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ON SEQUENCES OF σ-ALGEBRAS

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0. Introduction. Let $(A_n)_{n=0}^{\infty}$ be a sequence of σ-algebras in a probability space (Ω, A, P) such that

(i) $A_{n+1} \subseteq A_n$,

(ii) for each n, there is a σ-algebra C_n independent of A_{n+1} and together with A_{n+1} generating A_n,

(iii) $\bigcap_{n=0}^{\infty} A_n$ is trivial in the sense that its elements have the probability zero or one.

When is the σ-algebra generated by $(C_n)_{n=1}^{\infty}$ equal, up to sets of measure zero, to the σ-algebra A_1?

This is one of the questions which presents itself in certain formulations of non-linear prediction theory due to Kallianpur and Wiener [2] and Rosenblatt [7] and [8]. Actually, in their formulation, the σ-algebra A_n is taken to be the future σ-algebra from time n on of a purely non-deterministic strictly stationary stochastic process. A succinct account of this theory is given by Masani in [4], p. 89. In this paper we address ourselves to the question raised above, but we do not bring in the notions of probability and of independence of σ-algebras into considerations and instead only require that $C_n \cap A_{n+1} = \{\emptyset, \Omega\}$ and $C_n \cup A_{n+1}$ generate A_n. The results, however, do have relevance to the question raised above in the context of probability. The paper contains mainly counter-examples to plausible conjectures.

1. Let A be a σ-algebra of subsets of a set Ω. Two subsets A and B of Ω are *separated* in A if there exists a subset $C \in A$ such that $A \subseteq C$ and $B \subseteq \Omega \setminus C$. We say that A *separates points* if any two distinct points are separated in A. A non-empty subset $A \in A$ is called an *atom* if there is no proper non-empty subset of A which belongs to A, and A is called *atomic* if every non-empty element of A is a union of atoms of A. For any finite collections $A_0, A_1, \dots, A_n$ of σ-algebras on Ω, we write $A_0 \vee A_1 \vee \vee \dots \vee A_n$ and $A_0 \wedge A_1 \wedge \dots \wedge A_n$ to denote the σ-algebras generated by $A_0 \cup A_1 \cup \dots \cup A_n$ and $A_0 \cap A_1 \cap \dots \cap A_n$, respectively. $\bigvee_{n=1}^{\infty} A_n$ and $\bigwedge_{n=1}^{\infty} A_n$ have a similar meaning. We shall need the following theorem due to Blackwell [1] and Mackey [3]:

BLACKWELL-MACKEY THEOREM. *Let Ω be an analytic set in complete separable metric space and let B denote its Borel σ-algebra. If A is a countably generated sub-σ-algebra of B which separates points, then $A = B$.*

PROPOSITION 1. *Let C_n and D_n $(n \geqslant 0)$ be two sequences of σ-algebras on a set Ω, and assume that the sequence D_n $(n \geqslant 0)$ is decreasing. Let*

$$C = \bigvee_{n=1}^{\infty} C_n \quad \text{and} \quad D = \bigcap_{n=1}^{\infty} D_n,$$

and let x and y be two distinct points of Ω.

(i) *If x and y are separated in $C_n \vee D_n$ for every n, then x and y are separated in $C \vee D$.*

(ii) *If x and y are separated in $C \vee D$, then x and y are separated in $C_n \vee D_n$ for some n.*

Proof. (i) If x and y are separated in $C_n \vee D_n$ for every n, then, for any given n, x and y are separated either in C_n or in D_n. If x and y are separated in C_n for some n, then, clearly, x and y are separated in C, and hence in $C \vee D$. Suppose that x and y are not separated in any C_n. Then x and y are separated in D_n for every n, and so, for each $n \geqslant 0$, there is an $A_n \in D_n$ such that $x \in A_n$ and $y \notin A_n$. Let $A = \limsup A_n$. Then $x \in A$ and $y \notin A$, and since D_n, $n \geqslant 0$, is a decreasing sequence, $A \in D_n$ for every n. Hence $A \in D$. Thus x and y are separated in D, and hence in $C \vee D$.

(ii) If x and y are separated in $C \vee D$, then they are separated either in C or in D. If they are separated in C, they are separated in some C_n. (For if not, then the collection $\{A\colon x, y \in A \text{ or } x, y \in A^c\}$ is a σ-algebra which contains every C_n, whence C, and thus x and y are not separated in C.) Hence x and y are separated in $C_n \vee D_n$. If they are separated in D, then, in view of $D \subseteq D_n$, x and y are separated in D_n, and hence in $C_n \vee D_n$ for every n.

