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GRZEGORZ LEWICKI

Best approximation in spaces of bounded linear operators

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CONTENTS

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Chapter 0

0.1. Introduction. The first 30-years of our century is a period of time during which the foundations of the theory of best approximation in normed linear spaces were laid. The name forever connected with the origin of that theory is that of S. Banach's [Ba]. Later the ideas of Banach were developed in the works of S. Mazur, M. G. Krein, S. N. Nikol'skiĭ, N. I. Achiezer, J. Walsh, A. N. Kolmogorov, A. I. Markushevich, R. James, R. Phelps, A. L. Garkavi, I. Singer, E. W. Cheney, S. B. Stechkin, and others (see [Che1, Lin2, Su] for detailed references).

The problem of best approximation in a normed linear space X by elements of a nonempty set $V \subset X$ is formulated as follows. Given $x \in X$ find an element $v_0 \in V$ for which the lower bound

$$\operatorname{dist}(x, V) = \inf\{\|x - v\| : v \in V\} \tag{0.1.1}$$

is attained. The number $\operatorname{dist}(x, V)$ is called the distance to x from V. The element v_0 (which need not be unique) is called an *element of best approximation* (briefly *best approximation* or best approximant) from V to x. Define

$$\mathcal{P}_V(x) = \{v \in V : \|x - v\| = \operatorname{dist}(x, V)\}\,. \tag{0.1.2}$$

It is worth saying that, in general, the problem of finding an element of best approximation effectively is very complicated. For this reason in approximation theory the following five principal problems are posed:

(0.1.3) existence of best approximation ($\mathcal{P}_V(x) \neq \emptyset$);

(0.1.4) uniqueness of best approximation ($\operatorname{card} \mathcal{P}_V(x) = 1$);

(0.1.5) characterization of elements of best approximation;

(0.1.6) estimation of the constant $\operatorname{dist}(x, V)$;

(0.1.7) construction of algorithms for seeking best approximation.

The aim of this paper is to present some results concerning the problems (0.1.4)–(0.1.7) in the case of the space of all linear, continuous mappings from a normed space B into a normed space D equipped with the operator norm (we will denote it by $\mathcal{L}(B, D)$ ($\mathcal{L}(B)$ if $B = D$)). These results, except the preliminary Section 0.2, were obtained by the author in [Bar1, LG1–LG4] (see also [OdL]). Sections II.1, II.2 and Chapter 3 contain unpublished material. The problems considered here have their origin in the results of J. Blatter, E. W. Cheney, C. Franchetti, W. A. Light, W. Odyniec, P. D. Morris, K. H. Price concerning minimal linear projections [Che2–Che8, Fr, Od].

In this paper we mainly treat the case of convex subsets of $\mathcal{K}(B,D)$ (the space of all compact operators going from B into D) and, if $D\subset B$, the case $V=\mathcal{L}|_D(B,D)$, where

(0.1.8) $$\mathcal{L}|_D(B,D)=\{L\in\mathcal{L}(B,D): L|_D=0\}\,.$$

Note that for $A\in\mathcal{L}(D)$ and $L_0\in\mathcal{L}(B,D)$, $L_0|_D=A$, each linear extension of A onto the whole space B having minimal norm belongs to $\mathcal{P}_{\mathcal{L}|_D(B,D)}(L_0)$. In particular, if $A=\mathrm{Id}|_D$ and $D\subset B$, the set $\mathcal{P}_{\mathcal{L}|_D(B,D)}(\mathrm{Id})$ contains all linear, continuous projections from B onto D of minimal norm. The projections of minimal norm play an essential role in approximation theory because of the following well known inequality (see e.g. [Che8, p. 263]):

(0.1.9) $$\|b-Pb\|\le\|\mathrm{Id}-P\|\operatorname{dist}(b,D)\le(1+\|P\|)\operatorname{dist}(b,D)$$

for any $b\in B$ and $P\in\mathcal{L}(B,D)$, $P|_D=\mathrm{Id}|_D$. Hence the quality of approximation of $b\in B$ by $Pb\in D$ depends, in fact, on the norm of P. For more information about projections the reader is referred to [Bar2, Bl, Cha, Che2–Che8, Fr, Is, Ja, Ki, Ko, Lev, Lew, Od, OdL, Ro, So, Wu] and references in [OdL].

Now we describe the contents of this paper.

In Chapter I we prove various Kolmogorov type characterizations of best approximation in the operator case ($X=\mathcal{L}(B,D)$) in which only the extremal points of the unit sphere in B and D^* are involved.

In Section I.1 we will be concerned with the case $X=\mathcal{L}(B,D)$ and V being a finite-dimensional subspace of X.

In Section I.2 we restrict our attention to $X=\mathcal{K}(B,D)$ (the space of all compact operators from B into D) and V being a convex subset of X.

In Section I.3 we specialize our results to the case of compact operators from $C_K(T)$ into itself having carriers contained in a given set $F\subset T$. (Here $C_K(T)$ denotes the space of all continuous K-valued functions ($K=\mathbb{R}$ or $K=\mathbb{C}$) defined on a compact set T with the supremum norm). Next, we will be concerned with the case of operators having finite carriers.

In Section I.4 we consider compact operators defined on the sequence spaces c_0, l_1, l_∞.

The results of this chapter have its origin in the following theorem due to Cheney:

THEOREM 0.0.1 [Che2]. *Let $B=C_{\mathbb{R}}(T)$ and let $D\subset B$ be a finite-dimensional subspace. Set*

(0.1.10) $$\mathcal{P}(B,D)=\{P\in\mathcal{L}(B,D): P|_D=\mathrm{Id}|_D\}$$

(*the set of all linear projections from B onto D*) *and*

(0.1.11) $$\mathcal{I}(B,D)=\Big\{P\in\mathcal{P}(B,D): P=\sum_{i=1}^{\dim D}\widehat{t_i}\cdot y_i,\ t_i\in T,\ y_i\in D\Big\}$$

($\mathcal{I}(B,D)$ is the set of all interpolating projections from B onto D). Then $P_0 \in \mathcal{I}(B,D)$ is an element of minimal norm in $\mathcal{P}(B,D)$ if and only if for every $P \in \mathcal{P}(B,D)$ there is $\widehat{t} \in \mathrm{crit}^*(P_0)$ (the symbol $\widehat{t}$ denotes the evaluation functional $\widehat{t}(y) = y(t)$) such that

$$\inf\{((P - P_0)x)(t) : x \in A_t\} \le 0\,. \tag{0.1.12}$$

(Here $\mathrm{crit}^*(P_0) = \{t \in T : \|\widehat{t} \circ P_0\| = \|P_0\|\}$ and $A_t = \{x \in B : \|x\| = 1, (P_0x)(t) = \|\widehat{t} \circ P_0\|.\}$)

The main tool in our approach in this chapter is a theorem due to H. S. Collins and W. Ruess [Co] which characterizes the extremal points of the unit sphere in $\mathcal{K}^*(B,D)$ in terms of extremal points of the unit spheres in B^{**} and D^* (see Theorem 0.2.7 in the next section). The other important tools are a generalization of the classical Kolmogorov criterion proved by Brosowski and Wegmann in [Br] (see Theorem 0.2.8 in the next section) and a Kolmogorov type criterion for strong unicity given by Wójcik in [Wó] (see Theorem 0.2.12 in the next section). For the convenience of the reader we present the proofs of the above mentioned results in Section 0.2.

In Chapter II we deal with some applications of the results proved in Chapter I for seeking extensions of operators of minimal norm. In other words, for given $A \in \mathcal{L}(D)$, we concentrate on the problem of approximation of a given extension L_0 of A by elements of the set $\mathcal{L}|_D(B,D)$ (see (0.1.8)). We mainly consider the case $A = \mathrm{Id}|_D$, i.e. the problem of looking for a projection of minimal norm. Throughout this chapter, except Section II.6, $B = l_1^{(n)}$ or $B = l_\infty^{(n)}$ and D will be a subspace of B.

In Section II.1, for some special cases of $A \in \mathcal{L}(D)$ and D being a hyperplane of $l_\infty^{(n)}$, we present explicit formulas for extensions of A having minimal norm. In particular, we show a formula for minimal projections going from $l_\infty^{(n)}$ onto D (the method of proof presented here is different from that of [Bl] and [Od, Th. 2.2]).

In Section II.2 we calculate the norm of a minimal projection going from $l_1^{(n)}$ onto a hyperplane D. The method presented here is different from that of [Bl] and [OdL, Th. II.4.9, p. 62].

In Section II.3 the unicity and strong unicity of minimal projections in the situations considered in Sections II.1 and II.2 will be investigated. (The notion of strong unicity will be defined in Section 0.2 of this chapter.) In particular, we present a different proof of the result obtained in [Od] (see also [OdL, Th. II.5.2, p. 70]).

In Section II.4 we solve partially the problem of finding a projection of minimal norm (so called minimal projection) onto D which is a subspace of codimension two in $l_\infty^{(n)}$.

Section II.5 will be devoted to a characterization of unicity and strong unicity of a minimal projection in the case considered in Section II.4.

The main result of Section II.6 states that if B is a three-dimensional real

Banach space, D is one of its hyperplanes and $A \in \mathcal{L}(D)$ is so chosen that there is no extension of A onto B which preserves the norm of A, then A has exactly one extension L_0 of minimal norm. Moreover, this extension is strongly unique.

In Chapter III we deal with the problem of calculing the $\operatorname{dist}(L_0, \mathcal{L}_D(B, D))$ for a given $L_0 \in \mathcal{L}(B, D)$, where B is an infinite-dimensional Banach space and D one of its finite-dimensional subspaces. We present some results which permit to reduce this problem to the case of B being a finite-dimensional Banach space.

In Section III.1 we consider the case of Orlicz–Musielak spaces (in particular L^p spaces) defined on sets with nonatomic measure.

In Section III.2 we will be concerned with the spaces $C_K(T)$, $C^{(k)}[a, b]$ and sequence Orlicz–Musielak spaces.

In Section III.3, adapting the reasoning presented in [Che6], we show how to apply the classical Remez algorithm for estimation of the constant $\operatorname{dist}(L_0, \mathcal{L}_D(B, D))$ in the cases described in Sections III.1 and III.2.

The results of this chapter have its origin in the following two theorems:

Theorem 0.1.2 [Che5]. *Assume $B = C_{\mathbb{R}}(T)$ and let $D \subset B$ be an n-dimensional subspace. Put*

$$\mathcal{D}(B, D) = \Big\{ P \in \mathcal{P}(B, D) : Px = \sum_{i=1}^{k} x(t_i) y_i,\ y_1, \ldots, y_k \in D \Big\}. \tag{0.1.13}$$

Then for every $P \in \mathcal{P}_{\mathrm{Id}}(B, D)$ (see (0.1.10)) *there exists a net $\{P_\beta\} \subset \mathcal{D}(B, D)$ with*

$$\|Px - P_\beta x\| \to 0 \quad \text{for every } x \in B$$

and $\|P_\beta\| \to \|P\|$.

In particular, this means that the norm of a minimal projection can be calculated as the limit of norms of operators with finite carriers (when T is metrizable we can use sequences).

Theorem 0.1.3 [Che6]. *Let T be a metrizable compact set and let $\{t_i\}$ be a countable dense set in T. Assume $D \subset C_{\mathbb{R}}(T)$ is an n-dimensional subspace. For $m \ge n$, $f \in B$ and $P \in \mathcal{D}(B, D)$ put*

$$\Delta_m(f) = \max_{i=1,\ldots,m} \{|f(t_i)|\}, \tag{0.1.14}$$

$$N_m(P) = \sup_{\Delta_m(f) \le 1} \Delta_m(Pf), \tag{0.1.15}$$

$$N_m = \inf\{N_m(P) : P \in \mathcal{D}(B, D),\ \operatorname{car}(P) \subset \{t_1, \ldots, t_m\}\} \tag{0.1.15}$$

(*the symbol* $\operatorname{car}(P)$ *denotes the carrier of* P). *Then*

$$\lambda_{\mathrm{Id}}(B, D) = \lim_{m \to \infty} N_m .$$

(*Here* $\lambda_{\mathrm{Id}}(B, D) = \inf\{\|P\| : P \in \mathcal{P}(B, D)\}$).

Now we present some basic notations which will be of use later. Throughout the paper $\mathbb{C}$ ($\mathbb{R}$) denotes the field of complex (real) numbers. The symbol $S_B(x, r)$

$(B_B(x,r))$ stands for a sphere (a closed ball resp.) in a normed space B with center x and radius r. If $x=0$ and $r=1$ we will write S_B (B_B) for brevity. By ext_B we will denote the set of all extremal points of the unit sphere in a normed space B. We also denote classical Banach spaces in a standard way. Namely, we write $C(T)$ (resp. $C_{\mathbb{R}}(T)$) for the space of complex-valued (resp. real-valued) continuous functions defined on a compact set T with the supremum norm. By $L_p(T,\Sigma,\mu)$ we denote as usual the space of μ-measurable, complex (or real) -valued functions f such that $\int_T |f(t)|^p\,d\mu(t) < \infty$. The symbol $l_p^{(n)}$ stands for the euclidean space $\mathbb{C}^n$ $(\mathbb{R}^n)$ equipped with the l_p norm $(p=1,2,\dots,\infty)$. We will write $\mathcal{L}(B,D)$ $(\mathcal{L}(B)$ if $B=D)$ for the space of all linear, continuous mappings from a normed space B into a normed space D equipped with the operator norm. Similarly, by $\mathcal{K}(B,D)$ $(\mathcal{K}(B)$ if $B=D)$ we denote the space of all compact operators from B into D. If W is a subset of a linear topological vector space X then $\mathrm{conv}(W)$ (resp. $\mathrm{conv}(W)^-$) will be the smallest convex (resp. the smallest convex closed set) containing W. Moreover, by $\mathrm{int}(W)$ we denote the interior of the set W.

Now, suppose $D \subset B$ is a closed subspace of a Banach space B and let $A \in \mathcal{L}(D)$ be given. Then the following notation will be frequently used:

$$\mathcal{P}_A(B,D) = \{L \in \mathcal{L}(B,D) : P|_D = A\}\,; \tag{0.1.17}$$

$$\lambda_A(B,D) = \inf\{\|P\| : P \in \mathcal{P}_A(B,D)\}\,; \tag{0.1.18}$$

$(\lambda_A(B,D) = \infty$ if $\mathcal{P}_A(B,D) = \emptyset)$;

$$\mathcal{P}_A^0(B,D) = \{P \in \mathcal{P}_A(B,D) : \|P\| = \lambda_A(B,D)\}\,. \tag{0.1.19}$$

Note that for any $A \in \mathcal{L}(D)$,

$$\lambda_A(B,D) = \mathrm{dist}(L_0, \mathcal{L}_D(B,D))\,, \tag{0.1.20}$$

where $L_0 \in \mathcal{P}_A(B,D)$ is a fixed extension of A.

Other special notations will be individually explained.

All items (definitions, theorems, propositions, corollaries, lemmas, examples, remarks) are numbered consecutively within any section. And thus Remark 0.2.1 opens the second section of Chapter 0 (Section 0.2) and is followed by Theorem 0.2.2, Theorem 0.2.3 and so on. When the reference is made to an item within the same chapter, the chapter number is omitted (for example we write (see Theorem 2.2)). Separate numbering is employed for formulas. For example the reference (see (0.1.2)) means the second formula in Section 1 of Chapter 0.

0.2. Preliminary results. In this section, for the convenience of the reader, we present the proofs of some results which will be of use later. We start with the proof of a theorem due to Collins and Ruess, crucial for our later investigations. The above mentioned result permits one to express the points from the set $\mathrm{ext}_{\mathcal{K}^*(B,D)}$ using elements from $\mathrm{ext}_{B^{**}}$ and ext_{D^*}. To do this, let us introduce some notation.

Let (X, Y) denote a dual pair [Al, p. 335]. For $W \subset X$ set

$$W^0 = \{y \in Y : |(y,x)| \le 1 \text{ for every } x \in W\} \tag{0.2.1}$$

and

$$W^{00} = \{x \in X : |(y,x)| \le 1 \text{ for every } y \in W^0\}\,. \tag{0.2.2}$$

Remark 0.2.1. It is well known (see e.g. [Al, p. 338]) that for any Banach space B the pair (B^*, B) is a dual pair if in B^* we consider the weak* topology.

In the sequel we will need the following results:

THEOREM 0.2.2 (see e.g. [Al, p. 320]). *Assume B is a locally convex linear topological space and let $W \subset B$ be a compact subset. If* $\operatorname{conv}(W)^-$ *is compact then*

$$\operatorname{ext}(\operatorname{conv}(W)^-) \subset W \tag{0.2.3}$$

where $\operatorname{conv}(W)^-$ *denotes the smallest closed convex set containing W.*

THEOREM 0.2.3 (see e.g. [Al, p. 339]). *Let (X, Y) be a dual pair and let $W \subset X$ be a balanced set. Then*

$$\operatorname{conv}(W)^- = W^{00}\,. \tag{0.2.4}$$

Now, for given Banach spaces B, D, let us denote by $\mathcal{L}_e(B^*, D)$ the space of all weak*-weak continuous compact operators from B^* into D endowed with the operator norm.

PROPOSITION 0.2.4 [Co, Ex. (0.2)]. *The space $\mathcal{K}(B, D)$ is linearly isometric to the space $\mathcal{L}_e(B^{**}, D)$.*

Proof. Fix $T \in \mathcal{K}(B, D)$ and $\phi \in B^{**}$. By Goldstine's Theorem there exists a net $(x_u) \subset B_B(0, \|\phi\|)$ such that $x_u \to \phi$ weak* in B^{**}. Since $\|x_u\| \le \|\phi\|$ for each u, the set $\{Tx_u\}_{u \in U}$ is relatively compact. Set $C_u = \operatorname{cl}\{Tx_w\}_{w \ge u}$ for $u \in U$ (the closure is taken with respect to the norm topology in D). Note that $\{C_u\}_{u \in U}$ is a centered family of closed sets. By the compactness of T, $\bigcap_{u \in U} C_u \ne \emptyset$. Take $y \in \bigcap_{u \in U} C_u$. We show that Tx_u tends to y weakly in D. To do this, fix $\varepsilon > 0$ and $f \in D$. Let $V = \{z \in D : |f(y)| < \varepsilon/2\}$. Since T is continuous with respect to the norm topologies in B and D, T is weak*-weak continuous. Hence we may select an open neighbourhood W of 0 (in the weak topology in B) such that $T(W) \subset V$. Since x_u tends to ϕ weak* in B^{**}, $x_w - x_z \in W$ for $z, w \ge u_0$. Choose $w \ge u_0$ with $\|Tx_w - y\| \le \varepsilon/2$ (the existence of such a w is guaranteed by the definition of C_u). Then for each $z \ge u_0$,

$$|f(Tx_z) - y| \le |f(T(x_z - x_w))| + |f(Tx_w - y)| \le \varepsilon/2 + \|f\|\varepsilon/2\,,$$

since the first term belongs to V. Consequently, $Tx_u \to y$ weakly in D. Now we shall show that for every net $(y_w) \subset B_B(0, \|\phi\|)$ tending weak* in B^{**} to ϕ, the net (Ty_w) tends to y weakly in D. Assume this is not true and select a net $(y_w) \subset B_B(0, \|\phi\|)$, $y_w \to \phi$ weak* in B^{**} with $Ty_w \to x \ne y$ weakly in D. Take $f \in S_{D^*}$ with $f(x - y) = \|x - y\|$ and let $V = \{z \in D : |f(z)| < \|x - y\|/2\}$. Reasoning as in the previous part of the proof, we obtain $|f(Tx_u - Ty_w)| < \|x-y\|/2$ for $u \ge u_0$ and $w \ge w_0$, and consequently $|f(x-y)| \le \|x-y\|/2$; a contradiction.

Now we define the required isometry by

$$T^{**}\phi = \lim_u Tx_u \tag{0.2.5}$$

for $\phi \in B^{**}$, where $\{x_u\} \subset B_B(0, \|\phi\|)$ is an arbitrary net tending to ϕ weak* in B^{**} (we have proved that the limit does not depend on the choice of the net $\{x_u\}$). In fact, T^{**} is the second adjoint operator for T. It is clear that T^{**} is linear and $T^{**}|_B = T$.

Now we prove that $T^{**} \in \mathcal{L}_e(B^{**}, D)$. Take $\{\phi_u\} \subset B^{**}$, and let ϕ_u tend to 0 weak* in B^{**}. Fix $\varepsilon > 0$, $f \in D^*$ and let $V = \{y \in D : |f(y)| < \varepsilon\}$. Since T is weak-weak continuous, there exists an open (in the weak topology of D) neighbourhood W of 0 with $T(W) \subset V$. Note that $\phi_u \in W$ for $u \ge u_0$ (W may be treated as a weak*-open set in B^{**}). For each $u \ge u_0$ select a net $(x_u^w) \subset B$ such that $\|x_u^w\| \le \|\phi_u\|$ and (x_u^w) tends weak* in B^{**} to ϕ_u. It is clear that $x_u^w \in W$ for $w \ge w_u$. Consequently, by (0.2.5), $T^{**}\phi_u = \lim_w Tx_u^w$ belongs to V, which proves the weak*-weak continuity of T^{**}.

To show that T^{**} is a compact operator, note that there exists a compact set $K \subset D$ with $T(B_B(0,1)) \subset K$. By (0.2.5), $T^{**}(B_{B^{**}}(0,1)) \subset \mathrm{conv}(K)^-$. By Mazur's Theorem the set $\mathrm{conv}(K)^-$ is also compact. Since $B_D(0, \|T\|)$ is a convex set, by (0.2.5), $T^{**}(B_{B^{**}}(0,1)) \subset B_D(0, \|T\|)$, which means $\|T^{**}\| = \|T\|$. It is clear that for every $S \in \mathcal{L}_e(B^{**}, D)$, $S = (S|_B)^{**}$. Consequently, the operator ** defined by (0.2.5) is a linear isometry, which completes the proof.

Remark 0.2.5. If $L \in \mathcal{K}(B, D)$ is a finite-dimensional operator then

$$L^{**}f = \sum_{i=1}^n f(x_i^*)y_i \quad \text{for } f \in B^{**}, \qquad \text{where } L = \sum_{i=1}^n x_i^*(\cdot)y_i\,. \tag{0.2.6}$$

To present a result which is crucial for our later investigations let us introduce some notation. For $x \in B^*$, $y \in D^*$ and $h \in \mathcal{L}_e(B^*, D)$ put

$$(x \otimes y)(h) = y(h(x))\,. \tag{0.2.7}$$

Set

$$B_B^0 \otimes B_D^0 = \{x \otimes y : x \in B_B^0,\ y \in B_D^0\}. \tag{0.2.8}$$

Then we have

Theorem 0.2.6. *Assume B, D are Banach spaces. Then*

$$\mathrm{ext}_{\mathcal{L}_e^*(B^*,D)} \subset \mathrm{ext}_{B^*} \otimes \mathrm{ext}_{D^*}\ . \tag{0.2.9}$$

Proof. First we show that $B_B^0 \otimes B_D^0$ is compact in the weak* topology of $\mathcal{L}_e^*(B^*, D)$. To do this, define

$$\Phi : B_B^0 \times B_D^0 \to B_B^0 \otimes B_D^0 \quad \text{by} \quad \Phi(x,y) = x \otimes y\,. \tag{0.2.10}$$

In view of the Banach–Alaoglu Theorem, B_B^0 (resp. B_D^0) is a weak*-compact set in B^* (resp. in D^*). Hence it suffices to show that Φ is continuous (in $B_B^0 \times B_D^0$ we consider the Tikhonov topology). Take $\{x_u\} \subset B_B^0$, $\{y_u\} \subset B_D^0$ with $x_u \to x$ and $y_u \to y$ and fix $h \in \mathcal{L}_e(B^*, D)$. First we prove that $\|h(x_u - x)\| \to 0$. Suppose, on the contrary, that $\|h(x_w - x)\| \geq d > 0$ for some subnet $\{x_w\} \subset \{x_u\}$. Set $C_w = \operatorname{cl}\{h(x_z - x)\}_{z \geq w, z \in W}$ for $w \in W$ (the closure is taken with respect to the norm topology in D). By the compactness of h, $\bigcap_{w \in W} C_w \neq \emptyset$. Take $y \in \bigcap_{w \in W} C_w$ and select $f \in S_{D^*}$ with $f(y) = \|y\|$. For each $n \in \mathbb{N}$ and for every $w \in W$ there exists $z_w \in W$, $z_w \geq w$ with $\|h(x_{z_w} - x) - y\| \leq 1/n$. But this means that $f(h(x_{z_w} - x)) \to f(y) = \|y\| \geq d > 0$; a contradiction, since $h(x_u - x) \to 0$ weakly in D. Note that

$$\begin{aligned} |\Phi(x_u, y_u)(h) - \Phi(x, y)(h)| &= |(x_u \otimes y_u)h - (x \otimes y)h| \\ &\leq |y_u(hx_u) - y_u(hx)| + |y_u(hx) - y(hx)| \\ &\leq \sup\{\|y_u\| : u \in U\} \cdot \|h(x_u - x)\| + |(y_u - y)(hx)| \,. \end{aligned}$$

Since $\sup\{\|y_u\| : u \in U\} < \infty$, $\|h(x_u - x)\| \to 0$ and $y_u \to y$ weakly in D^*, the proof of the continuity of Φ is complete. Consequently, $B_B^0 \otimes B_D^0$ is compact in the weak* topology in $\mathcal{L}_e^*(B^*, D)$. It is easy to show that $B_B^0 = B_{B^*}$, $B_D^0 = B_{D^*}$ and $(B_B^0 \otimes B_D^0)^{00} = B_{\mathcal{L}_e^*(B^*,D)}$. From Remark 2.1 and Theorem 2.3 it follows that

$$\operatorname{conv}(B_B^0 \otimes B_D^0)^- = (B_B^0 \otimes B_D^0)^{00} \,.$$

Hence, by the above reasoning, the Banach–Alaoglu Theorem and Theorem 2.2, $\operatorname{ext}_{\mathcal{L}_e^*(B^*,D)} \subset B_{B^*} \otimes B_{D^*}$, and consequently, $\operatorname{ext}_{\mathcal{L}_e^*(B^*,D)} \subset \operatorname{ext}_{B^*} \otimes \operatorname{ext}_{D^*}$ as required.

