X - A] \neq \emptyset$, a contradiction. Thus, we have proved $(*)$.

Hence, defining f by

$$f(t) = \sigma(t) \cup B, \quad 0 \leqslant t \leqslant 1,$$

we see that f is a continuous function from $[0, 1]$ into $2^X - \{A\}$ such that $f(0) = B$ and $f(1) = X$. It now follows that $2^X - \{A\}$ is arcwise connected. This proves Theorem 4.2.

In 8.2 of [7], Kelley proves that X is decomposable if and only if $C(X) - \{X\}$ is arcwise connected. The following lemma is the analogous result for 2^X.

(4.3) LEMMA. *A non-degenerate continuum E is decomposable if and only if $2^E - \{E\}$ is arcwise connected.*

Proof. Assume E is decomposable and let F_1 and F_2 be proper subcontinua of E such that $E = F_1 \cup F_2$. Let $A \in [2^E - \{E\}]$. We will show:

$(*)$ there is an arc in $2^E - \{E\}$ with A as one non-cut point and a member of $C(E)$ as the other non-cut point.

To prove $(*)$ we take two cases involving how A sits in E.

Case 1. $[(F_1 - F_2) \cup (F_2 - F_1)] \subset A$. Then, by (10.1) of [14], each component of A intersects $F_1 \cap F_2$. Hence, by 2.3 of [7], there is a segment $\sigma_1\colon [0, 1] \to 2^A$ from $A \cap [F_1 \cap F_2]$ to A. Again using 2.3 of [7], there is a segment $\sigma_2\colon [0, 1] \to 2^{F_1}$ from $A \cap [F_1 \cap F_2]$ to F_1. Clearly, $E \notin [\sigma_1([0, 1]) \cup \sigma_2([0, 1])]$ and inside $\sigma_1([0, 1]) \cup \sigma_2([0, 1])$ there is an arc with non-cut points A and F_1. We have proved $(*)$ for Case 1.

Case 2. $[(F_1 - F_2) \cup (F_2 - F_1)] \not\subset A$. Without loss of generality assume $(F_1 - F_2) \not\subset A$. If $A \subset F_1$, then by 2.3 of [7] there is a segment $\sigma\colon [0, 1] \to 2^{F_1}$ from A to F_1, and $(*)$ holds. Thus, we assume $A \not\subset F_1$. Hence $A \cap F_2 \neq \emptyset$ and therefore, since F_2 is connected, each component of $A \cup F_2$ intersects A. So, by 2.3 of [7], there is a segment $\sigma_1\colon [0, 1] \to 2^{[A \cup F_2]}$ from A to $A \cup F_2$. Also, by 2.3 of [7], there is a segment $\sigma_2\colon [0, 1] \to 2^{A \cup F_2}$ from $[A \cup F_2] \cap F_1$ to $A \cup F_2$ and a segment $\sigma_3\colon [0, 1] \to 2^{F_1}$ from $[A \cup F_2] \cap F_1$ to F_1. Clearly, since $(F_1 - F_2) \not\subset A$, $E \notin [\bigcup_{i=1}^{3} \sigma_i([0, 1])]$. Since

$\bigcup_{i=1}^{3} \sigma_i([0,1])$ contains an arc with non-cut points A and F_1, we have proved $(*)$.

Using 8.2 of [7] and $(*)$, it follows that $2^E - \{E\}$ is arcwise connected.

To prove the converse, assume E is indecomposable. Let x and y be points of different composants of E. Assume $f\colon [0,1] \to 2^E$ is a continuous function such that $f(0) = \{x\}$ and $f(1) = \{y\}$. We will show that $f(t_0) = E$ for some $t_0 \in [0,1]$. First note that, since f is continuous and $f(0) \in C(E)$, it follows from 1.2 of [7] that $[\bigcup f([0,t])] \in C(E)$ for each $t \in [0,1]$. Thus, letting

$$g(t) = \bigcup f([0,t]), \qquad 0 \leqslant t \leqslant 1,$$

we see that g is a continuous function from $[0,1]$ into $C(E)$. Since the composants of E are mutually disjoint (see Theorem 3–47 of [5], p. 140) and $x, y \in g(1)$, we have that $g(1) = E$. Let

$$t_0 = \text{g.l.b.}\ (\{t \in [0,1]\colon g(t) = E\}).$$

Clearly, $g(t_0) = E$ and $g(t)$ is a proper subcontinuum of E for all $t < t_0$ (note: since $g(0) = f(0) = \{x\}$ and since $g(t_0) = E$ is non-degenerate, $t_0 > 0$). Hence, since E is indecomposable, $g(t)$ is nowhere dense in E for all $t < t_0$ (see Theorem 3–41 of [5], p. 139). Now, $\bigcup f([t, t_0])$ is compact and

$$E = g(t_0) = g(t) \cup [\bigcup f([t, t_0])]$$

for all $t < t_0$. Therefore, $\bigcup f([t, t_0]) = E$ for all $t < t_0$. Thus, by continuity of f, $f(t_0) = E$. We now conclude that $2^E - \{E\}$ is not arcwise connected.

