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Torus embeddings, polyhedra, k^* -actions and homology

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CONTENTS

Introduction	5
1. General torus embeddings	7
1.1. Sets of subrings	7
1.2. Complex of cones and torus embeddings. Basic properties and notation	8
1.3. Jets of 1-p.s. at 0	12
1.4. An application of torus embeddings. Desingularization of plane cusps by blowings up of the plane	14
1.5. Some G_m -actions on torus embedding	18
2. Complex torus embeddings. Real and non-negative parts	20
2.1. Introduction	20
2.2. The real non-negative part of the variety $X_{\mathcal{F}}$	21
2.3. Bijection of $X_{\mathcal{F}}^{\geq 0}$ onto \mathcal{F}	29
2.4. Real part of $X_{\mathcal{F}}$. Reflexions	35
3. Projective torus embeddings	37
3.1. Polyhedra	37
3.2. Morse function	41
3.3. Filtrations, cycles of orbits and projectivity	46
4. Homology	50
4.1. Poincaré polynomial	50
4.2. Chow ring and l -adic cohomology	51
4.3. Cohomology ring of $X_{\mathcal{F}}(\mathbb{R})$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$	52
4.4. Orientation.	55
4.5. The 2-dimensional case, homology with integral coefficients	56
References	62
Index	64

Introduction

The theory of torus embeddings has been introduced (in other terminology) by Demazure in [Dem] and then developed in a number of works, among them the fundamental is [TE]. The aim of this paper is mainly to collect and present with complete proofs the author's results announced elsewhere. For the contents of sections 1.3, 1.4, 2.3.7 and 4.3–4.5, this is the first presentation.

The central object of the paper is the real non-negative part of a torus embedding, and in the projective case a polyhedron defined by an ample divisor, as it is pointed out, the two objects are essentially equivalent. The reader interested in the foundations of torus embeddings may find a complete exposition in [TE] or [TEA]. Another reference is [Dan].

Now, here is a brief survey of the contents of the present paper. Chapter 1 begins with a short description of the general torus embedding X_T determined by a complex of polyhedral cones Σ , the correspondence (cones of Σ) \leftrightarrow (orbits in X_T) etc. Then the jets of 1-parameter subgroups of the torus T , in X_T , are studied (Thm. 1.3.3). The chapter ends with some applications.

The case of the ground field \mathbb{C} is considered in Chapter 2. A decomposition of X_T into the non-negative and radial parts may be defined. This decomposition is used to describe the structure of the variety X_T (2.2.5, 2.2.7). In the affine case, the non-negative part of X_σ (the torus embedding defined by a cone σ) is shown to be essentially equal to the dual cone $\check{\sigma}$ (Thm. 2.3.2). The identification of the two objects results by means of a certain natural morphism of X_σ into the complex space \mathbb{C}^n (n being the dimension of the torus T).⁽¹⁾ This morphism is then studied in more detail. The chapter ends with an examination of $X_T(\mathbb{R})$, the real part of X_T .

Chapter 3 deals with projective torus embeddings. The related polyhedra (see 3.1) are used in the construction of a certain Morse function on the variety X_T with a \mathbb{C}^* -action (in the complex case). This function generalizes the standard Morse function on the complex projective space. The theory of torus embeddings is then used to give an example of a multiplicative group action on a complete variety with "cycles of orbits". It gives the negative answer to the conjecture of the existence of Morse functions on a compact

⁽¹⁾ cf. the comment 1° added in proof.

analytic manifold, compatible with a given C^* -action (see [C-S]). The last chapter is concerned with the ring of classes of cycles on a torus embedding $X_{\mathcal{L}}$ and its relation to the 1-adic cohomology, and in the complex case, also the cohomology ring of $X_{\mathcal{L}}(\mathbf{R})$, with coefficients in $\mathbf{Z}/2$. All those rings are expressed in terms of the complex of cones Σ (compare the note to 4.3).

As far as the integral coefficients are involved, the author has been able to elaborate the 2-dimensional case only.

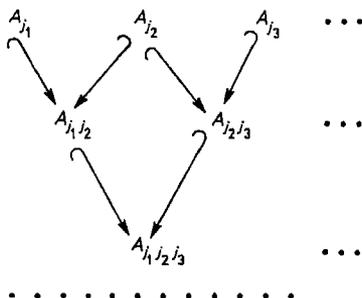
The terminology and notation is taken in general from [TE]. It is introduced mainly through Chapter 1 of the present paper and is collected in the index.

Several problems and ideas considered here have been suggested to me by A. Białyński-Birula. I owe also some arguments to J. Konarski and K. Nowiński.

1. GENERAL TORUS EMBEDDINGS

1.1. Sets of subrings

Let X be a separable integral quasi-compact scheme and U_1, U_2, \dots, U_m its open affine finite covering, $U_j \cong \text{Spec } A_j$. Consider all the intersections $U_{j_1} \cap \dots \cap U_{j_r}$. Since they are affine open subschemes, say, of the form $\text{Spec } A_{j_1 j_2 \dots j_r}$, one obtains a commutative diagram of rings and inclusions i.e. a set of subrings of $A_{123\dots m}$, corresponding to the open immersions of Spec 's:



1.1.1. Clearly, the above set of subrings uniquely defines the scheme X which contains $\text{Spec } A_{12\dots m}$ as an open dense subscheme, namely $X = \text{inj lim Spec } A_{j_1 \dots j_r}$. If X has a structure of k -scheme where k is a field then one has to consider k -algebras A_j instead of rings.

The varieties called torus embeddings are, roughly speaking, defined in the above way. One considers the affine algebra A of the n -dimensional torus (see 1.2.5 for definition) over a fixed ground field k , i.e. $A = k[X_1, X_2, \dots, X_n, (X_1 X_2 \dots X_n)^{-1}]$ and a set of subalgebras of A . This set is described by a complex Σ of polyhedral rational cones in \mathbb{R}^n . The scheme obtained by glueing together the spectra of the above subalgebras is the torus embedding associated with Σ . There is a large list of applications of torus embedding based on the fact that the theory of those schemes may be reduced to the theory of complexes of cones (see references in the Introduction).

Basic definitions and theorems are listed below.

1.2. Complex of cones and torus embeddings. Basic properties and notation

1.2.1. Let N be a lattice of a fixed rank n , M the dual lattice, $N_{\mathbf{Q}} = N \otimes_{\mathbf{Z}} \mathbf{Q}$, $M_{\mathbf{Q}} = M \otimes_{\mathbf{Z}} \mathbf{Q}$, $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ and let $(\alpha, m) \mapsto \langle \alpha, m \rangle$ be the pairing $N_{\mathbf{R}} \times M_{\mathbf{R}} \rightarrow \mathbf{R}$. The set of positive (resp. non-negative) reals will be denoted by $\mathbf{R}^{>0}$ (resp. $\mathbf{R}^{\geq 0}$).

A *convex polyhedral cone* σ in $N_{\mathbf{R}}$ is the intersection of a finite number of half-spaces: $\sigma = H_1 \cap H_2 \cap \dots \cap H_m$. The cone σ is said to be *rational* if the half-spaces are defined over \mathbf{Q} , i.e. σ is described by a system of inequalities:

$$x \in \sigma \Leftrightarrow \langle x, m_i \rangle \geq 0 \text{ for } m_1, m_2, \dots, m_r \in M.$$

Equivalently, a *convex rational polyhedral cone* σ can be described by $\sigma = \{ \sum_i r_i \alpha_i \mid r_i \geq 0 \}$, for some fixed $\alpha_1, \alpha_2, \dots, \alpha_k \in N$. The same definitions apply for $M_{\mathbf{R}}$ in place of $N_{\mathbf{R}}$.

The cone in $M_{\mathbf{R}}$ dual to $\sigma \subset N_{\mathbf{R}}$ is defined by

$$\check{\sigma} = \{ x \in M_{\mathbf{R}} : \langle x, y \rangle \geq 0 \text{ for all } y \in \sigma \}.$$

The dimension of a cone is, by definition, the dimension of the vector space spanned by it; this dimension will be denoted by $\dim_{\mathbf{R}}$.

In the sequel, only convex rational polyhedral cones in $N_{\mathbf{R}}$ which do not contain any line will be considered. They will be simply called cones. The dual to such a cone is of dimension n . More precisely, if σ is a cone in $N_{\mathbf{R}}$, of codimension r , then $\check{\sigma}$ is the product of a linear space of dimension r and a cone of dimension $n-r$, not containing any line.

A *face* τ of a cone σ is a subset of σ which can be defined by an equation

$$\tau = \{ x \in \sigma : \langle x, m \rangle = 0, \text{ for some } m \in \check{\sigma} \}.$$

A face is itself a cone, and a face of a face of σ is itself a face of σ . Every cone of dimension d contains faces of dimension $0, 1, 2, \dots, d$ and a finite number of them.

Let τ be a face of the cone σ . Define

$$\tau^{\perp} = \{ x \in \check{\sigma} : \langle x, y \rangle = 0 \text{ for all } y \in \tau \}.$$

The face τ^{\perp} of $\check{\sigma}$ is said to be *orthogonal* to τ .

1.2.2. There is a useful construction concerning convex rational polyhedral cones (see [Dan], ch. 1, §1). If σ is a cone in $M_{\mathbf{R}}$ and τ is a face of σ , then $\sigma - \tau = \{ m_1 - m_2 : m_1 \in \sigma, m_2 \in \tau \}$ is the smallest cone containing σ and the linear hull of τ (see Fig. 1). In particular $\sigma - \sigma$ is the linear hull of the cone σ .

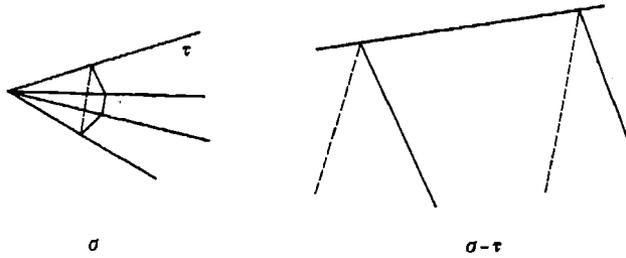


Fig. 1

1.2.3. A complex of cones in $N_{\mathbb{R}}$ is a set Σ of cones in $N_{\mathbb{R}}$ such that

- 1) if $\sigma \in \Sigma$ and τ is a face of σ then $\tau \in \Sigma$,
- 2) if $\sigma_1 \in \Sigma$, $\sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2 .

Let k be a fixed field, algebraically closed. We consider the group algebra $k[M]$ of the lattice M ; the algebra is isomorphic to $k[X_1, X_2, \dots, X_n, (X_1 \dots X_n)^{-1}]$. Given a complex of cones Σ in $N_{\mathbb{R}}$, we associate with it a set of subalgebras of $k[M]$. Namely, to each cone $\sigma \in \Sigma$ we attach the semi-group $M \cap \sigma$ and the subalgebra $A_{\sigma} = k[M \cap \sigma]$ of $k[M]$ spanned by $M \cap \sigma$.

1.2.4. THEOREM. Given a complex of cones Σ in $N_{\mathbb{R}}$, the corresponding set $\{A_{\sigma}\}_{\sigma \in \Sigma}$ of subalgebras of $k[M]$ determines, in the sense of 1.1.1, a k -algebraic scheme denoted by X_{Σ} . The set $\{\text{Spec } A_{\sigma}\}_{\sigma \in \Sigma}$ is an open affine covering of X_{Σ} (see [TE] ch. 1, §§ 1, 2 for details and proof).

1.2.5. An n -dimensional algebraic torus is by definition the group-scheme $T = G_m \times \dots \times G_m$, n times, where G_m is the multiplicative group having $\text{Spec } k[X, 1/X]$ as the underlying scheme. Hence the affine algebra of T is $k[X_1, X_2, \dots, X_n, (X_1 X_2 \dots X_n)^{-1}]$ and it will be in the sequel identified with the algebra $k[M]$. Hence $T = \text{Spec } k[M]$.

By Theorem 1.2.4, and since $\sigma = \{0\}$ is always a cone of Σ , we get that T is an open subscheme of the integral scheme X_{Σ} defined in 1.2.4.

1.2.6. DEFINITION. The k -scheme X_{Σ} of Theorem 1.2.4 together with the inclusion $T \hookrightarrow X_{\Sigma}$ is called the *torus embedding* associated with the complex of cones Σ .

If σ is a cone, then $\text{Spec } A_{\sigma}$ is a torus embedding associated with the complex of all faces of σ . This scheme is denoted by X_{σ} .

From now on, a *variety* means an algebraic separable reduced k -scheme. Point means a closed point.⁽¹⁾ A variety will be often identified with the set of its closed points.

1.2.7. Properties of X_{Σ} : X_{Σ} is an integral normal Cohen–Macaulay variety of dimension n (equal to the dimension of the lattice). It is rational (since it contains $T \cong k^* \times \dots \times k^*$ as an open subset).

⁽¹⁾ except for comment 2° added in proof.

Nonsingularity being a local question, it may be determined on the affine parts of X_{Σ} .

1.2.8. PROPOSITION. *Let σ be a cone. X_{σ} is non-singular if and only if $N \cap \sigma$ is generated by a subset of a basis of the lattice. In such a case, X_{σ} is of the form $A^d \times (A - \{0\})^{n-d}$, where $d = \dim_{\mathbb{R}} \sigma$ and $A = A^1$ the affine line. In particular, if σ is of dimension n , then X_{σ} is an affine space (see [TE]).*

The cone σ (resp. the complex Σ) is said to be *non-singular* if the variety X_{σ} (resp. X_{Σ}) is non-singular.

1.2.9. The following proposition provides a simple criterion for X_{Σ} to be complete.

PROPOSITION. *X_{Σ} is a complete variety if the union of cones of Σ is the whole space $N_{\mathbb{R}}$.*

Such a complex is called *complete*. The torus T acts on itself by translations. The induced action of T on its affine algebra $k[M]$ can be described as follows. An element m of M is a regular function on T (with values in k), which is denoted by \mathcal{X}^m . For a point y of T we shall write \mathcal{Y}_m instead of $\mathcal{X}^m(y)$. In this notation, an element y of T acts on the function \mathcal{X}^m by $y \cdot \mathcal{X}^m = y^m \mathcal{X}^m$. This action may be extended to the whole algebra $k[M]$ by linearity. For all $\sigma \in \Sigma$ the subalgebra A_{σ} of $k[M]$ is stable with respect to the action described above, hence we get the actions of $y \in T$ on all X_{σ} 's, and those actions coincide on the intersections. Therefore each y acts on the variety X_{Σ} . The mapping

$$T \times X_{\Sigma} \rightarrow X_{\Sigma}$$

obtained in such a way is an algebraic action of T on X_{Σ} .

Now, the basic result (Theorem 1.2.11 below, see also [TE]) concerning this action is that the decomposition of X_{Σ} into T -orbits (orbits with respect to T) is dual to the decomposition of the complex Σ into open faces. To describe this precisely, first consider each $\alpha \in N$ as a linear form $\alpha: M \rightarrow \mathbb{Z}$. It can be extended to a homomorphism of algebras $k[M] \rightarrow k[\mathbb{Z}]$ and hence induces the morphism

$$G_m = \text{Spec } k[\mathbb{Z}] \xrightarrow{\lambda_{\alpha}} \text{Spec } k[M] = T.$$

This morphism turns to be a homomorphism of algebraic groups. Generally, group homomorphisms $G_m \rightarrow T$ are called *1-parameter subgroups* (notation: 1-p.s.) of T . It follows that $\alpha \leftrightarrow \lambda_{\alpha}$ is an isomorphism: $N \cong$ (the group of 1-p.s. of T). For an element t of G_m , in place of $\lambda_{\alpha}(t)$ we shall write also t^{α} .

From now on, we fix the embedding $G_m \hookrightarrow A^1$, the affine line, so that $A^1 = G_m \cup \{0\}$.

1.2.10. We say that a 1-p.s. $\lambda_\alpha: t \mapsto t^\alpha$ has a limit in X_Σ for $t \rightarrow 0$ if there exists a morphism $\bar{\lambda}_\alpha$ such that the following diagram commutes:

$$\begin{array}{ccc} G_m & \xrightarrow{\lambda_\alpha} & T \\ \downarrow & & \downarrow \\ A^1 & \xrightarrow{\bar{\lambda}_\alpha} & X_\Sigma \end{array}$$

In this case we denote the point $\bar{\lambda}_\alpha(0)$ by $\lim_{t \rightarrow 0} \lambda_\alpha(t)$ or simply O^α . We say that a 1-p.s. λ_α has a limit in X_Σ for $t \rightarrow \infty$ if the 1-p.s. $\lambda_{(-\alpha)}: t \mapsto t^{-\alpha}$ has a limit for $t \rightarrow 0$. We denote this limit by $\lim_{t \rightarrow \infty} \lambda_\alpha(t)$.

1.2.11. THEOREM ([TE]) a) For $\alpha \in N$, the 1-p.s. λ_α has limit in X_Σ for $t \rightarrow 0$ iff $\alpha \in \bigcup_{\sigma \in \Sigma} \sigma$.

b) $O^\alpha = O^\beta$ if and only if α and β belong to the interior of the same face of Σ .

c) In each T -orbit in X_Σ there is exactly one point of the form O^α for some $\alpha \in \bigcup_{\sigma \in \Sigma} \sigma$, i.e. a limit point of a 1-p.s.

d) Denote by \mathcal{O}_τ the orbit of the point O^α , for $\alpha \in \text{int } \tau$. Then $\dim \mathcal{O}_\tau = n - \dim_{\mathbb{R}} \tau$.

e) For $\tau, \sigma \in \Sigma$, the cone τ is a face of σ iff $\mathcal{O}_\sigma \subset \overline{\mathcal{O}_\tau}$, the closure of \mathcal{O}_τ .

1.2.12. COROLLARY. There are one-to-one correspondences

$$(\text{cones of } \Sigma) \cong_{\tau \mapsto \mathcal{O}_\tau} (\text{orbits in } X_\Sigma) \cong_{T \cdot e \mapsto e} (\text{points of the form } O^\alpha).$$

If the cone τ is of dimension 1 and if $0 \neq \alpha \in \tau$, then we shall often write \mathcal{O}_α instead of \mathcal{O}_τ . For an element α of the interior of the face τ let us denote by ε_τ the limit point O^α (which does not depend of the choice of α , according to Theorem 1.2.11), so that $T \cdot \varepsilon_\tau = \mathcal{O}_\tau$. These distinguished points can be characterized as follows. Let σ be a cone.

1.2.13. PROPOSITION. a) A point $x \in X_\sigma$ is of the form O^α for some α iff $x^m = 0$ or 1 for every $m \in M \cap \check{\sigma}$. Namely, let $x = \varepsilon_\tau$. Then

b)

$$x^m = \begin{cases} 1 & \text{if } m \in \tau^\perp \quad (\text{see 1.2.1}) \\ 0 & \text{otherwise} \end{cases}$$

Proof of b): If $m \in M \cap \tau^\perp$, then for a point $\alpha \in N \cap \text{int } \tau$ we have $\langle \alpha, m \rangle = 0$; hence for $t \in G_m$, $(t^\alpha)^m = t^{\langle \alpha, m \rangle} = 1$; hence $x^m = 1$. Otherwise $\langle \alpha, m \rangle > 0$, and so $(t^\alpha)^m$ converges to 0, while $t \rightarrow 0$. Hence $x^m = 0$.

1.2.14. LEMMA. Let x_1, x_2 be points of the variety X_σ and suppose that for all $m \in M \cap \check{\sigma}$, $x_1^m = x_2^m$. Then $x_1 = x_2$.

(A proof follows directly from the definition of X_σ).

1.2.15. LEMMA. *If $x \in X_\sigma$, $y \in T$, $m \in M \cap \bar{\sigma}$ then $(y \cdot x)^m = y^m \cdot x^m$.*

Proof. For a fixed $y \in T$ consider two mappings $X_\sigma \xrightarrow{f} k$, $f(x) = (yx)^m$ and $g(x) = y^m \cdot x^m$. Both f and g are continuous and coincide on T , hence are equal.

1.2.16. PROPOSITION. *Let τ be a cone in Σ , \mathcal{O}_τ the corresponding orbit in X_Σ and ε_τ the distinguished point of this orbit (1.2.12). Then the stabilizer T_τ of ε_τ , which coincides with the stabilizer of \mathcal{O}_τ , is the subgroup of T described by the system of equations $y^m = 1$ for all $m \in M \cap \tau^\perp$.*

Proof. For a fixed $y \in T$ consider two mappings $X_\sigma \xrightarrow{f} k$, $f(x) = (yx)^m$ equivalent to the system of equalities $(y\varepsilon_\tau)^m = (\varepsilon_\tau)^m$ for all $m \in M \cap \bar{\sigma}$, by 1.2.14. If $m \notin M \cap \tau^\perp$, then $\varepsilon_\tau^m = 0$ and the equality holds. Otherwise, if $m \in M \cap \tau^\perp$, then $(y\varepsilon_\tau)^m = \varepsilon_\tau^m$ holds if and only if $y^m = 1$ (use Lemma 1.2.15).

1.2.17. COROLLARY. *The group T/T_τ acts faithfully on the orbit \mathcal{O}_τ and $\dim T_\tau = \dim_{\mathbb{R}} \tau$.*

1.3. Jets of 1-p.s. at 0

1.3.1. It might be of interest to generalize the statement b) of Theorem 1.2.11 by considering jets of 1-p.s. in place of limits. Theorem 1.3.3 below is an attempt towards such a generalization. If a 1-p.s. $\lambda_\alpha: G_m \rightarrow T$ has a limit in X_Σ for $t \rightarrow 0$, then by the r -jet of λ_α at $t = 0$ we mean the composed morphism

$$\text{Spec } k[t]/(t^{r+1}) \xrightarrow{j_r} \text{Spec } k[t] = A^1 \xrightarrow{\lambda_\alpha} X_\Sigma$$

where j_r is induced by the canonical epimorphism.

