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Derivatives of noninteger order and their applications

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CONTENTS

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Introduction

In this paper we define the derivatives of noninteger order for functions of several real variables, examine their properties and solve certain initial and boundary value problems for differential equations of noninteger order. In Chapter V we also present other applications of these derivatives.

G. W. Leibniz was the first who gave a definition of the derivative of noninteger order and since then the existence, properties and applications of such derivatives have been examined by many authors with the considerations focused mostly on the one-dimensional case (cf. [34] and [40]).

A number of the results discussed here, especially those concerning initial and boundary value problems for differential equations of noninteger order, have been published in the author's earlier papers (cf. [24]–[28]); however, in this paper these results are generalized and uniformly presented.

The paper consists of five chapters. In Chapter I we introduce the notation, and give the definition and basic properties of the derivatives of noninteger order.

Chapters II–IV deal with boundary value, multipoint and initial value problems for partial and ordinary differential equations of noninteger order, respectively.

Chapter II concerns a characteristic problem which generalizes the Darboux problem for the Mangeron polyvibrating equation (cf. [6], [21] and [23]). Reducing the problem to a nonlinear integral equation we give sufficient conditions for the existence and uniqueness of its solution. Moreover, we prove that the solution depends continuously on the initial data.

In Chapter III we consider a two-dimensional noncharacteristic boundary value problem of Z. Szmydt type and prove the existence of its solution.

In Chapter IV we examine a multipoint problem for ordinary differential equations of noninteger order.

Finally, in Chapter V we give further applications of the derivatives of noninteger order. First, basing on the properties of the derivatives given in Chapter I, we construct new examples of Mikusiński operators which are functions. Then, we generalize the Cauchy and Schwarz integral formulae for analytic functions of several complex variables defined in a polydisc.

Throughout the paper $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of reals, $\mathbb{R}_+$ the set of nonnegative numbers and $\mathbb{C}$ the set of complex numbers.

I. Derivatives of noninteger order

Let $x = (x_i)$, $t = (t_i)$ and $\alpha = (\alpha_i)$ belong to $\mathbb{R}^n$. We write $t < x$ ($t \le x$) if $t_i < x_i$ ($t_i \le x_i$) for $i = \overline{1,n}$ and set $|x|_+ := \sum_{i=1}^n x_i$, $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$, $\mathbf{0} := (0,\ldots,0) \in \mathbb{R}^n$ and $\mathbf{1} := (1,\ldots,1) \in \mathbb{R}^n$.

For $t < x$, the open n-dimensional interval $(t;x)$ is defined to be the subset of the form $\{\xi \in \mathbb{R}^n : t < \xi < x\}$. Other types of intervals are defined similarly. Finally, for $x \in \mathbb{R}^n$ and a nonempty domain $\Omega \subset \mathbb{R}^n$ we introduce the set $\Omega(x) := \{t \in \overline{\Omega} : t \le x\}$.

Let $f\colon \Omega \to \mathbb{R}$ be locally integrable (we write briefly $f \in L^1_{\mathrm{loc}}$), and $p = (p_i) \in \mathbb{N}_0^n$ such that $\alpha \le p$. We define the derivative $\mathcal{D}^\alpha f$ at the point x by

$$\mathcal{D}^\alpha f(x) := \begin{cases} D^{\mathbf{1}} \int_{\Omega(x)} (x-t)^{-\alpha} f(t)\,dt/\Gamma(\mathbf{1}-\alpha) & \text{for } \alpha \le \mathbf{0}\,; \\ D^p \mathcal{D}^{\alpha-p} f(x) & \text{for } \alpha \in \mathbb{R}^n \setminus (-\infty,0]^n \end{cases} \tag{1.1}$$

($\Gamma(\alpha) := \prod_{i=1}^n \Gamma(\alpha_i)$ with $\Gamma(\alpha_i)$ being the value of the Euler gamma function), where $D^{\mathbf{1}}$ and D^p are understood in the classical sense (we assume that for $p > \mathbf{0}$, $D^{p-\mathbf{1}}\mathcal{D}^{\alpha-p} f$ is absolutely continuous in each variable).

By the derivative $\mathcal{D}^\alpha f$ of f in Ω we mean the function which assigns the number $\mathcal{D}^\alpha f(x)$ to every point $x \in \Omega$.

Proposition 1.1 ([9], pp. 311, 312). *Let $I_1, I_2 \subset \mathbb{R}$, $I = I_1 \times I_2$, and let $\mu,\nu : I_1 \to I_2$ be monotonic and absolutely continuous. If $v : I \to \mathbb{R}$ is absolutely continuous in the first variable and measurable in the second, and if $|v(\xi,h)| \le M(\eta)$ with $M \in L^1(I_2)$ and $D_\xi v \in L^1(I)$, then*

$$\Big(\int_{\mu(\xi)}^{\nu(\xi)} v(\xi,\eta)\,d\eta\Big)' = \int_{\mu(\xi)}^{\nu(\xi)} D_\xi v(\xi,\eta)\,d\eta + \nu'(\xi)v(\xi,\nu(\xi)) - \mu'(\xi)v(\xi,\mu(\xi)) \quad \textit{a.e. in } I_1\,.$$

Let $A_i := \inf\{x_i : x \in \Omega\}$ and $B_i := \sup\{x_i : x \in \Omega\}$ and write $A := (A_i)$, $B := (B_i) \in \mathbb{R}^n$. If Ω is bounded then, without loss of generality, we set $A = \mathbf{0}$.

It easily follows from Definition 1.1 and Proposition 1.1 that, for $\alpha = p \in \mathbb{N}_0^n$,

$$\mathcal{D}^p f(x) = D^{p+\mathbf{1}} \int_{\Omega(x)} f(t)\,dt = D^p\Big(D^{\mathbf{1}} \int_{(A;x)} f(t)\chi_\Omega(t)\,dt\Big) = D^p f(x) \quad \text{a.e. in } \Omega$$

(where χ_Ω is the characteristic function of Ω), and for $\alpha > \mathbf{0}$,

$$\mathcal{D}^{-\alpha} f(x) = \int_{\Omega(x)} (x-t)^{\alpha-\mathbf{1}} f(t)\,dt/\Gamma(\alpha) \quad \text{a.e. in } \Omega\,.$$

Denote by $\|\cdot\|_m$ the $L^m(\Omega)$-norm ($m \ge 1$).

Lemma 1.1. *If* Ω *is a bounded domain in* $\mathbb{R}^n$, $\alpha \geq \mathbf{0}$ *and* $f \in L^m(\Omega)$, *then* $\mathcal{D}^{-\alpha} f \in L^m(\Omega)$ *and the linear operator* $\mathcal{D}^{-\alpha} : L^m(\Omega) \to L^m(\Omega)$ *is continuous. Moreover*, *for* $\alpha > \mathbf{0}$ *the operator* $\mathcal{D}^{-\alpha}$ *is completely continuous.*

Proof. $f \geq 0$ implies $\mathcal{D}^{-\alpha} f \geq 0$. Observe that $z \leq 1 - 1/m + z^m/m$ for $z \in \mathbb{R}_+$ and $m \geq 1$. Thus, $\mathcal{D}^{-\alpha}|f(x)| \leq \mathcal{D}^{-\alpha}(1-1/m) + \mathcal{D}^{-\alpha}|f(x)|^m/m$, whence and by the Fubini theorem we get

$$\int_\Omega |\mathcal{D}^{-\alpha} f(x)|\, dx \leq \operatorname{const}(1 + \|f\|_m^m)\,,$$

(here and in the sequel const denotes a positive constant).

This proves that $\mathcal{D}^{-\alpha} f \in L^1(\Omega)$.

Next, we prove that $\mathcal{D}^{-\alpha} f \in L^m(\Omega)$. Obviously this holds for $m = 1$. Let $m > 1$. By the Hölder inequality, for $\alpha > \mathbf{0}$ we obtain

$$\Big(\int_{\Omega(x)} (x-t)^{\alpha-\mathbf{1}} |f(t)|\, dt\Big)^m \leq \Big(\int_{(\mathbf{0};x)} (x-t)^{\alpha-\mathbf{1}}\, dt\Big)^{m/(m-1)} \int_{\Omega(x)} (x-t)^{\alpha-\mathbf{1}} |f(t)|^m\, dt$$

and hence $(\mathcal{D}^{-\alpha}|f(x)|)^m \leq \operatorname{const} \mathcal{D}^{-\alpha}|f(x)|$. As $|f|^m \in L^1(\Omega)$, we have $\mathcal{D}^{-\alpha}|f|^m \in L^1(\Omega)$ and, by the above inequality, $|\mathcal{D}^{-\alpha} f|^m \in L^1(\Omega)$. Moreover,

$$\|\mathcal{D}^{-\alpha} f\|_m \leq \operatorname{const} \|f\|_m \tag{1.2}$$

whence $\mathcal{D}^{-\alpha} : L^m(\Omega) \to L^m(\Omega)$ is continuous. The argument for $\alpha \geq \mathbf{0}$ is analogous. Thus, the proof of the first assertion is complete.

Now, we prove the complete continuity of $\mathcal{D}^{-\alpha}$ for $\alpha > \mathbf{0}$. First we consider the case $m = 1$. By (1.2) with $m = 1$, the operator $\mathcal{D}^{-\alpha}$ is bounded. Let $h = (h_i) > \mathbf{0}$. Then

$$\int_\Omega \Big| \int_{\Omega(x+h)} (x+h-t)^{\alpha-\mathbf{1}} f(t)\, dt - \int_{\Omega(x)} (x-t)^{\alpha-\mathbf{1}} f(t)\, dt \Big|\, dx \leq I_1 + I_2 + I_3,$$

where

$$I_1 := \int_\Omega dx \int_{(x;x+h)} (x+h-t)^{\alpha-\mathbf{1}} \chi_\Omega(t) |f(t)|\, dt\,,$$

$$I_2 := \int_\Omega dx \int_{\Omega_{x,h}} (x+h-t)^{\alpha-\mathbf{1}} \chi_\Omega(t) |f(t)|\, dt\,,$$

$$I_3 := \int_\Omega dx \int_{(\mathbf{0};x)} |(x+h-t)^{\alpha-\mathbf{1}} - (x-t)^{\alpha-\mathbf{1}}| \chi_\Omega(t) |f(t)|\, dt$$

and $\Omega_{x,h} := (\mathbf{0}; x+h) \setminus ((\mathbf{0};x) \cup (x; x+h))$.

The integral I_1 can be estimated by

$$I_1 = \int\limits_{(\mathbf{0};h)} (h-\sigma)^{\alpha-\mathbf{1}} \Big(\int\limits_{\Omega} \chi_{\Omega}(x+\sigma)|f(x+\sigma)|\,dx\Big)\,d\sigma \le h^{\alpha}\|f\|_1/\alpha^{\mathbf{1}}\,.$$

Now, observe that $\Omega_{x,h}$ is the sum of the sets

$$\Omega^{(i)}_{x,h} := \bigtimes_{k=1}^{i-1} (x_k; x_k+h_k) \times (0;x_k) \times \bigtimes_{k=i+1}^{n} (x_k;x_k+h_k)\,.$$

Repeating the argument used for I_1 we obtain

$$I_2 = \sum_{i=1}^{n} \int\limits_{\Omega} dx \int\limits_{\Omega^{(i)}_{x,h}} (x+h-t)^{\alpha-\mathbf{1}}\chi_{\Omega}(t)|f(t)|\,dt \le \text{const} \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\ne i}}^{n} h^{\alpha_j}\|f\|_1\,.$$

In order to estimate I_3 we use the following proposition whose inductive proof is omitted.

Proposition 1.2. *If $a_i, b_i \in \mathbb{R}$ $(i=\overline{1,n})$, then*

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i-b_i) \prod_{j=1}^{i-1} a_j \prod_{k=i+1}^{n} b_k$$

(*where here and in the sequel* $\prod_{j=q}^{p} a_j = 1$ *for* $p<q$).

By the Fubini theorem we have

$$I_3 \le \int\limits_{\Omega} \Big(\int\limits_{(t;B)} |(x+h-t)^{\alpha-\mathbf{1}} - (x-t)^{\alpha-\mathbf{1}}|\,dx\Big)\chi_{\Omega}(t)|f(t)|\,dt$$

and hence, by Proposition 1.2, we get

$$\begin{aligned} &\int\limits_{(t;B)} |(x+h-t)^{\alpha-\mathbf{1}} - (x-t)^{\alpha-\mathbf{1}}|\,dx \\ &\quad \le \sum_{i=1}^{n} \int\limits_{(t;B)} |(x_i+h_i-t_i)^{\alpha_i-1} - (x_i-t_i)^{\alpha_i-1}| \prod_{j=1}^{i-1} (x_j+h_j-t_j)^{\alpha_j-1} \\ &\quad\quad \times \prod_{k=i+1}^{n} (x_k-t_k)^{\alpha_k-1}\,dx \le \text{const} \sum_{i=1}^{n} h^{\tilde{\alpha}_i} \prod_{\substack{j=1\\ j\ne i}}^{n} B^{\tilde{\alpha}_j}\,, \end{aligned}$$

where $\widetilde{\alpha}_i := \min(1,\alpha_i)$. Thus

$$I_3 \le \text{const} \sum_{i=1}^{n} h^{\tilde{\alpha}_i}\|f\|_1\,.$$

Combining the estimates of I_1, I_2 and I_3, we obtain

$$\int\limits_{\Omega} |\mathcal{D}^{-\alpha} f(x+h) - \mathcal{D}^{-\alpha} f(x)|\,dx \le P(h)\|f\|_1\,,$$

where $P(h) \to 0$ as $h \to \mathbf{0}$.

Set $B^m_\varrho := \{f\in L^m(\Omega) : \|f\|_m \le \varrho\}$. From the M. Riesz compactness theorem (cf. [18], p. 242 and [12], p. 166), we conclude that the set $\mathcal{D}^{-\alpha}(B^1_\varrho)$ is relatively compact in $L^1(\Omega)$. Hence, the operator $\mathcal{D}^{-\alpha}$ is completely continuous, as required.

Now, assume that $m > 1$. For $\alpha \ge \mathbf{1}$ the operator $\mathcal{D}^{-\alpha} : L^m(\Omega) \to L^m(\Omega)$ is completely continuous since its kernel is continuous.

Define

$$[\mathcal{D}^{-\alpha} f]_n(x) := \int\limits_{\Omega(x)} \min\{(x-t)^{\alpha-\mathbf{1}}, n\} f(t)\, dt/\Gamma(\alpha)\,.$$

Clearly, the operator $[\mathcal{D}^{-\alpha}]_n : L^m(\Omega) \to L^m(\Omega)$ is completely continuous.

Assume that $\alpha_1 = \min(\alpha_1, \ldots, \alpha_n) < 1$ and fix $1 < r < 1/(1-\alpha_1)$. Then $c_1 := \int_{\Omega(x)} (x-t)^{r(\alpha-\mathbf{1})}\, dt < \infty$.

By the Hölder inequality, we have the estimate

$$|\mathcal{D}^{-\alpha} f(x) - [\mathcal{D}^{-\alpha} f]_n(x)| \le \Big(\int\limits_{\Omega(x)} k_n(x,t)\, dt\Big)^{1-1/m} \Big(\int\limits_{\Omega(x)} k_n(x,t)|f(t)|^m dt\Big)^{1/m} /\Gamma(\alpha)$$

with $k_n(x,t) = (x-t)^{\alpha-\mathbf{1}} - [(x-t)^{\alpha-\mathbf{1}}]_n$. Moreover,

$$\int\limits_{\Omega(x)} k_n(x,t)\, dt \le \int\limits_{A_n(x)} (x-t)^{\alpha-\mathbf{1}}\, dt$$

where $A_n(x) := \{t \in \Omega(x) : (x-t)^{\alpha-\mathbf{1}} > n\}$. Note that

$$(x-t)^{\alpha-\mathbf{1}} < n^{1-r}(x-t)^{r(\alpha-\mathbf{1})}$$

for $t \in A_n(x)$ and hence

$$\int\limits_{\Omega(x)} k_n(x,t)\, dt \le c_1 n^{1-r}\,.$$

Thus

$$\|\mathcal{D}^{-\alpha} f - [\mathcal{D}^{-\alpha} f]_n\|_m \le \mathrm{const}(c_1 n^{1-r})^{1-1/m} \|\mathcal{D}^{-\alpha}|f|^m\|_1^{1/m}\,.$$

Consequently, by (1.2) we have

$$\|\mathcal{D}^{-\alpha} f - [\mathcal{D}^{-\alpha} f]_n\|_m \le \mathrm{const}\, n^{(1-r)(1-1/m)} \|f\|_m\,,$$

which proves that $\mathcal{D}^{-\alpha}$, as a limit (in the space $L^m(\Omega)$) of completely continuous operators, is completely continuous.

Remark 1.1. Let $\gamma \in \mathbb{R}^n$. We say that f belongs to $C_\gamma(\Omega)$ if f is continuous in Ω and $f(t)(t-A)^\gamma$ is bounded for every $t \in \overline{\Omega}$. $C_\gamma(\Omega)$ with the norm $|f|_\gamma := \max\{|f(t)|(t-A)^\gamma : t \in \overline{\Omega}\}$ is a Banach space. Assume that $\alpha, \gamma \ge \mathbf{0}$, $\mathbf{0} \le \delta < \mathbf{1}$ and $\alpha + \gamma - \delta \ge \mathbf{0}$. Observe that $|\mathcal{D}^{-\alpha} f|_\gamma \le c_{\Omega,\alpha,\delta,\gamma}|f|_\delta$, with $c_{\Omega,\alpha,\delta,\gamma} := (B-A)^{\alpha+\gamma-\delta}\Gamma(\mathbf{1}-\delta)/\Gamma(\mathbf{1}+\alpha-\delta)$, and hence, $\mathcal{D}^{-\alpha}$ is a bounded linear operator from $C_\delta(\Omega)$ into $C_\gamma(\Omega)$, completely continuous for $\alpha > \mathbf{0}$.

