

POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSSERTATIONES
MATHematicae
(ROZPRAWY MATEMATYCZNE)

KOMITET REDAKCYJNY

ANDRZEJ BIALYNICKI-BIRULA, BOGDAN BOJARSKI,
ZBIGNIEW CIESIELSKI, JERZY ŁOŚ,
ZBIGNIEW SEMADENI, JERZY ZABCZYK redaktor,
WIESŁAW ŻELAZKO zastępca redaktora

CCCLXXII

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Borel spaces II

WARSZAWA 1998

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Published by the Institute of Mathematics, Polish Academy of Sciences

Typeset using T_EX at the Institute

Printed and bound by

Publishing House of the Warsaw University of Technology

ul. Polna 50, 00-644 Warszawa

P R I N T E D I N P O L A N D

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ISSN 0012-3862

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1991 *Mathematics Subject Classification*: 28A05, 04A15, 03E15.

Received 23.12.1996.

Acknowledgements

Most of this work was completed over several years during visits by K.P.S. to Wesleyan University. He wishes to record his thanks to the Department of Mathematics at Wesleyan University for inviting him to visit.

Professors J. K. Ghosh and S. B. Rao as Directors of the Indian Statistical Institute facilitated the visits by K.P.S. to Wesleyan and he would like to acknowledge this. K.P.S. also wishes to record his appreciation of his colleagues at Stat-Math Unit of ISI, Bangalore for the camaraderie shown during January '94–May '94 while K.P.S. was visiting Wesleyan. Surekha has gone through a torturous time during this period. We appreciate her forbearance.

Introduction

The theory of Borel spaces has come of age. Several problems in this theory have attracted the attention of mathematicians and many interesting problems are now being looked into.

The first “house of its own” for Borel spaces was built by K. P. S. Bhaskara Rao and B. V. Rao in 1981. The problems posed in [7] spurred an enormous amount of research activity among set theorists and many new theories have been developed. (This includes work done by the first author of this memoir.)

Writing the present memoir is like building “a whole new floor” for the house of Borel spaces. We present many of the results obtained over the last fifteen years and several new problems are raised. We hope that this memoir will help build a “palazzo” for Borel spaces in the near future.

We begin with a short review of some concepts in Borel spaces.

A *Borel structure* on a non-empty set X is a collection \mathcal{B} of subsets of X containing the empty set, closed under countable unions and complements. If \mathcal{B} is a Borel structure on X then (X, \mathcal{B}) is known as a *Borel space*.

A Borel structure \mathcal{B} on a set X is said to be *countably generated* (c.g.) if there is a countable collection of sets \mathcal{G} such that the smallest Borel structure containing \mathcal{G} is \mathcal{B} . A Borel structure \mathcal{B} is said to be *separable* if it is countably generated and has singleton atoms. If \mathcal{C} is a c.g. structure on a set X , a *Marczewski function of \mathcal{C}* is any real-valued measurable function $f : X \rightarrow \mathbb{R}$ such that the spectral Borel structure of f , i.e. the collection $\{f^{-1}(B) : B \text{ is Borel in } \mathbb{R}\}$, is \mathcal{C} . The product of Borel structures (X, \mathcal{A}) and (Y, \mathcal{B}) is denoted by $(X \times Y, \mathcal{A} \times \mathcal{B})$.

A Borel space (S, \mathcal{B}) is called *standard Borel* if either S is countable and $\mathcal{B} = \mathcal{P}(S)$ or it is isomorphic to the real line with its usual Borel structure. Most of the notions used in this memoir are either well known or can be picked up from [7]. If $X \subset S$, then $\mathcal{B}(X)$ stands for the restriction of \mathcal{B} to X .

Since we will be referring to “Borel spaces” [7] frequently we shall abbreviate it as BS.

We refer the reader to BS for many basic and nice results in Borel spaces.

Throughout this memoir, (S, \mathcal{B}) will be reserved to indicate an uncountable standard Borel space S with its Borel structure \mathcal{B} .

1. Blackwell spaces

A separable Borel space (X, \mathcal{A}) is said to be a *Blackwell space* (or, to have the *Blackwell property*) if the only separable substructure of \mathcal{A} is itself. A separable Borel space

(X, \mathcal{A}) is said to be a *strongly Blackwell space* (or to have the *strong Blackwell property*) if any two c.g. substructures of \mathcal{A} with the same atoms coincide. Every analytic space is strongly Blackwell. In BS a strongly Blackwell space which is not analytic was constructed. A major problem from BS was to construct a Blackwell space which is not strongly Blackwell. In this chapter we shall give three such examples, Propositions 1, 2 and 36, under some set-theoretic hypotheses. We shall also look at various other developments in the theory of Blackwell spaces.

1.1. Blackwell spaces without the strong Blackwell property. In this section we exhibit, under some set-theoretic assumptions, examples of Blackwell spaces that are not strongly Blackwell. The first (Proposition 1) is due independently to D. H. Fremlin [20] and W. Bzyl and J. Jasiński [12]. It assumes MA and \neg CH (Martin's Axiom and the negation of the Continuum Hypothesis). The second (Proposition 2) is due to D. H. Fremlin [20] and assumes CH. Assuming CH, another example of a Blackwell space which is not strongly Blackwell, due to the present authors, will be presented in Proposition 36 *infra*. The question of whether such a space can be constructed in ZFC remains open (Problem PP1, Appendix I).

PROPOSITION 1 (MA + \neg CH). *Let B and Z be subsets of \mathbb{R} with $B \in \mathcal{B}(\mathbb{R})$, $Z \cap B = \emptyset$, and $\text{card}(Z) = \aleph_1 < c$. Then $X = B \cup Z$ is Blackwell, but not strongly Blackwell.*

DEMONSTRATION. Under MA + \neg CH the relative Borel structure on any set Z of cardinality $< c$ is simply the power set of Z and is countably generated. Hence the relative Borel structure on Z is Blackwell (see BS; in fact, strongly Blackwell). Now, $X = B \cup Z$ is Blackwell because of BS [2° on page 28].

Now, let A be an uncountable analytic non-Borel subset of B . Let $f : X \rightarrow A \cup Z$ be a measurable function with $f(B) = A$ and $f(z) = z$ for $z \in Z$. We assert that $A \cup Z$ is not a Blackwell space. This will establish that $X = B \cup Z$ is not strongly Blackwell because of the function f . To prove the assertion, let W be a subset of \mathbb{R} having exactly one point in common with each constituent of the co-analytic set $\mathbb{R} - A$, and let $g : A \cup Z \rightarrow A \cup W$ be a one-one function such that $g(a) = a$ for $a \in A$, and $g(Z) = W$. There is such a function because $\text{card}(Z) = \aleph_1$. The function g is measurable, because the relative Borel structure on $A \cup Z$ is simply the disjoint sum of the relative Borel structure on A and $\mathcal{P}(Z)$ (since $A \subset B$ and $B \cap Z = \emptyset$). But $g^{-1} : A \cup W \rightarrow A \cup Z$ is not measurable, because $A \in \mathcal{B}(A \cup Z)$ and $g^{-1}(A) = A \notin \mathcal{B}(A \cup W)$. Thus $A \cup Z$ is not Blackwell. ■

PROPOSITION 2 (CH). *There is a subset $X \subseteq [0, 1] \times [0, 1]$ which, in the relative Borel structure, is Blackwell but not strongly Blackwell.*

DEMONSTRATION. Enumerate the non-empty perfect subsets of $[0, 1]^2$ as $K_0, K_1, \dots, K_\alpha, \dots, \alpha < \omega_1$, and enumerate the Borel measurable functions $f : [0, 1]^2 \rightarrow \mathbb{R}$ as $f_0, f_1, \dots, f_\alpha, \dots, \alpha < \omega_1$. Let $\varrho : [0, 1]^2 \rightarrow [0, 1]$ be the projection to the first factor. For $A \subseteq [0, 1]^2$, define

$$W(A) = [0, 1]^2 - (\varrho(A \cap ([0, 1] \times \{0\})) \times (0, 1] \cup \varrho(A \cap ([0, 1] \times (0, 1])) \times \{0\}).$$

Call an $A \subseteq [0, 1]^2$ a *good set* if $(x, 0) \in A$ implies $(x, y) \notin A$ for every $y \neq 0$ and also $(x, y) \in A$ for some $y \neq 0$ implies $(x, 0) \notin A$.

We note:

1. an increasing union of good sets is a good set;
2. a set A is good if and only if $A \subseteq W(A)$;
3. $W(\bigcup A_t) = \bigcap W(A_t)$ for any family of sets $\{A_t\}$; and
4. if A is countable, then $W(A)$ is Borel.

We now define Borel sets $Y_\alpha \subseteq [0, 1]^2$ and countable sets $X_\alpha \subseteq [0, 1]^2$ so that:

- (i) $X_0 = \emptyset$ and $Y_0 = [0, 1]^2$;
- (ii) if $\beta \leq \alpha < \omega_1$, then $X_\beta \subseteq X_\alpha \subseteq Y_\alpha \subseteq Y_\beta$;
- (iii) for each $\alpha < \omega_1$, we have $Y_\alpha \subseteq W(X_\alpha)$, so that $X_\alpha \subseteq W(X_\alpha)$ is a good set; each vertical section of $W(X_\alpha) - Y_\alpha$ is countable and does not contain 0;
- (iv) for each $\alpha < \omega_1$, $X_{\alpha+1}$ meets both $K_\alpha \times \{0\}$ and $K_\alpha \times (0, 1]$;
- (v) For each $\alpha < \omega_1$, if f_α is not injective on $Y_{\alpha+1}$, neither is it injective on $X_{\alpha+1}$.

CONSTRUCTION. (a) Given X_α and Y_α satisfying (iii), take two points x_0 and x_1 in $K_\alpha - \varrho(X_\alpha)$; set $X'_\alpha = X_\alpha \cup \{(x_0, 0), (x_1, y_1)\}$, where $(x_1, y_1) \in Y_\alpha$ and $y_1 > 0$; put $Y'_\alpha = Y_\alpha \cap W(X'_\alpha)$. Then X'_α and Y'_α satisfy (iii) and X'_α meets both $K_\alpha \times \{0\}$ and $K_\alpha \times (0, 1]$.

(b) Suppose that there are points (x, y) and (x', y') in Y'_α such that $f_\alpha(x, y) = f_\alpha(x', y')$ and either $x \neq x'$, or y and y' are distinct points of $(0, 1]$. Then set $X_{\alpha+1} = X'_\alpha \cup \{(x, y), (x', y')\}$ and $Y_{\alpha+1} = Y'_\alpha \cap W(X_{\alpha+1})$. Then $X_{\alpha+1}$ is a good set, so that $X_\alpha \subseteq X_{\alpha+1} \subseteq Y_{\alpha+1} \subseteq Y_\alpha$, and the pair $X_{\alpha+1}, Y_{\alpha+1}$ satisfies conditions (iii) and (iv).

(c) Otherwise, consider the set

$$U = \{(x, y) \in Y'_\alpha : (x, 0) \in Y'_\alpha, f_\alpha(x, y) = f_\alpha(x, 0), \text{ and } y > 0\}.$$

Then U is a Borel set meeting each vertical line in at most one point. Put $X_{\alpha+1} = X'_\alpha$ and $Y_{\alpha+1} = Y'_\alpha - U$, so that f_α is one-one on $Y_{\alpha+1}$.

(d) At countable limit ordinals $\alpha > 0$, take $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$ and $Y_\alpha = \bigcap\{Y_\beta : \beta < \alpha\}$. Then X_α, Y_α for $\alpha < \omega_1$ satisfy conditions (iii) to (v).

Having completed the construction, set $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$. To see that X is a Blackwell set, let $h : X \rightarrow \mathbb{R}$ be a one-one measurable function. Then there is some $\alpha < \omega_1$ such that f_α extends h . Since f_α is one-one on $X_{\alpha+1} \subseteq X$, it is one-one on $Y_{\alpha+1} \supseteq X$. Since $Y_{\alpha+1}$ is a Borel set, f_α is an isomorphism on $Y_{\alpha+1}$, and h is an isomorphism as required.

To see that X is not strongly Blackwell, let $g : X \rightarrow [0, 1]$ be the projection to the second factor and put $E = g^{-1}(0)$. By condition (iii), we see that X is a good set, so that $E = \varrho^{-1}(\varrho(E))$. But by (iv), $\varrho(E)$ and $\varrho(X - E)$ both meet every non-void perfect set in $[0, 1]$ and so cannot be separated by Borel sets; E cannot be expressed as $\varrho^{-1}(H)$ for any Borel set $H \subseteq [0, 1]$, and yet E is a union of atoms of $\varrho^{-1}(\mathcal{B}[0, 1])$. Thus X is not strongly Blackwell. ■

The following curious result (Proposition 3) was observed by V. V. Srivatsa: under the axiom of determinacy (AD), every Blackwell set is analytic. We sketch the argument. We need a result which is known as the Wadge lemma.

LEMMA (Wadge). *Let 2^ω be the Cantor space. Under AD, if A and B are subsets of 2^ω , then either*

- (1) *there is a continuous function $f : 2^\omega \rightarrow 2^\omega$ such that $f^{-1}(B) = A$, or*
- (2) *there is a continuous function $g : 2^\omega \rightarrow 2^\omega$ such that $B = g^{-1}(2^\omega - A)$.*

INDICATION. Use Lemma 5.2.1 of [32, p. 427]. It is enough to assume that the game $G(A, B)$ is determined. ■

PROPOSITION 3 (AD). *Every Blackwell subset of a standard Borel space is analytic.*

DEMONSTRATION. Suppose that $A \subseteq 2^\omega$ is Blackwell, but not analytic. We show that this implies the existence of a co-analytic, non-Borel Blackwell set C . Fix a co-analytic, non-Borel set $B \subseteq 2^\omega$ and apply Wadge's lemma to the sets $2^\omega - A$ and B . Clearly, there cannot exist a continuous function $f : 2^\omega \rightarrow 2^\omega$ such that $f^{-1}(B) = 2^\omega - A$. So let $g : 2^\omega \rightarrow 2^\omega$ be a continuous function such that $B = g^{-1}(A)$. Put $C = g(2^\omega) \cap A$. Being the intersection of a Borel set ($g(2^\omega)$ is actually compact) and a Blackwell set, C is Blackwell. Since $B = g^{-1}(C)$, and B is non-Borel, C must be non-Borel. But $C = g(2^\omega) - g(2^\omega - B)$, which implies that C is analytic. By a result of Steel (see [49]), AD implies that all co-analytic, non-Borel sets are isomorphic. But there are co-analytic, non-Blackwell sets [Proposition 14 of BS]. This is a contradiction. ■

Thus, construction of non-analytic Blackwell sets requires something like the axiom of choice. The proof above also yields

PROPOSITION 4. *Under projective determinacy (PD), every projective Blackwell set is analytic.* ■

Under the axiom of constructibility ($V = L$), H. Sarbadhikari [34] has shown the existence of a projective strongly Blackwell set that is not analytic. The question of whether such a set can be proved to exist in ZFC remains open (Problem PP2, Appendix I).

1.2. Blackwell properties and density. In view of the examples of §1.1, the question arises as to when a Blackwell space becomes strongly Blackwell. In order to find useful sufficient conditions for this, we develop the concepts of density, and reticulated sets. In terms of the idea of density, it results that Blackwell sets are, in a certain sense, relatively large, and strongly Blackwell sets larger yet. The technique allows us to prove a separation principle for isomorphic Blackwell sets and answer some combinatorial questions from [BS].

In the following, we shall discuss strongly Blackwell spaces since our message is more transparent with such spaces.

In the discussion below, note that if \mathcal{I} is a σ -ideal in a Borel space (S, \mathcal{B}) and \mathcal{C} is a substructure of \mathcal{B} then $\mathcal{C} \vee \sigma(\mathcal{I})$, the smallest substructure of \mathcal{B} containing \mathcal{C} and the Borel structure generated by \mathcal{I} , is $\{C \Delta I : C \in \mathcal{C} \text{ and } I \in \mathcal{I}\}$. This was made evident in Proposition 38 of BS. Also,

Let \mathcal{I}_0 stand for the σ -ideal of all countable subsets of the basic set in context.

Let X be a set on which $\mathcal{E} \subset \mathcal{F}$ are two c.g. Borel structures. If \mathcal{E} and \mathcal{F} have the same atoms, one way of guaranteeing $\mathcal{E} = \mathcal{F}$ is that $\mathcal{F} \subset \mathcal{E} \vee \sigma(\mathcal{I}_0)$:

PROPOSITION 5. *If \mathcal{E} and \mathcal{F} are two Borel structures on a set X such that \mathcal{E} and \mathcal{F} have the same atoms and $\mathcal{E} \subset \mathcal{F} \subset \mathcal{E} \vee \sigma(\mathcal{I}_0)$, then $\mathcal{E} = \mathcal{F}$.*

DEMONSTRATION. For an $F \in \mathcal{F}$, let $E \in \mathcal{E}$ be such that $F = E \Delta I$ where I is a countable set. Then $F \Delta E = I \in \mathcal{F}$. Since I is countable, I is a countable union of atoms of \mathcal{F} . Since \mathcal{E} and \mathcal{F} have the same atoms, $I \in \mathcal{E}$. Thus $F = E \Delta I \in \mathcal{E}$. Hence $\mathcal{E} = \mathcal{F}$. ■

Hence let us fix a σ -ideal \mathcal{I} in (S, \mathcal{B}) and let $X \subset S$ be such that $X \cap I$ is countable for all $I \in \mathcal{I}$. Let us call such an X an \mathcal{I} -Lusin set. Now we introduce the following property:

- (I) Whenever $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ are c.g. structures such that \mathcal{D} is not a substructure of $\mathcal{C} \vee \sigma(\mathcal{I})$ then $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ do not have the same atoms.

PROPOSITION 6. *Let \mathcal{I} be a σ -ideal in (S, \mathcal{B}) and $X \subset S$ be an \mathcal{I} -Lusin set. If X has also property (I), then, $(X, X \cap \mathcal{B})$ is strongly Blackwell.*

DEMONSTRATION. If X has the above properties and $\mathcal{E} \subset \mathcal{F} \subset \mathcal{B}(X)$ are two c.g. structures with the same atoms then take two c.g. structures $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ such that $\mathcal{C}(X) = \mathcal{E}$ and $\mathcal{D}(X) = \mathcal{F}$. Then $\mathcal{D} \subset \mathcal{C} \vee \sigma(\mathcal{I})$. Restricting to X we get $\mathcal{E} \subset \mathcal{F} \subset \mathcal{E} \vee \sigma(\mathcal{I}_0)$. By Proposition 5 we get $\mathcal{E} = \mathcal{F}$. ■

Thus we have given a sufficient condition for an $X \subset S$ to be a strongly Blackwell space.

On the other hand, if $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ are two c.g. structures and if \mathcal{D} is not a substructure of $\mathcal{C} \vee \sigma(\mathcal{I})$, i.e., \mathcal{C} and \mathcal{D} are not \mathcal{I} -equivalent, and if X is, in a sense, of outer measure one, then $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ are not equal. We shall make this precise in Proposition 8.

PROPOSITION 7. *Let $\mathcal{C} \subseteq \mathcal{D}$ be c.g. Borel structures on a set X . If $\mathcal{C} \cap Y = \mathcal{D} \cap Y$ for some $Y \subseteq X$, then there is a set $D \in \mathcal{D}$ such that $Y \subseteq D$ and $\mathcal{C} \cap D = \mathcal{D} \cap D$. For general \mathcal{C} and \mathcal{D} , the D can be chosen to be in $\sigma(\mathcal{C}, \mathcal{D})$.*

DEMONSTRATION. Let $g : X \rightarrow \mathbb{R}$ be a Marczewski function of \mathcal{D} . The restriction of g to Y is $\mathcal{C} \cap Y$ -measurable, so that there is a \mathcal{C} -measurable $f : X \rightarrow \mathbb{R}$ with $f(x) = g(x)$ for $x \in Y$. Put $D = \{x : f(x) = g(x)\}$. The result follows. ■

PROPOSITION 8. *Let \mathcal{I} be a σ -ideal in (S, \mathcal{B}) . Let $X \subset S$ be such that whenever $B \in \mathcal{B}$ and $B \supset X$, then $S - B \in \mathcal{I}$. If $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ are two c.g. structures such that \mathcal{D} is not a substructure of $\mathcal{C} \vee \sigma(\mathcal{I})$ then $\mathcal{C} \cap X \neq \mathcal{D} \cap X$.*

DEMONSTRATION. If $\mathcal{C} \cap X = \mathcal{D} \cap X$ there is a $D_0 \in \mathcal{D}$, $D_0 \supset X$, such that $\mathcal{C} \cap D_0 = \mathcal{D} \cap D_0$ by Proposition 7. From the hypothesis, $(S - D_0) \in \mathcal{I}$. If $D \in \mathcal{D}$, then there is a $C \in \mathcal{C}$ such that $D \cap D_0 = C \cap D_0$, i.e. $D \Delta C \subset S - D_0$. Hence $D = C \Delta (C \Delta D) \in \mathcal{C} \vee \sigma(\mathcal{I})$, since $S - D_0 \in \mathcal{I}$. Thus $\mathcal{D} \subset \mathcal{C} \vee \sigma(\mathcal{I})$. ■

Let us say that X is \mathcal{I} -dense if X satisfies the hypothesis of Proposition 8, i.e., $X \cap B \neq \emptyset$ whenever $B \in \mathcal{B}$ and $B \notin \mathcal{I}$. If X is \mathcal{I} -Lusin and \mathcal{I} -dense we say that X is an \mathcal{I} -Sierpiński set. Then we have shown that:

PROPOSITION 9. *Let \mathcal{I} be a σ -ideal in (S, \mathcal{B}) . Let X be an \mathcal{I} -Sierpiński subset of S . Then X is strongly Blackwell if and only if whenever $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ are c.g. structures such that \mathcal{D} is not a substructure of $\mathcal{C} \vee \sigma(\mathcal{I})$, then $X \cap \mathcal{C}$ and $X \cap \mathcal{D}$ do not have the same*

atoms. Analogously, X is Blackwell if and only if whenever $\mathcal{C} \subset \mathcal{B}$ is a c.g. structure such that $\mathcal{C} \vee \sigma(\mathcal{I}) \neq \mathcal{B}$, then $X \cap \mathcal{C}$ does not have all singleton atoms. ■

Let us also observe that, if $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ are two c.g. structures and $X \subset S$, then the statements that $X \cap \mathcal{C}$ and $X \cap \mathcal{D}$ have the same atoms and “ $X \cap \mathcal{C} = X \cap \mathcal{D}$ ” can be expressed as simple set-theoretic statements.