Hence the 0-jet of λ_α at 0 is \mathcal{O}^α , the limit point, and the 1-jet is the tangent vector to $\lambda_\alpha(t)$ at $t = 0$. Suppose that $\text{char } k = 0$. If we embed an affine neighbourhood of \mathcal{O}^α in an affine space k^s , then the r -jet of λ_α at 0 can be identified with the sum of the first r terms of the Taylor expansion of λ_α at $t = 0$: $\lambda_\alpha(0) + \lambda'_\alpha(0)t + \dots + \frac{1}{r!} \lambda_\alpha^{(r)}(0)t^r$.

1.3.2. Let us restrict our considerations to the affine case X_σ , where σ is a cone of dimension n . For $r \geq 0$ we shall say that the elements α and β of $N \cap \sigma$ are r -equivalent if the r -jets of 1-p.s. λ_α and λ_β in 0 coincide. Of course, if α and β are r -equivalent, they are also $(r-1)$ -equivalent. We assume that all elements of $N \cap \sigma$ are (-1) -equivalent. We state (proof omitted):

1.3.3. THEOREM. *Let m_1, m_2, \dots, m_s be generators of the semigroup $M \cap \bar{\sigma}$, let $\alpha \in N \cap \sigma$ and $r = -1, 0, 1, 2, \dots$. By choosing suitable numeration,*

one can assume that the element $x = \alpha$ satisfies the system of equations and inequalities

$$\begin{aligned} \langle x, m_1 \rangle &\geq r+1, \\ \dots \dots \dots \\ \langle x, m_k \rangle &\geq r+1, \\ \langle x, m_{k+1} \rangle &= c_{k+1}, \\ \dots \dots \dots \\ \langle x, m_s \rangle &= c_s, \end{aligned}$$

where c_{k+1}, \dots, c_s are some integers $\leq r$.

Let P be the polyhedron in $N_{\mathbb{R}}$ described by the above system. Then

- a) the element β of $N \cap \sigma$ is r -equivalent to α iff $\beta \in P$,
- b) elements β_1 and β_2 of $P \cap N$ are $(r+1)$ -equivalent iff they belong to the interior of the same face of the polyhedron P .

Conjecture.⁽¹⁾ The polyhedron P defined in the above theorem is the convex hull of the r -equivalence class of α . That is to say, $P = \text{convex}(P \cap N)$.

Assuming the conjecture we should have the following description of r -equivalence classes (r -e.c. for short) in the semi-group $N \cap \sigma$. The (-1) -e.c. is the whole of $N \cap \sigma$. The 0 -e.c. have the form $K = N \cap \text{int } \tau$ where τ are the faces of σ (this is precisely statement b) of Theorem 1.2.11). Given a 0 -e.c. K , we take the polyhedron $\text{convex}(K)$, decompose it into interiors of faces $\text{int } F$, and then the sets $N \cap \text{int } F$ are 1 -e.c., etc. for $r = 2, 3 \dots$

1.3.4. Examples of decomposition into r -equivalence classes:

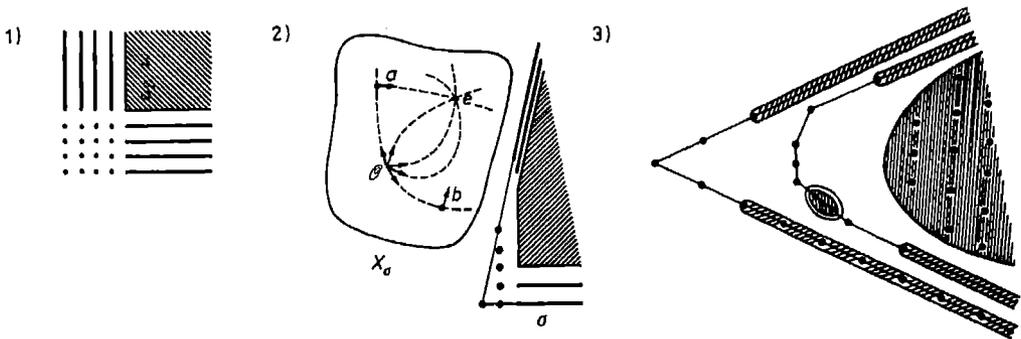


Fig. 2. 1) $\sigma = \langle e_1, e_2 \rangle$, $r = 3$; there are $(r+2)^2$ of r -equivalence classes.

Fig. 2. 2) $\sigma = \langle e_1, e_1 + 4e_2 \rangle$, $r = 1$; there are 11 of 1-e.c. (i.e. tangent vectors) on the surface X . The 0-vectors in the points $0, a, b, c$, are not drawn.

Fig. 2. 3) $\sigma = \langle e_1, 4e_1 + 2e_2 \rangle$, $r = 1$; 14 of 1-e.c. (symbolic picture). Notice the vector tangent to precisely three of 1-p.s.!

⁽¹⁾ Added in proof: The conjecture is true for surfaces only.

1.4. An application of torus embeddings. Desingularization of plane cusps by blowings up of the plane (compare Hirzebruch [H])

1.4.1. Consider the case of $n = 2$. We identify T with $k^* \times k^* \subset k^1 \times k^1$. Suppose r and s are two coprime positive integers and consider the 1-parameter subgroup $\lambda: t \mapsto (t^r, t^s)$ in T . Then $\lambda(t)$ converges to the point $(0, 0)$ when $t \rightarrow 0$, and the closure of $\lambda(k^*)$ is the curve C with the equation $x^s = y^r$ in k^2 . Our aim is to describe, using methods of torus embeddings, a sequence of blowing up's of the plane k^2 , which desingularize the curve C . The idea is the following: Under the blowing up $X \rightarrow k^2$ with center at the origin, the proper inverse image of C is, speaking unprecisely, the same 1-p.s. λ of T .

1.4.2. THEOREM (Hirzebruch). *Let C be the plane curve $(t^r, t^s)_{t \in k}$, $r > s \geq 1$, $\theta = 0 \times k^1$ be the line strictly tangent to C in the origin (see [W]) and*

$$r/s = a_m + \frac{1}{a_{m-1} + \frac{1}{\dots \frac{1}{a_1 + \frac{1}{a_0}}}}$$

$a_0 \geq 2, a_1, \dots, a_m \geq 1$, the expansion into continued fraction. Then there exists a sequence of blowing up's $g: X_s \rightarrow X_{s-1} \rightarrow \dots \rightarrow X_0 = k^2$ such that the proper inverse image C' of C (that is to say, the closure of $g^{-1}(C \setminus (0, 0))$) in X_s is the affine line, and the exceptional curve $g^{-1}((0, 0))$ consists of a chain of projective lines intersecting transversally, with the self-intersection numbers (abbrev. s.i.n.) and the inverse image θ' of θ indicated on the graph (Fig. 3).

The above sequence of blowing up's is the minimal desingularizing the curve C .

1.4.3. Remark. Since the sequence a_0, a_1, \dots, a_m depends only on the ratio r/s , all the numbers occurring in the graph depend only on the curve C and not on its particular parametrization (t^r, t^s) .

Before proving, Theorem 1.4.2 will be translated into the language of complexes of cones (Thm. 1.4.4). Let $\varepsilon_1, \varepsilon_2$ be a basis of the lattice N and σ the cone spanned by $\varepsilon_1, \varepsilon_2$. We can make the following identifications:

$$T = k^* \times k^*, \quad X_\sigma = k^1 \times k^1, \quad G_m = k^*,$$

for $\alpha = r\varepsilon_1 + s\varepsilon_2$ and $t \in k^*$ there is $t^\alpha = (t^r, t^s)$. If σ' is a cone generated by a basis $\varepsilon'_1, \varepsilon'_2$ of N , that is to say, if $X_{\sigma'}$ is an affine plane (see [TE], Thm. 4), then the blowing up with center at the origin is a torus embedding corresponding to the polyhedral cone defined by division of σ' by the half-line $l = \mathbf{R}^{\geq 0} \cdot (\varepsilon'_1 + \varepsilon'_2)$. We shall refer to it as the elementary division.

Observe that the curve $(t^r, t^s)_{t \in k}$ is non-singular iff r divides s or

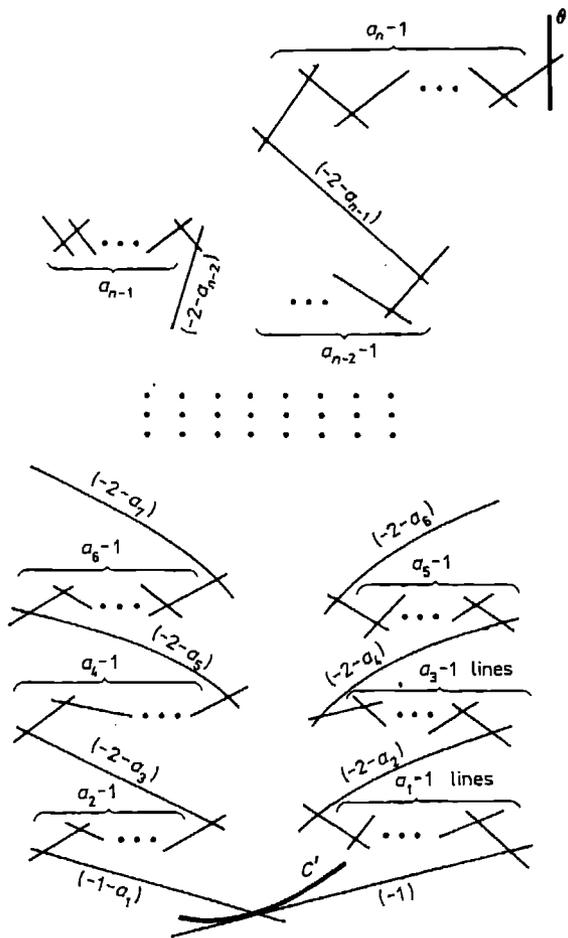


Fig. 3. The self-intersection numbers are in (-). The lines without indications have self-intersection numbers (-2). θ' is the proper inverse image of the tangent θ to C .

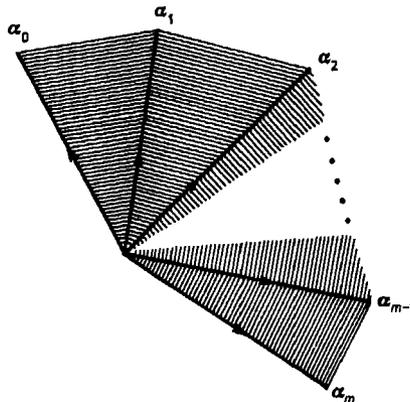


Fig. 4

vice-versa. Finally the self intersection numbers can be calculated as follows. Consider the plane complex of cones as shown in Figure 4, where the half-lines are spanned subsequently by primitive elements $\alpha_0, \alpha_1, \dots, \alpha_m$ of N . Suppose that the complex is non-singular (1.2.8). Then the elements $\alpha_{i-1} + \alpha_{i+1}$ and α_i are proportional for all i . Let D_i be the closure of the orbit corresponding to the cone $\alpha_i R^{\geq 0}$. Then the self-intersection number

$$D_i \cdot D_i = -\frac{\alpha_{i-1} + \alpha_{i+1}}{\alpha_i} \quad ([TEA], \text{ Prop. 6.7}).$$

1.4.4. THEOREM. Let $p = r\varepsilon_1 + s\varepsilon_2$ be a point of the non-singular cone σ spanned by ε_1 and ε_2 , $r > s \geq 1$. There exists a sequence of non-singular complexes $\Sigma_0 = \{\text{the set of faces of } \sigma\}$, $\Sigma_1, \Sigma_2, \dots, \Sigma_s$ such that Σ_i is obtained from Σ_{i-1} by elementary division of a cone of Σ_{i-1} , with the following property: p belongs to the interior of a 2-dimensional face of Σ_s spanned by f_1, f_2 and $p = r'f_1 + s'f_2$ where $r'|s'$ or $s'|r'$.

If the sequence $\Sigma_1, \Sigma_2, \dots, \Sigma_s$ is minimal with the above property, and $\varepsilon_2, \alpha_1, \dots, \alpha_s, \varepsilon_1$ are the primitive elements of the successive 1-dimensional faces of Σ_s , then the s.i. n.'s of the faces $\alpha_i R^{\geq 0}$ are the successive s.i. n.'s of the graph (beginning with the projective line intersecting C'). (See Figure 5.)

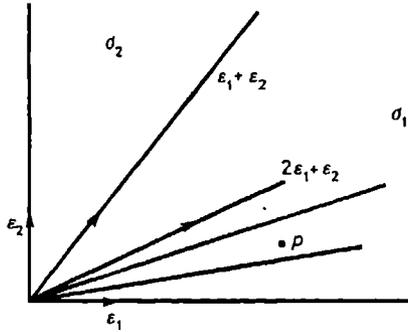


Fig. 5

Proof (by recurrency). a) Let $m = 1$, $a_0 \geq 2$, $a_1 = 1$, that is to say, $r/s = 1 + 1/a_0$. In this case neither $s|r$ nor $r|s$, but after a single elementary division of σ we get a complex Σ_1 consisting of 2-dimensional simplices σ_1 and σ_2 where σ_1 is spanned by $\varepsilon_1, \varepsilon_1 + \varepsilon_2$, and in this basis, p has the form $p = (r-s)\varepsilon_1 + s(\varepsilon_1 + \varepsilon_2)$ and $(r-s)|s$ as required. s.i.n. of the half-line spanned by $\varepsilon_1 + \varepsilon_2$ is $-\frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = -1$.

b) Suppose the theorem is true for some m , arbitrary a_0, a_1, \dots, a_{m-1} and $a_m < i$ for some $i > 1$. We shall prove it for $a_m = i$. As in the previous part, after elementary division of σ , the point p belongs to the cone σ_1 and, in the basis spanning σ_1 , it has the form

$$p = (r-s)\varepsilon_1 + s(\varepsilon_1 + \varepsilon_2).$$

Now, $r/s = a_m + \dots > 2$, hence $r - s > s$. We shall apply the recurrency assumption to σ_1 and p . We have

$$\frac{r-s}{s} = (a_m - 1) + \frac{1}{\dots \frac{1}{a_1 + \frac{1}{a_0}}}$$

and $a_m - 1 < i$, so that the assumption is satisfied.

The sequence of half-lines "desingularizing" p in σ is obtained from the sequence desingularizing p in σ_1 by adjoining the half-line $(\varepsilon_1 + \varepsilon_2) \mathbf{R}^{\geq 0}$ at the beginning. The s.i.n. of the adjoined line is $-\frac{(2\varepsilon_1 + \varepsilon_2) + \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = -2$. This proves that, under the previous supposition, the theorem holds for $a_m = i$.

c) Now suppose it holds from $m = i - 1$ and arbitrary a_0, a_1, \dots, a_{m-1} . We are to prove that the theorem is true for $m = i$, arbitrary a_0, \dots, a_{m-1} and $a_m = 1$. We have

$$r/s = 1 + \frac{1}{a_{m-1} + \frac{1}{a_{m-2} + \dots}}$$

As before, apply the elementary division of σ . Now the coordinates of p in the basis of N spanning σ_1 are $(s, r - s)$ where $s > r - s$ and

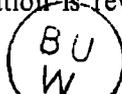
$$\frac{s}{r-s} = a_{m-1} + \frac{1}{a_{m-2} + \dots}$$

Interchanging the roles of r and s we may apply the recurrency assumption to the cone σ_1 and the point p . It remains to calculate the s.i.n. of the adjoined half-line $(\varepsilon_1 + \varepsilon_2) \mathbf{R}^{\geq 0}$. This s.i.n. depends on the superior half-line l dividing σ_1 . To find l we restrict ourselves to the cone σ_1 . Reversing orientation we have the basis φ_1, φ_2 of σ_1 , where $\varphi_1 = \varepsilon_1 + \varepsilon_2, \varphi_2 = \varepsilon_1$. The point p has the form $p = r' \varphi_1 + s' \varphi_2$, where $r' = s, s' = r - s$, hence

$$r'/s' = a_{m-1} + \frac{1}{a_{m-2} + \dots}$$

We may suppose $m > 1$. The sequence of half-lines dividing σ_1 during the process of desingularization of p in σ_1 begins with $(\varphi_1 + \varphi_2) \mathbf{R}^{\geq 0}, (2\varphi_1 + \varphi_2) \mathbf{R}^{\geq 0}, \dots, (j\varphi_1 + \varphi_2) \mathbf{R}^{\geq 0}, \dots$ until $j/1 \geq r'/s'$. It results that $j = a_{m-1} + 1$.

The half-line spanned by $(a_{m-1} + 1)\varphi_1 + \varphi_2$ is therefore the lowest in the division of σ_1 , while the orientation is reversed. Hence this is just the half-



line l . Now, the half-lines the closest to $\varphi_1 = \varepsilon_1 + \varepsilon_2$ are l and $\varepsilon_2 \cdot \mathbf{R}^{\geq 0}$, and we have

$$(\text{s.i.n. of } (\varepsilon_1 + \varepsilon_2) \mathbf{R}^{\geq 0}) = -\frac{(a_{m-1} + 1)(\varepsilon_1 + \varepsilon_2) + \varepsilon_1 + \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = -2.$$

This concludes induction.

The remaining parts of the theorem are obvious from the construction of the division of σ .

1.5. Some G_m -actions on torus embedding

Let Σ be a complex of cones, X_Σ the associated torus embedding and $\alpha \in N$. Then the 1-p.s. $\lambda_\alpha: G_m \rightarrow T$ induces an action of G_m on X_Σ , of the form $(t, x) \mapsto t^\alpha \cdot x$. Therefore to any pair (Σ, α) consisting of a complex of cones and of an element of N there corresponds a variety equipped with a G_m -action.

In this way one can obtain interesting examples of non-projective varieties equipped with a G_m -action (see 3.3, and [K]). In [J₁] the following lemma is proved ⁽¹⁾ about G_m -fixed points and invariant curves in X_Σ .

1.5.1. LEMMA. *Let σ be an n -dimensional cone in $N_{\mathbf{R}}$ and τ its $(n-1)$ -face.*

- a) *If $\alpha \in \tau - \tau$ then all the points of the curve $\bar{\mathcal{O}}_\tau$ are fixed by G_m . If $\alpha \notin \tau - \tau$, then \mathcal{O}_τ is a G_m -orbit and then:*
- b) *if $\alpha \in \sigma - \tau$ then the fixed point \mathcal{O}_σ is the limit of the G_m -orbit \mathcal{O}_τ in 0,*
- c) *if $\alpha \notin \sigma - \tau$, then the fixed point \mathcal{O}_σ is the limit of the orbit \mathcal{O}_τ in ∞ .*

This lemma may be generalized as follows.

1.5.2. DEFINITION. Let X be a variety with a G_m -action $(t, x) \mapsto t \cdot x$, F a set consisting of the fixed points of X . We say that a point $x \in X$ is *repelled* (resp. *attracted*) by F , if a point of F is the limit of the orbit of x , for $t \rightarrow 0$ (resp. for $t \rightarrow \infty$) (see 1.2.10).

1.5.3. THEOREM (see also [K]). *Let σ be a cone in $N_{\mathbf{R}}$, α an element of N and consider the induced G_m -action on X_σ . Then*

- a) *The G_m -fixed points form a set in X_σ which is the union of closures of T -orbits $\bar{\mathcal{O}}_\tau$, where τ are faces of σ , such that $\alpha \in \tau - \tau$.*
- b) *Let τ be a face such that $\alpha \in \tau - \tau$. Then the set of points repelled by the set $\bar{\mathcal{O}}_\tau$ is the union of T -orbits \mathcal{O}_ω such that*
 - 1) ω is a face of τ ,
 - 2) $\alpha \in \text{int}(\tau - \omega)$.

⁽¹⁾ With a misprint in the proof. It should read $R = \bigoplus_{r \geq \text{ton } \sigma} \mathfrak{R}^r$.

The description of the set of attracted points is obtained by considering $(-\alpha)$ in place of α . The proof is based on the following remark: If τ is a face of $\sigma \subset N_{\mathbb{R}}$ and τ^\perp the face of $\check{\sigma}$ orthogonal to τ , then the spaces $N_{\mathbb{R}}/(\tau-\tau)$ and $\tau^\perp-\tau^\perp$ are dual to each other, and the cone $\sigma' = (\sigma-\tau)/(\tau-\tau)$ in $N_{\mathbb{R}}/(\tau-\tau)$ is dual to $\tau^\perp \subset \tau^\perp-\tau^\perp$. Now, considering a suitable quotient torus T' of T one finds that the embedding of T' corresponding to the cone σ' coincide with the closure of the T -orbit \mathcal{O}_τ in X_σ . This allows us to reduce the proof to the case of $\alpha \in \text{int } \sigma$: hence the unique face τ satisfying the condition $\alpha \in \tau-\tau$ is σ itself. All faces occur therefore as ω and the proof in that case is obvious.

1.5.4. In [B-B₁] and [B-B₂] A. Białynicki-Birula considers the following decomposition of a non-singular complete variety X with G_m -action: The fixed point set F of X is non-singular, closed, non-connected in general, $F = F_1 \cup F_2 \cup \dots \cup F_r$, where F_i are the (irreducible) components. If X_i is the set of points repelled by F_i , $i = 1, 2, \dots, r$ then X is the disjoint union $X = \bigcup_{i=1}^r X_i$ of locally closed subvarieties X_i and each X_i is a G_m -invariant algebraic bundle over F_i , with affine spaces as the fibres ([B-B₁], § 4). The statement about the decomposition into the "attracted" parts is similar.

Theorem 1.5.3 gives a description of the above decomposition in case of a variety of the form X_Σ (which may be singular) with a G_m -action defined by a 1-parameter subgroup of T .

Consider the particular case of complex Σ in $N_{\mathbb{R}}$ and $\alpha \in N$ such that α is not contained in any linear hull $\tau-\tau$ of an $(n-1)$ -dimensional cone τ of Σ . Then the G_m -fixed points of X_Σ coincide with the T -fixed points.