As immediate consequences of proposition 1 we have the following

COROLLARY 1. *Assume, in addition to the hypothesis of proposition 1, that C_n, $n \geqslant 0$, is an increasing sequence.*

(i) *x and y are separated in $C_n \vee D_n$ for all but finitely many n if and only if x and y are separated in $C \vee D$.*

(ii) *If $C_n \vee D_n$ separates points for all but finitely many n, then $C \vee D$ separates points.*

COROLLARY 2. *Assume, in addition to the hypothesis of proposition 1, that Ω is an analytic subset of a complete separable metric space and that, for each n, $\boldsymbol{C}_n$ and $\boldsymbol{D}_n$ are countably generated sub-σ-algebras of the Borel σ-algebra $\boldsymbol{B}$ of Ω.*

(i) *If $\boldsymbol{C}_n \vee \boldsymbol{D}_n = \boldsymbol{B}$ for all n, then $\boldsymbol{C} \vee \boldsymbol{D}$ separates points.*

(ii) *If $\boldsymbol{C}_n \vee \boldsymbol{D}_n = \boldsymbol{B}$ for all n and $\boldsymbol{D}$ is countably generated, then $\boldsymbol{C} \vee \boldsymbol{D} = \boldsymbol{B}$.*

Part (ii) of this corollary follows from the Blackwell-Mackey theorem.

It can be conjectured that if $\boldsymbol{C}_n$ and $\boldsymbol{D}_n$ $(n \geqslant 0)$ are increasing and decreasing sequences, respectively, of countably generated sub-σ-algebras of the Borel σ-algebra $\boldsymbol{B}$ of an analytic set Ω such that $\boldsymbol{C}_n \vee \boldsymbol{D}_n = \boldsymbol{B}$ for all n, then $\boldsymbol{C} \vee \boldsymbol{D} = \boldsymbol{B}$. This, however, is not true as illustrated by the following example, the method of which is connected with the problem of complementation of σ-algebras considered by Rao [5]:

Example 1. Let $\Omega = \{0, 1\}^{N_0}$, where $\{0, 1\}$ is the additive group of two elements with the discrete topology, and N_0 is the set of non-negative integers. Regard Ω as a compact group with coordinatewise addition and product topology.

Let Q be a subgroup of Ω consisting of elements with finitely many ones. Let $\boldsymbol{D}_n$ be the smallest σ-algebra with respect to which all coordinate functions f_k, $k \geqslant n$, are measurable. Then $\boldsymbol{D}_{n+1} \subseteq \boldsymbol{D}_n$, and

$$\boldsymbol{D} = \bigcap_{n=0}^{\infty} \boldsymbol{D}_n$$

is the sub-σ-algebra of Borel sets of Ω, invariant under translation by elements of Q. The algebra $\boldsymbol{D}$ is not countably generated, it is atomic, and its atoms are precisely the cosets of Q. Further, by the Kolmogorov zero-one law, every element of $\boldsymbol{D}$ has Haar measure zero or one.

Let $\boldsymbol{F}$ be any other sub-σ-algebra of $\boldsymbol{B}$. Then

$$\wedge \boldsymbol{F}\boldsymbol{D} \subseteq \{C\colon C \text{ Borel, and there exists a } D \in \boldsymbol{F} \text{ such that } h(C \Delta D) = 0\},$$

where h stands for Haar measure on Ω. Now let $a \in \Omega \setminus Q$, say $a = (1, 1, 1, \ldots)$, and let $A = \{\omega\colon \omega_0 = 0\}$. Then

$$A + a = \{\omega\colon \omega_0 = 1\} = A^c,$$

where ω_0 denotes the 0-th coordinate of $\omega \in \Omega$. Now let

$$\boldsymbol{C} = \{B \cup B + a\colon B \text{ is a Borel set in } A\}.$$

Then $\boldsymbol{C}$ is a countably generated sub-σ-algebra of Ω, and its atoms are of type $\{x, x + a\}$, $x \in A$. Since $a \notin Q$, x and $x + a$ are separated in $\boldsymbol{D}$. Thus $\boldsymbol{C} \vee \boldsymbol{D}$ separates points. But every element of $\boldsymbol{C} \vee \boldsymbol{D}$ differs from a set in $\boldsymbol{C}$ by a set of Haar measure zero. Thus the set A can never belong to $C \vee D$. Now let $C_n = C$ for all n. Then $C_n \vee D_n$ separates points (since $D \subseteq D_n$), and is countably generated (since C_n and D_n are countably generated). Hence $C_n \vee D_n = B \neq C \vee D$ for each n.