PROPOSITION 10. Let $\mathcal{C} \subset \mathcal{D}$ be two c.g. structures in (S, \mathcal{B}) . Let $T(\mathcal{C}, \mathcal{D}) = \{(x, y) : x, y \text{ belong to an atom of } \mathcal{C} \text{ but belong to different atoms of } \mathcal{D}\}$. Then

- (a) $T(\mathcal{C}, \mathcal{D}) \in \mathcal{B} \otimes \mathcal{B}$,
- (b) $X \cap \mathcal{C}$ and $X \cap \mathcal{D}$ have the same atoms if and only if $(X \times X) \cap T(\mathcal{C}, \mathcal{D}) = \emptyset$, and
- (c) $X \cap \mathcal{C} = X \cap \mathcal{D}$ if and only if there is a $B \in \mathcal{B}$ such that $B \cap X = \emptyset$ and $T(\mathcal{C}, \mathcal{D}) \subset (B \times S) \cup (S \times B)$.

DEMONSTRATION. (a) If f and g are Marczewski functions of \mathcal{C} and \mathcal{D} respectively, then

$$T(\mathcal{C}, \mathcal{D}) = \{(x, y) : f(x) = f(y) \text{ but } g(x) \neq g(y)\}.$$

Hence $T(\mathcal{C}, \mathcal{D}) \in \mathcal{B} \otimes \mathcal{B}$.

(b) is clear.

(c) By Proposition 7 there is a $D \in \mathcal{D}$ such that $D \supset X$ and $D \cap \mathcal{C} = D \cap \mathcal{D}$. But since $D \in \mathcal{B}$, we see that D is a strongly Blackwell set. Hence $D \cap \mathcal{C} = D \cap \mathcal{D}$ if and only if $D \cap \mathcal{C}$ and $D \cap \mathcal{D}$ have the same atoms. By (b) above this is equivalent to $T(\mathcal{C}, \mathcal{D}) \subset ((S - D) \times S) \cup (S \times (S - D))$. ■

Hence an $X \subset S$ is strongly Blackwell if and only if whenever $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ are c.g. structures such that $(X \times X) \cap T(\mathcal{C}, \mathcal{D}) = \emptyset$, then there exists $B \in \mathcal{B}$, $B \cap X = \emptyset$, such that $T(\mathcal{C}, \mathcal{D}) \subset (B \times S) \cup (S \times B)$. Also,

PROPOSITION 11. Let $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ be two c.g. structures. Let \mathcal{I} be a σ -ideal in (S, \mathcal{B}) . Then the following are equivalent:

- (a) $\mathcal{D} \subset \mathcal{C} \vee \sigma(\mathcal{I})$.
- (b) There is an $I \in \mathcal{I}$ such that $(S - I) \cap \mathcal{D} = (S - I) \cap \mathcal{C}$.
- (c) There is an $I \in \mathcal{I}$ such that $T(\mathcal{C}, \mathcal{D}) \subset (I \times S) \cup (S \times I)$.

DEMONSTRATION. (a) \Rightarrow (b). If $\{D_1, D_2, \dots\}$ is a generator for \mathcal{D} then there exist C_1, C_2, \dots from \mathcal{C} such that $D_n \Delta C_n \in \mathcal{I}$ for all $n \geq 1$. Let $I = \bigcup (D_n \Delta C_n)$. Then $(S - I) \cap D_n = (S - I) \cap C_n$ for all $n \geq 1$. Thus $(S - I) \cap \mathcal{D} \subset (S - I) \cap \mathcal{C}$. But $\mathcal{C} \subset \mathcal{D}$. Hence $(S - I) \cap \mathcal{D} = (S - I) \cap \mathcal{C}$.

(b) \Leftrightarrow (c). Since $I \in \mathcal{I}$, we see that $S - I$ is a strongly Blackwell set. Hence $(S - I) \cap \mathcal{D} = (S - I) \cap \mathcal{C}$ if and only if $(S - I) \cap \mathcal{D}$ and $(S - I) \cap \mathcal{C}$ have the same atoms. By (b) of Proposition 10 we have $((S - I) \times (S - I)) \cap T(\mathcal{C}, \mathcal{D}) = \emptyset$, i.e., $T(\mathcal{C}, \mathcal{D}) \subset (I \times S) \cup (S \times I)$.

(b) \Rightarrow (a). If $D \in \mathcal{D}$, let $C \in \mathcal{C}$ be such that $D \cap (S - I) = C \cap (S - I)$. Then $D \Delta C \subset I$. Hence $D \Delta C \in \mathcal{I}$. Then $D = C \Delta (C \Delta D) \in \mathcal{C} \vee \sigma(\mathcal{I})$. ■

The above considerations give rise to the following definition introduced in [35] and [36]: For $n = 1, 2, \dots$, we consider the n -fold product space S^n . Under the product Borel structure, S^n is also standard. If N_1, \dots, N_n are subsets of S , then $\langle N_1, \dots, N_n \rangle$ denotes

the set of all (s_1, \dots, s_n) in S^n such that $s_k \in N_k$ for some k . Thus, $\langle N_1, \dots, N_n \rangle$ is the complement in S^n of $N_1^c \times \dots \times N_n^c$. Call a set $R \subseteq S^n$ \mathcal{I} -reticulate if there exist $I_1, \dots, I_n \in \mathcal{I}$ such that $R \subseteq \langle I_1, \dots, I_n \rangle$. From Propositions 9–11 we get:

PROPOSITION 12. *Let \mathcal{I} be a σ -ideal in (S, \mathcal{B}) . Let X be an \mathcal{I} -Sierpiński set. Then X is strongly Blackwell if and only if whenever $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{B}$ are c.g. structures such that $T(\mathcal{C}, \mathcal{D}) \cap (X \times X) = \emptyset$, then $T(\mathcal{C}, \mathcal{D})$ is \mathcal{I} -reticulate. Analogously, X is Blackwell if and only if whenever $\mathcal{C} \subseteq \mathcal{B}$ is a c.g. structure such that $T(\mathcal{C}, \mathcal{B}) \cap (X \times X) = \emptyset$, then $T(\mathcal{C}, \mathcal{B})$ is \mathcal{I} -reticulate. ■*

Let us say that a set $X \subseteq S$ is \mathcal{I} -dense of order n (in S) if X^n intersects each R in $\mathcal{B}(S^n)$ that is not \mathcal{I} -reticulate. This neat definition tells us that, if X is \mathcal{I} -Sierpiński and \mathcal{I} -dense of order 2, then X is strongly Blackwell.

There are three natural σ -ideals for which the above concepts and results yield fruitful consequences. Starting with an $X \subseteq S$, we define $\mathcal{I}(X) = \{B \in \mathcal{B} : B \cap X \text{ is countable}\}$. Then clearly X is $\mathcal{I}(X)$ -Lusin and $\mathcal{I}(X)$ -dense, i.e., X is $\mathcal{I}(X)$ -Sierpiński. We shall analyse this σ -ideal now. If \mathcal{I} is a σ -ideal which has all singleton sets and X is \mathcal{I} -Sierpiński then $\mathcal{I}(X) = \mathcal{I}$. Hence the σ -ideal $\mathcal{I}(X)$ is the most general σ -ideal one can examine for the study of \mathcal{I} -Sierpiński sets with respect to the Blackwell properties.

We shall also deal in this section with the σ -ideal of all countable sets. In the next section we shall deal with the σ -ideal of all sets of measure zero with respect to a given measure.

We now deal with $\mathcal{I}(X)$. Note that if X is uncountable, $\mathcal{I}(X)$ is a proper σ -ideal. Let us say that a set $R \subseteq S^n$ is X -reticulate if R is $\mathcal{I}(X)$ -reticulate.

Given $n \geq 1$, we say that $X \subseteq S$ is dense of order n (in S) if X^n intersects each R in $\mathcal{B}(S^n)$ that is not X -reticulate. Clearly, every $X \subseteq S$ is dense of order 1. A simple induction argument establishes:

PROPOSITION 13. *For $X \subseteq S$, the following statements are equivalent:*

- (1) X is dense of order n .
- (2) Whenever $B \in \mathcal{B}(S^n)$ is disjoint from X^n , there are sets N_1, \dots, N_n in \mathcal{B} which are disjoint from X and such that $B \subseteq \langle N_1, \dots, N_n \rangle$. ■

We say that $X \subseteq S$ is analytically dense of order n (in S) if X^n intersects each analytic set $A \subseteq S^n$ that is not X -reticulate. Thus, X is analytically dense of order 1 if and only if for each analytic A with $A \cap X = \emptyset$, there is some $B \in \mathcal{B}$ with $X \subseteq B$ and $A \subseteq B^c$. As above we have

PROPOSITION 14. *For $X \subseteq S$, the following statements are equivalent:*

- (1) X is analytically dense of order n .
- (2) Whenever $A \subseteq S^n - X^n$ is an analytic set, there are sets N_1, \dots, N_n in \mathcal{B} disjoint from X and such that $A \subseteq \langle N_1, \dots, N_n \rangle$. ■

EXAMPLE 1. Every analytic subset of S is analytically dense of order 1. This is simply another formulation of Lusin's first separation principle. In fact, Proposition 17 below will imply that analytic sets are analytically dense of order n for each n .

EXAMPLE 2. Let $X \subseteq S$ be a co-analytic set. Then X is analytically dense of order 1 if and only if X is a Borel subset of S .

It is not hard to show that density (resp. analytic density) of order $n + 1$ implies density (resp. analytic density) of order n . Clearly, analytic density of order n implies density of order n .

Proposition 12 raises the problem of looking at those sets in $\mathcal{B} \otimes \mathcal{B}$ which are not X -reticulate and arise out of $T(\mathcal{C}, \mathcal{D})$. Our next result is going to say, roughly, that graphs of isomorphisms correspond to $T(\mathcal{C}, \mathcal{B})$ and graphs of measurable functions correspond to $T(\mathcal{C}, \mathcal{D})$.

Let us say that a subset T of $S \times S$ is an X -thread if

- (1) T is the graph of a Borel isomorphism between sets A and B in \mathcal{B} , and
- (2) T is not X -reticulate.

Let us also call a subset G of $S \times S$ an X -graph if

- (1') G is the graph of a measurable function $g : A \rightarrow S$, where $A \in \mathcal{B}$, and
- (2') G is not X -reticulate.

(Note that G or T is allowed to be of either form $\{(s, t) : t = g(s)\}$ or $\{(s, t) : s = g(t)\}$.)

Now, consider the following conditions on a set $X \subseteq S$:

- (1) X is dense of order 2;
- (2) X has the strong Blackwell property;
- (3) X has the Blackwell property;
- (4) $X \times X$ intersects every X -graph in $S \times S$;
- (5) $X \times X$ intersects every X -thread in $S \times S$;

Then we have

PROPOSITION 15. For each $X \subseteq S$, the following diagram of implications obtains:

$$\begin{array}{ccccc} (1) & \Rightarrow & (2) & \Rightarrow & (3) \\ & & \Downarrow & & \Downarrow \\ & & (4) & \Rightarrow & (5) \end{array}$$

DEMONSTRATION. (1) \Rightarrow (2). Suppose that $\mathcal{C}(X) \subseteq \mathcal{D}(X)$ are c.g. Borel structures on X with the same atoms. Let \mathcal{C} and \mathcal{D} be c.g. sub-structures of $\mathcal{B}(S)$ such that $\mathcal{C} \cap X = \mathcal{C}(X)$ and $\mathcal{D} \cap X = \mathcal{D}(X)$. We consider the set $T(\mathcal{C}, \mathcal{D}) \in \mathcal{B}(S \times S)$. Since $\mathcal{C}(X)$ and $\mathcal{D}(X)$ have the same atoms, we have $(X \times X) \cap T(\mathcal{C}, \mathcal{D}) = \emptyset$. Since X is dense of order 2, there is, by Proposition 13, a set $N \in \mathcal{B}$ such that $N \cap X = \emptyset$ and $T(\mathcal{C}, \mathcal{D}) \subseteq \langle N, N \rangle$. Then $\mathcal{C} \cap N^c$ and $\mathcal{D} \cap N^c$ are c.g. substructures of $\mathcal{B} \cap N^c$ with the same atoms. Since N^c is standard, it has the strong Blackwell property. Thus $\mathcal{C} \cap N^c = \mathcal{D} \cap N^c$ and $\mathcal{C}(X) = \mathcal{D}(X)$.

(2) \Rightarrow (3). Obvious.

(2) \Rightarrow (4). We assume that $S = (0, 1)$ under the usual linear order and Borel structure. We shall demonstrate the contrapositive, so suppose that there is an X -graph $G \subseteq S \times S$ with $(X \times X) \cap G = \emptyset$. We assume without loss of generality that there is some $A \in \mathcal{B}(S)$

and a Borel function $k : A \rightarrow S$ such that $G = \text{graph}(k) = \{(s, k(s)) : s \in A\}$. Define

$$\Delta = \{(s, s) : s \in S\}, \quad \Delta_+ = \{(s, t) : s < t\}, \quad \Delta_- = \{(s, t) : s > t\}$$

as subsets of $S \times S$. It is easy to see that $G \cap \Delta$ is X -reticulate: it is contained in $\langle N, N \rangle$, where $N = \text{proj}(G \cap \Delta)$ is a Borel set disjoint from X . Thus one of the sets $G \cap \Delta_+$ or $G \cap \Delta_-$ is an X -graph. Without loss of generality, we assume that $G \cap \Delta_-$ is an X -graph.

For $\varepsilon > 0$, put $\Delta_-(\varepsilon) \equiv \{(s, t) : s > t + \varepsilon\}$. Then $G \cap \Delta_-(\varepsilon)$ is an X -graph for some $\varepsilon > 0$, and there is some open interval I of length ε such that $G \cap \Delta_-(\varepsilon) \cap (I \times S)$ is an X -graph. This set is the graph of a Borel function h defined on a set $B \in \mathcal{B}(S)$:

$$G \cap \Delta_-(\varepsilon) \cap (I \times S) = \{(s, h(s)) : s \in B\}.$$

Now, if $s, t \in B$, then $h(s) < s - \varepsilon < t$, so that $h(B) \cap B = \emptyset$. Define functions $f : S \rightarrow \mathbb{R}$ and $g : S \rightarrow \mathbb{R}^2$ by the formulae

$$f(s) = \begin{cases} h(s), & s \in B, \\ s, & s \in B^c, \end{cases} \quad g(s) = \begin{cases} (h(s), 0), & s \in B, \\ (s, s), & s \in B^c. \end{cases}$$

Let \mathcal{C} and \mathcal{D} be the spectral Borel structures on S generated by f and g , respectively. We see that $\mathcal{C} \subseteq \mathcal{D}$ and that

$$T(\mathcal{C}, \mathcal{D}) = \{(s, h(s)) : s \in B\} \cup \{(h(s), s) : s \in B\}$$

is not X -reticulate. Also, $(X \times X) \cap T(\mathcal{C}, \mathcal{D}) = \emptyset$, so that $\mathcal{C} \cap X \subseteq \mathcal{D} \cap X$ have the same atoms.

We now assert that $\mathcal{C} \cap X \neq \mathcal{D} \cap X$, which will imply that X does not have the Blackwell property, as required. Suppose, *a contrario*, that $\mathcal{C} \cap X = \mathcal{D} \cap X$. By Proposition 7, there is some $D \in \mathcal{D}$ with $\mathcal{C} \cap D = \mathcal{D} \cap D$ and $X \subseteq D$. Then $T(\mathcal{C}, \mathcal{D}) \subseteq \langle D^c, D^c \rangle$, so that $T(\mathcal{C}, \mathcal{D})$ is X -reticulate, a contradiction.

(3) \Rightarrow (5). Essentially the same proof as for (2) \Rightarrow (4) applies, taking G to be an X -thread and noting that, in this case, \mathcal{D} as defined above equals $\mathcal{B}(S)$. ■

We call a set $X \subset S$ *uniformisable* if every $R \in \mathcal{B}(S \times S)$ that is not X -reticulate contains an X -thread. Then

PROPOSITION 16. *Let $X \subseteq S$ be a uniformisable set. Then the following are equivalent:*

- (1) X is dense of order 2;
- (2) X has the strong Blackwell property;
- (3) X has the Blackwell property.

DEMONSTRATION. (1) \Rightarrow (2): This is part of Proposition 15.

(2) \Rightarrow (3). Obvious.

(3) \Rightarrow (1). From Proposition 15, $X \times X$ must intersect every X -thread in $S \times S$. Since X is uniformisable, each $B \in \mathcal{B}(S \times S)$ that is not X -reticulate contains an X -thread. It follows that X is dense of order 2. ■

The relation between analytic density and Blackwell properties is given by the following proposition.

PROPOSITION 17. *Let S be an uncountable standard space and suppose that $X \subseteq S$ is analytically dense of order 1 and A is an uncountable analytic subset of S . Then for each $n \geq 1$, the following are equivalent:*

- (1) $X^n \times S$ is strongly Blackwell;
- (2) $X^n \times A$ is strongly Blackwell;
- (3) $X^n \times A$ is Blackwell;
- (4) $X^n \times S$ is Blackwell;
- (5) X is analytically dense of order $2n$.

(We note that each of these conditions implies that X^n is strongly Blackwell.)

DEMONSTRATION. Since A is a measurable image of S , and S is isomorphic with a measurable subset of A , we get the implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).

(4) \Rightarrow (5). Suppose that A_0 is an analytic subset of $S^{2n} - X^{2n}$. Then there is some measurable function h defined on S with $h(S) = A_0$. We define a Borel subset T of $(S^n \times S) \times (S^n \times S)$ as

$$T = \{(s_1, \dots, s_n, s_{n+1}, t_1, \dots, t_n, t_{n+1}) : \\ s_{n+1} = t_{n+1} \text{ and } h(s_{n+1}) = (s_1, \dots, s_n, t_1, \dots, t_n)\}.$$

The set T is disjoint from $(X^n \times S) \times (X^n \times S)$. Furthermore, each section of T (in either direction) over a point in $S^n \times S$ is either null or a singleton.

Since $X^n \times S$ is Blackwell, it follows from Proposition 15 that $(X^n \times S) \times (X^n \times S)$ meets every $(X^n \times S)$ -thread in $(S^n \times S) \times (S^n \times S)$. Since the one-one graph T is not met, there is some N in $\mathcal{I}(X^n \times S)$ with $T \subseteq \langle N, N \rangle$. Let f be the natural projection map from $S^n \times S$ onto S^n . Then $M = f(N)$ is an analytic subset of S^n such that

- (a) $M \cap X^n$ is countable, and
- (b) $A_0 \subseteq \langle M, M \rangle$.

We have proved that to each analytic $A_0 \subseteq S^{2n} - X^{2n}$ there corresponds an analytic $M \subseteq S^n$ with properties (a) and (b).

The proof now proceeds by induction on n . For $n = 1$, take M corresponding to A_0 above. Since X is analytically dense in S there is some set $N \in \mathcal{I}(X)$ with $M \subseteq N$. Then $A_0 \subseteq \langle N, N \rangle$ as desired. Now we assume that the result has been established for dimensions $k \leq n$ and consider the case $n + 1$. Since $X^{n+1} \times S$ is Blackwell, so too is $X^n \times S$, hence (by the induction hypothesis) X is analytically dense of order $2n$ and therefore also of order $n + 1$. Again, given $A_0 \subseteq S^{2(n+1)} - X^{2(n+1)}$, take an analytic $M \subseteq S^{n+1}$ corresponding to A_0 as before. There are sets N_1, \dots, N_{n+1} in $\mathcal{I}(X)$ with $M \subseteq \langle N_1, \dots, N_{n+1} \rangle$. So $A_0 \subseteq \langle N_1, \dots, N_{n+1}, N_1, \dots, N_{n+1} \rangle$ as desired.

(5) \Rightarrow (1). From Proposition 15, it suffices to show that $X^n \times S$ is dense of order 2 in $S^n \times S$. Suppose that $R \in \mathcal{B}(S^n \times S \times S^n \times S)$ is contained in $(S^n \times S \times S^n \times S) - (X^n \times S \times X^n \times S)$. Let $f : S^n \times S \times S^n \times S \rightarrow S^n \times S^n$ be defined as the projection

$$f(s_1, \dots, s_n, s_{n+1}, t_1, \dots, t_n, t_{n+1}) = (s_1, \dots, s_n, t_1, \dots, t_n).$$

Then $f(R)$ is an analytic subset of $(S^n \times S^n) - (X^n \times X^n)$. Since X is analytically dense of order $2n$, there are, by Proposition 14, sets $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{B}(S)$ contained

in $S - X$ with $f(R) \subseteq \langle A_1, \dots, A_n, B_1, \dots, B_n \rangle$. So

$$R \subseteq \langle \langle A_1, \dots, A_n, \phi \rangle, \langle B_1, \dots, B_n, \phi \rangle \rangle.$$

Since $\langle A_1, \dots, A_n, \emptyset \rangle$ and $\langle B_1, \dots, B_n, \emptyset \rangle$ do not intersect $X^n \times S$, we see that R is $(X^n \times S)$ -reticulate, as desired. ■

We now prove a separation theorem for Blackwell sets and use it to deduce that a direct sum of n copies of a Blackwell space is again Blackwell.

PROPOSITION 18. *Let X_1, X_2, \dots be a (finite or countably infinite) sequence of subsets of a standard space S . Suppose that the sets X_n are pairwise disjoint and Borel isomorphic Blackwell sets. Then there is a sequence B_1, B_2, \dots of pairwise disjoint sets in $\mathcal{B}(S)$ with $X_n \subseteq B_n$ for each n .*

DEMONSTRATION. We first prove the case where the sequence is of length two. Let f be a Borel isomorphism of X_1 onto X_2 . Then f extends to a Borel isomorphism g between Borel subsets of S . From Proposition 15, we see that $X_1 \times X_1$ intersects every X_1 -thread in $S \times S$. Now $T = \text{graph}(g)$ is disjoint from $X_1 \times X_1$, so there must be some N in $\mathcal{I}(X_1)$ with $T \subseteq \langle N, N \rangle$. Then $B = (N - X_1) \cup g(N \cap X_1)$ is a Borel subset of S such that $X_2 \subseteq B$ and $X_1 \subseteq S - B$.