2. COMPLEX TORUS EMBEDDINGS. REAL AND NON-NEGATIVE PARTS

2.1. Introduction

In the case of the complex ground field $k = \mathbb{C}$, there exists an additional structure in the torus embeddings. It allows us, in particular, to distinguish the real non-negative part $X_{\Sigma}^{\geq 0}$ defined by F. Ehlers in the non-singular case (see below for definitions). One obtains in this way a real variety with singular corners $X_{\Sigma}^{\geq 0}$, which has some handy properties. First, $X_{\Sigma}^{\geq 0}$ can be identified with the quotient of X_{Σ} by a topological torus $c-T$, acting in a natural way on X_{Σ} . On the other hand, in the affine case, $X_{\Sigma}^{\geq 0}$ can be identified with the cone $\check{\sigma}$ dual to σ , and in the projective case, $X_{\Sigma}^{\geq 0}$ can be identified with some polyhedron Δ in $N_{\mathbb{R}}$, composed of translated cones dual to the cones of Σ . A simple example of such a polyhedron can be constructed

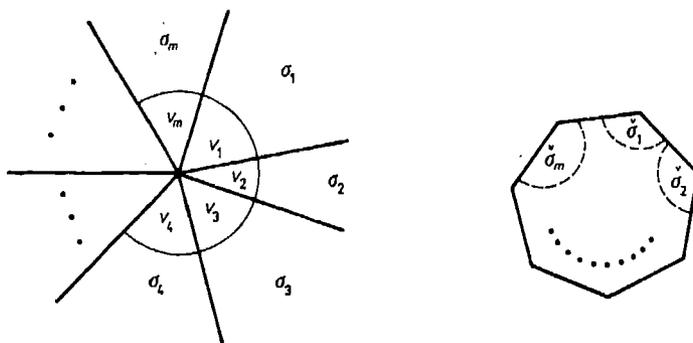


Fig. 6

as follows. Let $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_m, \tau_1, \tau_2, \dots, \tau_m, 0\}$ be a 2-dimensional complete complex in \mathbb{R}^2 (2.2.8). Then the sum of angles v_1, v_2, \dots, v_m occurring in Σ is 2π . If one considers the usual scalar product in \mathbb{R}^2 , then the dual cones to σ_j can be described as having the angles $\pi - v_1, \pi - v_2, \dots, \pi - v_m$. Therefore they can be glued together and form a polyhedron. Later on, we will see how this can be generalized to higher dimensions. It follows from the previous remarks that both for affine and projective torus

embeddings there are natural mappings onto polyhedra in \mathbf{R}^n and on the other hand, $X_{\mathcal{Z}}$ is a variety defined over \mathbf{Z} and therefore its real part $X_{\mathcal{Z}}(\mathbf{R})$ is well defined. It is a real algebraic variety with singularities and consists of 2^n copies of $X_{\mathcal{Z}}^{\geq 0}$, glued together along their faces. Taking into account the previous remarks one obtains that, at least in the projective case, $X_{\mathcal{Z}}(\mathbf{R})$ can be obtained by glueing 2^n polyhedra. In chapter 4 we use this fact to calculate homologies of $X_{\mathcal{Z}}(\mathbf{R})$.

Both $X_{\mathcal{Z}}^{\geq 0}$ and $X_{\mathcal{Z}}(\mathbf{R})$ are equipped with actions of suitable subgroups of the torus T . For example, $X_{\mathcal{Z}}^{\geq 0}$, under the action of $T^{>0}$, is decomposed into orbits which are precisely the real non-negative parts of the T -orbits in $X_{\mathcal{Z}}$. On the other hand, in the affine and projective case, under the identification of $X_{\mathcal{Z}}^{\geq 0}$ with a suitable polyhedron as mentioned above, the decomposition of $X_{\mathcal{Z}}^{\geq 0}$ into orbits coincides with the decomposition of the corresponding polyhedron into "open faces". Therefore many facts concerning the variety $X_{\mathcal{Z}}$ and its orbits can be illustrated and studied on the corresponding polyhedron in \mathbf{R}^n (the dual cone σ in the affine case X_{σ}).

2.2. The real non-negative part of the variety $X_{\mathcal{Z}}$

2.2.1. Let $k = \mathbf{C}$. Through this section we refer to the ordinary (non-Zariski) topology of varieties. Consider the decomposition of the multiplicative group of complex numbers into the product of two subgroups, the compact factor and the non-negative (radial) factor:

$$\mathbf{C}^* = \{z: |z| = 1\} \times \mathbf{R}^{>0}.$$

The first factor is also denoted by $U(1)$. Suppose the decomposition $T = \mathbf{C}^* \times \dots \times \mathbf{C}^*$ is given; then T decomposes into

$$\{(z_1, \dots, z_n): \forall_i |z_i| = 1\} \times \{(z_1, \dots, z_n): \forall_i z_i \in \mathbf{R}^{>0}\}.$$

This decomposition may be done intrinsically. Let $c-T = \{x \in T: |x^m| = 1 \text{ for all } m \in M\}$ ("compact torus" — see [SC]) and $T^{>0} = \{x \in T: x^m \in \mathbf{R}^{>0} \text{ for all } m \in M\}$ ("radial factor"). The compact torus coincides with the first factor of the above decomposition. Indeed, if $x = (z_1, z_2, \dots, z_n)$, $m = (m_1, m_2, \dots, m_n)$ then $|x^m| = |z_1|^{m_1} \dots |z_n|^{m_n}$. If $m = (0, 0, \dots, 1, \dots, 0)$, 1 on i -th place, for $i = 1, 2, \dots, n$, then $|x^m| = |z_i|$. Hence if $|x^m| = 1$ for every m , then $|z_i| = 1$ and the opposite implication is obvious. The group $T^{>0}$ coincides with the second factor of the decomposition, the proof is similar. Hence $T = (c-T) \times T^{>0}$.

The subgroup $(c-T)$ of T may be characterized also as the closure in T of the torsion subgroup of T . The projection onto the first factor is denoted by $y \mapsto \arg y$, and onto the second factor by $y \mapsto |y|$. Therefore $y = \arg y \cdot |y|$.

2.2.1.1. Let $\alpha \in \mathbf{N}$. If $t \in \mathbf{C}^*$, $|t| = 1$, then $|t^\alpha| = 1$ (the neutral element of T), and hence $t^\alpha \in (c - T)$. If now $t \in \mathbf{R}^{>0}$, then $t^\alpha \in T^{>0}$. It follows that both characters and 1-p.s. in T can be decomposed into their “compact” and “radial” factors. The sequence

$$\mathbf{C}^* \xrightarrow{\lambda_\alpha} T \xrightarrow{\mathcal{A}^m} \mathbf{C}^*$$

is the product “place by place” of two sequences

$$U(1) \rightarrow (c - T) \rightarrow U(1)$$

and

$$\mathbf{R}^{>0} \rightarrow T^{>0} \rightarrow \mathbf{R}^{>0}.$$

If $\langle \alpha, m \rangle = n$, then the first composed homomorphism is the n -fold wrapping, and the second, the n -th power.

2.2.1.2. The above decomposition induces the decompositions of algebraic subtori and quotient tori of T . Namely, if $T_0 \subset T$ is a subtorus, then $T_1 = T/T_0$ is also a torus and the exact sequence

$$1 \rightarrow T_0 \rightarrow T \rightarrow T_1 \rightarrow 1$$

can be decomposed into exact sequences

$$1 \rightarrow (c - T_0) \rightarrow (c - T) \rightarrow (c - T_1) \rightarrow 1$$

and

$$1 \rightarrow T_0^{>0} \rightarrow T^{>0} \rightarrow T_1^{>0} \rightarrow 1$$

(identify T with $\mathbf{C}^* \times \dots \times \mathbf{C}^*$ for the proof).

Let now Σ be a complex of cones and $T \hookrightarrow X_\Sigma$ the corresponding torus embedding. The decomposition of the torus T can be extended to a certain decomposition of X_Σ into a fibred product which will be now described.

2.2.2. DEFINITION. The non-negative part $X_\Sigma^{\geq 0}$ of a torus embedding $T \hookrightarrow X_\Sigma$ is the closure of $T^{>0}$ in X_Σ (it will be denoted also by $\overline{T^{>0}}$, if the embedding is fixed).

In the non-singular case $X_\Sigma^{\geq 0}$ is a real variety with corners (see [B-S]), since locally it has the form $\mathbf{R}^{\geq 0} \times \dots \times \mathbf{R}^{\geq 0} \times \mathbf{R}^{>0} \times \dots \times \mathbf{R}^{>0}$. In the general case we may call $X_\Sigma^{\geq 0}$ a variety with singular corners. More information about the structure of $X_\Sigma^{\geq 0}$ is contained in Theorem 2.3.2. Consider a cone τ of Σ , the corresponding orbit \mathcal{O}_τ and the distinguished point $\varepsilon_\tau \in \mathcal{O}_\tau$. Observe that by 1.2.13 all distinguished points belong to $X_\Sigma^{\geq 0}$.

2.2.3. PROPOSITION. Let $x \in X_\Sigma$. The following statements are equivalent:

- (i) $x \in X_\Sigma^{\geq 0}$,
- (ii) if $\sigma \in \Sigma$, $x \in X_\sigma$ and $m \in M \cap \check{\sigma}$ then $x^m \in \mathbf{R}^{\geq 0}$,
- (iii) $x = y \cdot \varepsilon_\tau$ for some $y \in T^{>0}$ and a cone τ of Σ .

Proof: (i) \Rightarrow (ii). Since \mathcal{A}^m is a continuous function positive in $T^{>0}$, it is non-negative in the closure of $T^{>0}$.

(ii) \Rightarrow (iii). We can restrict the proof to the case of an affine variety X_{σ} . Let \mathcal{O}_{τ} , where τ is a face of σ , be the orbit through x , and ε , the distinguished point of this orbit. Then $x = y \cdot \varepsilon$ for some $y \in T$. By Lemma 1.2.15 there is $x^m = y^m \cdot \varepsilon^m$; hence by Proposition 1.2.13

$$x^m = \begin{cases} y^m & \text{if } m \in \tau^{\perp} \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

But then, in the first case $y^m \in \mathbf{R}^{>0}$. Since $|y|^m = |y^m|$, we get $|y|^m = y^m$. In both cases we get $x^m = |y|^m \cdot \varepsilon^m$ for each $m \in M \cap \check{\sigma}$. It follows by Lemma 1.2.14 that $x = |y| \cdot \varepsilon$. Moreover $|y| \in T^{>0}$.

(iii) \Rightarrow (i). The point ε_{τ} belongs to $X_{\mathcal{F}}^{\geq 0} = \overline{T^{>0}}$, hence it is the limit in 0 of the "radial component" (see 2.2.1.1) of a 1-ps.

$$(\lambda_{\alpha})|_{\mathbf{R}^{>0}}: \mathbf{R}^{>0} \rightarrow T^{>0}, \quad \alpha \in \text{int } \tau.$$

In order to prove that $x \in \overline{T^{>0}}$, consider the continuous translation $X_{\mathcal{F}} \rightarrow X_{\mathcal{F}}$, $x \mapsto y \cdot x$, and notice that $T^{>0}$ is stable with respect to that translation, if $y \in T^{>0}$.

In the non-singular case considered in [E], F. Ehlers gives the following description of $X_{\mathcal{F}}^{\geq 0}$. The variety $X_{\mathcal{F}}$ is the union of r copies of \mathbf{C}^n and the glueing maps have the form $f(z_1, z_2, \dots, z_n) = (z^{p_1}, z^{p_2}, \dots, z^{p_n})$, where $z^{p_i} = z_1^{p_{i1}} \cdot z_2^{p_{i2}} \cdot \dots \cdot z_n^{p_{in}}$, p_{ij} are integers. Then $X_{\mathcal{F}}^{\geq 0}$ is obtained by glueing together r copies of $(\mathbf{R}^{\geq 0})^n$, using the restrictions of the same maps f .

EXAMPLES. 1) $X = \mathbf{P}^2$, the complex projective space, with the embedding of $T = (\mathbf{C}^*)^2$ given by $(z_1, z_2) \mapsto (1 : z_1 : z_2)$. Here $X_{\mathcal{F}}^{\geq 0}$ can be presented in the form of a triangle (Fig. 7).

2) $X = \mathbf{P}^1 \times \mathbf{P}^1$, $T = \mathbf{C}^* \times \mathbf{C}^*$ with the standard embedding. Then $X_{\mathcal{F}}^{\geq 0}$ is a square (Fig. 8).

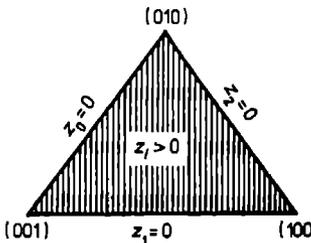


Fig. 7

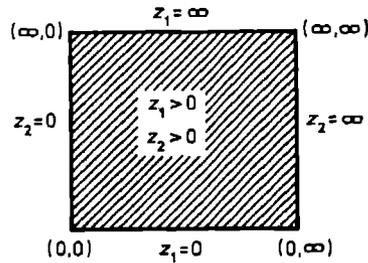


Fig. 8

The set $X_{\mathcal{F}}^{\geq 0}$ is stable under the action of $T^{>0}$ on $X_{\mathcal{F}}$, since for $y \in T^{>0}$, $x \in X_{\mathcal{F}}^{\geq 0}$ and any $m \in M \cap \check{\sigma}$ we have $(y \cdot x)^m = y^m x^m \geq 0$. Now, the decomposition of $X_{\mathcal{F}}^{\geq 0}$ into $T^{>0}$ -orbits essentially coincides with the decomposition of $X_{\mathcal{F}}$ into T -orbits in the following sense.

2.2.4. PROPOSITION. a) The intersections $\mathcal{O}_\tau \cap X_\Sigma^{\geq 0}$, for each τ , are $T^{>0}$ -orbits in $X_\Sigma^{\geq 0}$.

b) The mapping

$$i: \{T\text{-orbits in } X_\Sigma\} \rightarrow \{T^{>0}\text{-orbits in } X_\Sigma^{\geq 0}\}, \quad \mathcal{O} \mapsto \mathcal{O} \cap X_\Sigma^{\geq 0},$$

is bijective.

c) The bijection i preserves stratification, that is to say,

$$\mathcal{O}_\tau \subset \overline{O_\omega} \quad \text{iff} \quad i(O_\tau) \subset \overline{i(O_\omega)}$$

d) $i(O_\tau) = T^{>0} \cdot \varepsilon_\tau$ (compare 1.2.12).

Proof. a) and b) follow Proposition 2.2.3, since both T -orbits in X_Σ and $T^{>0}$ -orbits in $X_\Sigma^{\geq 0}$ contain precisely one distinguished point. c) follows from a) and b); d) is obvious.

By assertion d) and by 2.2.1.2 the T -orbit \mathcal{O}_τ may be decomposed into the compact and radial factors

$$\mathcal{O}_\tau = (c - T_\tau) \cdot \varepsilon_\tau \times T_\tau^{>0} \cdot \varepsilon_\tau$$

which will be denoted by $c - \mathcal{O}_\tau$ and $\mathcal{O}_\tau^{\geq 0}$, respectively. In particular, the compact factor of the dense orbit T is $c - T$ and the radial one is $T^{>0}$.

The intersection of the compact and radial part of an orbit \mathcal{O}_τ is just ε_τ . This distinguished point allows us to identify the compact part with $c - T_\tau$ and the radial with $T_\tau^{>0}$. We have $\dim_{\mathbb{R}}(c - \mathcal{O}_\tau) = \dim_{\mathbb{R}}(\mathcal{O}_\tau^{\geq 0}) = \dim_{\mathbb{R}} \tau$.

The above identifications give rise to the mappings $(c - \mathcal{O}_\tau) \xleftarrow{|\cdot|} \mathcal{O}_\tau \xrightarrow{\text{arg}} \mathcal{O}_\tau^{\geq 0}$. Let us study the properties of the mapping $|\cdot|$. If $x = y\varepsilon_\tau$ for $y \in T$, then $|x| = |y|\varepsilon_\tau$. It follows that for $x \in X_\Sigma$, $m \in M \cap \check{\sigma}$ we have

$$|x^m| = |(y\varepsilon_\tau)^m| = |y^m \varepsilon_\tau^m| = \begin{cases} |y^m| & \text{if } m \in M \cap \tau^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $|x^m| = |x|^m$.

2.2.4.1. We have the decomposition $X_\Sigma^{\geq 0} = \bigcup_{\tau \in \Sigma} \mathcal{O}_\tau^{\geq 0}$ (disjoint union). On

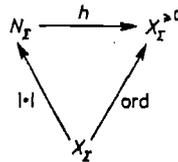
the other hand, $X_\Sigma = \bigcup_{\tau \in \Sigma} (c - \mathcal{O}_\tau) \times \mathcal{O}_\tau^{\geq 0}$. The compact torus acts transitively on the compact factors and trivially on the radial factors, and the radial part of the torus does conversely. It follows that $X_\Sigma^{\geq 0}$ is a fundamental set for the action of $c - T$ on X_Σ , that is to say, each $(c - T)$ -orbit in X_Σ intersects the set $X_\Sigma^{\geq 0}$ at exactly one point. Therefore $X_\Sigma^{\geq 0}$ may be identified in a natural way with the orbit space $N_\Sigma = X_\Sigma / (c - T)$ of X_Σ under the action of $c - T$, introduced in [SC] (see also [TEA] § 10). The projection $X_\Sigma \rightarrow X_\Sigma^{\geq 0}$ defined as the union of the projections $x \mapsto |x|$ on each orbit can be described globally by the following.

2.2.5. PROPOSITION. (cf. [J₃], Prop. 1). *There exists a continuous projection*

$$X_x \rightarrow X_x^{\geq 0}, \quad \tilde{x} \mapsto |x|,$$

such that

- a) $|\cdot|$ is $T^{>0}$ equivariant;
- b) fibres of $|\cdot|$ are precisely the $(c-T)$ -orbits in X_x ;
- c) $|\cdot|$ restricted to the dense orbit T coincides with the projection $T \rightarrow T^{>0}$, $y \mapsto |y|$;
- d) for each $x \in X_x$ and character \mathcal{X} regular in x , $\mathcal{X}(|x|) = |\mathcal{X}(x)|$ (the ordinary absolute value);
- e) for $y \in T$, $|y \cdot x| = |y| \cdot |x|$, in particular;
- f) $|y \cdot e_\tau| = |y| e_\tau$ for each cone τ in Σ ;
- g) (see [1], Ch. 1.1 for the definitions of $N_x = N_{(\sigma)_x}$ and ord). Fix the identification of G_m with C^* . Consider the homomorphism $N = \text{Hom}(G_m, T) \rightarrow T^{>0}$ given by $\lambda \mapsto \lambda(e^{-2\pi})$ and the induced isomorphism $h: N_{\mathbb{R}} = N \otimes \mathbb{R} \cong T^{>0}$. Then h extends to the (unique) homeomorphism $N_x \cong X_x^{\geq 0}$ and the diagram



is commutative.

Under the identification $\text{ord}: T^{>0} \cong N_{\mathbb{R}}$, the actions of $T^{>0}$ on $X_x^{\geq 0}$ and of $N_{\mathbb{R}}$ on N_x coincide.

Proof. Statements a)–g) follows from the previous considerations.

It remains to prove that $x \mapsto |x|$ is a continuous map. We may restrict our considerations to the affine case X_σ . Let $x_0 = \lim_{n \rightarrow \infty} x_n$; then for any $m \in M \cap \check{\sigma}$ we have $x_0^m = \lim_{n \rightarrow \infty} x_n^m$, the convergence of a sequence of complex numbers. Then the usual absolute value fulfils $|x_0^m| = \lim_{n \rightarrow \infty} |x_n^m|$. By the above mentioned properties of $|\cdot|$ we have $|x_0|^m = \lim_{n \rightarrow \infty} |x_n|^m$. We can consider the mappings $x \mapsto x^m$ as the coordinate mappings and then we get that $|x_0| = \lim_{n \rightarrow \infty} |x_n|$.

2.2.6. The fibred product. One can also define the “compact” part of X_x by:

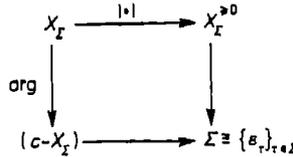
$$(c - X_x) = \bigcup_{\tau \in \Sigma} (c - \mathcal{O}_\tau).$$

Locally, in the affine coordinates the set $c - X_x$ may be characterized by the condition:

$$x \in (c - X_x) \Leftrightarrow \text{for all } m \in M \cap \check{\sigma}: \text{either } x^m = 0 \text{ or } |x^m| = 1.$$

This set, however, is less interesting than the non-negative part $X_{\Sigma}^{\geq 0}$, since $c - X_{\Sigma}$ is a disjoint union of topological tori. If the complex Σ is regarded as the set of its cones, the mapping $i: X_{\Sigma} \rightarrow \Sigma$ sends each point x into the cone τ such that $x \in \mathcal{O}_{\tau}$; then we have the fibred product diagram of sets:

2.2.7.



where the unlabelled arrows are the restrictions of i .

In the affine case there is a description of the variety X_{σ} which uses homomorphisms of semigroups (compare [TEA], Ch. 1). Namely, consider $M \cap \sigma$ as a sub-semigroup of the lattice M and C as a semigroup with respect to multiplication. Then

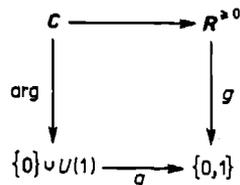
2.2.8.

$$x_{\sigma} \cong \text{Hom}_{C\text{-alg.}}(C[M \cap \sigma], C) \cong \text{Hom}_{\text{s.g.}}(M \cap \sigma, C)$$

(semigroup homomorphisms).

2.2.9. Remark. It follows that the variety X_{σ} (the affine case!) has semigroup structure given by the formula $(h_1 \cdot h_2)(m) = h_1(m) \cdot h_2(m)$ for $h_1, h_2 \in \text{Hom}(M \cap \sigma, C)$. This operation is not so much interesting, nevertheless, it allows us to define the regular mappings $X_{\sigma} \rightarrow X_{\sigma}$ of the form $x \mapsto x^n$, where n is a positive integer. Now, if the general non-affine case is considered, then, though multiplication cannot be defined for two points of distinct affine parts, still the “ n -th power mapping” is well defined all over the variety X_{Σ} . This mapping will be used in Chapter 3.