(4.4) THEOREM. *Let E be a non-degenerate proper subcontinuum of X. Consider the following three statements:*

(4.4.1) *If Y is a subcontinuum of X such that*

$$Y \cap E \neq \emptyset \neq Y \cap [X - E],$$

then $Y \supset E$;

(4.4.2) *$2^X - \{E\}$ is not arcwise connected;*

(4.4.3) *$C(X) - \{E\}$ is not arcwise connected.*

Then (4.4.1) *implies* (4.4.2), (4.4.1) *implies* (4.4.3), *and, if E is decomposable, all three statements are equivalent.*

Proof. Assume (4.4.1) holds. We simultaneously show that (4.4.2) and (4.4.3) each holds. Let $\begin{Bmatrix} f\colon [0,1] \to 2^X \\ f\colon [0,1] \to C(X) \end{Bmatrix}$ be continuous such that $f(0) \in [C(E) - \{E\}]$ and $f(1) \not\subset E$. Let

$$t_0 = \text{l.u.b.}\ (\{t \in [0,1]\colon \bigcup f([0,t]) \subset E\}).$$

Note that, since $f(1) \not\subset E$, $t_0 < 1$. Let t_1 be such that $t_0 < t_1 \leqslant 1$. We have:

(i) $\bigcup f([0, t_0]) \subset E$;

(ii) $\bigcup f([0, t_1]) \not\subset E$;

(iii) $\bigcup f([0, t_1])$ is a subcontinuum of X (this follows from 1.2 of [7] using that $f(0) \in C(X)$).

Hence, by (4.4.1), $\bigcup f([0, t_1]) \supset E$. Since t_1 was arbitrarily chosen in $(t_0, 1]$, a simple continuity argument together with (i) gives us that $\bigcup f([0, t_0]) = E$. Now, from Theorem 1.4 above, there is a segment from $f(t_0)$ to $\bigcup f([t_0, t_1])$. Therefore, by 2.3 of [7], each component of $\bigcup f([t_0, t_1])$ intersects $f(t_0)$. By (ii) above, there is a component K of $\bigcup f([t_0, t_1])$ such that $K \not\subset E$. We have

$$K \cap E \neq \emptyset \neq K \cap [X - E].$$

Hence, by (4.4.1), $K \supset E$ from which we conclude that $\bigcup f([t_0, t_1]) \supset E$. It now follows from a continuity argument that $f(t_0) \supset E$. Thus, by (*i*), $f(t_0) = E$. This proves that $\begin{Bmatrix}(4.4.2)\\(4.4.3)\end{Bmatrix}$ holds. Next, assume E is decomposable and (4.4.1) does not hold. We simultaneously show neither (4.4.2) nor (4.4.3) holds. Since (4.4.1) does not hold, there is a subcontinuum Y_0 of X such that

$$Y_0 \cap E \neq \emptyset \neq Y_0 \cap [X - E]$$

and

$$Y_0 \not\supset E.$$

Let $A \in \begin{Bmatrix}2^E - \{E\}\\ C(E) - \{E\}\end{Bmatrix}$. By $\begin{Bmatrix}\text{Lemma 4.3}\\ \text{8.2 of [7]}\end{Bmatrix}$, there is an arc in $\begin{Bmatrix}2^E - \{E\}\\ C(E) - \{E\}\end{Bmatrix}$ with non-cut points A and $\begin{Bmatrix}Y_0 \cap E\\ \text{a component } L \text{ of } Y_0 \cap E\end{Bmatrix}$. By 2.3 of [7], there is a segment $\begin{Bmatrix}\sigma\colon [0, 1] \to 2^{Y_0}\\ \sigma\colon [0, 1] \to C(Y_0)\end{Bmatrix}$ from $\begin{Bmatrix}Y_0 \cap E\\ L\end{Bmatrix}$ to Y_0. Note that, since $Y_0 \not\supset E$ and $(t) \subset Y_0$ for all $t \in [0, 1]$, $\sigma(t) \neq E$ for any $t \in [0, 1]$. Therefore, it follows that there is an arc in $\begin{Bmatrix}2^X - \{E\}\\ C(X) - \{E\}\end{Bmatrix}$ with non-cut points A and Y_0. Since A was an arbitrary member of $\begin{Bmatrix}2^E - \{E\}\\ C(E) - \{E\}\end{Bmatrix}$ and $Y_0 \in \begin{Bmatrix}2^X - 2^E\\ C(X) - C(E)\end{Bmatrix}$, it follows from Lemma 4.1 that $\begin{Bmatrix}(4.4.2)\\(4.4.3)\end{Bmatrix}$ does not hold. This completes the proof of the theorem.

The following example shows that without E being decomposable, (4.4.1) and (4.4.2) are not necessarily equivalent.

(4.5) Example. Let $X' = E' \cup Y$, where E' is a non-degenerate indecomposable continuum, Y is a non-degenerate continuum, and $E' \cap Y$ consists of only a single point p. It is clear that $X = X'$ and $E = E'$ do not have property (4.4.1). However, (4.4.2) does hold. This can be verified directly or the next theorem can be used by taking D in (4.6.1) to be any composant of E' not having p in it.