For the case where the sequence X_1, X_2, \dots is longer, use the preceding construction to find Borel sets $B(n, m)$ with $X_n \subseteq B(n, m)$ and $X_m \subseteq S - B(n, m)$ whenever $m \neq n$. The sets $B_n = \bigcap \{B(n, m) : m \neq n\}$, $n \geq 1$, satisfy the conclusion of the proposition. ■

PROPOSITION 19. *Let X be a Blackwell subset of S such that X and $S - X$ are Borel isomorphic. Then X is a Borel set.* ■

For strongly Blackwell sets, a somewhat improved separation can be entertained:

PROPOSITION 20. *Let X be a strongly Blackwell subset of S and suppose that $f : X \rightarrow S$ is a measurable function. If X and $f(X)$ are disjoint, then there is a Borel subset B of S such that $f(X) \subseteq B$ and $X \subseteq B^c$.*

DEMONSTRATION. The function f extends to a Borel mapping $g : S \rightarrow S$. Since $G = \text{graph}(g)$ does not meet $X \times X$, it follows from Proposition 15 that there is some N in $\mathcal{I}(X)$ such that $G \subseteq \langle N, N \rangle$. As in the proof of Proposition 18, put $B = (N - X) \cup g(N \cap X)$. Then $f(X) \subseteq B$ and $X \subseteq B^c$. ■

As an application of the separation principle of Proposition 18, we answer a question from [BS, p. 29]. We prove that the direct sum of n (or ω) copies of a Blackwell space is again Blackwell.

PROPOSITION 21. *Let X be a Blackwell subset of S and suppose that F is a finite or countably infinite set. Then $X \times F$ is a Blackwell space.*

DEMONSTRATION. Write $F = \{1, 2, \dots\}$ and let f be a one-one measurable map from $X \times F$ into S . For each n , define $X_n = f(X \times \{n\})$ and let f_n be the restriction of f to $X \times \{n\}$. Use Proposition 18 to produce a sequence of pairwise disjoint Borel subsets B_1, B_2, \dots of S with $X_n \subseteq B_n$. Using the fact that each f_n is of necessity a Borel isomorphism, for each n , extend f_n to a Borel isomorphism g_n between a Borel subset of

$S \times \{n\}$ and a Borel subset of B_n . Then the amalgam $g = \bigcup g_n$ is a Borel isomorphism extending f . So f is itself an isomorphism. ■

Whether the strong Blackwell property persists for direct sums is still not known (Problem PP3, Appendix I).

Now let us attend to the σ -ideal \mathcal{I}_0 of all countable sets in (S, \mathcal{B}) .

Clearly, every $X \subset S$ is \mathcal{I}_0 -Lusin and an $X \subset S$ is \mathcal{I}_0 -dense if and only if $S - X$ contains no uncountable Borel set, i.e., $S - X$ is totally imperfect. The next main result says essentially that this σ -ideal does not distinguish Blackwell spaces and strongly Blackwell spaces. It also says that Blackwell spaces as constructed in BS are strongly Blackwell.

Let us set up some notation, following BS. If \mathcal{C} and \mathcal{D} are two c.g. substructures of $\mathcal{B}(S)$ we say that \mathcal{C} is *proper* in \mathcal{D} if

- (i) $\mathcal{C} \subset \mathcal{D}$, and
- (ii) uncountably many atoms of \mathcal{C} are not atoms of \mathcal{D} .

We shall call a subset X of S a *(*)-Blackwell space* if whenever \mathcal{C} is a c.g. substructure of $\mathcal{B}(S)$ that is proper in $\mathcal{B}(S)$, there is an atom C of \mathcal{C} such that $C \cap X$ contains at least two distinct points. Similarly we call a subset X of S a *strongly (*)-Blackwell space* if whenever \mathcal{C} and \mathcal{D} are c.g. substructures of $\mathcal{B}(S)$ with \mathcal{C} proper in \mathcal{D} , there is an atom C of \mathcal{C} such that $C \cap X$ is not an atom of $\mathcal{D} \cap X$.

Clearly, as in BS, Section 9, every *(*)-Blackwell set* is Blackwell and every strongly *(*)-Blackwell set* is strongly Blackwell. Further, every strongly *(*)-Blackwell set* is *(*)-Blackwell*. Also,

PROPOSITION 22. *Every (*)-Blackwell set is \mathcal{I}_0 -dense in S .*

DEMONSTRATION. Let X be a *(*)-Blackwell set*. Let B be an uncountable Borel subset of S such that $X \cap B = \emptyset$. Let \mathcal{C} be a c.g. substructure of $\mathcal{B}(S)$ which has as atoms $S - B$ and pairs of points of B . This \mathcal{C} is proper in $\mathcal{B}(S)$ but violates the *(*)-Blackwell property* of X . ■

We need another result for our main proposition.

PROPOSITION 23. *Let $X \subset S$. The following are equivalent:*

- (a) *For every c.g. $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ if $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ have the same atoms then $\mathcal{D} \subset \mathcal{C} \vee \sigma(\mathcal{I}_0)$.*
- (b) *For every c.g. $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ if $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ have the same atoms then every atom of \mathcal{C} is an atom of \mathcal{D} for all but countably many atoms of \mathcal{C} .*

The corresponding equivalence for the case of $\mathcal{D} = \mathcal{B}$ also holds.

DEMONSTRATION. (a) \Rightarrow (b). Let $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ be c.g. such that $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ have the same atoms. Then $\mathcal{D} \subset \mathcal{C} \vee \sigma(\mathcal{I}_0)$ by (a). By Proposition 11 there exists a countable set $I \in \mathcal{I}_0$ such that $(S - I) \cap \mathcal{C} = (S - I) \cap \mathcal{D}$. Let $C = \bigcup_{x \in I} \{C_x : C_x \text{ is an atom of } \mathcal{C} \text{ and } x \in C_x\}$. Then $C \in \mathcal{C}$ and $C \supset I$. Hence $(S - C) \cap \mathcal{C} = (S - C) \cap \mathcal{D}$ and $C \in \mathcal{D}$ as well. Hence every atom of \mathcal{C} which is $\subset S - C$ is an atom of \mathcal{D} . Hence (b).

(b) \Rightarrow (a). Let $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ be c.g. such that $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ have the same atoms. Let C_0 be the union of the countable set of atoms of \mathcal{C} , say $\{C_n : n \geq 1\}$, which are not

atoms of \mathcal{D} . Then by the strong Blackwell nature of $S - C_0$, we have $\mathcal{C} \cap (S - C_0) = \mathcal{D} \cap (S - C_0)$. For C_n there is an atom D_n of \mathcal{D} , $D_n \subset C_n$, such that $C_n \cap X = D_n \cap X$. Now $C_n - D_n$ is a countable set because X is \mathcal{I} -dense. Hence $D_n = C_n - (C_n - D_n)$. Thus $D_n \in \mathcal{C} \vee \sigma(\mathcal{I}_0)$. And $\bigcup_{n \geq 1} (C_n - D_n)$ is a countable set. Putting together the facts that $\mathcal{C} \cap (S - C_0) = \mathcal{D} \cap (S - C_0)$, $D_n \in \mathcal{C} \vee \sigma(\mathcal{I}_0)$ for all $n \geq 1$ and that $\bigcup_{n \geq 1} (C_n - D_n)$ is countable, we get $\mathcal{D} \subset \mathcal{C} \vee \sigma(\mathcal{I}_0)$. ■

Now we are ready for our main result regarding \mathcal{I}_0 .

PROPOSITION 24. *Let X be a subset of a standard space (S, \mathcal{B}) . Let \mathcal{I}_0 be the σ -ideal of countable sets. Then the following statements are equivalent:*

- (1) X is Blackwell and \mathcal{I}_0 -dense in S (i.e., $S - X$ is totally imperfect).
- (2) For every c.g. $\mathcal{C} \subset \mathcal{B}$, if $\mathcal{C} \cap X$ has singleton atoms then $\mathcal{C} \vee \sigma(\mathcal{I}_0) = \mathcal{B}$.
- (3) X is $(*)$ -Blackwell.
- (4) X is strongly $(*)$ -Blackwell.
- (5) For every c.g. $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$, if $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ have the same atoms then $\mathcal{D} \subset \mathcal{C} \vee \sigma(\mathcal{I}_0)$.
- (6) X is strongly Blackwell and \mathcal{I}_0 -dense in S .

DEMONSTRATION. (1) \Rightarrow (2) follows from Proposition 7.

(2) \Rightarrow (3) follows from Proposition 23.

(3) \Rightarrow (4). Let $\mathcal{C} \subset \mathcal{D}$ be two c.g. substructures of $\mathcal{B}(S)$ such that uncountably many atoms of \mathcal{C} are not atoms of \mathcal{D} . Then clearly there exists $D \in \mathcal{D}$ such that D splits uncountably many atoms of \mathcal{C} .

(Definition: D splits a set C if $D \cap C$ and $D^c \cap C$ are both non-empty.)

Let f be the Marczewski function of \mathcal{C} . Since D splits uncountably many atoms of \mathcal{C} , $\text{range}(f|_D) \cap \text{range}(f|_{D^c})$ is uncountable. Let P be a perfect subset of this uncountable (analytic) set.

Let $P_1 \subset f^{-1}(P) \cap D$ be a perfect set such that f on P_1 is 1-1 and let $P_2 \subset f^{-1}(f(P_1)) \cap D^c$ be a perfect set such that f on P_2 is 1-1. This is possible because every measurable function g on a standard Borel space with an uncountable range is 1-1 on an uncountable perfect set. Let \mathcal{E} be the c.g. substructure of $\mathcal{B}(S)$ which has as atoms

$$\begin{cases} \{x\}, & x \in S - [P_2 \cup (f^{-1}(f(P_2)) \cap P_1)], \\ \{x\} \cup \{f^{-1}(f(x)) \cap P_1\}, & x \in P_2. \end{cases}$$

Observe that $f^{-1}(f(x)) \cap P_1$ for $x \in P_2$ is a singleton set. Then \mathcal{E} has uncountably many non-singleton atoms. By the $(*)$ -Blackwell property there exist x and y from X such that $x \neq y$ which are not separated in \mathcal{E} . Thus x and y belong to the same atom of \mathcal{C} and are separated (by D) in \mathcal{D} . Hence X is strongly $(*)$ -Blackwell.

(4) \Leftrightarrow (5) follows from Proposition 23.

(5) \Rightarrow (6). That X is strongly Blackwell follows from Proposition 5. We have already seen that (5) \Rightarrow (4) and (4) \Rightarrow (3). From these and Proposition 22 it follows that X is \mathcal{I}_0 -dense. Actually, it is easy to show that X is \mathcal{I}_0 -dense from (5) directly.

(6) \Rightarrow (1) is clear. ■

We remark that the above (1)–(6) are equivalent to X being \mathcal{I}_0 -dense of order 2. See [47] or [41].

1.3. Sierpiński sets and the Blackwell property. Once more, we reserve the letter S to represent an uncountable standard Borel space with Borel structure \mathcal{B} . Let m be a finite, non-negative non-zero countably additive measure on (S, \mathcal{B}) . Call a set $X \subseteq S$ an m -Sierpiński set if it is an \mathcal{I} -Sierpiński set with respect to the σ -ideal of all m -null sets, i.e., for each $N \in \mathcal{B}$, $N \cap X$ is countable if and only if $m(N) = 0$. Under CH, m -Sierpiński sets exist whenever m is a non-atomic measure. For such sets, we shall observe that the implications (2) \Rightarrow (4) and (3) \Rightarrow (5) of Proposition 15 are reversible. We shall also construct (under CH) an m -Sierpiński set that is Blackwell but not strongly Blackwell. At the root of our analysis lie certain measure-theoretic ideas, in particular, some measurable selection theorems.

Let m as before be a measure on (S, \mathcal{B}) and let $\mathcal{I}(m)$ be the σ -ideal of all $B \in \mathcal{B}$ with $m(B) = 0$. We call a set $R \subseteq S^n$ m -reticulate (in S^n) if there are sets $\{N_i : 1 \leq i \leq n\}$ in $\mathcal{I}(m)$ with $R \subseteq \langle N_1, \dots, N_n \rangle$. Let $L(S)$ be the set of all measurable functions from S to $[0, 1]$. For each n , we define set functions l_n and k_n on subsets of S^n as follows:

$$k_n(R) = \inf\{m(B_1) + \dots + m(B_n) : B_1, \dots, B_n \in \mathcal{B}(S) \text{ and } R \subseteq \langle B_1, \dots, B_n \rangle\},$$

$$l_n(R) = \inf \left\{ \int h_1 dm + \dots + \int h_n dm : \right. \\ \left. h_1, \dots, h_n \in L(S) \text{ and } I_R(s_1, \dots, s_n) \leq h_1(s_1) + \dots + h_n(s_n) \right\}.$$

Here, I_R is the indicator function of the set R . The set functions l_n and k_n have been considered by H. G. Kellerer in his work [28] on marginal problems. (See 1.3 and 1.8 therein for an equivalent definition of l_n .)

PROPOSITION 25. *With notation as above, we have:*

- (1) *The set function l_n is a regular (approximable from inside by l_n of compact sets) Choquet capacity on S^n .*
- (2) *A subset R of S^n is m -reticulate if and only if $l_n(R) = 0$.*
- (3) *A subset R of S^n is m -reticulate if and only if $k_n(R) = 0$.*
- (4) *For $n = 2$, one has $l_n = k_n$.*
- (5) *Suppose $R \subseteq S^n$ is an analytic set. Then $l_n(R) > \varepsilon$ if and only if there is some measure ν on S^n each of whose (univariate) marginals on S is m and for which $\nu(R) > \varepsilon$.*

INDICATION. (1) See 1.28 and 1.30 of [28].

(2) See part (b) in the proof of 1.15 in [28].

(3) The “only if” direction is clear. For the “if” part, choose sets $N_i^{(k)}$ with $R \subseteq \langle N_1^{(k)}, \dots, N_n^{(k)} \rangle$ and $m(N_1^{(k)}) + \dots + m(N_n^{(k)}) < 2^{-k}$. Put $N_i = \limsup N_i^{(k)}$. Then $R \subseteq \langle N_1, \dots, N_n \rangle$ and, by the Borel–Cantelli lemma, $m(N_i) = 0$. Thus R is m -reticulate.

(4) See [28, 3.3].

(5) This is the duality theorem of Kellerer. See [28, 3.13]. ■

Again, let m be a measure as before on S . Call a subset T of $S \times S$ an m -thread if

- (1) T is the graph of a Borel isomorphism between sets in $\mathcal{B}(S)$, and

(2) T is not m -reticulate.

A subset G of $S \times S$ is an m -graph if

(1') G is the graph of a measurable function $g : A \rightarrow S$, where $A \in \mathcal{B}(S)$, and

(2') G is not m -reticulate.

(Note that G or T is allowed to be of either form $\{(s, t) : t = g(s)\}$ or $\{(s, t) : s = g(t)\}$.)

The following selection theorems are fundamental to our approach. They show that every $T(\mathcal{C}, \mathcal{B})$ which is not m -reticulate contains an m -thread and every $T(\mathcal{C}, \mathcal{D})$ that is not m -reticulate contains an m -graph. Demonstrations of these, making use of various topological and measure-theoretic methods (hinted at in Proposition 25), may be found in [47].

PROPOSITION 26. *Let m be a finite, non-zero Borel measure on a standard space S . Suppose that $f : S \rightarrow \mathbb{R}$ is a measurable function generating the substructure $\mathcal{C} \subseteq \mathcal{B}$. If $T(\mathcal{C}, \mathcal{B})$ is not m -reticulate, then it contains an m -thread. ■*

PROPOSITION 27. *Let m be a finite, non-zero Borel measure on a standard space S . Suppose that $f : S \rightarrow \mathbb{R}$ and $g : S \rightarrow \mathbb{R}$ are measurable functions generating the Borel structures \mathcal{C} and \mathcal{D} , respectively, with $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{B}$. If $T(\mathcal{C}, \mathcal{D})$ is not m -reticulate, then it contains an m -graph. ■*

Turning now to Blackwell properties for m -Sierpiński sets, we note that for m -Sierpiński sets $X \subseteq S$ we have $\mathcal{I}(X) = \mathcal{I}(m)$, so that the notions of “ X -reticulate” and “ m -reticulate” coincide; the same is true of the pairs “ X -thread” and “ m -thread”, and “ X -graph” and “ m -graph”.

PROPOSITION 28. *For an m -Sierpiński set X , the notions of density of order n and analytic density of order n are coincident.*

DEMONSTRATION. Since each l_n is a regular capacity, Proposition 25 implies that every analytic set which is not m -reticulate contains a compact set with the same property. The proposition follows. ■

This immediately gives us:

PROPOSITION 29. *Let $X \subseteq S$ be an m -Sierpiński set and let A be an uncountable analytic set. The following are equivalent ($n \geq 1$):*

- (1) $X^n \times S$ is strongly Blackwell;
- (2) $X^n \times A$ is strongly Blackwell;
- (3) $X^n \times A$ is Blackwell;
- (4) $X^n \times S$ is Blackwell;
- (5) X is dense of order $2n$.

DEMONSTRATION. This follows from Proposition 28 applied to Proposition 17. ■

We shall now give a neat characterization of Blackwell and strongly Blackwell sets for m -Sierpiński sets.

PROPOSITION 30. *Let $X \subseteq S$ be an m -Sierpiński set. Then we have:*

- (1) *X is strongly Blackwell if and only if $X \times X$ meets every m -graph in $S \times S$.*
- (2) *X is Blackwell if and only if $X \times X$ meets every m -thread in $S \times S$.*

DEMONSTRATION. (1) The “only if” direction follows from Proposition 15. For the “if” direction, suppose that $\mathcal{C} \subseteq \mathcal{D}$ are c.g. substructures of $\mathcal{B}(S)$. Consider $T(\mathcal{C}, \mathcal{D}) \subseteq S \times S$. If $T(\mathcal{C}, \mathcal{D})$ is not m -reticulate, then it contains some m -graph G (Proposition 27). But then $(X \times X) \cap G \neq \emptyset$ and $(X \times X) \cap T(\mathcal{C}, \mathcal{D}) \neq \emptyset$, whence it follows that $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ cannot have the same atoms. If, however, $T(\mathcal{C}, \mathcal{D})$ is m -reticulate, then there is some Borel N with $m(N) = 0$ and $T(\mathcal{C}, \mathcal{D}) \subseteq \langle N, N \rangle$. In fact, we may assume that $X \cap N = \emptyset$ (as in Proposition 13). Then $\mathcal{C} \cap N^c$ and $\mathcal{D} \cap N^c$ are c.g. substructures of $\mathcal{B} \cap N^c$ with the same atoms. Thus $\mathcal{C} \cap N^c = \mathcal{D} \cap N^c$ and $\mathcal{C} \cap X = \mathcal{D} \cap X$.

(2) The proof here is much the same as in part (1), with “ m -graph” and “ $T(\mathcal{C}, \mathcal{D})$ ” replaced by “ m -thread” and “ $T(\mathcal{C}, \mathcal{B})$ ”. Also, Proposition 26 is to be used. ■

PROPOSITION 31. *Suppose that X_1 and X_2 are m -Sierpiński sets with $X_1 \subseteq X_2$ and $m^*(X_1) = m^*(X_2)$. (Here m^* stands for the outer measure).*

- (1) *If X_1 is Blackwell, then so is X_2 .*
- (2) *If X_1 is strongly Blackwell, then so is X_2 .*
- (3) *If $X_1^n \times S$ is Blackwell or strongly Blackwell then so is $X_2^n \times S$.*

DEMONSTRATION. This is clear from Proposition 30. ■

In Proposition 30 we have seen the effect of every $T(\mathcal{C}, \mathcal{B})$ which is not m -reticulate containing an m -thread and every $T(\mathcal{C}, \mathcal{D})$ that is not m -reticulate containing an m -graph. Let us propose:

HYPOTHESIS (G). Every Borel set $B \subseteq S \times S$ that is not m -reticulate contains an m -graph.

Though we have not been able to determine the validity of Hypothesis (G) (Problem PP4, Appendix I), we shall see a good consequence of it in Proposition 32.

Let us remark that:

(A) In Proposition 27 we have already seen that Hypothesis (G) holds for certain types of sets.

(B) Hypothesis (G) does not hold if “ m -graph” is replaced with “ m -thread”. Take $S = [0, 1]$, take m to be the Lebesgue measure and take a measure preserving map $h : S \rightarrow S \times S$. We write $h(x) = (h_1(x), h_2(x))$. Then $B = \text{graph}(h_1)$ is not m -reticulate, is an m -graph but contains no m -thread.

(C) Under CH, Hypothesis (G) is equivalent to a statement involving the strong Blackwell property, for products of certain special sets. This is discussed below.

PROPOSITION 32. *Let $X \subseteq S$ be an m -Sierpiński set. Suppose Hypothesis (G) holds. Then, if X is strongly Blackwell, so is $X \times S$.*

DEMONSTRATION. By Proposition 29, it suffices to prove that X is dense of order 2. By Hypothesis (G), every $R \in \mathcal{B}(S \times S)$ that is not m -reticulate contains an m -graph G . Since X is strongly Blackwell, $(X \times X) \cap G \neq \emptyset$. Density of order 2 follows. ■

In fact, under CH, a converse theorem is available. Hypothesis (G) seems to be connected with various questions involving doubly stochastic measures. For these points, see the discussion in [47]. We are left with the question of whether $X \times S$ is (strongly) Blackwell whenever X is a strongly Blackwell set (Problem PP5, Appendix I). See also Proposition 40, *infra*, in this connection.

We now turn to the question of the existence of m -Sierpiński sets of various types and the construction of one which is Blackwell but not strongly Blackwell.

PROPOSITION 33 (CH). *Let m be a non-atomic Borel probability on a standard space S . For each $n \geq 1$, there is an m -Sierpiński set $X \subseteq S$ dense of order n but not of order $n + 1$.*

DEMONSTRATION. We take $S = (0, 1)$ with Lebesgue measure m . Define $f_k(s) = s^k$. Well-order the class of all m -null sets in $\mathcal{B}(S)$ as $N_0, N_1, \dots, N_\alpha, \dots, \alpha < \aleph_1$, as well as all sets in $\mathcal{B}(S^n)$ not m -reticulate as $B_0, B_1, \dots, B_\alpha, \dots, \alpha < \aleph_1$. Put $M_\alpha = N_0 \cup N_1 \cup \dots \cup N_\alpha$. Select n -tuples $(x_\alpha(1), \dots, x_\alpha(n))$ in $B_\alpha - A_\alpha - \langle M_\alpha, \dots, M_\alpha \rangle$, where A_α is the union of all 1-slices of S^n over points $f_k^{-1}(f_l(x_\beta(j)))$ for $1 \leq j \leq n, 1 \leq k \leq n + 1, 1 \leq l \leq n + 1$, and $\beta < \alpha$.