In order to interpret the non-negative and compact part of X_{σ} and the fibred product diagram above, consider two sub-semigroups in C : $\{0\} \cup U(1)$ and $\mathbb{R}^{\geq 0}$. Their intersection is the sub-semigroup $\{0, 1\}$. Consider the mapping $g: C \rightarrow \{0, 1\}$ given by $g(0) = 0$, $g(C^*) = 1$. We get the fibred product diagram



(N. B. this is a particular case of the preceding diagram, see Fig. 9.1)). Now apply the functor $\text{Hom}_{\text{s.g.}}(M \cap \sigma, -)$ to the above diagram. We get

2.2.10.

$$\begin{array}{ccc}
 \text{Hom}(M \cap \check{\sigma}, \mathbb{C}) & \longrightarrow & \text{Hom}(M \cap \check{\sigma}, \mathbb{R}^{\geq 0}) \\
 \downarrow & & \downarrow \\
 \text{Hom}(M \cap \check{\sigma}, \{0\} \cup U(1)) & \longrightarrow & \text{Hom}(M \cap \check{\sigma}, \{0, 1\})
 \end{array}$$

2.2.11. PROPOSITION. *Diagrams 2.2.7 and 2.2.10 coincide under the identification 2.2.8.*

Proof. $X_\sigma^{\geq 0} = \text{Hom}_{\text{s.g.}}(M \cap \check{\sigma}, \mathbb{R}^{\geq 0})$ by statement (ii) of Proposition 2.2.3. For a similar reason,

$$(c - X_\sigma) = \text{Hom}_{\text{s.g.}}(M \cap \check{\sigma}, \{0\} \cup U(1)), \quad \text{e.t.c.}$$

2.2.12. Examples are presented in Figure 9 (p. 28).

The variety with singular corners $X_\sigma^{\geq 0}$ may also be described as a partial compactification of a real vector space (compare [SC], [TEA]). Consider first the affine case. Fix a basis e_1, e_2, \dots, e_n of M . Denote by \mathcal{X}_i the character \mathcal{X}^{e_i} . Let m_1, m_2, \dots, m_s be a system of generators of the semigroup $M \cap \check{\sigma}$. Then $X_\sigma = \text{Spec } \mathbb{C}[\mathcal{X}^{m_1}, \mathcal{X}^{m_2}, \dots, \mathcal{X}^{m_s}]$ is the closed subvariety in the affine space $A^s = \text{Spec } \mathbb{C}[Y_1, Y_2, \dots, Y_s]$ where Y_i are variables, the inclusion being given by $Y_i \mapsto \mathcal{X}^{m_i}$. On the other hand, we have the canonical embedding

$$T = \text{Spec } \mathbb{C}[\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n, (\mathcal{X}_1 \cdot \mathcal{X}_2 \cdot \dots \cdot \mathcal{X}_n)^{-1}] \hookrightarrow X_\sigma.$$

Identify, via the coordinate functions introduced above, T with $\mathbb{C}^* \times \dots \times \mathbb{C}^*$ and A^s with \mathbb{C}^s . The composed morphism $T \hookrightarrow X_\sigma \hookrightarrow A^s$ then can be written in the form

$$h: \mathbb{C}^* \times \dots \times \mathbb{C}^* \rightarrow \mathbb{C}^s, \quad y = (y_1, y_2, \dots, y_n) \mapsto (y^{m_1}, y^{m_2}, \dots, y^{m_s})$$

and the variety X_σ may be identified with the closure of the image of h , both in Zariski and ordinary topology.

Now, $X_\sigma^{\geq 0}$ is the closure of $T^{>0}$, that is to say, the closure of $h(\mathbb{R}^{>0} \times \dots \times \mathbb{R}^{>0})$ (in ordinary topology). Observe that this closure is contained in the topological space $(\mathbb{R}^{\geq 0})^n$.

Using logarithms we shall now pass to linear spaces and linear mappings. Let the diffeomorphism $\text{Exp}: \mathbb{R}^n \rightarrow (\mathbb{R}^{>0})^n$ be defined by

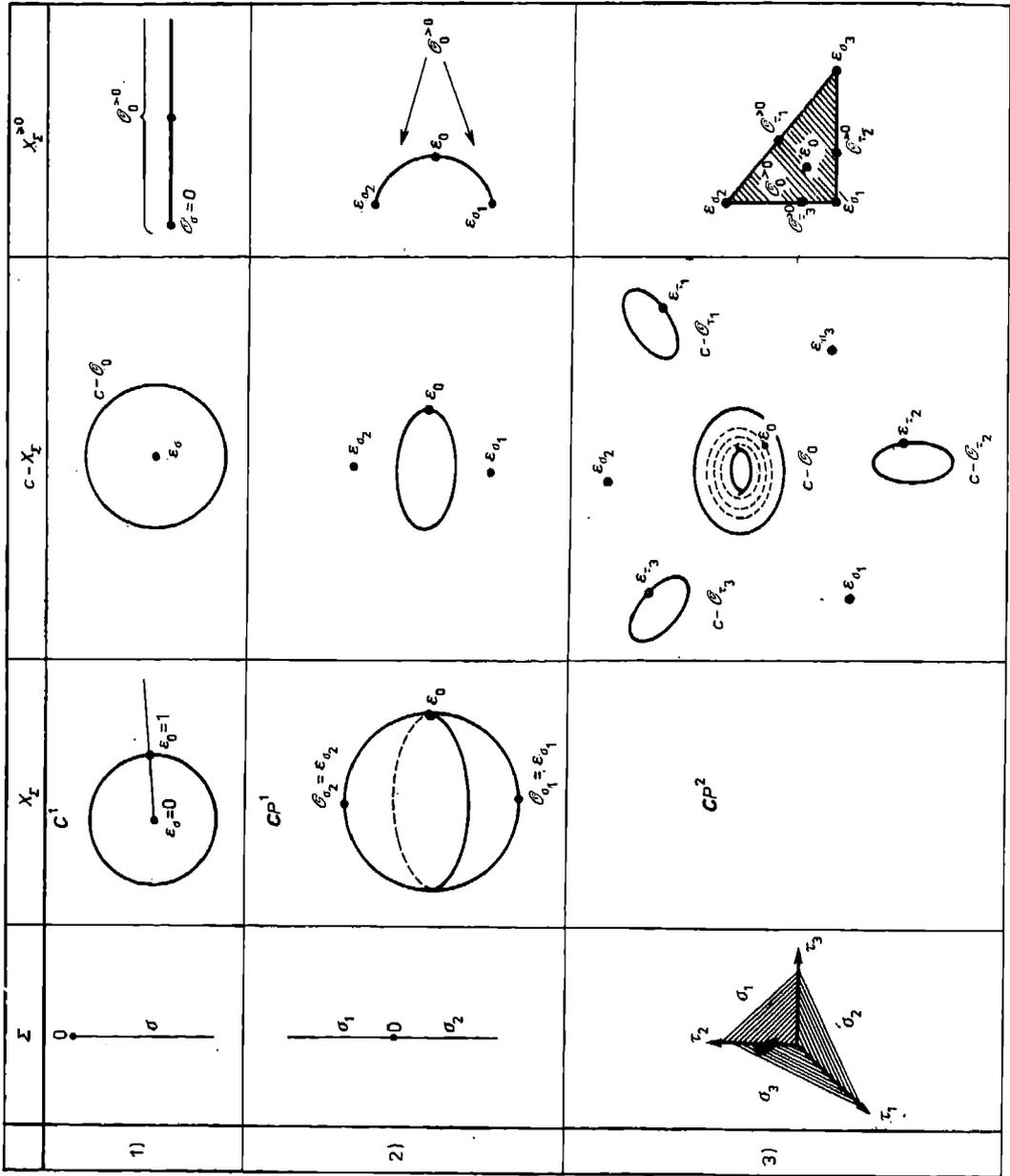


Fig. 9

$(x_1, \dots, x_n) \rightarrow (e^{x_1}, \dots, e^{x_n})$ and let Log be the inverse of Exp . We want to extend Log to the space $(\mathbb{R}^{\geq 0})^n$ in order to transform $X_\sigma^{\geq 0}$ into a partial compactification of \mathbb{R}^n . We first consider the partial compactification $[-\infty, \infty[$ of the line \mathbb{R}^1 and assume $e^{-\infty} = 0$. There is a commutative diagram

$$\begin{array}{ccc}
 (\mathbb{R}^{\geq 0})^n & \xrightarrow{h} & (\mathbb{R}^{\geq 0})^s \\
 \text{Exp} \uparrow & & \uparrow \text{Exp} \\
 \mathbb{R}^n & \xrightarrow{k} & ([-\infty, \infty[)^s
 \end{array}$$

where k is of the form $x \mapsto (\langle x, m_1 \rangle, \langle x, m_2 \rangle, \dots, \langle x, m_s \rangle)$ (here M is identified with Z^n via the basis e_1, e_2, \dots, e_n and $\langle x, m_i \rangle$ is the ordinary scalar product in \mathbb{R}^n). Since both Exp 's are homeomorphisms, we can identify $X_\sigma^{\geq 0}$ with the closure of the linear space $k(\mathbb{R}^n)$ in $([-\infty, \infty[)^s$.

Another way of describing the above space is the following. For $i = 1, 2, \dots, s$ let $m_i = (m_{i1}, m_{i2}, \dots, m_{in})$ (under the above identification). Let now β_j be the j -th row of the matrix (m_{ij}) , $\beta_j = (m_{1j}, m_{2j}, \dots, m_{sj})$. If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then $k(x) = \sum x_j \beta_j$. It follows that $k(\mathbb{R}^n)$ is the subspace of \mathbb{R}^s spanned by $\beta_1, \beta_2, \dots, \beta_n$ and finally we obtain the identification of $X_\sigma^{\geq 0}$ with the closure \bar{V} in $([-\infty, \infty[)^s$ of the space $V = \text{lin}(\beta_1, \dots, \beta_n)$. The group $T^{>0}$ is then identified with the space V itself, multiplication in $T^{>0}$ corresponding to addition in V and the action of V on its closure is the continuous extension of translations on V .

In the general non-affine case, the real non-negative part $X_x^{\geq 0}$ of X_x can be obtained by glueing together the spaces of the form \bar{V} and the glueing maps are extensions of linear isomorphisms.

The detailed description of the topology in the variety with singular corners \bar{V} is given in [SC].

2.3. Bijection of $X_\sigma^{\geq 0}$ onto $\bar{\sigma}$

We now show how one can obtain an identification of $X_\sigma^{\geq 0}$ with the cone $\bar{\sigma}$ (see the author's paper [J₃]) in the case of an n -dimensional cone σ , that is, when X_σ has a fixed point with respect to the action of T . This identification depends on the choice of a system of elements of the lattice M , spanning the cone $\bar{\sigma}$.

Under the assumption imposed on σ , the cone $\bar{\sigma}$ does not contain any line, thus there exists a distinguished minimal system spanning $\bar{\sigma}$, namely the system of primitive elements of the 1-dimensional faces of $\bar{\sigma}$.

2.3.1. LEMMA. *Let m_1, m_2, \dots, m_s be a subset of $M \cap \check{\sigma}$ such that $\check{\sigma} = \mathbf{R}^{\geq 0} \cdot m_1 + \dots + \mathbf{R}^{\geq 0} \cdot m_s$. Then every $m \in M \cap \check{\sigma}$ can be written in the form $m = q_1 m_1 + \dots + q_s m_s$ where all $q_i \in \mathbf{Q}^{\geq 0}$.*

Proof. Let $m = r_1 m_1 + \dots + r_l m_l$, $r_i \in \mathbf{R}^{> 0}$, $l \leq s$, vanishing summands omitted. Then the equation

$$m = X_1 m_1 + \dots + X_l m_l$$

has solutions in Q and those solutions form an affine subspace $V < Q^l$. By computing the respective rational and real dimensions one gets that the space of real solutions is just the closure \bar{V} of V in \mathbf{R}^l . Since $r = (r_1, r_2, \dots, r_l)$ belongs to \bar{V} and $r_i > 0$, it suffices to choose a point (q_1, \dots, q_l) in V , close enough to r . Then the inequalities $q_i > 0$ will hold and $q_1 m_1 + \dots + q_l m_l = m$.

The ordinary real analytic structure on the group $T^{> 0}$ induces real analytic structures on all $T^{> 0}$ -orbits in $X_{\check{\sigma}}^{\geq 0}$. We have

2.3.2. THEOREM (see [J₃], Thm. 1). *Let m_1, m_2, \dots, m_s span the cone $\check{\sigma}$ which is supposed not to contain any line. Then:*

a) *The regular morphism of varieties*

$$i: X_{\sigma} \rightarrow M_{\mathbf{C}} \cong \mathbf{C}^n, \quad x \mapsto x^{m_1} m_1 + x^{m_2} m_2 + \dots + x^{m_s} m_s$$

induces a homeomorphism of spaces

$$i: X_{\sigma}^{\geq 0} \cong \check{\sigma}$$

mapping $T^{> 0}$ -orbits onto corresponding open faces of $\check{\sigma}$.

b) *For each face τ of σ , the restricted homeomorphism $i_{\tau}: \mathcal{O}_{\tau}^{\geq 0} \xrightarrow{\sim} \text{int}(\tau^{\perp})$, where τ^{\perp} is the face of $\check{\sigma}$ orthogonal to τ , is a real analytic isomorphism.*

Proof follows in steps 1°–5°.

1° i is injective on $X_{\check{\sigma}}^{\geq 0}$: Let $x, y \in X_{\sigma}^{\geq 0}$ be such that $i(x) = i(y)$. Then using the linear structure of $M_{\mathbf{C}}$ we can write

$$\sum_{j=1}^s (x^{m_j} - y^{m_j}) m_j = 0.$$

Denote x^{m_j} by x_j and y^{m_j} by y_j . Then

$$1 = x^{\sum (x_j - y_j) m_j} = \prod x_j^{(x_j - y_j) m_j} = \prod x_j^{(x_j - y_j)}$$

and similarly $1 = \prod y_j^{(x_j - y_j)}$. Notice that the formula $x^{cm} = (x^m)^c$ for $c \in \mathbf{R}$, $m \in M$ holds if we assume $0^0 = 1$. All $x_j, y_j \in \mathbf{R}^{\geq 0}$. For two reals $a, b \geq 0$ we always have $a^{a-b} \geq b^{a-b}$ and equality holds iff $a = b$. Hence the equality $\prod x_j^{(x_j - y_j)} = \prod y_j^{(x_j - y_j)}$ implies $x_j = y_j$ for all j . It remains to show that $x = y$. Although x_1, x_2, \dots, x_s are not all the coordinates of x , since the system m_1, m_2, \dots, m_s needs not generate $M \cap \check{\sigma}$, nevertheless, by Lemma 2.3.1, for any $m \in M \cap \check{\sigma}$

there exist non-negative integers a_0, a_1, \dots, a_s such that $a_0 m = \sum a_j m_j$. Then the non-negative real number x^m equals to $(x^{\sum a_j m_j})^{1/a_0} = (\prod x_j^{a_j})^{1/a_0}$ and similarly $y^m = (\prod y_j^{a_j})^{1/a_0}$. It follows that $x^m = y^m$ for each $m \in M \cap \delta$. By Lemma 1.2.14, $x = y$.

2° For each face τ of σ , $i(\mathcal{O}_\tau^{\geq 0})$ is contained in $\text{int } \tau^\perp$. Indeed, if $x \in \mathcal{O}_\tau^{\geq 0}$ then $x = y\varepsilon_\tau$ (the distinguished point of \mathcal{O}_τ), where $y \in T^{>0}$ and

$$i(x) = \sum_j (y\varepsilon_\tau)^{m_j} \cdot m_j = \sum y^{m_j} \cdot \varepsilon_\tau^{m_j} \cdot m_j.$$

Since

$$\varepsilon_\tau^{m_j} = \begin{cases} 1 & \text{for } m_j \in \tau^\perp, \\ 0 & \text{otherwise,} \end{cases}$$

then $i(x) = \sum y^{m_j} \cdot m_j$, where all m_j belong to τ^\perp and all y^{m_j} are non-negative.

3° For each τ the map i_τ induces isomorphisms of tangent spaces at each point: Indeed, by the definition of the analytic structure of the orbit \mathcal{O}_τ we are to prove that the composed map

$$T^{>0}/T_\tau^{>0} \xrightarrow{\sim} \mathcal{O}_\tau^{\geq 0} \rightarrow \text{int}(\tau^\perp), \quad \tilde{y} \mapsto \tilde{y}\varepsilon_\tau,$$

is a local isomorphism. Since the dimensions are equal, it suffices to show that the map

$$T^{>0} \rightarrow \text{int}(\tau^\perp), \quad y \mapsto i_\tau(y\varepsilon_\tau),$$

has a surjective derivative at each $y \in T^{>0}$. Assume that $m_1, m_2, \dots, m_d \in \tau^\perp$ and $m_{d+1}, \dots, m_s \notin \tau^\perp$ for some d . Fix a basis in M and consider the resulting identifications $M = \mathbf{Z}^n$, $M_\mathbf{R} = \mathbf{R}^n$, $T = (\mathbf{C}^*)^n$. As in the previous step, the map $f: T^{>0} \rightarrow \text{int}(\tau^\perp)$ is given by $f(y) = \sum_{j=1}^d y^{m_j} m_j$. In order to differentiate f , apply the following decomposition:

$$f(y) = \sum_j \exp(\langle m_j, \log y \rangle) m_j.$$

Let A be the matrix with rows m_1, m_2, \dots, m_d and $D(\dots)$ the diagonal matrix. The Jacobian matrix of f at the point y is then

$$J(f) = \frac{\partial f_j}{\partial y_k} = A^T \cdot D(y^{m_1}, \dots, y^{m_d}) \cdot A \cdot D(1/y_1, \dots, 1/y_n).$$

Let $B = A^t D(y^{m_1/2}, \dots, y^{m_d/2})$. Then

$$J(f) = B \cdot B^T \cdot D(1/y_1, \dots, 1/y_n).$$

Since the last matrix is non-singular for $y \in T^{>0}$, it can be omitted in the rank calculation. We have:

$$\text{rank } J(f) = \text{rank } (B \cdot B^T) = \text{rank } (B) = \text{rank } A^T = \dim \tau^\perp,$$

since τ^\perp is spanned by m_1, \dots, m_d .

4° (J. Konarski). i and all i_τ are closed maps: Let $F \subset X_\sigma^{\geq 0}$ be a closed set; we are to prove that $i(F)$ is closed in $\check{\sigma}$. Let y belong to the closure of $i(F)$ and let (x_r) be the sequence in F such that $i(x_r) = \sum x_r^{m_j} m_j$ converges to y .

2.3.3. LEMMA. For all j , the sequences $(x_r^{m_j})_r$ are bounded.

Proof. If, for example, $(x_r^{m_1})_r$ would not be bounded, then taking a subsequence we could suppose that $(x_r^{m_1})_r$ converges to ∞ (since $x_r^{m_1}$ are positive), then $m_1 + (1/x_r^{m_1}) \cdot (x_r^{m_2} m_2 + \dots) \xrightarrow{r \rightarrow \infty} 0$, and finally $(1/x_r^{m_1}) \cdot (x_r^{m_2} m_2 + \dots) \rightarrow (-m_1)$. But the cone $\check{\sigma}$ is closed, and hence $-m_1 \in \check{\sigma}$, which contradicts the assumption that $\check{\sigma}$ does not contain any line.

Proof of Theorem 2.3.2, p. 4°, continuation. Replacing the sequence (x_r) by a subsequence, we may assume that each sequence $(x_r^{m_j})_r$ converges to some $c_j \geq 0$. We seek a point $x \in X_\sigma^{\geq 0}$ such that $x^{m_j} = c_j$ for $j = 1, 2, \dots, s$.

Now, such a point x can be described as a semigroup homomorphism $h: M \cap \check{\sigma} < \mathbf{R}^{\geq 0}$ (see 2.2.8). We shall define h by the formula $h(m) = \lim_r (x_r)^m$.

The map h is well defined, for if m can be written in the form $\sum q_i m_i$, $q_i \in \mathbf{Q}^{\geq 0}$, then $x_r^m = \prod (x_r^{m_i})^{q_i}$ and this sequence is convergent when $r \rightarrow \infty$. More precisely, we obtain $h(m) = \prod c_i^{q_i}$. The map h is a homomorphism, since $h(m+m') = \lim_r (x_r)^{m+m'} = \lim_r (x_r^m x_r^{m'}) = h(m) h(m')$ and $h(0) = \lim_r x_r^0 = 1$.

So the homomorphism h defines a point $x \in X_\sigma^{\geq 0}$ such that $x^m = h(m)$. Therefore $x^{m_j} = c_j$, $x^m = \lim_r x_r^m$ for all $m \in M \cap \check{\sigma}$; hence $x = \lim_r x_r$ and thus x

belongs to F . Now

$$i(x) = \sum_j x^{m_j} m_j = \sum_j \lim_r x_r^{m_j} m_j = \lim_r i(x_r) = y,$$

and finally we get that $y \in i(F)$, which proves that i is a closed map. Now, given a face τ of σ , if F_1 is a closed set in $\mathcal{O}_\tau^{\geq 0}$, then $F_1 = F \cap \mathcal{O}_\tau^{\geq 0}$ with an F closed in $X_\sigma^{\geq 0}$. We get

$$i_\tau(F_1) = i(F_1) = i(F \cap \mathcal{O}_\tau^{\geq 0}) = i(F) \cap i(\mathcal{O}_\tau^{\geq 0})$$

by the injectivity of i . Hence $i_\tau(F_1)$ is closed in $i(\mathcal{O}_\tau^{\geq 0})$.

By 3° the image of i_τ is open in $\text{int } \tau^\perp$ and by 4° it is closed; hence i_τ are surjective for all τ . The theorem follows from 1° to 5°.