By Proposition 2.4 we immediately get

THEOREM 0.2.7 [Co]. *Let B, D be Banach spaces. Then*

$$\operatorname{ext}_{\mathcal{K}^*(B,D)} \subset \operatorname{ext}_{B^{**}} \otimes \operatorname{ext}_{D^*} \tag{0.2.11}$$

where $(x \otimes y)L = y(L^{**}x)$ for every $x \in B^{**}$, $y \in D^*$ and $L \in \mathcal{K}(B, D)$ (L^{**} is defined by (0.2.5)).

Now we present the proofs of two generalizations of the Kolmogorov criterion. Recall that if $D \subset C_K(T)$ is a linear subspace, $f \in C_K(T)$ and $d_0 \in D$ then by the classical Kolmogorov result $d_0 \in \mathcal{P}_D(f)$ if and only if for every $d \in D$,

$$\inf\{\operatorname{re}((f(t) - d_0(t))\overline{d(t)}) : t \in C(f - d_0)\} \leq 0 \,,$$

where $C(f - d_0) = \{t \in T : |f(t) - d_0(t)| = \|f - d_0\|\}$.

For a Banach space B set

$$E(b) = \{f \in \operatorname{ext}_{B^*} : f(b) = \|b\|\} \tag{0.2.12}$$

and recall that a set $D \subset B$ is called a *sun* iff for every $b \in B$, $d_0 \in \mathcal{P}_D(b)$ (see (0.1.2)) and $t \geq 0$,

$$d_0 \in \mathcal{P}_D(d_0 + t(b - d_0)) \,. \tag{0.2.13}$$

THEOREM 0.2.8 [Br]. *Let $D \subset B$ be a sun and let $b \in B$. Then $d_0 \in \mathcal{P}_D(b)$ if and only if for every $d \in D$,*

(0.2.14) $$\text{there exists } f \in E(b - d_0) \text{ such that } \operatorname{re}(f(d - d_0)) \le 0\,.$$

Proof. Suppose $d_0 \notin \mathcal{P}_D(b)$. Then there is a $d_1 \in D$ with $\|b-d_1\| < \|b-d_0\|$. Take any $f \in E(b - d_0)$. Then

$$\operatorname{re}(f(d_1 - d_0)) = \operatorname{re}(f(b - d_0)) - \operatorname{re}(f(b - d_1)) \ge \|b - d_0\| - \|b - d_1\| > 0$$

and consequently, $\inf\{\operatorname{re}(f(d_1 - d_0)) : f \in E(b - d_0)\} > 0$.

To prove the converse, assume that $d_0 \in \mathcal{P}(b)$ and that there is a $d_1 \in D$ such that $\operatorname{re}(f(d_1 - d_0)) > 0$ for every $f \in E(b - d_0)$. We show that $d_0 \notin \mathcal{P}_D(d_0 + t_0(b - d_0))$ for some $t_0 > 0$. Set

$$E_1 = \{f \in B_{B^*} : f(b - d_0) = \|b - d_0\|\}\,.$$

It is clear that E_1 is a weak*-closed subset of B_{B^*} and hence, by the Banach–Alaoglu Theorem, E_1 is a weak*-compact set. Since the function $E_1 \ni f \to \operatorname{re}(f(d_1 - d_0))$ is weak*-continuous and linear, by the Krein–Milman Theorem, there is an $f_0 \in E(b - d_0)$ with

$$0 < \operatorname{re}(f_0(d_1 - d_0)) = c = \inf\{\operatorname{re}(f(d_1 - d_0)) : f \in E_1\}\,.$$

Now, set $U = \{f \in B_{B^*} : \operatorname{re}(f(d_1 - d_0)) > c/2\}$ and $V = B_{B^*} \setminus U$. It is clear that U is a weak*-open set in B_{B^*} and consequently, by the Banach–Alaoglu Theorem, V is a weak*-compact set. Since the function $B_{B^*} \ni f \to |\operatorname{re}(f(b-d_0))|$ is weak*-continuous and $E_1 \subset U$,

$$\sup\{|\operatorname{re}(f(b - d_0))| : f \in V\} = M < \|b - d_0\|\,.$$

Now take $t_0 > 0$ such that $t_0(\|b - d_0\| - M) > \|d_1 - d_0\|$. We show that $d_0 \notin \mathcal{P}_D(d_0 + t_0(b - d_0))$. To do this, fix $f \in B_{B^*}$ with $f(d_0 + t_0(b - d_0) - d_1) = \|d_0 + t_0(b - d_0) - d_1\|$. If $f \in V$ then

$$\begin{aligned}\|d_0 + t_0(b - d_0) - d_1\| &= |\operatorname{re}(f(d_0 + t_0(b - d_0) - d_1))| \\ &\le t_0|\operatorname{re}(f(b - d_0))| + |\operatorname{re}(f(d_1 - d_0))| \\ &\le t_0 M + \|d_1 - d_0\| < t_0 M + t_0\|b - d_0\| - t_0 M \\ &= t_0\|b - d_0\| = \|d_0 + t_0(b - d_0) - d_0\|\,.\end{aligned}$$

If $f \in U$, then

$$\begin{aligned}\|d_0 + t_0(b - d_0) - d_1\| &= \operatorname{re}(f(d_0 + t_0(b - d_0) - d_1)) \\ &= \operatorname{re}(f(t_0(b - d_0) - (d_1 - d_0))) \le \|t_0(b - d_0)\| - c/2 \\ &< t_0\|b - d_0\| = \|d_0 + t_0(b - d_0) - d_0\|\,.\end{aligned}$$

Consequently, $d_0 \notin \mathcal{P}_D(d_0 + t_0(b - d_0))$, a contradiction.

Remark 0.2.9. It is easy to show that any convex set $D \subset B$ is a sun, so the above characterization of best approximation holds true for convex subsets of B.

Remark 0.2.10. If D is a linear subspace of B then (0.2.14) is equivalent to

$$\text{(0.2.15)} \qquad \text{for every } d \in D \text{ there is } f \in E(b-d_0) \text{ such that } \operatorname{re}(f(d)) \le 0\,.$$

Now we present a similar Kolmogorov type characterization for the case of strong uniqueness (Theorem 2.12). This result was obtained by Wójcik [Wó]. In order to do this, let us recall that an element $d_0 \in D$ is called a *strongly unique best approximation* (briefly SUBA) to $b \in B$ if and only if there is $r > 0$ such that for every $d \in D$,

$$\text{(0.2.16)} \qquad \|b-d\| \ge \|b-d_0\| + r\|d-d_0\|\,.$$

The theory of strong unicity has its origin in the following result of Newman and Shapiro [Ne]. To present it, recall that an n-dimensional subspace D of $C_K(T)$ is called a *Haar subspace* if and only if for any set $\{t_1, \ldots, t_n\}$ of distinct points from T, 0 is the only element in D which vanishes on $\{t_1, \ldots, t_n\}$.

Theorem 0.2.11 [Ne]. *Assume $D \subset C_K(T)$ is a Haar subspace of $C_K(T)$. Then for every $b \in C_K(T)$ there is a constant $r > 0$ such that the best approximation $d_0 \in \mathcal{P}_D(b)$ satisfies one of the following inequalities*:

$$\text{(0.2.17)} \quad \|b-d\| \ge \|b-d_0\| + r\|d-d_0\| \quad \textit{for every } d \in D \textit{ in the case } K = \mathbb{R}$$

and

$$\text{(0.2.18)} \quad \|b-d\|^2 \ge \|b-d_0\|^2 + r\|d-d_0\|^2 \textit{ for every } d \in D \textit{ in the case } K = \mathbb{C}\,.$$

It is obvious that if u is the SUBA to b in D then card $\mathcal{P}_D(b) = 1$. The converse is not true (see e.g. [Che1]). The significance of this notion can be illustrated by E. W. Cheney's observation [Che1, p. 82] that the strong unicity of best approximation yields the continuity of the metric projection

$$\text{(0.2.19)} \qquad \mathcal{P}_D : B \ni b \to \mathcal{P}_D(b) \in D\,.$$

Also one can see that the proof of the Remez algorithm depends, in fact, on strong unicity. For more detailed information about strong unicity the reader is referred to [SW]. Now we can show the next important result for our later investigations.

Theorem 0.2.12 ([Wó] or [SW, Th. 4.2]). *Let $D \subset B$, $b \in B \setminus D$ and $d_0 \in D$. For $t > 0$ set $b_t = d_0 + t(b - d_0)$. Then the following statements are equivalent*:

(0.2.20) *for every $t \ge 0$, d_0 is a SUBA to $b_t \in D$ with a constant $r > 0$ independent of t*;

(0.2.21) *for every $d \in D$, $\operatorname{re}(f(d-d_0)) \le -r\|d-d_0\|$ for some $f \in E(b-d_0)$.*

Proof. Fix $t > 0$ and note that $E(b-d_0) = E(b_t - d_0)$. If (0.2.21) is satisfied then there are $r > 0$ and $f \in E(b-d_0)$ such that for every $d \in D$,

$$r\|d-d_0\| \le \operatorname{re}(f(b_t-d)) - \operatorname{re}(f(b_t-d_0)) \le \|b_t-d\| - \|b_t-d_0\|\,,$$

which proves (0.2.20).

To prove the converse, assume that

$$\|b_t-d\| - \|b_t-d_0\| = (\|b-d_0+(1/t)(d_0-d)\| - \|b-d_0\|)/(1/t) \ge r\|d-d_0\|\,.$$

It is well known [Du, Th. V.9.5] that

$$\lim_{\varepsilon\searrow 0}(\|x+\varepsilon y\|-\|x\|)/\varepsilon=\sup\{\mathrm{re}(f(y)) : f\in E(x)\}\,.$$

Hence, by the above inequality, we get

$$\inf\{\mathrm{re}(f(d-d_0)) : f\in E(b-d_0)\}\le -r\|d-d_0\|\,.$$

Since the function $B_{B^*}\ni f\to \mathrm{re}(f(d-d_0))$ is weak*-continuous and linear, the above infimum is attained at some point $f\in E(b-d_0)$, as required.

R e m a r k 0.2.13. In [Wó] it was shown that if $D\subset B$ is a starlike set with respect to some $d_0\in D$ then (0.2.20) is equivalent to the fact that d_0 is a SUBA to b. Hence Theorem 0.2.12 gives Kolmogorov's type characterization of strong uniqueness for convex subsets of B. Moreover, if D is a subspace of B then (0.2.21) can be replaced by:

(0.2.22) for every $d\in D$ there is an $f\in E(b-d_0)$ with $\mathrm{re}(f(d))\le -r\|d\|\,.$

By [Si, Th. 1.1, p. 170] and Theorems 2.8 and 2.12 one can easily get

COROLLARY 0.2.14. *Let $b\in B\setminus D$ and let D be a finite-dimensional subspace of D. Then $d_0\in\mathcal{P}_D(b)$ (d_0 is a SUBA to b resp.) if and only if* $0\in\mathrm{conv}(E(b-d_0)|_D)$ ($0\in\mathrm{int}(\mathrm{conv}(E(b-d_0)|_D))$ *resp.*), *where* $E(b-d_0)|_D=\{f|_D : f\in E(b-d_0)\}$.

To end this section we show three results concerning the case of X being a real, finite-dimensional space with $\mathrm{card}\,\mathrm{ext}_X<\infty$. First we prove the following

PROPOSITION 0.2.15. *Suppose X is a finite-dimensional real Banach space such that* $\mathrm{card}\,\mathrm{ext}_X$ *is finite. Let $D\subset X$ be a proper n-dimensional subspace and let $b\in X\setminus D$. Put $Z=D\oplus[b]$. Then $z\in\mathrm{ext}_Z$ if and only* $\dim\mathrm{Span}\,E(z)|_Z=n+1$.

P r o o f. Assume there is a $z\in S_Z\setminus\mathrm{ext}_Z$ such that $\dim\mathrm{Span}\,E(z)|_Z=n+1$. Then $z=\frac{1}{2}(z_1+z_2)$ for $z_1\ne z_2$, $z_1,z_2\in S_Z$. Hence, $f(z_i)=1$ for $i=1,2$ and every $f\in E(z)$. Since $\dim\mathrm{Span}\,E(z)|_Z=\dim Z$, $z_1=z_2$; a contradiction.

To prove the converse, let $z\in\mathrm{ext}_Z$ and let $\dim\mathrm{Span}\,E(z)\le n$. Hence there is a $z_1\in S_Z$ with $f(z_1)=0$ for every $f\in E(z)$. Since the set ext_Z is finite, so is ext_{Z^*}. Consequently, $M=\max\{f(z) : f\in\mathrm{ext}_{Z^*}\setminus E(z)\}<1$. But this gives $\|z+\alpha z_1\|\le 1$ for $|\alpha|$ sufficiently small. Since $z=\frac{1}{2}(z+\alpha z_1+z-\alpha z_1)$, we get a contradiction.

COROLLARY 0.2.16. *Let X, D, b, Z be as in Proposition* 2.15. *Then there are $d_0\in\mathcal{P}_D(b)$ and $f_1,\dots,f_{n+1}\in\mathrm{ext}_{X^*}$ linearly independent on Z such that*

$$f_j(b-d_0)=\mathrm{dist}(b,D)\quad \textit{for } j=1,\dots,n+1\,.$$

P r o o f. Note that $\mathcal{P}_D(b)$ is convex and compact with respect to the norm topology. Hence, by the Krein–Milman theorem, there is a $d_0\in\mathcal{P}_D(b)$ which is an extremal point of this set. Consequently, $(b-d_0)/\mathrm{dist}(b,D)\in\mathrm{ext}_Z$. Applying Proposition 2.15 we get the desired result.

PROPOSITION 0.2.17 [Ma]. *Let X, D, b, Z have the same meaning as in Proposition* 2.15. *Then* $\operatorname{card}\mathcal{P}_D(b)=1$ *if and only if b has a SUBA element $d_0\in D$.*

Proof. Suppose $\operatorname{card}\mathcal{P}_D(b)=1$ and let $d_0\in\mathcal{P}_D(b)$. If d_0 is not a SUBA to b in D then, by Corollary 2.14, $0\in\operatorname{conv}(E(b-d_0)|_D)\setminus\operatorname{int}(\operatorname{conv}(E(b-d_0)|_D))$. Hence there is a $d\in S_D$ such that $f(d)\ge 0$ for every $f\in E(b-d_0)$. Since ext_X is a finite set, reasoning as in Proposition 2.15, we get $d_0-\alpha d\in\mathcal{P}_D(b)$ for $\alpha>0$ sufficiently small; a contradiction.

Chapter I

I.1. Best approximation in finite-dimensional subspaces of $\mathcal{L}(B,D)$. In this section we consider the case of $X=\mathcal{L}(B,D)$ and $\mathcal{V}\subset X$ being a finite-dimensional subspace. We prove Kolmogorov's type characterization of best approximants (Theorem 1.1) which involves only elements from the sets S_B and ext_{D^*}. We also present a result concerning strong unicity.

Next we characterize finite-dimensional Chebyshev subspaces in the space $\mathcal{K}(c_0)$ of all compact operators from c_0 into c_0. (Recall that a set $D\subset B$ is called a *Chebyshev set* iff $\operatorname{card}\mathcal{P}_D(b)=1$ for every $b\in B$.)

Now we formulate the main result of this section.

THEOREM 1.1.1. *Let B, D be arbitrary normed linear spaces* (*we consider real and complex case*) *and let $\mathcal{V}\subset\mathcal{L}(B,D)$ be a finite-dimensional subspace. Assume $L\in\mathcal{L}(B,D)\setminus\mathcal{V}$ and $V_0\in\mathcal{V}$. Then $V_0\in\mathcal{P}_{\mathcal{V}}(L)$ if and only if for every $\varepsilon>0$ there exist $m\in\mathbb{N}$, $\varphi_1,\dots,\varphi_m\in\operatorname{ext}_{D^*}$ and $w_1,\dots,w_m\in S_B$ such that*

$$0\in\operatorname{conv}\{\varphi_1\otimes w_1|_{\mathcal{V}},\dots,\varphi_m\otimes w_m|_{\mathcal{V}}\}\tag{1.1.1}$$

and

$$\Big|\sum_{i=1}^{m}\lambda_i(\varphi_i\otimes w_i)(L-V_0)-\|L-V_0\|\Big|\le\varepsilon\,,\tag{1.1.2}$$

where $\lambda_i\ge0$, $\sum_{i=1}^m\lambda_i=1$. (*We write $(\varphi_i\otimes w_i)(L)=\varphi_i(Lw_i)$.*)

Proof. Fix $\varepsilon>0$ and let $\mathcal{L}=[L]\oplus\mathcal{V}$. Since $\mathcal{L}$ is finite-dimensional, $S_{\mathcal{L}}$ is a compact set. Hence there exist $C_1,\dots,C_m\in S_{\mathcal{L}}$ such that $S_{\mathcal{L}}\subset\bigcup_{i=1}^m B_{\mathcal{L}}(C_i,\varepsilon/3)$. For each $i\in\{1,\dots,m\}$ select $\varphi_i\in\operatorname{ext}_{D^*}$ and $w_i\in S_B$ with

$$|\,\|C_i\|-\varphi_i(C_iw_i)|\le\varepsilon/3\,.\tag{1.1.3}$$

Set $Z_1=\{\varphi_i\otimes w_i : i=1,\dots,m\}$ and $T=\{\varphi\otimes w : \varphi\in\operatorname{ext}_{D^*},\ w\in S_B\}$. Note that $T|_{\mathcal{L}}$ is a total set over $\mathcal{L}$. Hence we can choose $Z_2\subset T|_{\mathcal{L}}$, $Z_2=\{(\gamma_1\otimes u_1)|_{\mathcal{L}},\dots,(\gamma_{\dim\mathcal{L}}\otimes u_{\dim\mathcal{L}})|_{\mathcal{L}}\}$ which forms a basis of $\mathcal{L}^*$. Put

$$Z=Z_1\cup Z_2\tag{1.1.4}$$

and let $\mathcal{M}=\operatorname{conv}(\bigcup_{\alpha\in K,|\alpha|=1}\alpha Z)$. Since $\mathcal{M}$ is an absolutely convex, absorbing set, we can define $\|\cdot\|_{\mathcal{M}}$, the Minkowski functional of the set $\mathcal{M}$, which is a norm in

$\mathcal{L}^*$. Hence we can equip the space $(\mathcal{L}^*)^* = \mathcal{L}$ with a norm $\|A\|_\varepsilon = \max_{\gamma\in Z} |\gamma(A)|$. It is easy to observe that $\|\gamma\|_{\mathcal{M}} = \sup_{\|A\|_\varepsilon \le 1} |\gamma(A)|$ and consequently, $\| \ \|_{\mathcal{M}}$ is the dual norm for $\| \ \|_\varepsilon$ in $\mathcal{L}^*$.

Now we will show that for every $A \in \mathcal{L}$, $|\,\|A\| - \|A\|_\varepsilon| \le \varepsilon\|A\|$. Of course, we can assume $A \ne 0$. Then

$$\begin{aligned}|1 - \|(A/\|A\|)\|_\varepsilon| &= |\,\|C_i\| - \|(A/\|A\|)\|_\varepsilon| \\ &\le |\,\|C_i\| - \|C_i\|_\varepsilon| + |\,\|C_i\|_\varepsilon - \|(A/\|A\|)\|_\varepsilon|\,,\end{aligned}$$

where $C_i \in S_{\mathcal{L}}$ is so chosen that $\|(A/\|A\|) - C_i\| \le \varepsilon/3$. Hence

$$\begin{aligned}&|1 - \|(A/\|A\|)\|_\varepsilon| \\ &\quad \le \|C_i - (A/\|A\|)\|_\varepsilon + |\,\|C_i\| - \|C_i\|_\varepsilon| \le \|C_i - (A/\|A\|)\| + \varepsilon/3 \le (2/3)\varepsilon\,,\end{aligned}$$

since by (1.1.3),

$$\|C_i\|_\varepsilon \le \|C_i\| \le \varphi_i(C_i w_i) + \varepsilon/3 \le \|C_i\|_\varepsilon + \varepsilon/3$$

(by (1.1.4), $\varphi_i \otimes w_i \in Z_1 \subset Z$). Consequently,

$$|\,\|(A/\|A\|)\| - \|(A/\|A\|)\|_\varepsilon| \le 2\varepsilon/3$$

and

$$|\,\|A\| - \|A\|_\varepsilon| < \varepsilon\|A\|\,.$$

Hence we get immediately

$$(1-\varepsilon)\|A\| \le \|A\|_\varepsilon \le (1+\varepsilon)\|A\|\,.$$

From this it is easy to deduce that

$$|\mathrm{dist}(L,\mathcal{V}) - \mathrm{dist}_\varepsilon(L,\mathcal{V})| \le \varepsilon\|L\|\,.$$

($\mathrm{dist}_\varepsilon$ denotes the distance to L from $\mathcal{V}$ with respect to the $\| \ \|_\varepsilon$.) Now let $V_0 \in \mathcal{P}_{\mathcal{V}}(L)$ and let $V_\varepsilon \in \mathcal{P}^\varepsilon_{\mathcal{V}}(L)$ (the set of best approximants with respect to $\| \ \|_\varepsilon$). By Corollary 0.2.14, $0 \in \mathrm{conv}(E_\varepsilon(L - V_\varepsilon)|_{\mathcal{V}})$. It is evident by the definition of $\| \ \|_\varepsilon$ that $E_\varepsilon(L-V_\varepsilon) \subset \bigcup_{\alpha\in K, |\alpha|=1} \alpha Z$. Hence $E_\varepsilon(L-V_\varepsilon) = \{\varphi_1\otimes w_1|_{\mathcal{L}}, \ldots, \varphi_l\otimes w_l|_{\mathcal{L}}\}$, where $\varphi_i \in \mathrm{ext}_{D^*}$ and $w_i \in S_w$ for $i = 1, \ldots, l$. Note that

$$\begin{aligned}\varepsilon\|L\| \ge |\mathrm{dist}(L,\mathcal{V}) - \mathrm{dist}_\varepsilon(L,\mathcal{V})| &= |\|L - V_0\| - \|L - V_\varepsilon\|_\varepsilon| \\ &= \Big|\,\|L - V_0\| - \sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_\varepsilon)\Big| \\ &= \Big|\,\|L - V_0\| - \sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_0)\Big|\,,\end{aligned}$$

where $\lambda_i \ge 0$, $\sum_{i=1}^l \lambda_i = 1$ and $\sum_{i=1}^l \lambda_i(\varphi_i \otimes w_i)|_{\mathcal{V}} = 0$. This proves the first part of the theorem (if $\|L\| \ne 1$ we can start from $\varepsilon/\|L\|$). Now suppose, on the contrary, that $V_0 \notin \mathcal{P}_{\mathcal{V}}(L)$ and condition (1.1.2) holds. Put $\varepsilon = (\|L - V_0\| -$

$\operatorname{dist}(L,\mathcal{V}))/2$ and let $V_1 \in \mathcal{P}_{\mathcal{V}}(L)$. Then

$$\varepsilon + \|L - V_1\| < \|L - V_0\| \le \Big|\sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_0)\Big| + \varepsilon\,,$$

which by (1.1.1) gives

$$\|L - V_1\| < \Big|\sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_1)\Big|\,;$$

a contradiction.

Remark 1.1.2. In Theorem 1.1 the set ext_{D^*} can be replaced by any norming set $C \subset S_{D^*}$ and S_B by any norming set $W \subset S_{B^{**}}$. (A set $F \subset S_{D^*}$ is called a *norming set* iff $\|v\| = \sup_{f\in F} |f(v)|$ for every $v \in D$.)

Applying Theorem 1.1 we can prove a necessary condition for $\mathcal{V}$ to be a non-Chebyshev subspace. The method of proof is the same as in [Mal2].

THEOREM 1.1.3. *Assume $\mathcal{V} \subset \mathcal{L}(B,D)$ is a non-Chebyshev finite-dimensional subspace* (*we consider the real case*). *Then there exists $V \in \mathcal{V}$, $\|V\| = 1$, such that for every $\varepsilon > 0$ there are $f_1,\ldots,f_m \in \operatorname{ext}_{D^*}$ and $w_1,\ldots,w_m \in B$, $\sum_{i=1}^m \|w_i\| = 1$, such that*:

(a) $G = \sum_{i=1}^m (f_i \otimes w_i)|_{\mathcal{V}} = 0$;
(b) *if $F \in \mathcal{L}^*(B,D)$ and $\|G \pm F\| \le 1$ then $|F(V)| < \varepsilon$.*
(c) $\sum_{i=1}^m |(f_i \otimes w_i)(V)| < \varepsilon$.

Proof. Since $\mathcal{V}$ is a non-Chebyshev subspace, there exists $L \in \mathcal{L}(B,D)$ such that $0, \pm V \in \mathcal{P}_{\mathcal{V}}(L)$, $\|V\| = 1$. This will be the required V. Now fix $\varepsilon > 0$. Applying Theorem 1.1, we can find $f_1,\ldots,f_m \in \operatorname{ext}_{D^*}$, $u_1,\ldots,u_m \in S_B$ and $\lambda_1,\ldots,\lambda_m > 0$, $\sum_{i=1}^m \lambda_i = 1$ such that

$$\sum_{i=1}^{m} \lambda_i(f_i \otimes u_i)|_{\mathcal{V}} = 0 \tag{1.1.5}$$

and

$$\Big|\sum_{i=1}^{m} \lambda_i(f_i \otimes u_i)(L) - \|L\|\Big| < \varepsilon/2\,. \tag{1.1.6}$$

For $i = 1,\ldots,m$ put $w_i = \lambda_i u_i$. Now we check that $f_1,\ldots,f_m, w_1,\ldots,w_m$ satisfy (a)–(c). Note that condition (a) is guaranteed by (1.1.5). To prove (b), fix $F \in \mathcal{L}^*(B,D)$, $\|F \pm G\| \le 1$. Hence $(G \pm F)(L) \le \|L\|$. Since $G|_{\mathcal{V}} = 0$, $G(L) \pm F(L - V) \le \|L - V\| = \|L\|$. By (1.1.6), $|F(L)| < \varepsilon/2$ and $|F(L - V)| < \varepsilon/2$. Hence $|F(V)| < \varepsilon$. To show (c), put

$$\begin{cases} P = \{i : (f_i \otimes w_i)(V) \ge 0\}\,, \\ P_1 = \{i : (f_i \otimes w_i)(V) > 0\}\,, \\ N = \{i : (f_i \otimes w_i)(V) < 0\}\,. \end{cases} \tag{1.1.7}$$

If N (resp. P_1) is empty, then by (1.1.5) P_1 (resp. N) is empty and (c) holds true. So assume that N and P_1 are nonempty. Hence, by ((1.1.5),

$$\sum_{i\in P}|(f_i\otimes w_i)(V)| = \sum_{i\in N}|(f_i\otimes w_i)(V)|\,.$$

Now suppose that (c) does not hold. Then

$$\sum_{i\in P}|(f_i\otimes w_i)(V)|\ge\varepsilon/2 \quad\text{and}\quad \sum_{i\in N}|(f_i\otimes w_i)(V)|\ge\varepsilon/2\,. \tag{1.1.8}$$

Put $\gamma_P=\sum_{i\in P}\|w_i\|$ and $\gamma_N=\sum_{i\in N}\|w_i\|$. By (1.1.7), $\gamma_P>0$, $\gamma_N>0$ and $\gamma_N+\gamma_P=1$. Set

$$S_1=\sum_{i\in P}(f_i\otimes w_i)(L),\qquad S_2=\sum_{i\in N}(f_i\otimes w_i)(L)\,.$$

By (1.1.6), $S_1+S_2>\|L\|-\varepsilon/2$. Thus either $S_1>\gamma_P(\|L\|-\varepsilon/2)$ or $S_2>\gamma_N(\|L\|-\varepsilon/2)$. Suppose that $S_1>\gamma_P(\|L\|-\varepsilon/2)$. Then, by (1.1.7) and (1.1.8),

$$\sum_{i\in P}(f_i\otimes w_i)(L+V)>\gamma_P\|L\|=\gamma_P\|L+V\|\,,$$

since $0<\gamma_P<1$. But for each $i\in P$,

$$(f_i\otimes w_i)(L+V)\le\|w_i\|\cdot\|L+V\|\,. \tag{1.1.9}$$

By summing both sides of (1.1.9) over i we get a contradiction. If $S_2>\gamma_N(\|L\|-\varepsilon/2)$ then a similar argument using N and $L-V$ provides a contradiction. The proof of Theorem 1.1.3 is complete.