In the sequel we shall use

PROPOSITION 1.3 (generalized Minkowski inequality, [29], pp. 27, 28, [30], pp. 158–160). *Let* $\Omega_k \subset \mathbb{R}^{n_k}$ *with* $n_k \in \mathbb{N}$, $k = 1, 2$, *be measurable sets and let* $m \geq 1$. *Then for any function* $v : \Omega_1 \times \Omega_2 \to \mathbb{R}$,

$$\Big(\int\limits_{\Omega_1} \Big| \int\limits_{\Omega_2} v(\xi, \eta)\, d\eta \Big|^m d\xi \Big)^{1/m} \leq \int\limits_{\Omega_2} \Big(\int\limits_{\Omega_1} |v(\xi, \eta)|^m \, d\xi \Big)^{1/m} d\eta \,,$$

provided that the integrals on the right-hand side exist.

LEMMA 1.2. *Let* $f \in L^m(\Omega)$ *with* $\Omega \subset \mathbb{R}^n$ *being bounded. The function* $\mathbb{R}^n_+ \ni \alpha \to \mathcal{D}^{-\alpha} f \in L^m(\Omega)$ *is continuous. Moreover, the function* $\alpha \to \mathcal{D}^{-\alpha}$, *mapping* $\mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ *into the space of linear continuous operators from* $L^m(\Omega)$ *into itself* (cf. Lemma 1.1), *is also continuous.*

Proof. By Proposition 1.3, for $f \in L^m(\Omega)$ and $\alpha \in \mathbb{R}^n_+$,

$$\|\mathcal{D}^{-\alpha} f\|_m \leq B^\alpha \|f\|_m / \Gamma(\mathbf{1} + \alpha) \,. \tag{1.3}$$

1°. Consider the case $\alpha > \mathbf{0}$. We have

$$\mathcal{D}^{-\alpha} f(x) - \mathcal{D}^{-\beta} f(x) = T_1 f(x) + T_2 f(x) \,,$$

where

$$T_1 f(x) := \int\limits_{\Omega(x)} (x - t)^{\alpha - \mathbf{1}} f(t)\, dt / (1/\Gamma(\alpha) - 1/\Gamma(\beta)) \,,$$

$$T_2 f(x) := \int\limits_{\Omega(x)} [(x - t)^{\alpha - \mathbf{1}} - (x - t)^{\beta - \mathbf{1}}] f(t)\, dt / \Gamma(\alpha)$$

and $\beta > \mathbf{0}$.

Inequality (1.3) immediately yields

$$\|T_1 f\|_m \leq |1 - \Gamma(\alpha)/\Gamma(\beta)| B^\alpha \|f\|_m / \Gamma(\mathbf{1} + \alpha) \,.$$

To estimate the norm of $T_2 f$, by changing the variable we have

$$|T_2 f(x)| \leq \int\limits_{(\mathbf{0}; B)} t^{\alpha - \mathbf{1}} |1 - t^{\beta - \alpha}| |f(x - t) \chi_\Omega(x - t)| \, dt / \Gamma(\alpha) \,.$$

Proposition 1.3 now gives

$$\begin{aligned} \|T_2 f\|_m &\leq \int\limits_{(\mathbf{0}; B)} t^{\alpha - \mathbf{1}} |1 - t^{\beta - \alpha}| \Big(\int\limits_{\Omega} |f(x - t) \chi_\Omega(x - t)|^m \, dx \Big)^{1/m} / \Gamma(\alpha) \\ &\leq \|f\|_m \int\limits_{(\mathbf{0}; B)} t^{\alpha - \mathbf{1}} |1 - t^{\beta - \alpha}| \, dt / \Gamma(\alpha) \,. \end{aligned}$$

Summarizing, we obtain

$$\begin{aligned} \|\mathcal{D}^{-\alpha} f - \mathcal{D}^{-\beta} f\|_m / \|f\|_m \leq |1 - \Gamma(\alpha)/\Gamma(\beta)| B^\alpha / \Gamma(\mathbf{1} + \alpha) \\ + \int\limits_{(\mathbf{0}; B)} t^{\alpha - \mathbf{1}} |1 - t^{\beta - \alpha}| \, dt / \Gamma(\alpha) \,. \end{aligned}$$

Since the above integral is convergent as $\beta \to \alpha$, and since Γ is continuous for $\beta > \mathbf{0}$ and $\Gamma(\beta) \neq 0$, the right-hand side converges to 0 as $\beta \to \alpha$.

2°. For $\alpha \geq \mathbf{0}$, $\alpha \neq \mathbf{0}$, the above considerations are valid with the n-dimensional integrals replaced by κ-dimensional ones, where $\kappa := |(\operatorname{sgn} \alpha_i)|_+$ $(0 < \kappa < n)$.

Hence, the second assertion holds. Moreover, it implies

$$\|\mathcal{D}^{-\alpha} f - \mathcal{D}^{-\beta} f\|_m \to 0 \quad \text{as } \beta \to \alpha \,.$$

3°. Consider the case $\alpha = \mathbf{0}$. We shall prove that $\|\mathcal{D}^{-\beta} f - f\|_m \to 0$ as $\beta \to \mathbf{0}$. Without loss of generality we can assume that $\beta > \mathbf{0}$. Indeed, otherwise

$$\|\mathcal{D}^{-\beta} f - f\|_m \leq \|\mathcal{D}^{-\beta} f - \mathcal{D}^{-\gamma} f\|_m + \|\mathcal{D}^{-\gamma} f - f\|_m \,,$$

where $\gamma > \mathbf{0}$, and the first term of the right-hand side tends to 0 as $\gamma \to \beta$ (cf. the proof above).

We have

$$\begin{aligned} |\mathcal{D}^{-\beta} f(x) - f(x)| &\leq \Big| \int_{(\mathbf{0};x)} t^{\beta-\mathbf{1}} f(x-t)\chi_\Omega(x-t)\, dt/\Gamma(\beta) - f(x) \Big| \\ &\leq T_3 f(x) + T_4(x) \end{aligned}$$

where

$$T_3 f(x) := \int_{(\mathbf{0};x)} t^{\beta-\mathbf{1}} [f(x-t)\chi_\Omega(x-t) - f(x)]\, dt/\Gamma(\beta)$$

and

$$T_4 f(x) := f(x)[x^\beta/\Gamma(\mathbf{1}+\beta) - 1] \,.$$

Obviously,

$$\|T_4 f\|_m^m \leq \int_\Omega |f(x)|^m |x^\beta/\Gamma(\mathbf{1}+\beta) - 1|^m \, dx$$

and the Lebesgue dominated convergence theorem immediately shows that $\|T_4 f\|_m \to 0$ as $\beta \to \mathbf{0}$.

For given $\varepsilon > 0$, let P be a polynomial such that $\|f - P\|_m < \varepsilon$. Hence $\|T_3 f\|_m \leq \|T_3(f-P)\|_m + \|T_3 P\|_m$. Applying Proposition 1.3 to the first term on the right-hand side we have

$$\|T_3(f-P)\|_m \leq 2B^\beta \|f-P\|_m/\Gamma(\mathbf{1}+\beta) < \operatorname{const} \varepsilon \,.$$

For the second term we have

$$\begin{aligned} \|T_3 P\|_m^m &\leq \int_\Omega \Big(\int_{(\mathbf{0};x)} t^{\beta-\mathbf{1}} \max_{1\leq i\leq n} (\max_{t\in\Omega} |D_i P(t)|)|t|_+ \, dt/\Gamma(\beta)\Big)^m dx \\ &\leq \operatorname{const}(|\beta|_+/\Gamma(\mathbf{1}+\beta))^m \end{aligned}$$

with const being independent of β, and so $T_3 P \to 0$ as $\beta \to \mathbf{0}$. Hence, the proof is complete.

In the sequel we need the notation

$$R_{\partial\Omega} = \{x \in \overline{\Omega} : \Omega(x) = \{x\}\} \quad \text{and} \quad \Omega_{\mathrm{b}} = \overline{\Omega} \setminus R_{\partial\Omega} .$$

LEMMA 1.3 (cf. [47], p. 126). *Let* $f \in L^1(\Omega)$ *with* Ω *bounded*, $\alpha \geq \mathbf{0}$, $\lambda > 0$ *and assume that*

(i) *there is a neighbourhood* U *of* $R_{\partial\Omega}$ *in* $\mathbb{R}^n$ *such that* $f < 0$ *a.e. in* $U \cap \Omega_{\mathrm{b}}$,
(ii) $f < \lambda \mathcal{D}^{-\alpha} f$ *a.e. in* Ω_{b}.

Then $f \leq 0$ *a.e. in* Ω_{b}.

Proof. Assume that f satisfying (i) and (ii) is continuous, and that $\Lambda = \{x \in \Omega_{\mathrm{b}} : f(x) \geq 0\} \neq \emptyset$. Set $s^0 = \inf\{|x|_+ : x \in \Lambda\} = |x^0|_+$. Obviously $x^0 \in \overline{\Lambda}$. Moreover, $f(x^0) \geq 0$ by the continuity of f. If $x^0 \in \Lambda$, then $|x|_+ < |x^0|_+ = s^0$ for every $x \in \Omega(x^0) \setminus \{x^0\}$ and this implies $f < 0$ in $\Omega(x^0) \setminus \{x^0\}$. Hence, by (ii) and definition (1.1), we have $f(x^0) < \lambda \mathcal{D}^{-\alpha} f(x^0) < 0$, contrary to the assumption $x^0 \in \Lambda$. Thus, $x^0 \in \overline{\Lambda} \setminus \Lambda \subset R_{\partial\Omega}$ and $U \cap \Lambda \neq \emptyset$. But (i) shows that $f < 0$ in U, and, by the definition of Λ, $f \geq 0$ in Λ. Therefore $\Lambda = \emptyset$ and the assertion holds for f continuous.

Let f be an integrable function satisfying (i) and (ii). By the Lusin theorem, there exists a sequence $(f_k)_{k=1}^{\infty}$ of continuous functions such that $f_k \leq f$ and $\lim_{k\to\infty} f_k = f$ a.e. in Ω. We will show that the functions f_k ($k \in \mathbb{N}$) also satisfy (i) and (ii). Obviously, $f_k < 0$ in $U \cap \Omega_{\mathrm{b}}$ for every k. Assume that $f_k \geq \lambda \mathcal{D}^{-\alpha} f_k$ on a subset of Ω_{b} having a nonzero measure. By the Lebesgue dominated convergence theorem, $f = \lim_{k\to\infty} f_k \geq \lambda \lim_{k\to\infty} \mathcal{D}^{-\alpha} f_k = \lambda \mathcal{D}^{-\alpha} f$, which contradicts (ii) for f. Since the assertion of our lemma holds for every continuous function, $f_k \leq 0$ in Ω_{b} (more precisely, $f_k < 0$) and hence $f \leq 0$ a.e. in Ω_{b}, which ends the proof.

The following corollary is an immediate consequence of the above lemma.

COROLLARY 1.1. *Let* Ω *be a bounded domain and* $v \in L^1(\Omega)$, $\alpha \geq \mathbf{0}$ *and* $\lambda > 0$. *Assume that* $f, g \in L^1(\Omega)$ *and that*

(i) *there is a neighbourhood* U *of* $R_{\partial\Omega}$ *in* $\mathbb{R}^n$ *such that* $f < g$ *a.e. in* $U \cap \Omega_{\mathrm{b}}$,
(ii) $f < v + \lambda \mathcal{D}^{-\alpha} f$ *and* $g > v + \lambda \mathcal{D}^{-\alpha} g$ *a.e. in* Ω_{b}.

Then $f \leq g$ *a.e. in* Ω_{b}.

We say that a domain Ω satisfies *condition* (V) if for every $x \in \Omega$ and $t \in \Omega(x)$ the interval $(t; x)$ belongs to Ω.

LEMMA 1.4. *If* $f \in L^1(\Omega)$ *and* $\alpha, \beta \geq \mathbf{0}$, *then*

$$\mathcal{D}^{-\alpha}(\mathcal{D}^{-\beta}|f|) \leq \mathcal{D}^{-(\alpha+\beta)}|f| \quad \text{a.e. in } \Omega . \tag{1.4}$$

Moreover, *if* Ω *satisfies condition* (V), *then* $\mathcal{D}^{-\alpha}(\mathcal{D}^{-\beta} f) = \mathcal{D}^{-(\alpha+\beta)} f$ *a.e. in* Ω.

Proof. We begin with the case $\alpha, \beta > \mathbf{0}$. By (1.1) and the Fubini theorem,

$$\mathcal{D}^{-\alpha}(\mathcal{D}^{-\beta}|f(x)|) = \int\limits_{\Omega(x)} (x-t)^{\alpha-\mathbf{1}} \Big(\int\limits_{\Omega(t)} (t-\tau)^{\beta-\mathbf{1}} |f(\tau)|\, d\tau \Big)\, dt / (\Gamma(\alpha)\Gamma(\beta))$$

$$= \int_{\Omega(x)} \Big(\int_{(\tau;x)\cap\Omega} (x-t)^{\alpha-\mathbf{1}}(t-\tau)^{\beta-\mathbf{1}}\,dt\Big)|f(\tau)|\,d\tau/(\Gamma(\alpha)\Gamma(\beta))$$

$$\le \int_{\Omega(x)} (x-\tau)^{\alpha+\beta-\mathbf{1}}|f(\tau)|\,d\tau/\Gamma(\alpha+\beta) = \mathcal{D}^{-(\alpha+\beta)}|f(x)|$$

with equality if $(\tau;x)\subset\Omega$ for all $\tau,x\in\Omega$ such that $\tau\le x$ (this happens for example if Ω satisfies condition (V)).

By Lemma 1.2, we obtain (1.4) for $\alpha\ge\mathbf{0}$ and $\beta\ge\mathbf{0}$, $\beta\ne\mathbf{0}$. Clearly (1.4) is also satisfied when $\beta=\mathbf{0}$.

The proof of the second assertion is similar.

Thus, the proof is complete.

As an immediate consequence of Lemma 1.4 and definition (1.1), we obtain

LEMMA 1.5. *If $\alpha,\beta\ge\mathbf{0}$ and $\mathcal{D}^{\alpha-\beta}f$ exists, and if the domain Ω satisfies condition* (V), *then*

$$\mathcal{D}^{\alpha}\mathcal{D}^{-\beta}f(x) = \mathcal{D}^{\alpha-\beta}f(x) \quad a.e.\ in\ \Omega\,. \tag{1.5}$$

In the sequel we equip the space $L^1(\Omega)$ with the norm

$$\|f\|_{1,\tau} := \int_{\Omega} |f(t)|\exp(-\tau|t|_+)\,dt \tag{1.6}$$

where $\tau\in\mathbb{R}_+$ is fixed. Since $\exp(-\tau|B|_+)\le\exp(-\tau|x|_+)\le 1$ in Ω, the norms $\|\cdot\|_{1,\tau}$ and $\|\cdot\|_{1,0}$ are equivalent if Ω is bounded. Hence, the space of integrable functions with norm (1.6) is complete (cf. [37], Secs. 1.7, 1.26).

LEMMA 1.6. *Assume that Ω is bounded, $g\in L^m(\Omega)$ with $m\ge 1$, $r\in\mathbb{N}$, $\alpha^1,\dots,\alpha^r\in\mathbb{R}^n_+\setminus\{\mathbf{0}\}$ and $\lambda^1,\dots,\lambda^r\in\mathbb{R}_+$. If $f\in L^m(\Omega)$ and*

$$0\le f\le g+\sum_{j=1}^{r}\lambda^j\mathcal{D}^{-\alpha^j}f \tag{1.7}$$

a.e. in Ω, then

$$f\le g+\sum_{\nu=1}^{\infty}\sum_{1\le i_1,\dots,i_\nu\le r}\prod_{j=1}^{\nu}\lambda^j\mathcal{D}^{-\alpha^{(i,\nu)}}g \tag{1.8}$$

where $\alpha^{(i,\nu)}=\alpha^{i_1}+\ldots+\alpha^{i_\nu}$. Moreover, if Ω satisfies condition (V), *then the right-hand side of* (1.8) *is a solution of the equation*

$$f=g+\sum_{j=1}^{r}\lambda^j\mathcal{D}^{-\alpha^j}f \tag{1.9}$$

with $\lambda^1,\dots,\lambda^r\in\mathbb{R}$. This solution is unique in the space of integrable functions.

Proof. Applying (1.7) k times and using the properties of $\mathcal{D}^{-\alpha}$ (cf. Lemma 1.4), we get

$$0 \le f \le g + \sum_{\nu=1}^{k} \sum_{1\le i_1,\ldots,i_\nu\le r} \prod_{j=1}^{\nu} \lambda^j \mathcal{D}^{-\alpha^{(i,\nu)}} g + \sum_{1\le i_1,\ldots,i_{k+1}\le r} \prod_{j=1}^{\nu} \lambda^j \mathcal{D}^{-\alpha^{(i,k+1)}} f$$

a.e. in Ω.

Set $\lambda := \max_j \lambda^j$, $\alpha := (\min_{i,j} \alpha_i^j) \in \mathbb{R}_+^n$ and choose τ so that $q := r\lambda\tau^{-|\alpha|_+} < 1$. By direct calculation one can prove

$$\Big\| \sum_{1\le i_1,\ldots,i_\nu\le r} \prod_{j=1}^{\nu} \lambda^j \mathcal{D}^{-\alpha^{(i,k+1)}} f \Big\|_{1,\tau} \le q^{k+1} \|f\|_{1,\tau}\,,$$

whence the series

$$\sum_{\nu=1}^{\infty} \sum_{1\le i_1,\ldots,i_\nu\le r} \prod_{j=1}^{\nu} \lambda^j \mathcal{D}^{-\alpha^{(i,\nu)}} g$$

is convergent in $L^1(\Omega)$. As $g \in L^m(\Omega)$ the series is also convergent in $L^m(\Omega)$ (cf. Lemma 1.1). By (1.7), this yields (1.8).

Observe that (1.7) with $g = 0$ implies $|f| = 0$, which means that equation (1.9) has at most one integrable solution. One can prove the existence of such a solution in a way analogous to that above by using condition (V).

Thus, the proof of Lemma 1.6 is complete.