2.3.4. Remark. If the cone $\check{\sigma}$ does not fulfil the assumption of Theorem 2.3.2 then $\check{\sigma} = \mathbf{R}^k \times \check{\sigma}_0$, where $\check{\sigma}_0$ is a cone of dimension $n-k$, satisfying the above assumption. On the other hand, $X_\sigma = \text{Spec}(C[X_1, \dots, X_k] \otimes C[M_0 \cap \check{\sigma}_0])$ where M_0 is a lattice of dimension $n-k$. If we denote by X_{σ_0, M_0} the $(n-k)$ -dimensional torus embedding associated with σ_0 (the dual cone of $\check{\sigma}_0$), then

$$X_\sigma = \mathbf{C}^k \times X_{\sigma_0, M_0}, \quad X_\sigma^{\geq 0} = \mathbf{R}^k \times X_{\sigma_0, M_0}^{\geq 0}$$

Applying the previous theorem to the cone σ_0 , we obtain again the identification of $X_\sigma^{\geq 0}$ and $\check{\sigma}$.

2.3.5. Remark. Since in the affine case the real non-negative part of X_σ can be identified with the cone $\check{\sigma}$, one might expect that in the general case, $X_\sigma^{\geq 0}$ is a polyhedron obtained by glueing together the translated cones dual to the cones of Σ . However, those dual cones cannot be translated in such a way unless X_Σ is a quasi-projective variety (see 3.3 and 3.1).

2.3.6. Remark. Step 3° of the above proof shows that the morphism $x \mapsto \sum x^{m_j} m_j$ is étale at all points of $T^{>0}$. We show below some other properties of the morphism defined in Theorem 2.3.2.

Following Danilov [Dan], we shall call the variety X_σ quasi-smooth if the cone σ is simplicial, that is, if it is a cone over a simplex. Obviously, the dual cone $\check{\sigma}$ is also simplicial in this case.

2.3.7. PROPOSITION. Assume the usual identifications $M = \mathbf{Z}^n$ etc. If the variety X_σ is quasi-smooth, $\check{\sigma}$ spanned by primitive vectors $m_1, m_2, \dots, m_n \in M$, then the morphism

$$i: X_\sigma \rightarrow \mathbf{C}^n, \quad x \mapsto \sum x^{m_j} m_j,$$

is finite, of degree $|\det(m_1, m_2, \dots, m_n)|$.

Proof. $X_\sigma = \text{Spec } C[\mathcal{X}^{m_1}, \mathcal{X}^{m_2}, \dots, \mathcal{X}^{m_n}, \dots, \mathcal{X}^{m_j}, \dots]$ where for all $j > n$ we have $m_j = \sum_{1 \leq k \leq n} (c_{jk}/c_j) m_k$ with c_{jk}, c_j non-negative integers. The morphism i has, in the basis m_1, m_2, \dots, m_n of \mathbf{C}^n , the form $x \mapsto (x^{m_1}, x^{m_2}, \dots, x^{m_n})$. The corresponding homomorphism of affine algebras is

$$C[Y_1, Y_2, \dots, Y_n] \rightarrow C[\mathcal{X}^{m_1}, \mathcal{X}^{m_2}, \dots, \mathcal{X}^{m_n}, \dots, \mathcal{X}^{m_j}, \dots], \quad Y_i \mapsto \mathcal{X}^{m_i}.$$

It is a monomorphism. Indeed, (m_1, m_2, \dots, m_n) is a basis of a sublattice of $M = \mathbf{Z}^n$; hence the algebra $C[\mathcal{X}^{m_1}, \mathcal{X}^{m_2}, \dots, \mathcal{X}^{m_n}]$ is isomorphic to the polynomial algebra and $\mathcal{X}^{m_1}, \dots, \mathcal{X}^{m_n}$ are algebraically independent. Now, for $j > n$ the elements \mathcal{X}^{m_j} are integral over $C[\mathcal{X}^{m_1}, \dots, \mathcal{X}^{m_n}]$, since $(\mathcal{X}^{m_j})^{c_j} = \prod_i (\mathcal{X}^{m_i})^{c_{ij}}$. It follows that the extension under consideration is integral.

In order to find the degree of the morphism i , we restrict it to the torus T (leaving the same symbol i). Observe that $i(T) \subset C^{*n}$ hence, identifying T with another copy of C^{*n} , we have

$$i: C^{*n} \rightarrow C^{*n}$$

where

$$i(y) = i(y_1, y_2, \dots, y_n) = (y_1^{m_1}, \dots, y_n^{m_n}).$$

We pass to the endomorphism of the lattice Z^n . Observe that to each group endomorphism $g: C^{*n} \rightarrow C^{*n}$ there corresponds the group endomorphism

$$\tilde{g}: Z^n = \text{Hom}_{\text{gr}}(C^*, C^{*n}) \xrightarrow{\text{Hom}(C^*, g)} \text{Hom}(C^*, C^{*n}) = Z^n.$$

Moreover, the correspondence $g \rightarrow \tilde{g}$ is an isomorphism of semigroups $\sim: \text{End}(C^{*n}) \cong \text{End}(Z^n)$.

The image of the element i , that is, the endomorphism $\tilde{i}: Z^n \rightarrow Z^n$, is given by the matrix A with the rows m_1, m_2, \dots, m_n . It is injective since rank $A = n$. By the structural theorem on sublattices, there are automorphisms \tilde{j} and \tilde{k} of both copies of Z^n such that the composite $Z^n \xrightarrow{\tilde{j}} Z^n \xrightarrow{\tilde{i}} Z^n \xrightarrow{\tilde{k}} Z^n$ is diagonal, $(a_1, a_2, \dots, a_n) \mapsto (d_1 a_1, d_2 a_2, \dots, d_n a_n)$ and $|d_1 \cdot d_2 \cdot \dots \cdot d_n| = |\det A|$. It follows that the endomorphism

$$j \cdot i \cdot k: C^{*n} \rightarrow C^{*n}$$

has the form $(x_1, x_2, \dots, x_n) \mapsto (x_1^{d_1}, \dots, x_n^{d_n})$, and hence $\deg(j \cdot i \cdot k) = |d_1 \cdot d_2 \cdot \dots \cdot d_n|$. On the other hand, j and k are automorphisms, whence $\deg(j \cdot i \cdot k) = \deg(i)$. Finally $\deg i = |\deg A|$.

Now let us look at the simplest non-quasi-smooth example.

2.3.8. EXAMPLE. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be a basis of N and let $\tilde{\sigma}$ be the cone spanned by $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1 + \varepsilon_2 - \varepsilon_3$. These four vectors are also generators of the semigroup $M \cap \tilde{\sigma}$. Indeed, denote for short by $\langle \alpha_1, \alpha_2, \dots, \alpha_m \rangle$ the cone spanned by the elements $\alpha_1, \alpha_2, \dots, \alpha_m$ of N . In this notation $\sigma = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \cup \langle \varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2 - \varepsilon_3 \rangle$. Let $m = c_1 \varepsilon_1 + c_2 \varepsilon_2 + c_3 \varepsilon_3 \in M \cap \tilde{\sigma}$. If $c_3 \geq 0$ then $m \in \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ and the conclusion follows. If $c_3 < 0$ then $m \in \langle \varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2 - \varepsilon_3 \rangle$ and these three vectors form a basis of M etc.

Hence $X_\sigma = \text{Spec } C[X_1, X_2, X_3, X_1 X_2 / X_3]$ and $i: X_\sigma \rightarrow C^3$ has the form

$$i = \begin{bmatrix} X_1 + X_1 X_2 / X_3 \\ X_2 + X_1 X_2 / X_3 \\ X_3 - X_1 X_2 / X_3 \end{bmatrix}.$$

It is a birational morphism with the inverse

$$i^{-1} = (Y_1 + Y_2 + Y_3)^{-1} \cdot \begin{bmatrix} Y_1(Y_1 + Y_3) \\ Y_2(Y_2 + Y_3) \\ (Y_1 + Y_3)(Y_2 + Y_3) \end{bmatrix},$$

Y_1, Y_2, Y_3 are coordinates in \mathbb{C}^3 . There is a closed immersion $X_\sigma \hookrightarrow \mathbb{C}^4$ defined by the functions $x = X_1, y = X_2, z = X_3$ and X_σ becomes a hypersurface with the equation $xy = zt$. Using this coordinates, we can write i in the form $i(x, y, z, t) = (x+t, y+t, z-t)$. Then X_σ is isomorphic to the cone over the projective surface $xy = zt$ in P^3 , which is isomorphic to $P^1 \times P^1$.

Also \mathbb{C}^3 may be regarded as the cone over the projective plane P^2 . Moreover, the morphism $i: X_\sigma \rightarrow \mathbb{C}^3$ can be factorized through the morphism of those projective surfaces in the following way ($a_0: a_1$ and $b_0: b_1$ are coordinates in $P^1 \times P^1$):

$$\begin{array}{ccc} (x, y, z, t) \in X_\sigma \setminus \{0\} & \xrightarrow{i} & \mathbb{C}^3 \setminus \{0\} \\ \downarrow & & \downarrow \\ (z: x, t: x) \in P^1 \times P^1 & \xrightarrow{j} & P^2 \end{array}$$

where

$$j((a_0: a_1, b_0: b_1)) = (a_1(b_0 + b_1): b_0(a_0 + a_1): (a_0 b_1 - a_1 b_0)).$$

This diagram is commutative. The morphism $P^1 \times P^1 \xrightarrow{j} P^2$ is a blowing up of a point and contracting two lines, as one can check by choosing convenient coordinates.

2.4. Real part of X_σ . Reflexions

The variety X_σ is defined over \mathbb{Z} , hence its real part $X_\sigma(\mathbb{R})$ is well defined. It is an algebraic real variety, of dimension n . In order to decompose it in a form similar to 2.2.7 observe that $X_\sigma(\mathbb{R}) \supset X_\sigma^{\geq 0}$. We have the following fibred product diagram:

$$\begin{array}{ccc} X_\sigma(\mathbb{R}) & \xrightarrow{|\cdot|} & X_\sigma^{\geq 0} \\ \text{arg} \downarrow & & \downarrow \\ (c - X_\sigma) \cap X_\sigma(\mathbb{R}) & \longrightarrow & \mathcal{I} \end{array}$$

The elements of $(c - X_{\Sigma}) \cap X_{\Sigma}(\mathbf{R})$ are precisely the elements x of X with the following property: For each $m \in M$, we have $x^m = 0, 1$ or -1 , whenever x^m is defined.

2.4.1. The real part $T(\mathbf{R})$ of the torus may be decomposed into $T(\mathbf{R}) = T^{>0} \times (\text{torsion subgroup of } T(\mathbf{R}))$. The second factor is the set of elements in T of order ≤ 2 . Those elements will be called *reflexions* (the terminology refers to their action on X_{Σ}) and the group of reflexions will be denoted by μ . Hence μ has 2^n elements. If T is identified with $\mathbf{C}^* \times \dots \times \mathbf{C}^*$ then μ consists of elements of the form $(\pm 1, \pm 1, \dots, \pm 1)$.

Reflexions are also characterized as the elements of X_{Σ} of the form $(-1)^{\alpha}$, for $\alpha \in N$. The set $X_{\Sigma}(\mathbf{R})$ is invariant under the action of μ (the restriction of the action of T on X_{Σ}). If we consider the variety $X_{\Sigma}(\mathbf{R})$ as the fibred product of $(c - X_{\Sigma}) \cap X_{\Sigma}(\mathbf{R})$ and $X_{\Sigma}^{\geq 0}$, then each point $x \in X_{\Sigma}(\mathbf{R})$ can be written in the form $(\varrho \varepsilon_{\tau}, |x|)$ where ϱ is a reflexion, τ is a cone in Σ such that $x \in \mathcal{O}_{\tau}$ and ε_{τ} is the distinguished point of the orbit \mathcal{O}_{τ} . The action of μ on $X_{\Sigma}(\mathbf{R})$ can be expressed by the formula

$$\varrho' \cdot (\varrho \varepsilon_{\tau}, u) = (\varrho' \varrho \varepsilon_{\tau}, u)$$

where $u \in X_{\Sigma}^{\geq 0}$, $\varrho' \in \mu$. Also we have

$$X_{\Sigma}(\mathbf{R}) = \bigcup_{\varrho \in \mu} \varrho(X_{\Sigma}^{\geq 0}).$$

Since all the images by reflexions of the element $1 \in T^{>0} \subset X_{\Sigma}^{\geq 0}$ are distinct, the set $X_{\Sigma}(\mathbf{R})$ is the union of 2^n copies of the "variety with singular corners" $X_{\Sigma}^{\geq 0}$, glued together along boundaries. The orbit space $X_{\Sigma}(\mathbf{R})/\mu$ may be identified with $X_{\Sigma}^{\geq 0}$.

Let $\varrho = (-1)^{\alpha}$, $\alpha \in N$. Then the subset $\varrho \cdot X_{\Sigma}^{\geq 0}$ of X_{Σ} , the " ϱ -th copy of $X_{\Sigma}^{\geq 0}$ " can be characterized as follows: for an element x of X_{Σ} :

$$x \in \varrho \cdot X_{\Sigma}^{\geq 0} \Leftrightarrow x^m \begin{cases} \geq 0 & \text{if } \langle \alpha, m \rangle \text{ is even,} \\ \leq 0 & \text{otherwise.} \end{cases}$$

Indeed, if $x = (-1)^{\alpha} \cdot y$ and $y \in X_{\Sigma}^{\geq 0}$, then $x^m = (-1)^{\langle \alpha, m \rangle} \cdot y^m$, etc.

2.4.2. EXAMPLE. Let again $T = \mathbf{C}^* \times \dots \times \mathbf{C}^*$, let m_1, m_2, \dots, m_n be the corresponding basis of M and consider the reflexion $\varrho = (-1, 1, \dots, 1)$. Then

$$x \in \varrho X_{\Sigma}^{\geq 0} \Leftrightarrow \begin{cases} x^{m_1} \leq 0, \\ x^{m_2} \geq 0, \\ \dots \\ x^{m_n} \geq 0, \end{cases}$$

at all points where the rational functions x^{m_i} are defined.

3. PROJECTIVE TORUS EMBEDDINGS

3.1. Polyhedra

Let Σ be a complete complex of cones. Then by 1.2.9 the variety X_Σ is complete. The projectivity of X_Σ can be characterized by the following criterion.

3.1.1. THEOREM. *A complete variety X_Σ is projective iff there exists a function $f: N_{\mathbf{R}} \rightarrow \mathbf{R}$ such that for each n -dimensional cone $\sigma \in \Sigma$ there exists an element m_σ of M satisfying*

$$\langle y, m_\sigma \rangle \begin{cases} > f(y) & \text{for each } y \notin \sigma, \\ = f(y) & \text{for } y \in \sigma. \end{cases}$$

This theorem is a corollary from [TE] (ch. 1, § 2 Thm. 9 and § 3 Thm. 13). The function f above may be described more intuitively by means of a polyhedron Δ in $M_{\mathbf{R}}$ given by

$$\Delta = \bigcap_{\substack{\sigma \in \Sigma \\ \dim \sigma = n}} (m_\sigma + \bar{\sigma})$$

(compare [Dan] 6.3). Its vertices are the elements m_σ , and Δ is orthogonal to the complex Σ in the following sense: for each d -dimensional cone τ of Σ choose a cone $\sigma \in \Sigma$ of dimension n , containing τ (the choice of σ is irrelevant). Let τ^\perp be the face of $\bar{\sigma}$ orthogonal to τ . Now, $(m_\sigma + \tau^\perp) \cap \Delta$ is a face of Δ , which will be also denoted by τ^\perp . It has dimension $(n - \dim \tau)$ and the correspondence $\tau \rightarrow \tau^\perp$ is a bijection between the cones of Σ and the faces of Δ . Each element of τ considered as a linear form on $M_{\mathbf{R}}$ has a constant value on τ^\perp and it is its minimal value on the whole of Δ . Identifying the spaces $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$ via scalar product we can illustrate a polyhedron Δ for a complex of dimension 2 as follows (see Fig.10).

Actually, the polyhedron Δ (and the function f to which it corresponds) describe an ample Cartier divisor D on X_Σ ; locally on X_σ , for a cone $\sigma \in \Sigma$ of dimension n , this is the principal divisor described by the rational function \mathcal{X}^{-m_σ} . The functions \mathcal{X}^m , for $m \in \Delta \cap M$, form a basis of the linear system $\mathcal{L}(D)$.

Suppose that the polyhedron Δ has the following property:

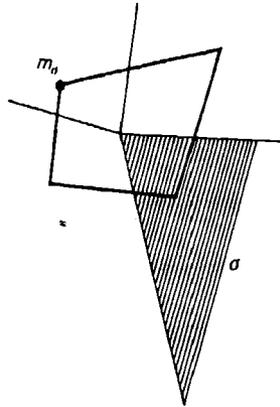


Fig. 10

3.1.2. For each cone $\sigma \in \Sigma$ of dimension n the set $M \cap (\Delta - m_\sigma)$ generates the semigroup $M \cap \sigma$ (if Δ has not this property, then $k \cdot \Delta$ has, for k large enough). Then the morphism

$$3.1.3. \quad X_{\Sigma} \rightarrow P^s, \quad x \mapsto (x^{m_0} : x^{m_1} : \dots : x^{m_s}),$$

where $m_0, m_1, \dots, m_s = \Delta \cap M$, is a closed immersion. This suggests the following simple definition of the projective torus embedding associated with a polyhedron Δ fulfilling 2.1.2 (or with a function f , as above):

Identify M with \mathbf{Z}^n and T with $k^* \times \dots \times k^*$. Then X_{Σ} is the Zariski closure in the projective space P^N of the image of T by the map 3.1.3.

Now, roughly speaking, non-projectivity of a complete torus embedding consist in the lack of possibility of constructing any polyhedron Δ orthogonal to Σ , that is to say, possibility of gluing together the translated dual cones.

In analogy to Theorem 2.3.2, we shall show that, over \mathbf{C} , the non-negative part $X_{\Sigma}^{\geq 0}$ of a projective torus embedding may be identified (in several ways) with the polyhedron Δ .

3.1.4.⁽¹⁾ THEOREM (see [J₃], Thm. 2). *Let X_{Σ} be a torus embedding over the complex numbers field, f a function such as in Theorem 3.1.1 and Δ the corresponding polyhedron in $M_{\mathbf{R}}$. Let $\{m_0, m_1, \dots, m_s\}$ be a subset of $\Delta \cap M$ containing all vertices of Δ . Then the rational morphism of varieties*

$$j: X_{\Sigma} \rightarrow M_{\mathbf{C}}, \quad j(x) = \left(\sum_{i=0}^s x^{m_i} \right)^{-1} \cdot \sum_{i=0}^s x^{m_i} \cdot m_i;$$

- a) is regular at each point of $X_{\Sigma}^{\geq 0}$;
- b) induces a homeomorphism;

$$j: X_{\Sigma}^{\geq 0} \xrightarrow{\sim} \Delta$$

mapping $T^{>0}$ -orbits onto corresponding open faces of Δ ;

⁽¹⁾ cf. the comment 1° added in proof.

c) $j: \mathcal{O}_\tau^{\geq 0} \xrightarrow{\sim} \text{int}(\tau)$ is a real analytic isomorphism for each cone τ in Σ .

Proof. The argument is to span the $(n+1)$ -dimensional cone over the polyhedron Δ and apply Theorem 2.3.2 to that cone.

a) Let σ_0 be a cone of dimension n in Σ . We shall show that j is regular on $X_{\sigma_0}^{\geq 0}$. Let m_0 be the vertex of Δ corresponding to σ_0 . The map j may be written in the form

$$j(x) = \frac{\sum_{i=1}^s x^{(m_i - m_0)} \cdot m_i}{1 + \sum_{i=1}^s x^{(m_i - m_0)}}.$$

All the vectors $m_i - m_0$ belong to σ_0 , and so $x^{(m_i - m_0)}$ are regular on X_{σ_0} and non-negative on $X_{\sigma_0}^{\geq 0}$. Hence j is regular.

b) Let D be the ample divisor corresponding to Δ . Consider the graded ring $A = \bigoplus_r H^0(X_{\mathbb{P}^1}, rD)$. Then $X_{\mathbb{P}^1} \cong \text{Proj } A$. Now, $\text{Spec } A$ is also a torus embedding, namely for the $(n+1)$ -dimensional torus $T' = T \times \mathbb{C}^*$. Take the lattice $M' = M \times \mathbb{Z}$ and the polyhedral cone $\sigma = \{(m, r) \in M_{\mathbb{R}} \times \mathbb{R} : m \in r\Delta\}$, contained in $M'_{\mathbb{R}} = M_{\mathbb{R}} \times \mathbb{R}$. Then $A = \mathbb{C}[M' \cap \sigma]$ and $\text{Spec } A = X_{\mathbb{P}^1}$ (compare [Dan]). Let $m'_i = (m_i, 1)$, elements of $M' \cap \sigma$. Since every element of Δ is a convex combination of m_0, m_1, \dots, m_s (with real coefficients), m'_0, m'_1, \dots, m'_s span the cone σ and therefore we are in the situation of Theorem 2.3.2. The gradation in the algebra A is given by $\text{deg}(m, r) = r$. For $x' \in X_{\mathbb{P}^1}$ let $i(x') = \sum x'^{m'_k} \cdot m'_k$ and let 0 be the (only) fixed point of $X_{\mathbb{P}^1}$. We have the following diagram of continuous maps:

$$\begin{array}{ccc}
 T'^{>0} \times \mathbb{R}^{>0} = T'^{>0} \subset X_{\mathbb{P}^1}^{>0} \setminus \{0\} & \longrightarrow & \sigma \setminus \{0\} \\
 \downarrow & \downarrow \alpha & \downarrow \beta \\
 T'^{>0} \subset X_{\mathbb{P}^1}^{>0} & \longrightarrow & \Delta
 \end{array}$$

3.1.5.

where α is the restriction of the canonical morphism $(\text{Proj } A) \setminus 0 \rightarrow \text{Spec } A$ and $\beta(m, r) = m/r$. This diagram is commutative. For instance, to prove that $\beta \cdot i = j \cdot \alpha$ it suffices to check this on the open $T'^{>0}$. Here for $y' = (y, c) \in T'^{>0}$ we have

$$\begin{aligned}
 i(y') &= \sum_k (y, c)^{(m_k, 1)} \cdot (m_k, 1) = c \cdot \sum y^{m_k} (m_k, 1) \\
 &= (c \sum y^{m_k} m_k, c \sum y^{m_k}),
 \end{aligned}$$

hence

$$\beta \cdot i(y') = (\sum y^{m_k} \cdot m_k) \cdot (\sum y^{m_k})^{-1} = j\alpha(y').$$

The fibres of α are precisely the $\mathbf{R}^{>0}$ -orbits and for $r \in \mathbf{R}^{>0}$, $x \in X_\sigma^{\geq 0}$ we have $i(rx) = ri(x)$. It follows that i maps each fibre of α onto the corresponding fibre of β , and hence j is a bijection.