Now we consider the case of strong unicity (see (0.2.16)). To present the next result, let us recall that an n-dimensional subspace $D\subset B$ is called an *interpolating subspace* iff for each set $\{d_1,\dots,d_n\}$ which is a basis of D and each set $\{f_1,\dots,f_n\}$ of linearly independent functionals from ext_{B^*},

$$\det[f_i(d_j)]_{i,j=1,\dots,n}\neq 0\,.$$

THEOREM 1.1.4. *Let X be a normed real space and let $V\subset X$ be an n-dimensional subspace with a basis $v_1,\dots,v_n$. Let $S\subset S_{X^*}$ be a norming set. Assume furthermore that there is a $\delta>0$ such that for every set $f_1,\dots,f_n$ of linearly independent functionals from S,*

$$|\det[f_i(v_j)]_{i,j=1,\dots,n}|>\delta>0\,. \tag{1.1.10}$$

Then each $x\in X$ has a strongly unique best approximation in V.

Proof. Fix $x\in X\setminus V$ and consider $Z=[x]\oplus V$. Since Z, being a finite-dimensional subspace, is separable, we may assume that S is countable. By the totality of S over Z, we can (for $k\ge k_0$) equip Z with the norm

$$\|z\|_k=\max_{i=1,\dots,k}|s_i(z)|\qquad(S=\{s_1,s_2,\dots\})\,.$$

From (1.1.10), V with $\| \|_k$ is an interpolating subspace of Z. Hence for $k \geq k_0$ there exists $v_k \in V$ which is a SUBA to x with respect to $\| \|_k$. By Corollary 0.2.14, $0 \in \text{int}(\text{conv}(E_k(x-v_k)|_V))$. (We consider the set $E_k(x-v_k)$ with respect to $\| \|_k$.) By Carathéodory's Theorem, $0 = \sum_{i=1}^{n+1} \lambda_i^k f_i^k|_V$, where $f_1^k, \ldots, f_{n+1}^k \in E_k(x-v_k)$, $\lambda_i^k > 0$ and $\sum_{i=1}^{n+1} \lambda_i^k = 1$. Passing to a subsequence if necessary, we may assume $v_k \to v_0$, $\lambda_i^k \to \lambda_i$, and $f_i^k \to f_i \in S_{Z^*}$. It is clear that $f_i(x - v_0) = \|x - v_0\|$ and $\sum_{i=1}^{n+1} \lambda_i f_i|_V = 0$. Now we show that $\lambda_i > 0$ for $i = 1, \ldots, n+1$. Note that $\lambda_{i_0} > 0$ for some $i_0 \in \{1, \ldots, n + 1\}$, since $\sum_{i=1}^{n+1} \lambda_i = 1$, $\lambda_i \geq 0$ for $i = 1, \ldots, n + 1$. We can assume $i_0 = n + 1$. By the Cramer rule

$$\lambda_i^k = \lambda_{n+1}^k \Delta_i^k / \Delta_{n+1}^k \quad \text{for } i = 1, \ldots, n\,, \tag{1.1.11}$$

where

$$\Delta_i^k = (-1)^{i+1} \det[f_l^k(v_j)]_{j=1,\ldots,n,l=1,\ldots,n+1,l\neq i}\,.$$

Hence, by (1.1.11), $1/|\lambda_i^k| \leq (M/\delta) \cdot 2/|\lambda_{n+1}|$ for k sufficiently large and $M > 0$ independent of k. Consequently, $\lambda_i = \lim_{k\to\infty} \lambda_i^k > 0$. Now take $w \in V \setminus \{0\}$. Since the set $\{f_1|_V, \ldots, f_{n+1}|_V\}$ is total over V, $f_{i_0}(w) < 0$ for some $i_0 \in \{1, \ldots, n+1\}$. From this we derive that $f(w) < 0$ for some $f \in \text{ext}\{g \in S_{Z^*} : g(x - v_0) = \|x - v_0\|\}$. Since each $f \in \text{ext}_{Z^*}$ can be extended to $f_1 \in \text{ext}_{X^*}$, we can assume that $f \in E(x - v_0)$ (see (0.2.12)). Note that a function $G : S_V \ni w \to \inf\{g(w) : g \in E(x-v_0)\}$ is upper-semicontinuous and, by the above reasoning, $G(w) < 0$ for every $w \in S_V$. By the compactness of S_V, we get $\sup\{G(w) : w \in S_V\} = -r < 0$. Now fix $v \in V \setminus \{0\}$ and take $f \in E(x - v_0)$ with $f(v/\|v\|) < G(v/\|v\|) + r/2$. Hence $f(v/\|v\|) < -r/2$ and consequently $f(v) < -(r/2)\|v\|$. In view of Remark 0.2.13, v_0 is a SUBA to x in V, which completes the proof of the theorem.

Remark 1.1.5. By [Mal1, Th. 3.3] the existence of δ in (1.1.10) is essential.

Example 1.1.6. Assume $B = D = c_0$. Let $A \in \mathcal{L}(B)$ be so chosen that for every $i \in \mathbb{N}$, $x \in \text{ext}_{l_\infty}$,

$$|(Ax)_i| > \delta > 0\,.$$

Then, by Theorem 1.4 , each $L \in \mathcal{L}(B)$ has a strongly unique best approximation in Span A. In this case

$$S = \{e_i \otimes x : x \in \text{ext}_{l_\infty},\ e_i \in \text{ext}_{l_1}\}\,.$$

Example 1.2.7. Assume $B = l_1$, $D = c_0$. Let $A \in \mathcal{L}(B, D)$ be represented as an infinite matrix $[A(i,j)]_{i,j=1,2,\ldots}$. If there exists $\delta > 0$ such that for every $i, j \in \mathbb{N}$, $|A(i,j)| > \delta > 0$, then each $L \in \mathcal{L}(B, D)$ has a strongly unique best approximation in Span A. In this case

$$S = \{e_i \otimes e_j : i, j = 1, 2, \ldots,\ e_i, e_j \in \text{ext}_{l_1}\}\,.$$

In the second part of this section we restrict ourselves to the case $\mathcal{K}(c_0)$. Namely, we prove

THEOREM 1.1.8. *Let* $\mathcal{V} \subset \mathcal{K}(c_0)$ (*we consider the real case*) *be a finite-dimensional Chebyshev subspace* (*i.e.* $\operatorname{card} \mathcal{P}_{\mathcal{V}}(x) = 1$ *for every* $x \in \mathcal{K}(c_0)$). *Then each* $L \in \mathcal{K}(c_0)$ *has a strongly unique best approximation in* $\mathcal{V}$.

Proof. Assume that there exists $L_0 \in \mathcal{K}(c_0) \setminus \mathcal{V}$ such that $V_0 \in \mathcal{P}_{\mathcal{V}}(L_0)$ is not a SUBA to L_0 in $\mathcal{V}$. Put

$$I = \{i \in \mathbb{N} : \|e_i \circ (L_0 - V_0)\| = \|L_0 - V_0\|\}. \tag{1.1.12}$$

(We write $e_i(x) = x_i$ for $x \in c_0$.) By Theorem 0.2.7,

$$\operatorname{ext}_{\mathcal{K}^*(c_0)} = \operatorname{ext}_{l_\infty} \otimes \operatorname{ext}_{l_1}. \tag{1.1.13}$$

Hence $\|L_0 - V_0\| = (x^i \otimes e_i)(L_0 - V_0)$ for some $x^i \otimes e_i \in E(L_0 - V_0)$ (see (0.2.12)). Consequently, the set I is nonempty. For each $i \in I$ define

$$Z_i = \{x \in \operatorname{ext}_{l_\infty} : (x \otimes e_i)(L_0 - V_0) = \|L_0 - V_0\|\}. \tag{1.1.14}$$

Since $V_0 \in \mathcal{P}_{\mathcal{V}}(L_0)$, by Remark 0.2.10, for every $V \in \mathcal{V}$ there exist $i \in I$ and $x^i \in Z_i$ such that

$$(x^i \otimes e_i)(V) \le 0. \tag{1.1.15}$$

Since V_0 is not a SUBA for L_0 and $\mathcal{V}$ is finite-dimensional, by Corollary 0.2.14, there exists $V_1 \in S_{\mathcal{V}}$ such that for every $i \in I$ and $x \in Z_i$,

$$(x \otimes e_i)(V_1) \ge 0. \tag{1.1.16}$$

Now assume that we have constructed $L \in \mathcal{K}(c_0)$ such that

$$\|L - \alpha V_1\| \le \|L\| \tag{1.1.17}$$

for $\alpha \in [0, \alpha_0)$ and

$$(x \otimes e_i)(L) = \|L\| \tag{1.1.18}$$

for every $i \in I$ and $x \in Z_i$. In view of Remark 0.2.10, (1.1.15) and (1.1.17), $\alpha V_1 \in \mathcal{P}_{\mathcal{V}}(L)$ for every $\alpha \in [0, \alpha_0)$, which contradicts the fact that $\mathcal{V}$ is a Chebyshev subspace. So to finish the proof, it is necessary to construct an $L \in \mathcal{K}(c_0)$ satisfying (1.1.17) and (1.1.18). To do this, fix $i \in I$ and $x = (x_1, x_2, \ldots) \in Z_i$. If $\sum_{k=1}^{\infty} |V_1(i,k)| = 0$ (V_1 is represented by the matrix $[V_1(i,k)]_{i,k=1,2,\ldots}$) then define

$$L_i = (L(i,k))_{k=1,2,\ldots} \tag{1.1.19}$$

where

$$L(i,k) = L_0(i,k) - V_0(i,k).$$

($[L_0(i,k)]_{i,k=1,2,\ldots}$ denotes the matrix corresponding to L_0 and $[V_0(i,k)]_{i,k=1,2,\ldots}$ the matrix corresponding to V_0.) If $\sum_{k=1}^{\infty} |V_1(i,k)| > 0$, then put

$$U_i = \{k \in \mathbb{N} : L_0(i,k) - V_0(i,k) = 0\}. \tag{1.1.20}$$

Since $\|e_i \circ (L_0 - V_0)\| = \operatorname{dist}(L_0, \mathcal{V}) > 0$, $\mathbb{N} \setminus U_i \ne \emptyset$. Put

$$F_i = \{k \in \mathbb{N} \setminus U_i : x_k = \operatorname{sgn} V_1(i,k)\}, \tag{1.1.21}$$

$$E_i = \mathbb{N} \setminus (U_i \cup F_i)\,. \tag{1.1.22}$$

Take $y = (y_1, y_2, \ldots) \in \mathrm{ext}_{l_\infty}$ given by

$$y_k = \begin{cases} x_k & \text{for } k \in F_i \cup E_i\,, \\ -\operatorname{sgn} V_1(i,k) & \text{for } k \in U_i\,. \end{cases} \tag{1.1.23}$$

By (1.1.20) and (1.1.23), $(y \otimes e_i)(L_0 - V_0) = \|L_0 - V_0\|$. By (1.1.16),

$$(y \otimes e_i)(V_1) = \sum_{k \in F_i} |V_1(i,k)| - \sum_{k \in U_i \cup E_i} |V_1(i,k)| \ge 0\,. \tag{1.1.24}$$

From this we derive $F_i \neq \emptyset$, since $\sum_{k=1}^{\infty} |V_1(i,k)| > 0$. For $k \in \mathbb{N}$ define

$$L(i,k) = \begin{cases} V_1(i,k) & \text{for } k \in F_i\,, \\ 0 & \text{for } k \in \mathbb{N} \setminus F_i\,, \end{cases} \tag{1.1.25}$$

and set $L_i = (L(i,1), L(i,2), \ldots)$. We show that for $\alpha \in [0,1)$, $\beta \ge 1$,

$$\sum_{k=1}^{\infty} |\beta L(i,k) - \alpha V_1(i,k)| \le \beta \sum_{k=1}^{\infty} |L(i,k)|\,. \tag{1.1.26}$$

To do this, take any $z \in \mathrm{ext}_{l_\infty}$. If $z_k = x_k$ for every $k \in F_i$ then

$$(z \otimes e_i)(V_1) = \sum_{k=1}^{\infty} V_1(i,k) z_k = \sum_{k \in F_i} |V_1(i,k)| + \sum_{k \in E_i \cup U_i} V_1(i,k) z_k \ge 0$$

by (1.1.24). Hence

$$\begin{aligned} \sum_{k=1}^{\infty} (\beta L(i,k) - \alpha V_1(i,k)) z_k &= \sum_{k=1}^{\infty} \beta L(i,k) z_k - \alpha \sum_{k=1}^{\infty} V_1(i,k) z_k \\ &= \beta \sum_{k \in F_i} |L(i,k)| - \alpha \sum_{k=1}^{\infty} V_1(i,k) z_k \le \beta \sum_{k=1}^{\infty} |L(i,k)|\,. \end{aligned}$$

If $z_k = -x_k$ for some $k \in F_i$, then the set $F_i^1 = \{k \in F_i : x_k = -z_k\}$ is nonempty. Note that

$$\begin{aligned} &\sum_{k=1}^{\infty} (\beta L(i,k) - \alpha V_1(i,k)) z_k \\ &\quad = \sum_{k \in F_i^1} (\beta L(i,k) - \alpha V_1(i,k)) z_k + \sum_{k \in (F_i \setminus F_i^1) \cup E_i \cup U_i} (\beta L(i,k) - \alpha V_1(i,k)) z_k \\ &\quad = \sum_{k \in F_i^1} (\alpha - \beta) |V_1(i,k)| + \sum_{k \in (F_i \setminus F_i^1) \cup E_i \cup U_i} (\beta L(i,k) - \alpha V_1(i,k)) z_k \\ &\quad \le \sum_{k \in F_i^1} (\beta - \alpha) |V_1(i,k)| + \sum_{k \in F_i \setminus F_i^1} (\beta - \alpha) |V_1(i,k)| + \sum_{k \in E_i \cup U_i} -\alpha V_1(i,k) z_k \\ &\quad = \sum_{k \in F_i} \beta |V_1(i,k)| - \alpha \Big(\sum_{k \in F_i} |V_1(i,k)| + \sum_{k \in E_i \cup U_i} V_1(i,k) z_k \Big) \end{aligned}$$

$$\leq \beta \sum_{k\in F_i} |V_1(i,k)| = \beta \sum_{k=1}^{\infty} |L(i,k)|$$

(see (1.1.24)). Now if $i \notin I$ then we define $L_i = (L(i,1), L(i,2), \ldots)$ by

$$L(i,k) = L_0(i,k) - V_0(i,k) \quad \text{for } k = 1,2,\ldots \tag{1.1.27}$$

Finally, observe that by the Schur Theorem [Ed, p. 864] for $i \geq i_0$,

$$\|e_i \circ (L_0 - V_0)\| \leq \operatorname{dist}(L_0, \mathcal{V})/2\,.$$

Hence the set I is finite and

$$M = \sup_{i\in\mathbb{N}\setminus I} \|e_i \circ (L_0 - V_0)\| < \|L_0 - V_0\|\,. \tag{1.1.28}$$

From (1.1.26) it follows that for $i \in I$ we can modify, if necessary, the rows L_i defined by (1.1.19) and (1.1.25), multiplying them by constants $\beta_i \geq 1$ such that

$$\|L_i - \alpha V_1(i,\cdot)\|_1 \leq \|L_i\|_1 = a > \|L_0 - V_0\|$$

for $\alpha \in [0,1)$. By (1.1.28), for $0 \leq \alpha \leq \alpha_0$ and $i \in \mathbb{N} \setminus I$,

$$\|L_i - \alpha V_1(i,\cdot)\|_1 < \|L_0 - V_0\|\,.$$

Consequently, the operator L defined by (1.1.19), (1.1.25) and (1.1.27) satisfies (1.1.17) and (1.1.18). The proof of Theorem 1.1.8 is complete.

Note that the unicity of best approximation for given $L \in \mathcal{K}(c_0)$ in $\mathcal{V}$ does not force the strong unicity because of

EXAMPLE 1.1.9. Let $L = [L(i,k)]_{i,k=1,2,\ldots}$ and $V = [V(i,k)]_{i,k=1,2,\ldots}$ be defined by

$$L(i,k) = \begin{cases} 0 & \text{if } i \neq 1\,, \\ 1/k^3 & \text{if } i = 1\,, \\ 0 & \text{if } i \neq 1\,, \end{cases} \qquad V(i,k) = \begin{cases} (-1)^k/k^2 & \text{for } i=1,\ k>1\,, \\ -\sum_{l=2}^{\infty} (-1)^l/l^2 & \text{for } i=1,\ k=1\,. \end{cases}$$

Let $\mathcal{V} = [V]$. We show that 0 is a unique best approximation for L in $\mathcal{V}$. Take $\alpha \in \mathbb{R} \setminus \{0\}$. If $\alpha > 0$, choose an even number k_0 such that $\alpha/k_0^2 > 1/k_0^3$. Let $z = (z_1, z_2, \ldots) \in \operatorname{ext}_{l_\infty}$ be given by

$$z_k = \begin{cases} 1 & \text{if } l \neq k_0\,, \\ -1 & \text{if } l = k_0\,. \end{cases}$$

Then

$$\|L - \alpha V\| \geq (z \otimes e_1)(L - V) = \sum_{l=1}^{\infty} z_k (L(1,l) - \alpha V(1,l))$$

$$= \sum_{l=1}^{\infty} (L(1,l) - \alpha V(1,l)) + 2(\alpha/k_0^2 - 1/k_0^3) > \sum_{l=1}^{\infty} L(1,l) = \|L - 0\|\,,$$

since $\sum_{l=1}^{\infty} V(1,l) = 0$. If $\alpha < 0$, choose an odd number k_0 such that $-\alpha/k_0^2 > 1/k_0^3$. Reasoning as above we get $\|L - \alpha V\| > \|L\|$. Hence $0 \in \mathcal{P}_{\mathcal{V}}(L)$ is the unique best approximation. However, $E(L-0)=\{(1,1,\ldots)\otimes e_1\}$ (see (1.1.13)). Since $(1,1,\ldots)\otimes e_1(V)=0$, by Remark 0.2.13, 0 is not a SUBA for L in $\mathcal{V}$.

Remark 1.1.10. If we replace c_0 by $l_\infty^{(m)}$, then by Theorem 0.2.7, the set $\mathrm{ext}_{\mathcal{K}^*(l_\infty^{(m)})}$ is finite. In view of Proposition 0.2.17 if $\mathcal{V}$ is a subspace of $\mathcal{K}(l_\infty^{(m)})$ then $L\in\mathcal{K}(l_\infty^{(m)})$ has a unique best approximation in $\mathcal{V}$ if and only if L has a strongly unique best approximation in $\mathcal{V}$.

COROLLARY 1.1.11. *If $V\in S_{\mathcal{K}(c_0)}$ then $\mathcal{V}=[V]$ is a Chebyshev subspace if and only if for every $i\in\mathbb{N}$ and $x\in\mathrm{ext}_{l_\infty}$,*

$$(x\otimes e_i)(V)\neq 0\,. \tag{1.1.29}$$

Comparing Corollary 1.11 with Theorem 3.3 of [Mal1] we get

PROPOSITION 1.1.12. *There exists*

$$\varphi\in\mathrm{ext}_{\mathcal{L}^*(c_0)}\setminus\{x\otimes e_i : i=1,2,\ldots,\ x\in\mathrm{ext}_{l_\infty}\}\,.$$

Proof. If $\mathrm{ext}_{\mathcal{L}^*(c_0)}\subset\{x\otimes e_i : i=1,2,\ldots,\ x\in\mathrm{ext}_{l_\infty}\}$ then by Remarks 0.2.10 and 0.2.13 each V satisfying (1.1.29) defines a Chebyshev subspace in $\mathcal{L}(c_0)$, which contradicts [Mal1, Th. 3.3].

Proposition 1.12 shows that Theorem 0.2.7 cannot be generalized from the case of compact operators to the case of linear operators.

To end this section we present an example of a two-dimensional Chebyshev subspace in $\mathcal{K}(c_0)$. The reasoning presented here is similar to that of [Bi]. First we recall after [Bi]

LEMMA 1.1.13. *Let $M>1$ be given. Assume $f(r)=\sum_{n=0}^{\infty}a_nr^n$ is a power series whose coefficients are not all 0. Assume that if $a_n\neq 0$ then $1\le|a_n|\le M$. Then $f(r)\neq 0$ for every $r\in(0,1/(M+1))$.*

Proof. Let N denote the smallest index n such that $a_n\neq 0$. Then

$$\begin{aligned}|f(r)|&=\Big|\sum_{n=N}^{\infty}a_nr^n\Big|\ge|a_Nr^N|-\sum_{n=N+1}^{\infty}|a_n|\cdot|r|^n\\&\ge|r|^N-M|r|^{N+1}/(1-|r|)=|r|^N/(1-|r|)(1-|r|(1+M))>0\,.\end{aligned}$$

EXAMPLE 1.1.14. Let $c\in(0,1)$, $r\in(0,1/4)$. Define

$$V_1(i,k)=c^ir^{2^{2k+1}}\quad\text{for } i,k=1,2,\ldots\,; \tag{1.1.30}$$

$$V_2(i,k)=(c/2)^ir^{2^{2k+2}}\quad\text{for } i,k=1,2,\ldots \tag{1.1.31}$$

We show that V_1,V_2 form a two-dimensional interpolating (hence Chebyshev) subspace in $\mathcal{K}(c_0)$. To do this, take two linearly independent functionals $\varphi_1=x_1\otimes e_{i_1}$, $\varphi_2=x_2\otimes e_{i_2}$ from $\mathrm{ext}\,S_{\mathcal{K}^*(c_0)}$. We prove that $\det[\varphi_i(V_j)]_{i,j=1,2}\neq 0$. Let

$x_j = (\sigma_{1j}, \sigma_{2j}, \ldots)$ for $j = 1, 2$ ($\sigma_{ij} = \pm 1$). Note that

$$\begin{aligned}\det[\varphi_i(V_j)]_{i,j=1,2} &= \det \begin{bmatrix} \sum_{j=1}^{\infty} \sigma_{1j} V_1(i_1, j) & \sum_{j=1}^{\infty} \sigma_{1j} V_2(i_1, j) \\ \sum_{j=1}^{\infty} \sigma_{2j} V_1(i_2, j) & \sum_{j=1}^{\infty} \sigma_{2j} V_2(i_2, j) \end{bmatrix} \\ &= \sum_{j_1, j_2 = 1}^{\infty} \det \begin{bmatrix} \sigma_{1j_1} V_1(i_1, j_1) & \sigma_{1j_2} V_2(i_1, j_2) \\ \sigma_{2j_1} V_1(i_2, j_1) & \sigma_{2j_2} V_2(i_2, j_2) \end{bmatrix} \\ &= \sum_{j_1, j_2 = 1}^{\infty} \det \begin{bmatrix} \sigma_{1j_1} c^{i_1} r^{2^{2j_1+1}} & \sigma_{1j_2} (c/2)^{i_1} r^{2^{2j_2+2}} \\ \sigma_{2j_1} c^{i_2} r^{2^{2j_1+1}} & \sigma_{2j_2} (c/2)^{i_2} r^{2^{2j_2+2}} \end{bmatrix} \\ &= \sum_{j_1, j_2 = 1}^{\infty} r^{2^{2j_1+1} + 2^{2j_2+2}} \det \begin{bmatrix} \sigma_{1j_1} c^{i_1} & \sigma_{1j_2} (c/2)^{i_1} \\ \sigma_{2j_1} c^{i_2} & \sigma_{2j_2} (c/2)^{i_2} \end{bmatrix}.\end{aligned}$$

If $2^{2j_1+1} + 2^{2j_2+2} = 2^{2k_1+1} + 2^{2k_2+2}$, because of the uniqueness of binary expression of each integer we get $j_1 = k_1$ and $j_2 = k_2$. In particular then, distinct pairs (j_1, j_2) give distinct powers of r. Hence the above determinant can be regarded as a power series with coefficients

$$A_{j_1, j_2} = \det \begin{bmatrix} \sigma_{1j_1} c^{i_1} & \sigma_{1j_2} (c/2)^{i_1} \\ \sigma_{2j_1} c^{i_2} & \sigma_{2j_2} (c/2)^{i_2} \end{bmatrix}.$$

If $i_1 = i_2$ then

$$\det[\varphi_i(V_j)]_{i,j=1,2} = (c^2/2)^{i_1} \sum_{j_1, j_2 = 1}^{\infty} r^{2^{2j_1+1} + 2^{2j_2+2}} B_{j_1, j_2},$$

where

$$B_{j_1, j_2} = \det \begin{bmatrix} \sigma_{1,j_1} & \sigma_{1j_2} \\ \sigma_{2,j_1} & \sigma_{2,j_2} \end{bmatrix}. \tag{1.1.32}$$

Since $x_1 \otimes e_{i_1}$, $x_2 \otimes e_{i_2}$ are linearly independent, not all B_{j_1, j_2} are equal to 0. Note that if $B_{j_1, j_2} \neq 0$ then $|B_{j_1, j_2}| = 2$. If $i_1 \neq i_2$ (we may assume $i_1 < i_2$) then

$$\det[\varphi_i(V_j)]_{i,j=1,2} = c^{i_1 + i_2} [(1/2)^{i_1} - (1/2)^{i_2}] \sum_{j_1, j_2 = 1}^{\infty} r^{2^{2j_1+1} + 2^{2j_2+2}} B_{j_1, j_2},$$

where

$$B_{j_1, j_2} = (1/[(1/2)^{i_1} - (1/2)^{i_2}]) \det \begin{bmatrix} \sigma_{1j_1} & \sigma_{1j_2} (1/2)^{i_1} \\ \sigma_{2j_1} & \sigma_{2j_2} (1/2)^{i_2} \end{bmatrix}. \tag{1.1.33}$$

It is clear that

$$\begin{aligned}1 \leq |B_{j_1, j_2}| &\leq [(1/2)^{i_1} + (1/2)^{i_2}]/[(1/2)^{i_1} - (1/2)^{i_2}] \\ &= [1 + (1/2)^{i_2 - i_1}]/[1 - (1/2)^{i_2 - i_1}] \leq [1 + (1/2)]/[1 - (1/2)] = 3.\end{aligned}$$

Applying Lemma 3.6 to the series

$$\sum_{j_1, j_2 = 1}^{\infty} B_{j_1, j_2} r^{2^{2j_1+1} + 2^{2j_2+2}},$$

where B_{j_1,j_2} are defined by (1.1.32) or (1.1.33) we get $\det[\varphi_i(V_j)]_{i,j=1,2} \neq 0$ as required.