Setting $r = 1$, we have the following consequence of the above lemma:

COROLLARY 1.2. *Let Ω be bounded, $g \in L^m(\Omega)$ with $m \ge 1$, $\alpha \ge \mathbf{0}$, $\alpha \ne \mathbf{0}$ and $\lambda \in \mathbb{R}_+$. If $f \in L^m(\Omega)$ and (1.7) holds with $r = 1$, then*

$$f(x) \le g(x) + \lambda D^{\mathbf{1}}\Big(\int_{\Omega(x)} (x-t)^{\alpha} E_{\alpha,\mathbf{1}+\alpha}(\lambda(x-t)^{\alpha}) g(t)\,dt\Big) \tag{1.10}$$

where

$$E_{\varrho,\mu}(z) = \sum_{k=0}^{\infty} z^k / \Gamma(k\varrho + \mu) \qquad (z \in \mathbb{C};\ \mu, \varrho \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}) \tag{1.11}$$

is the Mittag-Leffler function (cf. [13], p. 117). *Moreover, if Ω satisfies condition* (V), *then the right-hand side of* (1.10) *is a solution of equation* (1.9) *with* $r = 1$. *This solution is unique in the space of integrable functions.*

Let us point out an interesting property of the Mittag-Leffler function $E_{\varrho,\mu}$. Set $\alpha \ge \mathbf{0}$, $\alpha \ne \mathbf{0}$, $\Omega = [A;\infty) := \{x \in \mathbb{R}^n : A \le x\}$ and define $f_l(x) := (x-A)^{\alpha-l} E_{\alpha,\mathbf{1}+\alpha-l}(\lambda(x-A)^{\alpha})$, $x \in \Omega$, where $l \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}$ is chosen so that $\mathbf{1} + \alpha - l > \mathbf{0}$ and $\lambda \in \mathbb{R}$. Using the relation

$$\mathcal{D}^{\beta}((x-A)^{\gamma}/\Gamma(\mathbf{1}+\gamma)) = (x-A)^{\gamma-\beta}/\Gamma(\mathbf{1}+\gamma-\beta) \tag{1.12}$$

($\beta \in \mathbb{R}^n$, $\gamma > -\mathbf{1}$), which can be easily obtained by direct calculation, we notice that f_l satisfies the equation $\mathcal{D}^{\alpha} f_l = \lambda f_l$ ($\alpha \ge \mathbf{0}$, $\alpha \ne \mathbf{0}$).

In the remainder of this section we assume that the domain Ω is bounded and satisfies the condition $\Omega(x) = (\mathbf{0}; x)$ (hence, Ω satisfies also condition (V)), and that $f \in L^1(\Omega)$. As a consequence (cf. Lemma 1.4), for $\alpha \le \mathbf{0}$ we have

$$\mathcal{D}^\alpha f(x) = \mathcal{D}_1^{\alpha_1} \ldots \mathcal{D}_n^{\alpha_n} f(x)\,, \tag{1.13}$$

where $\mathcal{D}_i^{\alpha_i}$ denotes the derivative of order α_i with respect to x_i (cf. Definition (1.1)). Using Proposition 1.1 n times, we extend (1.13) to arbitrary $\alpha \in \mathbb{R}^n$.

For $l = (l_i) \in \mathbb{N}_0^n$ and $x = (x_i) \in \mathbb{R}^n$, set $\nabla_l(x) := (x_i \operatorname{sgn} l_i)$ and $\Delta_l(x) := (x_i \delta_{0 l_i})$, where δ is the Kronecker symbol.

We have the following lemma which generalizes the Taylor formula:

Lemma 1.7. *Assume that $\mathcal{D}^\alpha f$, where $\alpha \ge \mathbf{0}$, exists and is integrable. Then, for every $\beta \ge \mathbf{0}$, the derivative $\mathcal{D}^{\alpha-\beta} f$ exists and*

$$\mathcal{D}^{\alpha-\beta} f(x) = \sum_{l \le p} \mathcal{D}^{-\Delta_l(\beta)} \mathcal{D}^{\alpha - l} f(\Delta_l(x)) \times x^{\nabla_l(\beta - l)} / \Gamma(\mathbf{1} + \nabla_l(\beta - l)) \quad \text{a.e. in } \Omega \tag{1.14}$$

(*with $p \in \mathbb{N}^n$ chosen so that $p - \mathbf{1} < \alpha \le p$*).

Proof. First, we prove (1.14) for $\alpha = \beta$.

We use induction on n. For $n = 1$ formula (1.14) holds (cf. [13], p. 570). Assume its validity for $n = k-1$ ($k \ge 2$). Then

$$\mathcal{D}_k^{\alpha_k} f(x) = \sum_{r \le q} \mathcal{D}^{-\Delta_r(\alpha')} \mathcal{D}^{\alpha' - r} \mathcal{D}_k^{\alpha_k} f(\Delta_r(x'), x_k) x'^{\nabla_r(\alpha' - r)} / \Gamma(\mathbf{1}' + \nabla_r(\alpha' - r)),$$

where $x = (x', x_k) \in \mathbb{R}^k$, $\alpha = (\alpha', \alpha_k) \in \mathbb{R}_+^k$, $l = (r, l_k) \in \mathbb{N}_0^k$ and $p = (q, p_k)$, $\mathbf{1} = (\mathbf{1}', 1) \in \mathbb{N}^k$. Applying $\mathcal{D}_k^{-\alpha_k}$ to both sides, we obtain

$$f(x) = \sum_{l_k=1}^{p_k} \mathcal{D}_k^{\alpha_k - l_k} f(x', 0) x^{\alpha_k - l_k} / \Gamma(1 + \alpha_k - l_k) + \sum_{r \le q} \mathcal{D}_k^{-\alpha_k} \mathcal{D}^{-\Delta_r(\alpha')} \mathcal{D}^{\alpha' - r} \mathcal{D}_k^{\alpha_k} f(\Delta_r(x'), x_k) x'^{\nabla_r(\alpha' - r)} / \Gamma(\mathbf{1}' + \nabla_r(\alpha' - r))\,.$$

By the inductive assumption for $\mathcal{D}_k^{\alpha_k - l_k} f(x', 0)$, the first term on the right-hand side is

$$\sum_{l_k=1}^{p_k} \Big(\sum_{r \le q} \mathcal{D}^{-\Delta_r(\alpha')} \mathcal{D}^{\alpha' - r} \mathcal{D}_k^{\alpha_k - l_k} f(\Delta_r(x'), 0) \times x'^{\nabla_r(\alpha' - r)} / \Gamma(\mathbf{1}' + \nabla_r(\alpha' - r)) \Big) x_k^{\alpha_k - l_k} / \Gamma(1 + \alpha_k - l_k)\,.$$

Bearing in mind the properties of Δ_l and ∇_l, we have

$$f(x) = \sum_{l\le p; l_k\ge 1} \mathcal{D}^{-\Delta_l(\alpha)}\mathcal{D}^{\alpha-l} f(\Delta_l(x)) x^{\nabla_l(\alpha-l)}/\Gamma(\mathbf{1}+\nabla_l(\alpha-l))$$
$$+ \sum_{l\le p; l_k\ge 0} \mathcal{D}^{-\Delta_l(\alpha)}\mathcal{D}^{\alpha-l} f(\Delta_l(x)) x^{\nabla_l(\alpha-l)}/\Gamma(\mathbf{1}+\nabla_l(\alpha-l))\,.$$

Thus, if (1.14) is valid for $n = k-1$ then it also holds for $n = k$. Hence we get the assertion for $\alpha = \beta$.

Applying $\mathcal{D}^{\alpha-\beta}$ to both sides of (1.14) with $\alpha = \beta$, we get

$$\mathcal{D}^{\alpha-\beta} f(x) = \sum_{l\le p} \mathcal{D}^{\Delta_l(\alpha-\beta)}\mathcal{D}^{-\Delta_l(\alpha)}\mathcal{D}^{\alpha-l} f(\Delta_l(x))$$
$$\times \mathcal{D}^{\nabla_l(\alpha-\beta)}(x^{\nabla_l(\alpha-l)}/\Gamma(\mathbf{1}+\nabla l(\alpha-l)))\,.$$

Now (1.5) and (1.12) yield (1.14) for every $\alpha > \mathbf{0}$ and $\beta \ge \mathbf{0}$, which completes the proof of Lemma 1.7.

As a consequence of Lemmas 1.4 and 1.7 we have

COROLLARY 1.3. *If $\mathcal{D}^\alpha f$ is integrable, and if $\beta \le \alpha$, then $\mathcal{D}^\beta f$ exists and is given by*

(a) $\mathcal{D}^\beta f(x) = \mathcal{D}^{\beta-\alpha}\mathcal{D}^\alpha f(x)$ *for* $\alpha \le \mathbf{0}$,

(1.15) (b) $$\mathcal{D}^\beta f(x) = \sum_{l\le p} \mathcal{D}^{-\Delta_l(\alpha-\beta)}\mathcal{D}^{\alpha-l} f(\Delta_l(x)) \times x^{\nabla_l(\alpha-\beta-l)}/\Gamma(\mathbf{1}+\nabla_l(\alpha-\beta-l)) \quad \textit{for } \alpha \ge \mathbf{0},\ \alpha \ne \mathbf{0}$$

(*with p as in* (1.14)). *Moreover*, *$\mathcal{D}^\beta f$ is integrable in case* (a) (cf. Lemma 1.1).

Observe that Corollary 1.3 generalizes the result obtained by J. D. Tamarkin (cf. [45], Th. 6).

R e m a r k 1.2. It is worth noticing that in case $\alpha \ge \mathbf{0}$, the derivatives of order $\beta = \alpha - k$ (where $\mathbf{0} \le k \le p$) are integrable. Moreover, if $\alpha = p$, then for arbitrary β the derivative $\mathcal{D}^\beta f$ can be expressed in terms of derivatives of integer orders not higher than p (that is, partial derivatives).

R e m a r k 1.3. If the assumptions of Lemma 1.7 are satisfied and the derivative $\mathcal{D}^\alpha f$ does not depend on the order of differentiations, then (1.14) with $\alpha = \beta$ can be rewritten as

(1.16) $$f(x) = \mathcal{D}^{-\alpha}\mathcal{D}^\alpha f(x) - \sum_{l\le p; l\ne \mathbf{0}} \varepsilon_l \mathcal{D}^{\nabla_l(\alpha-l)} f(\Delta_l(x)) x^{\nabla_l(\alpha-l)}/\Gamma(\mathbf{1}+\nabla_l(\alpha-l))\,,$$

where $\varepsilon_l := (-1)^{|(\operatorname{sgn} l_i)|_+}$.

The following lemma, resulting from Lemma 1.7, gives necessary and sufficient conditions for the solvability of the n-dimensional Abel–Liouville equation of first kind (for the case $n = 1$, cf. [45], Th. 4).

LEMMA 1.8. *Let* $p - \mathbf{1} < \alpha \le p \in \mathbb{N}^n$. *If*

(i) $\mathcal{D}^\alpha f$ *exists and is integrable*,
(ii) $\mathcal{D}^{\alpha - l} f(\Delta_l(x)) = 0$ *for* $\mathbf{0} \le l \le p$, $l \ne \mathbf{0}$,

then there exists exactly one function $g = \mathcal{D}^\alpha f \in L^1(\Omega)$ *which is a solution of the equation*

$$\mathcal{D}^{-\alpha} g = f \,. \tag{1.17}$$

Conversely, *if* $g \in L^1(\Omega)$ *is a solution of* (1.17) *then* f *satisfies* (i) *and* (ii).

Proof. 1°. By (i), (ii) and Lemma 1.7, we have $f = \mathcal{D}^{-\alpha}\mathcal{D}^\alpha f$. Hence, the function $g := \mathcal{D}^\alpha f$ satisfies (1.17).

2°. Assume that $g \in L^1(\Omega)$ is a solution of (1.17). Then, by Lemma 1.6, $\mathcal{D}^{-\alpha} f = g \in L^1(\Omega)$ and $\mathcal{D}^{\alpha - l} f = \mathcal{D}^{-l} g$ for $\alpha - l \ge \mathbf{0}$. Hence $\mathcal{D}^{-l} g(\Delta_l(x)) = 0$ for $\mathbf{0} \le l \le \alpha$, $l \ne \mathbf{0}$. If there exists a positive integer i such that $\alpha_i - l_i < 0$, then, by definition (1.1) and Proposition 1.1, we get $\mathcal{D}^{\alpha - l} f(\Delta_l(x)) = 0$.

II. Characteristic problem for the Mangeron polyvibrating equation of noninteger order

1. The problem. Let $n, r \in \mathbb{N}$ and $B = (B_i) \in \mathbb{R}^n_+$ be fixed, and $\Omega = (\mathbf{0}; B)$. Also fix $\alpha = (\alpha_i) > \mathbf{0}$ and $\beta^j = (\beta^j_i) \in \mathbb{R}^n$ $(j = \overline{1,r})$ so that $\beta^j \le \alpha$ and $|\beta^j|_+ < |\alpha|_+$.

We consider the differential equation

$$\mathcal{D}^\alpha u(x) = F(x, \{\mathcal{D}^{\beta^j} u(x)\}), \qquad x \in \Omega \,, \tag{2.1}$$

where F is a given function, $\mathcal{D}^\alpha u$ and $\mathcal{D}^{\beta^j} u$ are derivatives of noninteger orders α and β^j, respectively, and $\{\mathcal{D}^{\beta^j} u\}$ denotes the r-element sequence of derivatives $\mathcal{D}^{\beta^j} u$.

(2.1) is an ordinary differential equation for $n = 1$ and a partial differential equation if $n \ge 2$. In the latter case it is a noninteger counterpart of the Mangeron polyvibrating equation (cf. [6], [21] and [23]). Let us note that equation (2.1) with $n = 1$ was examined in a few papers (cf. [35] and [40]), and with $n = 2$ in [7] (cf. Remark 2.4 below) and [24]. To the best of our knowledge the case $n \ge 3$ has not been considered so far.

Let $l = (l_i) \in \mathbb{N}^n_0$ be a multiindex and $p = (p_i) \in \mathbb{N}^n$ such that $p - \mathbf{1} < \alpha \le p$ with α being the multiindex from (2.1), and set

$$(x)^l := \begin{cases} 0 & \text{if } l > \mathbf{0}, \\ (x_{k_1}, \ldots, x_{k_\nu}) & \text{if } l_{k_1} = \ldots = l_{k_\nu} = 0 \\ & \text{for some } \nu \text{ where } 1 \le k_1 < \ldots < k_\nu \le n. \end{cases}$$

In this chapter we examine the following characteristic problem (P) for equation (2.1): Find a solution u of equation (2.1) in Ω (i.e. a function $u : \Omega \to \mathbb{R}$ such that $\mathcal{D}^\alpha u \in L^1(\Omega)$ and u satisfies (2.1) a.e. in Ω) which satisfies the boundary conditions

$$\mathcal{D}^{\alpha-l}u(\Delta_l(x)) = M_l((x)^l) \quad \text{for } l \le p,\ l \ne \mathbf{0}\,, \tag{2.2}$$

for given M_l. Observe that since $\mathcal{D}^\alpha u$ exists and is integrable, the derivatives $\mathcal{D}^{\alpha-l}u$ are at least absolutely continuous in x_i, provided that $l_i \ge 1$, $i = \overline{1,n}$.

Assume that u is a solution of problem (P). By Lemma 1.7, u is of the form

$$u(x) = \widehat{T}s(x) := w(x) + \mathcal{D}^{-\alpha}s(x)\,, \tag{2.3}$$

where

$$w(x) := \sum_{l\le p; l\ne\mathbf{0}} \mathcal{D}^{-\Delta_l(\alpha)} M_l((x)^l) x^{\nabla_l(\alpha-l)}/\Gamma(\mathbf{1}+\nabla_l(\alpha-l)) \tag{2.4}$$

and s, standing for the derivative of order α of u, is an integrable solution of the nonlinear integral equation

$$s(x) = Ts(x) := F(x, \{\mathcal{D}^{\beta^j} w(x) + \mathcal{D}^{\beta^j-\alpha}s(x)\}), \quad x \in \Omega\,. \tag{2.5}$$

By (2.3), $\mathcal{D}^\alpha u$ does not depend on the order of differentiations. Moreover, (2.3) establishes a one-to-one correspondence between the solutions u of problem (P) and the integrable solutions s of (2.5).

Remark 2.1. By (2.4) and Lemma 1.7, for $\beta \ge \mathbf{0}$ we have

$$\mathcal{D}^\beta w(x) = \sum_{l\le p; l\ne\mathbf{0}} \mathcal{D}^{-\Delta_l(\alpha-\beta)} M_l((x)^l) x^{\nabla_l(\alpha-\beta-l)}/\Gamma(\mathbf{1}+\nabla_l(\alpha-\beta-l))\,.$$

2. Existence of solutions. In this section we give sufficient conditions for the set of solutions of problem (P) to be nonempty and relatively compact in $L^1(\Omega)$. Since the transformation $\widehat{T}$ (cf. (2.3)) is completely continuous (cf. Lemma 1.1), it is enough to find sufficient conditions for the image of any bounded set under transformation T (cf. formula (2.5)) to be also bounded.

We assume the following:

I. The function $F : \Omega \times \mathbb{R}^r \to \mathbb{R}$ satisfies the Carathéodory conditions (cf. [16], Def. 12.2) and the inequality

$$|F(x, \{z_j\})| \le \sum_{j=1}^{r} \sum_{k=1}^{r_j} K_{jk}(x)|z_j|^{\kappa_{jk}} \quad \text{a.e. in } \Omega,\ z_j \in \mathbb{R}\,, \tag{2.6}$$

where $r_j \in \mathbb{N}$, $0 \le \kappa_{jk} \le 1$ are fixed numbers and $K_{jk} : \Omega \to \mathbb{R}_+$ are in $L^{1/(1-\kappa_{jk})}$ (here and in the sequel $1/0 := \infty$).

II. The functions $\mathcal{D}^{\beta^j} w : \Omega \to \mathbb{R}$ are integrable.

III. The functions $M_l : \mathbb{R}^\nu \to \mathbb{R}$ ($1 \le \nu := |(\operatorname{sgn} l_i)|_+ < n$; $l \le p$) are integrable.