By Theorem 2.3.2 applied to the cone $\tilde{\sigma}$ and the elements m'_0, m'_1, \dots, m'_s , the map i is homeomorphic. Consider i as an identification of topological spaces. Then α and β coincide and we conclude that j is a homeomorphism, too.

Let τ be a cone in Σ and let $\sigma \in \Sigma$ be a cone of dimension n containing τ . Write the map j as in the prof of a). An argument similar to that of step 2° in the proof of Theorem 2.3.2 shows that $j(\mathcal{O}_\tau^{\geq 0}) \subset \text{int } \tau^\perp$.

c) Take $x \in \mathcal{O}_\tau^{\geq 0} \subset X_\Sigma^{\geq 0}$ and a point $y \in \alpha^{-1}(x)$. Both maps i and α restricted to $T' \cdot y$ are smooth. The diagram 3.1.2 induces the following commutative diagram of suitable tangent spaces:

$$\begin{array}{ccc}
 \theta_{T', y, y} & \xrightarrow{Di} & \theta_{\text{int } \tau', i(y)} \\
 D\alpha \downarrow & & \downarrow D\beta \\
 \theta_{T', x} & \xrightarrow{Dj} & \theta_{\text{int } \tau, j(x)}
 \end{array}$$

where τ' is the face of $\tilde{\sigma}$ spanned over the face τ^\perp of Δ . By Theorem 2.3.2b) the derivated map Di is an isomorphism, $D\beta$ is obviously surjective. Comparing the dimensions of the vector spaces in question we conclude that Dj is an isomorphism.

3.1.6. Remark. Given a polyhedron Δ , one can distinguish two sets $\{m_0, m_1, \dots, m_s\}$ fulfilling the assumption of Theorem 3.1.4: the minimal one consisting of the vertices of Δ and the maximal one, $M \cap \Delta$. Then we get two distinguished homeomorphisms $X_\Sigma^{\geq 0} \cong \Delta$.

From Theorems 2.3.2 and 3.1.4 we can deduce the form of some natural bijections between the real vector spaces, open polyhedra and polyhedral cones. Namely, we have

3.1.7. COROLLARY. a) Let σ be a polyhedral cone in \mathbf{R}^n spanned by integral vectors m_1, m_2, \dots, m_s . Then the mapping

$$x \mapsto \sum_{i=1}^s e^{\langle x, m_i \rangle} \cdot m_i$$

is a real analytic diffeomorphism of \mathbf{R}^n onto the interior of σ .

b) Let Δ be a compact polyhedron in \mathbf{R}^n with integral vertices m_0, m_1, \dots, m_s . Then the map

$$x \mapsto \left(\sum_{i=0}^s e^{\langle x, m_i \rangle} \cdot m_i \right) \cdot \left(\sum_{i=0}^s e^{\langle x, m_i \rangle} \right)^{-1}$$

is a real analytic diffeomorphism of \mathbf{R}^n onto $\text{int } \Delta$.

Proof. Observe that for $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ we have $e^{\langle x, m \rangle} = (e^{x_1}, e^{x_2}, \dots, e^{x_n})^{m_i}$; hence the maps in question are compositions of \exp and i of Theorem 2.3.2 or j of Theorem 3.1.4.

3.2. Morse function

3.2.1. As in 2.2 let $c-T$ denote the compact torus contained in the algebraic torus T over the complex numbers field. Given a projective torus embedding X and a polyhedron Δ orthogonal to Σ as in 3.1 and a subset $\{m_0, m_1, \dots, m_s\}$ of $M \cap \Delta$ containing all the vertices of Δ , we have the bijections

$$X_{\Sigma}/(c-T) \cong X_{\Sigma}^{\geq 0} \cong \Delta.$$

We can rewrite the fibred product diagram 2.2.7 in the form

$$\begin{array}{ccc} X_{\Sigma} & \longrightarrow & \Delta \\ \downarrow & & \downarrow \\ c-X_{\Sigma} & \longrightarrow & \Sigma \end{array}$$

The polyhedron Δ reflects quite well the properties of X_{Σ} , by means of the bijections above. Some structures on the variety X_{Σ} are more intuitive when restricted to Δ . Open d -dimensional faces of Δ are real non-negative parts of orbits in X_{Σ} of complex dimension d . The closed face of Δ is the real non-negative part of the closure of the corresponding orbit etc.

Let us restrict our considerations to the non-singular case.

An element α of the lattice N can be interpreted as

- 1° a linear functions on Δ ,
- 2° a Morse function (up to a smoothing) on X_{Σ} and $X_{\Sigma}(\mathbf{R})$,
- 3° an action of the multiplicative group G_m on X_{Σ} .

We now explain the sense of 2° and we establish relation between the three concepts.

3.2.2. For $\alpha \in N$, regard the linear function $\alpha: M_{\mathbf{R}} \rightarrow \mathbf{R}$ as the "altitude" in $M_{\mathbf{R}}$. Next consider the G_m -action on X_{Σ} given by the 1-p.s. $\lambda_{\alpha}: G_m \rightarrow T$ (see 1.5). Then the Białynicki decomposition of X_{Σ} , restricted to $\Delta \cong X_{\Sigma}^{\geq 0}$, may be described as follows. The set of fixed points with respect to $G_m^{\geq 0} \cong \mathbf{R}^{\geq 0}$ is the union of closed horizontal faces of Δ (i.e. faces on which α is constant; among them, all vertices). The components of the set of fixed points are the maximal (with respect to inclusion) closed faces among the horizontal ones.

The non-horizontal faces decompose into 1-dimensional $G_m^{\geq 0}$ -orbits which are non-horizontal open arcs ending in horizontal faces. The locally

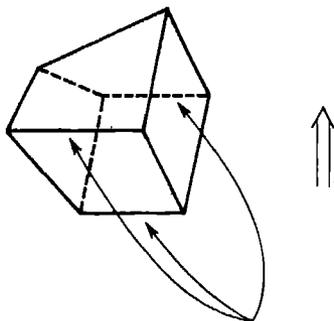
Horizontal edges of $\dim > 0$

Fig. 11

closed subset A^+ of Δ attracted by a maximal horizontal face A is the union of open faces U culminating in A (that means A intersects \bar{U} and $\alpha(A) = \sup_{x \in U} \alpha(x)$).

3.2.3. Consider for a moment an arbitrary action of G_m on a complete, non-singular variety X and an isolated fixed point of X . Let r_x be the induced representation of G_m on the tangent space to X at x .

DEFINITION. The *index* of x is the number of characters of r_x , with positive weights.

In other words: suppose that in a certain basis of the tangent space, r_x acts through the matrix

$$\begin{bmatrix} t^{m_1} & & 0 \\ & \ddots & \\ 0 & & t^{m_n} \end{bmatrix}$$

with $m_1 \geq m_2 \geq \dots \geq m_i > 0 > m_{i+1} \geq \dots \geq m_n$. Then i is the index of x .

In what follows $\alpha \in N$ is supposed not to belong to the linear hull of any $(n-1)$ -dimensional cone in Σ ; that is, no face of Δ is horizontal (except vertices). Then by Theorem 1.5.3, the only G_m -fixed points of X_x are the vertices of Δ .

3.2.4. PROPOSITION. Under the previous assumptions, the index of a vertex x of Δ , considered as a fixed point with respect to the G_m -action, is equal to the number of edges of Δ such that x is their lower end.

Proof. In the standard local coordinates in the neighbourhood of x , the torus T acts by

$$(y_1, y_2, \dots, y_n) \cdot (x_1, x_2, \dots, x_n) = (y_1 x_1, \dots, y_n x_n).$$

If $\alpha = (a_1, a_2, \dots, a_n)$ and, for all i , $a_i \neq 0$ then the action is of the form $t \cdot (x_1, x_2, \dots, x_n) = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n)$ and 0 is the only fixed point.

The edges of Δ beginning in x are $(u, 0, \dots, 0)_{u \in \mathbb{R}}$, $(0, u, 0, \dots)$ etc. Hence the number of edges such that x is their lower end coincides with the number of positive a_i .

To define the Morse function corresponding to α , take $\{m_0, m_1, \dots, m_s\} = \Delta \cap M$ (compare Remark 3.1.6) and the corresponding homeomorphism $j: X_{\Sigma}^{\geq 0} \xrightarrow{\sim} \Delta$. Next consider the morphism $X_{\Sigma} \rightarrow X_{\Sigma}$, $x \mapsto x^2$, (see 2.2.9). Locally it is given by $(x_1, x_2, \dots, x_n)^2 = (x_1^2, x_2^2, \dots, x_n^2)$. It follows that the map $x \mapsto x^2$ is a homeomorphism $X_{\Sigma}^{\geq 0} \xrightarrow{\sim} X_{\Sigma}^{\geq 0}$. This homeomorphism preserves T -orbits and is introduced here in order to modify the differential structure of $X_{\Sigma}^{\geq 0}$.

3.2.5.⁽¹⁾ THEOREM (see [J₃], Prop.). *Let $\alpha \in N$ be an element such that all $\langle \alpha, m_i \rangle$ are distinct and let f_{α} be the composite*

$$f_{\alpha}: X_{\Sigma}^{\geq 0} \xrightarrow{j} X_{\Sigma}^{\geq 0} \xrightarrow{x \mapsto x^2} X_{\Sigma}^{\geq 0} \xrightarrow{j} \Delta \xrightarrow{\alpha} \mathbb{R}$$

Then

- a) f_{α} is a Morse function both on X_{Σ} and its real part $X_{\Sigma}(\mathbb{R})$;
- b) f_{α} is constant on $(c-T)$ -orbits in X_{Σ} and increasing along $\mathbb{R}^{>0}$ -orbits;
- c) critical points with respect to f_{α} are precisely the fixed points with respect to the G_m -action corresponding to α and they all have distinct critical values.

d) Let i_x be the index of a G_m -fixed point x . Then the index of f_{α} in x is: i_x if f_{α} is considered as a Morse function on $X_{\Sigma}(\mathbb{R})$, $2i_x$ if f_{α} is considered as a Morse function on X_{Σ} .

Proof. Let σ_0 be a cone of dimension n in Σ . We shall express f_{α} in the standard local coordinates of X_{σ_0} in a neighbourhood of the fixed point x_0 . We can identify x_0 with the vertex m_0 of Δ , by the map j . Then we have

$$f_{\alpha}(x) = \frac{\sum_{i=0}^s |x|^{2m_i} \langle \alpha, m_i \rangle}{\sum_{i=0}^s |x|^{2m_i}} = \frac{\sum_{i=1}^s x^{2(m_i - m_0)} \langle \alpha, m_i - m_0 \rangle}{1 + \sum_{i=1}^s |x|^{2(m_i - m_0)}} + \langle \alpha, m_0 \rangle.$$

Observe that, by the non-singularity of X_{Σ} , among the vectors $m_i - m_0$ there is a basis of M , say $m_1 - m_0, m_2 - m_0, \dots, m_n - m_0$, and all the remaining $m_i - m_0$ have non-negative real coordinates in this basis. Assume $z_i = x^{m_i - m_0}$ for $i = 1, 2, \dots, n$, the coordinates of x in the affine space X_{σ_0} , and $a_i = \langle \alpha, m_i - m_0 \rangle$. Then

$$f_{\alpha}(x) = \frac{\sum_{i=1}^n |z_i|^2 a_i + \mathcal{O}_2}{1 + \sum_{i=1}^n |z_i|^2 + \mathcal{O}_2} + \langle \alpha, m_0 \rangle$$

⁽¹⁾ cf. comment 1° added in proof.

where $\mathcal{O}_2, \mathcal{O}'_2$ are polynomials in $|z_1|^2, |z_2|^2, \dots, |z_n|^2$ not containing members of degree < 2 . It follows easily that

$$3.2.6. \quad f_\alpha(x) = \langle \alpha, m_0 \rangle + \sum_{i=1}^n |z_i|^2 \cdot a_i + \mathcal{O}'_2.$$

Using the real coordinates $z_i = u_i + \sqrt{-1} w_i$ we have $f_\alpha(x) = \sum_{i=1}^n (u_i^2 + w_i^2) a_i + \mathcal{O}'_2(u_i, w_i)$. The restriction to $X_\Sigma(\mathbf{R})$ gives $f_\alpha(x) = \sum u_i^2 a_i + (\text{monomials in } u_i, \text{ of degree } \geq 4)$, and so statement a) is proved.

b) The first statement is true, because $f_\alpha(x)$ depends only on $|x|$. To prove the second one, we differentiate the function f_α with respect to the parameter of an $\mathbf{R}^{>0}$ -orbit. Fix $x \in X_\Sigma$. If $t \in \mathbf{C}^*$ then the image of x by t is $t^\alpha x$. Then

$$f_\alpha(t^\alpha x) = \left(\sum_{i=0}^s |t^\alpha x|^{2m_i} \right)^{-1} \sum_{i=0}^s |t^\alpha x|^{2m_i} \langle \alpha, m_i \rangle.$$

We consider it as a real function in t , defined on $\mathbf{R}^{>0}$. Write for short $y = |t^\alpha x|$. Observe that

$$\frac{dy^m}{dt} = \frac{1}{t} \langle \alpha, m \rangle y^m.$$

Hence

$$t \cdot \frac{df_\alpha(t^\alpha x)}{dt} = 2 \left(\sum_{i=0}^s y^{2m_i} \right)^{-1} \cdot \left(\left(\sum y^{2m_i} \langle \alpha, m_i \rangle^2 \right) \cdot \left(\sum y^{2m_i} \right) - \left(\sum y^{2m_i} \langle \alpha, m_i \rangle \right)^2 \right).$$

Now assume that x is not a fixed point. Then y^{2m_i} are non-negative reals and at least one of them is positive. The second factor has the form from "Schwartz Lemma" (substitute $b_i = y^{m_i} \cdot \langle \alpha, m_i \rangle$ and $c_i = y^{m_i}$). Since all $\langle \alpha, m^i \rangle$ are distinct by assumption and some y^{2m_i} are non-zero, it follows that b_i and c_i are not proportional, and the second factor under discussion is positive. Also $t > 0$; hence $\frac{df_\alpha(t^\alpha \cdot x)}{dt} > 0$ for all $t \in \mathbf{R}^{>0}$.

c) The first statement is clear from the formula 3.2.6. The second one follows from the equality $f_\alpha(x) = \langle \alpha, m_i \rangle$ for some i , where x is a fixed point, and from the assumption on α .

d) follows from Proposition 3.2.4.

3.2.7. EXAMPLE AND REMARK. Consider the projective space as a torus embedding. Here Σ consists of all cones spanned by the elements $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, (-\varepsilon_1 - \varepsilon_2 - \dots - \varepsilon_n)$ where $\varepsilon_1, \dots, \varepsilon_n$ is a basis of M . Take as Δ the simplex $\text{conv}(0, \varepsilon_1^*, \dots, \varepsilon_n^*)$ where $(\varepsilon_i^*)_i$ is the dual basis in M , and

assume $m_0 = 0$, $m_1 = \varepsilon_1^*$, ..., $m_n = \varepsilon_n^*$. Let $a = a_1 \varepsilon_1 + \dots + a_n \varepsilon_n$. If $x \in X_\Sigma$ is a point with homogeneous coordinates $(z_0 : z_1 : \dots : z_n)$, then

$$f_\alpha(x) = \frac{\sum_{i=0}^n |x|^{2m_i} \cdot \langle \alpha, m_i \rangle}{\sum |x|^{2m_i}} = \frac{|z_1|^2 a_1 + \dots + |z_n|^2 a_n}{|z_0|^2 + |z_1|^2 + \dots + |z_n|^2}.$$

It is the well-known Morse function on the projective space, corresponding to the C^* -action of the form

$$t \cdot (z_0 : z_1 : \dots : z_n) = (z_0 : t^{a_1} z_1 : t^{a_2} z_2 : \dots : t^{a_n} z_n).$$

On the other hand, it turns out that the general form of the Morse function given by Theorem 3.2.5 may be obtained by restriction of the particular one written above, if the projective variety X_Σ is suitably immersed in P^s and equipped with a C^* -action. To see this, use the notation of Theorem 3.2.5 and consider the closed immersion $\varphi: X_\Sigma \hookrightarrow P^s$ given by the basis m_0, m_1, \dots, m_s of the linear system $\mathcal{L}(D)$ of the ample divisor D corresponding to Δ . Let C^* act on P^s by $t \cdot (z_0 : z_1 : \dots : z_s) = (t^{b_0} z_0 : t^{b_1} z_1 : \dots : t^{b_s} z_s)$, where $b_i = \langle \alpha, m_i \rangle$.

Let $g: P^s \rightarrow \mathbf{R}$ be given by

$$g(z_0, \dots, z_n) = \frac{\sum |z_i|^2 b_i}{\sum |z_i|^2}.$$

Then

1° X_Σ is a C^* -invariant subvariety of P^s , since

$$\begin{aligned} \varphi(t^\alpha x) &= ((t^\alpha x)^{m_0} : \dots : (t^\alpha x)^{m_s}) \\ &= (t^{b_0} x^{m_0} : \dots : t^{b_s} x^{m_s}) \\ &= t \cdot (x^{m_0} : \dots : x^{m_s}) = t \cdot \varphi(x); \end{aligned}$$

2° f_α is equal to g on X_Σ , since for $x \in X_\Sigma$

$$g(\varphi(x)) = g(x^{m_0} : \dots : x^{m_s}) = \frac{\sum |x|^{2m_i} \cdot \langle \alpha, m_i \rangle}{\sum |x|^{2m_i}}.$$

The Morse function defined in Theorem 3.2.5 will be used in Chapter 4, to homology calculations.

The problem of existence of a Morse function increasing along $R^{>0}$ -orbits on a torus embedding (with a distinguished 1-parameter subgroup is related to the notions of "cycles of C^* -orbits" and to the filtrations on the variety with C^* -action. The above concepts will be considered in the next section.

3.3. Filtrations, cycles of orbits and projectivity

In [B-B₃] A. Białynicki-Birula deals with filtrations on varieties with cell decomposition (compare the notion of spherical complex used by Ehlers, [E], 4.1.2). This notion is useful to apply the recurrency principle. By a cell we mean a variety isomorphic to affine space. A disjoint decomposition of a variety X

$$X = W_1 \dot{\cup} W_2 \dot{\cup} \dots \dot{\cup} W_m$$

is said to be *cellular* if W_i are cells and locally closed subvarieties of X . A cellular decomposition of X is said to be *filtrable* if there exists a sequence

$$X = X_0 \supset X_1 \supset \dots \supset X_m = \emptyset$$

of closed subvarieties of X such that $X_j \setminus X_{j+1}$ is a cell of the decomposition for $j = 0, 1, \dots, m-1$.

3.3.1. EXAMPLE. Let W_i be the subset of the projective space P^n , consisting of the points $(x_0 : x_1 : \dots : x_n)$ such that $x_{i-1} \neq 0$ and $x_j = 0$ for $j < i - 1$. Then W_i is an affine space of dimension $n - i + 1$. The filtration is given by $X = X_0 \supset X_1 \supset \dots \supset X_{n+1} = \emptyset$ where X_i is given by $x_0 = \dots = x_i = 0$.

The notion of filtrability is applied in [B-B₃] to the case of decompositions determined by the multiplicative group actions on a variety (see also 1.5.4). We restrict here our consideration to the case of a finite fixed point set.

The decomposition in question has as cells the repelled (or alternatively: attracted) parts of the variety, with respect to the given G_m -action. The obstruction for a decomposition to be filtrable may occur in the form of a "cycle of orbits", that is to say, a sequence of non-fixed points $y_1, y_2, \dots, y_m, y_{m+1} = y_1$ such that $\lim_{t \rightarrow \infty} t \cdot y_i = \lim_{t \rightarrow 0} t \cdot y_{i+1}$ for all i .

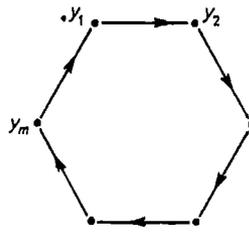


Fig. 12

Indeed, suppose that a cycle of orbits exists in X . If $F \subset X$ is a union of cells closed in X and if F contains one of the fixed points of the form $\lim_{t \rightarrow \infty} t \cdot x_i$, then it must contain all remaining fixed points occurring in the cycle. On the other hand, two subsequent closed subvarieties in a given

filtration always differ by one fixed point, since each cell contains one fixed point. Therefore a filtration does not exist.

In [B-B₃] the following is proved.

3.3.2. THEOREM. *Let X be a projective smooth variety. Then the decomposition into the repelled (or attracted) parts is filtrable.*

Nevertheless, using the method described in 1.5, one can construct examples of G_m -actions on complete smooth varieties which admit cycles of orbits and therefore both decompositions determined by the given action are not filtrable. An example has been reported in [J₁] and will be now described in more detail. In order to simplify terminology, we think of the vector $\alpha \in N$ as directed vertically upwards in $N_{\mathbb{R}}$ (compare 3.2.2). By Lemma 1.5.1, a cycle of orbits will occur in X_{Σ} if there is a sequence of cones of dimension n in Σ : $\sigma_1, \sigma_2, \dots, \sigma_m, \sigma_{m+1} = \sigma_1$, such that

- 1° $\sigma_i \neq \sigma_j$ for $i \neq j$,
- 2° $\dim \sigma_i \cap \sigma_{i+1} = n - 1$,
- 3° no one of the hyperplanes $\text{lin}(\sigma_i \cap \sigma_{i+1})$ is vertical,
- 4° σ_{i+1} is above $\text{lin}(\sigma_i \cap \sigma_{i+1})$ (and consequently σ_i is below $\text{lin}(\sigma_i \cap \sigma_{i+1})$, for all i).