Remark 1.1.15. For the special cases $B = C_{\mathbb{R}}(T)$, $D = C_{\mathbb{R}}(S)$ and $B = D = c_0$, Theorems 1.1 and 1.3 were proved by a different method in [Mal1, Mal2]. The reults presented in this section were obtained by the author in [LG3].

I.2. Kolmogorov's type criteria for spaces of compact operators; general case. In this section we prove two principal criteria concerning approximation in convex subsets of the space $\mathcal{K}(B, D)$. We start with the following

LEMMA 1.2.1. *Let B and D be Banach spaces, both over the same field K ($K = \mathbb{R}$ or $\mathbb{C}$). For $L \in \mathcal{K}(B, D)$ put*

$$\text{crit}^*(L) = \{f \in \text{ext}_{D^*} : \|f \circ L\| = \|L\|\} \tag{1.2.1}$$

Then the set $\text{crit}^*(L)$ *is nonvoid for every $L \in \mathcal{K}(B, D)$.*

Proof. Fix $L \in \mathcal{K}(B, D)$ and consider the function $\phi(f) = \|f \circ L\|$ for $f \in S_{D^*}$. We show that ϕ is weak*-continuous on S_{D^*}. By the compactness of L the space $L(B)$ is separable. Since $f \circ L = f|_{L(B)} \circ L$, we may restrict ourselves to the case of D being separable. By [Du, Th. 1, p. 426]), the space S_{D^*} with the weak* topology is metrizable in this case. Now suppose, on the contrary, that $\{f_n\} \subset S_{D^*}$ tends weak* to $f \in S_{D^*}$ and $\phi(f_n - f) \geq \varepsilon > 0$. Then $(f_n - f)(Lx_n) > \varepsilon/2$ for some $\{x_n\} \subset S_B$. By the compactness of L, we can assume $\|Lx_n - y\| \to 0$ for some $y \in D$. Note that

$$\begin{aligned} |(f_n - f)(Lx_n)| &\leq |(f_n - f)(Lx_n - y)| + |(f_n - f)(y)| \\ &\leq 2\|Lx_n - y\| + |(f_n - f)(y)| \leq \varepsilon/2 \quad \text{for } n \geq n_0\,; \end{aligned}$$

a contradiction. Applying the Banach–Alaoglu Theorem we deduce that there is $f \in S_{D^*}$ with $\|f \circ L\| = \|L\|$. Put $\text{crit}_1^*(L) = \{f \in S_{D^*} : \|f \circ L\| = \|L\|\}$. Since φ is weak*-continuous, by the Banach–Alaoglu Theorem, the sets $\text{crit}_1^*(L)$ and $\text{conv}(\text{crit}_1^*(L))^-$ (with respect to the weak* topology) are weak*-compact. By the Krein–Milman Theorem the set $\text{ext}(\text{conv}(\text{crit}_1^*(L))^-) \neq \emptyset$ and by Theorem 0.2.2, $\text{ext}(\text{conv}(\text{crit}_1^*(L))^-) \subset \text{crit}_1^*(L)$. Hence, it is easy to show that

$$\text{ext}(\text{conv}(\text{crit}_1^*(L))^-) \subset \text{ext}_{D^*}$$

and consequently $\text{crit}^*(L) \neq \emptyset$.

Now we prove one of the main results of this section.

THEOREM 1.2.2. *Let B, D be as in Lemma* 2.1. *Assume $\mathcal{V} \subset \mathcal{K}(B, D)$ is a convex set. Let $K \in \mathcal{K}(B, D)$ and $V \in \mathcal{V}$. Then we have*:

(a) *$V \in \mathcal{P}_{\mathcal{V}}(K)$ if and only if for every $U \in \mathcal{V}$ there exists $y \in \text{crit}^*(K - V)$ such that $\|\text{re}(y^* \circ (K - U))\| \geq \|K - V\|$.*

(b) *V is a SUBA to K in $\mathcal{V}$ with a constant $r > 0$ if and only if for every $U \in \mathcal{V}$ there exists $y^* \in \text{crit}^*(K - V)$ such that $\|\text{re}(y^* \circ (K - U))\| \geq \|K - V\| + r\|K - U\|$.*

Proof. (a) Fix $U \in \mathcal{V}$. Since $\|\mathrm{re}(y^* \circ (K-U))\| \geq \|K-V\|$ for some $y^* \in \mathrm{crit}^*(K-V)$, $V \in \mathcal{P}_{\mathcal{V}}(K)$. To prove the converse, assume that there exists $U \in \mathcal{V}$ such that $\|\mathrm{re}(y^* \circ (K-U))\| < \|K-V\|$ for every $y^* \in \mathrm{crit}^*(K-V)$. Take an arbitrary $f \in E(K-V)$. In view of Theorem 0.2.7, $f = x^{**} \otimes y^*$ for some $x^{**} \in \mathrm{ext}_{B^{**}}$ and $y^* \in \mathrm{ext}_{D^*}$. According to Goldstine's Theorem select a net $(x_u) \subset S_B$ such that x_u tends to x^{**} weak* in B^{**}. Since, by (0.2.5), $\mathrm{re}(y^* \circ (K-V)x_u)$ tends to $\mathrm{re}(y^* \circ (K-V)^{**}x^{**}) = \mathrm{re}((x^{**} \otimes y^*)(K-V)) = \mathrm{re}(f(K-V))$, $y^* \in \mathrm{crit}^*(K-V)$. Hence

$$\begin{aligned}\mathrm{re}(f(U-V)) &= \mathrm{re}(f(K-V)) - \mathrm{re}(f(K-U)) \\ &= \|K-V\| - \mathrm{re}(y^*((K-U)^*x^{**})) \\ &= \|K-V\| - \lim_u \mathrm{re}(y^*((K-U)x_u)) \\ &\geq \|K-V\| - \|\mathrm{re}(y^* \circ (K-U))\| > 0\,.\end{aligned}$$

By Theorem 0.2.8 and Remark 0.2.10, $V \notin \mathcal{P}_{\mathcal{V}}(K)$; a contradiction.

By the same reasoning, applying Remark 0.2.13, we can prove part (b) of our theorem.

Remark 1.2.3. In Theorem 2.2 the set $\mathrm{crit}^*(K-V)$ can be replaced by any set $C \subset \mathrm{crit}^*(K-V)$ such that $\bigcup_{|a|=1} aC = \mathrm{crit}^*(K-V)$ ($C \cup -C = \mathrm{crit}^*(K-V)$ in the real case) and $aC \cap bC = \emptyset$ for $a \neq b$, $|a| = |b| = 1$ ($C \cap -C = \emptyset$ in the real case).

Now fix $K \in \mathcal{K}(B,D)$ and for $y^* \in \mathrm{crit}^*(K)$ put

$$A_{y^*} = \{x \in S_B : y^*(Kx) = \|K\|\} \tag{1.2.2}$$

One can show the following

Remark 1.2.4. Let $K \in \mathcal{K}(B,D)$ and $y^* \in \mathrm{crit}^*(K)$. Then for any $a \in \mathbb{C}$, $|a| = 1$,

$$aA_{ay^*} = A_{y^*}\,. \tag{1.2.3}$$

Now we can prove

THEOREM 1.2.5. *Assume B is a reflexive space and let $D, \mathcal{V}, K, V$ be as in Theorem 2.2. Then we have*:

(a) *$V \in \mathcal{P}_{\mathcal{V}}(K)$ if and only if for every $U \in \mathcal{V}$ there exists $y^* \in \mathrm{crit}^*(K-V)$ such that*

$$\inf\{\mathrm{re}(y^*(U-V)x) : x \in A_{y^*}\} \leq 0\,.$$

(b) *V is a SUBA to K in $\mathcal{V}$ with a constant $r > 0$ if and only if for every $U \in \mathcal{V}$ there exists $y^* \in \mathrm{crit}^*(K-V)$ with*

$$\inf\{\mathrm{re}(y^*(U-V)x) : x \in A_{y^*}\} \leq -r\|U-V\|\,.$$

P r o o f. (a) Assume $V \notin \mathcal{P}_{\mathcal{V}}(K)$. Then $\|K-U\| < \|K-V\|$ for some $U \in \mathcal{V}$. Take an arbitrary $y^* \in \mathrm{crit}^*(K-V)$ and $x \in A_{y^*}$. Note that

$$\mathrm{re}(y^*(U-V)x) = \mathrm{re}(y^*(K-V)x) - \mathrm{re}(y^*(K-U)x) \geq \|K-V\| - \|K-U\| > 0$$

and consequently, $\inf\{\mathrm{re}(y^*(U-V)x) : x \in A_{y^*}\} > 0$.

To prove the converse, suppose that $\inf\{\mathrm{re}(y^*(U-V)x) : x \in A_{y^*}\} > 0$ for every $y^* \in \mathrm{crit}^*(K-V)$ (the set A_{y^*} is nonvoid by the reflexivity of D). Take an arbitrary $f \in E(K-V)$ (see 0.2.12). In view of Theorem 0.2.7, $f = x^{**} \otimes y^*$ for some $y^* \in \mathrm{ext}_{D^*}$ and $x^{**} \in \mathrm{ext}_{B^{**}}$. Since B is reflexive, $x^{**} = x$ for some $x \in S_B$. It is clear that $y^* \in \mathrm{crit}^*(K-V)$ and $x \in A_{y^*}$. Consequently, $\mathrm{re}(f(U-V)) > 0$ and, by Theorem 0.2.8, $V \notin \mathcal{P}_{\mathcal{V}}(K)$. Applying Theorem 0.2.12 and Remark 0.2.13, by the same reasoning, we can prove the part (b) of our theorem.

R e m a r k 1.2.6. In Theorem 2.5, by Remark 2.4, the set $\mathrm{crit}^*(K-V)$ may be replaced by any set $C \subset \mathrm{crit}^*(K-V)$ satisfying the requirements of Remark 2.3.

R e m a r k 1.2.7. If B is an arbitrary Banach space it may occur that the set A_{y^*} is empty. Take for example $B = C_0^{2\pi}$, the space of all real, 2π-periodic continuous functions, and let D_n be the space of all trigonometric polynomials of degree $\leq n$. Put $\mathcal{V} = \mathcal{P}_{\mathrm{Id}}(B, D_n)$ (see (0.1.17)), the space of all linear, continuous projections from B onto D_n. It is well known that the classical Fourier projection F_n is minimal among all projections, which means $F_n \in \mathcal{P}_{\mathcal{V}}(0)$ (see e.g. [Che1, p. 212]) According to [OdL, Lemma I.2.7], F_n cannot attain its norm at any point of S_{B^*}. Consequently, for every $y^* \in \mathrm{crit}^*(F_n)$ the set A_{y^*} is empty.

Now we restrict our interest to the case of extensions of operators of minimal norm. Applying Theorem 2.2 we get

THEOREM 1.2.8. *Let B be a Banach space and let D be a finite-dimensional linear subspace. Let $A \in \mathcal{L}(D)$. Assume that $P_0 \in \mathcal{P}_A^0(B, D)$* (see (0.1.19)), *i.e. P_0 is an extension of A of minimal norm. Assume furthermore that $\|P_0\| > \|A\|$. Then every set $C \subset \mathrm{crit}^*(P_0)$* (*we treat $\mathcal{L}_D(B, D)$ as a subset of $\mathcal{K}(B)$*) *such that*

$$\mathrm{crit}^*(P_0) = \bigcup_{|a|=1} aC \quad \textit{and} \quad aC \cap bC = \emptyset \textit{ for } a \neq b,\ |a| = |b| = 1 \tag{1.2.4}$$

(*resp. $C \cup -C = \mathrm{crit}^*(P_0)$, $C \cap -C = \emptyset$ in the real case*) *is linearly dependent on D.*

P r o o f. Take $C \subset \mathrm{crit}^*(P_0)$ satisfying (1.2.4) and assume that C is linearly independent on D. We can write $C = \{f_1, \ldots, f_k\}$, $k \leq n$. If $k < n$ we can add $f_{k+1}, \ldots, f_n \in S_{B^*}$ such that $\{f_1|_D, \ldots, f_n|_D\}$ form a basis of D^*. It is clear that the set $\{f_1, \ldots, f_n\}$ is total on D. By [Al, p. 74] there exist $y_1, \ldots, y_n \in D$ with $f_i(y_j) = \delta_{ij}$. Define an operator $P : B \to D$ by

$$Px = \sum_{i=1}^{n} g_i(x) y_i \quad \text{for } x \in B\,.$$

where for $i = 1, \dots, n$, g_i is a norm-preserving extension of the functional $f_i|_D \circ A$ onto B. To show that $P|_D = A$ fix $i \in \{1, \dots, n\}$. Note that for any $j \in \{1, \dots, n\}$, $f_i(Py_j - Ay_j) = 0$, by the definition of P. Since $f_1|_D, \dots, f_n|_D$ form a basis of D, this means $Py_j = Ay_j$ for $j \in \{1, \dots, n\}$ and consequently $P|_D = A$. Now we show that $\|P\| < \|P_0\|$. In view of Theorem 2.2 and Remark 2.3 it suffices to prove that $\|\mathrm{re}(f_i \circ P)\| < \|P_0\|$ for $i = 1, \dots, n$. For any $x \in S_B$ we have

$$\begin{aligned}\mathrm{re}((f_i \circ P)(x)) &= \mathrm{re}\Big(f_i\Big(\sum_{j=1}^n g_j(x) y_j\Big)\Big) = \mathrm{re}(g_i(x)) \le \|g_i\| \\ &= \|f_i|_D \circ A\| \le \|f_i\| \cdot \|A\| = \|A\| .\end{aligned}$$

Hence $\|\mathrm{re}(f_i \circ P)\| \le \|A\| < \|P_0\|$, which completes the proof.

Remark 1.2.9. If we consider the set $\mathcal{L}_D(B, D)$ as a subset of $\mathcal{K}(B, D)$ then Theorem 2.7 remains true.

If $\|P_0\| = 1$, then the set C given by (1.2.4) can be linearly dependent or independent on D. Consider the following

EXAMPLE 1.2.10. Let $B = l_\infty^{(n)}$ $(n \ge 3)$ and let $D = \ker(1/2, 1/4, 1/4, 0, \dots, 0)$. Take $y = (2, 0, \dots, 0)$ and define $P_y x = x - f(x) y$ for $x \in B$. It is easy to verify that $\|P_y\| = 1$ and consequently $P_y \in \mathcal{P}^0_{\mathrm{Id}}(B, D)$ (see (0.1.19)). We note that for every $i \in \{1, \dots, n\}$,

$$\begin{aligned}(1.2.5)\qquad \|e_i \circ P_y\| &= \sup\Big\{\Big|x_i - \Big(\sum_{j=1}^n f_j x_j\Big) y_i\Big| : x \in S_B\Big\} \\ &= |1 - f_i y_i| + |y_i|(1 - |f_i|) .\end{aligned}$$

Hence $\mathrm{crit}^*(P_y) = \{\pm e_i\}_{i=1}^n$. If we put $C = \{e_i\}_{i=1}^n$ then C is linearly dependent on D, since $e_1|_D = (-1/2) e_2|_D + (-1/2) e_2|_D$. However, if we take $D = \ker(2/3, 1/3, 0, \dots, 0)$ then, by the same reasoning, $\mathrm{crit}^*(P_y) = \{\pm e_i\}_{i=2}^n$ (we take $y = (3/2, 0, \dots, 0)$). It is easy to verify that $C = \{e_i\}_{i=2}^n$ satisfies (1.2.4) and it is linearly independent on D.

Now we consider the case $B = C_K(T)$.

COROLLARY 1.2.11. *Let $B = C_K(T)$ and let $D \subset B$ be an n-dimensional subspace. If $P_0 \in \mathcal{P}^0_A(B, D)$ for a given $A \in \mathcal{L}(D)$, $\|P_0\| > \|A\|$, then the set $\{t \in T : \|\widehat{t} \circ P_0\| = \|P_0\|\}$ is linearly dependent on D. In particular, if D is a Haar subspace, then*

$$\mathrm{card}\{t \in T : \|\widehat{t} \circ P_0\| = \|P_0\|\} \ge n + 1 .$$

Remark 1.2.12. Theorems 2.2 and 2.5 were proved by the author in [LG1]. In the real case for $A = \mathrm{Id}$ Theorem 2.5 and Corollary 2.11 were proved by E. W. Cheney and P. D. Morris in [Che5]. These results are connected with the problem when an interpolating projection (see (0.1.11)) belongs to $\mathcal{P}^0_{\mathrm{Id}}(B, D)$, i.e. is a minimal projection in the class of all linear projections.

In the case of $T = [a,b]$ and $D = P_n$ (the space of all polynomials of degree $\le n$ restricted to $[a,b]$), by [Ki], if $P \in \mathcal{I}(C_{\mathbb{R}}([a,b]), D)$ (see (0.1.11)) then the set $\{t \in [a,b] : \|\widehat{t} \circ P\| = \|P\|\}$ consists of at most n points. Note that, by [Che5], if $D \subset C_{\mathbb{R}}([a,b])$ is an n-dimensional Haar subspace containing constants ($n \ge 3$) then the norm of a minimal, linear projection onto P_n is greater than one. Hence, by Corollary 2.11, we get

Corollary 1.2.13 [Che2]. *In the set* $\mathcal{I}(C_{\mathbb{R}}([a,b]), P_n)$ ($n \ge 3$) *there is no minimal projection from* $C([a,b],\mathbb{R})$ *onto* P_n.

I.3. Criteria for the space $\mathcal{K}(C_K(T))$. Throughout this section, unless otherwise stated, $B = C_K(T)$ and $D \subset B$ is a linear subspace. We specialize the results from Section I.2 to the case of $\mathcal{K}(B)$. First we introduce some notations. Recall from [Che5]

Theorem 1.3.1. *For* $L \in \mathcal{L}(B)$ *set*

$$\mathcal{F}_L = \{F \subset T : F \text{ is closed and for every } x \in C_K(T),\ Lx = 0 \text{ if } x|_F = 0\}. \tag{1.3.1}$$

Then there exists a smallest (*in the sense of inclusion*) *set* $F_0 \in \mathcal{F}$.

Definition 1.3.2. Let $L \in \mathcal{L}(B)$. Then the smallest (in the sense of inclusion) set $F_0 \in \mathcal{F}_L$ is called the *carrier* of the operator L (we write $\mathrm{car}(L)$ for brevity.)

If the set $\mathrm{car}(L)$ is finite, then L is called a *discrete operator.* The set of all discrete operators from B into D will be denoted by $\mathcal{D}(B,D)$ ($\mathcal{D}(B)$ if $D = B$). For $F \subset T$ we denote by $\mathcal{K}_F(B)$ the space of all compact operators from B to B with carriers contained in F. For $t \in T$ the symbol $\widehat{t}$ stands for the evaluation functional.

We start with the following

Lemma 1.3.3. *Assume that* $V \in \mathcal{K}(B) \setminus \{0\}$ *and let* $\mathrm{card}\,\mathrm{car}(V) < \infty$, *i.e.* $V \in \mathcal{D}(B)$. *For* $\widehat{t} \in \mathrm{crit}^*(V)$ (see (1.2.1)) *put*

$$A_t = \{x \in S_B : (Vx)t = \|V\|\}. \tag{1.3.2}$$

Then for every $\widehat{t} \in \mathrm{crit}^*(V)$ *and every* $\{x_n\} \subset S_B$ *with* $(Vx_n)t \to \|V\|$, *there exists* $\{z_n\} \subset A_t$ *with* $\|z_n - x_n\| \to 0$ *as* $n \to \infty$.

Proof. Since $V \in \mathcal{D}(B)$, $V = \sum_{i=1}^k \widehat{t_i}(\cdot) y_i$, where $y_i \in B$, $t_i \in T$ for $i = 1,\ldots,k$. By the Tietze–Urysohn Theorem $\|V\| = \|\sum_{i=1}^k |y_i|\|$. Fix $\widehat{t} \in \mathrm{crit}^*(V)$, $\{x_n\} \subset S_B$ with $(Vx_n)t \to \|V\|$ and let $A = \{i \in \{1,\ldots,k\} : y_i(t) \ne 0\}$. First we will show that $x_n(t_i) \to \overline{y_i(t)}/|y_i(t)| = \mathrm{sgn}(y_i(t))$ for $i \in A$. Since $\sum_{i=1}^k |y_i(t)| = \sum_{i\in A} \mathrm{sgn}(y_i(t)) \cdot y_i(t)$, $|x_n(t_i)| \to 1$ for each $i \in A$. Assume that for some $i_0 \in A$ there exists a subsequence (x_{n_k}) with

$$|\mathrm{sgn}(y_{i_0}(t)) - x_{n_k}(t_{i_0})/|x_{n_k}(t_{i_0})|| \ge d > 0 \quad \text{for } k \ge k_0\,.$$

By the uniform convexity of $\mathbb{C}$ over $\mathbb{R}$,

$$|\operatorname{sgn}(y_{i_0}(t)) + x_{n_k}(t_{i_0})/|x_{n_k}(t_{i_0})||/2 \le 1-\delta \quad \text{for some } \delta > 0\,.$$

Note that

$$\begin{aligned}\Big|\frac{1}{2}\Big(\sum_{i\in A}|y_i(t)| + \sum_{i\in A}(x_{n_k}(t_i)/|x_{n_k}(t_i)|)y_i(t)\Big)\Big| \\ \le \sum_{i\in A\setminus\{i_0\}}|y_i(t)| + |\tfrac{1}{2}\operatorname{sgn}(y_{i_0}(t) + x_{n_k}(t_{i_0})/|x_{n_k}(t_{i_0})|)|\cdot|y_{i_0}(t)| \\ \le \sum_{i\in A\setminus\{i_0\}}|y_i(t)| + (1-\delta)|y_{i_0}(t)| < \|V\|\,.\end{aligned}$$

But, passing to a subsequence if necessary, $\sum_{i\in A}(x_{n_k}(t_i)/|x_{n_k}(t_i)|)y_i(t)$ tends to $\|V\|$ as $k\to\infty$; a contradiction.

Now we construct the sequence (z_n). For each $n\in\mathbb{N}$ set

$$\varepsilon_n = \max\{|x_n(t_i) - \operatorname{sgn}(y_i(t))| : i\in A\}\,.$$

Fix $n\in\mathbb{N}$ and for every $i\in A$ select an open neighbourhood U_i of t_i such that $\overline{U}_i\cap\overline{U}_j=\emptyset$ for $i\ne j$ and $|x_n(s)-x_n(t_i)|\le\varepsilon_n$ for $s\in\overline{U}_i$, $i\in A$. Fix $i\in A$. An easy calculation shows that for every $s\in\overline{U}_i$,

$$\operatorname{re}(x_n(s)) \in [\operatorname{re}(\operatorname{sgn}(y_i(t))) - 2\varepsilon_n, \operatorname{re}(\operatorname{sgn}(y_i(t))) + 2\varepsilon_n]\cap[-1,1] = [B,C]$$

and

$$\operatorname{im}(x_n(s)) \in [\operatorname{im}(\operatorname{sgn}(y_i(t))) - 2\varepsilon_n, \operatorname{im}(\operatorname{sgn}(y_i(t))) + 2\varepsilon_n]\cap[-1,1] = [D,E]\,.$$

Set $S_i = \sigma(U_i)\cup\{t_i\}$ and define for $s\in S_i$,

$$f_i(s) = \begin{cases}\operatorname{re}(x_n(s)), & s\in\sigma(U_i)\,,\\ \operatorname{re}(\operatorname{sgn}(y_i(s))), & s=t_i\,,\end{cases}\qquad g_i(s) = \begin{cases}\operatorname{im}(x_n(s)), & s\in\sigma(U_i)\,,\\ \operatorname{im}(\operatorname{sgn}(y_i(s))), & s=t_i\,.\end{cases}$$

By the Tietze–Urysohn Theorem, we can extend continuously the functions f_i and g_i to the whole set $\overline{U}_i$ so that $f_i(s)\in[B,C]$ and $g_i(s)\in[D,E]$ for every $s\in\overline{U}_i$. It is easy to show that

$$|(f_i+ig_i)(s) - \operatorname{sgn}(y_i(t))| \le 2\sqrt{2}\cdot\varepsilon_n\,.$$

Let $\pi_i : B_d(\operatorname{sgn}(y_i(t)), \sqrt{2}\cdot 2\varepsilon_n) \to B_d(\operatorname{sgn}(y_i(t)), \sqrt{2}\cdot 2\varepsilon_n)\cap B_d(0,1)$ ($B_d(x,r) = \{y\in\mathbb{C} : |x-y|\le r\}$) be a continuous function with

$$\pi_i|_{B_d(\operatorname{sgn}(y_i(t)),r)\cap B_d(0,1)} = \mathrm{Id} \qquad (r=\sqrt{2}\cdot 2\varepsilon_n)\,.$$

Put $z_i^n = \pi_i\circ(f_i+ig_i)$. We note that z_i^n is continuous, $z_i^n(t_i) = \operatorname{sgn}(y_i(t))$ and $\sup\{|z_i^n(s)| : s\in\overline{U}_i\} = 1$. Now define a function $z_n : T\to\mathbb{C}$ by

$$z_n(s) = \begin{cases} x_n(s), & s\in T\setminus\sum_{i\in A}\overline{U}_i\,,\\ z_i^n(s), & s\in\overline{U}_i\,.\end{cases}$$

Since for every $i\in A$ and $s\in\sigma(U_i)$, $z_n(s) = x_n(s)$, z_n is continuous. Moreover, $\|z_n\| = 1$ and $z_n(t_i) = \operatorname{sgn}(y_i(t))$ for $i\in A$, which means that $z_n\in A_t$. To finish the proof, it is sufficient to show that $\|z_n - x_n\| \to 0$. Fix $s \in T$. If $s \in T \setminus \bigcup_{i \in A} U_i$, then $|(x_n - z_n)(s)| = 0$. If $s \in \overline{U}_i$ for some $i \in A$, then

$$|x_n(s) - z_n(s)| \le |x_n(s) - x_n(t_i)| + |x_n(t_i) - \operatorname{sgn}(y_i(t))| + |\operatorname{sgn}(y_i(t)) - z_n(s)| \le (2 + \sqrt{2} \cdot 2)\varepsilon_n .$$

But this gives $\|z_n - x_n\| \to 0$, since $\varepsilon_n \to 0$. The proof is complete.