R e m a r k 2.2. By Assumption I, $\mathcal{F}(s_1, \ldots, s_r)(x) := F(x, \{s_j(x)\})$ is integrable (cf. also [42]) for integrable $s_1, \ldots, s_r$. The substitution operator $\mathcal{F} : (L^1(\Omega))^r \to L^1(\Omega)$ is bounded and continuous (cf. [22], pp. 161–163).

By Assumptions I and II (cf. also Remark 2.2) and Lemma 1.1, the transformation T is continuous. In the case $\beta^j < \alpha$, it is completely continuous.

We will find a bounded and convex set $\Lambda \subset L^1(\Omega)$ such that $T(\Lambda) \subset \Lambda$ and $T(\Lambda)$ is relatively compact.

By (2.6), the inequality $(y+z)^\kappa \le y^\kappa + z^\kappa$ $(y, z \in \mathbb{R}_+;\ 0 \le \kappa \le 1)$ and the well known properties of the Lebesgue integral, we have

$$|Ts(x)| \le \sum_{j=1}^{r} \sum_{k=1}^{r_j} K_{jk}(x)[|\mathcal{D}^{\beta^j} w(x)|^{\kappa_{jk}} + (\mathcal{D}^{\beta^j - \alpha}|s(x)|)^{\kappa_{jk}}] \quad \text{a.e. in } \Omega. \tag{2.7}$$

Equip the space $L^1(\Omega)$ with the norm (1.6). By the Hölder inequality, the functions $x \to K_{jk}(x)|\mathcal{D}^{\beta^j} w(x)|^{\kappa_{jk}}$ are integrable. Then, taking into account (2.7) and (1.6), we have

$$\begin{aligned} \|Ts\|_{1,\tau} \le {} & \Big\| \sum_{j=1}^{r} \sum_{k=1}^{r_j} K_{jk} |\mathcal{D}^{\beta^j} w|^{\kappa_{jk}} \Big\|_{1,\tau} \\ & + \sum_{j=1}^{r} \sum_{k=1}^{r_j} \int_\Omega K_{jk}(x)(|\mathcal{D}^{\beta^j - \alpha} s(x)|)^{\kappa_{jk}} \exp(-\tau |x|_+)\, dx\,. \end{aligned} \tag{2.8}$$

Consider the integral in (2.8). Using the Hölder inequality, and the properties of K_{jk} and the exponential function, for $\kappa_{jk} > 0$ we have

$$\begin{aligned} &\int_\Omega K_{jk}(x)(|\mathcal{D}^{\beta^j - \alpha} s(x)|)^{\kappa_{jk}} \exp(-\tau |x|_+)\, dx \\ &\qquad \le \Big(\int_\Omega (K_{jk}(x))^{1/(1-\kappa_{jk})} \exp(-\tau |x|_+)\, dx \Big)^{1-\kappa_{jk}} \\ &\qquad \quad \times \Big(\int_\Omega \mathcal{D}^{\beta^j - \alpha} |s(x)| \exp((-\tau/\kappa_{jk})|x|_+)\, dx \Big)^{\kappa_{jk}} \\ &\qquad \le \text{const} \Big(\int_\Omega \mathcal{D}^{\beta^j - \alpha} |s(x)| \exp(-\tau |x|_+)\, dx \Big)^{\kappa_{jk}} \end{aligned}$$

with const independent of s. For $\kappa_{jk} = 0$, we obtain

$$\int_\Omega K_{jk}(x)(|\mathcal{D}^{\beta^j - \alpha} s(x)|)^{\kappa_{jk}} \exp(-\tau |x|_+)\, dx \le \text{const}\,.$$

Changing the order of integration one gets the estimate

$$\int_\Omega K_{jk}(x)(|\mathcal{D}^{\beta^j - \alpha} s(x)|)^{\kappa_{jk}} \exp(-\tau |x|_+)\, dx \le \text{const}\, \tau^{\kappa_{jk} |\beta^j - \alpha|_+} \|s\|_{1,\tau}^{\kappa_{jk}}\,.$$

Using this estimate together with (2.8), we have

$$\|Ts\|_{1,\tau} \le \text{const}\Big(1 + \sum_{j=1}^{r}\sum_{k=1}^{r_j} \tau^{\kappa_{jk}|\beta^j-\alpha|_+}\|s\|_{1,\tau}^{\kappa_{jk}}\Big). \tag{2.9}$$

Write

$$B_\tau(R) := \{s \in L^1(\Omega) : \|s\|_{1,\tau} \le R\}, \tag{2.10}$$

where R is a positive constant. Clearly, $B_\tau(R)$ is a closed, bounded and convex set.

Since $0 \le \kappa_{jk} \le 1$ (cf. (2.6)), we can choose R and τ so that

$$\text{const}\Big(1 + \sum_{j=1}^{r}\sum_{k=1}^{r_j} \tau^{\kappa_{jk}|\beta^j-\alpha|_+}R^{\kappa_{jk}}\Big) \le R. \tag{2.11}$$

As a result T maps $B_\tau(R)$ into itself if R and τ are sufficiently large to satisfy (2.11). Furthermore, if s is a solution of equation (2.3), then s belongs to $B_\tau(R)$ with R and τ satisfying (2.11). In the case when $\beta^j < \alpha$, the set $T(B_\tau(R))$ is, obviously, relatively compact. In the more general case $\beta^j \le \alpha$, define

$$Z_\tau(R) := \Big\{s \in B_\tau(R) : \forall\varepsilon > 0\ \exists\delta > 0\ \forall h = (h_i) > \mathbf{0}\quad |h|_+ < \delta \Rightarrow \int_\Omega |s(x+h) - s(x)|\,dx \le \varepsilon\Big\}. \tag{2.12}$$

This set is convex and relatively compact. Since T is continuous, $T(A) \subset T(\overline{A}) \subset \overline{T(A)}$ for every set A. Thus $T(\overline{A}) = \overline{T(A)}$ and $T(A)$ is relatively compact, provided that so is A. So, $T(Z_\tau(R))$ is relatively compact.

Thus, all assumptions of the Schauder fixed point theorem are satisfied (cf. [12], p. 57 and [22], pp. 125–126) and so, by this theorem, we have

LEMMA 2.1. *Under Assumptions* I *and* II, *the set of solutions of equation* (2.5) *is nonempty and bounded. Moreover*, *if* $\beta^j < \alpha$ *then this set is relatively compact in* $L^1(\Omega)$.

Remark 2.3. Replace Assumptions I and II by the following:

I_1. The function $F : \Omega \times \mathbb{R}^r \to \mathbb{R}$ satisfies the Carathéodory conditions and is subadditive in the second variable.

II_1. The function $\Omega \ni x \to F(x, \{\mathcal{D}^{\beta^j} w(x)\})$ is integrable.

It is easily seen that Assumption I_1 yields the estimate

$$|Ts(x)| \le |F(x, \{\mathcal{D}^{\beta^j} w(x)\})| + \sum_{j=1}^{r}\sum_{k=1}^{r_j} K_{jk}(x)(\mathcal{D}^{\beta^j-\alpha}|s(x)|)^{\kappa_{jk}}.$$

Hence, bearing in mind Assumptions I_1 and II_1, and using an argument analogous to that subsequent to inequality (2.7), we can show that equation (2.5) has a solution.

By the complete continuity of the transformation defined by (2.3), and Lemma 2.1, we have

THEOREM 2.1. *If Assumptions* I–III (*or* I_1, II_1 *and* III) *are satisfied*, *then problem* (P) *has a solution. This solution is of the form* (2.3), *where* s *is an integrable solution of the nonlinear integral equation* (2.5). *Moreover*, *the set of solutions of problem* (P) *is relatively compact in* $L^1(\Omega)$.

Remark 2.4. Let $\Omega = \mathbb{R}^n_+$. There exists an increasing sequence $(B^k)_{k=1}^{\infty}$ of points from Ω such that $\Omega = \bigcup_{k\in\mathbb{N}} \Omega^k$, where $\Omega^k = (\mathbf{0}; B^k)$. By Lemma 2.1, equation (2.5) has a solution in Ω^k for $k \in \mathbb{N}$. As a consequence, the function $s(x) = s^k(x)$, where $x \in \Omega^k \setminus \Omega^{k-1}$ (we set $\Omega^0 := \emptyset$), $k \in \mathbb{N}$, is locally integrable and it is a solution of (2.5) in Ω. Taking into account the one-to-one correspondence between the solutions of (2.5) and those of problem (P) shows that (P) has a solution u in Ω, given by (2.3), such that $\mathcal{D}^\alpha u \in L^1_{\mathrm{loc}}$.

Remark 2.5. Consider equation (2.1) with the characteristic conditions

$$\mathcal{D}^{\nabla_l(\alpha-l)} u(\Delta_l(x)) = N_l((x)^l) \quad \text{for } l \le p,\ l \ne \mathbf{0}\,, \tag{2.13}$$

where N_l are given functions. Seeking a solution u of (2.1) such that the derivative $\mathcal{D}^\alpha u$ does not depend on the order of differentiations, and assuming that the N_l ($l \le p$; $1 \le |(\operatorname{sgn} l_i)|_+ < n$) have the derivatives $\mathcal{D}^{\nabla_l(\alpha)} N_l$ and Assumptions I and II are satisfied, one can prove, using the Schauder fixed point theorem, that problem (2.1), (2.13) has a solution.

Let us point out that problem (2.1), (2.13) with $n = 2$, $0 < \alpha_1 \le 1$, $1 < \alpha_2 \le 2$ and a less general right-hand side was examined in [7].

3. Uniqueness of the solution. Let us replace Assumption I by

I_2. The function $F : \Omega \times \mathbb{R}^r \to \mathbb{R}$ is measurable in $x \in \Omega$ and, for almost all $x \in \Omega$ and $z_j^1, z_j^2 \in \mathbb{R}$,

$$|F(x, \{z_j^1\}) - F(x, \{z_j^2\})| \le c_L \sum_{j=1}^{r} x^{-\mu^j} |z_j^1 - z_j^2|\,, \tag{2.14}$$

where c_L is a positive constant independent of $x \in \Omega$, with $\mu^j \in \mathbb{R}^n_+$ being fixed and such that $\alpha - \beta^j - \mu^j > \mathbf{0}$ ($j = \overline{1, r}$).

We have

LEMMA 2.2. *If Assumptions* I_2 *and* II_1 *are satisfied*, *then equation* (2.5) *has a unique integrable solution in* Ω.

Proof. By Assumptions I_2 and II_1, the transformation T (cf. (2.5)) maps $L^1(\Omega)$ into a set of measurable functions.

Choose τ so that

$$2c_L \sum_{j=1}^{r} (\Gamma(\alpha - \beta^j - \mu^j)/\Gamma(\alpha - \beta^j))\tau^{|\beta^j + \mu^j - \alpha|_+} \le 1 \tag{2.15}$$

(the possibility of such a choice follows from the assumption $\alpha - \beta^j - \mu^j > \mathbf{0}$ ($j = \overline{1,r}$)). We show that T is contractive. To this end let $s^1, s^2 \in L^1(\Omega)$. By Assumption I_2, in the case $\beta^j < \alpha$ we obtain

$$\|Ts^1 - Ts^2\|_{1,\tau} \le c_L \sum_{j=1}^{r} \int_{\Omega} dx \int_{\Omega(x)} (x-t)^{\alpha-\beta^j-\mathbf{1}} x^{-\mu^j} \times |s^1(t) - s^2(t)| \exp(-\tau|x|_+)\, dt / \Gamma(\alpha - \beta^j)\,.$$

Changing the order of integration gives

$$\|Ts^1 - Ts^2\|_{1,\tau} \le c_L \sum_{j=1}^{r} \int_{\Omega} |s^1(t) - s^2(t)| \exp(-\tau|t|_+) \times \Big(\int_{(t;B)} (x-t)^{\alpha-\beta^j-\mathbf{1}} \exp(-\tau|x-t|_+) x^{-\mu^j}\, dx \Big)\, dt / \Gamma(\alpha - \beta^j)\,.$$

Note that

$$\int_{(t;B)} (x-t)^{\alpha-\beta^j-\mathbf{1}} \exp(-\tau|x-t|_+) x^{-\mu^j}\, dx \le \int_{\mathbb{R}^n_+} \zeta^{\alpha-\beta^j-\mu^j-\mathbf{1}} \exp(-\tau|\zeta|_+)\, d\zeta$$

and hence,

$$\|Ts^1 - Ts^2\|_{1,\tau} \le c_L \|s^1 - s^2\|_{1,\tau} \times \sum_{j=1}^{r} \Gamma(\alpha - \beta^j - \mu^j) \tau^{|\beta^j + \mu^j - \alpha|_+} / \Gamma(\alpha - \beta^j) \le \frac{1}{2} \|s^1 - s^2\|_{1,\tau}\,.$$

The argument in the remaining cases is analogous.

Thus, T is a contraction with contraction constant $1/2$. Moreover, $\|Ts\|_{1,\tau} \le \frac{1}{2}\|s\|_{1,\tau}$, which proves that $Ts \in L^1(\Omega)$ provided that $s \in L^1(\Omega)$.

The proof is completed by using the Banach fixed point theorem.

Lemma 2.2 immediately implies

THEOREM 2.2. *If Assumptions* I_2, II_1 *and* III *are satisfied*, *then problem* (P) *has exactly one solution.*

Remark 2.6 (cf. also Remark 2.4). Let $\Omega = \mathbb{R}^n_+$ and let $(B^k)_{k=1}^{\infty}$ be the increasing sequence defined in Remark 2.4. By Theorem 2.2, for every $k \in \mathbb{N}$ there exists exactly one solution u^k of problem (P) in Ω^k and hence the function $u(x) := u^k(x)$ for $x \in \Omega^k \setminus \Omega^{k-1}$, $k \in \mathbb{N}$, is a solution of (P) in Ω such that $\mathcal{D}^\alpha u$ is locally integrable. Moreover, $\mathcal{D}^\alpha u$ belongs to the Banach space

$$(2.16)\qquad L^1_\tau(\mathbb{R}^n_+) := \Big\{f : \mathbb{R}^n_+ \to \mathbb{R} : f \text{ measurable and } \|s\|_{1,\tau} := \int_{\mathbb{R}^n_+} |s(x)|\exp(-\tau|x|_+)\,dx < \infty\Big\}$$

with τ being defined by (2.15). Note that $L^1(\Omega_*) \subset L^1_\tau(\Omega_*) \subset L^1_{\mathrm{loc}}(\Omega_*)$ for every $\Omega_* \subset \mathbb{R}^n_+$ and $\tau > 0$.

Conversely, if u is a solution of problem (P) in Ω such that $\mathcal{D}^\alpha u$ is locally integrable then, by the uniqueness of the solution of (P) in Ω^k, the function $u^k(x) := u(x)$ for $x \in \Omega^k$, $k \in \mathbb{N}$, is the unique solution of (P) in Ω^k such that $\mathcal{D}^\alpha u^k$ is integrable. It being so for every k, the uniqueness of the solution of (P) in Ω is proved.

4. Continuous solutions. Set $s(x) := x^{p-\alpha}\mathcal{D}^\alpha u(x)$. In this section we examine the existence of solutions of problem (P) such that s is bounded and continuous in Ω. Observe that if u is a solution of (P), then s is a solution of the integral equation

$$(2.17)\qquad \begin{aligned} s(x) &= T_C s(x) \\ &:= x^{p-\alpha}F(x, \{\mathcal{D}^{\beta^j}w(x) + \mathcal{D}^{\beta^j-\alpha}(x^{\alpha-p}s(x))\}), \qquad x \in \Omega\,. \end{aligned}$$

Denote by $BC(\Omega)$ the space of bounded continuous functions defined on Ω, with the norm

$$(2.18)\qquad |s|_\tau := \sup_{x\in\Omega}\{|s(x)\exp(-\tau|x|_+)|\},$$

where τ is a fixed nonnegative number. $BC(\Omega)$ is a Banach space. Moreover, for positive τ, the norms $|\cdot|_0$ and $|\cdot|_\tau$ are equivalent.

We assume the following:

I_3. The function F is continuous, satisfies the Lipschitz condition in the last r variables (cf. Assumption I_1) and there exists a constant c_F, independent of $x \in \Omega$, such that

$$(2.19)\qquad |F(x, \{\mathcal{D}^{\beta^j}w(x)\})| \le c_F x^{\alpha-p}, \qquad x \in \Omega\,,$$

where w is given by (2.4).

III_1. The functions M_l ($1 \le |(\operatorname{sgn} l_i)|_+ < n$; $l \le p$) are continuous.

We need

PROPOSITION 2.1. *If* $\mu, \nu, \tau > 0$ *and* $z > 0$, *then*

$$(2.20)\qquad z^{1-\nu}\int_0^z (z-\zeta)^{\nu-1}\zeta^{\mu-1}\exp(-\tau\zeta)\,d\zeta \le \mathrm{const}\,\tau^{-\mu}\,,$$

where const *is independent of* z.

Proof. Denoting by $I(z)$ the left-hand side of (2.20) and changing variables, we have

$$I(z) = z^{\mu} \int\limits_0^1 (1-\xi)^{\nu-1}\xi^{\mu-1}\exp(-\tau z\xi)\,d\xi\,.$$

Observe that

$$z^{\mu}(1-\xi)^{\nu-1}\xi^{\mu-1}\exp(-\tau z\xi) \le \begin{cases} \max(1,2^{1-\nu})z^{\mu}\xi^{\mu-1}\exp(-\tau z\xi) & \text{for } 0\le\xi<1/2,\\ 2(1-\xi)^{\nu-1}\Gamma(\mu+1)\tau^{-\mu} & \text{for } 1/2<\xi\le 1.\end{cases}$$

As a result, $I(z)\le \max(1,2^{1-\nu})\Gamma(\mu)(1+\mu/\nu)\tau^{-\mu}$, and the proof is complete.

LEMMA 2.3. *If the functions F and M_l satisfy Assumptions* I_3 *and* III_1, *respectively, then equation* (2.17) *has exactly one solution of class $BC(\Omega)$. Moreover, if s is a solution of* (2.17), *then*

$$|s(x)|\le 2c_F\exp(\tau|x|_+),\qquad x\in\Omega\,, \tag{2.21}$$

where τ is a fixed nonnegative number.