2. Let (Ω, A, P) be a probability space and let A_1 be a sub-σ-algebra of A. In the theory of non-linear prediction one seeks, as a first step, conditions on A_1 under which there is a sub-σ-algebra A_2 of A such that A_1 and A_2 are independent and $A_1 \vee A_2 = A$ (this last equality can be required to hold only up to sets of measure zero in the sense that, for every set in $A_1 \vee A_2$, there is a set in A such that the symmetric difference of both has measure zero). This problem has been considered by Rosenblatt [7] and, earlier, by Rohlin [6] in his study of Lebesgue spaces. In this section we consider σ-algebras of types A_1 and A_2 in the special case where Ω can be identified as a product of two sets Ω_1 and Ω_2, and A_1 and A_2 as σ-algebras of coordinate functions [1]. Here Ω_1 and Ω_2 are equipped with σ-algebras, and $\Omega_1 \times \Omega_2$ is the given product σ-algebra.

By an *automorphism* of a σ-algebra A we mean a one-one mapping of A onto A which preserves countable union and complementations. Let A_1 and A_2 be sub-σ-algebras of a σ-algebra A. We say that A_1 is *isomorphic to A_1' modulo A_2 in A* if there is an automorphism T of A such that $TA_1 = A_1'$ and $TA_2 = A_2$. Suppose A_1 and A_2 are atomic σ-algebras of Ω. Following Rohlin, we say that A_1 and A_2 are *crossed* if an atom of A_1 and an atom of A_2 have always a non-empty intersection. It is easy to see that if A_1 and A_2 are crossed, then $A_1 \cap A_2 = \{\emptyset, \Omega\}$, and $A_1 \vee A_2$ is atomic and its atoms are obtained by taking intersections of atoms of A_1 with atoms of A_2.

PROPOSITION 2. *Let A_1 and A_2 be crossed σ-algebras and let A_1 be isomorphic to A_1' modulo A_2 in $A_1 \vee A_2$. Then*

(i) *A_1' is atomic;*

(ii) *A_1' and A_2 are crossed;*

(iii) *$A_1' \vee A_2 = A_1 \vee A_2$;*

(iv) *there is an automorphism T of $A_1 \vee A_2$ such that $TA_1 = A_1'$ and T is an identity on A_2.*

Proof. Since A_1 is isomorphic to A_1' and A_1 is atomic, A_1' is atomic. This proves (i).

Let T be an isomorphism of A_1 onto A_1' modulo A_2 on $A_1 \vee A_2$. Let ξ and η be atoms of A_1' and A_2, respectively. Then $T^{-1}\xi$ and $T^{-1}\eta$

[1] It should be emphasized that, given a probability space (Ω, A, P) and independent σ-algebras A_1 and A_2 such that $A_1 \vee A_2 = A$, it is not necessarily true that (Ω, A, P) can be identified by a bimeasurable measure preserving-point map T with a product space $(\Omega_1 \times \Omega_2, B_1 \times B_2, P_1 \times P_2)$ (minus a $(P_1 \times P_2)$-null set, if necessary) such that $T^{-1}(B_1 \times \Omega_2) = A_1$ and $T^{-1}(\Omega_1 \times B_2) = A_2$.

are atoms of A_1 and A_2, respectively, and since A_1 and A_2 are crossed, they intersect. Since T is an automorphism, $TT^{-1}\xi = \xi$ and $TT^{-1}\eta = \eta$ intersect. This proves (ii).

To prove (iii) we note that

$$A_1 \vee A_2 = T(A_1 \vee A_2) = TA_1 \vee TA_2 = A_1' \vee A_2.$$

Finally, to prove (iv) fix an atom ξ of A_2 and let

$$E = \{\xi \cap a\colon \ a \text{ is an atom of } A_1\}.$$

Now an atom a of $A_1 \vee A_2$ is the intersection of an atom a_1 of A_1 and an atom a_2 of A_2. Let a_1' be the atom of A_1' such that $a_1 \cap \xi = a_1' \cap \xi$, and write $T(a_1 \cap a_2) = a_1' \cap a_2$. It is easy to check that $Ta_2 = a_2$ and $Ta_1 = a_1'$. Thus T is an identity on A_2. Since A_1, A_1' and $A_1 \vee A_2$ induce the same σ-algebra on ξ, we see that $TA_1 = A_1'$. Finally, if $A_1 \in A_1$ and $A_2 \in A_2$, then

$$T(A_1 \cap A_2) = T(A_1) \cap T(A_2)$$

from which it follows that T is an automorphism of $A_1 \vee A_2$.