Now we will prove the main result of this section.

THEOREM 1.3.4. *Let $\mathcal{V} \subset \mathcal{K}_F(B)$ be a convex set. Take $K \in \mathcal{K}_F(B), V \in \mathcal{V}$ and assume $K - V \in \mathcal{D}(B)$. Then we have*:

(a) *$V \in P_{\mathcal{V}}(K)$ if and only if for every $U \in \mathcal{V}$ there exists $\widehat{t} \in \operatorname{crit}^*(K - V)$ such that*

$$\inf\{\operatorname{re}(((U - V)x)t) : x \in A_t\} \le 0 ,$$

where A_t is defined by (1.3.2).

(b) *V is a SUBA to K in $\mathcal{V}$ with a constant $r > 0$ if and only if for every $U \in \mathcal{V}$ there exists $\widehat{t} \in \operatorname{crit}^*(K - V)$ such that*

$$\inf\{\operatorname{re}(((U - V)x)t) : x \in A_t\} \le -r\|U - V\| .$$

P r o o f. (a) Assume that $V \notin P_{\mathcal{V}}(K)$.Then there exists $U \in \mathcal{V}$ with $\|K-U\| < \|K - V\|$. Take $\widehat{t} \in \operatorname{crit}^*(K - V)$ and $x \in A_t$. We note that

$$\operatorname{re}(((U - V)x)t) = \operatorname{re}(((K - V)x)t) - \operatorname{re}(((K - U)x)t) \ge \|K - V\| - \|K - U\| > 0$$

and consequently, $\inf\{\operatorname{re}(((U - V)x)t) : x \in A_t\} > 0$.

To prove the converse, suppose that for some $U \in \mathcal{V}$ and every $\widehat{t} \in \operatorname{crit}^*(K - V)$,

$$\inf\{\operatorname{re}(((U - V)x)t) : x \in A_t\} > 0 .$$

By Theorem 0.2.8, it is sufficient to show that $\operatorname{re}(f(U - V)) > 0$ for every $f \in E(K - V)$ (see (0.2.12)). So fix $f \in E(K - V)$. By Theorem 0.2.7, $f = x^{**} \otimes \widehat{t}$ for some $t \in T$ and $x^{**} \in \operatorname{ext} S(X^{**})$. Applying Goldstine's Theorem we may select a net $\{x_\beta\} \subset S_B$ tending weak* in B^{**} to x^{**}. From (0.2.5) it follows that

$$\|K - V\| \ge \operatorname{re}(((K - V)x_\beta)t) \to \operatorname{re}(((K - V)^{**}x^{**})t) = \operatorname{re}(f(K - V)) = \|K - V\|$$

and consequently, $\widehat{t} \in \operatorname{crit}(K - V)$.

Now set $f_\beta = x_\beta \otimes \widehat{t}$ and observe that for every $W \in \mathcal{K}(B)$

$$f_\beta(W) = \widehat{t}(W(x_\beta)) \to \widehat{t}(W^{**}(x^{**})) = (x^{**} \otimes \widehat{t})(W^{**}) = f(W) .$$

Hence we can select a sequence $\{f_n\} \subset \{f_\beta\}$ ($f_n = x_n \otimes \widehat{t}$) such that

$$f_n(K-V) \to f(K-V) = \|K-V\| \quad \text{and} \quad f_n(U-V) = ((U-V)x_n)t \to f(U-V) .$$

By Lemma 3.3, there exists a sequence $\{z_n\} \subset A_t$ with $\|z_n - x_n\| \to 0$. It is clear that $((U - V)z_n - (U - V)x_n)t \to 0$, which yields $((U - V)z_n)t \to f(U - V)$. Since for $n = 1, 2, \ldots,$ $z_n \in A_t$ and $t \in \operatorname{crit}^*(K - V)$, $\operatorname{re}(f(U - V)) > 0$, which according to Theorem 0.2.8 completes the proof of part (a).

By Remark 0.2.13, part (b) can be shown in the same way.

COROLLARY 1.3.5. *Assume* $\mathcal{V}$, $\mathcal{K}_F(B)$, K, V *are as in Theorem* 3.4. *Assume furthermore that* $\operatorname{card} F < \infty$. *Then we have*:

(a) $V \in P_{\mathcal{V}}(K)$ *if and only if for every* $U \in \mathcal{V}$ *there exists* $\widehat{t} \in \operatorname{crit}^*(K - V)$ *with*

$$\inf\{\operatorname{re}(((U - V)x)t) : x \in A_t\} \le 0 .$$

(b) V *is a SUBA to* K *in* $\mathcal{V}$ *with a constant* $r > 0$ *if and only if for every* $U \in \mathcal{V}$ *there exists* $\widehat{t} \in \operatorname{crit}^*(K - V)$ *with*

$$\inf\{\operatorname{re}(((U - V)x)t) : x \in A_t\} \le -r\|U - V\| .$$

Theorem 3.4 yields immediately the following result:

THEOREM 1.3.6. *Let* $D \subset B$ *be an n-dimensional linear subspace and let* $\mathcal{V} = \mathcal{P}(B, D)$ (see (0.1.10)). *Assume* $P_0 \in \mathcal{P}(B, D) \cap \mathcal{D}(B)$. *Then* P_0 *is an operator of minimal norm in* $\mathcal{P}(B, D)$ (*resp.* P_0 *is a SUBA to* 0 *in* $\mathcal{P}(B, D)$ *with a constant* $r > 0$) *if and only if for every* $P \in \mathcal{P}(B, D)$ *there exists* $\widehat{t} \in \operatorname{crit}^*(P_0)$ *such that*

$$\inf\{\operatorname{re}(((P_0 - P)x)t) : x \in A_t\} \le 0 \quad (\textit{resp. } \le -r\|P - P_0\|) .$$

Proof. Take $K = 0$, $V = P_0$ and note that $\operatorname{crit}^*(P_0) = \operatorname{crit}^*(-P_0)$. By Theorem 3.4, we derive the desired result.

In the next part of this section we apply Theorem 3.4 to obtain some criteria for the case of extensions of operators with finite carriers. First we introduce some notions. Let $D \subset B$ with $\dim D = n$ and let $F = \{t_1, \ldots, t_m\}$ with $t_i \ne t_j$ for $i \ne j, m \ge n + 1$. Assume furthermore that F is total over D, i.e. if $y \in D$ and $y(t_j) = 0$ for $j = 1, \ldots, m$ then $y = 0$. Since $\dim D = n$, we can number the points from F in such a way that $(\widehat{t}_1|_D, \ldots, \widehat{t}_n|_D)$ form a basis of D^*. For $i = n+1, \ldots, m$ put $B_i = \{1, \ldots, n, i\}$ and select for $j \in B_i$ the numbers τ_i^j such that

$$\sum_{j \in B_i} |\tau_i^j| = 1 \quad \text{and} \quad \sum_{j \in B_i} (\tau_i^j \widehat{t}_j)|_D = 0 . \tag{1.3.3}$$

For $A \in \mathcal{L}(D)$ define

$$\mathcal{P}_A(B, D, F) = \{P \in \mathcal{P}_A(B, D) : \operatorname{car}(L) \subset F\} . \tag{1.3.4}$$

Now take $P \in \mathcal{P}_D(B, D, F)$, $P = \sum_{j=1}^m \widehat{t}_j(\cdot)u_j$, where $u_j \in D$ for $j = 1, \ldots, m$. For $i = n + 1, \ldots, m$ define the functions $v_i^P : T \to \mathbb{C}$ by

$$v_i^p(s) = \sum_{j \in B_i} \tau_i^j \operatorname{sgn}(u_j(s)) \tag{1.3.5}$$

and the functionals ϕ_i by

$$\phi_i = \sum_{j \in B_i} \tau_i^j \widehat{t}_j . \tag{1.3.6}$$

Then we can prove the following

THEOREM 1.3.7. (a) *P is not an operator of minimal norm in $\mathcal{P}_A(B,D,F)$ if and only if for every $i \in \{n+1,\dots,m\}$ there exist $y_i \in D$ such that for every $\widehat{t} \in \mathrm{crit}^*(P)$,*

$$\text{(1.3.7)}\qquad \mathrm{re}\Big(\sum_{i=n+1}^{m} v_i^P(s)y_i(s) - \sum_{j\in B_s^P}\Big|\sum_{i=n+1}^{m}\tau_i^j y_i(s)\Big| - \sum_{j\in C_s^P}|y_j(s)\tau_j^j|\Big) > 0$$

where $B_s^P = \{j\in\{1,\dots,n\} : u_j(s)=0\}$, $C_s^P=\{j\in\{n+1,\dots,m\} : u_j(s)=0\}$, $\sum_{j\in B_s^P}=0$ (resp. $\sum_{j\in C_s^P}=0$) if $B_s^P=\emptyset$ (resp. $C_s^P=\emptyset$).

(b) *P is not a SUBA to 0 in $\mathcal{P}_A(B,D,F)$ with a constant $r>0$ if and only if for every $i=n+1,\dots,m$ there exists $y_i\in D$ such that for every $\widehat{t}\in\mathrm{crit}^*(P)$,*

$$\text{(1.3.8)}\quad \mathrm{re}\Big(\sum_{i=n+1}^{m} v_i^P(s)y_i(s) - \sum_{j\in B_s^P}\Big|\sum_{i=n+1}^{m}\tau_i^j y_i(s)\Big| - \sum_{j\in C_s^P}|\tau_j^j\cdot y_j(s)|\Big) > -r\|L\|\,,$$

where $L=\sum_{i=n+1}^{m}\varphi_i(\cdot)y_i$.

Proof. (a) Assume that condition (1.3.7) is satisfied and let $L = \sum_{i=n+1}^{m}\varphi_i(\cdot)y_i$. To prove that P is not an operator of minimal norm in $\mathcal{P}_A(B,D,F)$, in view of Corollary 3.5, it is sufficient to show that for each $\widehat{s}\in\mathrm{crit}^*(P)$,

$$\inf\{\mathrm{re}((Lx)s) : x\in A_s\} > 0\,.$$

For $i=n+1,\dots,m$ set $D_i=\{j\in B_i : u_j(s)\neq 0\}$ and $E_i=B_i\setminus D_i$. Fix $s\in\mathrm{crit}^*(P)$, $x\in A_s$ and compute

$$\begin{aligned}
(Lx)s &= \sum_{i=n+1}^{m}\varphi_i(x)y_i = \sum_{i=n+1}^{m}\Big(\sum_{j=1}^{n}\tau_i^j x(t_j) + \tau_i^i x(t_i)\Big)\cdot y_i\\
&= \sum_{i=n+1}^{m}\Big(\sum_{j\in D_i}\tau_i^j\,\mathrm{sgn}(u_j(s)) - \sum_{j\in E_i}\tau_i^j(-x(t_j))\Big)\cdot y_i\\
&= \sum_{i=n+1}^{m} v_i^P(s)y_i(s) - \sum_{i=n+1}^{m}\Big(\sum_{j\in E_i}\tau_i^j(-x(t_j)y_i(s))\Big)\\
&= \sum_{i=n+1}^{m} v_i^P(s)y_i(s) - \sum_{j\in B_s^P}\Big(\sum_{i=n+1}^{m}\tau_i^j y_i(s)\Big)(-x(t_j)) - \sum_{j\in C_s^P}\tau_j^j y_j(s)(-x(t_j)).
\end{aligned}$$

Consequently, since $\|x\|\le 1$, we get

$$\mathrm{re}((Lx)s) \ge \mathrm{re}\Big(\sum_{i=n+1}^{m} v_i^P(s)y_i(s) - \sum_{j\in B_s^P}\Big|\sum_{i=n+1}^{m}\tau_i^j y_i(s)\Big| - \sum_{j\in C_s^P}|\tau_j^j y_j(s)|\Big) > 0\,.$$

By Corollary 3.5, P is not an operator of minimal norm in $\mathcal{P}_A(B,D,F)$.

To prove the converse, assume that P is not an operator of minimal norm in $\mathcal{P}_A(B,D,F)$ and choose $P_0\in\mathcal{P}_A(B,D,F)$ with $\|P_0\|<\|P\|$. By [Che7], we can assume

$$P_0 = P + \sum_{i=n+1}^{m} \phi_i(\cdot) y_i \quad \text{for some } y_{n+1}, \ldots, y_m \in D\,. \tag{1.3.9}$$

We show that the functions $y_{n+1}, \ldots, y_m$ satisfy (1.3.7). Fix $\widehat{t} \in \mathrm{crit}^*(P)$. By the Tietze–Urysohn Theorem we can define a function $x \in S_D$ with the properties

$$x(t_j) = \begin{cases} \mathrm{sgn}(u_j(s)), & u_j(s) \neq 0\,, \\ -\,\mathrm{sgn}(\sum_{i=n+1}^{m} \tau_i^j y_i(s)), & u_j(s) = 0\,, \end{cases} \quad \text{for } j = 1, \ldots, n\,,$$

and

$$x(t_j) = \begin{cases} \mathrm{sgn}(u_j(s)), & u_j(s) \neq 0\,, \\ -\,\mathrm{sgn}(\tau_j^j y_j(s)), & u_j(s) = 0\,, \end{cases} \quad \text{for } j = n+1, \ldots, m\,.$$

Observe that

$$(Px)s = \sum_{j=1}^{m} x(t_j) u_j(s) = \sum_{j \in B_s^P \cup C_s^P} x(t_j) u_j(s) = \sum_{j=1}^{m} |u_j(s)| = \|P\|\,.$$

Calculating as in the previous part of the proof we obtain

$$((P_0 - P)x)s = \sum_{i=n+1}^{m} v_i^P(s) y_i(s) - \sum_{j \in B_s^P} \Big| \sum_{i=n+1}^{m} \tau_i^j y_i(s) \Big| - \sum_{j \in C_s^P} |\tau_j^j y_j(s)|\,.$$

Since $\mathrm{re}(((P_0 - P)x)s) > 0$, by Corollary 3.5, the proof of (a) is complete.

The same reasoning applies to (b), so we omit it.

Observe that in the real case if $m = n+1$ condition (1.3.5) reduces to

$$|y_{n+1}(s)| \Big(v_{n+1}^P(s)\, \mathrm{sgn}(y_{n+1}(s)) - \sum_{j \in B_s^P \cup C_s^P} |\tau_{n+1}^j| \Big) > 0 \tag{1.3.10}$$

Now we specialize our results to the case of D being an n-dimensional Haar subspace of $C_{\mathbb{R}}(T)$ and $F = (t_1, \ldots, t_{n+1})$. For $A \in \mathcal{L}(D)$ define

$$E_A = \sup\{\|Ad\| : d \in D,\ \|d\|_F \leq 1\} \tag{1.3.11}$$

where

$$\|d\|_F = \max_{k=1,\ldots,n+1} |d(t_k)| \tag{1.3.12}$$

and for $P \in \mathcal{P}_A(B, D, F)$, $s \in T$ put

$$u^P(s) = \sum_{j \in B_s^P \cup C_s^P} |\tau_{n+1}^j|\,. \tag{1.3.13}$$

For brevity, we will write Φ instead of φ_{n+1} and v^P instead of v_{n+1}^P. For $s \in T$ define $Z_s^P = B_s^P \cup C_s^P$.

Now we prove some preliminary lemmas.

Lemma 1.3.8. *$\|P\| \geq E_A$ for every $P \in \mathcal{P}_A(B, D, F)$.*

Proof. Take any $P \in \mathcal{P}_A(B,D,F)$. Since F is a finite set, by the Tietze–Urysohn Theorem,

$$\|P\| = \sup_{\|x\|_F \le 1} \|Px\| \ge \sup_{\|x\|_F \le 1, x \in D} \|Px\| = E_A\,,$$

as required.

Lemma 1.3.9. *Let $P \in \mathcal{P}_A(B,D,F)$, $\widehat{s} \in \mathrm{crit}^*(P)$, $y \in A_s$ (see (1.3.2)). If $|v^P(s)| > u^P(s)$, then $\Phi(y)v^P(s) > 0$.*

Proof. By the definition of u^P, v^P and Φ, for every $s \in \mathrm{crit}^*(P)$ and $y \in A_s$,

$$v^P(s) - u^P(s) \le \Phi(y) \le v^P(s) + u^P(s)\,.$$

Since $|v^P(s)| > u^P(s)$, $\mathrm{sgn}(v^P(s) \pm u^P(s)) = \mathrm{sgn}(\Phi(y)) = \mathrm{sgn}(v^P(s))$, which gives the result.

Lemma 1.3.10. *If $\|P\| > E_A$ for $P \in \mathcal{P}_A(B,D,F)$ then $|v^P(s)| > u^P(s)$ for every $\widehat{s} \in \mathrm{crit}^*(P)$.*

Proof. Suppose that there is an $s \in \mathrm{crit}^*(P)$ such that $|v^P(s)| \le u^P(s)$. Then, by the definitions of v^P, u^P and Φ, we can define $x_1, x_2 \in A_s$ such that $\Phi(x_1) = v^P(s) + u^P(s)$ and $\Phi(x_2) = v^P(s) - u^P(s)$. Hence there is a $w \in A_s$ (A_s is a convex set) such that $\Phi(w) = 0$. By (1.3.9) each $P \in \mathcal{P}_A(B,D,F)$ can be written as

$$P = P_D^A + \Phi(\cdot)d$$

for some $d \in D$. Here $P_D^A = A \circ P_D$, where P_D is the operator of de la Vallée-Poussin [Che7]. By the proof of [Che7, Lemma 1.1], $\|w - P_D w\|_F \le |\Phi(w)| = 0$. Consequently, $1 \ge \|w\|_F = \|P_D w\|_F$. Hence

$$\|P\| = \|Pw\| = \|P_D^A w\| = \|(A \circ P_D)w)\| \le E_A\,;$$

a contradiction.

Now we can prove the following

Theorem 1.3.11. *Let $D \in B$ be an n-dimensional Haar subspace. If P_0 is an operator of minimal norm in $\mathcal{P}_A(B,D,F)$ and $\|P_0\| > E_A$, then P_0 is a SUBA to* 0 *in $\mathcal{P}_A(B,D,F)$.*

Proof. Put $\mathcal{V} = \{L \in \mathcal{L}(B,D) : L = \Phi(\cdot)d,\ d \in D\}$. It is easily seen that $\dim \mathcal{V} = n$. Since $\mathcal{P}_0$ is an operator of minimal norm in $\mathcal{P}_D^A(B,D,F)$, $0 \in \mathcal{P}_{\mathcal{V}}(P_0)$. Hence, by Corollaries 0.2.14 and 3.5,

$$0 = \sum_{l=1}^{k} \lambda_l (s_l \otimes z_l)|_{\mathcal{V}}\,,$$

where $\lambda_l > 0$, $\sum_{l=1}^k \lambda_l = 1$. By the Carathéodory Theorem, $k \le n+1$. We show that $k = n+1$. Suppose that this is not true. Let $d_1, \ldots, d_n$ be a basis of D. It is clear that $\Phi(\cdot)d_1, \ldots, \Phi(\cdot)d_n$ is a basis of $\mathcal{V}$. Note that $(\lambda_1, \ldots, \lambda_k)$ satisfy the following system of equalities:

$$\sum_{j=1}^{k} \lambda_j \Phi(z_j) d_i(s_j) = 0, \quad i = 1, \ldots, n.$$

Since $\|P_0\| > E_A$, by Lemma 3.10 it follows that $|v^P(s)| > u^P(s)$ for any $s \in \mathrm{crit}^*(P_0)$. By Lemma 3.9, $\Phi(z) \neq 0$ for any $z \in \bigcup_{s \in \mathrm{crit}^*(P_0)} A_s$. Consequently, $\det[d_i(s_j)]_{i,j=1,\ldots,n} = 0$ (if $k < n$ we add $n-k$ arbitrary different points from T); a contradiction . Hence for any $L = \Phi(\cdot)d \in \mathcal{V}$ there is an $i \in \{1, \ldots, n+1\}$ with $\Phi(z_i)d(s_i) < 0$. Since $\mathcal{V}$ is finite-dimensional, reasoning as in Theorem 1.1.4 we deduce that there is an $r > 0$ such that for any $L \in \mathcal{V}$ there are $s \in \mathrm{crit}^*(P_0)$ and $z \in A_s$ with $(Lz)s < -r\|L\|$. By Corollary 3.5, P_0 is a SUBA for 0 in $\mathcal{V}$, as required.

THEOREM 1.3.12. *If $\|P_0 z\| < E_A$ for any z with $\Phi(z) \neq 0$ then P_0 is an operator of minimal norm in $\mathcal{P}_A(B, D, F)$ but not unique.*

Proof. Since P_0 is a discrete operator, $\|P_0\| = \max\{\|Pz\| : z \in Z\}$ where Z is a fixed finite subset of S_B such that for any $\varepsilon \in \{-1, 1\}^{n+1}$ there is a $z \in Z$ with $z(t_i) = \varepsilon_i$ ($i \in \{1, \ldots, n+1\}$). Note that the number $\delta = E_A - \max\{\|P_0 z\| : z \in Z,\ \Phi(z) \neq 0\}$ is positive. By the proof of [Che7, Lemma 1.1] if $z \in Z$ and $\Phi(z) = 0$ then $(P_0 z)t_i = z(t_i)$ for $i = 1, \ldots, n$. Hence, by Lemma 3.8,

$$\begin{aligned} E_A \le \|P_0\| &= \max\{\|P_0 z\| : z \in Z, \Phi(z) = 0\} \\ &\le \max\{\|P_0 z\| : z \in Z,\ (P_0 z)t_i = z(t_i),\ i = 1, \ldots, n+1\} \\ &= \max\{\|Ad\| : d \in D,\ \|d\|_F = 1\} = E_A. \end{aligned}$$

Now if $\|d\| < \delta$ and $\Phi(z) \neq 0$, then

$$\|P_0 z + \Phi(z)d\| \le \|P_0(z)\| + \|d\| \le (E_A - \delta) + \delta = E_A.$$

If $\Phi(z) = 0$ then $\|P_0 + \Phi(z)d\| = \|P_0 z\| \le E_A$. Consequently, for every $d \in B_D(0, \delta)$, $\|P_0 + \Phi(z)d\| \le E_A$. By Lemma 3.8, the proof is complete.

Now we restrict ourselves to the case $T = [a, b]$ and prove a version of the alternation theorem. First we recall

LEMMA 1.3.13 [Che7]. *Assume $F \subset [a, b], F$ is closed and let $D \subset B$ be an n-dimensional Haar subspace. Let $\gamma : F \to \mathbb{R}$ be a function which has no zeros on F and such that* $\operatorname{sgn}\gamma$ *is continuous. If no $y \in D$ has the property $\gamma \cdot y|_F > 0$ then there exist $n+1$ points $t_1, \ldots, t_{n+1}$ in F such that $t_1 < \ldots < t_{n+1}$ and $\gamma(t_{i-1})\gamma(t_i) < 0$ for $i = 2, \ldots, n+1$.*

Now we are able to prove

THEOREM 1.3.14. *In order that P be an operator of minimal norm in the set $\mathcal{P}_A(B, D, F)$ it is necessary and sufficient that either*

(a) $\|P\| = E_A$, *or*

(b) *there exist* $s_1 < \ldots < s_{n+1}$ *with* $\widehat{s}_i \in \operatorname{crit}^*(P)$ *for* $i = 1, \ldots, n+1$ *such that*

$$\operatorname{sgn}(v^P(s_i)) = -\operatorname{sgn}(v^P(s_{i-1})) \quad (2 \le i \le n+1).$$

Proof. First suppose that (a) holds. By Lemma 3.8, P is an operator of minimal norm $\mathcal{P}_A(B, D, F)$.

Next suppose that (b) holds. Then no element $y \in D$ can have the property $v^P(s)\operatorname{sgn}(y(s)) > u^P(s)$ for $\widehat{t} \in \operatorname{crit}^*(P)$, for this inequality would require y to have at least n roots. Hence, by (1.3.10) and Theorem 3.4, P is an operator of minimal norm in $\mathcal{P}_A(B, D, F)$.

Finally, suppose that P is an operator of minimal norm in $\mathcal{P}_A(B, D, F)$ and (a) is not true. Then $\|P\| > E_A$. By Theorem 3.4 and (1.3.10), no $y \in D$ satisfies $v^P(s)\operatorname{sgn}(y(s)) > u^P(s)$ for any $\widehat{s} \in \operatorname{crit}^*(P) = S$. By Lemma 3.10, $|v^P| > u^P$ on S. Hence no y in D can satisfy the inequality $yv^P > 0$ on S. Since $u^P \ge 0$, v^P does not vanish on S. Now we verify that $\operatorname{sgn} v^P$ is continuous on S. If $\operatorname{sgn} v^P$ is discontinuous on S then consider the two sets

$$S_1 = \{t \in S : v^P(s) > 0\}, \quad S_2 = \{t \in S : v^P(s) < 0\}.$$

One of them must contain an accumulation point of the other. But this is not possible, for as we now show, S_1 and S_2 are closed. Consider, for example, S_1. For each $\varepsilon \in \{-1, 1\}^n$ and $t \in S_1$ select $z_\varepsilon \in Z$ (the set Z is as in Theorem 3.12) with

$$(1.3.14) \qquad (\widehat{t} \circ P)z_\varepsilon = \|P\| \quad \text{and} \quad \sum_{i=1}^{n+1} z_\varepsilon(t_i)\tau_i^{n+1} = v^P(t) > 0.$$

Note that for each $z \in Z$ the set of all $t \in [a, b]$ satisfying (1.3.14) is closed. Since S_1 is the union of such sets and the set Z is finite, S_1 is also closed; a contradiction with discontinuity of $\operatorname{sgn} v^P$. By Lemma 3.13, the proof is complete.