Proof. By Assumptions I_3 and III_1, the function $T_C s$ (cf. (2.17)) is continuous for $s\in BC(\Omega)$. Let $s^1, s^2\in BC(\Omega)$, $\beta^j<\alpha$ $(j=\overline{1,r})$ and observe that, by Assumption I_3 and definition (2.18), we get

$$|T_C s^1(x)-T_C s^2(x)|\exp(-\tau|x|_+)\le c_L x^{p-\alpha}|s^1-s^2|_\tau \times\sum_{j=1}^{r}\int\limits_{(\mathbf{0};x)}(x-t)^{\alpha-p}t^{\alpha-\beta^j-\mathbf{1}}\exp(-\tau|t|_+)\,dt/\Gamma(\alpha-\beta^j),\qquad x\in\Omega\,.$$

It follows from Proposition 2.1 and definition (2.18) that

$$|T_C s^1-T_C s^2|_\tau\le \mathrm{const}\,|s^1-s^2|_\tau\sum_{j=1}^{r}\tau^{|\beta^j-\alpha|_+}\,,$$

where const is independent of s^1 and s^2. This inequality also holds for $\beta^j\le\alpha$, $|\beta^j|_+<|\alpha|_+$ $(j=\overline{1,r})$. Hence, for fixed τ such that (cf. the proof of Lemma 2.2)

$$\mathrm{const}\sum_{j=1}^{r}\tau^{|\beta^j+\mu^j-\alpha|_+}\le 1/2\,,$$

T_C is contractive. Moreover, $|T_C s|_\tau\le\frac12|s|_\tau+c_F$ and hence $|T_C s|_\tau$ is finite provided that $s\in BC(\Omega)$.

Notice that if s is a solution of (2.17), then $|s|_\tau\le 2c_F$ and so inequality (2.21) is valid.

Using the Banach fixed point theorem we easily conclude the proof.

Remark 2.7. The function $u(x):=w(x)+\mathcal{D}^{-\alpha}(x^{\alpha-p}s(x))$, with s being a continuous solution of equation (2.17), is a solution of problem (P). Moreover, if the M_l satisfy Assumption III_1 then the derivatives $\mathcal{D}^m u$ ($m = (m_i) \in \mathbb{N}_0^n$; $m_i = \overline{0, [\alpha_i]}$, $i = \overline{1, n}$) exist in Ω (cf. also Lemma 1.7), are continuous and

$$|\mathcal{D}^m u(x)| \le |\mathcal{D}^m w(x)| + \mathrm{const}\,\tau^{m-\alpha} x^{\alpha-p} \exp(\tau|x|_+), \qquad x \in \Omega\,. \tag{2.22}$$

The foregoing considerations yield

THEOREM 2.3. *If the functions F and M_l satisfy Assumptions* I_3 *and* III_1, *respectively, then the n-dimensional characteristic problem* (2.1), (2.2) *has a solution u. This solution is such that $x^{p-\alpha}\mathcal{D}^\alpha u \in BC(\Omega)$, $\mathcal{D}^m u \in C(\Omega)$ ($m = (m_i) \in \mathbb{N}_0^n$; $m_i = \overline{0, [\alpha_i]}$, $i = \overline{1, n}$) and estimates* (2.22) *hold.*

5. Continuous dependence of the solution on the boundary data. Let u^1 and u^2 be two solutions of problem (P) with M_l^1 and M_l^2 being the corresponding boundary data (cf. conditions (2.2)).

Assume the following:

I_4. The function F satisfies the Carathéodory conditions and

$$|F(x, \{z_j^1\}) - F(x, \{z_j^2\})| \le \sum_{j=1}^{r} a_j(x)|z_j^1 - z_j^2| \quad \text{a.e. in } \Omega,\ z_j^1, z_j^2 \in \mathbb{R}\,, \tag{2.23}$$

where $a_j : \Omega \to \mathbb{R}_+$ are bounded and such that $a_j(x)\mathcal{D}^{\beta^j}(w^1(x) - w^2(x))$ are integrable, where w^1 and w^2 are given by (2.4) with M replaced by M_l^1 and M_l^2, respectively.

II_2. The functions M_l^1 and M_l^2 satisfy Assumption III and

$$\|w^1 - w^2\|_{1,0} < \delta_0 \quad \text{and} \quad \|a_j \mathcal{D}^{\beta^j}(w^1 - w^2)\|_{1,0} < \delta_j\,,$$

where δ_j ($j = \overline{0, r}$) are given positive numbers.

Let $s^1 := \mathcal{D}^\alpha u^1$ and $s^2 := \mathcal{D}^\alpha u^2$. By (2.5) and (2.23), we have

$$|s^1(x) - s^2(x)| \le \sum_{j=1}^{r} |a_j(x)\mathcal{D}^{\beta^j}(w^1(x) - w^2(x))| + c\sum_{j=1}^{r} \mathcal{D}^{\beta^j-\alpha}|s^1(x) - s^2(x)|,$$

where $c := \max_{j,x} a_j(x)$. The above inequality implies

$$\mathcal{D}^{-\alpha}|s^1(x) - s^2(x)| \le V(x) + c\sum_{j=1}^{r} \mathcal{D}^{\beta^j-\alpha}(\mathcal{D}^{-\alpha}|s^1(x) - s^2(x)|)\,,$$

with

$$V(x) := \sum_{j=1}^{r} \mathcal{D}^{-\alpha}(a_j(x)|\mathcal{D}^{\beta^j}(w^1(x) - w^2(x))|)\,.$$

By Lemma 1.6, we have the estimate

$$\mathcal{D}^{-\alpha}|s^1(x) - s^2(x)| \le V(x) + \sum_{k=1}^{\infty} \sum_{1\le j^1,\ldots,j^k\le r} c^k \mathcal{D}^{\beta^{(j,k)}-k\alpha} V(x)$$

where $\beta^{(j,k)} := \beta^{j^1} + \ldots + \beta^{j^k}$, and the series is convergent. Formula (2.3) and the above estimate yield

$$|u^1(x) - u^2(x)| \le |w^1(x) - w^2(x)| + V(x) + \sum_{k=1}^{\infty} \sum_{1\le j^1,\ldots,j^k\le r} c^k \mathcal{D}^{\beta^{(j,k)}-k\alpha} V(x),$$

whence

$$\text{(2.25)} \quad \|u^1 - u^2\|_{1,0} \le \delta_0 + \sum_{j=1}^{r} \delta_j \Big(\sum_{k=1}^{\infty} \sum_{1\le j^1,\ldots,j^k\le r} c^k B^{k\alpha-\beta^{(j,k)}} / \Gamma(\mathbf{1} + k\alpha - \beta^{(j,k)}) \Big)$$

with the series appearing in (2.25) being convergent.

It follows from inequality (2.25) that the following theorem is valid:

THEOREM 2.4. *If Assumptions* I_4 *and* II_2 *are satisfied, then the solutions of problem* (P) *depend continuously on the boundary data, i.e. for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $\delta_j < \delta$ $(j = \overline{0,r})$ *then for any two solutions* u^1 *and* u^2 *of problem* (P) *with boundary functions* M_l^1 *and* M_l^2, *respectively, the estimate* $\|u^1 - u^2\|_{1,0} < \varepsilon$ *holds.*

Remark 2.8. One can observe that if Assumptions I_2, I_4, II_1, II_2, and III are satisfied then problem (P) is well posed in the space of integrable functions with integrable derivatives of order α.

III. Noncharacteristic boundary value problem

1. The problem. In this section, which extends earlier research of J. Conlan (cf. [7]) and the present author (cf. [28]), we deal with the counterpart of Z. Szmydt's problem (cf. [44]) for a partial differential equation of noninteger order with two independent variables.

Let $\Omega := (0, A) \times (0, B)$, where $0 < A, B < \infty$. Assume that $\alpha, \beta > 0$ and set $q_1 := -[-\alpha]$, $q_2 := -[-\beta]$.

For $i = \overline{1, q_1}$ and $j = \overline{1, q_2}$, let $g_j : [0, A] \to [0, B]$, $h_i : [0, B] \to [0, A]$, $G_j : (0, A) \to \mathbb{R}$, and $H_i : (0, B) \to \mathbb{R}$ be given functions (the curves of equations $y = g_j(x)$ and $x = h_i(y)$ will be denoted by l_j^1 and l_i^2, respectively), (x_{ij}, y_{ij}) fixed points of $\overline{\Omega}$ and u_{ij}^0 given numbers.

We deal with the partial differential equation

$$\text{(3.1)} \qquad \mathcal{D}_x^{\alpha} \mathcal{D}_y^{\beta} u(x, y) = F(x, y, \{\mathcal{D}_x^{\gamma} \mathcal{D}_y^{\lambda} u(x, y)\}), \qquad (x, y) \in \Omega\,,$$

where $\{\mathcal{D}_x^{\gamma} \mathcal{D}_y^{\lambda} u\}$ is the finite sequence of $\mathcal{D}_x^{\gamma} \mathcal{D}_y^{\lambda} u$ such that $\gamma \le \alpha$, $\lambda \le \beta$, and $\gamma + \lambda < \alpha + \beta$ (the total number of these derivatives will be denoted by r).

We study the problem (S) which consists in finding a solution of equation (3.1) in Ω (cf. Section II.1) satisfying

$$\mathcal{D}_x^{\alpha}\mathcal{D}_y^{\beta-j}u(x,g_j(x))=G_j(x),\qquad \mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta}u(h_i(y),y)=H_i(y)\,, \tag{3.2}$$

$$\mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta-j}u(x_{ij},y_{ij})=u_{ij}^0\,. \tag{3.3}$$

($i=\overline{1,q_1}$; $j=\overline{1,q_2}$). The above problem was considered by Z. Szmydt (in the paper quoted above), in the case when $\alpha,\beta\in\mathbb{N}$, as a generalization of the classical Goursat problem.

We assume the following:

I. The function $F:\Omega\times\mathbb{R}^r\to\mathbb{R}$ satisfies the Carathéodory conditions and

$$|F(x,y,\{z_{\gamma\lambda}\})| \le K(x,y)+\sum_{\gamma,\lambda}\sum_{l=1}^{r_{\gamma\lambda}}K_{\gamma\lambda l}(x,y)|z_{\gamma\lambda}|^{\kappa_{\gamma\lambda l}}\quad \text{a.e. in } \Omega,\ z_{\gamma\lambda}\in\mathbb{R}, \tag{3.4}$$

where $r_{\gamma\lambda}\in\mathbb{N}$, $0<\kappa_{\gamma\lambda l}\le 1$, and $K,K_{\gamma\lambda l}:\Omega\to\mathbb{R}_+$ are given functions of class $L^1(\Omega)$ and $L^{1/(1-\kappa_{\gamma\lambda l})}(\Omega)$, respectively.

II. The functions g_j and h_i ($i=\overline{1,q_1}$; $j=\overline{1,q_2}$) are continuous.

III. The functions G_j and H_i ($i=\overline{1,q_1}$; $j=\overline{1,q_2}$) are integrable.

IV. The functions $x^{\alpha-\gamma-k}/\Gamma(1+\alpha-\gamma-k)$ and $y^{\beta-\lambda-l}/\Gamma(1+\beta-\lambda-l)$ ($k=\overline{1,q_1}$; $l=\overline{1,q_2}$) are integrable.

2. Local solutions of the problem. Let $s=\mathcal{D}_x^{\alpha}\mathcal{D}_y^{\beta}u$. By an argument analogous to that in [6] (cf. also [23], p. 99) one can establish

LEMMA 3.1. *If u is a solution of equation* (3.1) *in Ω, then there are integrable functions $\varphi_j:(0,A)\to\mathbb{R}$ and $\psi_i:(0,B)\to\mathbb{R}$, and constants $c_{ij}\in\mathbb{R}$ ($i=\overline{1,q_1}$; $j=\overline{1,q_2}$) such that*

$$\begin{aligned} u(x,y)=&\sum_{k=1}^{q_1}\sum_{l=1}^{q_2}c_{kl}x^{\alpha-k}y^{\beta-l}/(\Gamma(1+\alpha-k)\Gamma(1+\beta-l))\\ &+\sum_{k=1}^{q_1}x^{\alpha-k}\psi_k^{(-\beta)}(y)/\Gamma(1+\alpha-k)\\ &+\sum_{l=1}^{q_2}\varphi_l^{(-\alpha)}(x)y^{\beta-l}/\Gamma(1+\beta-l)+\mathcal{D}_x^{-\alpha}\mathcal{D}_y^{-\beta}s(x,y)\end{aligned} \tag{3.5}$$

($\varphi^{(-\alpha)}(x):=\mathcal{D}_x^{-\alpha}\varphi(x)$; $\psi^{(-\beta)}(y):=\mathcal{D}_y^{-\beta}\psi(y)$).

Conversely, if u is given by (3.5) *with some integrable $\varphi_j:(0,A)\to\mathbb{R}$ and $\psi_i:(0,B)\to\mathbb{R}$, and $c_{ij}\in\mathbb{R}$, then u is a solution of* (3.1) *in Ω.*

Imposing on u (cf. (3.5)) conditions (3.2) and (3.3), we obtain

$$\varphi_j(x) = G_j(x) - \sum_{l=1}^{j-1} \varphi_l(x)[g_j(x)]^{j-l}/(j-l)! - \int_0^{g_j(x)} [g_j(x)-\eta]^{j-1} s(x,\eta)\,d\eta/(j-1)!\,, \tag{3.6}$$

$$\psi_i(y) = H_i(y) - \sum_{k=1}^{i-1} [h_i(y)]^{i-k}\psi_k(y)/(i-k)! - \int_0^{h_i(y)} [h_i(y)-\xi]^{i-1} s(\xi,y) d\xi/(i-1)! \tag{3.7}$$

and

$$c_{ij} = u^0_{ij} - \Big(\sum_{\substack{1\le k\le i, 1\le l\le j\\ k+l<i+j}} c_{kl} x_{ij}^{i-k} y_{ij}^{j-l}/((i-k)!(j-l)!)) + \sum_{k=1}^{i} x_{ij}^{i-k} \psi_k^{(-j)}(y_{ij})/(i-k)! + \sum_{l=1}^{j} \varphi_l^{(-i)}(x_{ij}) y_{ij}^{j-l}/(j-l)! + \int_0^{x_{ij}} d\xi \int_0^{y_{ij}} (x_{ij}-\xi)^{i-1}(y_{ij}-\eta)^{j-1} s(\xi,\eta)\,d\eta/((i-1)!(j-1)!)\Big) \tag{3.8}$$

$(i = \overline{1,q_1};\ j = \overline{1,q_2})$.

Note that φ_j, ψ_i and c_{ij} depend on s but each of them can be represented as the sum of two terms the first of which depends linearly on s and the second is independent of s (depends only on the boundary data).

Observe that

$$\int_0^A |\varphi_j(x)|\,dx \le \mathrm{const}(1+\|s\|_{1,\tau})\,, \qquad \int_0^B |\psi_i(y)|\,dy \le \mathrm{const}(1+\|s\|_{1,\tau}) \tag{3.9}$$

$(i = \overline{1,q_1};\ j = \overline{1,q_2})$, where $\|\cdot\|_{1,\tau}$ is the norm in $L^1(\Omega)$ defined in (1.6), and const, here and in the sequel, does not depend on s. Using the Fubini theorem and (3.9), we get

$$\int_0^A |\varphi_j^{(-\alpha)}(x)|\,dx \le \mathrm{const}\, A^\alpha(1+\|s\|_{1,\tau})\,, \qquad \int_0^B |\psi_i^{(-\beta)}(y)|\,dy \le \mathrm{const}\, B^\beta(1+\|s\|_{1,\tau})\,. \tag{3.10}$$

Thus (cf. (3.9) and (3.10)), we have

$$|c_{ij}| \le \text{const}(1 + \|s\|_{1,\tau}) . \tag{3.11}$$

Set

$$\begin{aligned} L_{\gamma\lambda}s(x,y) &= \sum_{k=1}^{q_1}\sum_{l=1}^{q_2} c_{kl}x^{\alpha-\gamma-k}y^{\beta-\lambda-l}/(\Gamma(1+\alpha-\gamma-k)\Gamma(1+\beta-\lambda-l)) \\ &+ \sum_{k=1}^{q_1} x^{\alpha-\gamma-k}\psi_k^{(\lambda-\beta)}(y)/\Gamma(1+\alpha-\gamma-k) \\ &+ \sum_{l=1}^{q_2} \varphi_l^{(\gamma-\alpha)}(x)y^{\beta-\lambda-l}/\Gamma(1+\beta-\lambda-l) + \mathcal{D}_x^{\gamma-\alpha}\mathcal{D}_y^{\lambda-\beta}s(x,y) \end{aligned} \tag{3.12}$$

with c_{ij}, φ_j and ψ_i ($i = \overline{1, q_1}$; $j = \overline{1, q_2}$) being given by (3.6)–(3.8), respectively. Observe that Problem (S) is equivalent to the integro-functional equation

$$s(x,y) = T_s s(x,y) := F(x, y, \{L_{\gamma\lambda}s(x,y)\}), \quad (x,y) \in \Omega . \tag{3.13}$$

Now, we prove that equation (3.13) has at least one solution. To this end consider the transformation T_s on $L^1(\Omega)$. Clearly, T_s is the composition of two transformations: the affine transformation $L_{\gamma\lambda}$ (cf. [39], pp. 170, 171) and the substitution operator $\mathcal{F}$ (cf. Remark 2.2).

It follows from Assumption IV, and from the inequality

$$\|L_{\gamma\lambda}s^1 - L_{\gamma\lambda}s^2\|_{1,\tau} \le \text{const}\, A^{\alpha-\gamma}B^{\beta-\lambda}\|s^1 - s^2\|_{1,\tau}, \quad s^1, s^2 \in L^1(\Omega), \tag{3.14}$$

resulting from (3.7)–(3.12), that $L_{\gamma\lambda}$ is a continuous mapping of $L^1(\Omega)$ into itself. By an argument based on the Riesz compactness theorem (cf. [12], p. 166) one can prove

LEMMA 3.2. *If $\gamma < \alpha$ and $\lambda < \beta$, then the transformation $L_{\gamma\lambda} : L^1(\Omega) \to L^1(\Omega)$ is completely continuous.*

By Assumptions I and III, the substitution operator is continuous from $L^1(\Omega)$ into itself (cf. [16], Th. 12.10), whence and by the continuity of $L_{\gamma\lambda}$, T_s is also continuous. Moreover, by Lemma 3.2, it is completely continuous in the case $\gamma < \alpha$, $\lambda < \beta$.