PROPOSITION 3. *Let A_1 and A_2 be crossed σ-algebras and let p and q be atoms of $A_1 \vee A_2$ which belong to distinct atoms of A_2. Then in $A_1 \vee A_2$ there exists an A_1' isomorphic to A_1 modulo A_2 such that p and q belong to the same atom of A_1'.*

Proof. Let π and χ be atoms of A_1 containing p and q, respectively. Let A_1' consist of those sets in $A_1 \vee A_2$ which are unions of atoms of A_1, except the atoms π and χ which are replaced by $(\pi \cup q) - \eta \cup \pi$ and $(\chi - q) \cup (\eta \cap \pi)$, respectively ($\eta$ is the atom of A_2 which intersects χ in q). Since p and q belong to distinct atoms of A_2, we see that

$$p,\, q \in (\pi \cup q) - \eta \cap \pi \quad \text{ and } \quad p,\, q \notin (\chi - q) \cup (\eta \cap \pi).$$

Thus p and q belong to the same atom of A_1'. Finally, the automorphism

$$Tx = \begin{cases} x & \text{if } x \neq q \text{ or } \eta \cap \pi, \\ \eta \cap \pi & \text{if } x = q, \\ q & \text{if } x = \eta \cap \pi, \end{cases}$$

sets up an isomorphism of A_1 onto A_1' modulo A_2 in $A_1 \vee A_2$.

Now let $A_0, A_1, \ldots$ be point-separating σ-algebras of $\Omega_0, \Omega_1, \ldots$, respectively. Let

$$\Omega = \Omega_0 \times \Omega_1 \times \ldots \quad \text{ and } \quad A = A_0 \times A_1 \times \ldots$$

We denote by A_k also the σ-algebra

$$\Omega_0 \times \Omega_1 \times \ldots \times \Omega_{k-1} \times A_k \times \Omega_{k+1} \times \ldots$$

Let $D_n = A_n \vee A_{n+1} \vee A_{n+2} \ldots$, and $C_n = A_0 \vee A_1 \vee \ldots \vee A_{n-1}$. Then $C_n \vee D_n = A$. Let A'_k be isomorphic to A_k modulo A_{k+1} in A_k, and write $C'_n = A'_0 \ldots A'_{n-1}$. Then $A = C'_n \vee D_n = C_n \vee D_n$. The proposition

$$\bigvee_{n=0}^{\infty} C_n = \bigvee_{n=0}^{\infty} C'_n$$

is, in general, not true. To see this we take $\Omega_n = \{0, 1\}$ and put A_n to be the discrete σ-algebra on $\{0, 1\}$. Now take $p, q \in \Omega$, p and q belonging to distinct atoms of D_∞, where

$$D_\infty = \bigcap_{n=1}^{\infty} D_n$$

(see example 1). Then p and q belong to distinct atoms of D_n for each n (since $D_n \supseteq D_\infty$, atoms of D_n are contained in atoms of D_∞). Let π_n and χ_n be atoms of D_n which contain p and q, respectively. Then π_n and χ_n are subsets of distinct atoms of D_{n+1}. Hence, by proposition 3, there exists in D_n a σ-algebra A'_n isomorphic to A_n modulo D_{n+1} such that π_n and χ_n belong to the same atom of A'_n. Thus, since A'_n is defined for each n, we see that p and q are not separated in $\bigvee_{n=0}^{\infty} A'_n$ whereas they are separated in $\bigvee_{n=0}^{\infty} A_n$. Hence

$$\bigvee_{n=0}^{\infty} A'_n \neq \bigvee_{n=0}^{\infty} A_n.$$

In the case of (Ω_i, A_i) being complete separable metric spaces with their Borel σ-algebras, we have the following

PROPOSITION 4. *We have*

$$\bigvee_{n=0}^{\infty} C_n = \bigvee_{n=0}^{\infty} C'_n \textit{ if and only if } D_\infty \subseteq \bigvee_{n=0}^{\infty} C'_n.$$