Remark 1.3.15. Theorems 3.4 and 3.7 were proved by the author in [LG1]. For the case of projections and $m = n + 1$ a result similar to Theorem 3.7 was obtained in [Che7]. Note that Theorem 3.6 extends Theorem 0.0.1 of [Che2], which is a background for the investigations presented in this section. The method of proving Theorems 3.11, 3.12 and 3.14 is the same as in [Che7].

I.4. The case of sequence spaces. First we present basic terminology conerning generalized sequence spaces. For an arbitrary set T we denote by $c_0(T)$, or c_0 for brevity, the space of all functions $x : T \to K$ such that the set $\{t \in T : |x(t)| > \varepsilon\}$ is finite for all $\varepsilon > 0$. The norm in c_0 is $\|x\|_\infty = \sup\{|x(t)| : t \in T\}$. The space $l_1(T)$ consists of all functions $x : T \to K$ which are zero except on a countable set in T and for which $\|x\|_1 = \sum_{t \in T} |x(t)| < \infty$. It is well known that the conjugate space of c_0 can be isometrically identified with $l_1(T)$ (written l_1 for brevity), and the conjugate space of l_1 with l_∞, where

$$(1.4.1) \qquad l_\infty = \{x : T \to K : \sup\{|x(t)| : t \in T\} < \infty\}.$$

We note that

$$\mathrm{ext}_{l_1} = \{\alpha f_t : t \in T,\ \alpha \in K,\ |\alpha| = 1\}\,, \tag{1.4.2}$$

where

$$f_t(s) = \begin{cases} 0 & \text{if } s \neq t\,, \\ 1 & \text{if } s = t\,, \end{cases}$$

and

$$\mathrm{ext}_{l_\infty} = \{f : T \to K : |f(t)| = 1 \text{ for every } t \in T\}\,. \tag{1.4.3}$$

By [Du, Th. 18, p. 274], the set $\mathrm{ext}_{l_\infty^*}$ has the following representation:

$$\mathrm{ext}_{l_\infty^*} = \mathrm{cl}\{\widehat{t} : t \in T\}\,, \tag{1.4.4}$$

where $\widehat{t}(f) = f(t)$ for every $f \in l_\infty$ and the closure is taken with respect to the weak* topology in l_∞^*.

Now assume $D \subset c_0$ is an n-dimensional subspace and let $y_1, \ldots, y_n$ be a basis of D. For $K \in \mathcal{K}(c_0, D)$, $K = \sum_{i=1}^n f_i(\cdot) y_i$ $(f_i \in l_1)$, put

$$K_K(s,t) = \sum_{i=1}^n f_i(s) y_i(t) \quad \text{for } s, t \in T\,. \tag{1.4.5}$$

As in the previous section for $t \in T$ the symbol $\widehat{t}$ stands for the evaluation functional. By [Che3, Lemma 1], $\widehat{t} \in \mathrm{crit}^*(K)$ if and only if t is a critical point of the function $\varLambda_K : T \to \mathbb{R}_+$ defined by

$$\varLambda_K(s) = \Big\| \sum_{i=1}^n y_i(s) f_i \Big\|_1 = \sum_{u \in T} |K_K(u,s)|$$

i.e.

$$\varLambda_K(t) = \sup\{\varLambda_K(s) : s \in T\} \tag{1.4.6}$$

(the symbol $\|\cdot\|_1$ denotes the norm in the space l_1).

Using these notations we may prove the following

THEOREM 1.4.1. *Let $\mathcal{V} \subset \mathcal{K}(c_0, D)$ be a convex set and let $K \in \mathcal{K}(c_0, D)$, $V \in \mathcal{V}$. Then we have*:

(a) *$V \in P_{\mathcal{V}}(K)$ if and only if for every $U \in \mathcal{V}$ there exists $\widehat{t} \in \mathrm{crit}^*(K - V)$ with*

$$\mathrm{re}\Big(\sum_{s \in T} K_{U-V}(s,t)\,\mathrm{sgn}(K_{K-V}(s,t)) - \sum_{s \in A_t} |K_{U-V}(s,t)| \Big) \le 0\,. \tag{1.4.7}$$

(b) *$V \in \mathcal{V}$ is a SUBA to K in $\mathcal{V}$ with a constant $r > 0$ if and only if for every $U \in \mathcal{V}$ there exists $\widehat{t} \in \mathrm{crit}^*(K - V)$ such that*

$$\mathrm{re}\Big(\sum_{s \in T} K_{U-V}(s,t)\,\mathrm{sgn}(K_{K-V}(s,t)) - \sum_{s \in A_t} |K_{U-V}(s,t)| \Big) \le -r\|U - V\|\,, \tag{1.4.8}$$

where $A_t = \{s \in T : K_{K-V}(s,t) = 0\}$.

Proof. Assume there exists $U \in \mathcal{V}$ such that for every $\widehat{t} \in \operatorname{crit}^*(K - V)$, (1.4.7) does not hold. In view of Theorem 0.2.8, it is sufficient to show that $\operatorname{re}(\phi(U - V)) > 0$ for every $\phi \in E(K - V)$ (see (0.2.12)). Since $\mathcal{K}(c_0, D) \subset \mathcal{K}(c_0)$, by Theorem 0.2.7 we have $\phi = \psi \otimes \gamma$ for some $\psi \in \operatorname{ext}_{c_0^{**}}$ and $\gamma \in \operatorname{ext}_{c_0^*}$. Applying (1.4.1) and (1.4.2), we can assume that $\psi \in l_\infty$, $|\psi(s)| = 1$ for every $s \in T$ and $\gamma = \widehat{t}$ for some $t \in$ T. Let

$$K - V = \sum_{i=1}^{n} f_i(\cdot) y_i$$

and

$$U - V = \sum_{i=1}^{n} g_i(\cdot) y_i \quad \text{for some } f_i, g_i \in l_1 .$$

In view of Remark 0.2.5 and (1.4.6), we note that

$$\begin{aligned} \|K - V\| &= \phi(K - V) = \widehat{t}(K - V)^{**}\psi = \sum_{i=1}^{n} \psi(f_i) y_i(t) \\ &= \sum_{i=1}^{n} \Big(\sum_{s \in T} f_i(s)\psi(s) \Big) y_i(t) = \sum_{s \in T} \psi(s) \Big(\sum_{i=1}^{n} f_i(s) y_i(t) \Big) \\ &\le \sum_{s \in T} |K_{K-V}(s,t)| = \|K - V\| . \end{aligned}$$

This means that $\psi(s) = \operatorname{sgn}(K_{K-V}(s,t))$ if $s \in T \setminus A_t$. Note that

$$\begin{aligned} \operatorname{re}(\phi(U - V)) &= \operatorname{re}\Big(\sum_{i=1}^{n} \psi(g_i) y_i(t) \Big) = \operatorname{re}\Big(\sum_{i=1}^{n} \Big(\sum_{s \in T} \psi(s) g_i(s) \Big) y_i(t) \Big) \\ &= \operatorname{re}\Big(\sum_{s \in T} \psi(s) \Big(\sum_{i=1}^{n} g_i(s) y_i(t) \Big) \Big) = \operatorname{re}\Big(\sum_{s \in T} \psi(s) K_{U-V}(s,t) \Big) \\ &= \operatorname{re}\Big(\sum_{s \in T} K_{U-V}(s,t) \cdot \operatorname{sgn}(K_{K-V}(s,t)) \\ &\quad - \sum_{s \in A_t} (-\psi(s)) K_{U-V}(s,t) \Big) . \end{aligned}$$

Since

$$\Big| \operatorname{re}\Big(\sum_{s \in A_t} (-\psi(s) K_{U-V}(s,t)) \Big) \Big| \le \sum_{s \in A_t} |K_{U-V}(s,t)| ,$$

we get

$$\operatorname{re}(\phi(U - V)) \ge \operatorname{re}\Big(\sum_{s \in T} K_{U-V}(s,t) \operatorname{sgn}(K_{K-V}(s,t)) - \sum_{s \in A_t} |K_{U-V}(s,t)| \Big) > 0 .$$

By Theorem 0.2.8, $V \notin P_{\mathcal{V}}(K)$.

To prove the converse, suppose $V \notin P_{\mathcal{V}}(K)$ and choose a $U \in \mathcal{V}$ with $\|U-K\| < \|V-K\|$. Fix $\widehat{t} \in \mathrm{crit}^*(K-U)$. Define a function $\psi \in l_\infty$ by

$$\psi(s) = \begin{cases} \mathrm{sgn}(K_{K-V}(s,t)), & K_{K-V}(s,t) \neq 0\,, \\ -\,\mathrm{sgn}(K_{U-V}(s,t)), & K_{K-V}(s,t) = 0, K_{U-V}(s,t) \neq 0\,, \\ 1, & \text{otherwise}\,. \end{cases}$$

Set $\phi = \psi \otimes \widehat{t}$. By [Li], $\phi \in \mathrm{ext}_{\mathcal{K}^*(c_0)}$. Observe that

$$\begin{aligned} \phi(K-V) &= \sum_{i=1}^n \psi(f_i) y_i(t) = \sum_{i=1}^n \Big(\sum_{s\in T} \psi(s) f_i(s)\Big) y_i(t) \\ &= \sum_{s\in T} \psi(s) \Big(\sum_{i=1}^n f_i(s) y_i(t)\Big) = \sum_{s\in T} |K_{K-V}(s,t)| = \|K-V\|\,. \end{aligned}$$

Hence $\phi \in E(K-V)$ and consequently, $\mathrm{re}(\phi(U-V)) > 0$. But

$$\begin{aligned} \mathrm{re}(\phi(U-V)) &= \mathrm{re}\Big(\sum_{s\in T} \psi(s)\cdot K_{U-V}(s,t)\Big) \\ &= \mathrm{re}\Big(\sum_{s\in T} K_{U-V}(s,t)\,\mathrm{sgn}(K_{K-V}(s,t)) - \sum_{s\in A_t} |K_{U-V}(s,t)|\Big)\,, \end{aligned}$$

which gives the desired result.

In view of Remark 0.2.13, part (b) can be proved in the same way.

Now we present a similar result for the space $\mathcal{K}(l_1, D)$. To do this, for $K \in \mathcal{K}(l_1, D)$, $K = \sum_{i=1}^n f_i(\cdot) y_i$, where $f_i \in l_\infty$ for $i = 1, \ldots, n$ and $y_1, \ldots, y_n$ is a fixed basis of D, put

$$K_K(\psi, t) = \sum_{i=1}^n \psi(f_i) y_i(t), \qquad \psi \in l_1^{**}, t \in T\,. \tag{1.4.9}$$

By the Banach–Alaoglu and Krein–Milman Theorems, and by the definition of the space $\mathcal{L}_e(l_1^{**}, D)$ (see Proposition 0.2.4), we note that the set

$$C_K = \{\psi \in \mathrm{ext}_{l_1^{**}} : K^{**}(\psi) = \|K\|\} \tag{1.4.10}$$

is nonvoid. Moreover,

$$\psi \in C_K \quad \text{if and only if} \quad \sum_{t\in T} |K_K(\psi, t)| = \|K\|\,. \tag{1.4.11}$$

Using the above notations we can prove

THEOREM 1.4.2. *Let $\mathcal{V} \subset \mathcal{K}(l_1, D)$ be a convex set and let $K \in \mathcal{K}(l_1, D)$, $V \in \mathcal{V}$. Then we have*:

(a) *$V \in P_{\mathcal{V}}(K)$ if and only if for every $U \in \mathcal{V}$ there exists $\psi \in C_{K-V}$ such* *that*

$$\mathrm{re}\Big(\sum_{t\in T} K_{U-V}(\psi,t)\,\mathrm{sgn}(K_{K-V}(\psi,t)) - \sum_{t\in A_\psi} |K_{U-V}(\psi,t)|\Big) \le 0\,. \tag{1.4.12}$$

(b) *V is a SUBA to K in $\mathcal{V}$ with a constant $r > 0$ if and only if for every $U \in \mathcal{V}$ there exists $\psi \in C_{K-V}$ with*

$$\mathrm{re}\Big(\sum_{t\in T} K_{U-V}(\psi,t)\,\mathrm{sgn}(K_{K-V}(\psi,t)) - \sum_{t\in A_\psi} |K_{U-V}(\psi,t)|\Big) \le -r\|U-V\|\,, \tag{1.4.13}$$

where $A_\psi = \{t \in T : K_{K-V}(\psi,t) = 0\}$.

Proof. (a) Fix $K \in \mathcal{K}(l_1, Y)$ and $V \in P_{\mathcal{V}}(K)$. Let $K-V = \sum_{i=1}^n f_i(\cdot)y_i$. Assume that for some $U \in \mathcal{V}$, (1.4.12) is not satisfied. Suppose $U-V = \sum_{i=1}^n g_i(\cdot)y_i$ and take $\phi \in E(K-V)$. We show that $\mathrm{re}(\phi(U-V)) > 0$. To do this, we note that by Theorem 0.2.7, $\phi = \psi\otimes\gamma$, where $\psi \in \mathrm{ext}_{l_1^{**}}$ and $\gamma \in \mathrm{ext}_{l_1^*}$. By (1.4.2), we can assume that $\gamma \in S_{l_\infty}$ and $|\gamma(t)| = 1$ for every $t \in T$. Observe that

$$\|K-V\| = \phi(K-V) = \gamma((K-V)^{**}\psi) = \gamma\Big(\sum_{i=1}^n \psi(f_i)\cdot y_i\Big)$$
$$= \sum_{t\in T}\gamma(t)K_{K-V}(\psi,t) \le \sum_{t\in T}|K_{K-V}(\psi,t)| \le \|K-V\|\,.$$

By (1.4.11), $\psi \in C_{K-V}$. Hence $\gamma(t) = \mathrm{sgn}(K_{K-V}(\psi,t))$ if $t \in T\setminus A_\psi$. Note that

$$\mathrm{re}(\phi(U-V)) = \mathrm{re}\Big(\gamma\Big(\sum_{i=1}^n\psi(g_i)y_i\Big)\Big) = \mathrm{re}\Big(\sum_{t\in T}\gamma(t)\cdot K_{U-V}(\psi,t)\Big)$$
$$= \mathrm{re}\Big(\sum_{t\in T}K_{U-V}(\psi,t)\,\mathrm{sgn}(K_{K-V}(\psi,t)) - \sum_{t\in A_\psi}K_{U-V}(\psi,t)(-\gamma(t))\Big)$$
$$\ge \mathrm{re}\Big(\sum_{t\in T}K_{U-V}(\psi,t)\,\mathrm{sgn}(K_{K-V}(\psi,t)) - \sum_{t\in A_\psi}|K_{U-V}(\psi,t)|\Big) > 0\,.$$

By Theorem 0.2.8, $V \notin P_{\mathcal{V}}(K)$.

Now suppose $V \notin P_{\mathcal{V}}(K)$ and take $U \in \mathcal{V}$ with $\|K-U\| < \|K-V\|$. Choose $\psi \in C_{K-V}$ and define $\gamma \in \mathrm{ext}_{l_\infty}$ by

$$\gamma(t) = \begin{cases} \mathrm{sgn}(K_{K-V}(\psi,t)), & K_{K-V}(\psi,t) \ne 0\,,\\ -\,\mathrm{sgn}(K_{U-V}(\psi,t)), & K_{K-V}(\psi,t) = 0 \text{ and } K_{U-V}(\psi,t) \ne 0\,,\\ 1 & \text{otherwise}\,. \end{cases}$$

Let $\phi = \psi\otimes\gamma$. In view of [Li], $\phi \in \mathrm{ext}_{\mathcal{K}^*(l_1)}$. Observe that, by Proposition 0.2.4

and (1.4.11),

$$\phi(K-V)=\gamma\Big(\sum_{i=1}^{n}\psi(f_i)y_i\Big)=\sum_{t\in T}\gamma(t)K_{K-V}(\psi,t)$$
$$=\sum_{t\in T\setminus A_\psi}|K_{K-V}(\psi,t)|=\|K-V\|\,.$$

Hence, by Theorem 0.2.8, $\mathrm{re}(\phi(U-V))>0$. But

$$\mathrm{re}(\phi(U-V))=\mathrm{re}\Big(\sum_{t\in T}\gamma(t)K_{U-V}(\psi,t)\Big)$$
$$=\mathrm{re}\Big(\sum_{t\in T}K_{U-V}(\psi,t)\,\mathrm{sgn}\,K_{K-V}(\psi,t)-\sum_{t\in A_\psi}|K_{U-V}(\psi,t)|\Big)\,,$$

which gives the desired result.

By (1.4.11), [Che3, Th. 4] and a reasoning similar to that in Theorem 4.2, we can prove

THEOREM 1.4.3. *Let* $\mathcal{V}=\mathcal{P}_A(l_1,D)$ *where* $A\in\mathcal{L}(D)$ *and* $K=0$. *Assume furthermore that* $\dim(D|_B)=\dim D$ *for every infinite set* $B\subset\{t\in T:y(t)\neq 0$ *for some* $y\in D\}$. *Then we have*:

(a) $V\in\mathcal{V}$ *is an operator of minimal norm in* $\mathcal{V}$ *if and only if for every* $U\in\mathcal{V}$ *there exists* $s\in T$ *with* $\widehat{s}\in C_{K-V}$ (compare with (1.4.11)) *such that*

$$\mathrm{re}\Big(\sum_{t\in T}K_{V-U}(\widehat{s},t)\,\mathrm{sgn}(K_{K-V}(\widehat{s},t))-\sum_{t\in A_s}|K_{V-U}(\widehat{s},t)|\Big)\le 0\,.$$

(b) $V\in\mathcal{V}$ *is a SUBA to* K *in* $\mathcal{V}$ *with a constant* $r>0$ *if and only if for every* $U\in\mathcal{V}$ *there exists* $s\in T$ *with* $\widehat{s}\in C_{K-V}$ *such that*

$$\mathrm{re}\Big(\sum_{t\in T}K_{V-U}(\widehat{s},t)\,\mathrm{sgn}(K_{K-V}(\widehat{s},t))-\sum_{t\in A_s}|K_{V-U}(\widehat{s},t)|\Big)\le -r\|U-V\|\,.$$

Remark 1.4.4. Theorems 4.1–4.3 were proved by the author in [LG1]. For the case of projections (see (0.1.10) part (a) of Theorems 4.1 and 4.3 (in the real case) has been proved (by a different method) in [Che3, Th. 1 and 5].

Chapter II

II.1. Extensions of linear operators from hyperplanes of $l_\infty^{(n)}$. In this section, unless otherwise stated, $B=l_\infty^{(n)}$ (we consider the real case) and $D\subset B$ be a hyperplane in B, i.e. $D=\ker(f)$ for some $f\in S_{B^*}$. (The symbol $\ker(f)$ denotes the kernel of the functional f.) We present a complete characterization of those $A\in\mathcal{L}(D)$ for which $\lambda_A(B,D)=\|A\|$ (see (0.1.20)). Next we consider the case

$\lambda_A(B,D) > \|A\|$. For brevity, for given $A \in \mathcal{L}(D)$ we will write $\mathcal{L}_D$, $\mathcal{P}_A$, λ_A, $\mathcal{P}_A^0$ instead of $\mathcal{L}_D(B,D)$ (see (0.1.8)), $\mathcal{P}_A(B,D)$ (see (0.1.17)), $\lambda_A(B,D)$, $\mathcal{P}_A^0(B,D)$ (see (0.1.19)). Throughout this section we assume that $f = (f_1,\dots,f_n) \in S_{B^*}$ is such that $|f_i| < 1/2$ for $i = 1,\dots,n$. By [Bl, Th. 1] this condition is equivalent to $\lambda_{\mathrm{Id}} > 1$. If $\lambda_{\mathrm{Id}} = 1$ and $P_0 \in \mathcal{P}^0_{\mathrm{Id}}$, then for any $A \in \mathcal{L}(D)$, $\|A \circ P_0\| = \|A\|$ and consequently, $A \circ P_0 \in \mathcal{P}_A^0$. We start with the following

DEFINITION 2.1.1 [SW]. Let $b \in B$. A set $U = \{g_1,\dots,g_k\} \subset E(b)$ is called an *I-set* if and only if

$$0 = \sum_{i=1}^{k} \lambda_i g_i|_D, \qquad \lambda_i > 0,\ \sum_{i=1}^{n} \lambda_i = 1\,, \tag{2.1.1}$$

and no proper subset of U has this property. If $k = n$ the I-set U is called *regular*.

The notion of I-set was introduced in [SW]. The role of regular I-sets is illustrated by

THEOREM 2.1.2 [SW, Th. 5.8]. *Assume B is a normed space and let $D \subset B$ be an n-dimensional subspace. Let $b \in B\backslash D$ and let $d_0 \in D$ be the best approximation to b in D. If $E(b-d_0)$ contains a regular I-set then d_0 is a SUBA to b in D.*

PROPOSITION 2.1.3. *Let $f = (f_1,\dots,f_n) \in S_{B^*}$ satisfy $|f_i| > 0$. Assume $A \in \mathcal{L}(D)$ and $\lambda_A = \|A\|$. For all $i_0 \in \{1,\dots,n\}$ and $i \in \{1,\dots,n\} \setminus \{i_0\}$ define $y_i^{i_0} = (y_i^{i_0}(1),\dots,y_i^{i_0}(n))$ by*

$$y_i^{i_0}(j) = \begin{cases} 0 & \text{if } j \neq i_0, i\,,\\ 1 & \text{if } j = i\,,\\ -f_i/f_{i_0} & \text{if } j = i_0\,. \end{cases} \tag{2.1.2}$$

If $L \in \mathcal{P}_A^0$, then for each $i_0 \in \{1,\dots,n\}$,

$$L = \sum_{i \neq i_0} g_i(\cdot) y_i^{i_0}\,, \tag{2.1.3}$$

where $g_i \in B^*$, $g_i|_D = e_i \circ A$ and $\|g_i\| \le \|A\|$ for $i \neq i_0$. Moreover, if L satisfies (2.1.3) for some $i_0 \in \{1,\dots,n\}$ and $\|e_{i_0} \circ L\| \ \le \|A\|$ then $L \in \mathcal{P}_A^0$.

Proof. Fix $i_0 \in \{1,\dots,n\}$, $L \in \mathcal{P}_A^0$ and let $U_{i_0} = \sum_{i\neq i_0} (e_i \circ L) y_i^{i_0}$. We show that $U_{i_0} = L$. Note that for $i, j \neq i_0$ and $x \in B$,

$$(e_j \circ U_{i_0})x = \sum_{k \neq i_0} e_k(Lx) e_j(y_k^{i_0}) = e_j(Lx)\,.$$

Since $f_{i_0} \neq 0$ and $U_{i_0}x \in D$, $U_{i_0}x = L$. It is clear that $(e_i \circ L)|_D = e_i \circ A$. Since $L \in \mathcal{P}_A^0$ and $\lambda_A = \|A\|$, $\|e_i \circ L\| \le \|A\|$ for $i \neq i_0$. Now assume that L satisfies (2.1.3) and $\|e_{i_0} \circ L\| \le \ \|A\|$. Hence for $i, j \neq i_0$,

$$(e_j \circ L) y_i^{i_0} = \sum_{k \neq i_0} g_k(y_i^{i_0}) e_j(y_k^{i_0}) = e_j(A y_i^{i_0})$$

and consequently, $L \in \mathcal{P}_A$. Since $\|L\| = \max_{i=1,\dots,n} \|e_i \circ L\|$, we immediately see that $L \in \mathcal{P}_A^0$.

Note that $\mathcal{P}_A^0$ is a compact convex set. Hence, by the Krein–Milman Theorem, the set $\mathrm{ext}(\mathcal{P}_A^0)$ is nonempty. Moreover, we have the following

PROPOSITION 2.1.4. *Let $A \in \mathcal{L}(D)$ and let $\lambda_A = \|A\|$. If $L \in \mathrm{ext}(\mathcal{P}_A^0)$ then*

$$\mathrm{card}\{i : \|e_i \circ L\| = \|L\|\} \geq n - 1\,.$$

Proof. Suppose that there exists $L \in \mathrm{ext}(\mathcal{P}_A^0)$ such that

$$\mathrm{card}\{i : \|e_i \circ L\| = \|L\|\} < n - 1\,.$$

Let $\|e_{i_1} \circ L\| < \|L\| = \|A\|$ and $\|e_{i_2} \circ L\| < \|A\|$ for $i_1, i_2 \in \{1, \dots, n\}$, $i_1 \neq i_2$. It is easy to check that $L = \sum_{i \neq i_1} (e_i \circ L) y_i^{i_1}$. For $\lambda \in \mathbb{R}$ define $L_\lambda = \sum_{i \neq i_1} g_i(\cdot) y_i^{i_1}$, where

$$g_i = \begin{cases} e_i \circ L & \text{if } i \neq i_2\,, \\ e_i \circ L + \lambda f & \text{if } i = i_2\,. \end{cases} \tag{2.1.4}$$

Note that $L_\lambda \in \mathcal{P}_A$, $L_\lambda \neq L$ for $\lambda \neq 0$ and $L = (L_{-\lambda} + L_\lambda)/2$ for every $\lambda \in \mathbb{R}$. We show that $L_\lambda \in \mathcal{P}_A^0$ for $|\lambda|$ sufficiently small. It is clear that for $j = i_1, i_2$,

$$\|e_j \circ L_\lambda\| = \|e_j \circ (L + \lambda f(\cdot) y_{i_2}^{i_1})\| \leq \|e_j \circ L\| + |\lambda| \|y_{i_2}^{i_1}\|\,.$$

For $j \neq i_1, i_2$,

$$\|e_j \circ L_\lambda\| = \|e_j \circ L\| \leq \|A\|\,.$$

Since $\|e_{i_1} \circ L\| < \|A\|$ and $\|e_{i_2} \circ L\| < \|A\|$, the proof is complete.

PROPOSITION 2.1.5. *Assume $D = \ker(f)$, $|f_i| > 0$ for $i = 1, \dots, n$ and*

$$f(x) \neq 0 \quad \textit{for every } x \in \mathrm{ext}_{B^*}\,. \tag{2.1.5}$$

Let $A \in \mathcal{L}(D)$. If $\|e_i \circ A\| = \|A\|$, then there exists exactly one $g \in S_{B^*}(0, \|A\|)$ with $g|_D = e_i \circ A$. If $\|e_i \circ A\| < \|A\|$ then there exist exactly two functionals $g_1, g_2 \in S_{B^*}(0, \|A\|)$ with $g_i|_D = e_i \circ A$ for $i = 1, 2$.