Our next theorem shows that arcwise disconnecting 2^X is always equivalent to arcwise disconnecting $C(X)$. It also gives a complete characterization of subcontinua of X which arcwise disconnect. I am grateful to B. J. Ball for helping to formulate condition (4.6.1), which was not in the original write-up.

(4.6) Theorem. *If E is a non-degenerate proper subcontinuum of X, then the following three statements are equivalent:*

(4.6.1) *There is a dense subset D of E such that if Y is a subcontinuum of X satisfying*

$$Y \cap D \neq \emptyset \neq Y \cap [X - E],$$

then $Y \supset E$;

(4.6.2) $2^X - \{E\}$ *is not arcwise connected;*

(4.6.3) $C(X) - \{E\}$ *is not arcwise connected.*

Proof. We take two cases.

Case 1. E is decomposable. By Theorem 4.4 we need only show (4.6.1) is equivalent to (4.4.1). Clearly (take $D = E$), (4.4.1) implies (4.6.1). So, assume (4.6.1) holds and let Y be a subcontinuum of X such that

$$Y \cap E \neq \emptyset \neq Y \cap [X - E].$$

Since E is decomposable, there are proper subcontinua A and B of E such that $E = A \cup B$. Without loss of generality, assume $Y \cap A \neq \emptyset$. Since D is dense in E and $A - B$ is a non-empty open subset of E, $[Y \cup A] \cap D \neq \emptyset$. Thus, by (4.6.1), $[Y \cup A] \supset E$. Hence, $Y \supset [B - A]$. Therefore, since $B - A$ is a non-empty open subset of E, $Y \cap D \neq \emptyset$. Thus, by (4.6.1), $Y \supset E$. This proves (4.4.1) holds.

Case 2. E is indecomposable. Assume (4.6.1) holds. We simultaneously show (4.6.2) and (4.6.3) each holds. Let $\left\{\begin{matrix} f\colon [0,1] \to 2^X \\ f\colon [0,1] \to C(X) \end{matrix}\right\}$ be continuous such that $f(0) = \{p\}$, $p \in D$, and $f(1) \not\subset E$. Then, using (4.6.1) and 1.2 of [7], $\bigcup f([0,1]) \supset E$. Let

$$t_0 = \text{g.l.b.}\ \left(\left\{t \in [0,1]\colon \bigcup f([0,t]) \supset E\right\}\right).$$

Note that $t_0 > 0$. Since $f(0) \subset D$ and $\bigcup f([0,t])$ is a continuum (by 1.2 of [7]), it follows from (4.6.1) and the definition of t_0 that $\bigcup f([0,t])$ is a proper subcontinuum of E for all $t < t_0$ and $\bigcup f([0,t_0]) = E$. The last part of the proof of Lemma 4.3 applies giving us that $f(t_0) = E$. Hence $\begin{Bmatrix}(4.6.2)\\(4.6.3)\end{Bmatrix}$ holds. Next, assume (4.6.1) does not hold. We prove:

(*) *If $p \in E$, then there is a subcontinuum Z of X such that $p \in Z$, $Z \cap [X - E] \neq \emptyset$, and $Z \not\supset E$.*

Proof of (*). Suppose (*) is false for some point $p \in E$. Let C_p denote the composant of E determined by p, i.e., $C_p = \bigcup\{K\colon K$ is a proper subcontinuum of E and $p \in K\}$. Let Y be a subcontinuum of X satisfying

$$Y \cap C_p \neq \emptyset \neq Y \cap [X - E].$$

Let K be a proper subcontinuum of E such that $p \in K$ and $Y \cap K \neq \emptyset$. Since we are supposing that (*) is false, $[Y \cup K] \supset E$. Since K is nowhere dense in E (Theorem 3–41 of [5], p. 139), it now follows that $Y \supset E$. Thus, since C_p is a dense subset of E (Theorem 3–44 of [5], p. 140), we have that (4.6.1) holds, a contradiction. Hence, (*) is true.

Now, we simultaneously show that neither (4.6.2) nor (4.6.3) holds. Let $M \in \begin{Bmatrix}2^E - \{E\}\\ C(E) - \{E\}\end{Bmatrix}$. Let $p \in M$ and let Z be as in (*). By 2.3 of [7] (also see 2.6 of [7]), there is a segment $\sigma\colon [0,1] \to C(Z)$ from $\{p\}$ to Z. Let $\begin{Bmatrix}k\colon [0,1] \to 2^X\\ k\colon [0,1] \to C(X)\end{Bmatrix}$ be given by

$$k(t) = M \cup \sigma(t), \qquad 0 \leqslant t \leqslant 1.$$

Clearly, k is continuous, $k(0) = M$, and $k(1) = [M \cup Z] \not\subset E$. Suppose there exists s_0 such that $k(s_0) = E$. Then, since $Z \not\supset E$, $\sigma(s_0)$ would be a proper subcontinuum of E. Hence, $\sigma(s_0)$ would be nowhere dense in E (by Theorem 3–41 of [5], p. 139). Therefore, since M is compact and $M \cup \sigma(s_0) = E$, $M = E$, a contradiction. Thus, $k(t) \neq E$ for any $t \in [0,1]$. Since M was an arbitrary member of $\begin{Bmatrix}2^E - \{E\}\\ C(E) - \{E\}\end{Bmatrix}$, it follows from Lemma 4.1 that $\begin{Bmatrix}(4.6.2)\\(4.6.3)\end{Bmatrix}$ does not hold.

