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SOME REMARKS ON STABLE SEQUENCES OF RANDOM VARIABLES

Z. RYCHLIK

Institute of Mathematics, Maria Curie-Skłodowska University, Lublin, Poland

Stable and mixing properties of a randomly indexed sequence are studied. No assumption concerning the interdependence between random indices and the considered random variables is made. The obtained results generalize the known limit theorems with random indices. In particular case of a sequence of normed sums of independent random variables the conditions of the basic results are satisfied.

1. Introduction

Let $\{Y_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{A}, P)$. Let $\{N_t, t > 0\}$ be a family of positive integer-valued random variables defined on the same probability space $(\Omega, \mathscr{A}, P)$.

Several authors (see, e.g., [16], [20], [17] and [18]) investigated the asymptotic distribution of Y_{N_t} for $t \to \infty$. In all these investigations it has been supposed that N_t is for any $t > 0$ independent of the random variables Y_n, $n \geq 1$, and N_t converges in probability to infinity for $t \to \infty$. This case will not be considered here except as a particular case.

It seems more interesting to allow the random variables N_t, $t > 0$, to be dependent on Y_n, $n \geq 1$. In this case, a general and very useful theorem has been proved by Anscombe ([3]). Generalizations of his result have been given in [11], [13], [14], [1], [7] and [8]. The obtained results are not only of theoretical interest but are also very important in various applications, e.g., in the theory of Markov chains, in sequential analysis, in random walk problems, in connection with Monte Carlo methods and in the theory of queues (cf. [10], [5], [6] and [8]).

In the present paper we establish some theorems concerning the limit behaviour of sequences of random variables with random indices in the case where no assumption concerning the interdependence between the random indices $\{N_t, t > 0\}$ and the terms $\{Y_n, n \geq 1\}$ is made. The basic results are given in Theorems 1, 2 and 3. Theorems 1 and 2 generalize the main results presented in [3], [11], [1], [7] and [19], while Theorem 3 generalizes the main results given in [13]. The results presented in Theorems 1 and 2 may be especially useful in the study of the limit behaviour of sequences of random variables with random indices under different normalizations.

Finally, we want to emphasize that in the particular case of the sequence of normed sums of independent random variables the conditions of the basic results are satisfied (Theorem 3).

2. Notations and definitions

Let $\{k_n, n \geq 1\}$ be a sequence of positive numbers. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables.

We begin with the following definitions.

DEFINITION 1. A sequence $\{Y_n, n \geq 1\}$ of random variables is said to satisfy the *generalized Anscombe condition* with a norming sequence $\{k_n, n \geq 1\}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} P(\max_{i \in D_n(\delta)} |Y_i - Y_n| \geq \varepsilon) \leq \varepsilon, \tag{1}$$

where, here and in the sequel, $D_n(\delta) = \{i\colon |k_i^2 - k_n^2| \leq \delta k_n^2\}$.

DEFINITION 2. A sequence of random variables $\{Y_n, n \geq 1\}$ is called *stable* if for every event $A \in \mathscr{A}$, with $P(A) > 0$, there exists a distribution function F_A such that

$$\lim_{n\to\infty} P(Y_n < x|A) = F_A(x) \tag{2}$$

for every x which is continuity point of the distribution function F_A.

Let us observe that the set of the discontinuity points of F_Ω contains the set of the discontinuity points of F_A and therefore the set of the discontinuity points for all random events A is denumerable. Furthermore, if $\{Y_n, n \geq 1\}$ is a stable sequence of random variables then, for every fixed continuity point x, the measure $Q_x(\cdot)$ defined as follows

$$Q_x(A) = F_A(x) P(A), \qquad A \in \mathscr{A}$$

is absolutely continuous with respect to the probability measure P. According to the Radon–Nikodym theorem the derivative

$$dQ_x/dP = \mathscr{L}_x(w), \qquad w \in \Omega, \tag{3}$$

exists and it is determined uniquely modulo P. The random variable $\mathscr{L}_x(w)$, $w \in \Omega$, is called the *local density* of the stable sequence $\{Y_n, n \geq 1\}$.

DEFINITION 3. A sequence of random variables $\{Y_n, n \geq 1\}$ is said to be *mixing* with density $F(x)$, if for every $A \in \mathscr{A}$ with $P(A) > 0$

$$\lim_{n \to \infty} P(Y_n < x | A) = F(x) \tag{4}$$

for every x which is continuity point of the distribution function F.