Put $X = \{x_\alpha(j) : 1 \leq j \leq n, \alpha < \aleph_1\}$. Then X is an m -Sierpiński set dense of order n but not of order $n + 1$: The set X^{n+1} misses

$$T = \{s \in S^{n+1} : f_1(s(1)) = f_2(s(2)) = \dots = f_{n+1}(s(n+1))\},$$

which is not m -reticulate. ■

PROPOSITION 34 (CH). *For each $n \geq 1$, there is a Sierpiński set X such that $X^n \times S$ is strongly Blackwell but $X^{n+1} \times S$ is not Blackwell.*

DEMONSTRATION. Apply Propositions 29 and 33. ■

PROPOSITION 35 (CH). *There is a Sierpiński set X without the Blackwell property.*

DEMONSTRATION. Take X as in Proposition 33 with $n = 1$. Then T is an m -thread in $S \times S$ missed by $X \times X$, and Proposition 30 applies. ■

PROPOSITION 36 (CH). *There is a Sierpiński set X which is Blackwell but not strongly Blackwell.*

DEMONSTRATION. Put $S = [0, 1] \times (0, 1)$ under the usual Borel structure. Put $A_1 = \{0\} \times (0, 1)$ and $A_2 = S - A_1 = (0, 1) \times (0, 1)$. Define a measure m on S as follows:

- (i) on subsets of A_1 , m agrees with the usual linear Lebesgue measure;
- (ii) on subsets of A_2 , m agrees with the usual planar area measure.

Define functions f and g on S by setting $f(x, y) = y$ and

$$g(x, y) = \begin{cases} y, & x = 0, \\ y + 1, & x > 0. \end{cases}$$

Let \mathcal{C} and \mathcal{D} be the Borel structures on S generated by f and g , respectively. Clearly $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{B}(S)$.

CLAIM 1. *The set $T(\mathcal{C}, \mathcal{D})$ is not m -reticulate in $S \times S$.*

DEMONSTRATION OF CLAIM. If $T(\mathcal{C}, \mathcal{D}) \subseteq \langle N, N \rangle$ with $m(N) = 0$, then $f(A_1 \cap N)$ is a Lebesgue null set in $(0, 1)$, and from Fubini's Theorem, almost all horizontal sections of $A_2 \cap N$ have linear measure zero. It follows that for some $y \in (0, 1)$, we have $(0, y) \notin N$ and $(x, y) \notin N$ for some $x > 0$. Now $\mathcal{C} \cap N^c$ and $\mathcal{D} \cap N^c$ separate the same pairs of points, yet the pair $(0, y), (x, y)$ is separated by \mathcal{D} and not \mathcal{C} , a contradiction.

CLAIM 2. *For every m -thread $R \subseteq S \times S$, the intersection $R \cap T(\mathcal{C}, \mathcal{D})$ is m -reticulate.*

DEMONSTRATION OF CLAIM. We partition $T(\mathcal{C}, \mathcal{D})$ into the Borel sets $U_1 = \{((0, y), (x', y)) : x' > 0\}$ and $U_2 = \{((x, y), (0, y)) : x > 0\}$. It suffices to show that $R \cap U_1$ is m -reticulate, the case for $R \cap U_2$ being symmetrical. Now $R \cap U_1$ is the graph of some function $h = C \rightarrow S$, where $C = \{0\} \times D$ for some Borel $D \subseteq (0, 1)$. Define $k : D \rightarrow (0, 1)$ by $h(0, y) = (k(y), y)$ and consider $E = \{(k(y), y) : y \in D\} \subseteq A_2$. We have $m(E) = 0$ and $R \cap U_1 = \text{graph}(h) \subseteq S \times E$, proving the claim.

With the two claims at hand, a simple transfinite construction yields an m -Sierpiński set $X \subseteq S$ such that $X \times X$ intersects every m -thread in $S \times S$ but $(X \times X) \cap T(\mathcal{C}, \mathcal{D}) = \emptyset$. By Proposition 30, X has the Blackwell property.

Now $(X \times X) \cap T(\mathcal{C}, \mathcal{D}) = \emptyset$ implies that $\mathcal{C} \cap X$ and $\mathcal{D} \cap X$ have the same atoms. Yet if $\mathcal{C} \cap X = \mathcal{D} \cap X$, then there is (Proposition 7) some $D \in \mathcal{D}$ with $X \subseteq D$ and $\mathcal{C} \cap D = \mathcal{D} \cap D$. Then $N = S - D$ has $m(N) = 0$, and also $T(\mathcal{C}, \mathcal{D}) \subseteq \langle N, N \rangle$, a contradiction. ■

Let us remark that another construction is possible of an m -Sierpiński set which is Blackwell but not strongly Blackwell, again by using transfinite induction and Proposition 9. See [47].

1.4. Products and intersections of Blackwell sets. In this section, several examples are presented to illustrate the irregular behaviour of the Blackwell property under set-theoretic operations such as product and intersection. Some of these results stand in contrast to results such as Proposition 17. The first result is due to J. Jasiński [24].

PROPOSITION 37 (MA + ¬CH). *There is a Blackwell set $X \subseteq \mathbb{R}$ and an analytic set $A \subseteq \mathbb{R}$ such that $X \cap A$ is not Blackwell.*

DEMONSTRATION. Let $B \subseteq \mathbb{R}$ be an uncountable Borel set and $Z \subseteq \mathbb{R} - B$ a set with $\omega_0 < \text{card}(Z) < c$. Put $X = B \cup Z$. As in the proof of Proposition 1, X is a Blackwell set. Let $A \subseteq B$ be an analytic, non-Borel set. Then $X \cap (B^c \cup A) = A \cup Z$ is not Blackwell. Again, see the proof of Proposition 1. ■

Under MA and ¬CH, is it the case that if X is strongly Blackwell and A is analytic then $X \cap A$ is Blackwell (or even strongly Blackwell)? (Problem PP6, Appendix I.)

The next construction, also due to J. Jasiński [24], requires a couple of preliminary results. Demonstrations of these are to be found in [24].

PROPOSITION 38 (MA). *Let S be the space of irrationals and let $X \subseteq S$ be a set of power less than the continuum. Then $B \in \mathcal{B}(X \times S)$ if and only if there is some $\alpha < \omega_1$ such that for each $x \in X$, the section $B_x = \{s : (x, s) \in B\}$ is of additive class α . ■*

PROPOSITION 39 (MA). *Let S be the space of irrationals and let $X \subseteq S$ be a set of power less than the continuum. A function $f : X \times S \rightarrow S$ is measurable if and only if*

there is some $\alpha < \omega_1$ such that for each $x \in X$, the section f_x defined by $f_x(s) = f(x, s)$ is of class α . ■

PROPOSITION 40 (MA + \neg CH). *There is a strongly Blackwell set $X \subseteq S$ such that $X \times S$ is not Blackwell.*

DEMONSTRATION. Without loss of generality, we may take S to be the space of irrational numbers. Let $X \subseteq S$ be a set with $\text{card}(X) = \aleph_1$. Write $X = \{x_\alpha : \alpha < \aleph_1\}$. By a result of Hausdorff [23] we may write S as a disjoint union of Borel sets E_α ($\alpha < \aleph_1$) of additive class 3. There is, from Kuratowski [29, p. 450, Theorem 2 of Section 37 of Ch. III], for every α a Borel isomorphism f_α ($\alpha < \aleph_1$) mapping S onto E_α , each of class α . By Proposition 39, the one-one function $h : X \times S \rightarrow S$ defined by $h(x_\alpha, s) = f_\alpha(s)$ is measurable. Since $X \times S$ is not standard, h cannot be an isomorphism. Thus $X \times S$ is not Blackwell. ■

As complement to this stands the following result, originally in [42]. Again, let S be an uncountable standard space.

PROPOSITION 41 (CH). *There is a strongly Blackwell set $X \subseteq S$ and an analytic set $A \subseteq S$ such that neither $X \times S$ nor $X \cap A$ is Blackwell.*

DEMONSTRATION. We take $S = (-1, 0) \cup (0, 1)$ under the usual Borel structure and let $f : S \rightarrow S$ be the map $f(s) = -s$. Let A^+ be an analytic, non-Borel subset of $(0, 1)$ and put $A = A^+ \cup f(A^+)$. Let A_0 be the graph of the restriction of f to A .

Define \mathcal{I} to be the σ -ideal in $\mathcal{B}(S)$ generated by all standard Borel subsets of A and of $S \setminus A$. Each set in \mathcal{I} decomposes as the union of two such sets. Let $R_0, R_1, \dots, R_\alpha, \dots$, $\alpha < c$, be a transfinite listing of all sets in $\mathcal{B}(S \times S)$ not contained in a set of the form $\langle N, N \rangle$ for $N \in \mathcal{I}$. Call a subset P of A *trivial* if there is some $N \in \mathcal{I}$ with $P \subseteq N$. List in transfinite series $P_0, P_1, \dots, P_\alpha, \dots$, $\alpha < c$, all non-trivial analytic subsets of A . Also list the sets in \mathcal{I} as $N_0, N_1, \dots, N_\alpha, \dots$, $\alpha < c$, and define $M_\alpha = N_0 \cup N_1 \cup \dots \cup N_\alpha$.

Choose a point (x_0, y_0) in $R_0 - \langle M_0, M_0 \rangle - A_0$, define $U_0 = \{x_0, y_0\} \cap A$ and select $z_0 \in P_0 - f(U_0)$. Put $X_0 = \{x_0, y_0, z_0\}$.

We assume that for each $\beta < \alpha$, a countable set X_β has been defined so that $(X_\beta \times X_\beta) \cap R_\beta$ and $X_\beta \cap P_\beta$ are non-void, whilst $(X_\beta \times X_\beta) \cap A_0 = \emptyset$. Also, we assume that for $\gamma \leq \beta < \alpha$, one has $X_\beta \cap M_\gamma = X_\gamma \cap M_\gamma$.

Define $Y_\alpha = \bigcup \{X_\beta : \beta < \alpha\}$ and $Z_\alpha = Y_\alpha \cap A$.

Noting the assumption of CH, we select a point $(x_\alpha, y_\alpha) \in R_\alpha - \langle M_\alpha \cup f(Z_\alpha), M_\alpha \cup f(Z_\alpha) \rangle - A_0$. Define $U_\alpha = Z_\alpha \cup (\{x_\alpha, y_\alpha\} \cap A)$ and select an element z_α from $P_\alpha - f(U_\alpha)$. Put $X_\alpha = Y_\alpha \cup \{x_\alpha, y_\alpha, z_\alpha\}$. Then $(X_\alpha \times X_\alpha) \cap R_\alpha$ and $X_\alpha \cap P_\alpha$ are non-null, whereas $(X_\alpha \times X_\alpha) \cap A_0 = \emptyset$. Also, if $\gamma \leq \alpha$, then $X_\alpha \cap M_\gamma = X_\gamma \cap M_\gamma$.

Define $X = \bigcup \{X_\alpha : \alpha < c\} = \{x_\alpha, y_\alpha, z_\alpha : \alpha < c\}$. It is easy to see that $\mathcal{I} = \mathcal{I}(X)$. Also, X is dense of order 2, and so is strongly Blackwell.

CLAIM 1. *$X \times S$ is not Blackwell.*

DEMONSTRATION OF CLAIM. Let $g : S \rightarrow A_0$ be a measurable function onto A_0 and consider the subset of $(S \times S) \times (S \times S)$ defined by

$$T = \{(s_1, s_2, t_1, t_2) : s_2 = t_2 \text{ and } g(s_2) = (s_1, t_1)\}.$$

We see that $[(X \times S) \times (X \times S)] \cap T = \emptyset$: this follows because $(X \times X) \cap A_0 = \emptyset$. However, we assert that T is an $(X \times S)$ -thread. The claim will follow from this. Suppose that $T \subseteq \langle N, N \rangle$ for some Borel set N with $N \cap (X \times S) = \emptyset$. Let N_0 be the (analytic) projection of N on the first factor of $S \times S$. Since $N_0 \cap X = \emptyset$, we see that N_0 is a trivial analytic set and is contained in a set N_{00} in \mathcal{I} . Then $T \subseteq \langle N_{00} \times S, N_{00} \times S \rangle$, implying that $A_0 \subseteq \langle N_{00}, N_{00} \rangle$ and $A \subseteq N_{00} \cup f(N_{00})$, a contradiction, as desired.

CLAIM 2. $X \cap A$ is not Blackwell.

DEMONSTRATION OF CLAIM. We assert that $\text{graph}(f)$ is an $(X \cap A)$ -thread in $S \times S$. Since $[(X \cap A) \times (X \cap A)] \cap \text{graph}(f) = (X \times X) \cap A_0 = \emptyset$, this will establish the claim. Suppose that $N \in \mathcal{B}(S)$ with $N \cap (X \cap A) = \emptyset$ and $\text{graph}(f) \subseteq \langle N, N \rangle$. The first of these implies that N is trivial. Then so is $f(N)$. The second containment implies that $S = N \cup f(N)$, a contradiction. ■

We now list a number of unsolved problems concerning some combinatorial aspects of Blackwell properties.

- Is there a set $X \subseteq \mathbb{R}$ such that $X \times \mathbb{R}$ is Blackwell but not strongly Blackwell? (Problem PP7, Appendix I.) Compare with Proposition 17.
- Can one produce (in ZFC) a Blackwell (or strongly Blackwell) set X and an analytic set A such that $X \cap A$ and $X \times A$ are not Blackwell (or strongly Blackwell)? (Problem PP8, Appendix I.)
- Can one obtain, under CH, a Blackwell set $X \subseteq \mathbb{R}$ and an analytic set $A \subseteq \mathbb{R}$ such that $X \cup A$ is not Blackwell? (Problem PP9, Appendix I.)
- Suppose that $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \dots$ is a sequence of c.g. Borel structures on a set X with (X, \mathcal{C}_n) Blackwell. If $\mathcal{C} = \bigcap \mathcal{C}_n$ is c.g., then is (X, \mathcal{C}) Blackwell? (Problem PP10, Appendix I. This is a problem of D. H. Fremlin.)

Let us remark at the end that D. H. Fremlin, in 1983, has constructed an example of a separable space which gives a negative answer to the problem: Let (X, \mathcal{B}) be a separable space. Is there a separable $\mathcal{C} \subseteq \mathcal{B}$ with (X, \mathcal{C}) Blackwell?

2. Lattice of subalgebras

In this chapter we shall study some aspects of the lattice $L_{\mathcal{B}}$ of all substructures of a given Borel structure (X, \mathcal{B}) . The complementation problem for $L_{\mathcal{B}}$ was studied in BS. We shall study some more aspects of this lattice.

If \mathcal{B} is a Borel structure on a set X , the set of all substructures of \mathcal{B} is denoted by $L_{\mathcal{B}}$. $L_{\mathcal{B}}$ has a natural lattice structure: if $\mathcal{C}, \mathcal{D} \in L_{\mathcal{B}}$, then $\mathcal{C} \vee \mathcal{D}$ stands for the smallest structure containing \mathcal{C} and \mathcal{D} and $\mathcal{C} \wedge \mathcal{D}$ stands for the largest structure contained in \mathcal{C} and \mathcal{D} . Also, $L_{\mathcal{B}}$ has a smallest element and a largest element. This makes the concept of a complement in $L_{\mathcal{B}}$ meaningful. Also, concepts such as conjugates, maximal conjugates, minimal complements and minimal weak complements become interesting. We shall study some of these concepts in this chapter.

2.1. Complements. In §19 of BS, the problem of characterizing Borel structures \mathcal{B} for which the lattice $L_{\mathcal{B}}$ is complemented was studied in detail and several results were obtained. In particular, it was shown that, if $L_{\mathcal{B}}$ is complemented, then every c.g. substructure of \mathcal{B} is given by a countable partition. Problem P13 of BS asked if the above property characterizes Borel structures \mathcal{B} for which $L_{\mathcal{B}}$ is complemented.

We shall give an example to show that the above property does *not* characterize \mathcal{B} for which $L_{\mathcal{B}}$ is complemented.

PROPOSITION 42. *Let X be an uncountable set. Suppose that for each ordinal $\alpha < \omega_1$, \mathcal{P}_α is a countable partition of X such that for $\alpha < \beta$, \mathcal{P}_β is a refinement of \mathcal{P}_α . Let \mathcal{C} be the countable-cocountable structure on X and μ be the 0-1 measure on \mathcal{C} giving measure 0 to countable sets. Let $\mathcal{B} = \sigma(\mathcal{C}, \bigcup_\alpha \mathcal{P}_\alpha)$. Then every c.g. substructure of \mathcal{B} is given by a countable partition. Also, μ can be extended as a 0-1 measure on \mathcal{B} if and only if for each α , there is an uncountable set A_α in \mathcal{P}_α such that $A_\beta \subset A_\alpha$ for all $\alpha < \beta$.*

DEMONSTRATION. That every c.g. substructure of \mathcal{B} is given by a countable partition is clear.

If μ can be extended as a 0-1 measure $\tilde{\mu}$ on \mathcal{B} then for each α there is exactly one set A_α in \mathcal{P}_α such that $\tilde{\mu}(A_\alpha) = 1$. These sets satisfy our requirements.

To prove the converse, suppose that A_α , $\alpha < \omega_1$, exist as stated. In case there is an α_0 such that $A_\beta = A_{\alpha_0}$ for all $\beta \geq \alpha_0$, the structure $\mathcal{B} \cap A_{\alpha_0}$ is the countable-cocountable structure on A_{α_0} . Define $\tilde{\mu}$ on \mathcal{B} by $\tilde{\mu}(B) = 0$ if $B \cap A_{\alpha_0}$ is countable and $\tilde{\mu}(B) = 1$ if $B \cap A_{\alpha_0}$ is cocountable in A_{α_0} . Then $\tilde{\mu}$ extends μ .

In case there is a $\beta > \alpha$, for every α , such that $A_\beta \neq A_\alpha$, we proceed as follows. First observe that for every $B \in \mathcal{B}$ there is an α such that $A_\alpha - \bigcap_\beta A_\beta \subset B$ or $A_\alpha - \bigcap_\beta A_\beta \subset B^c$. Also observe that exactly one of the above should hold, for otherwise, there are α_1 and α_2 such that

$$A_{\alpha_1} - \bigcap_\beta A_\beta \subset B \quad \text{and} \quad A_{\alpha_2} - \bigcap_\beta A_\beta \subset B^c,$$

so that if $\alpha = \max(\alpha_1, \alpha_2)$ then $A_\alpha = \bigcap_\beta A_\beta$, which cannot occur in the case under consideration. Now define a 0-1 measure $\tilde{\mu}$ on \mathcal{B} by $\tilde{\mu}(B) = 1$ if $A_\alpha - \bigcap_\beta A_\beta \subset B$ for some α , and $\tilde{\mu}(B) = 0$ if $A_\alpha - \bigcap_\beta A_\beta \subset B^c$ for some α . To see that $\tilde{\mu}$ is countably additive, let B_1, B_2, \dots be sets in \mathcal{B} such that $\tilde{\mu}(B_i) = 0$ for every i . Then for each i there is an α_i such that $A_{\alpha_i} - \bigcap_\beta A_\beta \subset B_i^c$. Taking $\gamma > \alpha_i$ for all i observe that $A_\gamma - \bigcap_\beta A_\beta \subset (\bigcup_i B_i)^c$, proving that $\tilde{\mu}(\bigcup_i B_i) = 0$.

Since $\bigcap_\alpha (A_\alpha - \bigcap_\beta A_\beta) = \emptyset$, for every x there is an α_0 such that $x \notin A_{\alpha_0} - \bigcap_\beta A_\beta$, proving that $\tilde{\mu}\{x\} = 0$. This shows that $\tilde{\mu}$ is an extension of μ . ■

PROPOSITION 43. *Let X be a set of power \aleph_1 . Let \mathcal{C} be the countable-cocountable structure on X and μ the 0-1 measure on \mathcal{C} giving measure 0 to singletons. Then there is a structure \mathcal{B} on X such that*

- (i) $\mathcal{C} \subset \mathcal{B}$,
- (ii) every c.g. substructure of \mathcal{B} is given by a countable partition, and
- (iii) μ cannot be extended as a 0-1 measure on \mathcal{B} .

DEMONSTRATION. Let \leq be a partial order on X such that (X, \leq) is an Aronszajn tree [27, §1], that is,

- (a) for every x in X , $\{y : y \leq x\}$ is well ordered with order type, say, $\varrho(x)$,
- (b) for every x , $\varrho(x) < \omega_1$,
- (c) for every $\alpha < \omega_1$ the set $\{x : \varrho(x) = \alpha\}$ is countable, and
- (d) there is no linearly ordered subset of power \aleph_1 .

For each $\alpha < \omega_1$, let \mathcal{P}_α be the partition of X consisting of the singleton sets $\{x\}$ for each x with $\varrho(x) < \alpha$ and the sets $\{y : x \leq y\}$ for each x with $\varrho(x) = \alpha$. This is a countable partition and for $\alpha < \beta$, \mathcal{P}_β is a refinement of \mathcal{P}_α . Let $\mathcal{B} = \sigma(\bigcup_\alpha \mathcal{P}_\alpha)$. (i) clearly holds. Proposition 42 above proves (ii). To prove (iii) suppose for each α that A_α is an uncountable set in \mathcal{P}_α such that $A_\beta \subset A_\alpha$ whenever $\alpha < \beta$. Then by the definition of \mathcal{P}_α it must be the case that $A_\alpha = \{y : x_\alpha \leq y\}$ for some x_α with $\varrho(x_\alpha) = \alpha$. Then $\{x_\alpha : \alpha < \omega_1\}$ is a linearly ordered set of power \aleph_1 , contradicting (d). So, by Proposition 42, μ cannot be extended as a 0-1 measure on \mathcal{B} . ■

Now we are ready to give our example.