(4.7) Remark. By choosing $p \in D$ one shows easily that (4.6.1) implies

(4.7.1) There exists a point $p \in E$ such that if Z is a subcontinuum of X such that $p \in Z$ and $Z \cap [X - E] \neq \emptyset$, then $Z \supset E$.

Easy examples show that (4.7.1) does not imply (4.6.1). However, as shown in the proof of Theorem 4.6 (see (*)), (4.6.1) and (4.7.1) are equivalent when E is indecomposable.

The next result gives relationships between the arc components of $2^X - \{E\}$ and those of $C(X) - \{E\}$. In particular, it shows that if Γ is an arc component of $2^X - \{E\}$ and $\Gamma \cap C(X) \neq \emptyset$, then $\Gamma \cap C(X)$ is an arc component of $C(X) - \{E\}$.

(4.8) COROLLARY. *Let E, A, and B be subcontinua of X. Then, the following two statements are equivalent:*

(4.8.1) *If a is an arc in $C(X)$ such that $A, B \in a$, then $E \in a$;*

(4.8.2) *If a is an arc in 2^X such that $A, B \in a$, then $E \in a$.*

Proof. Throughout the proof assume $A \neq E \neq B$. Assume (4.8.1) holds. Then, by Lemma 4.1, at least one of A and B, say A, is contained in E. We take two cases.

Case 1. E is decomposable. Then, by 8.2 of [7] and (4.8.1), $B \not\subset E$. Also, by (4.8.1), $C(X)-\{E\}$ is not arcwise connected (recall: $A \neq E \neq B$). Hence, by Theorem 4.6, $2^X-\{E\}$ is not arcwise connected. Therefore, since $A \subset E$ and $B \not\subset E$, (4.8.2) follows using Lemma 4.1 and Lemma 4.3.

Case 2. E is indecomposable. Let f be a continuous function from $[0,1]$ into 2^X such that $f(0) = A$ and $f(1) = B$. We will show that $f(t_0) = E$ for some $t_0 \in [0,1]$. Since A and B are continua, we have (by 1.2 of [7]) that $\bigcup f([0,t])$ and $\bigcup f([t,1])$ are each continua for any $t \in [0,1]$. Thus, letting

$$g(t) = \begin{cases} \bigcup f([0,t]), & 0 \leqslant t \leqslant 1, \\ \bigcup f([t-1,1]), & 1 \leqslant t \leqslant 2, \end{cases}$$

we see that g is a continuous function from $[0,2]$ into $C(X)$ such that $g(0) = A$ and $g(2) = B$. Hence, (4.8.1) implies that there is a $t \in [0,2]$ such that $g(t) = E$. Let

$$t_0 = \text{g.l.b.}\ \big(\{t \in [0,2]\colon g(t) = E\}\big).$$

If $t_0 \leqslant 1$, the last part of the proof of Lemma 4.3 applies giving us that $f(t_0) = E$. If $t_0 > 1$, then let

$$s_0 = \text{l.u.b.}\ \big(\{t \in [1,2]\colon g(t) = E\}\big);$$

use the last part of the proof of Lemma 4.3 (except for very minor changes) to obtain that $f(s_0) = E$. This proves (4.8.2) holds for Case 2 and completes the proof of the corollary (clearly, (4.8.2) implies (4.8.1)).

We state the next result here for use in the next section.

(4.9) COROLLARY. *If E is a decomposable subcontinuum of X such that $2^X-\{E\}$ is not arcwise connected, then E is nowhere dense in any subcontinuum of X properly containing E.*

Proof. Assume there is a subcontinuum Z of X such that Z properly contains E and E is not nowhere dense in Z. Let Y be the closure of a component of $Z-E$. By (10.2) of [14], p. 16, $Y \cap E \neq \emptyset$. Since E is not nowhere dense in Z, $Y \not\supset E$. Thus, (4.4.1) is violated and therefore, since E is decomposable, $2^X-\{E\}$ is arcwise connected by Theorem 4.4.

(4.10) **Remark.** See Example 4.5 –though $2^X-\{E\}$ is not arcwise connected, E is not nowhere dense in X.

It is possible to have every non-degenerate subcontinuum of X arcwise disconnect 2^X. In fact, as the next result of this section shows, the class of continua X with this property is precisely the class of non-degenerate hereditarily indecomposable continua.