3.3.3. EXAMPLE OF A CYCLE OF G_m -ORBITS. Let T be the torus of dimension 3, N the corresponding lattice with a basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$. We define in $N_{\mathbb{R}}$ the following complex of cones Σ . Begin with the polyhedron as in Figure 13 and consider the cones from the origin 0, over the faces of the

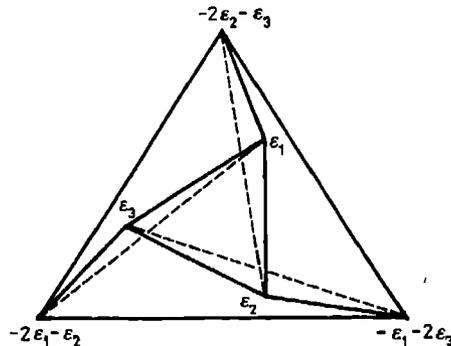


Fig. 13

polyhedron. Next decompose the inferior cone as showed in Figure 14. The set consisting of all cones thus obtained is a complex of cones, covering the whole space $N_{\mathbb{R}}$.

Indeed, it suffices to check that the cone B (see Fig. 15) intersects properly (that means, along faces) the cones G and C . All the remaining intersections either may be obtained by rotation of the above, or are obviously proper. Therefore we have to verify that: 1) the plane $\text{lin}(\varepsilon_1,$

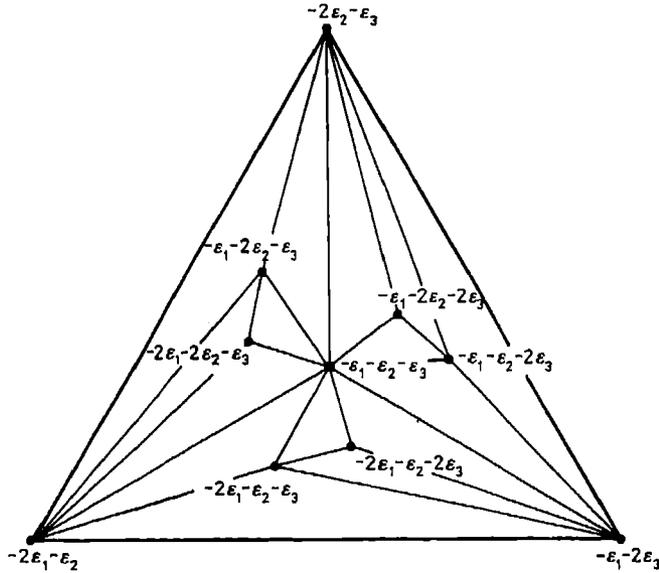


Fig. 14

$-2\varepsilon_1 - \varepsilon_2$) separates the vectors $(-2\varepsilon_2 - \varepsilon_3)$ and ε_3 , 2) the plane $\text{lin}(\varepsilon_1, -2\varepsilon_2 - \varepsilon_3)$ separates the vectors ε_2 and $(-2\varepsilon_1 - \varepsilon_2)$. Indeed (Fig. 16),

$$\det \begin{bmatrix} 0 & -2 & 1 \\ -2 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} < 0, \quad \det \begin{bmatrix} 0 & -2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} > 0, \quad \det \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} > 0$$

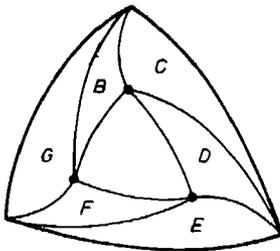


Fig. 15

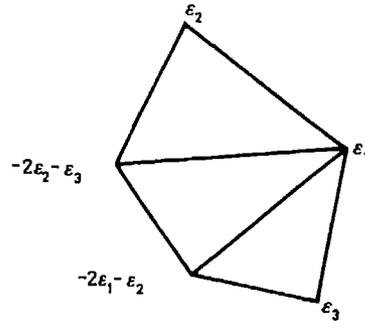


Fig. 16

(columns express vectors). The complex Σ is non-singular, as one can verify directly. Now let $\alpha' = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Consider the circular sequence of six n -dimensional cones of Σ : B, C, D, E, F, G, B (Fig. 15). The determinants

describing the situation of the vector α' with respect to the faces $B \cap C$, $C \cap D$ etc. are respectively

$$d_1 = \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad d_2 = \det \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad d_3 = \det \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

and so on, cyclically. Then

3.3.4. $d_1 < 0$, $d_2 < 0$, ..., $d_6 < 0$ (use rotation around the line $\langle \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \rangle$). By the remark preceding the present example, the sequence B, C, \dots, G makes a cycle of orbits. Unfortunately, α' belongs to the line spanned by the vector $(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3)$, and consequently X_Σ has non-isolated G_m -fixed points. Therefore we choose a vector $\alpha = n_1 \varepsilon_1 + n_2 \varepsilon_2 + n_3 \varepsilon_3$, which forms a sufficiently small angle with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and such that α does not belong to any 2-dimensional cone of Σ . The inequalities 3.3.4 with the column $[n_1, n_2, n_3]^T$ in place of $[1, 1, 1]^T$ hold. Now consider the G_m -action defined by α on the variety X_Σ associated with the complex Σ . By Lemma 1.5.1 the G_m -orbits determined by the cones $B \cap C$, $C \cap D$, ..., $G \cap B$ form a cycle of orbits.

3.3.5. Remark. If one does not require the non-singularity of the whole of X_Σ , one can disregard the triangulation of the face spanned by $-2\varepsilon_1 - \varepsilon_2$, $-\varepsilon_1 - 2\varepsilon_3$, $-2\varepsilon_2 - \varepsilon_3$ and take $\alpha' = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ in place of α . Still, any point occurring in the cycle will not be singular and the example becomes much simpler.

3.3.6. COROLLARY. a) *The variety X_Σ constructed in Example 3.3.3 is complete non-projective, smooth.*

b) *The decomposition of X_Σ into repelled (or attracted) parts determined by the vector α is non-filtrable.*

3.3.7. Remark. Take \mathbb{C} as the ground field. Then X_Σ constructed as in 3.3.3 can be regarded as an analytic manifold equipped with a \mathbb{C}^* -action. There does not exist any continuous real function on X_Σ increasing along the $\mathbb{R}^{>0}$ -orbits.

On the other hand, on any Kähler manifold with \mathbb{C}^* -action there exists even a Morse function satisfying the above condition (see [F]). In the case of projective torus embedding such a function is constructed in 3.2.

Another application of torus embedding in construction of pathological example of a G_m -action is given by J. Konarski in [K].

4. HOMOLOGY

4.1. Poincaré polynomial

We begin this chapter with calculation of the l -adic Betti numbers of X_{Σ} , that is, the dimensions b_i of $H_{\text{ét}}^i(X, \mathbf{Q}_l)$, by means of the complex Σ , in smooth complete case. The formula for ordinary Betti numbers in case of complex ground field was first found by E. Ehlers in [E]. Next, Danilov ([Dan]) gave a few demonstrations of this formula. Curiously, the Poincaré polynomial of X_{Σ} , $P(t) = \sum b_{2i} t^{2i}$ is expressed by a formula much simpler than the Betti numbers themselves cf. 4.1.1. We give a proof of the formula using an interpretation of the polynomial $P(t)$ by means of the complex Σ (see also [J₂]).⁽¹⁾

Consider the complex vector space $N_{\mathbf{C}}$ and its standard immersion into the complex projective n -dimensional space P . The Chow ring of P is the truncated polynomial ring $\mathbf{Z}[U]/U^{n+1}$ where the coset u of U is the class of a hyperplane. To any finite set of cones $\{\sigma_i\}$ in $N_{\mathbf{R}}$ we attach an element \sum_i (class of the projective subspace spanned by σ_i) of the Chow ring of P . Let a complete non-singular complex of cones be given. Let d_i the number of cones of codimension i in Σ . Then obviously the element of the Chow ring of P corresponding to the whole Σ is $\sum_i d_i u^i \in \mathbf{Z}[U]/U^{n+1}$. On the other hand, the above element may be calculated using the l -adic Betti numbers of X_{Σ} . Consider the G_m -action on X_{Σ} determined by $\alpha \in N$ such that the fixed points set is finite and let $X_{\Sigma} = \bigcup X_j$ be the decomposition of X_{Σ} into the repelled parts (1.5). By Theorem 1.5.3 applied to the non-singular case, X_j are of the form $X_j = \bigcup \mathcal{O}_{\tau}$, τ running over the set $C_j = \{\text{faces of } \omega_j \text{ containing } \sigma_j\}$ where σ_j and ω_j are cones of Σ fixed for a given j , and $\dim \omega_j = n$.

By "Weil conjectures",

$$b_j = \begin{cases} \text{number of cells of dim } j/2 \text{ in } X_{\Sigma}, & \text{for } j \text{ even,} \\ 0 & \text{otherwise} \end{cases}$$

(see [B-B₂], [G], [Del]). (Notice that the above formula holds whenever the cells are locally closed, no matter in which way they are glued together. This

⁽¹⁾ A shorter argument is added in proof, comment 2°.

allows to go over the difficulties of the non-projective case.) It follows that

$$b_{2i} = \# \{j: n\text{-dim } \sigma_j = i\}.$$

Observe that $\bigcup_{\tau \in C_j} \tau$ is the intersection of r half-spaces and $(n-r)$ hyperplanes, where r is the codimension of σ_j (see [J₂] for details). It follows that the element $(u+1)^r$ in the Chow ring of P corresponds to the set of cones C_j , and finally, the whole complex Σ determines the element $\sum_{i=0}^n b_{2i}(u+1)^i$.

Comparing this with the previous formula we get

4.1.1. PROPOSITION. *The Poincaré polynomial of X_{Σ} is*

$$P(t) = \sum d_j(t-1)^j$$

and consequently

$$b_{2i} = \sum_j (-1)^{j-i} \binom{j}{i} d_j.$$

The odd Betti numbers are equal to zero.

4.2. Chow ring and l -adic cohomology

Let Σ be a non-singular complete complex of cones, $\alpha_1, \alpha_2, \dots, \alpha_k \in N$ the primitive vectors spanning the 1-dimensional cones in Σ and let D_i be the closure of the $(n-1)$ -dimensional orbit O_{α_i} for $i = 1, 2, \dots$

4.2.1. LEMMA. *Let $m \in M$. Then the order of zero of the rational function \mathcal{X}^m on D_i is $\langle \alpha_i, m \rangle$ (see for instance [TEA]). Then the principal divisor on X_{Σ} determined by \mathcal{X}^m has the form*

$$(\mathcal{X}^m) = \sum_{i=1}^k \langle \alpha_i, m \rangle \cdot D_i.$$

Passing to the Chow ring of X_{Σ} we get that $\sum \langle \alpha_i, m \rangle \text{cl } D_i = 0$, where $\text{cl } D_i$ stands for the class of rational equivalence of the cycle D_i . Another obvious relation between the above classes is $\text{cl } D_{i_1} \cdot \text{cl } D_{i_2} \cdot \dots \cdot \text{cl } D_{i_s} = 0$, whenever $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_s}$ span no cone in Σ .

It turns out that the elements $\text{cl } D_i$ generate the Chow ring of X_{Σ} and the above two types of relations generate all the relations between $\text{cl } D_i$. The system of relations can be reduced in such a way that we get

4.2.2. THEOREM. *In a fixed coordinate system in N let $\alpha_i = (a_{i1}, a_{i2}, \dots, a_{in})$ for $i = 1, 2, \dots, k$. Let I be the ideal in the polynomial ring $Z[U_1, U_2, \dots, U_k]$ generated by linear polynomials $\sum_{i=1}^k a_{ij} U_i$ and J the ideal generated by all monomials $U_{i_1} \cdot U_{i_2} \cdot \dots \cdot U_{i_s}$ such that $(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_s})$ are*

minimal sequences spanning no face in Σ . Then the natural homomorphism

$$\mathbf{Z}[U_1, \dots, U_k]/(I+J) \rightarrow (\text{Chow ring of } X_\Sigma)$$

is an isomorphism of graded rings. Those rings are torsion-free, and hence free \mathbf{Z} -modules. If l is an integer prime to the characteristic of the ground field then the natural homomorphism

$$\mathbf{Z}[U_1, \dots, U_p]/(I+J) \otimes \mathbf{Q}_l \rightarrow H_{\text{ét}}^*(X_\Sigma, \mathbf{Q}_l)$$

is an isomorphism. In case of the complex ground field,

$$\mathbf{Z}[U_1, \dots, U_k]/(I+J) \cong H^*(X_\Sigma, \mathbf{Z}).$$

The ‘‘Chow ring’’ and ‘‘ l -adic’’ part was proved by the author in [J₂] in the following cases:

- 1) when X is projective,
- 2) when X is complete of $\dim \leq 3$.

Then Danilov proved the general version in [Dan], Ch. 3.

4.3. Cohomology ring of $X_\Sigma(\mathbf{R})$ with coefficients in $\mathbf{Z}/2\mathbf{Z}$ ⁽¹⁾

See [M] for the basic relations between Morse functions and homology.

In this section $*$ denotes reduction mod 2. We consider the case of the ground field C , and assume Σ to be projective, non-singular. We shall find formulas analogous to Theorem 4.2.2 for the real part $X_\Sigma(\mathbf{R})$ of the complex variety X_Σ . The author is able to calculate cohomology with integral coefficients only for torus embedding of dimension 2 (see 4.5). In the general projective case we restrict our considerations to $H^*(X_\Sigma(\mathbf{R}), \mathbf{Z}_2)$.

For the rest of this paper let V denote the real part of X_Σ for a given complex Σ . We shall often omit the coefficient ring \mathbf{Z}_2 in the notation of homology groups. By Poincaré duality, we identify $H^k(V, \mathbf{Z}_2)$ with $H_{n-k}(V, \mathbf{Z}_2)$. For $i = 1, 2, \dots, h$ let E_i be the closure of $\mathcal{O}_{\alpha_i}(\mathbf{R})$. This is a closed subvariety of V , of codimension 1. The corresponding element of $H^1(V, \mathbf{Z}_2)$ will be denoted by $\text{cl}(E_i)$.

4.3.1. THEOREM. *The homomorphism*

$$p: \mathbf{Z}_2[U_1, \dots, U_h] \rightarrow H(V, \mathbf{Z}_2), \quad U_i \mapsto \text{cl}(E_i),$$

induces an isomorphism of graded rings

$$i: \mathbf{Z}_2[U]/(I^* + J^*) \cong H^*(V, \mathbf{Z}_2), \quad \text{where } \deg U_1 = \deg U_2 = \dots = 1.$$

⁽¹⁾ F. Ehlers in [E] gives a formula for the groups $H_k(X_\Sigma(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$, for an arbitrary complete torus embedding. However, it seems that there is an error in his proof. As far as I understand, the Theorems 2 and 3 are not correct if X_Σ is not assumed to be projective (compare 3.3 of the present paper or [J₂]). After all, my proof does need the same assumption.

Proof. The subvarieties E_1, E_2, \dots, E_h intersect transversally (use local coordinates). The intersections $E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_s}$ (where i_1, \dots, i_s are all distinct) are also closed subvarieties, and hence they define the elements $\text{cl}(E_{i_1} \cap \dots \cap E_{i_s}) = \text{cl}(E_{i_1}) \cdot \dots \cdot \text{cl}(E_{i_s})$ of $H^*(V, \mathbb{Z}_2)$. We shall describe some relations between these elements. Consider the decomposition of V into $T^{>0}$ -orbits (Fig. 17). By 2.4.1, V consists of 2^n copies of the variety with corners

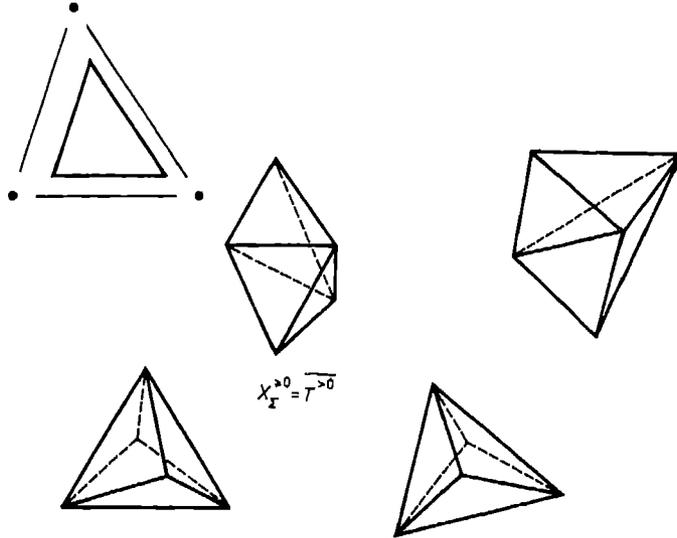


Fig. 17

$X_{\mathbb{R}}^{\geq 0}$, identified with a polyhedron Δ (see 3.1), glued together along faces of dimension $n-1$, in the way indicated by the complex. The $T^{>0}$ -orbits are the open faces (compare 4.4 and Fig. 21).

This is a cellular decomposition and each variety $E_{i_1} \cap \dots \cap E_{i_s}$ is a union of cells (not confound them with the cells defined in 3.3). For a given $m \in M$ the character \mathcal{X}^m is regular and not null on n -cells, for they are all contained in $T(\mathcal{R})$. We consider the chain \mathcal{V}_m , with coefficients in \mathbb{Z}_2 , of those n -cells in which \mathcal{X}^m has a positive value. The boundary $\partial \mathcal{V}_m$ of this chain consists of $(n-1)$ -cells on which the rational function \mathcal{X}^m changes sign; hence the order of zero of this function is odd. From Lemma 4.2.1 we conclude that for $i = 1, 2, \dots, m$:

$$E_i \text{ occurs in } \partial \mathcal{V}_m \Leftrightarrow \langle \alpha_i, m \rangle \equiv 1 \pmod{2}.$$

Passing to cohomology classes, we get

$$\sum_i \langle \alpha_i, m \rangle^* \cdot \text{cl}(E_i) = 0, \quad \text{for all } m.$$

In particular,

$$(i) \quad \sum_i a_{ij} \text{cl}(E_i) = 0, \quad \text{for } j = 1, 2, \dots, n.$$

On the other hand, if $(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_s})$ is a minimal sequence spanning no face, then $E_{i_1} \cap \dots \cap E_{i_s} = \emptyset$ and hence

$$(ii) \quad \text{cl}(E_{i_1}) \cdot \dots \cdot \text{cl}(E_{i_s}) = 0.$$

It follows from (i) and (ii) that the homomorphism p induces the homomorphism i of the theorem. It remains to prove that it is an isomorphism. For this purpose, using the Morse function on V defined in Theorem 3.2.5, for some fixed α , we shall show the way in which a basis of $H^*(V, \mathbf{Z}_2)$ can be chosen from among elements of the form $\text{cl}(E_{i_1}) \cdot \text{cl}(E_{i_2}) \cdot \dots \cdot \text{cl}(E_{i_s})$.

Let x_i be a fixed point and \mathcal{O} a $T(\mathbf{R})$ -orbit in V . We shall say that \mathcal{O} culminates in x_i if $x_i \in \mathcal{O}$ and $f(x_i) \geq f(x)$ for all $x \in \mathcal{O}$ (compare 3.2.1). We assume that the critical points x_1, x_2, \dots, x_r (or equivalently, $T(\mathbf{R})$ -fixed points) are enumerated in such a way that $f(x_1) < f(x_2) < \dots < f(x_r)$. Let λ_i be the index of the function f at the critical point x_i . Among the orbits culminating in x_i there is precisely one of maximal dimension, equal to $n - \lambda_i$. The closure of this orbit, which is a subvariety of V , will be denoted by V_i^+ ⁽¹⁾. Let V_l denote $f^{-1}(]-\infty, f(x_l)[$ (the inverse image of the open half-line), for $l = 1, 2, \dots, r$, and $V_{r+1} = V$. Then $V_l^+ \subset V_{l+1}$. We shall express the homology of V by relative homology groups $H_k(V_{l+1}, V_l)$. By the standard properties of a Morse function,

$$H_k(V_{l+1}, V_l; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{for } k = n - \lambda_l, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the standard homomorphism

$$H_{n-\lambda_l}(V_{l+1}, \mathbf{Z}_2) \rightarrow H_{n-\lambda_l}(V_{l+1}, V_l; \mathbf{Z}_2).$$

4.3.2. LEMMA. *The image of $\cdot \text{cl}(V_i^+)$ is not null (hence it is the generator).*

Proof. Consider the formula 3.2.6 for the Morse function f , in the local coordinates at the point x_i . We may suppose that $a_1, a_2, \dots, a_{\lambda_i}$ are positive and $a_{\lambda_i+1}, \dots, a_n$ negative. Then in a neighbourhood of x_i , V_i^+ is the subvariety defined by $z_{\lambda_i+1} = \dots = z_n = 0$, where z_i is the coordinate corresponding to a_i . This proves the V_i^+ may be taken as the "lover disc" for the point x_i .

4.3.3. COROLLARY. *The classes of V_1^+, V_2^+, \dots form a basis of $H_*(V, \mathbf{Z}_2)$.*

Proof. Consider the exact sequence

$$H_{n-\lambda_{l+1}}(V_{l+1}, V_l) \rightarrow H_{n-\lambda_l}(V_l) \rightarrow H_{n-\lambda_l}(V_{l+1}) \rightarrow H_{n-\lambda_l}(V_{l+1}, V_l)$$

⁽¹⁾ It coincides with the real part of the closure of cells defined in [E], 4.1.2 with α in place of v .