PROPOSITION 44. *For the (X, \mathcal{B}) as in Proposition 43, $L_{\mathcal{B}}$ is not complemented.*

DEMONSTRATION. Let \mathcal{C} be as in Proposition 43. Suppose that there is a structure \mathcal{D} such that $\mathcal{C} \vee \mathcal{D} = \mathcal{B}$ and $\mathcal{C} \wedge \mathcal{D} = \{\emptyset, X\}$. Fix a point x_0 in X . Define $\tilde{\mu}$ on \mathcal{B} by $\tilde{\mu}(B) = 1$ if there is a D in \mathcal{D} such that $B \Delta D$ is countable and $x_0 \notin D$. Then $\tilde{\mu}$ is a 0-1 measure on \mathcal{B} extending μ of Proposition 43 (details are as in the proof of Proposition 39 of BS). Again by Proposition 43, such an extension does not exist. So \mathcal{C} has no complement in \mathcal{B} . ■

If $\mathcal{C} \in L_{\mathcal{B}}$, we say that \mathcal{D} is a *weak complement* of \mathcal{C} if $\mathcal{C} \vee \mathcal{D} = \mathcal{B}$. Call a weak complement \mathcal{D} a *minimal weak complement* if there is no $\mathcal{E} \subset \mathcal{D}$, $\mathcal{E} \neq \mathcal{D}$, which is a complement of \mathcal{C} . This concept is dual to the concept of maximal conjugate. A study of minimal weak complements must be interesting (Problem PP11, Appendix I).

2.2. Maximal conjugates. If $\mathcal{C}, \mathcal{D} \in L_{\mathcal{B}}$, we say that \mathcal{D} is a *conjugate* of \mathcal{C} if $\mathcal{C} \cap \mathcal{D} = \{\emptyset, X\}$. If there is no other $\mathcal{D}' \supseteq \mathcal{D}$ such that $\mathcal{C} \cap \mathcal{D}' = \{\emptyset, X\}$ then we say that \mathcal{D} is a *maximal conjugate* of \mathcal{C} . Various problems arise. Should there always be maximal conjugates? Is every maximal conjugate a complement? We shall obtain several results in these directions and pose several problems.

Let us start with an example of a Borel structure in an $L_{\mathcal{B}}$ which has no maximal conjugate.

EXAMPLE 3. *Let \mathcal{B} be the usual Borel structure on \mathbb{R} and let \mathcal{C} be the σ -field generated by Borel sets of Lebesgue measure zero. Then \mathcal{C} has no maximal conjugate in \mathcal{B} .*

DEMONSTRATION. Suppose that \mathcal{D} is a conjugate for \mathcal{C} . Every non-empty set in \mathcal{D} has positive Lebesgue measure. Thus every family of disjoint sets in \mathcal{D} is at most countable. Since \mathcal{D} is a σ -field, \mathcal{D} is given by a countable partition. Let D be an atom of \mathcal{D} . Then D is a disjoint union of sets B_1 and B_2 in \mathcal{B} each of positive measure. Clearly, $\mathcal{D}' = \sigma(\mathcal{D}, B_1)$ is also a conjugate of \mathcal{C} and it is strictly larger than \mathcal{D} . ■

The same argument applies when \mathcal{C} is replaced with the σ -field generated by any other σ -ideal \mathcal{I} of \mathcal{B} with the property that the quotient \mathcal{B}/\mathcal{I} satisfies c.c.c. For \mathcal{C} the σ -field of countable sets and their complements, the situation seems more complicated.

PROPOSITION 45 (Baumgartner [5]). *Let X be a set of cardinality κ and let \mathcal{C} be the countable-cocountable structure on X . If \mathcal{C} has a maximal conjugate in $\mathcal{P}(X)$, then there is a family $\{\mu_x : x \in X\}$ of κ many continuous 0-1 valued outer measures on X with the property that for each $A \subseteq X$, there is some $x \in X$ such that A is μ_x -completion measurable.*

DEMONSTRATION. Suppose that \mathcal{D} is a maximal conjugate of \mathcal{C} . Consider $\sigma(\mathcal{D}, \mathcal{C})$. Each set in $\sigma(\mathcal{D}, \mathcal{C})$ may be written as $D \Delta C$ for some $D \in \mathcal{D}$ and a countable set C . Since $\mathcal{C} \cap \mathcal{D} = \{\emptyset, X\}$, this representation is unique. For $x \in X$, define μ_x on $\sigma(\mathcal{D}, \mathcal{C})$ by

$$\mu_x(D \Delta C) = \begin{cases} 1, & x \in D, \\ 0, & x \notin D. \end{cases}$$

Then μ_x is a 0-1 measure on $\sigma(\mathcal{D}, \mathcal{C})$ and so extends to a 0-1 outer measure on X .

Each set $A \in \sigma(\mathcal{D}, \mathcal{C})$ is μ_x -completion measurable for each x . Given $A \notin \sigma(\mathcal{D}, \mathcal{C})$, we see that $\sigma(\mathcal{D}, A)$ is not a conjugate for \mathcal{C} , so that for some $D \in \mathcal{D}$, either $D \cap A$ or $D \cap (X - A)$ is countable and non-empty. Both of these cases are similar. Suppose $D \cap A$ is countable and take $x \in D - (D \cap A)$. Then, since $D - (D \cap A) \subseteq X - A$ we have $\mu_x(X - A) \geq \mu_x(D - (D \cap A)) = 1$; so $X - A$ (also A) is μ_x -completion measurable. ■

NOTE. A result of Prikry shows that the conclusion of Proposition 45 is consistent with ZFC + GCH. Jensen later showed that it in fact follows from $V = L$. Prikry also showed that it is consistent with ZFC + CH that this conclusion is false. Thus one will not be able to show in ZFC that the countable-cocountable structure has no maximal conjugate in $\mathcal{P}(X)$.

Now, let us show that the simplest possible substructures always have maximal conjugates, in fact, nice maximal conjugates.

PROPOSITION 46. *Let (X, \mathcal{B}) be a Borel space and let C_1, C_2, \dots be a countable partition of X into sets in \mathcal{B} . Put $\mathcal{C} = \sigma(C_1, C_2, \dots)$. Then*

- (1) \mathcal{C} has a maximal conjugate in \mathcal{B} .
- (2) Every maximal conjugate for \mathcal{C} is a (maximal) complement.
- (3) If \mathcal{B} is c.g., then every maximal conjugate of \mathcal{C} is c.g.
- (4) Let $\mu_1, \mu_2 \dots$ be 0-1 countably additive measures on $C_1 \cap \mathcal{B}, C_2 \cap \mathcal{B}, \dots$ respectively.

Then $\mathcal{D} = \{B \in \mathcal{B} : \mu_i(B \cap C_i) = 0 \text{ for all } i \text{ or } \mu_i(B \cap C_i) = 1 \text{ for all } i\}$ is a maximal conjugate of \mathcal{C} and every maximal conjugate of \mathcal{C} arises out of some $\mu_1, \mu_2 \dots$ as above.

DEMONSTRATION. (1) Fix points $x_1 \in C_1, x_2 \in C_2, \dots$. Let $\mathcal{D} = \{B \in \mathcal{B} : B \supset \{x_1, x_2, \dots\} \text{ or } B \cap \{x_1, x_2, \dots\} = \emptyset\}$. Then \mathcal{D} is a substructure of \mathcal{B} ; we show that \mathcal{D} is a maximal conjugate of \mathcal{C} .

Indeed, if $B \in \mathcal{C} \cap \mathcal{D}$, assume without loss of generality that $B \supset \{x_1, x_2, \dots\}$. But the only set in \mathcal{C} which contains $\{x_1, x_2, \dots\}$ is X . Thus $\mathcal{C} \cap \mathcal{D} = \{\emptyset, X\}$, i.e., \mathcal{D} is a conjugate of \mathcal{C} .

To show that \mathcal{D} is a maximal conjugate of \mathcal{C} , let $B \notin \mathcal{D}$. Then

$$B = \left(B \cap \bigcup_{\{i:x_i \in B\}} C_i \right) \cup \left(B \cap \bigcup_{\{i:x_i \notin B\}} C_i \right).$$

So the second set $B \cap \bigcup_{\{i:x_i \notin B\}} C_i$ is a set which has empty intersection with $\{x_1, x_2, \dots\}$ and hence belongs to \mathcal{D} . Thus,

$$B \cap \bigcup_{\{i:x_i \in B\}} C_i = B - \left(B \cap \bigcup_{\{i:x_i \notin B\}} C_i \right) \in \sigma(\mathcal{D}, B).$$

Also, $\left(\bigcup_{\{i:x_i \in B\}} C_i \right) - \left(B \cap \bigcup_{\{i:x_i \in B\}} C_i \right)$ is a set which has empty intersection with $\{x_1, x_2, \dots\}$ and hence belongs to \mathcal{D} . Now

$$\bigcup_{\{i:x_i \in B\}} C_i = \left(\bigcup_{\{i:x_i \in B\}} C_i \right) - \left(B \cap \bigcup_{\{i:x_i \in B\}} C_i \right) \cup \left(B \cap \bigcup_{\{i:x_i \in B\}} C_i \right)$$

and this set belongs to $\sigma(\mathcal{D}, B)$ and also to \mathcal{C} . Further, $\bigcup_{\{i:x_i \in B\}} C_i$ equals neither \emptyset nor X because $B \notin \mathcal{D}$. Thus $\mathcal{C} \cap \sigma(\mathcal{D}, B) \neq \{\emptyset, X\}$. This being true for every $B \notin \mathcal{D}$, we find that \mathcal{D} is a maximal conjugate of \mathcal{C} .

(2) Let us show that every maximal conjugate of \mathcal{C} is a complement.

Let \mathcal{D} be a maximal conjugate of \mathcal{C} . To show that \mathcal{D} is a complement of \mathcal{C} it is sufficient to show that every $A \subset C_1$, $A \in \mathcal{B}$, also belongs to $\mathcal{C} \vee \mathcal{D}$. So, let $A \in \mathcal{B}$ be such that $A \subset C_1$.

From the hypothesis that \mathcal{D} is a maximal conjugate of \mathcal{C} let us first show that for every $B \subset C_1$, $B \in \mathcal{B}$, with $B \notin \mathcal{D}$ there is a $D \in \mathcal{D}$ such that

(*) $B \cup D \in \mathcal{C}$, $B \cup D \neq \emptyset$, $B \cup D \neq X$ and $C_1 \subset B \cup D$ with the further property that if $E \in \mathcal{D}$ is such that $B \cup E \in \mathcal{C}$ and $C_1 \subset B \cup E$ then $B \cup D \subset B \cup E$ (i.e., $B \cup D$ is smallest such).

In fact, since \mathcal{D} is a maximal conjugate of \mathcal{C} and $B \notin \mathcal{D}$, we see that $\mathcal{C} \cap \sigma(\mathcal{D}, B)$ is a non-trivial ($\neq \{\emptyset, X\}$) substructure of \mathcal{C} and (hence) is also atomic. Hence there is an atom C of $\mathcal{C} \cap \sigma(\mathcal{D}, B)$ such that $C_1 \subset C$. Since $C \in \sigma(\mathcal{D}, B)$, and since $\sigma(\mathcal{D}, B) =$ the field generated by $\{\mathcal{D}, B\}$, there exist $D_1, D_2 \in \mathcal{D}$ such that $C = (D_1 \cap B) \cup (D_2 \cap B^c)$. Since $B \subset C_1 \subset C$, $D_1 \cap B$ must equal B . Thus $C = B \cup (D_2 \cap B^c) = B \cup D_2$. Now, if E is another set in \mathcal{D} such that $B \cup E \in \mathcal{C}$ and $C_1 \subset B \cup E$ then $B \cup E \in \mathcal{C} \cap \sigma(\mathcal{D}, B)$ and E should contain the atom (containing C_1) $B \cup D_2$ of $\mathcal{C} \cap \sigma(\mathcal{D}, B)$.

Now, let $A \in \mathcal{B}$ be such that $A \subset C_1$. Let us suppose that $A \notin \mathcal{C} \vee \mathcal{D}$. We shall arrive at a contradiction to the hypothesis that \mathcal{C} is given by a countable partition.

Since $A \notin \mathcal{D}$ there exists a $D \in \mathcal{D}$ such that $A \cup D \in \mathcal{C}$ and having all the other properties in (*). Let $A_1 = A \cap D$. Then $A = (A - A_1) \cup A_1$ and $A - (A \cap D) = (A \cup D) - D \in \mathcal{C} \vee \mathcal{D}$. If $A_1 \in \mathcal{C} \vee \mathcal{D}$ then A would also belong to $\mathcal{C} \vee \mathcal{D}$. Hence $A_1 \notin \mathcal{C} \vee \mathcal{D}$.

Again, there would exist a $D_1 \in \mathcal{D}$ such that $A_1 \cup D_1 \in \mathcal{C}$ and has all the properties as in (*). Since $A_1 \subset A \subset C_1 \subset A_1 \cup D_1$, we have $A \cup D_1 = A_1 \cup D_1 \in \mathcal{C}$. By the minimal property of A and D as in (*), we have $A \cup D \subset A \cup D_1 = A_1 \cup D_1$. If $A \cup D = A_1 \cup D_1$ then, since $A_1 \subset D$, it would follow that $A_1 \cup D_1 \subset D \cup D_1 \subset A_1 \cup D_1$. Thus $A \cup D = A_1 \cup D_1 = D \cup D_1 \in \mathcal{C} \cap \mathcal{D}$, which is not possible since $A \cup D \neq \emptyset$ and $A \cup D \neq X$. Thus $A \cup D \subsetneq A_1 \cup D_1$ and $A \cup D \neq A_1 \cup D_1$.

Let $A_1 \cap D_1 = A_2$. If $A_2 \in \mathcal{C} \vee \mathcal{D}$, we have $A = (A - A_1) \cup (A_1 - A_2) \cup A_2 \in \mathcal{C} \vee \mathcal{D}$. Thus $A_2 \notin \mathcal{C} \vee \mathcal{D}$. We again proceed as before and manufacture a D_3 and an A_3 . Then we will have $A \supset A_1 \supset A_2 \supset A_3$ and $D, D_1, D_2, D_3 \in \mathcal{D}$ with the property that $A \cup D \subset A_1 \cup D_1 \subset A_2 \cup D_2 \subset A_3 \cup D_3$, all these sets are distinct and belong to \mathcal{C} . Also, A, A_1, A_2 and $A_3 \notin \mathcal{C} \vee \mathcal{D}$.

Proceeding in this way, for the first countable ordinal ω , we set $A_\omega = \bigcap_{n \geq 1} A_n$. Then $A = (A - A_1) \cup (A_1 - A_2) \cup \dots \cup A_\omega$ and each of $A - A_1, A_1 - A_2, \dots$ belongs to $\mathcal{C} \vee \mathcal{D}$. Hence if $A_\omega \in \mathcal{C} \vee \mathcal{D}$, we will have $A \in \mathcal{C} \vee \mathcal{D}$. So $A_\omega \notin \mathcal{C} \vee \mathcal{D}$.

Again, we can get $D_\omega \in \mathcal{D}$ such that $A_\omega \cup D_\omega \in \mathcal{C}$ and having all the properties of (*). As in the previous case, we can show that $A \cup D \subset A_1 \cup D_1 \subset \dots \subset A_\omega \cup D_\omega$. Then we write $A_{\omega+1} = A_\omega \cap D_\omega$.

Proceeding in this way we get for every countable ordinal $\alpha < \omega_1$ an $A_\alpha \notin \mathcal{C} \vee \mathcal{D}$ and a $D_\alpha \in \mathcal{D}$ such that A_α 's are decreasing, $A_\alpha \cup D_\alpha$'s are strictly increasing and $A_\alpha \cup D_\alpha \in \mathcal{C}$. But \mathcal{C} being given by a countable partition there cannot exist strictly increasing chains of length ω_1 . So, our fundamental assumption that $A \notin \mathcal{C} \vee \mathcal{D}$ is not valid.

Thus $A \in \mathcal{C} \vee \mathcal{D}$.

(3) Let \mathcal{D} be a maximal conjugate of \mathcal{C} . We have already seen above that for every $B \in \mathcal{B}$ with $B \subset C_1$, either $B \in \mathcal{D}$ or there is a $D \in \mathcal{D}$ such that $B \cup D \in \mathcal{C}$, $B \cup D \neq \emptyset$ and $B \cup D \neq X$.

For a $B \in \mathcal{B}$ with $B \subset C_1$, if $B \in \mathcal{D}$ and if also there is a $D \in \mathcal{D}$ such that $B \cup D \in \mathcal{C}$, $B \cup D \neq \emptyset$ and $B \cup D \neq X$, then $B \cup D \in \mathcal{C} \cap \mathcal{D}$ and $\mathcal{C} \cap \mathcal{D} \neq \{\emptyset, X\}$. Hence for a $B \in \mathcal{B}$ with $B \subset C_1$, exactly one of “ $B \in \mathcal{D}$ ” and “there is a $D \in \mathcal{D}$ such that $B \cup D \in \mathcal{C}$, $B \cup D \neq \emptyset$ and $B \cup D \neq X$ ” holds. Taking the cue from here let us define $\mathcal{I} = \{B \subset C_1 : B \in \mathcal{B} \text{ and } B \in \mathcal{D}\}$.

Let us see that \mathcal{I} is a σ -ideal in the Borel structure $(C_1, C_1 \cap \mathcal{B})$ and that \mathcal{I} is a maximal σ -ideal.

Definitely, \mathcal{I} is closed under countable unions. If $B \in \mathcal{I}$, $B_1 \in \mathcal{B}$, $B_1 \subset B$ and $B_1 \notin \mathcal{I}$ then there would exist a $D_1 \in \mathcal{D}$ such that $B_1 \cup D_1 \in \mathcal{C}$, $B_1 \cup D_1 \neq \emptyset$ and $B_1 \cup D_1 \neq X$. Then since $B_1 \subset B \subset C_1$ it would follow that $B \cup D = B_1 \cup D_1$ and this would mean that $\mathcal{C} \cap \mathcal{D} \neq \{\emptyset, X\}$. Hence $B_1 \in \mathcal{I}$. Thus \mathcal{I} is a σ -ideal in $(C_1, C_1 \cap \mathcal{B})$.

To show that \mathcal{I} is a maximal σ -ideal, it is sufficient to show that, for every $B \in \mathcal{B}$, $B \subset C_1$, either $B \in \mathcal{I}$ or $C_1 - B \in \mathcal{I}$. Suppose that $B \notin \mathcal{I}$ and also that $C_1 - B \notin \mathcal{I}$. Then there would exist $D \in \mathcal{D}$ and $D_1 \in \mathcal{D}$ such that $B \cup D \in \mathcal{C}$, $(C_1 - B) \cup D_1 \in \mathcal{C}$, $B \cup D \neq \emptyset$, $B \cup D \neq X$, $(C_1 - B) \cup D_1 \neq \emptyset$ and $(C_1 - B) \cup D_1 \neq X$. Then, since $C_1 \subset B \cup D$ and $C_1 \subset (C_1 - B) \cup D_1$, we have $B \subset D_1$ and $C_1 - B \subset D$. Hence $(B \cup D) \cap [(C_1 - B) \cup D_1] = D \cap D_1$, $C_1 \subset D \cap D_1$ and $D \cap D_1 \neq X$. Hence $D \cap D_1 \in \mathcal{C} \cap \mathcal{D}$. This implies that $\mathcal{C} \cap \mathcal{D} \neq \{\emptyset, X\}$. Hence either $B \in \mathcal{I}$ or $C_1 - B \in \mathcal{I}$. Thus \mathcal{I} is a maximal σ -ideal.

Implementing this procedure for C_1, C_2, \dots , we get points $x_1 \in C_1, x_2 \in C_2 \dots$ such that $C_1 - \{x_1\}, C_2 - \{x_2\}, \dots \in \mathcal{D}$. This would mean that $\bigcup_{i=1}^{\infty} (C_i - \{x_i\}) = X - \{x_1, x_2, \dots\} \in \mathcal{D}$. Thus $\{x_1, x_2, \dots\} \in \mathcal{D}$ and from the property of \mathcal{I} being a σ -ideal it would also follow that every $B \in \mathcal{B}$ with $B \cap \{x_1, x_2, \dots\} = \emptyset$ belongs to \mathcal{D} . If we define $\mathcal{D}^* = \{B \in \mathcal{B} : B \supset \{x_1, x_2, \dots\} \text{ or } B \cap \{x_1, x_2, \dots\} = \emptyset\}$ then as in (1) above

we deduce that \mathcal{D}^* is a maximal conjugate and since $\mathcal{D}^* \subset \mathcal{D}$ we have $\mathcal{D}^* = \mathcal{D}$. But $\mathcal{D} \cap [X - \{x_1, x_2 \dots\}] = \mathcal{B} \cap [X - \{x_1, x_2 \dots\}]$ and $\{x_1, x_2 \dots\}$ is an atom of \mathcal{D} . Thus \mathcal{D} is countably generated.

(4) This is essentially included in the demonstrations of (1), (2) and (3). ■

Shortt's original proof (see [43]) of the above proposition used hypergraphs.

For analytic spaces, every c.g. substructure has a maximal conjugate as the following proposition shows.

PROPOSITION 47. *Let (X, \mathcal{B}) be analytic. Then every c.g. substructure \mathcal{C} of \mathcal{B} has a maximal conjugate in \mathcal{B} .*

DEMONSTRATION. Our proof will be divided into three cases depending on the behaviour of \mathcal{C} .

CASE 1: There is an uncountable atom C_0 of \mathcal{C} . Let φ be any choice function for the atoms of \mathcal{C} : for each atom C of \mathcal{C} we have $\varphi(C) \in C$. Let ψ be a one-one correspondence between a set $P \subset C_0$ and the collection of all atoms of \mathcal{C} other than C_0 . (If C_0 is the only atom of \mathcal{C} the problem is trivial: $C_0 = X$, and $\mathcal{D} = \mathcal{B}$ is a maximal conjugate.) Define $f : P \rightarrow X$ by $f(p) = \varphi(\psi(p))$ and put

$$\mathcal{D} = \{B \in \mathcal{B} : \{p, f(p)\} \subset B \text{ or } \{p, f(p)\} \subset X - B, \text{ for all } p \in P\}.$$

Any $D \in \mathcal{D}$ containing C_0 must meet all atoms of \mathcal{C} . Thus $\mathcal{D} \cap \mathcal{C}$ is trivial. Now suppose that $B \in \mathcal{B} - \mathcal{D}$. Taking a complement if necessary, we assume that there is some $p \in P$ such that $p \notin B$ and $f(p) \in B$. Now $D_1 = \{p, f(p)\}$ and $D_2 = \psi(p) - \{f(p)\}$ are sets in \mathcal{D} . We see that $(B \cap D_1) \cup D_2 = \psi(p)$ is a set in \mathcal{C} . This shows that \mathcal{D} is a maximal conjugate.