(4.11) COROLLARY. *The following three statements are equivalent:*

(4.11.1) *X is hereditarily indecomposable (and non-degenerate);*

(4.11.2) *For any non-degenerate subcontinuum E of X, $2^X-\{E\}$ is not arcwise connected;*

(4.11.3) *For any non-degenerate subcontinuum E of X, $C(X)-\{E\}$ is not arcwise connected.*

Proof. Assume (4.11.1) holds and let E be a non-degenerate subcontinuum of X. If $E = X$, then $2^X-\{E\}$ and $C(X)-\{E\}$ not being arcwise connected follows from Lemma 4.3 and 8.2 of [7], respectively. So, assume $E \neq X$ Then E satisfies the initial conditions of Theorem 4.4. A formulation of hereditary indecomposability is: if two subcontinua intersect, then one of them is contained in the other. From this we see that E also satisfies (4.4.1). Therefore, by Theorem 4.4, $2^X-\{E\}$ and $C(X)-\{E\}$ are not arcwise connected. This proves (4.11.2) and (4.11.3) each holds. Next, assume (4.11.1) does not hold and let M be a decomposable subcontinuum of X. Let A and Y be proper subcontinua of M such that $M = A\cup Y$. Let U be an open subset of M such that $U \supset A$ and $\overline{U} \neq M$. Let E be the component of $\overline{U}$ containing A. By (10.1) of [14], p. 16, A is a *proper* subcontinuum of E. Also, since $E \subset M$, $A-Y$ is a non-empty open subset of E. This proves that A is a proper subcontinuum of E with interior in E. Hence, E is decomposable (see Theorem 3–41 of [5], p. 139). Now

$$Y\cap E \neq \emptyset$$

and, since $\overline{U} \neq M$, it follows that

$$Y\cap[X-E] \neq \emptyset.$$

However,

$$Y \not\supset E.$$

Thus, (4.4.1) is violated. Therefore, since E is decomposable, we have by Theorem 4.4 that $2^X-\{E\}$ and $C(X)-\{E\}$ are each arcwise connected. This proves that neither (4.11.2) nor (4.11.3) holds.

(4.12) **Remark.** We could have shown (4.6.1) was violated in the second half of the proof of Corollary 4.11 by observing that $Y \supset [E-A]$, a non-empty open subset of E.

Our next result, (4.13), gives a direct relationship between arcwise accessibility of singletons and sets which arcwise disconnect a hyperspace. We justify the restriction to hereditarily decomposable continua in two major ways: (1) the usefulness of (4.13) in (3.12) and (2) the host of spaces one obtains by letting E', in (4.5), be hereditarily indecomposable. Also, in light of Theorem 3.3, the restriction seems appropriate. We mention that we do not know if the converse of (4.13) is valid (see Section 6).

(4.13) THEOREM. *Let M be an hereditarily decomposable continuum and let $x_0 \in M$. If x_0 belongs to arbitrarily small subcontinua (of M) each of which arcwise disconnects 2^M, then $\{x_0\}$ is not arcwise accessible from $2^M - C(M)$.*

Proof. Assume $\{x_0\}$ is arcwise accessible from $2^M - C(M)$ and let $f\colon [0,1] \to 2^M$ be continuous such that $f(t) \in [2^M - C(M)]$ for all $t < 1$ and $f(1) = \{x_0\}$. Let

$$\eta = \text{diameter of } \bigcup f([0,1]).$$

Let K be a non-degenerate subcontinuum of M such that $x_0 \in K$ and the diameter of K is (strictly) less than η. Clearly, $\bigcup f([0,1]) \not\subset K$. Hence, there exists $s_0 \in [0,1]$ such that $f(s_0) \not\subset K$. Then, since $f(1) = \{x_0\} \subset K$ and $f(t) \neq K$ for any $t \in [0,1]$, it follows from Lemma 4.1 and Lemma 4.3 (applied to K) that $2^M - \{K\}$ is arcwise connected. This completes the proof.

(4.14) Remark. For completeness we make some technical comments relating to (3.3) and (4.13). Assume $x_0 \in X$ such that $\{x_0\}$ is arcwise accessible from $2^X - C(X)$. Let $f\colon [0,1] \to 2^X$ be continuous such that $f(t) \in [2^X - C(X)]$ for all $t < 1$ and $f(1) = \{x_0\}$. By Theorem 3.3, $\bigcup f([t,1])$ is a decomposable subcontinuum of X for all $t < 1$. Let $s < 1$ such that $\bigcup f([s,1]) \neq \bigcup f([0,1])$. Then, there exists $u_0 < s$ such that $f(u_0) \not\subset \bigcup f([s,1])$. Also, recall that $f(1) \subset \bigcup f([s,1])$. Therefore, since $\bigcup f([s,1])$ is decomposable and $f(t) \neq \bigcup f([s,1])$ for any $t \in [0,1]$, it follows from Lemma 4.1 and Lemma 4.3 that $\bigcup f([s,1])$ does not arcwise disconnect 2^X. Thus, we have proved the following