Proof. We can assume without loss of generality that $\|A\| = 1$. First we consider the case $\|e_i \circ A\| = \|A\|$. Since $1/2 > |f_i| > 0$ for $i \in \{1, \dots, n\}$, $\|e_i|_D\| = 1$ (the element $y = (y_1, \dots, y_n)$, where

$$y_j = \begin{cases} -\operatorname{sgn}(f_j / \sum_{k \neq i} |f_k|) f_i & \text{if } j \neq i\,, \\ 1 & \text{if } j = i\,, \end{cases}$$

has norm 1 and belongs to D). It is clear that $\mathrm{ext}_{D^*} \subset \{\pm e_j|_D\}_{j=1,\dots,n}$. Now take $y^0 = (y_1^0, \dots, y_n^0) \in \mathrm{ext}_D$ with $(e_i \circ A) y^0 = \|e_i \circ A\| = 1$. By (2.1.5) , there exists exactly one $i_0 \in \{1, \dots, n\}$ with $|y_{i_0}^0| < 1$. In view of [Si, Lemma 1.1, p. 166],

$$e_i \circ A = \sum_{j \in J_i \subset \{1,\dots,n\} \setminus \{i_0\}} \lambda_j y_j^0 e_j|_D\,, \tag{2.1.6}$$

where $\lambda_j > 0$ and $\sum_{j \in J_i} \lambda_j = 1$. We show that (2.1.6) is the unique expression of $e_i \circ A$ as a convex combination of points from the set ext_{D^*} (with strictly positive coefficients). Indeed, let $e_i \circ A = \sum_{j\in J_1} \gamma_j e_j|_D$ where $0 < |\gamma_j|$, $\sum_{j\in J_1} |\gamma_j| = 1$. Since $|y_{i_0}| < 1$, $J_1 \subset \{1,\ldots,n\} \setminus \{i_0\}$. Hence, because $\{e_j|_D\}_{j\neq i_0}$ is a basis of D^*, $J_1 = J_i$ and $\gamma_j = \lambda_j y_j^0$. Now define

$$g = \sum_{j\in J_i} \lambda_j y_j^0 e_j\,.$$

It is clear that $\|g\| = 1$ and $g|_D = e_i \circ A$. We show that g is the unique norm-preserving extension of $e_i \circ A$. To do this, take $h \in S_{B^*}$, $h|_D = e_i \circ A$. By [Si, Lemma 1.1, p. 166],

$$h = \sum_{j\in Z} \gamma_j y_j^0 e_j\,,$$

where $0 < \gamma_j$, $\sum_{j\in Z} \gamma_j = 1$. Since $h(y^0) = (e_i \circ A)(y^0)$, $Z \subset \{1,\ldots,n\} \setminus \{i_0\}$. Consequently, reasoning as above, we get $Z = J_i$ and $\lambda_j = \gamma_j$ for $j \in J_i$. Now assume $\|e_i \circ A\| < \|A\| = 1$. Applying the first part of the proof, we can show that there exists exactly one $h_i \in B^*$ with $\|h_i\| = \|e_i \circ A\|$ and $h_i|_D = e_i \circ A$. Note that if $g \in B^*$ and $g|_D = e_i \circ A$, then $g = h_i + \lambda f$ for some $\lambda \in \mathbb{R}$. Since $\|h_i\| < \|A\| = 1$, the line $h_i + \lambda f$ intersects S_{B^*} in exactly two points g_1, g_2. The proof is complete.

Now for given $A \in \mathcal{L}(D)$ and $i \in \{1,\ldots,n\}$ define

$$\text{crit}_A = \{i \in \{1,\ldots,n\} : \|e_i \circ A\| = \|A\|\} \tag{2.1.7}$$

and

$$\mathcal{E}_i = \{g \in S_{B^*}(0,\|A\|) : g|_D = e_i \circ A\}\,. \tag{2.1.8}$$

By Proposition 2.5, $\operatorname{card}\mathcal{E}_i = 1$ if $i \in \text{crit}_A$, and $\operatorname{card}\mathcal{E}_i = 2$ in the opposite case. Set

$$\mathcal{D} = \Big\{L \in \mathcal{L}(B,D) : L = \sum_{i\neq i_0} g_i(\cdot) y_i^{i_0} \text{ for some } i_0 \in \{1,\ldots,n\},\ g_i \in \mathcal{E}_i\Big\}\,. \tag{2.1.9}$$

Now we can state

THEOREM 2.1.6. *Suppose* $D = \ker(f)$, $f = (f_1,\ldots,f_n)$, *where* f *satisfies* (2.1.5), $f_i \neq 0$ *for* $i = 1,\ldots,n$. *Let* $A \in \mathcal{L}(D)$. *Then* $\lambda_A = \|A\|$ *if and only if there exists* $L \in \mathcal{D}$ *with* $\|L\| = \|A\|$.

Proof. It is easy to check that $\mathcal{D} \subset \mathcal{P}_A$. Hence if $\|L\| = \|A\|$ for some $L \in \mathcal{D}$, then $\lambda_A = \|A\|$. If $\lambda_A = \|A\|$ take any $L \in \text{ext}(\mathcal{P}_A^0)$. By Proposition 1.3, there exists $i_0 \in \{1,\ldots,n\}$ such that $\|e_i \circ L\| = \|A\|$ for $i \neq i_0$. It is clear that $L = \sum_{i\neq i_0}(e_i \circ L) y_i^{i_0}$. Hence $L \in \mathcal{D}$. The proof is complete.

Remark 2.1.7. Theorem 1.6 introduces a method which permits one to check if $\lambda_A = \|A\|$ or $\lambda_A > \|A\|$ for any $A \in \mathcal{L}(A)$. This method consists of the following steps:

(a) calculating the set ext_D;
(b) calculating the norm of $e_i \circ A$ for $i = 1,\ldots,n$ using the set ext_D;

(c) choosing, for each $i \in \{1, \ldots, n\}$, $y_i \in \mathrm{ext}_D$ satisfying $(e_i \circ A)y_i = \|A\|$;

(d) finding, for $i = 1, \ldots, n$, the unique functional $h_i \in X^*$ such that $h_i|_Y = e_i \circ A$ and $\|e_i \circ A\| = \|h_i\|$;

(e) finding the set $\mathcal{E}_i$ for each $i \in \{1, \ldots, n\} \setminus \mathrm{crit}_A$;

(f) seeking an operator of minimal norm in $\mathrm{conv}(\mathcal{D})$; here the convex programming method can be applied (see [Che1, p. 54 and Ex. 3, p. 51]).

However, note that there exist operators $A \in \mathcal{L}(D)$ for which we can check in a simpler way if $\lambda_A = \|A\|$.

Example 2.1.8. Assume $\|e_i \circ A\| = \|A\|$ for each $i \in \{1, \ldots, n\}$. Then the set $\mathcal{D}$ consists of exactly one element.

Example 2.1.9. Assume $L \in \mathcal{L}(B, D)$ is represented by a matrix $[l(i,j)]_{i,j=1,\ldots,n}$. Put $A = L|_D$ and assume that there exists $i_0 \in \{1, \ldots, n\}$ such that for every $j \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{n} |(-f_i/f_{i_0}) l(j, i_0) + l(j, i)| \le \|A\| . \tag{2.1.10}$$

Then $\lambda_A = \|A\|$.

Proof. Fix $i_0 \in \{1, \ldots, n\}$ satisfying (2.1.10). Define $L_1 = \sum_{i \ne i_0} e_i(\cdot) L y_i^{i_0}$. It is clear that $L_1|_D = L|_D = A$. Moreover, it is easy to check that

$$\|L_1\| = \max_{j=1,\ldots,n} \sum_{i \ne i_0} |e_j(L y_i^{i_0})| .$$

Observe that $|e_j(L y_i^{i_0})| = |(-f_i/f_{i_0}) l(j, i_0) + l(j, i)|$. By (2.1.10), the proof is complete.

Now we consider the situation $\lambda_A > \|A\|$.

Theorem 2.1.10. *Assume that $f \in S_{B^*}$, $f = (f_1, \ldots, f_n)$, satisfies* (2.1.5), *and $|f_i| > 0$ for $i = 1, \ldots, n$. Assume furthermore that $A \in \mathcal{L}(D)$ and let $\lambda_A > \|A\|$. If $L_0 \in \mathcal{P}_A^0$ then $E(L_0)$ contains a regular I-set* (see (2.1.1)).

Proof. Let $L_0 \in \mathcal{P}_A^0$. It is easy to verify that $\|L_0\| = \mathrm{dist}(L_0, \mathcal{L}_D)$. Hence, by Corollary 0.2.14, $0 \in \mathrm{conv}(E(L_0)|_{\mathcal{L}_D})$, i.e.

$$0 = \sum_{i=1}^{k} \lambda_i \varphi_i|_{\mathcal{L}_D} ,$$

where $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Assume $k \in \mathbb{N}$ is a minimal number for which the above equality is satisfied. If we show that $k = n$, then $\{\varphi_1, \ldots, \varphi_n\}$ will be the required regular I-set. By the Carathéodory Theorem, we can assume $k \le n$ ($\dim \mathcal{L}_D = n-1$). By Theorem 0.2.7, $\varphi_i = x_i \otimes e_{j(i)}$, where $j(i) \in \{1, \ldots, n\}$ and $x_i \in \mathrm{ext}_B$. There is no loss of generality in assuming $j(1) \le \ldots \le j(k)$. First we show that $j(1) = 1$. Suppose, on the contrary, that $j(1) > 1$ and put

$$E_1 = \{i : j(i) = j(1)\} .$$

Then

$$0=\sum_{i=1}^{k}\lambda_i(x_i\otimes e_{j(i)})|_{\mathcal{L}_D}=\sum_{i\in E_1}\lambda_i(x_i\otimes e_{j(1)})|_{\mathcal{L}_D}+\sum_{i\notin E_1}\lambda_i(x_i\otimes e_{j(i)})|_{\mathcal{L}_D}\,.$$

Put

$$L_{j(1)}=f(\cdot)y^1_{j(1)}\,. \tag{2.1.11}$$

Note that if $i\notin E_1$, then $j(1)<j(i)$. Hence for each $i\in E_1$,

$$(x_i\otimes e_{j(i)})(L_{j(1)})=f(x_i)e_{j(i)}(y^1_{j(1)})=0\,.$$

Consequently,

$$0=\sum_{i\in E_1}\lambda_i(x_i\otimes e_{j(1)})(L_1)=\sum_{i\in E_1}\lambda_i f(x_i)\,,$$

since $e_{j(1)}(y^1_{j(1)})=1$. To get a contradiction, we show that either $f(x_i)>0$ for every $i\in E_1$ or $f(x_i)<0$ for every $i\in E_1$. By (2.1.5) $f(x_i)\neq 0$ for every $i\in E_1$. Suppose that there exist $i_1,i_2\in E_1$ with $f(x_{i_1})<0$ and $f(x_{i_2})>0$. Then it is easy to show that

$$(y\otimes e_{j(1)})(L_0)=\|L_0\|$$

for some $y\in S_D$. But this contradicts the assumption $\lambda_A>\|A\|$. So we proved $j(1)=1$.

To end the proof of the theorem, we check that the map $i\to j(i)$ is surjective. If not, there exists $i_0\in\{1,\dots,n\}$ with $j(i)\neq i_0$ for $i=1,\dots,k$. Since $j(1)=1$, $i_0>1$. Put $I_1=\{i\in\{1,\dots,k\}:j(i)=1\}$. An easy computation shows that

$$0=\sum_{i=1}^{k}(x_i\otimes e_{j(i)})(L_{i_0})=(-f_{i_0}/f_1)\sum_{i\in I_1}\lambda_i f(x_i)\,.$$

Reasoning as in the first part of the proof we get either $f(x_i)>0$ for each $i\in I_1$ or $f(x_i)<0$ for each $i\in I_1$; a contradiction. Hence the map $i\to j(i)$ is surjective and consequently $k=n$. The proof is complete.

Reasoning as in Theorem 2.10 we can prove

THEOREM 2.1.11. *Let* $L\in\mathcal{L}(B)$ *and let* $L_0\in\mathcal{P}_{\mathcal{L}_D}(L)$ (*the set of all best approximants for* L *in* $\mathcal{L}_D$). *Assume* $\operatorname{dist}(L,\mathcal{L}_D)>\|L|_D\|$. *Then the set* $E(L-L_0)$ *contains a regular I-set.*

By Theorem 1.2 we get immediately

COROLLARY 2.1.12. *Let* A, L_0, f *be as in Theorem* 1.11. *Then there exists* $r>0$ *such that for every* $L\in\mathcal{P}_A$,

$$\|L\|\geq\|L_0\|+r\|L-L_0\|\,.$$

In particular, *the set* $\mathcal{P}^0_A$ *consists of exactly one element.*

Note that the assumption $\lambda_A>\|A\|$ in Theorem 1.10 and Corollary 1.12 is essential because of

EXAMPLE 2.1.13. Let $n = 3$ and let $f = (1/3, 1/3, 1/3)$, $D = \ker(f)$. Define $L \in \mathcal{L}(B, D)$ to be the matrix

$$L = \begin{pmatrix} a & -a & 0 \\ -a/2 & a/2 & 0 \\ -a/2 & a/2 & 0 \end{pmatrix}$$

where a is a fixed positive number. Put $A = L|_D$. It is easy to verify that

$$\mathrm{ext}_D = \{\pm(1,-1,0), \pm(1,0,-1), \pm(0,1,-1)\}\,.$$

Hence $\|L\| = \|A\|$ and consequently $\lambda_A = \|A\|$. For $\delta \in \mathbb{R}$ consider the operator L_δ defined by the matrix

$$L_\delta = \begin{pmatrix} a & -a & 0 \\ -a/2+\delta & a/2+\delta & \delta \\ -a/2-\delta & a/2-\delta & -\delta \end{pmatrix}.$$

Note that

$$L_\delta(-1,1,0) = (-2a, a, a) = L(-1,1,0)$$

and

$$L_\delta(-1,0,1) = (-a, a/2, a/2) = L(-1,0,1)\,.$$

Hence $L_\delta|_D = L|_D = A$. It is easy to verify that $L_\delta \in \mathcal{L}(B, D)$ and $\|L_\delta\| = \|A\|$ for $|\delta|$ sufficiently small. Hence the set $\mathcal{P}^0_A$ consists of more than one element.

Theorems 1.10 and 1.11 lead to an effective method of calculating $\mathrm{dist}(L, \mathcal{L}_D)$ for given $L \in \mathcal{L}(B)$ if $\mathrm{dist}(L, \mathcal{L}_D) > \|L|_D\|$. To do this, for given $x_1, \ldots, x_n \in \mathrm{ext}_B$ consider the following system of equations:

$$\text{(2.1.12)} \qquad \begin{cases} (x_i \otimes e_i)(L - f(\cdot)(y_1, \ldots, y_n)) = z \quad \text{for } i = 1, \ldots, n\,, \\ \displaystyle\sum_{i=1}^n f_i y_i = 0\,, \end{cases}$$

with unknowns $y_1, \ldots, y_n, z$. Assume additionally that

$$\text{(2.1.13)} \qquad 0 \in \mathrm{conv}((x_1 \otimes e_1)|_{\mathcal{L}_D}, \ldots, (x_n \otimes e_n)|_{\mathcal{L}_D})\,.$$

Let $L_0 = f(\cdot)y^0 \in \mathcal{P}(\mathcal{L}_D)(L)$. Then, in view of Theorem 1.11, if f satisfies (2.1.5) and $f_i \neq 0$ for $i = 1, \ldots, n$ there exist $x_1, \ldots, x_n \in \mathrm{ext}_B$ such that $y^0_1, \ldots, y^0_n$, $\mathrm{dist}(L, \mathcal{L}_Y)$ are the solution of (2.1.12) for the above $x_1, \ldots, x_n$. So to find $L_0 \in \mathcal{P}_{\mathcal{L}_D}$ and $\mathrm{dist}(L, \mathcal{L}_D)$ it is sufficient to solve a finite number of equations (2.1.12) for $x_1, \ldots, x_n$ satisfying (2.1.13). For verifying (2.1.13) we can apply the following

PROPOSITION 2.1.14. *Assume $x_1, \ldots, x_n \in \mathrm{ext}_B$. Let $f \in S_{B^*}$ satisfy* (2.1.5) *and let $f_i \neq 0$ for $i = 1, \ldots, n$. Put $D = \ker(f)$. Then*

$$\text{(2.1.14)} \qquad 0 \in \mathrm{conv}((x_1 \otimes e_1)|_{\mathcal{L}_D}, \ldots, (x_n \otimes e_n)|_{\mathcal{L}_D}) \quad \textit{iff} \\ \mathrm{sgn}(f(x_j) f_1) = \mathrm{sgn}(f(x_1) f_j) \quad \textit{for } j = 1, \ldots, n\,.$$

Proof. Fix $x_1, \ldots, x_n \in \mathrm{ext}_B$ and suppose

$$0 = \sum_{i=1}^{k} \lambda_i (x_i \otimes e_{j(i)})|_{\mathcal{L}_D} .$$

Reasoning as in Theorem 1.10 we get $k = n$ and $j(i) = i$ for $i = 1, \ldots, n$. Now for $j = 2, \ldots, n$ take the map $L_j \in \mathcal{L}_D$ defined by (2.1.1). Note that for $j = 2, \ldots, n$,

$$0 = \sum_{i=1}^{n} \lambda_i (x_i \otimes e_i)(L_j) = \lambda_1(-f_j/f_1) f(x_1) + \lambda_j f(x_j) .$$

Consequently, $\lambda_1/\lambda_j = f(x_j) f_1 / f(x_1) f_j$, which completes the proof.

To end this section, for some $L \in \mathcal{L}(B)$, we calculate an explicit formula for $\mathrm{dist}(L, \mathcal{L}_D)$ ($\lambda_{L|_D}$ if $L \in \mathcal{L}(B, D)$) and determine an element of best approximation in this case.

THEOREM 2.1.15. *Let f satisfy the assumptions of Theorem* 1.11, $f > 0$. *Let $L \in \mathcal{L}(B)$. Assume furthermore that there is a $d_0 \in f^{-1}(1)$ such that the matrix $[d_{ij}]_{i,j=1,\ldots,n}$ of the map $L_1 = L + f(\cdot) d_0$ satisfies the following conditions*:

$$(2.1.15) \qquad d_{ij} \le 0 \quad \textit{or} \quad d_{ij} \ge 1 ,$$

$$(2.1.16) \qquad \sum_{j \in N_i} f_j < \sum_{j \in P_i \cup Z_i} f_j \quad \textit{for every } i \in \{1, \ldots, n\} ,$$

and

$$(2.1.17) \qquad \sum_{j \in P_i} f_j < \sum_{j \in N_i \cup Z_i} f_j \quad \textit{for every } i \in \{1, \ldots, n\}$$

(*here* $Z_i = \{j : d_{ij} = 0\}$, $P_i = \{j : d_{ij} > 0\}$, $N_i = \{j : d_{ij} < 0\}$ *for* $i \in \{1, \ldots, n\}$), *and*

$$(2.1.18) \qquad P_i \subset \{j : f_j \le f_i\} .$$

Then $\mathrm{dist}(L, \mathcal{L}_D) = \|L|_D\|$ *or*

$$\mathrm{dist}(L, \mathcal{L}_D) = \Big(1 + \sum_{i=1}^{n} f_i B_i / A_i\Big) \Big/ \Big(\sum_{i=1}^{n} f_i / A_i\Big) ,$$

where $A_i = \sum_{j \in N_i \cup Z_i} f_j - \sum_{j \in P_i} f_j$ *and* $B_i = \sum_{j \in P_i} d_{ij} - \sum_{j \in Z_i \cup N_i} d_{ij}$ *for* $i \in \{1, \ldots, n\}$.

Proof. Suppose $\mathrm{dist}(L, \mathcal{L}_D) > \|L|_D\|$. It is clear that $\mathrm{dist}(L, \mathcal{L}_D) = \mathrm{dist}(L_1, \mathcal{L}_D^1)$, where

$$\mathcal{L}_D^1 = \{L \in \mathcal{L}(B, D) : L = f(\cdot) y : y \in f^{-1}(1)\} .$$

In view of Theorem 1.10, $\mathrm{card}\, \mathcal{P}_{\mathcal{L}_D^1}(L_1) = 1$. By the definition of $\mathcal{L}_D^1$ an element $L^0 \in \mathcal{P}_{\mathcal{L}_D^1}(L_1)$ is represented in a unique way by some $y^0 = (y_1^0, \ldots, y_n^0) \in f^{-1}(1)$.

By the proof of Theorem 1.10, the numbers $d = \mathrm{dist}(L_1, \mathcal{L}_D), y_1^0, \ldots, y_n^0$ must satisfy the following system of equations:

$$(2.1.19)\qquad \begin{cases} \sum_{j=1}^{n} |d_{ij} - f_j y_i^0| = d \quad \text{for } i = 1, \ldots, n\,, \\ \sum_{i=1}^{n} f_i y_i = 1\,. \end{cases}$$

Now suppose that if $L^0 = f(\cdot)y^0 \in \mathcal{P}_{\mathcal{L}_D^1}(L^1)$ then $y_i^0 \geq 0$ for $i \in \{1, \ldots, n\}$. Then, by (2.1.15)–(2.1.18), the system (2.1.19) reduces to the following system of linear equations:

$$(2.1.20)\qquad \begin{cases} \sum_{j\in P_i} d_{ij} - \sum_{j\in N_i\cup Z_i} d_{ij} \\ \qquad\qquad + y_i^0\Big(\sum_{j\in N_i\cup Z_i} f_j - \sum_{j\in P_i} f_j\Big) = d \quad \text{for } i = 1, \ldots, n\,, \\ \sum_{i=1}^{n} f_i y_i^0 = 1\,. \end{cases}$$

Hence after simple calculations

$$(2.1.21)\qquad \begin{cases} d = \Big(1 + \sum_{i=1}^{n} f_i B_i/A_i\Big)\Big/\Big(\sum_{i=1}^{n} f_i/A_i\Big)\,, \\ y_i^0 = (d - B_i)/A_i \quad \text{for } i \in \{1, \ldots, n\}\,, \end{cases}$$

as required. So to finish the proof, it is sufficient to show that the vector y^0 corresponding to the best approximation for L^1 has nonnegative coordinates Assume that this is not true. Hence there is an $i_0 \in \{1, \ldots, n\}$ with $y_{i_0}^0 < 0$. Since $y^0 \in f^{-1}(1)$, there is an $i_1 \in \{1, \ldots, n\}$ with $y_{i_1}^0 > 0$. For $\theta > 0$ define $w^\theta = (w_1^\theta, \ldots, w_n^\theta)$ by

$$(2.1.22)\qquad w_i^\theta = \begin{cases} y_i^0 & \text{if } i \neq i_0, i_1\,, \\ y_i^0 + \theta/f_i & \text{if } i = i_0\,, \\ y_i^0 - \theta/f_i & \text{if } i = i_1\,. \end{cases}$$

It is clear that $f(w^\theta) = 1$. We show that the operator $W^\theta \in \mathcal{L}_D^1$ corresponding to w^θ satisfies

$$\|e_j \circ (L^1 - W^\theta)\| < \|e_j \circ (L^1 - L^0)\| \leq \mathrm{dist}(L^1, \mathcal{L}_D^1)$$

for $j = i_0, i_1$ and $\theta > 0$ sufficiently small. For $j = i_0$, by (2.1.16) and a simple calculation, we have

$$(2.1.23)\qquad \|e_j \circ (L^1 - W^\theta)\| = \sum_{k=1}^{n} |d_{jk} - f_k w_j^\theta|$$

$$= \sum_{k=1}^{n} |d_{jk} - y_j^0 f_k| + \theta\Big(\sum_{k\in N_j} f_k - \sum_{k\in P_j\cup Z_j} f_k\Big)/f_j$$
$$< \|e_j \circ (L^1 - L^0)\| \le \operatorname{dist}(L^1, \mathcal{L}_D^1)\,.$$

If $j = i_1$, then by (2.1.17) for $\theta > 0$ sufficiently small

$$\|e_j \circ (L^1 - W^\theta)\| = \sum_{k=1}^{n} |d_{jk} - y_j^0 f_k| + \theta\Big(\sum_{k\in P_j} f_k - \sum_{k\in N_j\cup Z_j} f_k\Big)/f_j$$
$$< \|e_j \circ (L^1 - L^0)\| = \operatorname{dist}(L^1, \mathcal{L}_D^1)\,.$$

Consequently, $W^\theta \in \mathcal{P}_{\mathcal{L}_D^1}(L^1)$ for θ sufficiently small, which by (2.1.23) leads to a contradiction with Theorem 1.11. The proof is complete.

COROLLARY 2.1.16. *Assume that* $f \in S_{B^*}$ *satisfies the requirements of Theorem* 1.15. *Put* $L^1 = \mathrm{Id}$. *Then* $\lambda_{\mathrm{Id}|_D} = 1 + (\sum_{i=1}^n |f_i|/(1-2|f_i|))^{-1}$.

This reproves the result of Blatter and Cheney [Bl, Th. 2].

COROLLARY 2.1.17. *Let* f, L^1, $\mathcal{L}_D^1$ *be as in Theorem* 1.15. *Let* $\operatorname{dist}(L^1, \mathcal{L}_D^1) = \|L^1|_D\|$. *Then* (by Proposition 1.4) *there is an* $i_0 \in \{1, \ldots, n\}$ *such that the vector* $y = (y_1^0, \ldots, y_n^0)$, *where*

$$y_i^0 = \begin{cases} (\|L^1|_D\| - B_i)/A_i & \text{if } i \ne i_0\,,\\ (1 - \sum_{j\ne i} f_j y_j^0)/f_i & \text{if } i = i_0\,,\end{cases}$$

defines the best approximation to L^1 *in* $\mathcal{L}_D^1$.

II.2. Minimal projections onto hyperplanes of $l_1^{(n)}$. Throughout this section, unless otherwise stated, $B = l_1^{(n)}$ and $D \subset B$ is a hyperplane, i.e. $D = \ker(f)$ for some $f \in S_{B^*}$. We determine the constant $\lambda_{\mathrm{Id}}(B, D)$ and $P_0 \in \mathcal{P}^0_{\mathrm{Id}}(B, D)$ (see (0.1.18), (0.1.19)). We will write λ_{Id} for $\lambda_{\mathrm{Id}}(B, D)$, $\mathcal{P}$ for $\mathcal{P}_{\mathrm{Id}}(B, D)$ and $\mathcal{P}^0$ for $\mathcal{P}^0_{\mathrm{Id}}(B, D)$ for brevity. In other words, we obtain a formula for a minimal projection from B onto D and we calculate its norm. The method presented here is based on Theorem 1.2.8 and it is different from that of [OdL, Th. II.4.9] and [Bl]. First we state some preliminary results.