Let us observe that a stable sequence of random variables $\{Y_n, n \geq 1\}$ with the density $\mathscr{L}_x(w)$, $w \in \Omega$, is mixing if and only if $\mathscr{L}_x(w)$ is constant in w with probability one.

A survey of stable and mixing sequences of random variables may be found in [15] and [2].

DEFINITION 4. A family $\{N_t, t > 0\}$ of positive integer-valued random variables is said to *satisfy the condition* ($\triangle$) with norming sequence $\{k_n, n \geq 1\}$, if for every $\varepsilon > 0$ and $\delta > 0$ there exist a finite and measurable partition $\{A_1, A_2, \ldots, A_M\}$ of Ω and a family $\{a_{tj}, 1 \leq j \leq M, t > 0\}$ of positive integers such that $a_{tj} \to \infty$ as $t \to \infty$, and

$$\limsup_{t \to \infty} \sum_{j=1}^{M} P_{A_j}(|k_{N_t}^2 - k_{a_{tj}}^2| > \delta k_{a_{tj}}^2) \leq \varepsilon, \tag{5}$$

where $P_A(B) = P(A \cap B)$.

3. Limit behaviour of sequences of random variables with random indices

Let Z_t, $t \geq 0$, be d-dimensional random vectors such that

$$Z_t \xrightarrow{P} Z_0 \quad \text{as} \quad t \to \infty \quad (P\text{-in probability}). \tag{6}$$

THEOREM 1. *Let $\{Y_n, n \geq 1\}$ be a stable sequence of random variables with local density $\mathscr{L}_x(w)$, and let Z_t, $t \geq 0$, be d-dimensional random vectors for which* (6) *holds. Suppose g is a continuous function of* $(d+1)$ *variables. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $A \in \mathscr{A}$*

$$\limsup_{n \to \infty} P_A(\max_{i \in D_n(\delta)} |Y_i - Y_n| \geq \varepsilon) \leq \varepsilon P(A), \tag{7}$$

then the family $\{g(Y_{N_t}, Z_t), t > 0\}$ of random variables is stable with local density

$$\beta_x(w) = \int_{[g(y, Z_0) < x]} d_y \mathscr{L}_y(w) \tag{8}$$

for every family $\{N_t, t > 0\}$ of positive integer-valued random variables satisfying the condition ($\triangle$).

Proof. Let $\varepsilon > 0$ and a set $A \in \mathscr{A}$, with $P(A) > 0$, be given. Choose $\delta > 0$ as in (7). Then there exist a measurable partition $\{A_1, A_2, \ldots, A_M\}$ of Ω

and positive integers a_{tj}, $1 \leqslant j \leqslant M$, $t > 0$, such that (5) holds true. Let F_A be the distribution function satisfying (2). Then, for every x which is a continuity point of F_A, by (5) and (7), we have

$$\limsup_{t\to\infty} P(Y_{N_t} < x;\ A) \leqslant \limsup_{t\to\infty} \sum_{j=1}^{M} P(Y_{N_t} < x;\ A;\ |k_{N_t}^2 - k_{a_{tj}}^2|$$
$$\leqslant \delta k_{a_{tj}}^2;\ \max_{i\in D_{a_{tj}}(\delta)} |Y_i - Y_{a_{tj}}| < \varepsilon;\ A_j) + 2\varepsilon,$$

where

$$D_{a_{tj}}(\delta) = \{i\colon\ |k_i^2 - k_{a_{tj}}^2| \leqslant \delta k_{a_{tj}}^2\}.$$

Furthermore, the first term on the right-hand side of the last inequality can be estimated from above by

$$\limsup_{t\to\infty} \sum_{j=1}^{M} P(Y_{a_{tj}} < x+\varepsilon;\ A;\ A_j).$$

But by stability, for every $1 \leqslant j \leqslant M$, we have

$$\limsup_{t\to\infty} P(Y_{a_{tj}} < x+\varepsilon;\ A;\ A_j) = \int_{A\cap A_j} \mathscr{L}_{x+\varepsilon}\, dP.$$

Finally, we get

$$\limsup_{t\to\infty} P(Y_{N_t} < x;\ A) \leqslant \int_A \mathscr{L}_{x+\varepsilon}\, dP + 2\varepsilon.$$