CASE 2: Every atom of \mathcal{C} is countable and there is a \mathcal{B} -measurable selector for the \mathcal{C} -atoms: there is some $G \in \mathcal{B}$ intersecting each atom of \mathcal{C} in precisely one point. Then put $\mathcal{D} = \{B \in \mathcal{B} : G \subseteq B \text{ or } G \subseteq B^c\}$. Any set in \mathcal{D} intersecting G intersects every atom of \mathcal{C} . So \mathcal{D} is a conjugate of \mathcal{C} . Suppose now that $B \in \mathcal{B} - \mathcal{D}$. Let $p : X \rightarrow \mathbb{R}$ be a measurable function generating \mathcal{C} , i.e., $\mathcal{C} = p^{-1}(\mathcal{B}(\mathbb{R}))$. Then $p(G \cap B)$ and $p(G \cap B^c)$ are disjoint, non-empty analytic subsets of \mathbb{R} . By the first separation principle of Lusin, there is some linear Borel set A with $p(G \cap B) \subset A$ and $p(G \cap B^c) \subset \mathbb{R} - A$. Then $D = p^{-1}(A) - G$ and G are both sets in \mathcal{D} . We see that $D \cup (G \cap B) = p^{-1}(A)$ is a non-trivial set in \mathcal{C} . This proves that \mathcal{D} is maximal.

CASE 3: Every atom of \mathcal{C} is countable, and there is no full selector as in Case 2. Let $p : X \rightarrow \mathbb{R}$ generate \mathcal{C} as above. A selection theorem (see p. 11 of BS) says that in this case X may be partitioned into sets A_1, A_2, \dots in \mathcal{B} in such a way that p is one-one on each A_n but not on any union $A_n \cup A_m$ for $n \neq m$. Since there is no full selector for the atoms of \mathcal{C} it must be that for some pair n, m the intersection $p(A_n) \cap p(A_m)$ is uncountable. Let $B_0 \subseteq p(A_n) \cap p(A_m)$ be an uncountable Borel set. Noting that p , restricted to either A_n or A_m , is a Borel isomorphism, we see that $B_1 = p^{-1}(B_0) \cap A_n$ and $B_2 = p^{-1}(B_0) \cap A_m$ are uncountable standard Borel sets.

Using once more the fact that there is no full selector for the atoms of \mathcal{C} we see that $B_0 = p(B_1)$ is a proper subset of $p(X)$. Now $p(X) - B_0$ is analytic, and so there is a

Borel isomorphism $\psi : K \rightarrow p(X) - B_0$ taking some analytic set $K \subseteq B_2$ onto $p(X) - B_0$. Let φ be a choice function for the atoms of \mathcal{C} as in Case 1. Define $f : K \rightarrow X$ by $f(a) = \varphi(p^{-1}(\psi(a)))$. Let \mathcal{D} be the Borel structure

$$\mathcal{D} = \{B \in \mathcal{B} : B_1 \subseteq B \text{ or } B_1 \subseteq B^c\} \\ \cap \{B \in \mathcal{B} : \{a, f(a)\} \subseteq B \text{ or } \{a, f(a)\} \subseteq B^c \text{ for each } a \in K\}.$$

Every set in \mathcal{D} intersecting B_1 must contain it. Every set in \mathcal{C} containing B_1 must contain $p^{-1}(B_0) \supseteq B_2 \supseteq K$. Every set in \mathcal{D} containing K must contain $f(K)$. Every set in \mathcal{C} containing $f(K)$ contains $p^{-1}(\mathbb{R} - B_0) = X - p^{-1}(B_0)$. Thus $\mathcal{C} \cap \mathcal{D}$ is trivial.

Now suppose that $B \in \mathcal{B} - \mathcal{D}$. Suppose first that for some $a \in K$, we have $f(a) \in B$ and $a \notin B$. The sets $D_1 = \{a, f(a)\}$ and $D_2 = p^{-1}(\psi(a)) - \{f(a)\}$ belong to \mathcal{D} . We see that $(D_1 \cap B) \cup D_2 = p^{-1}(\psi(a))$ is a non-trivial set in \mathcal{C} . Next, suppose that both $B_1 \cap B$ and $B_1 - B$ are non-void. The set $p(B_1 \cap B)$ is standard Borel, so that $p^{-1}(p(B_1 \cap B))$ is a set in \mathcal{C} . Put $K_0 = p^{-1}(p(B_1 \cap B)) \cap K$ and note that $\psi(K_0)$ is relatively Borel in $p(X) - B_0$ and hence also in $p(X)$. (It is possible that $K_0 = \emptyset$.) So $p^{-1}(\psi(K_0))$ is a set in \mathcal{C} . Also, $D = [p^{-1}(p(B_1 \cap B)) - B_1] \cup p^{-1}(\psi(K_0))$ and B_1 are sets in \mathcal{D} . We see that $C = D \cup (B_1 \cap B) = p^{-1}(p(B_1 \cap B)) \cup p^{-1}(\psi(K_0))$ is a set in \mathcal{C} . Since $B_1 \cap B \subseteq C$ and $B_1 \cap B^c \subseteq C^c$, the set C is non-trivial. We have proved that \mathcal{D} is a maximal conjugate. ■

Are there Borel spaces, other than analytic spaces, with the above property? More precisely,

QUESTION. For which separable Borel spaces (X, \mathcal{B}) does every c.g. $\mathcal{C} \subseteq \mathcal{B}$ have a maximal conjugate? (Problem PP12, Appendix I.)

But there are separable spaces (X, \mathcal{B}) such that not every substructure of \mathcal{B} has a maximal conjugate. We know this only under CH.

EXAMPLE 4. Assuming CH, there is a separable space (X, \mathcal{B}) and a c.g. $\mathcal{C} \subseteq \mathcal{B}$ such that \mathcal{C} has no maximal conjugate in \mathcal{B} .

DEMONSTRATION. A standard transfinite induction argument (using CH) establishes the existence of a subset X of the square $S = [0, 1] \times [0, 1]$ with the properties:

- (1) X is uncountable;
- (2) every horizontal and vertical section of X is either a singleton or void;
- (3) $X \cap N$ is countable whenever N is a Borel subset of S of planar Lebesgue measure zero.

Let \mathcal{B} be the Borel structure on X inherited from S and let $p : X \rightarrow [0, 1]$ be the projection onto the first factor. Define $\mathcal{C} \subseteq \mathcal{B}$ to be the spectral Borel structure for p , i.e., $\mathcal{C} = p^{-1}(\mathcal{B}[0, 1])$, noting that \mathcal{C} contains all countable subsets of X . Note also that, by condition (3), any family of pairwise disjoint uncountable sets in \mathcal{B} is countable. Let us now see that \mathcal{C} is a proper substructure of \mathcal{D} . In fact, we prove

CLAIM. Let D be an uncountable set in \mathcal{B} . Then $\mathcal{C} \cap D$ is a proper substructure of $\mathcal{B} \cap D$.

DEMONSTRATION OF CLAIM. Suppose instead that $\mathcal{C} \cap D = \mathcal{B} \cap D$ for some uncountable $D \in \mathcal{D}$.

Then the restriction p_0 of p to D is a Borel isomorphism. Let $r : p(D) \rightarrow D$ be the inverse of p_0 and let $g : X \rightarrow [0, 1]$ be the projection to the second factor of S . Then $f = g \circ r$ is a measurable mapping of $p(D)$ onto $g(D)$ extending to a Borel function $\bar{f} : [0, 1] \rightarrow [0, 1]$. But then $G = \text{graph}(\bar{f})$ has measure zero and yet $G \cap X$ contains the uncountable set D , contradicting condition (3).

Suppose now that \mathcal{D} is a maximal conjugate of \mathcal{C} . Definitely, every set in \mathcal{D} is uncountable. Now, \mathcal{D} cannot be atomless, because, if it is, there would exist uncountably many pairwise disjoint non-empty sets in \mathcal{D} [BS; p. 29], contradicting our earlier observation. So let D be an atom of \mathcal{D} . Necessarily, D is uncountable, and the claim implies that $\mathcal{C} \cap D$ is a proper substructure of $\mathcal{B} \cap D$. Select $B \in (\mathcal{B} \cap D) - (\mathcal{C} \cap D)$ and form $\mathcal{D}_0 = \sigma(\mathcal{D}, B)$, a strict enlargement of \mathcal{D} . The supposed maximality of \mathcal{D} implies that there are sets D_1 and D_2 in \mathcal{D} such that $(D_1 \cap B) \cup (D_2 \cap B^c) (= C, \text{ say})$ is a non-trivial set in \mathcal{C} . Without loss of generality we may assume that $D \subseteq D_1$. Then $C = B \cup (D_2 \cap B^c)$. If now $D \subseteq D_2$, then $C = D_2$ is a non-trivial set in $\mathcal{C} \cap \mathcal{D}$, a contradiction. If, on the other hand, $D \subseteq X - D_2$, then $C \cap D = B \in \mathcal{C} \cap D$, another contradiction. Thus \mathcal{C} has no maximal conjugate in \mathcal{B} . ■

In the next two propositions we shall state some more results about maximal conjugates of the Borel substructures of analytic spaces.

PROPOSITION 48. *Let (X, \mathcal{B}) be an analytic space and let \mathcal{C} and \mathcal{D} be c.g. substructures of \mathcal{B} . If \mathcal{D} is a maximal conjugate of \mathcal{C} , then \mathcal{D} is a (maximal) complement of \mathcal{C} .*

DEMONSTRATION. Let \mathcal{D} be a maximal conjugate of \mathcal{C} . Since \mathcal{C} and \mathcal{D} are both countably generated, if we show that $\mathcal{C} \cup \mathcal{D}$ separates points of X , by the Blackwell nature of every analytic space, it will follow that \mathcal{D} is a complement of \mathcal{C} . Since \mathcal{D} is a maximal conjugate of \mathcal{C} , it will follow that \mathcal{D} is a maximal complement of \mathcal{C} .

Let us show that $\mathcal{C} \cup \mathcal{D}$ separates points of X . Suppose not; let $p, q \in X$ be such that p and q are not separated by $\mathcal{C} \cup \mathcal{D}$. Let $\mathcal{D}_0 = \sigma(\mathcal{D}, \{p\})$. Since \mathcal{D} is a maximal conjugate of \mathcal{C} , and \mathcal{D}_0 properly contains \mathcal{D} , $\mathcal{C} \cap \mathcal{D}_0$ has a non-trivial ($\neq \emptyset, \neq X$) set, say C . Hence there are sets $D_1, D_2 \in \mathcal{D}$ such that $(D_1 \cap \{p\}) \cup (D_2 \setminus \{p\}) = C$. Without loss of generality assume that $p \in C$. Hence $\{p\} \cup (D_2 \setminus \{p\}) = C$. Since $C \in \mathcal{C}$, also q must belong to C . Hence $q \in D_2$. So $p \in D_2$. Thus $D_2 = C$, a contradiction. Therefore $\mathcal{C} \cup \mathcal{D}$ separates points of X . ■

The following is an interesting result.

PROPOSITION 49. *Let (X, \mathcal{B}) be an analytic space and let \mathcal{C} be a c.g. substructure of \mathcal{B} . Then \mathcal{C} has a c.g. maximal conjugate if and only if there is a measurable full selector for the atoms of \mathcal{C} , i.e. there is some $S \in \mathcal{B}$ such that $\text{card}(S \cap C) = 1$ for each atom $C \in \mathcal{C}$. If \mathcal{D} is such a c.g. maximal conjugate, then \mathcal{D} also has a measurable full selector.* ■

DEMONSTRATION. See [43]. ■

QUESTIONS. Let (X, \mathcal{B}) be an analytic space. If a substructure \mathcal{C} has a maximal conjugate, is \mathcal{C} necessarily c.g.? If \mathcal{C} is c.g., is every maximal conjugate of \mathcal{C} c.g.? (Problem PP13, Appendix I.)

In general, not every maximal conjugate is a complement. Here is an example.

EXAMPLE 5. Let (X, \mathcal{B}) be an uncountable standard space. There are substructures \mathcal{C} and \mathcal{D} of \mathcal{B} such that

- (1) \mathcal{C} is c.g.;
- (2) \mathcal{D} is a maximal conjugate for \mathcal{C} ;
- (3) \mathcal{D} is not a complement for \mathcal{C} .

DEMONSTRATION. We realise X as the union of two line segments in the plane: $X = L_1 \cup L_2$, where

$$L_1 = \{(x, 0) : 0 < x < 1\}, \quad L_2 = \{(0, y) : 0 < y < 1\}.$$

Let \mathcal{C} be the structure generated by the projection $p : X \rightarrow \mathbb{R}$ onto the first factor. Partition $L_1 = R_1 \cup R_2$ into two non-Borel sets of power c . Partition $L_2 = P_1 \cup P_2$ into two uncountable Borel sets. Let $f : L_2 \rightarrow L_1$ be a one-one correspondence such that $f(P_1) = R_1$ and $f(P_2) = R_2$. Define

$$\mathcal{D} = \{B \in \mathcal{B} : \{a, f(a)\} \subseteq B \text{ or } \{a_1 f(a)\} \subseteq B^c \text{ for each } a \in L_2\}.$$

Then the only set in \mathcal{D} containing the \mathcal{C} -atom L_2 is X , so that \mathcal{D} is a conjugate of \mathcal{C} . Also, suppose that for some $B \in \mathcal{B} - \mathcal{D}$ we have $a \notin B$ and $f(a) \in B$ for some $a \in L_2$. Put $D = \{a, f(a)\}$ and note that $D \in \mathcal{D}$ and $D \cap B = \{f(a)\} \in \mathcal{C}$. Thus, \mathcal{D} is a maximal conjugate for \mathcal{C} .

We now show that \mathcal{D} is not a complement by proving that $P_1 \notin \sigma(\mathcal{C}, \mathcal{D})$. Suppose contrariwise that $P_1 \in \sigma(\mathcal{C}, \mathcal{D})$. Then $P_1 \in \sigma(\mathcal{C}, \mathcal{D}) \cap L_2 = \mathcal{D} \cap L_2$, since L_2 is an atom of \mathcal{C} . So, for some $D \in \mathcal{D}$, we have $P_1 = D \cap L_2$. Since D contains P_1 , it follows that D also contains $f(P_1) = R_1$. Since R_1 is not Borel, D must intersect R_2 and therefore also P_2 , a contradiction. ■

Several questions are of interest.

QUESTION. Let (X, \mathcal{B}) be a separable space and let \mathcal{C} be a proper substructure of \mathcal{B} . If \mathcal{C} is separable, can \mathcal{C} have a maximal conjugate in \mathcal{B} ? (Problem PP14, Appendix I.)

QUESTION. Find complements, minimal complements and maximal conjugates for the tail, invariant and symmetric Borel structures on $\{0, 1\}^\omega$. (Problem PP15, Appendix I.)

QUESTION. Let X be the union of the lines $y = 0$ and $y = 1$ in the plane. Let \mathcal{C} be the substructure of \mathcal{B} generated by the projection onto the x -axis. Is every maximal conjugate for \mathcal{C} c.g.? (Problem PP16, Appendix I.)

2.3. Minimal complements. In Proposition 53 of BS minimal complements in the lattice $L_{\mathcal{B}}$ were studied. A complement \mathcal{C}' of \mathcal{C} in $L_{\mathcal{B}}$ is said to be a *minimal complement* of \mathcal{C} if no proper substructure of \mathcal{C}' is a complement of \mathcal{C} in $L_{\mathcal{B}}$. We shall now obtain some more results on minimal complements.

PROPOSITION 50. Let (X, \mathcal{B}) be a separable space and suppose that C_1, \dots, C_n is a partition of X into n (non-empty) members of \mathcal{B} . Put $\mathcal{C} = \sigma(C_1, \dots, C_n)$. Then a substructure \mathcal{D} of \mathcal{B} is a minimal complement of \mathcal{C} (in \mathcal{B}) if and only if

- (1) $\mathcal{D} \cap C_i = \mathcal{B} \cap C_i$ for $i = 1, \dots, n$, and
 (2) the union of no two pseudo-atoms of \mathcal{D} is a partial selector for the partition C_1, \dots, C_n .

If the partition is an infinite partition C_1, C_2, \dots , then the “only if” direction of the analogous statement is also true.

DEMONSTRATION. If \mathcal{D} is any complement of \mathcal{C} , then $\mathcal{D} \cap C_i = \mathcal{B} \cap C_i$ for each i . This implies that no pseudo-atom (pseudo-atoms of a Borel structure \mathcal{D} are defined to be the equivalence classes under the equivalence relation: x is equivalent to y if x and y are not separated by \mathcal{D}) of \mathcal{D} contains more than n points. If \mathcal{D} is a minimal complement of \mathcal{C} , since \mathcal{B} is countably generated we see that \mathcal{D} is countably generated, and so each such pseudo-atom is an atom (it belongs to \mathcal{D}). Now suppose, for the sake of deriving a contradiction, that A_1 and A_2 are atoms of \mathcal{D} such that $A = A_1 \cup A_2$ is a partial selector for the partition C_1, \dots, C_n . Let \mathcal{D}' be the quotient Borel structure of \mathcal{D} wherein the atoms A_1 and A_2 are identified. Put $C = \bigcup\{C_i : C_i \cap A_1 \neq \emptyset\}$. Then $C \cap A = A$ shows that $\sigma(C, \mathcal{D}') = \sigma(C, \mathcal{D}) = \mathcal{B}$. Thus \mathcal{D}' is a complement of C , contradicting minimality of \mathcal{D} . Condition (2) follows.

Now suppose that conditions (1) and (2) hold for some Borel structure $\mathcal{D} \subseteq \mathcal{B}$. Given $B \in \mathcal{B}$, write

$$B = \bigcup_i (B \cap C_i) = \bigcup_i (D_i \cap C_i)$$

for some $D_i \in \mathcal{D}$ (condition (1)). Thus $\sigma(\mathcal{C}, \mathcal{D}) = \mathcal{B}$. If $B \in \mathcal{C} \cap \mathcal{D}$, then let A_1 and A_2 be the pseudo-atoms of \mathcal{D} with $A_1 \subseteq B$ and $A_2 \subseteq B^c$. Then $A_1 \cup A_2$ is a partial selector for the partition C_1, \dots, C_n , a contradiction. It must be that $B = \emptyset$ or $B = X$. So \mathcal{D} is a complement of \mathcal{C} .

Suppose that $\mathcal{D}' \subseteq \mathcal{D}$ is also a complement for \mathcal{C} . We see that \mathcal{D}' and \mathcal{D} have the same pseudo-atoms: if A is a pseudo-atom of \mathcal{D} and not a pseudo-atom of \mathcal{D}' , it follows that then A contains pseudo-atoms A_1 and A_2 of \mathcal{D} ; since $\sigma(\mathcal{C}, \mathcal{D}') = \mathcal{B}$, it follows that A is a partial selector and so too is $A_1 \cup A_2$, contradicting condition (2). If A is a pseudo-atom of \mathcal{D} , define $S(A) = \{i : C_i \cap A \neq \emptyset\}$ and let \mathcal{S} be the (finite) collection of all such $S(A)$ as A ranges over all pseudo-atoms of \mathcal{D} . Notice that $S(A) \cap S(A') \neq \emptyset$ for any two pseudo-atoms A and A' : this restates condition (2).

Given $D \in \mathcal{D}$, we must show that $D \in \mathcal{D}'$ to establish minimality. Since \mathcal{D}' is a complement of \mathcal{C} , we have $\mathcal{D}' \cap C_i = \mathcal{B} \cap C_i = \mathcal{D} \cap C_i$ for each i . So write

$$D = \bigcup_i (D \cap C_i) = \bigcup_i (D'_i \cap C_i)$$

for certain $D'_i \in \mathcal{D}'$. That $D \in \mathcal{D}'$ will follow from

$$\text{CLAIM. } D = \bigcup_{S \in \mathcal{S}} \bigcap_{i \in S} D'_i.$$

DEMONSTRATION OF CLAIM. Note that if $x \in D$ and A is the pseudo-atom of \mathcal{D} (or \mathcal{D}') containing x , then $A \cap C_i \neq \emptyset$ for $i \in S(A)$. Thus A meets and is therefore contained in the sets D'_i for $i \in S(A)$. We have proved the inclusion \subseteq .

Now suppose that for some pseudo-atom A , we have $x \in \bigcap\{D'_i : i \in S(A)\}$. Let A' be the pseudo-atom containing x . Choose $j \in S(A) \cap S(A')$. Then A' meets and is contained

in D'_j , whilst $A' \cap C_j \subseteq D'_j \cap C_j = D \cap C_j$, so that A' meets and is contained in D . Thus $x \in D$. The claim follows.

For $\{C_1, C_2, \dots\}$ being a countably infinite collection, the argument for the “only if” direction requires no essential changes. ■

A sufficient condition for \mathcal{D} to be a minimal complement can be salvaged from the above proof as in the following proposition. We shall not prove this.

PROPOSITION 51. *Let (X, \mathcal{B}) be a separable space and suppose that C_1, C_2, \dots is a countable partition of X into sets in \mathcal{B} . Let \mathcal{C} be the Borel structure generated by this partition. A substructure \mathcal{D} of \mathcal{B} is a minimal complement of \mathcal{C} if*

- (1) $\mathcal{D} \cap C_i = \mathcal{B} \cap C_i$ for $i = 1, 2, \dots$, and
- (2) there is some n such that the union of no two pseudo-atoms of \mathcal{D} is a partial selector for $\{C_1, \dots, C_n\}$, i.e. the union of any two pseudo-atoms of \mathcal{D} meets one of the sets C_1, \dots, C_n in at least two points. ■

The necessary condition of Proposition 50 for infinite partitions cannot be made sufficient as the following example shows.

EXAMPLE 6. *Consider the power set of \mathbb{N} in its representation as a denumerable product $S = \{0, 1\} \times \{0, 1\} \times \dots$ with the standard Borel structure \mathcal{A} . Let U be a free ultra-filter over \mathbb{N} , and consider U as a subset of S in the usual way. Put $\mathcal{U} = \sigma(\mathcal{A}, U)$. For $i \in \mathbb{N}$, define $S_i \subseteq S$ by*

$$S_i = \{x : x(i) = 1 \text{ and } x \in U\} \cup \{x : x(i) = 0 \text{ and } x \notin U\}.$$

Also define $C_i = S_i \times \{i\}$ as a subset of $S \times \mathbb{N}$.