Proof. The necessity is obvious, since

$$D_\infty \subseteq \bigvee_{n=0}^{\infty} C_n.$$

To prove the sufficiency we note that, since $C'_n \vee D_n = A$, D_n and C'_n separate points. Hence, by proposition 1, $\bigvee_{n=0}^{\infty} C'_n$ and D_∞ separate points. If, in addition,

$$D_\infty \subseteq \bigvee_{n=0}^{\infty} C'_n,$$

then $\bigvee_{n=0}^{\infty} C'_n$ separates points and is countably generated. Hence, by the

Blackwell-Mackey theorem,

$$\bigvee_{n=0}^{\infty} \boldsymbol{C}'_n = \bigvee_{n=0}^{\infty} \boldsymbol{C}_n.$$

Though

$$\bigvee_{n=0}^{\infty} \boldsymbol{C}_n = \bigvee_{n=0}^{\infty} \boldsymbol{C}'_n$$

is not in general true, one can suspect that

$$\bigvee_{n=0}^{\infty} \boldsymbol{C}_n \vee \boldsymbol{D}_\infty = \bigvee_{n=0}^{\infty} \boldsymbol{C}'_n \vee \boldsymbol{D}_\infty.$$

Here we give an example to show that this also need not be true. We make use of example 1 given in section 1.

Take $\Omega_k = \{0, 1\}$ with $\boldsymbol{A}_k$ the discrete σ-algebra on Ω_k. Write also

$$\boldsymbol{A}_k = \Omega_0 \times \Omega_1 \times \ldots \times \Omega_{k-1} \times \boldsymbol{A}_k \times \Omega_k \times \ldots,$$

$$\boldsymbol{D}_k = \Omega_0 \times \Omega_1 \times \ldots \times \Omega_{k-1} \times \boldsymbol{A}_k \times \boldsymbol{A}_{k+1} \times \boldsymbol{A}_{k+2} \times \ldots$$

Take $\boldsymbol{A}'_k = \{\emptyset, \Omega, A_k, \Omega - A_k\}$, where $A_k = A_{k1} \cup A_{k2}$ with

$$A_{k1} = \Omega_0 \times \Omega_1 \times \ldots \times \Omega_{k-1} \times \{0\} \times \{1\} \times \Omega_{k+2} \times \Omega_{k+3} \times \ldots,$$

$$A_{k2} = \Omega_0 \times \Omega_1 \times \ldots \times \Omega_{k-1} \times \{1\} \times \{0\} \times \Omega_{k+2} \times \Omega_{k+3} \times \ldots$$

Define f_k by

$$f_k(x_0, x_1, \ldots, x_{k-1}, 0, 1, x_{k+2}, \ldots) = (x_0, x_1, \ldots, x_{k-1}, 1, 1, x_{k+2}, \ldots),$$

$$f_k(x_0, x_1, \ldots, x_{k-1}, 1, 0, x_{k+2}, \ldots) = (x_0, x_1, \ldots, x_{k-1}, 1, 0, x_{k+2}, \ldots),$$

$$f_k(x_0, x_1, \ldots, x_{k-1}, 1, 1, x_{k+2}, \ldots) = (x_0, x_1, \ldots, x_{k-1}, 0, 1, x_{k+2}, \ldots),$$

$$f_k(x_0, x_1, \ldots, x_{k-1}, 0, 0, x_{k+2}, \ldots) = (x_0, x_1, \ldots, x_{k-1}, 0, 0, x_{k+2}, \ldots).$$

It is easily checked that $f_k\colon \Omega \to \Omega$ sets up in $\boldsymbol{D}_k$ an isomorphism between $\boldsymbol{A}'_k$ and $\boldsymbol{A}_k$ modulo $\boldsymbol{D}_{k+1}$. Since $\boldsymbol{A}_k + a = \boldsymbol{A}_k$, the σ-algebra $\bigvee_{n=0}^{\infty} \boldsymbol{A}'_n$, which is the same as $\bigvee_{n=0}^{\infty} \boldsymbol{C}'_n$, is contained in $\boldsymbol{C}$, where a and $\boldsymbol{C}$ are as in example 1. Thus

$$\Big(\bigvee_{n=0}^{\infty} \boldsymbol{C}'_n\Big) \vee \boldsymbol{D}_\infty \subseteq \boldsymbol{C} \vee \boldsymbol{D}_\infty \neq \boldsymbol{B}.$$

It is worth noting that in this example the collection $\boldsymbol{A}'_n$, $n \geqslant 0$, is independent, and $\boldsymbol{A}'_n$ and $\boldsymbol{A}_n$ are both independent complements of $\boldsymbol{D}_{n+1}$ in $\boldsymbol{D}_n$.

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