Proceeding as above one gets

$$\liminf_{t\to\infty} P(Y_{N_t} < x;\ A) \geqslant \int_A \mathscr{L}_{x-\varepsilon}\, dP - 2\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the last two inequalities prove that the family $\{Y_{N_t}, t > 0\}$ is stable with the density $\mathscr{L}_x(w)$. Thus, by Theorem 2, [12], Theorem 1 is proved. ∎

Remark 1. If we suppose, in Theorem 1, that $\{Y_n, n \geqslant 1\}$ is mixing instead of being stable, then the local density (8) will be as follows

$$\beta_x(w) = \int_{[g(y,Z_0)<x]} dF(y).$$

THEOREM 2. *Let Z_t, $t \geqslant 0$, be a family of random variables such that $P(Z_0 > 0) = 1$ and (6) holds. Suppose h is a continuous and increasing function defined on the real line. If $\{Y_n, n \geqslant 1\}$ satisfies (7) and $\mathscr{L}_x(w)$, $w \in \Omega$, $x \in R$, is a continuous function of x, then the following conditions are equivalent*

(i) *$\{Y_n, n \geqslant 1\}$ is a stable sequence with local density $\mathscr{L}_x(w)$;*

(ii) *$\{h(Y_{N_t})Z_t, t > 0\}$ is a stable family of random variables with local density $\mathscr{L}_{h^{-1}(x/Z_0(w))}(w)$ for every family $\{N_t, t > 0\}$ of positive integer-valued* *random variables satisfying the condition* ($\triangle$), i.e., *for every* $A \in \mathscr{A}$

$$\lim_{t\to\infty} P\big(h(Y_{N_t})Z_t < x;\, A\big) = \int_A \mathscr{L}_{h^{-1}(x/Z_0(w))}(w)\,dP(w), \tag{9}$$

where h^{-1} *is the inverse function of* h.

Proof. Putting $g(x, y) = h(x)y$ in Theorem 1 and using (8) we see that (i) implies (ii). Thus suppose that (ii) holds. Let $\varepsilon > 0$ be given. Choose real numbers a_0 and a_m such that

$$P(a_0 \leqslant Z_0 \leqslant a_m) \geqslant 1-\varepsilon. \tag{10}$$

Let $a_0 < a_1 < a_2 < \ldots < a_m$ be a partition of the interval $[a_0, a_m]$, and let $A \in \mathscr{A}$ be given. Then, by (10), for every $x \in R$

$$\sum_{i=1}^{m} P[h(Y_{N_t})Z_0 < h(x)a_{i-1};\, a_{i-1} \leqslant Z_0 < a_i;\, A] \leqslant P(Y_{N_t} < x;\, A)$$
$$\leqslant \sum_{i=1}^{m} P[h(Y_{N_t})Z_0 < h(x)a_i;\, a_{i-1} \leqslant Z_0 < a_i;\, A]+\varepsilon. \tag{11}$$

It is obvious that if $\{X_n, n \geqslant 1\}$ and $\{Y_n, n \geqslant 1\}$ are sequences of random variables such that $X_n - Y_n \xrightarrow{P} 0$ as $n \to \infty$ and $\{X_n, n \geqslant 1\}$ is stable, then the sequence $\{Y_n, n \geqslant 1\}$ is also stable with the same local density (cf. Lemma 3, [13]). Thus, by our assumptions, the family $\{h(Y_{N_t})Z_0, t > 0\}$ of random variables is stable with the local density $\mathscr{L}_{h^{-1}(x/Z_0(w))}(w)$. Hence, by (11), we get

$$\sum_{i=1}^{m} E\mathscr{L}_{h^{-1}(a_{i-1}h(x)/Z_0)} I[a_{i-1} \leqslant Z_0 < a_i;\, A]$$
$$\leqslant \liminf_{t\to\infty} P(Y_{N_t} < x;\, A) \leqslant \limsup_{t\to\infty} P(Y_{N_t} < x;\, A)$$
$$\leqslant \sum_{i=1}^{m} E\mathscr{L}_{h^{-1}(a_i h(x)/Z_0)} I[a_{i-1} \leqslant Z_0 < a_i;\, A]. \tag{12}$$

Since the functions h, h^{-1} and $\mathscr{L}_x(w)$ as the functions of x are continuous therefore we can choose the splitting points a_i, $1 \leqslant i \leqslant m$, and the number m in such a way that

$$|\mathscr{L}_{h^{-1}(a_i h(x)/u)}(w) - \mathscr{L}_x(w)| < \varepsilon \tag{13}$$

and

$$|\mathscr{L}_{h^{-1}(a_{i-1} h(x)/u)}(w) - \mathscr{L}_x(w)| < \varepsilon \tag{14}$$

for every $a_{i-1} \leqslant u < a_i$, $1 \leqslant i \leqslant m$, and almost all $w \in \Omega$.