Let X be the union $X = C_1 \cup C_2 \cup \dots$ and let $f = X \rightarrow S$ be the projection onto the first factor of $S \times \mathbb{N}$. Note that f is surjective. Define the Borel structures

$$\mathcal{C} = \sigma(C_1, C_2, \dots), \quad \mathcal{D} = f^{-1}(\mathcal{U}), \quad \mathcal{D}' = f^{-1}(\mathcal{A}), \quad \mathcal{B} = \sigma(\mathcal{C}, \mathcal{D}).$$

Then

- (1) $\mathcal{D}' \cap C_i = \mathcal{D} \cap C_i = \mathcal{B} \cap C_i$ for $i = 1, 2, \dots$;
- (2) the union of no two atoms of \mathcal{D} is a partial selector for the partition C_1, C_2, \dots ;
- (3) \mathcal{D}' is a proper substructure of \mathcal{D} ;
- (4) both \mathcal{D} and \mathcal{D}' are complements of \mathcal{C} in \mathcal{B} , so that \mathcal{D} is not a minimal complement.

DEMONSTRATION. To prove (1), note that the sets in $\mathcal{D}' \cap C_i$ are of the form $(A \cap S_i) \times \{i\}$ for $A \in \mathcal{A}$ whilst those in $\mathcal{B} \cap C_i$ are of the form $(B \cap S_i) \times \{i\}$ for $B \in \mathcal{U}$. So we need to show only that $U \cap S_i$ is a member of $\mathcal{A} \cap S_i$; but this follows from the equality $U \cap S_i = \{x \in S_i : x(i) = 1\}$.

To prove (2), note that the atoms of \mathcal{D} are the sets of the form

$$f^{-1}(x) = \begin{cases} \{(x, i) : x(i) = 1\} & \text{if } x \in U, \\ \{(x, i) : x(i) = 0\} & \text{if } x \notin U, \end{cases}$$

as x ranges over S . To show that $f^{-1}(x) \cap f^{-1}(y)$ (for $x \neq y$) meets some C_i in two points, there are three cases to consider, according as neither, one, or both of x, y are in

U ; in each case, the result follows from properties of U such as closure under intersections or maximality as a filter.

With regard to (3) we assert that $f^{-1}(U) = (U \times \mathbb{N}) \cap X$ is a member of \mathcal{D} but not of \mathcal{D}' : if $f^{-1}(U) \in \mathcal{D}$, then $U = ff^{-1}(U) \in \mathcal{A}$. This last is an impossibility (as implied e.g. by the Kolmogorov 0-1 law).

Part (4) proceeds from (1) and (2) much as in the proof of Proposition 50. ■

If \mathcal{C} is given by a partition consisting of two sets, Proposition 50 can be greatly improved.

PROPOSITION 52. *Let (X, \mathcal{B}) be a separable space and suppose that $X = C_1 \cup C_2$ is a partition of X into sets C_1 and C_2 in \mathcal{B} . Then the Borel structure $\mathcal{C} = \sigma(C_1, C_2)$ has a minimal complement if and only if either*

- (1) C_1 is embedded in C_2 , or
- (2) C_2 is embedded in C_1 ,

DEMONSTRATION. Suppose that $\varphi : C_1 \rightarrow C_2$ is an embedding of C_1 into C_2 . Then define $f : X \rightarrow C_2$ by

$$f(x) = \begin{cases} x & \text{if } x \in C_2, \\ \varphi(x) & \text{if } x \in C_1. \end{cases}$$

(Compare this with the second part of the proof of Proposition 52 in BS.) Then $\mathcal{D} = f^{-1}(\mathcal{B} \cap C_2)$ is a minimal complement of \mathcal{C} in \mathcal{B} : this follows from Proposition 50, noting that the atoms of \mathcal{D} are of the form $\varphi^{-1}(x) \cup x$ as x ranges over C_2 .

Now suppose that \mathcal{D} is a minimal complement of \mathcal{C} . By Proposition 42 of BS, \mathcal{D} is c.g., and hence there is a function $f : X \rightarrow \mathbb{R}$ whose spectral Borel structure is \mathcal{D} . The atoms of \mathcal{D} are the non-empty sets of the form $f^{-1}(t)$ for $t \in \mathbb{R}$. Since \mathcal{D} is a complement of \mathcal{C} , these atoms are partial selectors for the partition C_1, C_2 , i.e. the cardinalities of $f^{-1}(t) \cap C_1$ and $f^{-1}(t) \cap C_2$ are 0 or 1. Since \mathcal{D} is minimal, Proposition 53 (condition (2)) of BS implies that for distinct atoms $f^{-1}(t_1)$ and $f^{-1}(t_2)$, at least one of the sets $(f^{-1}(t_1) \cup f^{-1}(t_2)) \cap C_1$ and $(f^{-1}(t_1) \cup f^{-1}(t_2)) \cap C_2$ has two elements.

From this we see that for one of the sets C_1 or C_2 , say C_2 , the intersection $f^{-1}(t) \cap C_2$ is a singleton set for each atom $f^{-1}(t)$. It follows that $f(C_2) = f(X) \supseteq f(C_1)$. Proposition 51 implies that $\mathcal{D} \cap C_2 = \mathcal{B} \cap C_2$, which is to say that f_0 , the restriction of f to C_2 , is an isomorphism of C_2 onto $f(C_2) = f(X)$. Define $\varphi : C_1 \rightarrow C_2$ by $\varphi(X) = f_0^{-1}(f(x))$. Because $\mathcal{D} \cap C_1 = \mathcal{B} \cap C_1$, we see that φ is an embedding of C_1 into C_2 . ■

With the above result at hand we shall now give an example of a Borel structure which has a complement but not a minimal complement, thus solving Problem P14 of BS.

EXAMPLE 7. *There is a separable space (X, \mathcal{B}) and a two-fold partition of X into sets C_1, C_2 in \mathcal{B} such that the Borel structure $\mathcal{C} = \sigma(C_1, C_2)$ has no minimal complement in \mathcal{B} .*

DEMONSTRATION. Let $C_1 \subset \mathbb{R}$ be a set such that both C_1 and $\mathbb{R} - C_1$ are strongly Blackwell and Borel dense in \mathbb{R} . Propositions 9 and 10 of BS, together, will witness such a C_1 . Put $X = \mathbb{R}$ and let \mathcal{B} be the Borel structure on X generated by the usual Borel structure on \mathbb{R} together with C_1 . Put $C_2 = X - C_1$.

From Proposition 52, our assertion will be proved once we establish that there is no embedding of C_1 into C_2 or of C_2 into C_1 . These cases are symmetrical, so we detail only the first. Suppose that φ is an embedding of C_1 into C_2 . Then φ extends to an isomorphism $\bar{\varphi}$ between Borel subsets of \mathbb{R} . But then $\text{graph}(\bar{\varphi})$ is a Borel set contained in $(\mathbb{R} \times \mathbb{R}) - (C_1 \times C_1)$ that cannot be covered by countably many sections of $\mathbb{R} \times \mathbb{R}$. This contradicts the choice of C_1 . ■

3. Further topics

In this chapter we shall deal with some general topics of interest.

3.1. Generators and minimal generators. A generator \mathcal{G} of a Borel structure \mathcal{B} is called a *minimal generator* (*m.g.* for short) if no proper subcollection of \mathcal{G} generates \mathcal{B} (see §6 of BS). In BS every c.g. Borel structure was shown to have an m.g. Problem P2 of BS asked if every Borel structure has a minimal generator.

In this section we shall present several examples (some in ZFC and some under additional set-theoretic assumptions) of Borel structures without minimal generators. All these are essentially due to Aniszczyk and Frankiewicz [4].

Let us first recall a few results which will be used later. Any \aleph_1 many subsets of a set are contained in a c.g. Borel structure (p. 18 of BS). Under MA any c many subsets of a set X with $|X| = c$ are contained in a c.g. Borel structure on X (Mauldin [31, p. 292]).

The following proposition gives a simple but extremely useful condition so that a Borel structure does not have an m.g.

PROPOSITION 53. *If \mathcal{B} is a Borel structure on a set X such that any \aleph_1 many sets in \mathcal{B} are contained in a c.g. substructure of \mathcal{B} and if \mathcal{B} is not c.g. then \mathcal{B} does not have an m.g.*

DEMONSTRATION. Let $\sigma(\mathcal{G}) = \mathcal{B}$. Suppose that \mathcal{B} is not c.g. Then $|\mathcal{G}| \geq \aleph_1$. Let $\mathcal{G}_0 \subset \mathcal{G}$ be such that $|\mathcal{G}_0| = \aleph_1$. By the hypothesis there is a countable family $\mathcal{H} \subset \mathcal{B}$ such that $\mathcal{G}_0 \subset \sigma(\mathcal{H})$. Since \mathcal{H} is countable there is another countable family $\mathcal{H}_1 \subset \mathcal{G}$ such that $\sigma(\mathcal{H}) \subset \sigma(\mathcal{H}_1)$. Hence

$$\mathcal{B} = \sigma(\mathcal{G}) = \sigma(\mathcal{G}_0 \cup (\mathcal{G} - \mathcal{G}_0)) \subset \sigma(\mathcal{H} \cup (\mathcal{G} - \mathcal{G}_0)) \subset \sigma(\mathcal{H}_1 \cup (\mathcal{G} - \mathcal{G}_0)) \subset \mathcal{B}.$$

Thus $\sigma(\mathcal{G}) = \sigma(\mathcal{H}_1 \cup (\mathcal{G} - \mathcal{G}_0))$. If we take a $G \in \mathcal{G}_0 - \mathcal{H}_1$, which is possible because $|\mathcal{G}_0| = \aleph_1$ and $|\mathcal{H}_1| = \aleph_0$, we have $G \in \sigma(\mathcal{G} - \{G\})$. Thus \mathcal{G} is not a minimal generator. Hence \mathcal{B} does not have a minimal generator. ■

This, together with the earlier paragraph, immediately gives:

PROPOSITION 54. *Either $\mathcal{P}(\aleph_1)$ is c.g. or it does not have an m.g. ■*

We shall now give a quick first example of a Borel structure without an m.g.

PROPOSITION 55 (ZFC). *The \aleph_1 , co- \aleph_1 Borel structure on a set X with $|X| \geq \aleph_2$ does not have a minimal generator.*

DEMONSTRATION. Let $\mathcal{C} = \{A \text{ and } X - A : A \subset X, |A| \leq \aleph_1\}$ be the \aleph_1 , co- \aleph_1 Borel structure.

\mathcal{C} is not c.g. because on any c.g. structure every 0-1 measure is concentrated at a point, whereas on \mathcal{C} there is a natural 0-1 measure which is not concentrated at any point. Thus, if $\sigma(\mathcal{G}) = \mathcal{C}$, then $|\mathcal{G}| \geq \aleph_1$. Let us see that \mathcal{C} satisfies the hypothesis of Proposition 53. Let $\mathcal{F} \subset \mathcal{C}$ be any \aleph_1 many sets. Assume without loss of generality that each $F \in \mathcal{F}$ satisfies $|F| \leq \aleph_1$. Let $Y = \bigcup\{F : F \in \mathcal{F}\}$. Then $|Y| \leq \aleph_1$. From the paragraph before Proposition 53 we deduce that there is a c.g. Borel structure \mathcal{D} on Y such that $\mathcal{F} \subset \mathcal{D}$. Let \mathcal{D}^* on X be defined by $\mathcal{D}^* = \{D \text{ and } X - D : D \in \mathcal{D}\}$. Then \mathcal{D}^* is a c.g. Borel substructure of \mathcal{C} such that $\mathcal{F} \subset \mathcal{D}^*$.

Thus \mathcal{C} satisfies the hypothesis of Proposition 53. Hence \mathcal{C} has no minimal generator. ■

A second example without an m.g. is given by the following proposition.

PROPOSITION 56 (ZFC). *The Borel structure \mathcal{B} of the ordered topological space $[0, \omega_1)$ has no m.g.*

DEMONSTRATION. \mathcal{B} is in fact the collection of all those subsets A of $[0, \omega_1)$ such that A or A^c contains a closed unbounded set or CUB for short (see [10] or BS).

Let us first see that if $\sigma(\mathcal{G}) = \mathcal{B}$ then $|\mathcal{G}| > \aleph_1$. If $|\mathcal{G}| \leq \aleph_1$, assume without loss of generality that each set in \mathcal{G} contains a CUB. Using a transfinite argument construct a CUB C such that $C - G$ is countable for all G in \mathcal{G} . This implies that $C \cap \sigma(\mathcal{G})$ is a substructure of the countable-cocountable structure on C . But inside the CUB C there is another CUB D such that $|C - D| = \aleph_1$. This D has to belong to $C \cap \sigma(\mathcal{G})$. This is a contradiction! So $|\mathcal{G}| > \aleph_1$.

Now, if $\sigma(\mathcal{G}) = \mathcal{B}$ find a $\mathcal{G}_1 \subset \mathcal{G}$ so that $|\mathcal{G}_1| = \aleph_1$ and all the singleton sets are in $\sigma(\mathcal{G}_1)$. Find a $\mathcal{G}_2 \subset \mathcal{G} \setminus \mathcal{G}_1$ such that $|\mathcal{G}_2| = \aleph_1$. Assuming without loss of generality that every set in \mathcal{G} contains a CUB, find a CUB C such that $C - G$ is countable for every G in \mathcal{G}_2 . The intersection $C^c \cap \mathcal{G}_2$ being a collection of \aleph_1 many subsets of C^c , since $C^c \cap \mathcal{B} = \mathcal{P}(C^c)$, from the paragraph before Proposition 53 we can find a family $\mathcal{H} \subset \mathcal{G}$ such that $|\mathcal{H}| = \aleph_0$, $C^c \cap \mathcal{G}_2 \subset \sigma(C^c \cap \mathcal{H})$ on C^c and $C \in \sigma(\mathcal{H})$. Now, for any $G \in \mathcal{G}_2$, we have $G = (G \Delta C) \Delta C \in \sigma(\mathcal{H} \cup \mathcal{G}_1)$.

Hence,

$$\sigma(\mathcal{G}) = \sigma((\mathcal{G} \setminus \mathcal{G}_2) \cup \mathcal{G}_2) = \sigma((\mathcal{G} \setminus \mathcal{G}_2) \cup \mathcal{H})$$

and thus \mathcal{G} is not an m.g. of \mathcal{B} . So \mathcal{B} does not have an m.g. ■

For more examples under some set-theoretic hypotheses, we present the following two propositions.

PROPOSITION 57 (MA). *The Lebesgue Borel structure \mathcal{L} does not have an m.g.*

DEMONSTRATION. Let \mathcal{B} be the usual Borel structure of the real line. Let $\sigma(\mathcal{G}) = \mathcal{L}$. Then $|\mathcal{G}| > c$. So, there are Borel sets B and C such that $B \subset C$, $\lambda(B) = \lambda(C)$ and $|\{G \in \mathcal{G} : B \subset G \subset C\}| > c$. The existence of such B and C can be seen as follows. Let ϕ be a map from \mathcal{G} defined by $\phi(G) = (B, C)$ where B and C are Borel sets such that $B \subset G \subset C$ and $\lambda(B) = \lambda(C)$. Then ϕ is a map from a set of cardinality $> c$ into a set of cardinality c . So the inverse image of at least one point is $> c$.

Now take a subfamily \mathcal{G}_1 of $\{G \in \mathcal{G} : B \subset G \subset C\}$ with $|\mathcal{G}_1| = c$. Since $\mathcal{L} \cap (C - B) = \mathcal{P}(C - B)$, and since $|C - B| = c$, under MA, we can find from the paragraph before Proposition 53 and as in the proof of Proposition 56, an $\mathcal{H} \subset \mathcal{G}$ such that $|\mathcal{H}| = \aleph_0$ and $\{\mathcal{G}_1 \cap (C - B), C, B\} \subset \sigma(\mathcal{H})$. Since $G = B \cup [G \cap (C - B)]$ for every G in \mathcal{G}_1 , we have $\mathcal{G}_1 \subset \sigma(\mathcal{H})$. Now,

$$\mathcal{B} = \sigma(\mathcal{G}) = \sigma(\mathcal{G}_1 \cup (\mathcal{G} - \mathcal{G}_1)) \subset \sigma(\mathcal{H} \cup (\mathcal{G} - \mathcal{G}_1)) \subset \mathcal{B}.$$

If we take a $G \in \mathcal{G}_1 - \mathcal{H}$ then $G \in \sigma(\mathcal{G} - \{G\})$. Hence \mathcal{G} is not an m.g. ■

PROPOSITION 58 (MA). $\mathcal{P}(\aleph)$ has an m.g. for all $\aleph < c$, and $\mathcal{P}(c)$ has no m.g.

DEMONSTRATION. Under MA for all $\aleph \leq c$, $\mathcal{P}(\aleph)$ being separable has an m.g. Under MA, any \aleph_1 sets in $\mathcal{P}(c)$ are contained in a c.g. structure, by the result of Mauldin quoted in the paragraph before Proposition 53, and $\mathcal{P}(c)$ is not c.g. Now by Proposition 53 it follows that $\mathcal{P}(c)$ has no m.g. ■

Since $\text{CH} \Rightarrow \text{MA}$, Propositions 57 and 58 hold under CH as well. In particular, under CH, $\mathcal{P}(\aleph_1)$ does not have an m.g. and under $\text{MA} + \neg\text{CH}$, $\mathcal{P}(\aleph_1)$ has an m.g. This shows that the statement “ $\mathcal{P}(\aleph_1)$ has an m.g.” is undecidable in ZFC. Also, the statement “ $\mathcal{P}(c)$ has an m.g.” cannot be shown in ZFC. But can we find in ZFC an $\aleph > \aleph_1$ so that $\mathcal{P}(\aleph)$ has no m.g. (Problem PP17, Appendix I)? Or, can we find in ZFC an $\aleph > \aleph_1$ so that $\mathcal{P}(\aleph)$ has an m.g. (Problem PP18, Appendix I)? The answers to these two problems would probably involve the construction of some kind of models for ZFC. The above proofs also suggest the conjecture: Any Borel structure generated by \aleph_1 many sets has an m.g. (Problem PP19, Appendix I).

Now we turn to another aspect of generators. In BS, an argument was given attempting to show that no minimal generator of the standard Borel structure \mathcal{B} on \mathbb{R} could be constructed using intervals solely. The underlying premise was that a family of intervals is a generator if and only if the set of corresponding interval end-points is dense in \mathbb{R} . This is not true as is witnessed by the family of all symmetric intervals. However, a family of intervals of the type (a, ∞) is a generator for \mathcal{B} if and only if the set of end-points is dense. Hence from this family no m.g. can be extracted. Thus a generator of \mathcal{B} need not contain a minimal generator.

This raises two problems: firstly, is there an m.g. consisting only of open intervals for \mathcal{B} ? There indeed is—as was observed by S. Solecki (see [9]). The family $\mathcal{F} = \{I_{n,k} = (n2^k, (n+1)2^k) : n \text{ and } k \text{ are integers}\}$ is a family of open intervals which separates all pairs of points of \mathbb{R} . Also, $(2n+1)2^{k-1}$ and $n2^k$ are separated by I_{nk} and by no other interval in \mathcal{F} . This \mathcal{F} is a minimal separating family. Since \mathcal{F} is countable, \mathcal{F} is an m.g. for \mathcal{B} by the Blackwell property of \mathcal{B} . Another m.g. for \mathcal{B} consisting of left-open, right-closed intervals was given in [9].

Secondly, when is a family of open intervals \mathcal{F} a generator for \mathcal{B} ? Surprisingly, \mathcal{F} is a generator for \mathcal{B} if and only if \mathcal{F} separates all pairs of points. In fact,

PROPOSITION 59. *If \mathcal{F} is any family of either open or closed, non-degenerate intervals, there is a countable subfamily \mathcal{F}_0 such that $\sigma(\mathcal{F}_0) = \sigma(\mathcal{F})$. In particular, for such an \mathcal{F} , we have $\sigma(\mathcal{F}) = \mathcal{B}$ if and only if \mathcal{F} separates all pairs of points.*

DEMONSTRATION. Let us first look at the case of \mathcal{F} consisting only of open non-empty intervals. For each pair of rational numbers $r < s$ let $\mathcal{F}_{r,s}$ be a countable subfamily of \mathcal{F} such that

- (a) $r, s \in I$ for each $I \in \mathcal{F}_{r,s}$, and
- (b) $\bigcap\{I : I \in \mathcal{F}_{r,s}\} = \bigcap\{I \in \mathcal{F} : r, s \in I\}$.

This is possible because any intersection of a family of open intervals equals the intersection of a countable subfamily.

If we let $\mathcal{F}_0 = \bigcup\{\mathcal{F}_{r,s} : r < s \text{ and } r \text{ and } s \text{ are rationals}\}$ then \mathcal{F}_0 separates the same pairs of points as \mathcal{F} does. That $\sigma(\mathcal{F}_0) = \sigma(\mathcal{F})$ follows from the Blackwell nature of any c.g. substructure of \mathcal{B} .

The case of non-degenerate closed intervals can be proved similarly by letting $\mathcal{F}_{r,s}$ be a countable subfamily of \mathcal{F} such that $\bigcup\{I : I \in \mathcal{F}_{r,s}\} = \bigcup\{I \in \mathcal{F} : r, s \notin I\}$. The case of non-degenerate intervals, open or closed, follows by putting together the above two results. ■

Here is a remarkable consequence: $\sigma(\mathcal{F})$ is c.g. for every \mathcal{F} as in the above proposition. For further reading see [9] and [19].

3.2. Rigid and strongly rigid Borel spaces. A separable structure on an uncountable set X is called a *rigid Borel structure* (following BS, p. 20) if every automorphism of X is identity except on a countable set. That there are rigid Borel spaces was shown in BS. Following Shortt and van Mill [48], we say that a Borel space (X, \mathcal{B}) is *strictly rigid* (called a “rigid Borel space” in BS) if the only automorphism of (X, \mathcal{B}) is the identity. In BS, it was conjectured that there are strictly rigid Borel spaces (Problem P3 of BS on p. 21).

This problem attracted considerable attention and we present here two of the solutions and comment on a third solution.

PROPOSITION 60 (Shortt and van Mill). *There is a strictly rigid Borel space.*

DEMONSTRATION. Call a completely regular topological space X a *P-space* if every G_δ subset of X is open. Clearly, in a *P-space* the collection \mathcal{B} of all clopen sets of X forms a Borel structure and the automorphisms of the Borel space (X, \mathcal{B}) are simply the homeomorphisms of the topological space X , since, for completely regular *P-spaces*, the clopen sets form a base. Thus, any 0-dimensional *P-space* which is rigid as a topological space would provide an example of a strictly rigid Borel space.