(4.14.1) THEOREM. *Let $x_0 \in X$. If $f\colon [0,1] \to 2^X$ is continuous such that $f(t) \in [2^X - C(X)]$ for all $t < 1$ and $f(1) = \{x_0\}$, then there exists $s_0 < 1$ such that, for any $s \in [s_0, 1)$, $\bigcup f([s,1])$ is a decomposable subcontinuum (with x_0 in it) which does not arcwise disconnect 2^X.*

Using Theorem 1.4, we can give a version of (4.14.1) without reference to a particular f as follows:

(4.14.2) THEOREM. *Let $x_0 \in X$. If $\{x_0\}$ is arcwise accessible from $2^X - C(X)$, then there is a segment $\sigma\colon [0,1] \to C(X)$, with $\sigma(0) = \{x_0\}$, such that $\sigma(t)$ is a decomposable subcontinuum of X and does not arcwise disconnect 2^X for any $t > 0$ (note: $\sigma(0)$ is not decomposable).*

Theorem 4.14.1 adds to the information in (3.3). Note that, for the class of hereditarily decomposable continua, (4.13) is a better result than (4.14.1) or (4.14.2). This is because (4.13) guarantees that if $\{x_0\}$ is arcwise accessible, then *no* continuum containing x_0 of sufficiently small diameter can arcwise disconnect 2^X. One final observation in regard to this: If a point $p \in X$ belongs to arbitrarily small decomposable subcontinua of X, then p belongs to arbitrarily small decomposable subcontinua which do not arcwise disconnect 2^X. To see this, let M be any decomposable subcontinuum of X such that $p \in M$. The continuum E, produced as in the second part of the proof of (4.11), is a decomposable subcontinuum of M which does not arcwise disconnect 2^X; also, $p \in E$ if we assume (as we may, without loss of generality) that $p \in A$ in the proof of (4.11).

5. Hereditary indecomposability and arcwise accessibility. In Section 4, (4.11), we characterized hereditarily indecomposable continua in terms of arcwise disconnection of their hyperspaces. In this section we characterize them in terms of certain arcwise accessibility properties in their hyperspaces (see Theorem 5.4). In addition, we obtain information about the structure of arcs in 2^X when X is hereditarily indecomposable.

Throughout this section we let G denote a non-degenerate hereditarily indecomposable continuum.

(5.1) LEMMA. *Let h: $[0,1]\to 2^G$ be a homeomorphism. If $a < b$ such that $h(a) \in C(G)$ and $h(b) \in C(G)$, then $h(t) \in C(G)$ for all $t \in [a,b]$.*

Proof. Let a_0 be an arc in $C(G)$ with non-cut points $h(a)$ and $h(b)$. Let $E \in a_0$. Suppose $E \notin h([a,b])$; then, by Corollary 4.8 above, there is an arc β in $C(G)$ such that $h(a), h(b) \in \beta$ and $E \notin \beta$, a contradiction to 8.4 of [7]. Hence, $a_0 \subset h([a,b])$. Thus, since $a_0 \subset h([a,b])$ are each arcs with the same non-cut points, $a_0 = h([a,b])$.

The following theorem is an immediate consequence of Lemma 5.1.

(5.2) THEOREM. *If Γ is an arcwise connected subset of 2^G, then $\Gamma\cap C(G)$ is arcwise connected.*

We use (5.1) to prove the following lemma which, in turn, we will use in the proof of Theorem 5.4.

(5.3) LEMMA. *Let h: $[0,1]\to 2^G$ be a homeomorphism such that $h([0,1])$ $\cap C(G) \neq \emptyset$. Let*

$$t_0 = \text{g.l.b.}\left(\{t \in [0,1]\colon h(t) \in C(G)\}\right).$$

Then $h(t_0)$ is the (unique) subcontinuum of G irreducible about $h(t)$ for any $t \in [0, t_0]$.

Proof. Let $t_1 \in [0, t_0]$. Since the lemma is clear for $t_1 = t_0$, we assume $t_1 < t_0$. Let A denote the unique subcontinuum of G irreducible about

$h(t_1)$; uniqueness is a consequence of the hereditary indecomposability of G. By 2.3 of [7], there exists a segment σ: $[0, 1] \to 2^G$ from $h(t_1)$ to A. Since $h(t_1) \neq A$ (because $t_1 < t_0$ implies $h(t_1) \notin C(G)$), it follows from the irreducibility of A that $\sigma(t) \in C(G)$ if and only if $t = 1$. Now, suppose $A \neq h(t_0)$. Then, let

$$s_0 = \text{l.u.b.}\ \big(\{t \in [0, 1]\colon\ \sigma(t) \in h([t_1, t_0])\}\big)$$

and let $t_2 \in [t_1, t_0]$ such that

$$h(t_2) = \sigma(s_0).$$

It is easy to verify that $s_0 < 1$ and $t_2 < t_0$. It follows that

$$\gamma = h([t_2, t_0]) \cup \sigma([s_0, 1])$$

is an arc in 2^G intersecting $C(G)$ only in its two non-cut points $h(t_0)$ and A. This contradicts Lemma 5.1; therefore, $A = h(t_0)$.