Note that in that case, Lemma 3.2 implies the complete continuity of the transformation defined by the right-hand side of (3.5), where c_{ij}, φ_i and ψ_i ($i = \overline{1, q_1}$; $j = \overline{1, q_2}$) are given by (3.6)–(3.8), respectively.

Consider the set $B_\varrho := B_0(\varrho + \varrho_K)$, where $\varrho_K := \|K\|_{1,\tau}$ and ϱ is a positive number, and its relatively compact subset $Z_\varrho := Z_0(\varrho + \varrho_K)$ (cf. (2.10), (2.12)).

Let $s \in L^1(\Omega)$. We have the estimate (cf. (3.4))

$$|T_s s(x,y)| \le K(x,y) + \sum_{\gamma,\lambda}\sum_{l=1}^{r_{\gamma\lambda}} K_{\gamma\lambda l}(x,y)|L_{\gamma\lambda}s(x,y)|^{\kappa_{\gamma\lambda l}}\,. \tag{3.15}$$

By the Hölder inequality and (3.14), (3.15), we get

$$\begin{aligned}\|T_s s\|_{1,\tau} \le \varrho_K + \sum_{\gamma,\lambda}\sum_{l=1}^{r_{\gamma\lambda}} \|K_{\gamma\lambda l}^{1/(1-\kappa_{\gamma\lambda l})}\|_{1,\tau}^{1-\kappa_{\gamma\lambda l}} \\ \times (A^{\alpha-\gamma}B^{\beta-\lambda}/(\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\lambda)))(1+\|s\|_{1,\tau}^{\kappa_{\lambda\gamma l}})\,,\end{aligned}$$

whence

$$\|Ts\|_{1,\tau} \le \varrho_K + c_T\mathcal{A}(1+\|s\|_{1,\tau}^{\kappa})\,, \tag{3.16}$$

where c_T is a positive constant independent of s, $\mathcal{A} := \max(A,B)$, and $\kappa := \max_{\gamma,\lambda,l}\kappa_{\gamma\lambda l}$.

Obviously, the condition

$$c_T\mathcal{A}(1+(\varrho_K+\varrho)^{\kappa}) \le \varrho\,. \tag{3.17}$$

is sufficient for the inclusion $T_s(B_\varrho) \subset B_\varrho$.

Let us distinguish two cases: (a) $\kappa < 1$, (b) $\kappa = 1$.

In case (a), inequality (3.17) is satisfied provided that

$$\varrho \ge \max(2c_T\mathcal{A} + \varrho_K, (2c_T\mathcal{A})^{1/(1-\kappa)} - \varrho_K)\,.$$

In case (b), (3.17) holds if

$$\mathcal{A} < 1/c_T \quad \text{and} \quad \varrho \ge c_T\mathcal{A}(1+\varrho_K)/(1-c_T\mathcal{A})\,.$$

By the continuity of T_s, we infer that $T_s(\overline{Z}_\varrho)$ is a compact subset of $L^1(\Omega)$ and hence it is the closure of a set which is uniformly continuous in average.

Thus, the inclusion $T_s(\overline{Z}_\varrho) \subset \overline{Z}_\varrho$ holds. By the Schauder fixed point theorem, equation (3.13) has an integrable solution. Moreover, if s is a solution to (3.13) then, by (3.16), $\|s\|_{1,\tau} \le \varrho_0$, where the real $\varrho = \varrho_0$ is a solution of (3.17).

The foregoing considerations establish

THEOREM 3.1. *If Assumptions* I–IV *are satisfied*, *then problem* (S) *has a global solution in the case* $\kappa < 1$ *and a local one in the case* $\kappa = 1$ (cf. the discussion subsequent to (3.17)). *These solutions are continuous when* $\alpha, \beta \in \mathbb{N}$.

3. Extension of the local solution. In this section we extend a local solution of problem (S) to a global one in the case $\kappa = 1$.

First of all we consider the problem (S_0), that is, problem (S) in which

$$\kappa = 1, \quad \lambda < \beta, \quad y_{ij} = 0, \quad g_j \equiv 0 \quad (i = \overline{1,q_1};\ j = \overline{1,q_2})\,. \tag{3.18}$$

As a result, $\varphi_j = G_j$ and $c_{ij} = u_{ij}^0$ ($i = \overline{1,q_1}$; $j = \overline{1,q_2}$). Clearly, (S_0) is a counterpart of the Picard problem.

Equip $L^1(\Omega)$ with the norm

$$|s|_\tau := \iint_\Omega |s(x,y)| \exp(-\tau y)\, dx\, dy\,, \tag{3.19}$$

where τ is a positive number. Using the Hölder inequality, we obtain

$$\begin{aligned}\iint_\Omega & K_{\gamma\lambda l}(x,y)|L_{\gamma\lambda}s(x,y)|^{\kappa_{\gamma\lambda l}} \exp(-\tau y)\, dx\, dy \\ &\le |K_{\gamma\lambda l}^{1/(1-\kappa_{\gamma\lambda l})}|_\tau^{1-\kappa_{\gamma\lambda l}} \Big| \iint_\Omega L_{\gamma\lambda}s(x,y)\exp(-\tau y)\, dx\, dy\Big|^{\kappa_{\gamma\lambda l}}.\end{aligned} \tag{3.20}$$

By direct calculation, we can easily show that

$$\iint_\Omega \exp(-\tau y) x^{\alpha-\gamma-k} |\psi_j^{(\lambda-\beta)}(y)|\, dx\, dy \le \mathrm{const}\, \tau^{\lambda-\beta} A^{1+\alpha-\gamma-k}(1+|s|_\tau), \tag{3.21}$$

$$\iint_\Omega \exp(-\tau y) \mathcal{D}_x^{\gamma-\alpha} \mathcal{D}_y^{\lambda-\beta} |s(x,y)|\, dx\, dy \le \mathrm{const}\, \tau^{\lambda-\beta} A^{1+\alpha-\gamma-k}(1+|s|_\tau). \tag{3.22}$$

Finally, we have

$$|T_s s|_\tau \le \varrho_K + \mathrm{const} \sum_{\gamma,\lambda} \sum_{l=1}^{r_{\gamma\lambda}} (1 + (\tau^{\lambda-\beta}(1+|s|_\tau))^{\kappa_{\gamma\lambda l}})\,. \tag{3.23}$$

Hence, for τ and ϱ sufficiently large to satisfy

$$\mathrm{const} \sum_{\gamma,\lambda} \sum_{l=1}^{r_{\gamma\lambda}} (1 + (\tau^{\lambda-\beta}(1+\varrho_K+\varrho))^{\kappa_{\gamma\lambda l}}) \le \varrho\,, \tag{3.24}$$

T_s continuously maps the compact $\overline{Z}_{\varrho,\tau}$ (i.e. $\overline{Z}_\varrho$ with $\|\cdot\|_{1,\tau}$ replaced by $|\cdot|_\tau$) into itself.

LEMMA 3.3. *If Assumptions* I–IV *and* (3.18) *are satisfied*, *then problem* (S_0) *has a global solution.*

Remark 3.1. Lemma 3.3 is valid if condition (3.18) is replaced by

$$\kappa = 1, \quad \lambda < \alpha, \quad x_{ij} = 0, \quad h_i \equiv 0 \quad (i = \overline{1,q_1};\ j = \overline{1,q_2})\,.$$

Remark 3.2. Since the right-hand sides of estimates (3.21) and (3.22) do not depend on B, one can show that if $\Omega = (0,A)\times(0,\infty)$, then equation (3.13) has a solution s in the class of measurable functions such that $|s|_\tau < \infty$ (the parameters τ in (3.19) and ϱ in the definition of B_ϱ are chosen so that inequality (3.24) is satisfied).

Replace Assumption II by

II_1. The functions g_j and h_i ($i = \overline{1,q_1}$; $j = \overline{1,q_2}$) are continuous, satisfy $g_j(0) = h_i(0) = 0$ and the curves l_j^1 and l_i^2 do not intersect each other in Ω.

Moreover, we also assume that $\gamma < \alpha$, $\lambda < \beta$ and $x_{ij} = y_{ij} = 0$.

Notice (cf. Theorem 3.1, (3.5) and (3.17)) that there exists a sufficiently small $\delta > 0$ such that Problem (S) has a solution, say u_1, in $(0,\delta)^2 \subset \Omega$.

Now, we will use Lemma 3.3 to extend the local solution of Problem (S) (cf. [9] and [10]). To this end we assume that $g_{q_2}(\delta) < \delta$ (in the opposite case $h_{q_1}(\delta) < \delta$) and define $a := \max\{x \in [\delta, A] : g_{q_2}(x) \le \delta\}$.

We seek a function $u : (0,a) \times (0,\delta) \to \mathbb{R}$ such that $u = u_1$ in $(0,\delta)^2$, which is a solution of equation (3.1) in $(\delta,a) \times (0,\delta)$ and satisfies

$$\mathcal{D}_x^{\alpha}\mathcal{D}_y^{\beta-j}u(x,g_j(x)) = G_j(x), \quad \mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta}u(\delta,y) = \mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta}u_1(\delta,y),$$
$$\mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta-j}u(\delta,0) = \mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta-j}u_1(\delta,0)$$

$(x \in (\delta,a);\ y \in (0,\delta);\ i = \overline{1,q_1};\ j = \overline{1,q_2})$.

It can be shown by an argument analogous to that in the proof of Lemma 3.3 that there is a solution, say u_2, of the above problem.

Set $b := \max\{y \in [\delta, B] : h_{q_1}(y) \le a\}$. Similarly to the above, we search for $u : (0,a) \times (0,b) \to \mathbb{R}$ such that $u = u_2$ in $(0,a) \times (0,\delta)$, u is a solution of (3.1) in $(0,a) \times (\delta,b)$ and

$$\mathcal{D}_x^{\alpha}\mathcal{D}_y^{\beta-j}u(x,\delta) = \mathcal{D}_x^{\alpha}\mathcal{D}_y^{\beta-j}u_2(x,\delta), \quad \mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta}u(h_i(y),y) = H_i(y),$$
$$\mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta-j}u(0,\delta) = \mathcal{D}_x^{\alpha-i}\mathcal{D}_y^{\beta-j}u_2(0,\delta)$$

$(x \in (0,a);\ y \in (\delta,b);\ i = \overline{1,q_1};\ j = \overline{1,q_2})$.

Denote by u_3 a solution of the above problem. Clearly, u_3 is a solution of (3.1) in $(0,a) \times (0,b)$ and, moreover, satisfies (3.2) for $x \in (0,a)$ and $y \in (0,b)$, and (3.3) with $x_{ij} = y_{ij} = 0$.

Continuing this process, we extend a local solution of problem (S) to get a global one.

Theorem 3.2. *If Assumptions* I, II_1, III *and* IV (*with* $\kappa = 1$, $\gamma < \alpha$ *and* $\lambda < \beta$) *are satisfied*, *then problem* (S) *with* $x_{ij} = y_{ij} = 0$ *has a global solution.*

IV. Some problems for ordinary differential equations

1. Multipoint problem

1.1. *The problem.* Set $n = 1$, $\Omega := (0; B)$ and let $r \in \mathbb{N}$, $\alpha > 0$, $\beta^l < \alpha$ $(l = \overline{1,r})$ be fixed numbers. As usual, we set $p := -[-\alpha]$ and denote by $\{y^{(\beta^l)}\}$ the r-element sequence of $y^{(\beta^l)}$.

Consider the following nonlinear multipoint problem (M): Find a solution of the differential equation of noninteger order

$$y^{(\alpha)}(x) = F(x, \{y^{(\beta^l)}(x)\}), \quad x \in \Omega, \tag{4.1}$$

where F is given, with the multipoint conditions

$$\sum_{i=1}^{p}\Big(\sum_{k=1}^{m_{ij}} z_{ijk}y^{(\alpha-i)}(x_{ijk}) + \int_0^B b_{ij}(x)y^{(\alpha-i)}(x)\,dx\Big) = \eta_j \tag{4.2}$$

($j = \overline{1,p}$), where $0 \le x_{ijk} \le B$, $m_{ij} \in \mathbb{N}$, $z_{ijk}, \eta_j \in \mathbb{R}$ are fixed numbers and $b_{ij} : \Omega \to \mathbb{R}$ given functions. This solution is understood analogously to the solution of (2.1).

A multipoint problem of similar type for a linear ordinary differential equation was posed by J. D. Tamarkin (cf. [43], p. 113 and references). If $b_{ij} \equiv 0$, $x_{ijk} = 0$ ($i, j = \overline{1,p}$; $k = \overline{1, m_{ij}}$), m_{ij} and z_{ijk} are such that $z_{ij1} + z_{ij2} + \ldots + z_{ijm_{ij}} = \delta_{ij}$ ($i, j = \overline{1,p}$), where δ_{ij} is the Kronecker symbol, then (M) becomes the Cauchy problem.

The multipoint problem (M) also generalizes the Bellman problem (cf. [5]). Namely, on setting $z_{ijk} = 0$ ($i, j = \overline{1,p}$; $k = \overline{1, m_{ij}}$) and $b_{ij} \equiv 0$ ($i = \overline{1, p-1}$; $j = \overline{1,p}$), conditions (4.2) are reduced to those of Bellman type

$$\int_0^B b_{pj}(x) y^{(\alpha-p)}(x)\, dx = \eta_j \qquad (j = \overline{1,p}),$$

where $\eta_j \in \mathbb{R}$ and b_{pj} are given functions.

In the sequel we examine problem (M) by using the Schauder fixed point theorem.

We assume the following:

I. The function $F : \Omega \times \mathbb{R}^r \to \mathbb{R}$ satisfies the Carathéodory conditions and

$$|F(x, \{z_l\})| \le \sum_{l=1}^{r} \sum_{i=1}^{r_l} K_{il}(x) |z_l|^{\kappa_{il}} \quad \text{a.e. in } \Omega,\ z_l \in \mathbb{R}, \tag{4.3}$$

where $r_l \in \mathbb{N}$, $0 \le \kappa_{il} \le \kappa < 1$ are fixed numbers and $K_{il} : \Omega \to \mathbb{R}$ given functions of class $L^{1/(1-\kappa_{il})}$.

II. The functions b_{ij} ($i, j = \overline{1,p}$) are integrable and the matrix

$$W = \Big[\sum_{i=m}^{p} \Big(\sum_{k=1}^{m_{ij}} z_{ijk} x_{ijk}^{i-m} + \int_0^B x^{i-m} b_{ij}(x)\, dx \Big) \Big/ (i-m)! \Big]_{1 \le j,m \le p} \tag{4.4}$$

is nonsingular (this assumption is satisfied for example in the case of the Cauchy and Bellman problems).

III. The functions $x \to w^{(\beta^l)}(x)$ ($l = \overline{1,r}$) are integrable.

1.2. *Solution of the problem.* Any solution y of (4.1) (cf. Lemma 1.7) is of the form

$$y(x) = w(x) + s^{(-\alpha)}(x), \tag{4.5}$$

where $s := y^{(\alpha)}$ satisfies the equation

$$s(x) = T_M s(x) := F(x, \{w^{(\beta^l)}(x) + s^{(\beta^l - \alpha)}(x)\}), \qquad x \in \Omega, \tag{4.6}$$

with

$$w(x) := \sum_{m=1}^{p} c_m x^{\alpha-m} / \Gamma(1 + \alpha - m) \tag{4.7}$$

and $c_1, \ldots, c_p$ being arbitrary constants.

Imposing on y (cf. (4.5)) the multipoint conditions (4.2), we have

$$(4.8)\quad \sum_{m=1}^{p} c_m \sum_{i=m}^{p} \Big(\sum_{k=1}^{m_{ij}} z_{ijk} x_{ijk}^{i-m} + \int_0^B x^{i-m} b_{ij}(x)\,dx\Big)/(i-m)! = \eta_j - \sum_{i=1}^{p} \Big(\sum_{k=1}^{m_{ij}} z_{ijk} s^{(-i)}(x_{ijk}) + \int_0^B b_{ij}(x) s(x)\,dx\Big)$$

for $j = \overline{1,p}$.

By Assumption II and the Cramer formulae, we easily get

$$(4.9)\quad c_m = \sum_{j=1}^{p} (\det W^{mj} / \det W) \times \Big(\eta_j - \sum_{i=1}^{p} \Big(\sum_{k=1}^{m_{ij}} z_{ijk} s^{(-i)}(x_{ijk}) + \int_0^B b_{ij}(x) s^{(-i)}(x)\,dx\Big)\Big)$$

($m = \overline{1,p}$), where W^{mj} is the matrix obtained from W (cf. (4.4)) by deleting its mth row and jth column and then multiplying the result by $(-1)^{m+j}$.

We examine the transform T_M (cf. (4.6)), where w is given by (4.7) with the constants $c_1, \ldots, c_p$ defined by (4.9). This transform is the superposition of the operators $N_l s := w^{(\beta^l)} + s^{(\beta^l - \alpha)}$ ($l = \overline{1,r}$) and the substitution operator $\mathcal{F}$.

Define $B_\varrho := \{s \in L^1(\Omega) : \|s\| \le \varrho\}$, where $\|\cdot\|$ denotes the (standard) norm in $L^1(\Omega)$.

Proposition 4.1. *Under Assumption* III, *the operators N_l are affine and completely continuous from $L^1(\Omega)$ into $L^1(\Omega)$.*

Proof. By Assumption II and Lemma 1.1, N_l ($l = \overline{1,r}$) maps $L^1(\Omega)$ into itself. Moreover, the second term of N_l is linear and completely continuous.

Let $s^1, s^2 \in L^1(\Omega)$. By direct calculation we get

$$|c_m^1 - c_m^2| \le \text{const}\, \|s^1 - s^2\| \sum_{i=1}^{p} \Big(\sum_{k=1}^{m_{ij}} |z_{ijk}| B^{i-1} + \int_0^B x^{i-1} b_{ij}(x)\,dx\Big)$$

($m = \overline{1,p}$), where c_m^1, c_m^2 are given by (4.9) with s replaced by s^1 and s^2, respectively, and const (here and in the sequel) is a constant independent of s.