Thus, by (11)–(14) and (ii), the family $\{Y_{N_t}, t > 0\}$ of random variables is stable, with local density $\mathscr{L}_x(w)$, for every family $\{N_t, t > 0\}$ of random variables satisfying the condition ($\triangle$).

Suppose the sequence $\{Y_n, n \geq 1\}$ is not stable. Then there exist a set $A \in \mathscr{A}$, and a continuous and bounded function f and subsequences $\{i_n, n \geq 1\}$, $\{j_n, n \geq 1\}$ of positive integers such that

$$\lim_{n\to\infty} Ef(Y_{i_n})I(A) \neq \lim_{n\to\infty} Ef(Y_{j_n})I(A).$$

For every $t > 0$ define $N_t^{(1)}$ and $N_t^{(2)}$ to be $[t]+1$ on $\Omega - A$ and $N_t^{(1)} = i_{[t]+1}$, $N_t^{(2)} = j_{[t]+1}$ on A, where $[t]$ denotes the integer part of the real number t. Then, for $i = 1, 2$, the random variables $\{N_t^{(i)}, t > 0\}$ satisfy the condition ($\triangle$). But

$$\lim_{t\to\infty} |Ef(Y_{N_t^{(1)}}) - Ef(Y_{N_t^{(2)}})| = \lim_{n\to\infty} |Ef(Y_{i_n})I(A) - Ef(Y_{j_n})I(A)| \neq 0.$$

However, the last relation is in contradiction with the fact that the family $\{Y_{N_t}, t > 0\}$ of random variables is stable with local density $\mathscr{L}_x(w)$ for every family $\{N_t, t > 0\}$ of positive integer-valued random variables satisfying the condition ($\triangle$). Thus the proof of Theorem 2 is completed. ∎

Remark 2. If we assume, in Theorem 2 (i), that $\{Y_n, n \geq 1\}$ is mixing with a continuous density $F(x)$ instead of being stable, then the family of random variables given in (ii) of Theorem 2 is stable with local density $F\big(h^{-1}(x/Z_0(w))\big)$.

Let $\{N_t, t > 0\}$ be a family of positive integer-valued random variables such that

$$k_{N_t}^2 / k_{a_t}^2 \xrightarrow{P} \lambda \quad \text{as} \quad t \to \infty, \tag{15}$$

where λ is a positive random variable, $\{a_t, t > 0\}$ is a family of positive integers, $a_t \to \infty$ as $t \to \infty$, and $\{k_n, n \geq 1\}$ is a sequence of positive numbers. Assume

$$k_{n-1}/k_n \to 1, \qquad k_n \to \infty \qquad \text{as} \quad n \to \infty. \tag{16}$$

Considering a sequence of simple random variables which approximates the random variable λ, we see that if (15) and (16) hold, then (5) holds too. Thus Theorems 1 and 2 do give generalizations of the main results presented in [3], [11], [1], [13], [14] and [19] even in the case $k_n^2 = n$, $n \geq 1$.

Let us observe that in accordance with the example given by Durrett and Resnick ([9]) it is not enough to assume that (1) holds instead of (7). This fact, in the case $k_n^2 = n$, $n \geq 1$, has not been noticed by Mogyoródi ([14]) and many others authors that used his result. One can easily note that if λ is an arbitrary positive random variable satisfying (15) with $k_n^2 = n$, $n \geq 1$, it is not possible to apply the Anscombe condition ([3]) (the condition (1) above with $k_n^2 = n$, $n \geq 1$) to the inequality (7) of the Mogyoródi's paper [14], hence the last inequality on page 467 is not true.

On the other hand, for some sequences $\{Y_n, n \geq 1\}$ of random variables the condition (7) is a consequence of (1). In fact, for example, condition (7) is a consequence of (1) any time when the tail σ-field of $\{Y_n, n \geq 1\}$ is trivial under P and $k_n \to \infty$ as $n \to \infty$. This is a consequence of the following lemma ([4]).