(5.4) THEOREM. *The following three statements are equivalent*:

(5.4.1) X *is hereditarily indecomposable*;

(5.4.2) *Given* $K \in [2^X - C(X)]$, *there exists one and only one* $A \in C(X)$ *which is arcwise accessible from* $2^X - C(X)$ *beginning with* K;

(5.4.3) *If* A *is a subcontinuum of* X *and* $K \in [2^X - C(X)]$, *then* A *is arcwise accessible from* $2^X - C(X)$ *beginning with* K *if and only if* A *is irreducible about* K.

Proof. Assume (5.4.1) holds and let $K \in [2^X - C(X)]$. Since 2^X is arcwise connected, surely there is an $A \in C(X)$ such that A is arcwise accessible from $2^X - C(X)$ beginning with K. The fact that there is *only* one such A is a consequence of Lemma 5.3. This proves (5.4.2) holds. Next, assume (5.4.2) holds. Let A be a subcontinuum of X, let $K \in [2^X - C(X)]$, and assume A is arcwise accessible from $2^X - C(X)$ beginning with K. Let A' denote a subcontinuum of X which is irreducible about K. Using 2.3 of [7] as we did in part of the proof of Lemma 5.3, we see that A' is arcwise accessible from $2^X - C(X)$ beginning with K. Hence, by the uniqueness in (5.4.2), $A' = A$ and, therefore, A is irreducible about K. Of course, as seen above, if A is irreducible about K, then A is arcwise accessible from $2^X - C(X)$ beginning with K. This proves (5.4.3) holds. Finally, assume (5.4.1) does not hold. Then there are subcontinua M and N of X such that $M \cap N \neq \emptyset$, $M \not\subset N$, and $N \not\subset M$. Let $a \in [M - N]$ and let $b \in [N - M]$. Let $K = \{a, b\}$. By 2.3 of [7], there are segments σ_i: $[0, 1] \to 2^X$, $i = 1, 2$, and 3, such that:

(1) $\sigma_1(0) = K$ and $\sigma_1(1) = M \cup \{b\}$;

(2) $\sigma_2(0) = L \cup \{b\}$ and $\sigma_2(1) = M \cup \{b\}$, where L is some component of $M \cap N$;

(3) $\sigma_3(0) = L\cup\{b\}$ and $\sigma_3(1)$ is some subcontinuum A of N which is irreducible about $L\cup\{b\}$.

The irreducibility condition in (3) guarantees that $\sigma_3(t)\in C(X)$ if and only if $t = 1$. Define f: $[0,1]\to 2^X$ by the formula

$$f(t) = \begin{cases} \sigma_1(3\cdot t), & 0\leqslant t\leqslant \frac{1}{3}, \\ \sigma_2(-3\cdot t+2), & \frac{1}{3}\leqslant t\leqslant \frac{2}{3}, \\ \sigma_3(3\cdot t-2), & \frac{2}{3}\leqslant t\leqslant 1. \end{cases}$$

It is easy to verify that f is a continuous function such that $f(t)\in C(X)$ if and only if $t = 1$. Hence, it follows that A is arcwise accessible from $2^X - C(X)$ beginning with K. Since $a\notin N$ and $A\subset N$, $a\notin A$, so clearly A is not irreducible about K. This proves (5.4.3) does not hold.

(5.5) COROLLARY. *The "unicoherence property" described in* (5.2) *characterizes hereditary indecomposability.*

(5.6) Remark. From results in this section we see that: If Γ is an arcwise connected subcontinuum of 2^G such that $\Gamma\cap C(X) = \{A\}$, then $B\subset A$ for all $B\in\Gamma$; geometrically, Γ can never be "above" A. However, this is not true for subcontinua of 2^G, in general, as (6.8.1) shows.

6. Problems. In this section we state and briefly discuss some unsettled questions related to the material in the previous sections. We also prove a result about continuumwise accessibility ((6.8.1)).

In Lemma 3.10 and Theorem 3.11 we showed that every non-degenerate rational continuum contains a subcontinuum with a cut point and, hence, "lots" of singletons which are arcwise accessible.

(6.1) For any hereditarily decomposable continuum M, must there be a point $x\in M$ such that $\{x\}$ is arcwise accessible from $2^M - C(M)$?

(6.2) Is there an hereditarily decomposable continuum such that no subcontinuum has a cut point?

(6.3) Is there a rationalless hereditarily decomposable continuum, i.e., is there an hereditarily decomposable continuum such that no (non-degenerate) subcontinuum is a rational continuum?

Note that a negative answer to (6.3) implies a negative answer to (6.2) which implies an affirmative answer to (6.1).

(6.4) What are conditions, which are at the same time both necessary and sufficient, in order that for a given point y_0 of a continuum (rational continuum, hereditarily decomposable continuum) Y, $\{y_0\}$ is arcwise accessible from $2^Y - C(Y)$? In particular, for hereditarily decomposable continua, is the converse of Theorem 4.13 valid?

(6.5) What rational continua Q have the property that *each* singleton is arcwise accessible from $2^Q - C(Q)$?

(6.6) What hereditarily decomposable continua M have the property that each (or some or no) singleton is arcwise accessible from $2^M - C(M)$?