As a consequence, assuming that w^1 and w^2 are given by (4.7) with s replaced by s^1 and s^2, respectively, we have $\|w^{1(\beta^l)} - w^{2(\beta^l)}\| \le \text{const}\, \|s^1 - s^2\|$, which proves the continuity of the second term of N_l.

In order to prove the complete continuity of N_l, we use the Riesz theorem. Let $s \in B_\varrho$. By an argument analogous to that used above one can show that

$$\int_0^B |w^{(\beta^l)}(x)|\,dx \le \sum_{m=1}^{p} |c_m| B^{1+\alpha-\beta^l-m} / \Gamma(1 + \alpha - \beta^l - m) \le \text{const}(1 + \varrho)\,.$$

Moreover, if $h \in \mathbb{R}$ then

$$\int_0^B |w^{(\beta^l)}(x+h) - w^{(\beta^l)}(x)|\,dx \le \operatorname{const}(1+\varrho)\sum_{m=1}^{p}(h^{1+\alpha-\beta^l-m} + h^{\min(1,1+\alpha-\beta^l-m)}) \to 0$$

for $h \to 0$, where const is independent of s and hence the convergence is uniform in $s \in B_\varrho$.

By the Riesz theorem, N_l maps B_ϱ into a relatively compact subset of $L^1(\Omega)$. Thus the proof is complete.

By Assumption I, we have

$$(4.10)\qquad |T_M s(x)| \le \sum_{l=1}^{r}\sum_{i=1}^{r_l} K_{il}(x) \times\Big(\sum_{m=1}^{p}|c_m x^{\alpha-\beta^l-m}/\Gamma(1+\alpha-\beta^l-m)|^{\kappa_{il}} + |s^{(\beta^l-\alpha)}(x)|^{\kappa_{il}}\Big)\quad \text{a.e. in } \Omega\,.$$

Notice that (4.9) yields $|c_m| \le \operatorname{const}(1+\|s\|)$ $(m=\overline{1,p})$.

By the Hölder inequality and the Fubini theorem, we have

$$\|K_{il}|s^{(\beta^l-\alpha)}|^{\kappa_{il}}\| \le \operatorname{const}\|s\|^{\kappa_{il}} \qquad (i=\overline{1,r_l};\ l=\overline{1,r})\,.$$

Finally, we obtain $\|T_M s\| \le c_*(1+\|s\|^\kappa)$, where c_* is a positive constant independent of s.

Thus, if $c_*(1+\varrho^\kappa) \le \varrho$ (it is sufficient to assume that $\varrho \ge \max(2c_*, (2c_*)^{1/(1-\kappa)})$), then T_M maps B_ϱ into itself. Moreover, it is completely continuous (cf. Proposition 4.1).

Using the Schauder fixed point theorem we conclude that the set of solutions of (4.6) is nonempty and relatively compact in $L^1(\Omega)$.

THEOREM 4.1. *If Assumptions* I–III *are satisfied*, *then the multipoint problem* (M) *has a solution given by* (4.5), *where s is a solution of* (4.6).

Remark 4.1. The operator T_M given by (4.5) is completely continuous (cf. Proposition 4.1 with $\beta_l = 0$) and hence the set of solutions of the considered multipoint problem is relatively compact.

2. Polarographic equation

2.1. *The Cauchy problem.* The equation

$$(4.11)\qquad y^{(1/2)}(x) - \nu x^\beta y(x) = x^{-1/2}, \qquad x > 0$$

$(-1/2 < \beta \le 0;\ \nu \in \mathbb{R}_+)$ plays an important role in polarography (for chemical background cf. [17] and [31]–[33]). Notice that the above equation was examined by K. Wiener in [48], [49] under the assumption that the derivative of noninteger order appearing in (4.11) is meant in the sense of Hadamard.

Let $\alpha > 0$. We consider the equation

$$y^{(\alpha)}(x) - \nu x^{\beta} y(x) = h(x), \qquad x > 0\,, \tag{4.12}$$

where h is a given function and $\nu \in \mathbb{R}_+$, $\beta \in \mathbb{R}$ are fixed. We examine the Cauchy problem which consists in finding a function $y \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ with $y^{(\alpha-1)}$ absolutely continuous, satisfying (4.12) a.e. in $\mathbb{R}_+$ and fulfilling the initial conditions

$$y^{(\alpha-k)}(0) = c_k, \qquad (k = \overline{1,p})\,, \tag{4.13}$$

where c_k are given constants.

It follows from Theorems 2.2 and 2.4 that, for $\beta \geq 0$, $x \in (0, A)$ with arbitrary positive A, and $h \in L^1(0, A)$, the Cauchy problem is well posed in a certain class of functions (cf. Remark 2.8 and Section 1.1).

In the sequel we examine (4.12), with negative β, in the Banach space $L^1_\tau(\mathbb{R}_+)$ (cf. Remark 2.6) with some τ.

Assume the following:

IV. The function h is in $L^1_\tau(\mathbb{R}_+)$ and the number $\beta \in (-\alpha, 0]$ is such that $x^\beta w(x)$ belongs to $L^1_\tau(\mathbb{R}_+)$ with τ sufficiently large to satisfy

$$q := \nu \Gamma(\alpha+\beta)\tau^{-(\alpha+\beta)}/\Gamma(\alpha) < 1\,. \tag{4.16}$$

Write, as in Section 1.2,

$$s(x) := y^{(\alpha)}(x) \quad \text{and} \quad w(x) := \sum_{k=1}^{p} c_k x^{\alpha-k}/\Gamma(1+\alpha-k)\,.$$

The function s is a solution of the equation

$$s(x) = \nu x^\beta w(x) + h(x) + \nu x^\beta s^{(-\alpha)}(x)\,, \tag{4.14}$$

whence $y = w + s^{(-\alpha)}$ is a solution of the Cauchy problem for (4.12) with conditions (4.13).

For $s \in L^1_\tau(\mathbb{R}_+)$, define a transformation T_P by the formula

$$T_P s(x) := \nu x^\beta w(x) + h(x) + \nu x^\beta s^{(-\alpha)}(x)\,. \tag{4.15}$$

Observe that

$$\begin{aligned}\|T_P s^1 - T_P s^2\|_{1,\tau} &\leq \nu \int_0^\infty x^\beta \exp(-\tau x)\Big(\int_0^x (x-t)^{\alpha-1}|s^1(t) - s^2(t)|\,dt\Big)\,dx/\Gamma(\alpha)\\ &\leq \nu \int_0^\infty |s^1(t) - s^2(t)| \exp(-\tau t)\\ &\quad\times \Big(\int_t^\infty (x-t)^{\alpha-1} x^\beta \exp(-\tau(x-t))\,dx\Big)\,dt/\Gamma(\alpha)\,.\end{aligned}$$

We examine the integral with respect to x. Changing variables and bearing in mind the definition of Euler's gamma function we have

$$\int_t^\infty (x-t)^{\alpha-1}x^\beta \exp(-\tau(x-t))\,dx$$
$$\le t^{\alpha+\beta}\int_0^\infty \xi^{\alpha-1}(1+\xi)^\beta \exp(-\tau t\xi)\,d\xi \le \Gamma(\alpha+\beta)\tau^{-(\alpha+\beta)}\,,$$

which implies

$$\|T_P s^1 - T_P s^2\|_{1,\tau} \le \nu\Gamma(\alpha+\beta)\tau^{-(\alpha+\beta)}\|s^1-s^2\|_{1,\tau}/\Gamma(\alpha) \le q\|s^1-s^2\|_{1,\tau}\,.$$

Thus, for τ satisfying (4.16), T_P is contractive with contraction constant q. Moreover, T_P maps $L^1_\tau(\mathbb{R}_+)$ into itself. Using the Banach fixed point theorem we conclude that there exists in $L^1_\tau(\mathbb{R}_+)$ exactly one solution of (4.14). Thus, $y = w + s^{(-\alpha)}$ is the only solution of the Cauchy problem for (4.12), (4.13) such that $y^{(\alpha)} \in L^1_\tau(\mathbb{R}_+)$.

By the above argument, the function

$$y(x) = \sum_{n=1}^{\infty}(\nu/\Gamma(\alpha))^{n-1}\int_0^x K_n(x,t)t^{-\beta}h(t)\,dt/\Gamma(\alpha)$$
$$+\sum_{k=1}^{p} c_k x^{\alpha-k}\Big(1+\sum_{l=1}^{\infty}(\nu x^{\alpha+\beta}/\Gamma(\alpha))^l$$
$$\times\prod_{j=1}^{l}B(\alpha, j(\alpha+\beta)+1-k)\Big)\Big/\Gamma(1+\alpha-k)\,, \tag{4.17}$$

where

$$K_n(x,t) = \begin{cases} (x-t)^{\alpha-1}t^\beta & \text{for } n=1,\\ \int_t^x K_n(x,\xi)K_1(\xi,t)\,d\xi & \text{for } n\ge 2,\end{cases}$$

and B is Euler's beta function, is a solution of the Cauchy problem for equation (4.12).

THEOREM 4.2. *If Assumption* IV *is satisfied, then there exists exactly one solution y of the Cauchy problem for equation* (4.12) *such that* $y^{(\alpha)} \in L^1_\tau(\mathbb{R}_+)$. *This solution is given by* (4.17).

Remark 4.2. The improper integral $\int_0^\infty t^{\kappa-1}\exp(-\tau t)\,dt$ ($\tau > 0$) is convergent if and only if $\kappa > 0$. Hence, the condition

$$c_k = 0 \text{ for } k \in \{1,\ldots,p\} \text{ such that } 1+\alpha+\beta-k \le 0$$

is necessary and sufficient for the function $x^\beta w(x)$ to be in $L^1_\tau(\mathbb{R}_+)$.

Remark 4.3. We point out two special cases. For $\beta = 0$ the solution of equation (4.12), say y^0, has the form

$$y^0(x) = \int\limits_0^x t^{\alpha-1} E_{\alpha,\alpha}(\nu t^\alpha) h(x-t)\,dt + \sum_{k=1}^{p} c_k x^{\alpha-k} E_{\alpha,1+\alpha-k}(\nu x^\alpha)\,, \tag{4.18}$$

where $E_{\varrho,\mu}$ is given by (1.11).

In the case $h(x) = \gamma x^{\kappa-1}$ ($\kappa > 0$), the function

$$\begin{aligned} y(x) = (\gamma/\Gamma(\alpha))x^{\alpha+\kappa-1} \sum_{n=0}^{\infty} (\nu x^{\alpha+\beta}/\Gamma(\alpha))^n \prod_{j=0}^{n} B(\alpha, j(\alpha+\beta)+\kappa) \\ + \sum_{k=1}^{p} c_k x^{\alpha-k}\Big(1 + \sum_{l=1}^{\infty} (\nu x^{\alpha+\beta}/\Gamma(\alpha))^l \cdot \\ \times \prod_{j=1}^{n} B(\alpha, j(\alpha+\beta)+1-k)\Big)\Big/\Gamma(1+\alpha-k)\,, \end{aligned}$$

is a solution of the Cauchy problem for (4.12). Moreover, $y(x) = \gamma x^{\alpha+\kappa-1} \times (\Gamma(\alpha+\kappa)/\Gamma(\kappa)-\nu)^{-1}$ is a solution of (4.12) with $\beta = -\alpha$ and $\nu \neq \Gamma(\alpha+\kappa)/\Gamma(\kappa)$ (cf. also [49], pp. 165–166).

By Theorem 4.2 we can formulate the following corollaries.

COROLLARY 4.1. *Let Assumption* IV *be satisfied and* $c_k \geq 0$ ($k = \overline{1,p}$), $\nu \geq 0$, *and* $h \geq 0$ *a.e. in* $(0,\delta) \subset (0,1)$. *If* y *and* y^0 *are two solutions of the Cauchy problem for equation* (4.12) *with* $\beta < 0$ *and* $\beta = 0$, *respectively*, *then* $y > y^0 > 0$ *a.e. in* $(0,\delta)$. *Moreover*, *if* $\alpha \notin \mathbb{N}$ *and* $c_p > 0$, *then* $\lim_{x\to 0+} y(x) = \infty$.

Proof. By induction, for every positive integer n,

$$K_n(x,t)t^{-\beta} > (\Gamma(\alpha))^n (x-t)^{n\alpha-1}/\Gamma(n\alpha)\,.$$

Furthermore, if $x < 1$, then

$$\begin{aligned} x^{\alpha+\beta} B(\alpha, j(\alpha+\beta)+1-k) &= x^{\alpha+\beta} \int\limits_0^1 t^{\alpha-1}(1-t)^{j(\alpha+\beta)-k}\,dt \\ &> x^{\alpha} \int\limits_0^1 t^{\alpha-1}(1-t)^{j\alpha-k}\,dt \\ &= x^\alpha B(\alpha, j\alpha+1-k)\,. \end{aligned}$$

Hence, by (4.17) and (4.18), we get our assertion.

COROLLARY 4.2. *If Assumption* IV *is satisfied and* $c_k \geq 0$ ($k = \overline{1,p}$), $\sum_{k=1}^{p} |c_k| > 0$, $\nu \geq 0$, $h \geq 0$ *a.e. in* $\mathbb{R}_+$, *and if* y *is a solution of* (4.12), *then* $y > 0$ *a.e. in* $\mathbb{R}_+$ *and* $\lim_{x\to\infty} y(x) = \infty$.

2.2. *Continuous dependence of the solution on the initial data.* If $s = y^{(\alpha)} \in L^1_\tau(\mathbb{R}_+)$, then $y = w + s^{(-\alpha)}$ is measurable and

$$\|y\|_{1,\tau} \le \tau^{-\alpha}\Big(\sum_{k=1}^{p} c_k \tau^{k-1} + \|s\|_{1,\tau}\Big),$$

whence $y \in L^1_\tau(\mathbb{R}_+)$.

THEOREM 4.3. *Let y^1 and y^2 be solutions of the Cauchy problem for equation* (4.12) *with the initial constants c_k^1 and c_k^2 ($k = \overline{1,p}$), and right-hand sides h^1 and h^2, and let Assumption* IV *be satisfied with h replaced by h^1 or h^2, respectively. For every $\varepsilon > 0$ there is a $\delta > 0$ such that $\max_{1\le k\le p} |c_k^1 - c_k^2| < \delta$ and $\|h^1 - h^2\|_{1,\tau} < \delta$ imply $\|y^1 - y^2\|_{1,\tau} < \varepsilon$.*

Proof. Let $\widehat{\delta} = \max_{1\le k\le p} |c_k^1 - c_k^2|$ and $h(t) = |h^1(t) - h^2(t)|$. For $n \ge 2$,

$$K_n(x,t)t^{-\beta} \le (x-t)^{n\alpha+\beta-1} \prod_{j=1}^{n-1} B(\alpha, j\alpha+\beta).$$

Hence, by (4.17) and (1.6), we have

$$\begin{aligned}\|y^1 - y^2\|_{1,\tau} \le \|h\|_{1,\tau}\Big(\tau^{-\alpha} + (\tau^{-\beta}/\nu)\sum_{l=2}^{\infty}(\nu\tau^{-\alpha}/\Gamma(\alpha))^l \Gamma((l+1)\alpha+\beta)\\ \times \prod_{j=1}^{l-1} B(\alpha, j\alpha+\beta)\Big) + \widehat{\delta}\sum_{k=1}^{p}\tau^{-(1+\alpha-k)}\Big(1 + \sum_{l=1}^{\infty}(\nu\tau^{-(\alpha+\beta)}/\Gamma(\alpha))^l\\ \times \Gamma(1+\alpha-k+l(\alpha+\beta))/\Gamma(1+\alpha-k)\prod_{j=1}^{l} B(\alpha, j(\alpha+\beta)+1-k)\Big).\end{aligned}$$

For every finite value of τ the series appearing in the above inequality are convergent. Thus, $\|y^1 - y^2\|_{1,\tau} \le \text{const}\,\widehat{\delta} = \varepsilon$ for $\max(\widehat{\delta}, \|h\|_{1,\tau}) < \delta$.

2.3. *The multipoint problem.* Let r be the greatest integer such that $1+\alpha+\beta-r > 0$. Consider equation (4.12) with the multipoint conditions

$$y^{(\alpha-m)}(x_m) = \eta_m \quad (m = \overline{1,r}), \tag{4.19}$$

where $x_m \in \mathbb{R}_+$ and $\eta_m \in \mathbb{R}$ are given.

Imposing (4.19) on the general solution (4.17) of (4.12), we get the following algebraic system with unknowns $c_1, \ldots, c_r$:

$$\begin{aligned}\sum_{k=1}^{r} c_k x_m^{m-k}\Big(1/\Gamma(1+m-k)\\ + \sum_{l=1}^{\infty}(\nu x_m^{\alpha+\beta}/\Gamma(\alpha))^l \prod_{j=1}^{l} B(\alpha, j(\alpha+\beta)+1-k)\end{aligned} \tag{4.20}$$

$$\times\,\Gamma(1+\alpha-k+l(\alpha+\beta)/(\Gamma(1+m-k+l(\alpha+\beta))\Gamma(1+\alpha-k))\Big)$$

$$= \eta_m - y_N^{(\alpha-m)}(x_m)$$

($m=\overline{1,r}$; by Remark 4.2, $c_k=0$ for $k=\overline{r+1,p}$), where y_N is a solution of the nonhomogeneous equation (4.12) with homogeneous initial conditions.

THEOREM 4.4. *If Assumption* IV *is satisfied and system* (4.20) *has a solution*, *then the multipoint problem for equation* (4.12) *has a solution in the class of locally integrable functions such that their derivatives of order* α *belong to* $L^1_\tau(\mathbb{R}_+)$. *Moreover*, *if the determinant of* (4.20) *is not* 0, *then the solution is unique.*

V. Further applications of the derivatives of noninteger order

In this chapter we give some further applications of the derivatives of noninteger order. In the first section we use the properties of those derivatives to construct some new examples of Mikusiński operators which are functions. In the second section we prove some integral formulae for analytic functions in a subset of $\mathbb{C}^n$. These formulae, generalizing the classical integral formulae of Cauchy and Schwarz, extend to the case of arbitrary n some result of Dzhrbashyan obtained for $n=1$.