(the coefficient ring \mathbf{Z}_2 is omitted). The first space is $\{0\}$ and the last homomorphism of \mathbf{Z}_2 -spaces is surjective. The resulting short exact sequence splits, the last space is spanned by $\text{cl}(V_i^+)$ and hence

$$H_{n-\lambda_l}(V_{i+1}) = H_{n-\lambda_l}(V_i) \oplus \text{cl}(V_i^+).$$

Homologies in the remaining dimensions do not change when the point x_i is exceeded. Finally,

$$H_r(V) = \bigoplus_{l:\lambda_l=n-r} \langle \text{cl}(V_l^+) \rangle.$$

Now, for fixed l , let \mathcal{O} be the maximal (with respect to the dimension) $T(\mathbf{R})$ -orbit culminating in x_i , and $j_1, j_2, \dots, j_{\lambda_l}$ be indices such that $\alpha_{j_1} \cdot \mathbf{R}^{\geq 0} + \alpha_{j_2} \cdot \mathbf{R}^{\geq 0} + \dots + \alpha_{j_{\lambda_l}} \cdot \mathbf{R}^{\geq 0}$ is the face of Σ corresponding to \mathcal{O} . Then the monomial $U_{j_1} \cdot U_{j_2} \cdot \dots \cdot U_{j_{\lambda_l}}$ is sent onto $\text{cl}(V_i^+)$ by the homomorphism p . It is shown in [J₂] that the set of monomials which arise in this way generates the \mathbf{Z} -module $\mathbf{Z}[U]/(I+J)$; hence also the \mathbf{Z}_2 -space $\mathbf{Z}_2[U]/(I^*+J^*)$. In this way have indicated the generators of $\mathbf{Z}_2[U]/(I^*+J^*)$ whose images under i form a basis of the space $H^r(V, \mathbf{Z}_2)$. It follows that i is bijective. Moreover, multiplications in the two rings coincide, as required.

4.4. Orientation

The variety V is equipped with the action of the group μ of reflexions (see 2.4.1). Also, for all k the group μ acts on the set of k -cells. Moreover, for $k = n$ it acts effectively and transitively. This action allows us to define orientations in cells: choose arbitrary orientations in the cells occurring in $X_{\Sigma}^{\geq 0}$ and transport them to all other cells by reflexions. This procedure is well defined in view of

4.4.1. LEMMA. *Let u and w be two reflexions and K_1, K_2 two cells such that $u \cdot K_1 = K_2$, $w \cdot K_1 = K_2$. Then the actions of u and w coincide on K_1 .*

Proof. The cell K_1 is of the form $y \cdot K^0$ where K^0 is contained in $X_{\Sigma}^{\geq 0}$ and y is a reflexion. Therefore we may suppose that K_1 itself is contained in $X_{\Sigma}^{\geq 0}$. Consider the reflexion $v = w^{-1}u$. Then $v \cdot K_1 = K_1$. Let ε be the distinguished point contained in K_1 . We shall find the image of ε . For any $m \in M$ such that the function \mathcal{R}^m is regular in ε we have $(v \cdot \varepsilon)^m = v^m \cdot \varepsilon^m = \pm 1$ or 0. Nevertheless, $(v \cdot \varepsilon)^m \geq 0$, since $v \cdot \varepsilon \in K_1 \subset X_{\Sigma}^{\geq 0}$. It follows that $v \cdot \varepsilon$ is the distinguished point, i.e. ε . Finally, v is the identity on the whole $T^{>0}$ -orbit through ε .

4.4.2. Remark. The orientations just defined have the following property: whenever a $(k-1)$ -cell K_1 is in the boundary of the k -cells K_2 and

K_3 , then the orientation of both these k -cells induces the same orientation on K_1 , for all k . Hence one gets a kind of a "completely disoriented complex".

4.4.3. THEOREM. *Let $\alpha_1, \alpha_2, \dots, \alpha_h$ be as in 4.2. Then the variety V is not orientable iff there exists a subset $\{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_s}\}$ of the set $\{\alpha_i\}$ such that s is odd and $\alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_s} \equiv 0 \pmod{2}$. (Other formulation of this condition: there exist integers c_1, c_2, \dots, c_h such that $c_1 + c_2 + \dots + c_h \equiv 1$ and $c_1 \alpha_1 + \dots + c_h \alpha_h = 0 \pmod{2}$).*

Sketch of proof. Let $(K_0, K_1, \dots, K_{r-1}, K_0)$ be a circular sequence of n -cells, such that K_j and K_{j+1} have common $(n-1)$ -cells in the boundary, $j = 0, 1, \dots, r-1$. The oriented cells $K_0, -K_1, K_2, \dots, (-1)^{r-1} K_{r-1}$ have compatible orientations, by Remark 4.4.2 (Fig. 18). The last orientation is

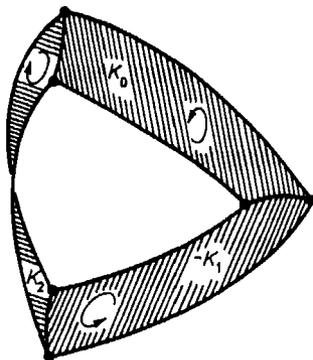


Fig. 18. Notice that not all points of contact of the cells K_0, K_1 and K_2 are indicated.

compatible with the orientation of K_0 iff r is even. For $j = 0, 1, \dots, r-1$ we choose a common $(n-1)$ -face of K_j and K_{j+1} , and let α_j be the corresponding vector among $\alpha_1, \dots, \alpha_m$. We consider the reflexion $(-1)^{\alpha_{j_1}}, \dots, (-1)^{\alpha_{j_r}}$ and conclude that their product equals to $1 \in T$; hence $\alpha_{j_1} + \dots + \alpha_{j_r} \equiv 0 \pmod{2}$.

Conversely, the sequence $\alpha_{j_1}, \dots, \alpha_{j_r}$ such that $\sum \alpha_{j_k} \equiv 0$ gives rise to a circular sequence of n -cells, for example

$$T^{>0}, (-1)^{\alpha_{j_1}} \cdot T^{>0}, (-1)^{\alpha_{j_1} + \alpha_{j_2}} \cdot T^{>0}, \dots,$$

which admit compatible orientations iff r is odd.

4.5. Two-dimensional case, homology with integral coefficients

Through this section for $c \in \mathbf{Z}$, we designate by c^* the integer (not the element of \mathbf{Z}_2 !) 0 if c is even and 1 otherwise. We consider the case of $n = 2$; then V is a real surface.

The complex Σ is shown in Figure 19.

Each subvariety E_i is the circle $P_1(\mathbb{R})$; hence, with a suitable orientation, it determines an element $\text{cl}(E_i)$ of $H_1(V, \mathbb{Z})$.

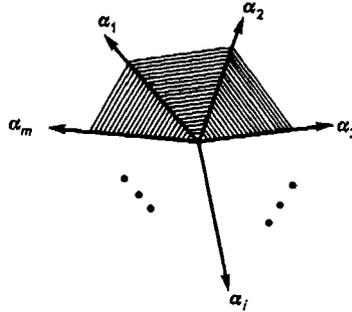


Fig. 19

4.5.1. THEOREM. In the notation of 4.2, let Σ be two-dimensional complex. Then

a) $H_1(V, \mathbb{Z})$ is generated by the elements $\text{cl}(E^i)$, $i = 1, \dots, h$, with the relations:

$$\sum \langle \alpha_i, m \rangle^{\#} \text{cl}(E_i) = 0, \quad \text{for all } m \in M.$$

b) This infinite system of relations can be replaced by three of them. In the coordinate system of 4.2,

$$H_1(V, \mathbb{Z}) \cong \mathbb{Z}^h/A$$

where A is the subgroup generated by the elements

$$(a_{11}^{\#}, a_{21}^{\#}, \dots, a_{h1}^{\#}), \quad (a_{12}^{\#}, a_{22}^{\#}, \dots, a_{h2}^{\#}), \quad ((a_{11} + a_{12})^{\#}, \dots, (a_{h1} + a_{h2})^{\#}).$$

c) $H_1(V, \mathbb{Z}) \cong \mathbb{Z}^{h-3} \oplus \mathbb{Z}_2$ (hence V is not orientable), except when the complex Σ looks as in Figure 20, with d even. In the last case V is homeomorphic to the torus $S^1 \times S^1$.

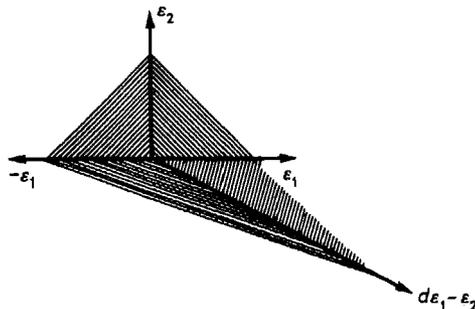


Fig. 20

Proof. a) There are 4 reflexions in this case. In the given coordinate system we can designate them by $(++)$, $(+-)$, $(-+)$ and $(--)$. For example, $(+-)$ stands for $(1, -1) \in T = \mathbf{C}^* \times \mathbf{C}^*$. Let W_1, W_2, \dots, W_h be the 1-dimensional $T^{>0}$ -orbits in $X_{\mathbb{R}}^{\geq 0}$. For each i , among the sets $W_i, (+-)\cdot W_i, (-+)\cdot W_i, (--)\cdot W_i$ there are two different ones. One of them is W_i and the other will be designated by W_i^∞ (Fig. 21). With this notation the cells are:

- 0-cells: x_1, x_2, \dots, x_h (fixed points),
- 1-cells: $W_1, W_1^\infty, W_2, W_2^\infty, \dots, W_h, W_h^\infty$,
- 2-cells: $T^{>0}, (+-)\cdot T^{>0}, (-+)\cdot T^{>0}, (--)\cdot T^{>0}$.

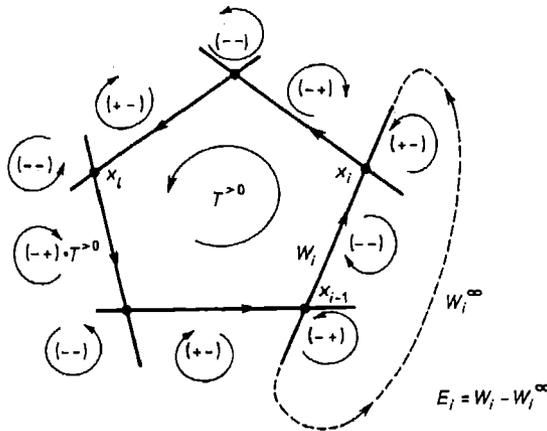


Fig. 21

The orientation is shown in Figure 21 (compare with 4.4). In analogy to 4.3 we define, for each $m \in M$, the 2-chain \mathcal{V}_m with coefficients in \mathbf{Z} , consisting of 2-cells on which \mathcal{X}^m is positive. The coefficients are chosen in such a way that the orientations of the cells are compatible; see Remark 4.4.2.

Namely, \mathcal{V}_m is either $T^{>0} - (+-)\cdot T^{>0}$ or $T^{>0} - (-+)\cdot T^{>0}$ or $T^{>0} - (--)\cdot T^{>0}$, depending on m .

The orientation cycle of E_i may now be written in the form $W_i - W_i^\infty$. The 0-chains corresponding to the fixed points will be denoted by the same symbols x_1, x_2, \dots, x_r . The group of chains with respect to the cellular decomposition under consideration can be written in the form convenient to homology computation:

$$C_2 = \sum_{m \in M} \langle \mathcal{V}_m \rangle \oplus \langle T^{>0} \rangle,$$

($\langle \mathcal{V}_m \rangle$ is the group of chains generated by \mathcal{V}_m , etc.),

$$C_1 = \langle E_1, E_2, \dots, E_h \rangle \oplus \langle W_1, W_2, \dots, W_h \rangle,$$

$$C_0 = \langle x_1, x_2, \dots, x_h \rangle.$$

To describe the boundary map ∂_2 , notice that if the cell W_i occurs in the boundary of the chain \mathcal{V}_m for some m , then also W_i^∞ occurs in it, and by the argument similar to that of § 1 we get

$$\partial_2(\mathcal{V}_m) = \sum_i \langle \alpha_i, m \rangle^* (W_i - W_i^\infty) = \sum \langle \alpha_i, m \rangle^* E_i.$$

Further,

$$\begin{aligned} \partial_2(T^{>0}) &= W_1 + W_2 + \dots + W_h, \\ \partial_1(E_i) &= 0, \quad \partial_1(W_i) = x_i - x_{i-1}. \end{aligned}$$

Hence we obtain:

$$\begin{aligned} H_1(V, \mathbf{Z}) &= H_1(\sum \langle \mathcal{V}_m \rangle \rightarrow \langle E_1, \dots, E_h \rangle \rightarrow 0) \oplus \\ &\oplus H_1(\langle T^{>0} \rangle \rightarrow \langle W_1, \dots, W_h \rangle \rightarrow \\ &\rightarrow \langle x_1, \dots, x_h \rangle) = \langle E_1, \dots, E_h \rangle / \sum_m \langle \partial \mathcal{V}_m \rangle, \end{aligned}$$

since the second sequence is exact.

b) It suffices to notice that the chains \mathcal{V}_m are all of the form either $\mathcal{V}_{(1,0)}$ or $\mathcal{V}_{(0,1)}$ or $\mathcal{V}_{(1,1)}$.

c) A priori, in the sequence $\alpha_1, \alpha_2, \dots, \alpha_h$ there can occur, mod 2, r distinct elements, where $r = 1$ or 2 or 3. Nevertheless, the case $r = 1$ cannot occur for any nonsingular complex Σ .

$r = 2$: If $X_{\mathcal{Z}}$ is not a relatively minimal surface, then it contains an exceptional divisor corresponding to the element α_i for some i . It results that $\alpha_i = \alpha_{i-1} + \alpha_{i+1}$ (see e.g. [TEA]). This excludes the case of $r = 2$. Otherwise, if $X_{\mathcal{Z}}$ is relatively minimal, one considers all possible cases to conclude that only the complex as shown in Figure 20 with d even, leads to $r = 2$. In that case, using assertion b), we obtain that $H_1(V, \mathbf{Z}) \cong \mathbf{Z}^2$.

$r = 3$: Since the order in the set $\{\alpha_1, \dots, \alpha_h\}$ is irrelevant in the proof, we may assume that $\alpha_1 = (1, 0)$, $\alpha_2 = (0, 1)$, $\alpha_3 = (1, 1)$. Then the subgroup A of \mathbf{Z}^h is spanned by three elements of the form

$$a = (1, 0, 1, *, *, \dots, *), \quad b = (0, 1, 1, *, *, \dots, *), \quad c = (1, 1, 0, *, *, \dots, *),$$

with $a + b \equiv c \pmod{2}$. Hence $\mathbf{Z}^h/A \cong \mathbf{Z}^{h-3} \oplus \mathbf{Z}_2$, which completes the proof.

4.5.2. Remark. Sketch of another proof of assertion c): $X_{\mathcal{Z}}$ can be obtained either from the complex surface $P^2(\mathbf{C})$ or $P^1(\mathbf{C}) \times P^1(\mathbf{C})$ or $F_d(\mathbf{C})$ by several blowing up's. It follows that $V = X_{\mathcal{Z}}(\mathbf{R})$ is obtained either from the surface $P_2(\mathbf{R})$ or the torus $S^1 \times S^1$ (for d even) or the Klein bottle (for d odd) by glueing in a suitable number of Moebius bands in place of points of V .

Theorem 4.4.3 is somewhat provisional. It might be interesting to express the ring $H(V, \mathbf{Z})$ in a similar form to that of Theorem 4.3.1, for arbitrary dimension.

Added in proof

1° Theorem 3.1.4 b), first published in 1981 in [J₃], was in fact a version of the „convexity theorem for the moment map”, cf. Atiyah [A] from 1982, a paper by L. Ness with the appendix by Mumford [N+M] from 84 and Guillemin and Sternberg.

Indeed, let $\Delta \cap M = \{m_0, m_1, \dots, m_s\}$ in the notation of Chapter 3. Take in C^{s+1} the standard norm $\|v\| = \sum |v_i|^2$. Consider the action of T on C^{s+1} given by

$$(*) \quad g \cdot v = (g^{m_0} \cdot v_0, \dots, g^{m_s} \cdot v_s)$$

and the corresponding action of T on P^s . Let $u: X_\Delta \hookrightarrow P^s$ be given by $u(x) = (x^{m_0}, \dots, x^{m_s})$. Then $u(X_\Delta)$ is a closed invariant subset of P^s and $u: T \hookrightarrow u(X_\Delta)$ is a torus embedding isomorphic to $(T \subset X_\Delta)$. According to [N+M] the moment map \mathcal{M} with respect to the action $(*)$ is defined as follows. Take $x \in P^s$, $v \in C^{s+1}$ lying over x , $g \in T$ and set $r_v(g) = \|g \cdot v\|^2$. The norm is invariant with respect to the compact torus $c = T$. The tangent space at the identity e to $T^{>0} = T/(c - T)$ is canonically isomorphic to $N_{\mathbf{R}} = N \otimes \mathbf{R}$; hence the differential $d_e(r_v)$ taken at the identity may be viewed as an element of $N_{\mathbf{R}}^* = M_{\mathbf{R}}$. By definition $\mathcal{M}: P^s \rightarrow M_{\mathbf{R}}$, $\mathcal{M}(x) = \|v\|^{-2} d_e(r_v)$. Observe that under the previous identification $d_e(g \mapsto |g^m|) = m$. It follows that $\mathcal{M}(x) = 2 \cdot \|v\|^{-2} \sum_i |v_i|^2 m_i$. In order to obtain the moment map on X_Δ

we have to precede \mathcal{M} by the immersion u . Finally we get

$$\mathcal{M} \cdot u: X_\Delta \rightarrow M_{\mathbf{R}}, \quad y \mapsto (\sum |y|^{2m_i} \cdot 2m_i) (\sum |y|^{2m_i})^{-1}.$$

This is (up to the factor 2) the map

$$X_\Delta \rightarrow X_\Delta^{\geq 0} \rightarrow X_\Delta^{\geq 0} \rightarrow \Delta$$

from page 26 of [J₃] (Thm. 3.2.5 of the present paper). Theorem 2 part b) of [J₃] (Thm 3.1.4 b)) obviously implies the convexity of the image of X_Δ by the moment map. Atiyah ([A]) proves the similar statement in the context of Kähler manifolds. Mumford's proof in [N+M] covers the general case of a reductive group action (and needs some refinement of the map \mathcal{M}).

2° One more proof of Ehlers formula (Prop. 4.1.1): For $j = 0, 1, \dots, n$ there are d_j orbits of dimension j in $X_{\mathbf{F}}$. The orbits are locally closed subschemes defined over \mathbf{Z} and isomorphic to some products $G_m \times \dots \times G_m$. For $r = 1, 2, \dots$ and the field F_{p^r} , the number of F_{p^r} -points of G_m is $p^r - 1$. It follows that the number N_r of F_{p^r} -points of $X_{\mathbf{F}}$ is $N_r = \sum_{j=0}^n d_j (p^r - 1)^j$. Let

a_0, a_1, \dots, a_n be integers such that $\sum a_j t^j = \sum d_j (t-1)^j$ (t is a variable). Then

$$(**) \quad N_r = \sum a_j p^{rj} = \sum a_j (p^j)^r.$$

$X_{\mathcal{Z}}$ is a scheme defined over \mathcal{Z} , smooth in all characteristics and complete. Then, by "Weil conjecture" (proved in [Del]), if $e_1, e_2, \dots, e_u, f_1, f_2, \dots, f_v$ are complex numbers such that $e_\alpha \neq f_\beta$ for all α, β and $N_r = \sum e_\alpha^r - \sum f_\beta^r$ for all r , then the Betti numbers are $b_{2i} = \#\{j: |e_j| = p^i\}$, $b_{2i+1} = \#\{j: |f_j| = p^{i+1/2}\}$. So by (**), $b_{2i}(X_{\mathcal{Z}}) = a_i$, $b_{2i+1}(X_{\mathcal{Z}}) = 0$ for all i .

Finally, I would like to thank James B. Carrell for suggesting to me the connection between the moment map and [J₃].

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Index

- | | |
|--|--|
| <p>Attracted point 1.5.2
 Complete complex 1.2.9
 Compact torus 2.2.1
 Dual cone 1.2.1
 Face 1.2.1
 —, orthogonal to 1.2.1, 3.1
 Index 3.2.3
 Limit of 1-p.s. 1.2.10</p> | <p>Non-singular cone, complex 1.2.8
 Radial factor 2.2.1
 Reflexion 2.4.1
 Repelled point 1.5.2
 r-equivalence 1.3.2
 r-e.c. 1.3.3
 Variety 1.2.6 -
 —, with corners 2.2.2</p> |
|--|--|

List of symbols

- | | | |
|---|--|---|
| <p>$N, N_{\mathbb{Q}}, N_{\mathbb{R}}$ 1.2.1
 $M, M_{\mathbb{Q}}, M_{\mathbb{R}}$ 1.2.1
 $\mathbb{R}^{>0}, \mathbb{R}^{\geq 0}$ 1.2.1
 $\sigma, \tau, \bar{\sigma}$, 1.2.1
 τ^{\perp} 1.2.1, 3.1
 $\sigma - \tau, \Sigma$ 1.2.2
 A_{σ} 1.2.2
 T 1.2.5
 $X_{\mathbb{F}}$ 1.2.4
 X_{σ} 1.2.6
 $\mathcal{X}, \mathcal{X}^m$ 1.2.9</p> | <p>y^m, λ_{α}, 1-p.s., t^{α} 1.2.9
 $0^{\alpha}, \varepsilon_r$ 1.2.10
 $\mathcal{O}_r, \bar{\mathcal{O}}_r$ 1.2.11
 \mathcal{O}_{α} 1.2.12
 $X_{\mathbb{F}}(\mathbb{R})$ 2.1, 2.4
 $U(1), c - T, T^{>0}$ 2.2.1
 $X_{\mathbb{F}}^{\geq 0}$ 2.2.2
 $\arg x, x$ 2.2.1, 2.2.4, 2.2.5
 $c - \mathcal{O}_r, \mathcal{O}_r^{\geq 0}$ 2.2.4
 $c - X_{\mathbb{F}}$ 2.2.6
 $\text{Hom}_{s,g}$ 2.2.8</p> | <p>Exp 2.2.12
 Log 2.2.12
 μ 2.4.1
 Δ 3.1
 $\#$ 4.1, 4.5
 $\text{cl}(E)$ 4.3
 $*$ 4.3
 V_r^+ 4.3.1</p> |
|---|--|---|