(6.7) Is there a member of $C(X)$ which is arcwise accessible from $2^X - C(X)$, but not arcwise accessible from $C_2(X) - C(X)$? Is there a member of $C(X)$ which is arcwise accessible from $2^X - C(X)$, but not arcwise accessible beginning with a two-point set? An affirmative answer to the second of these two questions yields an affirmative answer to the first. Also note that, by Theorem 2.2, the two questions are unsolved only for singletons.

(6.8) Remark. We have investigated *arcwise* accessibility in hyperspaces. A type of accessibility which has been of general interest to topologists is continuumwise accessibility. We define a member A of $C(X)$ to be *continuumwise accessible* from $2^X - C(X)$ if and only if there is a non-degenerate subcontinuum Γ of 2^X such that $\Gamma \cap C(X) = \{A\}$. The following result and (2.2) show that every member of $C(X)$ is continuumwise accessible from $2^X - C(X)$.

(6.8.1) THEOREM. *Given any* $A \in [C(X) - \{X\}]$, *there is a non-degenerate subcontinuum* Γ *of* $C_2(X)$ *such that*

(i) $\Gamma \cap C(X) = \{A\}$,

and

(ii) Γ *is a monotone continuous image of* X.

Furthermore, if A *is a singleton,* Γ *can be chosen so as to be homeomorphic with* X.

Proof. The proof is easy —just let

$$\Gamma = \{A \cup \{x\}: x \in X\}$$

and define $f: X \to \Gamma$ by

$$f(x) = A \cup \{x\}$$

for all $x \in X$. Letting d denote the metric for X and letting H denote the Hausdorff metric for 2^X, specifically defined by the formula in [7], p. 23, we see that

$$H(f(x), f(y)) \leqslant d(x, y)$$

for all $x, y \in X$. Hence, f is continuous (and, clearly, monotone and onto Γ). Finally, if A is a singleton, f is one-to-one.

Theorem 6.8.1 and its proof give specific examples of continua, other than arcs, which intersect $C(X)$ in just one point of $C(X)$.

(6.9) What subcontinua Λ of 2^X have the property that Λ intersects $C(X)$ in just one point of $C(X)$?

Note that Γ of Theorem 6.8.1 is a monotone upper semi-continuous decomposition of X.

(6.10) What monotone upper semi-continuous decompositions Λ of X have the property that Λ intersects $C(X)$ in just one point of $C(X)$?

Of particular interest in regard to (6.9) and (6.10) is the case where X is hereditarily indecomposable.

7. Added in proof. Part of (4.1) and the fact that (4.4.3) implies (4 4.1) when E is decomposable were also obtained by J. T. Rogers, Jr., *General topology and its applications* (1973), p. 284.

References

[0] R. D. Anderson and G. Choquet, *A plane continuum no two of whose non-degenerate subcontinua are homeomorphic: an application of inverse limits*, Proc. Amer. Math. Soc. 10 (1959), pp. 347–353.

[1] R. H. Bing, *Some characterizations of arcs and simple closed curves*, Amer. J. Math. 70 (1948), pp. 497–506.

[2] K. Borsuk and S. Mazurkiewicz, *Sur l'hyperespace d'un continu*, Warszawskie Towarzystwo Naukowe 24 (1931), pp. 149–152.

[3] C. E. Capel, *Inverse limit spaces*, Duke Math. J. 21 (1954), pp. 233–245.

[4] C. E. Eberhart and Sam B. Nadler, Jr., *The dimension of certain hyperspaces*, Bull. Pol. Acad. Sci. 19 (1971), pp. 1027–1034.

[5] J. G. Hocking and G. S. Young, *Topology*, Addison–Wesley, Reading, Mass., 1961.

[6] J. L. Kelley, *A metric connected with property S*, Amer. J. Math. 61 (1939), pp. 764–768.

[7] — *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. 52, (1942), pp. 22–36.

[8] J. Krasinkiewicz, *On the hyperspaces of snake-like and circle-like continua*, Fund. Math. 83 (1974), pp. 155–164.

[9] K. Kuratowski, *Topology*, vol. I, Academic Press, New York–London–Warszawa 1966.

[10] — *Topology*, vol. II, Academic Press, New York–London–Warszawa 1968.

[11] Sam B. Nadler, Jr., *Some basic connectivity properties of Whitney map inverses in $C(X)$*, Studies in Topology, edited by N. Stavrakas and K. Allen, Academic Press, 1975, pp. 393–410.

[12] H. Whitney, *Regular families of curves*, I, Proc. N.A.S. 18 (1932), pp. 275–278.

[13] G. T. Whyburn, *A continuum every subcontinuum of which separates the plane*, Amer. J. Math. 52 (1930), pp. 319–330.

[14] — *Analytic topology*, Amer. Math. Soc., Providence, Rhode Island, 1942.

[15] M. Wojdysławski, *Rétractes absolus et hyperespaces des continus*, Fund. Math. 32 (1939), pp. 184–192.

UNIVERSITY OF GEORGIA
ATHENS, GEORGIA