We shall first give the construction of a topological space on a set X which looks like a tree and then impose various conditions on the construction so that X becomes a rigid 0-dimensional *P-space*.

To start with, let $X_0 = \{0\}$.

If X_n has already been defined for a non-negative integer n let $\{B(x, n) : x \in X_n\}$ be a family of non-empty sets such that

- (a) $B(x, n) \cap X_n = \emptyset$, and
- (b) $B(x, n) \cap B(y, n) = \emptyset$ if $x, y \in X_n$ are distinct.

Let $X_{n+1} = \bigcup_{x \in X_n} B(x, n)$ and $X = \bigcup_{n=0}^{\infty} X_n$.

Clearly,

$$X_1 = B(0, 0), \quad X_2 = \bigcup_{x \in X_1} B(x, 1).$$

There is a natural partial order on X , namely $x \leq y$ for $x, y \in X$ if and only if either $x = y$ or $x \neq y$, $x \in X_n$, $y \in X_m$, $n < m$ and there are $x = x_0, x_1, \dots, x_{m-n} = y$ so that $x_1 \in B(x_0, n)$, $x_2 \in B(x_1, n+1) \dots$ and $x_{m-n} \in B(x_{m-n-1}, n+m-n-1)$.

For any x in X we define $T(x) = \{y \in X : x \leq y\}$, the tail from x . If $x \in X_n$ and $F \subset B(x, n)$ we define $T(x, F) = \{x\} \cup \bigcup \{T(y) : y \in B(x, n) \setminus F\}$, the tail from x avoiding F .

Now, let us describe a topology on X . Observe that, if $x \in X$, there is a unique n such that $x \in X_n$ and for this n and x we have a set $B(x, n)$. Fix ideals $\{I_x : x \in X\}$ where I_x is a proper ideal on $B(x, n)$. Equip X with a topology which has $\{T(x, F) : F \in I_x\}$ as the neighbourhood system at x for every $x \in X$. Since $\bigcap_i T(x, F_i) = T(x, \bigcup_i F_i)$, it follows that the topology is well-defined. If all the singletons of $B(x, n)$ are in I_x it follows that the topology is a 0-dimensional Hausdorff topology. If we can make sure that each of the ideals I_x is closed under countable unions, it follows that the topology is a P -space. This is because, if U is an open set and $x \in U$, then there is an $F \in I_x$ such that $T(x, F) \subset U$. Also, one can easily see that $T(x, F)$ is clopen for all x and F .

Now choose the $B(x, n)$'s so that they satisfy

- (c) $|B(x, n)|$ is regular and of uncountable cofinality,
- (d) $|B(x, n)| > |X_n|$, and
- (e) if $x, y \in X_n$ are distinct, then $|B(x, n)| \neq |B(y, n)|$.

Define $I_x = \{F \subset B(x, n) : |F| < |B(x, n)|\}$. This together with (c) will ensure that X is a P -space. Since $\{T(x, F) : F \in I_x\}$ is a base of clopen sets at x and because of (c) it will follow that the character at x (the least cardinality of a local base at x) equals $|B(x, n)|$. Of course, if we take a homeomorphism of X to X and if $f(x) = f(y)$, then the character at x equals the character at y . But (d) and (c) ensure that $\text{Char}(x)$, $x \in X$ are all distinct.

Thus X is a rigid 0-dimensional P -space. Hence (X, \mathcal{B}) , where \mathcal{B} is the Borel structure of clopen subsets of X , is a strongly rigid Borel space. ■

Unfortunately, the X of Proposition 60 is of huge cardinality. We shall now give a construction due to Aniszczyk of a strongly rigid Borel space on a set of cardinality \aleph_1 .

PROPOSITION 61 (Aniszczyk). *There is a strongly rigid Borel space on any set X of power \aleph_1 .*

DEMONSTRATION. Let X be as above with $B(x, n)$ being a copy of ω_1 , the first uncountable cardinal, for all x and all n . Then $|X| = \aleph_1$.

In the previous construction the character at x played a crucial role in showing the rigidity at x . Since we want $|X| = \aleph_1$, we have to devise another method depending on the ideals to distinguish between points in X .

If I is a proper non-principal σ -ideal on ω_1 , and $f : \omega_1 \rightarrow \omega_1$ is a function, we say that f is I -small if $f^{-1}(x) \in I$ for all $x \in \omega_1$ and we define $f_*(I) = \{A \subseteq \omega_1 : f^{-1}(A) \in I\}$. Then $f_*(I)$ is a proper non-principal σ -ideal on ω_1 .

For proper non-principal σ -ideals I and J on ω_1 , the Rudin–Kiesler preordering is defined by: $I \leq_{RK} J$ if and only if $I = f_*(J)$ for some J -small function $f : \omega_1 \rightarrow \omega_1$. We also define $I <_{RK} J$ if $I \leq_{RK} J$ and $J \not\leq_{RK} I$.

A theorem of Baumgartner–Taylor–Wagon [6] says that there is a family $\{I_\alpha : \alpha < \omega_1\}$ of proper non-principal σ -ideals on ω_1 such that $I_\alpha <_{RK} I_\beta$ if $\alpha < \beta < \omega_1$.

Fix such a family of σ -ideals, index them as $\{I_x : x \in X\}$ so that I_x is a σ -ideal on $B(x, n)$. Now, define the topology as in the previous proof; clearly, it is a 0-dimensional P -space. Let us show that it is topologically rigid.

Suppose that $f : X \rightarrow X$ is a homeomorphism with $f(p) = q$ and $p \neq q$. Let us assume that $p \in X_n$ and $q \in X_m$.

If we start with an $F_0 \in I_p$, then $T(p, F_0)$ is a clopen neighbourhood of p and so $f(T(p, F_0))$ is a clopen neighbourhood of q , and so there is a $G_0 \in I_q$ such that $f(T(p, F_0)) \supset T(q, G_0)$. By a similar argument we get an $F_1 \in I_p$ such that $f^{-1}(T(q, G_0)) \supset T(p, F_1)$. Continuing this process and taking $\bigcup_{i=0}^{\infty} F_i = F$ and $\bigcup_{i=0}^{\infty} G_i = G$, we see that there are $F \in I_p$ and $G \in I_q$ such that $F \supset F_0$ and $f(T(p, F)) = T(q, G)$.

Now let us define a function $g : B(p, n) \rightarrow B(q, m)$. Starting from $F_0 = \emptyset$, as in the previous paragraph find $F \in I_p$ and $G \in I_q$ so that $f(T(p, F)) = T(q, G)$. For $x \in B(p, n) - F$ define $g(x) = y$ where $y \in B(q, m)$ and $y \leq f(x)$. Such a y exists and is unique and $y \notin G$. For all $x \in F$ define $g(x) = y_0$, a fixed point of $B(q, m)$.

We claim that $g_*(I_p) = I_q$. First of all it is easily observed that if $F \subset F' \in I_p$, $G \subset G' \in I_q$ and $f(T(p, F)) = T(q, G)$ then $g(F') \subset G' \cup \{y_0\}$ and $F' \supset g^{-1}(G')$, which in turn implies that $g(F') \in I_q$ and $g^{-1}(G') \in I_p$.

Now if $A \in I_q$, using a procedure similar to the previous one find $F' \supset F$ and $G' \supset G \cup A$ so that $f(T(p, F')) = T(q, G')$. This would imply that $g^{-1}(A) \in I_p$. A similar argument also shows that $A \in I_q$ if $g^{-1}(A) \in I_p$. Thus $g_*(I_p) = I_q$.

Similarly we can define a function $h : B(q, m) \rightarrow B(p, n)$ so that $h_*(I_q) = I_p$. But by our choice, $I_p < I_q$ or $I_q < I_p$. Hence X is topologically rigid. ■

The above construction can be used for any uncountable regular cardinal κ (in place of ω_1). Indeed, by modifying Baumgartner–Taylor–Wagon’s proof one can obtain, for any regular κ , a family of ideals of size 2^κ non-comparable in \leq_{RK} -preordering. Since every κ -subset of this family gives a rigid Borel space, we can obtain 2^{2^κ} rigid Borel spaces which are pairwise non-isomorphic.

Here is a result of Droste: There are rigid Borel spaces on any set X of power $\geq \aleph_2$.

In the literature there are constructions of rigid Boolean algebras using linearly ordered sets. Using similar methods and devising new techniques of defining characters of points, Droste in [16] has constructed, for any X with $|X|^\kappa \geq \aleph_2$, a family of 2^κ incomparable Borel spaces. We shall not go into the details of this construction.

For further reading on this topic the reader is referred to [16], [17] and [42].

3.3. The extension property. It is well-known that every real-valued measurable function from a subspace of a Borel space (X, \mathcal{A}) can be extended to a measurable function from the whole space (X, \mathcal{A}) . This can be done by the usual limiting arguments starting with simple functions.

This leads us to the definition: We say that a Borel space (Y, \mathcal{B}) has the *extension property* (EP for short) if every measurable function defined on a subspace (A, \mathcal{A}) of a Borel space (X, \mathcal{A}) and taking values in (Y, \mathcal{B}) can be extended to a measurable function on (X, \mathcal{A}) taking values in (Y, \mathcal{B}) . In this section we shall comment on Borel spaces with EP.

As mentioned earlier, $(\mathbb{R}, \mathcal{B})$ has EP. Any Borel subspace of a (Y, \mathcal{B}) with EP has EP. For separable spaces, we have

PROPOSITION 62. *A separable Borel space has EP if and only if it is standard.*

DEMONSTRATION. If (Y, \mathcal{B}) has EP and is separable, embed (Y, \mathcal{B}) in $\{0, 1\}^{\aleph_0}$ and extend the identity map to a measurable function $f : \{0, 1\}^{\aleph_0} \rightarrow (Y, \mathcal{B})$. Now $\{y \in \{0, 1\}^{\aleph_0} : f(y) = y\} = Y$ and so it is in the Borel structure of $\{0, 1\}^{\aleph_0}$. Hence, (Y, \mathcal{B}) is standard. ■

Let us define: A subset Z of a Borel space (X, \mathcal{A}) is a *retract* of (X, \mathcal{A}) if there is a measurable function $f : (X, \mathcal{A}) \rightarrow (Z, \mathcal{Z} \cap \mathcal{A})$ which extends the identity map on Z .

Every Borel set in a Borel space is a retract. But the converse need not be true: Consider $X = \{0, 1\}^{\aleph_1}$ with the product Borel structure \mathcal{A} and let

$$Z = \{y \in \{0, 1\}^{\aleph_1} : y_\alpha = 0 \text{ for all limit ordinals } \alpha\}.$$

Then Z is a retract of (X, \mathcal{A}) but $Z \notin \mathcal{A}$.

PROPOSITION 63. (a) *Any product of spaces with EP has EP.*

(b) *A Borel space has EP if and only if it is isomorphic to a retract of $\{0, 1\}^{\aleph}$ for some \aleph .*

DEMONSTRATION. (a) is clear and (b) is routine. ■

Even though (b) gives a characterization of spaces with EP, it would be nice to have an internal characterization of spaces with EP which is readily applicable (Problem PP20, Appendix I).

Before we treat a few examples, we give some necessary conditions for a space to have EP which might help with the above problem.

PROPOSITION 64. (a) *If (Y, \mathcal{B}) has EP, then every measurable image of (Y, \mathcal{B}) into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is analytic.*

(b) *If (Y, \mathcal{B}) has EP, then any 0-1 measure on (Y, \mathcal{B}) is concentrated at a point of Y .*

(c) *If (Y, \mathcal{B}) has EP, then any family of pairwise disjoint non-empty sets in \mathcal{B} has cardinality $\leq c$.*

DEMONSTRATION. (a) By Proposition 63(b), (Y, \mathcal{B}) is a retract and hence a measurable image of $\{0, 1\}^{\aleph}$ for some \aleph . But any separable measurable image of $\{0, 1\}^{\aleph}$ is analytic. So, we are done.

(b) If μ is a 0-1 measure on (Y, \mathcal{B}) , define $Y^* = Y \cup \{\mu\}$ and

$$\mathcal{B}^* = \{B \cup \{\mu\} : B \in \mathcal{B} \text{ and } \mu(B) = 1\} \cup \{B : B \in \mathcal{B} \text{ and } \mu(B) = 0\}.$$

Then (Y^*, \mathcal{B}^*) is a Borel space and its trace on Y is (Y, \mathcal{B}) . Since (Y, \mathcal{B}) has EP, the identity function $f : Y \rightarrow Y$ extends to a measurable function $f^* : Y^* \rightarrow Y$. Let us show that μ is concentrated at the point $f^*(\mu)$ in Y .

If $B \in \mathcal{B}$, then $f^{*-1}(B)$ is either B or $B \cup \{\mu\}$. If $\mu(B) = 1$ then $B \notin \mathcal{B}^*$ and so $f^{*-1}(B) = B \cup \{\mu\}$ and this shows that $f^*(\mu) \in B$. On the other hand, if $\mu(B) = 0$ then $B \cup \{\mu\} \notin \mathcal{B}^*$ and so $f^{*-1}(B) = B$ and this shows that $f^*(\mu) \notin B$. Thus μ is concentrated at $f^*(\mu)$.

(c) By Proposition 63(b), it is sufficient to show that the product Borel structure \mathcal{B} of $\{0, 1\}^I$ for any index set I has the property that any family $\mathcal{C} = \{C_\alpha\}_{\alpha \in I}$ of pairwise disjoint non-empty sets in \mathcal{B} has cardinality $\leq c$. Since every set in \mathcal{B} depends on countably many coordinates, there are countable subsets $I_\alpha \subset I$ and Borel sets B_α in $\bigotimes_{\beta \in I_\alpha} \mathcal{P}\{0, 1\}$ such that $C_\alpha = B_\alpha \times \{0, 1\}^{I-I_\alpha}$. Now an application of the Erdős–Rado intersection theorem (see Roy Davies [15]) completes the demonstration. ■

Now we shall give some examples.

EXAMPLE 8. $(X, \mathcal{P}(X))$ has EP if and only if $|X| \leq \aleph_0$. Indeed, if $|X| \leq \aleph_0$, then clearly $(X, \mathcal{P}(X))$ has EP. To prove the converse, using a result on p. 19 of BS, take an $S \subset \mathbb{R}$, such that $|S| = \aleph_1$ and $(S, S \cap \mathcal{B}(\mathbb{R}))$ does not support any non-atomic measure. So, this S cannot be analytic. Now $(S, S \cap \mathcal{B}(\mathbb{R}))$ is a measurable image of $(X, \mathcal{P}(X))$ for any X with $|X| \geq \aleph_1$. Now use Proposition 64(a).

EXAMPLE 9. The countable-cocountable structure on a set X has EP if and only if $|X| \leq \aleph_0$. This can be seen from Proposition 64(b).

EXAMPLE 10. Let \mathcal{B} be the countable-cocountable structure on an uncountable set Y . Let μ be the standard 0-1 measure on (Y, \mathcal{B}) and (Y^*, \mathcal{B}^*) be as in the proof of Proposition 64(b). Then (Y^*, \mathcal{B}^*) has EP if $|Y| \leq \aleph_1$ and does not have EP if $|Y| > c$. To see the first part, let $|Y| \leq \aleph_1$ and let $f : (A, A \cap \mathcal{A}) \rightarrow (Y^*, \mathcal{B}^*)$ be measurable. For $y \in Y$, let $f^{-1}(\{y\}) = A \cap A_y$ with $A_y \in \mathcal{A}$. Since $|Y| \leq \aleph_1$, the A_y 's can be disjointified so that the new A_y 's are still in \mathcal{A} . Define $f^*(x) = y$ if $x \in A_y$ and $= \mu$ if $x \notin \bigcup_{y \in Y} A_y$. This f^* is measurable and extends f . The second part follows from Proposition 64(c).

EXAMPLE 11. (Y^*, \mathcal{B}^*) of Example 10 with $|Y| = \aleph_1$ has EP and is a retract of $\{0, 1\}^{\aleph_1}$.

As the last result let us characterize metric spaces with EP.

PROPOSITION 65. *A metric space X has EP if and only if it is Polish.*

DEMONSTRATION. We shall show that if a metric space X has EP then it must be separable as a metric space. Then, clearly, its Borel structure must be separable. Proposition 62 will complete the result.

If X is not separable as a metric space then, clearly, there exists an uncountable subset Y of X and an $\varepsilon > 0$ such that any two points of Y are at a distance $> \varepsilon$. Clearly, Y is a closed subset of X and hence is a Borel subset of a space X which has EP. Hence Y with its relative Borel structure, which is the power set of Y , itself has EP. This contradicts Example 8. Hence the result. ■

Appendix I

Here is the list of all the problems posed in this monograph along with page number references.

PP1. In ZFC, is it possible to construct a Blackwell space which is not strongly Blackwell? (Page 6.)

PP2. In ZFC, is it possible to show the existence of a projective strongly Blackwell set that is not analytic? (Page 8.)

PP3. Does the strong Blackwell property persist for direct sums (finite/countable)? (Page 16.)

PP4. Does every Borel set $B \subseteq S \times S$ that is not m -reticulate contain an m -graph? (Page 20.)

PP5. Is $X \times S$ (strongly) Blackwell whenever X is a strongly Blackwell set? (Page 21.)

PP6. Under MA and \neg CH, is it the case that if X is strongly Blackwell and A is analytic then $X \cap A$ is Blackwell (or even strongly Blackwell)? (Page 22.)

PP7. Is there a set $X \subseteq \mathbb{R}$ such that $X \times \mathbb{R}$ is Blackwell, but not strongly Blackwell? (Page 24.) Compare with Proposition 17.

PP8. Can one produce (in ZFC) a Blackwell (or strongly Blackwell) set X and an analytic set A such that $X \cap A$ and $X \times A$ are not Blackwell (or strongly Blackwell)? (Page 24.)

PP9. Can one obtain, under CH, a Blackwell set $X \subseteq \mathbb{R}$ and an analytic set $A \subseteq \mathbb{R}$ such that $X \cup A$ is not Blackwell? (Page 24.)

PP10. Suppose that $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \dots$ is a sequence of c.g. Borel structures on a set X with (X, \mathcal{C}_n) Blackwell. If $\mathcal{C} = \bigcap \mathcal{C}_n$ is c.g., then is (X, \mathcal{C}) Blackwell? (Page 24. This is a problem of D. H. Fremlin.)

PP11. Study the minimal weak complements. (Page 26.)

PP12. For which separable Borel spaces (X, \mathcal{B}) does every c.g. $\mathcal{C} \subseteq \mathcal{B}$ have a maximal conjugate? (Page 31.)

PP13. Let (X, \mathcal{B}) be an analytic space. If a substructure \mathcal{C} has a maximal conjugate, is \mathcal{C} necessarily c.g.? If \mathcal{C} is c.g., is every maximal conjugate of \mathcal{C} c.g.? (Page 32.)

PP14. Let (X, \mathcal{B}) be a separable space and let \mathcal{C} be a proper substructure of \mathcal{B} . If \mathcal{C} is separable, can \mathcal{C} have a maximal conjugate in \mathcal{B} ? (Page 33.)

PP15. Find complements, minimal complements and maximal conjugates for the tail, invariant and symmetric Borel structures on $\{0, 1\}^\omega$. (Page 33.)

PP16. Let X be the union of the lines $y = 0$ and $y = 1$ in the plane. Let \mathcal{C} be the substructure of \mathcal{B} generated by the projection onto the X-axis. Is every maximal conjugate of \mathcal{C} c.g.? (Page 33.)

PP17. In ZFC, can we find an $\aleph > \aleph_1$ so that $\mathcal{P}(\aleph)$ has no m.g.? (Page 39.)

PP18. In ZFC, can we find an $\aleph > \aleph_1$ so that $\mathcal{P}(\aleph)$ has an m.g.? (Page 39.)

PP19. Settle the conjecture: Any Borel structure generated by \aleph_1 sets has an m.g. (Page 39.)

PP20. Give an internal characterization of spaces with EP, i.e. the extension property. (Page 43.)

Appendix II

In this appendix we give the status of all the problems posed in BS. There was vigorous research activity on almost all of them. We shall go over the problems one by one.

P1. If κ is the number of sets in an infinite Borel structure, is it true that $\kappa^{\aleph_0} = \kappa$? Yes. In fact, it was shown much earlier than BS that the answer is “yes”; see [14].

P2. Is it true that every Borel space has a minimal generator? No. See the present memoir for various developments.

P3. Does there exist a rigid Borel space? Yes. See the present memoir for various developments.

P4. Are there Blackwell not strongly Blackwell spaces? The answer is not yet known in ZFC. Under CH and MA + \neg CH various examples have been constructed. See the present memoir.

P5. Is there a projective strongly Blackwell space which is not analytic? Only under the Axiom of Constructibility is such a space known to exist; see [33].

P6. If A is Blackwell and B is analytic, is $A \cap B$ Blackwell?

P7. If A is Blackwell and B is analytic, is $A \cup B$ Blackwell?

Bzyl and Jasiński in [12] have shown that under MA + \neg CH both P6 and P7 have negative answers. See the present memoir.

P8. If A is Blackwell and B is Borel, is $A \times B$ Blackwell? Jasiński [24] has shown that under MA + \neg CH the answer is negative. Shortt [39] has shown that if B is countable then the answer is in the affirmative. See the present memoir.

P9. Does every separable structure on an uncountable set contain an atomless substructure? It is known that (see BS) on a set of power \aleph_1 , this problem is undecidable. For higher powers this is still not settled.

P10. If a separable Borel structure contains an atomless structure, should it contain an atomless separated structure? There is no solution yet.

P11. Is there an interlacing sequence of length ω_1 of substructures of the usual Borel structure on $[0, 1]$ which are countably generated and atomless alternately? There is no solution yet.

P12. Find a neat characterization of those Borel structures \mathcal{B} for which $L_{\mathcal{B}}$ is anti-atomic. One such characterization was obtained by Aniszczyk [2].

P13. If B is a Borel structure such that every countably generated substructure of B is given by a countable partition, then is the lattice L_B complemented? No. See the present memoir.

P14. Is there a separable structure \mathcal{B} and a substructure \mathcal{C} of \mathcal{B} such that \mathcal{C} has a complement in \mathcal{B} but has no minimal complement? Yes, there is. See the present memoir, Example 7.

P15. Is the lattice L_B c.g. complemented for every \mathcal{B} ? There is no solution yet.

P16. What is the cardinality of L_X ? In BS the solution was given under GCH. Nothing beyond this is known.

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