1. An application to Mikusiński's operator theory. Let $M=\{f:\mathbb{R}_+\to\mathbb{C}: f'\in L^1_{\mathrm{loc}}(\mathbb{R}_+)\}$ and define in M the operation

$$f(t)\otimes g(t)=\Big(\int_0^t f(t-\tau)g(\tau)\,d\tau\Big)'.$$

Let $(M,+,\otimes)$, where $+$ is addition of functions, be the Mikusiński ring and $\mathcal{M}(M)$ the field of operators generated by M. Finally, define

$$P(\lambda):=\sum_{i=1}^m a_i\lambda^{\alpha_i}\quad\text{and}\quad Q(\lambda):=\sum_{j=1}^r b_j\lambda^{\beta_j}$$

with a_i, $b_j\in\mathbb{C}$, $\alpha_i,\beta_j\in\mathbb{R}_+$ ($i=1,\ldots,m$; $j=1,\ldots,r$).

Let $s:=\frac{1}{t}\in\mathcal{M}(M)$ be the differential operator. Then $R(s):=\frac{Q(s)}{P(s)}$ is called the *rational operator*.

Let α be a positive number, p a positive integer, $p-1<\alpha\le p$, and $f\in M$ such that $f^{(p)}$ exists. Then

$$\begin{aligned} s^\alpha\otimes f(t)&=\frac{\Gamma(1+\alpha)}{t^\alpha}\otimes f(t)=\Gamma(1+\alpha)\frac{t^{p-\alpha}\otimes f(t)}{t^{p-\alpha}\otimes t^\alpha}\\ &=\frac{p!}{\Gamma(1+p-\alpha)}\frac{t^{p-\alpha}\otimes f(t)}{t^p}=\frac{1}{\Gamma(1+p-\alpha)}s^p\otimes t^{p-\alpha}\otimes f(t)\,. \end{aligned}$$

Furthermore, assuming that the derivative of order $p-1$ of $t^{p-\alpha} \otimes f(t)$ is absolutely continuous, and bearing in mind the formula (cf. [11], p. 103)

$$s^p \otimes g(t) = g^{(p)}(t) + \sum_{k=1}^{p} s^k g^{(p-k)}(0)$$

(where, by assumption, $g^{(p)}$ exists) with $g(t) = t^{p-\alpha} \otimes f(t)$, we obtain

$$s^\alpha \otimes f(t) = f^{(\alpha)}(t) + \sum_{k=1}^{p} s^k f^{(\alpha-k)}(0)\,, \tag{5.1}$$

where $f^{(\alpha-k)}(t) := (t^{p-\alpha} \otimes f(t))^{(p-k)}/\Gamma(1+p-\alpha)$ $(k = \overline{0,p})$ is the one-dimensional derivative of order $\alpha - k$ (cf. def. (1.1)). Hence, s^α is called a *differential operator of noninteger order*.

Consider $f_j(t) := t^{\alpha-j}\,{}_1E_{\alpha,1+\alpha-j}(\lambda t^\alpha)$ $(\lambda \in \mathbb{C};\ j = \overline{1,p})$, where (cf. also formula (1.11))

$${}_kE_{\varrho,\mu}(z) := \sum_{m \geq \mathbf{0}} z^m \Big/ \Gamma\Big(\sum_{i=1}^{k} m_i \varrho_i + \mu\Big)$$

$(z = (z_i) \in \mathbb{C}^k$, $m = (m_i) \in \mathbb{N}_0^k$, $\varrho = (\varrho_i) \in \mathbb{R}_+^k \setminus \{\mathbf{0}\}$, $\mu > 0)$.

By Corollary 1.2 and Lemma 1.7, f_j is the only solution of $f_j^{(\alpha)} = \lambda f_j$ such that $f_j^{(\alpha-k)}(0) = \delta_{jk}$ $(j,k = \overline{1,p})$.

Hence, $s^\alpha \otimes f_j(t) = \lambda f_j(t) + s^j$, which implies

$$\frac{s^j}{s^\alpha - \lambda} = t^{\alpha-j}\,{}_1E_{\alpha,1+\alpha-j}(\lambda t^\alpha) \qquad (j = \overline{1,p})\,. \tag{5.2}$$

Moreover,

$$\frac{1}{s^\alpha - \lambda} = t \otimes t^{\alpha-1}{}_1E_{\alpha,\alpha}(\lambda t^\alpha) = \int_0^t \tau^{\alpha-1}{}_1E_{\alpha,\alpha}(\lambda \tau^\alpha)\,d\tau = t^\alpha{}_1E_{\alpha,1+\alpha}(\lambda t^\alpha),$$

which proves that (5.2) also holds for $j = 0$. Notice that $f_j \in M$ $(j = \overline{1,p-1})$ and $f_p \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap C^\infty(0,\infty)$.

We extend (5.2) to some noninteger values of j. By (5.2),

$$\frac{s^{j+\kappa}}{s^\alpha - \lambda} = s^\kappa \otimes f_j(t) \qquad (0 \leq \kappa < 1;\ j = \overline{0,p})\,.$$

Moreover, $f_j^{(\kappa)}(t) = t^{\alpha-j-\kappa}{}_1E_{\alpha,1+\alpha-j-\kappa}(\lambda t^\alpha)$ and $f_j^{(\kappa-1)}(0) = 0$ if $1+\alpha-j-\kappa > 0$ $(j = \overline{0,p})$. Hence, and by (5.1), we obtain (5.2) with j replaced by β such that $0 \leq \beta < 1 + \alpha$.

Basing on (5.2) one can construct examples of rational operators which are functions. By induction we can prove the equality

$$\frac{s^{\beta_1}}{s^{\alpha_1} - \lambda_1} \otimes \ldots \otimes \frac{s^{\beta_k}}{s^{\alpha_k} - \lambda_k} = t^\mu{}_kE_{(\alpha_i),1+\mu}((\lambda_i t^{\alpha_i}))\,.$$

with $\mu = |\alpha - \beta|_+$.

R e m a r k 5.1. $R(s)$ is a function if and only if $\max\beta_i \le \max\alpha_j$, provided that $\alpha_i, \beta_j \in \mathbb{N}_0$ (cf. [11], p. 118). Generally, this conclusion is not true for all $\alpha_i, \beta_j \in \mathbb{R}_+$ (cf. (5.2) with $\alpha < j = p$).

The above results imply the following final

THEOREM 5.1. *If*

$$Q(\lambda) = \prod_{i=1}^{k} \sum_{j=1}^{r_i} b_{ij} \lambda^{\beta_{ij}} \quad \textit{and} \quad P(\lambda) = a \prod_{i=1}^{l} (\lambda^{\gamma_i} - \lambda_i)$$

($a, b_{ij}, \lambda_i \in \mathbb{C}$; $\beta_{ij}, \gamma_i \in \mathbb{R}_+$, $k, l, r_i \in \mathbb{N}$ *are fixed*), $k \le l$ *and* $\beta_{ij} < 1 + \gamma_i$ $(i = \overline{1,k}$; $j = \overline{1,r_i})$, *then the operator* $R(s) := \frac{Q(s)}{P(s)}$ *is a function of class* $C^\infty(0,\infty)$. *If, moreover*, $\prod_{i=1}^{k}(\gamma_i - \beta_{ij}) \ge 0$ $(j = \overline{1,r_i})$ *then* $R(s) \in M$.

2. Integral representation of analytic functions. Let $n \in \mathbb{N}$, $\alpha = (\alpha_j)$, $\varrho = (\varrho_j)$, $R = (R_j) \in \mathbb{R}_+^n$, and introduce the set

$$B_R := \{ z = (z_j) \in \mathbb{C}^n : |z_j| < R_j \}, \tag{5.3}$$

and the functions

$$C_\alpha(z) := \Gamma(\mathbf{1} + \alpha)(\mathbf{1} - z)^{-\mathbf{1}-\alpha}, \tag{5.4}$$

$$S_\alpha(z) := 2C_\alpha(z) - C_\alpha(\mathbf{0}) \tag{5.5}$$

$(\alpha \ge -\mathbf{1}$, $z = (z_j) \in B_{\mathbf{1}})$. Assume that $f : B_R \to \mathbb{C}$ is analytic, whence it has, for $z \in B_R$, the representation by the uniformly convergent series (cf. [41], pp. 30–32)

$$f(z) = \sum_{k \ge \mathbf{0}} a_k z^k, \tag{5.6}$$

$(k = (k_j) \in \mathbb{N}_0^n)$, where $a_k = D^k f(\mathbf{0})/k! \in \mathbb{C}$.

The following theorem generalizes a result of Dzhrbashyan (cf. [13], p. 594).

THEOREM 5.2. (i) *For every* $\alpha > -\mathbf{1}$,

$$f_\alpha(z) := r^{-\alpha}\mathcal{D}^{-\alpha} f(z) = \sum_{k \ge \mathbf{0}} a_k (\Gamma(\mathbf{1} + k)/\Gamma(\mathbf{1} + k + \alpha)) z^k \tag{5.7}$$

$(z = (z_j)$, $z_j = r_j \exp(i\varphi_j)$, $\mathbf{0} \le r = (r_j) < R$, $\varphi = (\varphi_j)$; $k = (k_j) \in \mathbb{N}_0^n$ *and the differentiation* $\mathcal{D}^{-\alpha}$ *is performed with respect to* $r)$ *is analytic in* B_R.

(ii) *For every* $\alpha > -\mathbf{1}$, $\mathbf{0} < \varrho < R$ *and* $z \in B_R$,

$$f(z) = (2\pi)^{-n} \int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} C_\alpha(z_{\varrho\vartheta}) f_\alpha(\varrho_\vartheta)\, d\vartheta_n, \tag{5.8}$$

$$f(z) = i\Im f(\mathbf{0}) + (2\pi)^{-n} \int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} S_\alpha(z_{\varrho\vartheta}) \Re f_\alpha(\varrho_\vartheta)\, d\vartheta_n, \tag{5.9}$$

where $\varrho_\vartheta = (\varrho_j \exp(i\vartheta_j))$, $z_{\varrho\vartheta} = (\exp(-i\vartheta_j) z_j/\varrho_j) \in \mathbb{C}^n$ *and* $\Re$ *and* $\Im$ *denote the real and imaginary part, respectively.*

P r o o f. (i) The function f, being analytic in B_R, can be represented by the series (5.6), and if $z_j = r_j \exp(i\varphi_j)$ $(j = \overline{1,n})$ then $z^k = r^k \exp(i|(k_j\varphi_j)|_+)$ and

$$f(z) = \sum_{k\geq\mathbf{0}} a_k r^k \exp(i|(k_j\varphi_j)|_+)\,,$$

where $r = (r_j) \in \mathbb{R}^n_+$. Hence, f_α is of the form

$$f_\alpha(z) = \sum_{k\geq\mathbf{0}} a_k r^k (\Gamma(\mathbf{1}+\alpha)/\Gamma(\mathbf{1}+\alpha+k)) \exp(i|(k_j\varphi_j)|_+)\,,$$

which implies the series representation of f_α in B_R. Furthermore, for every $\alpha > -\mathbf{1}$,

$$\lim_{|k|_+\to\infty} (\Gamma(\mathbf{1}+k)/\Gamma(\mathbf{1}+\alpha+k))^{1/|k|_+} = 1$$

and hence f_α is analytic in B_R.

(ii) Clearly, C_α can be expanded in the uniformly convergent series

$$\begin{aligned} C_\alpha(z_{\varrho\vartheta}) &= \sum_{l\geq\mathbf{0}} (\Gamma(\mathbf{1}+\alpha+l)/\Gamma(\mathbf{1}+l)) z^l_{\varrho\vartheta} \\ &= \sum_{l\geq\mathbf{0}} (\Gamma(\mathbf{1}+\alpha+l)/\Gamma(\mathbf{1}+l)) \varrho^{-l} z^l \exp(-i|(l_j\vartheta_j)|_+) \end{aligned}$$

$(l = (l_j) \in \mathbb{N}^n_0)$ for every $z \in B_\varrho$ and $\vartheta \in [0, 2\pi)^n$.

Consider the integral appearing in (5.8). We have

$$\begin{aligned} &\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} C_\alpha(z_{\varrho\vartheta}) f_\alpha(\varrho_\vartheta)\, d\vartheta_n \\ &\quad = \int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} \sum_{l\geq\mathbf{0}} (\Gamma(\mathbf{1}+\alpha+l)/\Gamma(\mathbf{1}+l)) \varrho^{-l} z^l \exp(-i|(l_j\vartheta_j j)|_+) \\ &\qquad \times \sum_{k\geq\mathbf{0}} a_k \varrho^k (\Gamma(\mathbf{1}+k)/\Gamma(\mathbf{1}+\alpha+k)) \exp(i|(k_j\vartheta_j)|_+)\, d\vartheta_n\,. \end{aligned}$$

By the uniform convergence of the above series we can interchange the integrals and sums to obtain

$$\begin{aligned} &\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} C_\alpha(z_{\varrho\vartheta}) f_\alpha(\varrho_\vartheta)\, d\vartheta_n \\ &\quad = \sum_{k\geq\mathbf{0}} \sum_{l\geq\mathbf{0}} a_k [\Gamma(\mathbf{1}+k)\Gamma(\mathbf{1}+\alpha+l)/(\Gamma(\mathbf{1}+l)\Gamma(\mathbf{1}+\alpha+k))] \varrho^{k-l} z^l \\ &\qquad \times \int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} \exp(i|((l_j-k_j)\vartheta_j)|_+)\, d\vartheta_n\,. \end{aligned}$$

Using the equality

$$\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} \exp(i|((l_j - k_j)\vartheta_j)|_+)\, d\vartheta_n = (2\pi)^n \delta_{k,l}\,, \tag{5.10}$$

(the symbol $\delta_{k,l}$ is understood analogously to Kronecker's delta), which follows from the properties of exponential function, we get (5.8).

We now prove (5.9). Repeating the argument used above we have

$$\begin{aligned}\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} & C_\alpha(z_{\varrho_\vartheta})\overline{f_\alpha(\varrho_\vartheta)}\, d\vartheta_n \\ &= \sum_{k\geq \mathbf{0}}\sum_{l\geq \mathbf{0}} a_k[\Gamma(\mathbf{1}+k)\Gamma(\mathbf{1}+\alpha+l)/(\Gamma(\mathbf{1}+l)\Gamma(\mathbf{1}+\alpha+k))] \\ &\quad \times \varrho^{-k-l}z^l\delta_{-k,l} = \overline{f(\mathbf{0})}\,.\end{aligned}$$

Thus

$$f(z) + \overline{f(\mathbf{0})} = 2(2\pi)^{-n}\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} C_\alpha(z_{\varrho_\vartheta})\Re f_\alpha(\varrho_\vartheta)\, d\vartheta_n\,.$$

By the above equality and (5.5), we obtain

$$\begin{aligned} f(z) = & -\overline{f(\mathbf{0})} + (2\pi)^{-n}\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} C_\alpha(\mathbf{0})\Re f_\alpha(\varrho_\vartheta) d\vartheta_n \\ & + (2\pi)^{-n}\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} S_\alpha(z_{\varrho_\vartheta})\Re f_\alpha(\varrho_\vartheta)\, d\vartheta_n\,.\end{aligned}$$

Consider the second term on the right-hand side of the last relation. Since $\Re : \mathbb{C} \to \mathbb{R}$ is linear, we can interchange it with integration (cf. [39], p. 508). Hence using (5.8) with $z = \mathbf{0}$ we have

$$\Re f(\mathbf{0}) = (2\pi)^{-n}\int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} C_\alpha(\mathbf{0})\Re f_\alpha(\varrho_\vartheta)\, d\vartheta_n\,.$$

The above results yield (5.9), which completes the proof of Theorem 5.2.

Let $G \subset \mathbb{R}^{2n}$. A function $u : G \to \mathbb{R}$ is called *pluriharmonic* in G (cf. [19], p. 81, [41], p. 25) if it is in $C^2(G)$ and the equations

$$D_{x_k}D_{x_l}u + D_{y_k}D_{y_l}u = 0, \qquad D_{x_k}D_{y_l}u - D_{x_l}D_{y_k}u = 0$$

($k, l = \overline{1,n}$, $(x_j), (y_j) \in \mathbb{R}^n$) are satisfied for $(x,y) \in G$.

Define $G = B_R^* := \{(x,y) \in \mathbb{R}^{2n} : z = x + iy \in B_R\}$.

Corollary 5.1. (i) *If u is pluriharmonic in B_R and if $\alpha > -\mathbf{1}$, then*

$$u_\alpha(z) := r^{-\alpha}\mathcal{D}^{-\alpha}u(z) \tag{5.11}$$

($z_j = r_j \exp(i\varphi_j)$, $\mathbf{0} \le r = (r_j) < R$ *and the differentiation* $\mathcal{D}^{-\alpha}$ *is performed with respect to* r) *considered as a function of* (x, y) ($(x, y) \in \mathbb{R}^{2n}$, $z = x + iy$) *is pluriharmonic in* B_R^*.

(ii) *For every* $\mathbf{0} < \varrho < R$ *and* $z \in B_R$,

$$u(z) = (2\pi)^{-n} \int_0^{2\pi} d\vartheta_1 \ldots \int_0^{2\pi} \Re S_\alpha(z_{\varrho_\vartheta}) u_\alpha(\varrho_\vartheta)\, d\vartheta_n\,, \tag{5.12}$$

where $\varrho_\vartheta = (\varrho_j \exp(i\vartheta_j))$, $z_{\varrho_\vartheta} = (\exp(-i\vartheta_j) z_j/\varrho_j) \in \mathbb{C}^n$.

Proof. There exists f analytic in B_R such that $u = \Re f$ (cf. [19], p. 82). By Theorem 5.2, for every $\alpha > -\mathbf{1}$, $f_\alpha(z)$ (cf. (5.7)) is analytic, and hence its real part $u_\alpha = \Re f_\alpha$ (considered as a function of (x, y)) is pluriharmonic in B_R^*. The proof of (5.12) is completed by taking the real part of both sides of (5.9).

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