LEMMA. *Let $\{Y_n, n \geq 1\}$ be a sequence of random variables. Let $F^{(n)}$ be the smallest σ-field over which the Y_k, $k \geq n$, are measurable and let $F^{\infty} = \bigcap_{n=1}^{\infty} F^{(n)}$ denote the tail σ-field of $\{Y_n, n \geq 1\}$. Then the tail σ-field F^{∞} is trivial under P if and only if*

$$\lim_{n\to\infty} \sup_{A \in F^{(n)}} |P(A \cap B) - P(A)\,P(B)| = 0 \tag{17}$$

for each B in $\mathcal{A}$.

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with zero means and finite variances. Let $k_n^2 = \sigma^2 S_n$ and put $Y_n = S_n/k_n$, $n \geq 1$, where $S_n = X_1 + X_2 + \ldots + X_n$. Then $\{Y_n, n \geq 1\}$ satisfies the generalized Anscombe condition with the norming sequence $\{k_n = \sigma s_n, n \geq 1\}$ because, by Kolmogorov inequality, we have

$$P(\max_{i \in D_n(\delta)} |Y_i - Y_n| \geq \varepsilon) \leq (24\delta + 16\delta^2)/\varepsilon^2(1-\delta) \tag{18}$$

for every $\varepsilon > 0$ and $\delta > 0$, $\delta \neq 1$. Thus, by Lemma the sequence $\{S_n/k_n, n \geq 1\}$ satisfies (7) too if, in addition, $k_n \to \infty$ as $n \to \infty$.

Suppose

$$\lim_{n\to\infty} P(S_n < xk_n) = F(x), \tag{19}$$

where $F(x)$ is a distribution function. Then, by Theorem 4, [15], $\{S_n/k_n, n \geq 1\}$ is mixing with density $F(x)$. Furthermore, if (19) holds and X_n/k_n, $n \geq 1$, are uniformly asymptotically negligible, then the sequence $\{k_n = \sigma S_n, n \geq 1\}$ satisfies (16). Thus we see that any sequence of normed sums of independent random variables satisfies the conditions of the basic results. Hence, from Theorem 2, we get

THEOREM 3. *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with zero means and finite variances. Assume that*

$$k_n^2 = \sigma^2 S_n \to \infty, \qquad k_{n-1}/k_n \to 1 \qquad \text{as} \quad n \to \infty.$$

Let $\{M_t, t > 0\}$ be a family of positive integer-valued random variables such that

$$k_{M_t}/k_{a_t} \xrightarrow{P} \tau \qquad \text{as} \quad t \to \infty,$$

where $\{a_t, t > 0\}$ is a family of positive integers $a_t \to \infty$ as $t \to \infty$ and $P(\tau > 0) = 1$. If $F(x)$ is a continuous distribution function, then the following conditions *are equivalent*

(i) $\lim_{n\to\infty} P(S_n < xk_n) = F(x)$;

(ii) *the family* $\{S_{N_t} k_{a_t}/k_{N_t} k_{M_t},\ t > 0\}$ *is stable with local density* $F(x\tau)$.

As a consequence of Theorems 3 and 1 we have the following

COROLLARY. *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of independent random variables as in Theorem 3. Let* $\{M_t,\ t > 0\}$ *and* $\{N_t,\ t > 0\}$ *be families of positive integer-valued random variables such that*

$$k_{M_t}/k_{a_t} \xrightarrow{P} \tau \qquad \text{and} \qquad k_{N_t}/k_{a_t} \xrightarrow{P} \lambda \qquad \text{as} \quad t \to \infty, \tag{20}$$

where $\{a_t,\ t > 0\}$ *is as in Theorem 3 and*

$$P(\tau > 0) = P(\lambda > 0) = 1.$$

If $F(x)$ *is a continuous distribution function such that* (i) *in Theorem 3 holds, then* $\{S_{N_t}/k_{M_t},\ t > 0\}$ *is stable with local density* $F(x\tau/\lambda)$, i.e.,

$$\lim_{t\to\infty} (S_{N_t} < xk_{M_t}) = \int_{\Omega} F(x\tau/\lambda)\, dP.$$

Putting $M_t = a_t$, $t > 0$, in Corollary, we obtain

$$\lim_{t\to\infty} P(S_{N_t} < xk_{a_t}) = \int_0^{\infty} F(x/z)\, dP(\lambda < z).$$

Furthermore, putting $N_t = a_t$, $t > 0$, we get

$$\lim_{t\to\infty} P(S_{a_t} < xk_{M_t}) = \int_0^{\infty} F(xz)\, dP(\tau < z).$$

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