Remark 2.2.1 (see e.g. [Bl, Lemma 1]). Assume B is a normed space and let $D = \ker(f)$ for some $f \in S_{B^*}$. Then for every $P \in \mathcal{P}$ there exists a unique $d^P \in f^{-1}(1)$ such that

$$Pb = b - f(\cdot)d^P \quad \text{for every } b \in B\,. \tag{2.2.1}$$

Conversely, for any $d \in f^{-1}(1)$ the operator P_d defined by (2.2.1) belongs to $\mathcal{P}$.

Remark 2.2.2 (see e.g. [Bl]). For every $P \in \mathcal{P}$, $\|P\| = \max_{i=1,\ldots,n} \|Pe_i\|$ and

$$\|Pe_i\| = |1 - f_i d_i^P| + |f_i|(\|d^P\|_1 - |d_i^P|)\,. \tag{2.2.3}$$

Since B is a symmetric and reflection invariant space (see e.g. [Od, Def. 1.1]), by Remark 2.2, until the end of this section we can assume without loss of generality that

$$1 = f_1 \geq \ldots \geq f_n \geq 0\,. \tag{2.2.4}$$

Moreover, it is easy to show [Bl, Th. 3] that $\lambda_{\mathrm{Id}} = 1$ if and only if the functional f has at most two coordinates different from 0. So until the end of this section we assume that $f_3 > 0$. Let us introduce the following notations:

$$a_i = \sum_{j=1}^{i} f_j, \quad b_i = \sum_{j=1}^{i} f_j^{-1} \quad \text{and for } i \geq 3, \quad \beta_i = b_i/(i-2)\,, \tag{2.2.5}$$

$$E_f = \{i \geq 3 : \beta_i \geq f_i^{-1}\}, \quad G_f = \{i \geq 3 : f_i^{-1} \leq \beta_i \leq f_{i+1}^{-1}\} \tag{2.2.6}$$

(we define $f_{n+1} = 0$).

Now we can prove

LEMMA 2.2.3. *Let $i \in E_f$. Consider the system of equations*

$$\begin{cases} z = \sum_{j=1}^{i-1} d_j\,, \\ \sum_{j=1}^{i-1} f_j d_j = 1\,, \\ f_j(z - 2d_j) = x \quad \textit{for } j = 2, \ldots, i-1\,, \\ f_i z = x \end{cases} \tag{2.2.7}$$

with unknowns $z, x, d = (d_1, \ldots, d_{i-1})$. Let $\widehat{z}, \widehat{x}, \widehat{d}$ be a solution of (2.2.7). *Then* $\|P_d\| = 1 + \widehat{x}$ ($P_d \in \mathcal{P}$ is the projection induced by d; see Remark 2.1).

Proof. One can verify that the numbers

$$\begin{cases} \widehat{x} = 2(k_i + f_i^{-1} a_i - i)^{-1} \quad \text{where } k_i = b_i - (i-2) f_i^{-1}\,, \\ \widehat{d_1} = \widehat{x}(k_i + f_i^{-1} - 1)/2\,, \\ \widehat{d_j} = \widehat{x}(f_i^{-1} - f_j^{-1})/2 \quad \text{for } j = 2, \ldots, i-1\,, \\ \widehat{z} = \widehat{x} f^{-1} \end{cases} \tag{2.2.8}$$

form a solution of (2.2.7). By (2.2.4), $d_j \geq 0$ for $j \in \{2, \ldots, i-1\}$. Since $i \in E_f$, $d_1 \geq 0$. Put $d = (\widehat{d_1}, \ldots, \widehat{d}_{i-1}, 0, \ldots, 0)$. By Remark 2.2 and (2.2.7),

$$\|P_d e_j\| = 1 + f_j(\widehat{z} - 2\widehat{d_j}) = 1 + \widehat{x} \quad \text{for } j = 2, \ldots, i-1\,.$$

For $i \leq j \leq n$,

$$\|P_d e_j\| = 1 + f_j \widehat{z} \leq 1 + f_i \widehat{z} \leq 1 + \widehat{x}\,.$$

For $j = 1$, the inequality $\|P_d e_j\| = 1 + (\|d\|_1 - 2d_1) \leq 1 + x$ is equivalent to $(f_i^{-1} - k_i - f_i^{-1} + 1)\widehat{x} \leq \widehat{x}$, which gives $\beta_i \geq f_i^{-1}$. Since $i \in E_f$, the proof is complete.

LEMMA 2.2.4. *Let $i \in G_f$. Consider the system of linear equations*

$$(2.2.9)\qquad \begin{cases} f_j(z-2d_j) = x \quad \textit{for } j=1,\ldots,i, \\ z = \sum_{j=1}^{i} d_j\,, \\ \sum_{j=1}^{i} f_j d_j = 1\,. \end{cases}$$

Let $\widehat{x}, \widehat{z}, \widehat{d} = (\widehat{d}_1, \ldots, \widehat{d}_i)$ be a solution of (2.2.9). *Then $\|P_d\| = 1 + x$ where P_d is the projection induced by $\widehat{d}$.*

P r o o f. One can verify that the numbers

$$(2.2.10)\qquad \begin{cases} x = 2(a_i\beta_i - i)^{-1}\,, \\ \widehat{d}_1 = \widehat{x}(\beta_i - f_1^{-1})/2, \ldots, \widehat{d}_i = x(\beta_i - f_i^{-1})/2\,, \\ \widehat{z} = x \cdot \beta_i \end{cases}$$

form a solution of (2.2.9). Since $i \in G_f$, $\widehat{d}_j \ge 0$ for $j \in \{1, \ldots, i\}$. Put $d = (\widehat{d}_1, \ldots, \widehat{d}_i, 0\ldots, 0)$. Then, by Remark 2.2 and (2.2.10), for $j = 1, \ldots, i$,

$$\|P_d e_j\| = |1 - f_j\widehat{d}_j| + f_j(\|d\|_1 - \widehat{d}_j)\| = 1 + f_j(\widehat{z} - 2\widehat{d}_j) = 1 + x\,.$$

If $i < n$, then for $i < j \le n$,

$$\|P_d e_j\| = 1 + f_j\|d\|_1 = 1 + \widehat{z} f_j\,.$$

Since the inequality $1 + f_j\widehat{z} \le 1 + x$ is equivalent to $\beta_i \le f_j^{-1}$ and $i \in G_f$, $\|P_d e_j\| \le 1 + \widehat{x}$, as required.

In the next part of this section we show that there exists $P^0 \in \mathcal{P}^0$, $P^0 = P_d^0$ for some $d \in f^{-1}(1)$, such that $x = \|P_d^0\| - 1$, $z = \|d\|_1$ and $(d_1, \ldots, d_i)$ satisfy the system (2.2.7) for some $i \in E_f$ or (2.2.9) for some $i \in G_f$, and we determine this index. To do this, we need some preliminary results.

LEMMA 2.2.5 [Bl, Lemma 4]. *There exists $P^0 \in \mathcal{P}^0$ such that the vector $d \in f^{-1}(1)$ corresponding to P^0 has nonnegative coordinates.*

LEMMA 2.2.6. *Let $f \in S_{B^*}$, $f_2 < 1$. Let $P^0 = P_d^0 \in \mathcal{P}^0$ and let $d \ge 0$. Then for every $j \ge 2$ with $d_j > 0$, $\|P^0 e_j\| = \|P^0\|$.*

P r o o f. Suppose, on the contrary, that $\|P^0 e_j\| < \|P^0\|$ for some $j \ge 2$ with $d_j > 0$. Note that, by Remark 2.2, $\|P^0 e_j\| = 1 + f_j(\|d\|_1 - 2d_j)$. Select $r > 0$ with $1 + f_j(\|d\|_1 - 2(d_j - r/f_j)) < \|P^0\|$ and $d_j - r/f_j > 0$. Define a vector $d^1 = (d_1^1, \ldots, d_n^1) \in B$ by

$$d_k^1 = \begin{cases} d_k + r & \text{if } k = 1\,, \\ d_k - r/f_k & \text{if } k = j\,, \\ d_k & \text{if } k \ne 1, j\,. \end{cases}$$

Note that $d^1 \in f^{-1}(1)$ and $\|d^1\|_1 = \sum_{k=1}^n d_k + r - r/f_j < \|d\|_1$, since $f_j \le f_2 < 1$. Let P^1 denote the projection induced by d^1. We show that $\|P^1\| < \|P^0\|$. To do this, take $k \in \{1, \ldots, n\}$. If $k = j$, then

$$\|P^1 e_k\| = 1 + f_k(\|d^1\|_1 - 2(d_k - r/f_k)) < 1 + f_k(\|d\|_1 - 2(d_k - r/f_k)) < \|P^0\|$$

by the definition of r. If $k = 1$, then

$$\begin{aligned}\|P^1 e_k\| &= 1 + \|d^1\|_1 - 2d_1^1 = 1 + \|d\|_1 + r - r/f_j - 2(d_1 + r) \\ &= 1 + \|d\|_1 - 2d_1 - r(1 + f_j^{-1}) < 1 + \|d\|_1 - 2d_1 = \|P^0 e_1\| \le \|P^0\|.\end{aligned}$$

If $k \ne 1, j$, then

$$\begin{aligned}\|P^1 e_k\| &= 1 + f_k(\|d^1\|_1 - 2d_k^1) = 1 + f_k(\|d^1\|_1 - 2d_k) \\ &< 1 + f_k(\|d\|_1 - 2d_k) = \|P^0 e_k\| \le \|P^0\|,\end{aligned}$$

which contradicts the minimality of P^0.

COROLLARY 2.2.7. *Let $P^0 = P_d^0 \in \mathcal{P}^0$ and let $d \ge 0$. Let $f \in S_{B^*}$, $f_2 < 1$. Set $i_0 = \min\{i \ge 2 : d_i = 0\}$. Then $d_i = 0$ for every $i > i_0$.*

Proof. Suppose $d_i > 0$ for some $i > i_0$. In view of Lemma 2.6, $\|P^0 e_i\| = \|P^0\|$. Hence

$$\|P^0\| = 1 + f_i(\|d\|_1 - 2d_i) < 1 + f_i\|d\|_1 \le 1 + f_{i_0}\|d\|_1 \le \|P^0\|;$$

a contradiction.

In the sequel we will need the following notation. For $i \in \{3, \ldots, n\}$ and $j \in \{2, \ldots, i\}$ set

$$g_j^i = (-1, \ldots, 1_j, \ldots, -1_i). \tag{2.2.11}$$

DEFINITION 2.2.8. We say that $f \in B^*$ satisfies *condition* $(*)$ if and only if for every $i \in \{3, \ldots, n\}$ the vectors $\{g_j^i : j \in \{2, \ldots, i\}, f^i = (f_1, \ldots, f_i)\}$ form a linearly independent set in $\mathbb{R}^i$ (in other words, $\det A_i \ne 0$ for $i \in \{3, \ldots, n\}$, where A_i is the matrix with rows g_j^i, $j \in \{2, \ldots, i\}$, and f^i).

Let us set

$$F = \{f \in B^* : f \text{ satisfies condition } (*)\}. \tag{2.2.12}$$

LEMMA 2.2.9. *The set F is dense in B^*.*

Proof. Note that $B^* \setminus F = \bigcup_{i=3}^n G_i$, where $G_i = \{f \in B^* : \det A_i = 0\}$. It is obvious that each G_i is a linear subspace of B^*. So to finish the proof, it is sufficient to show that $G_i \subsetneq F$ for $i \ge 3$. Take $f = (1, \ldots, 1)$ and assume that $f \in G_i$ for some $i \ge 3$. This means that $\sum_{j=1}^{i-1} \alpha_j g_{j+1}^i + \alpha_i f^i = 0$ for some $\alpha = (\alpha_1, \ldots, \alpha_i) \in \mathbb{R}^i$, $\sum_{j=1}^{i} |\alpha_j| > 0$. In particular, taking the first coordinates we get $-\sum_{k=1}^{i-1} \alpha_k + \alpha_i = 0$. For fixed $j \in \{1, \ldots, i-1\}$, taking the $(j+1)$th coordinates we obtain $\alpha_j - \sum_{k=1, k\ne j}^{i-1} \alpha_k + \alpha_i = 0$. Subtracting the above equalities, we derive $\alpha_j = 0$ for $j = 1, \ldots, i-1$ and consequently $\alpha_i = 0$; a contradiction.

LEMMA 2.2.10. *Let* $P^0 = P^0_d \in \mathcal{P}$ *and let* $d_j > 0$ *for* $j = 1, \ldots, n$. *Set*

$$C = \{g^n_i : |(g^n_i \circ P^0)e_i| = \|P^0\|\}\,. \tag{2.2.13}$$

Then $C \cap -C = \emptyset$ *and* $C \cup -C = \operatorname{crit}^*(P^0)$ (see (1.2.1)).

Proof. It is clear that $C \cap -C = \emptyset$. Assume that $g \in \operatorname{crit}^*(P^0)$. Then $|(g \circ P^0)e_i| = \|P^0\|$ for some $i \in \{1, \ldots, n\}$. If $(g \circ P^0)e_i = \|P^0\|$ then, by (2.2.3), $g_i = \operatorname{sgn}(1 - f_i d_i) = 1$ and $g_j = \operatorname{sgn}(-f_i d_j) = -1$ for $j \neq i$. Consequently, $g = g^n_i$ and $g \in C$. In the opposite case $-g \in C$. The converse is obvious.

Now we can show

THEOREM 2.2.11. *Assume* $f \in S_{B^*}$. *Then there exists* $P^0 = P^0_d \in \mathcal{P}^0$ *such that* $x = \|P^0\| - 1$, $z = \|d\|_1$ *and* $d_1, \ldots, d_i$ *satisfy* (2.2.7) *for some* $i \in E_f$ *or* (2.2.9) *for some* $i \in G_f$ (see (2.2.6)).

Proof. First we assume additionally that $1 = f_1 > f_2 > f_3$ and $f \in S_{B^*} \cap F$ (see (2.2.10)). By Lemma 2.5, we can choose $P^0 = P^0_d \in \mathcal{P}^0$ such that d has nonnegative coordinates. Now we consider two cases.

Case I: $d_i > 0$ *for* $i = 1, \ldots, n$. In view of Lemma 2.6, $\|P^0 e_i\| = \|P^0\|$ for $i \geq 2$. Suppose that $\|P^0 e_1\| < \|P^0\|$. By Lemma 2.10, $\operatorname{crit}^*(P^0) = \{g^n_2, \ldots, g^n_n\} \cup \{-g^n_2, \ldots, -g^n_n\}$. Since $f \in F$, $\det(g^n_2, \ldots, g^n_n, f) \neq 0$. Hence the set $\{g^n_2, \ldots, g^n_n\}$ is linearly independent on $D = \ker(f)$; a contradiction with Theorem 1.2.8. Consequently, $\|P^0 e_i\| = \|P^0\|$ for $i = 1, \ldots, n$ and it is easy to verify that $\|P^0\| - 1, \|d\|_1, d$ satisfy the system (2.2.9) for $i = n$.

Case II: $d_i = 0$ *for some* $i \in \{1, \ldots, n\}$. First we show that $d_1, d_2 > 0$. If $d_1 = 0$, then $\|P^0\| \geq \|P^0 e_1\| = 1 + \|d\|_1 \geq 2$, since $f \in S_{B^*}$ and $f(d) = 1$. But, in view of [Lev], $\lambda_{\mathrm{Id}} < 2$; a contradiction. If $d_2 = 0$, then by Corollary 2.7, $d_j = 0$ for $j = 2, \ldots, n$. Consequently, $d_1 = 1$. Now select $r > 0$ with $1 + f_3(\|d\|_1 + (f_2^{-1} - 1)r) < \|P^0\|$ (this is possible, since $\|P^0\| = 1 + f_2\|d\|_1$ and $f_3 < f_2$). Put $d^1 = (1 - r, r/f_2, 0, \ldots, 0)$. It is clear that $f(d^1) = 1$ and it is easy to verify that $\|P_{d^1}\| < \|P^0\|$; a contradiction. Now put $i_0 = \min\{j \geq 3 : d_j = 0\}$ and assume $\|P^0 e_{i_0}\| = \|P^0\|$. In view of Lemma 2.6, $\|P^0 e_j\| = \|P^0\|$ for $2 \leq j \leq i_0$. Since $d \geq 0$, by Remark 2.2, the numbers $x = \|P^0\| - 1$, $z = \|d\|_1$, and $d_1, \ldots, d_{i_0}$ satisfy the system (2.2.7) for $i = i_0$. To finish this part of the proof, we show that $i_0 \in E_f$ (see (2.2.6)). Since the solution of (2.2.7) is unique, by (2.2.8), $d_1 = (k_{i_0} + f_{i_0}^{-1} - 1)(\|P^0\| - 1)/2$ and $\|d\|_1 = (\|P^0\| - 1)f_{i_0}^{-1}$. But the inequality $\|P^0 e_1\| \leq \|P^0\|$ yields $k_{i_0} \geq 0$ and consequently $\beta_{i_0} \geq f_{i_0}^{-1}$. Hence $i_0 \in E_f$.

Now suppose $\|P^0 e_{i_0}\| < \|P^0\|$. Then $i_0 > 3$. If not, then consider $d^2 = (d_1, d_2)$, $f^2 = (f_1, f_2)$ and let $\widehat{P}^0_{d_2} \in \mathcal{P}(l_1^{(2)}, \ker(f^2))$. In view of Remark 2.2, $\|P^0\| = \|\widehat{P}^0\|$ and $\|\widehat{P}^0 e_j\| = \|P^0 e_j\|$ for $j = 1, 2$. By [Bl, Th. 3], there exists $\widehat{P}^1 \in \mathcal{P}(l_1^{(2)}, \ker(f^2))$ with $\|\widehat{P}^1\| = 1$. Extending in a natural way $\widehat{P}^1$ to $P^1 \in \mathcal{P}$, define $P^\alpha = \alpha P^0 + (1 - \alpha)P^1$ for $\alpha \in (0, 1)$. Since $\|P^0 e_{i_0}\| < \|P^0\|$, it is easy to verify that $\|P^\alpha\| < \|P^0\|$ for α sufficiently close to 1; a contradiction.

Now assume $i_0 \ge 4$, $\|P^0 e_{i_0}\| < \|P^0\|$. By Lemma 2.6, $\|P^0 e_j\| = \|P^0\|$ for $2 \le j \le i-1$. We show that $\|P^0 e_1\| = \|P^0\|$. Suppose that this is not true. Define $\widehat{P}^0$, f^{i_0-1} as in the case $i_0 = 3$. By Lemma 2.10, $\text{crit}^*(\widehat{P}^0) = \{\pm g_2^{i_0-1}, \ldots, \pm g_{i_0-1}^{i_0-1}\}$. Since $f \in F$, the set $C = \{g_2^{i_0-1}, \ldots, g_{i_0-1}^{i_0-1}\}$ is total over $\ker(f^{i_0-1})$ and consequently it is linearly independent over $\ker(f^{i_0-1})$. By Theorem 1.2.8, there exists $\widehat{P}^1 \in \mathcal{P}_{\text{Id}}(l_1^{(i_0-1)}, \ker(f^{i_0-1}))$ with $\|\widehat{P}^1\| < \|\widehat{P}^0\|$. Reasoning as above we get $\|P^\alpha\| < \|P^0\|$ for α sufficiently close to one; a contradiction with minimality of P^0. Therefore, $\|P^0 e_j\| = \|P^0\|$ for $j = 1, \ldots, i_0 - 1$. Reasoning as in the case when $\|P^0 e_{i_0}\| = \|P^0\|$, we can show that the numbers $x = \|P^0\| - 1$, $z = \|d\|_1$ and $d_1, \ldots, d_{i_0-1}$ satisfy the system (2.2.9) for $i = i_0 - 1$. Since $\|P^0 e_{i_0}\| < \|P^0\|$ and $d_{i_0-1} = (\|P^0\| - 1)(\beta_{i_0-1} - f_{i_0-1}^{-1})/2$, by a simple calculation we get $\beta_{i_0-1} \ge f_{i_0-1}^{-1}$, $\beta_{i_0-1} < f_{i_0}^{-1}$ and consequently $i_0 - 1 \in G_f$.

To end the proof of the theorem note that by Lemma 2.9, the set

$$\{f : 1 = f_1 > f_2 > f_3 \ge \ldots \ge f_n,\ f \in F\}$$

is dense in $Z = \{f : 1 = f_1 \ge f_2 \ge \ldots \ge f_n\}$. Consequently, by the continuity of the function $Z \ni f \to \lambda_{\text{Id}}(B, \ker(f))$ [OdL, Lemma II.7.6, p. 83] and a compactness argument, the reasoning presented above holds true for any $f \in Z$, which completes the proof of the theorem.

To present the next results of this section note the following properties of the numbers b_i, β_i, a_i (see (2.2.5)):

R e m a r k 2.2.12. (a) For $3 \le i$ and $3 \le j \le i$ if $a_i \ge i - 2$ (resp. $\beta_i \ge f_i^{-1}$) then $a_j \ge j - 2$ (resp. $\beta_j \ge f_j^{-1}$).

(b) For $i \ge 3$, $f_i b_{i-1} \ge i - 3$ iff $\beta_i \ge f_i^{-1}$.

For $j = 3, \ldots, n$ define

$$c_j = \min\{f_j b_{j-1}, a_{j-1}\} \tag{2.2.14}$$

and set

$$i = i(f) = \max\{j \ge 3 : c_j \ge j - 3\}\,. \tag{2.2.15}$$

THEOREM 2.2.13. *Let $P^0 \in \mathcal{P}^0$. Then $\|P^0\| = 1 + x$, where*

$$x = \begin{cases} 2((\beta_i - f_i^{-1})(i-2) + a_i f_i^{-1} - i)^{-1} & \text{if } a_i < i - 2\,, \\ 2(a_i \beta_i - i)^{-1} & \text{if } a_i \ge i - 2\,. \end{cases}$$

P r o o f. By Theorem 2.11, there is a $P^0 = P_d^0 \in \mathcal{P}^0$ such that $\|P^0\| - 1$, $\|d\|_1$, $d_1, \ldots, d_{i_0}$ form a solution of (2.2.7) for some $i_0 \in E_f$ or (2.2.9) for some $i_0 \in G_f$. We will show that it is possible to choose $i_0 = i(f)$. Note that for any $i = 3, \ldots, n$ and $j = 3, \ldots, n-1$,

$$(\beta_i - f_i^{-1})(i-2) + a_i f_i^{-1} - i - (a_i \beta_i - i) = (\beta_i - f_i^{-1}) \cdot (i - 2 - a_i) \tag{2.2.16}$$

and

$$(2.2.17)\qquad \begin{aligned}&(\beta_{j+1}-f_{j+1}^{-1})(j-1)+a_{j+1}f_{j+1}^{-1}-(j+1)\\ &\qquad -(\beta_j-f_j^{-1})(j-2)-(a_jf_j^{-1}-j)\\ &\qquad = b_{j+1}-(j-1)f_{j+1}^{-1}-b_j+f_j^{-1}(j-2)+a_j(f_{j+1}^{-1}-f_j^{-1})\\ &\qquad = a_j(f_{j+1}^{-1}-f_j^{-1})+(j-2)(f_j^{-1}-f_{j+1}^{-1})\\ &\qquad = (a_j-(j-2))\cdot(f_{j+1}^{-1}-f_j^{-1}).\end{aligned}$$

Suppose that $i_0 < i(f)$. If $\|P^0\|-1$, $\|d\|_1$, $d_1,\ldots,d_{i_0}$ satisfy (2.2.9) for $i=i_0$ then $\beta_{i_0}\le f_{i_0+1}^{-1}$, which by (2.2.5) gives $\beta_{i_0+1}\le f_{i_0+1}^{-1}$. Since $i_0<i(f)$, by (2.2.15), $\beta_{i_0+1}\ge f_{i_0+1}^{-1}$ and consequently $\beta_{i_0+1}=f_{i_0+1}^{-1}=\beta_{i_0}$. Reasoning as above, by Remark 2.2.12 and (2.2.15), we get $\beta_j=f_j^{-1}=\beta_{j-1}$ for $j=i_0+1,\ldots,i(f)$. Consequently, $\beta_{i_0}=f_{i(f)}^{-1}=f_j^{-1}$ for $j=i_0+1,\ldots,i(f)$. Hence,

$$a_{i_0}\beta_{i_0}-i_0-(a_{i(f)}\beta_{i(f)}-i(f))=i(f)-i_0-\Big(\sum_{j=i_0+1}^{i(f)}f_j\Big)f_{i(f)}^{-1}=0\,.$$

Consequently, the projection P^1 defined by the solution of (2.2.9) for $i=i(f)$ satisfies $\|P^1\|\le\|P^0\|$, so we can choose $i(f)$ as i_0. (If $i(f)<n$ then $\beta_{i(f)}=f_{i(f)}^{-1}\le f_{i(f)+1}^{-1}$, which gives $i(f)\in G_f$.)

If $\|P^0\|-1$, $\|d\|_1$, $d_1,\ldots,d_{i_0}$ satisfy (2.2.7) for $i=i_0$, then by (2.2.17) we can also choose $i_0=i(f)$.

Now suppose that $i_0>i(f)$. Hence $\beta_j\ge f_j^{-1}$ and $a_j<j-2$ for $i(f)\le j\le i_0$. By (2.2.16), (2.2.17), the solution of (2.2.7) for $i=i(f)$ defines the projection P^1 such that $\|P^1\|\le\|P^0\|$, where P^0 is the minimal projection satisfying (2.2.7) for $i=i_0$. Consequently, we can choose $i_0=i(f)$. The formula for the norm of the minimal projection is a consequence of Lemmas 2.3 and 2.4. The proof is complete.

Corollary 2.2.14 [Od]. *Let $f\in S_{B^*}$ be as in Theorem* 2.2.13. *Put*

$$(2.2.18)\qquad m=m(f)=\begin{cases} i(f) & \text{if } \beta_{i(f)}>f_{i(f)}^{-1}\,,\\ u(f) & \text{if } \beta_{i(f)}=f_{i(f)}^{-1}\,,\end{cases}$$

where

$$(2.2.19)\qquad u(f)=\min\{j\le i(f)-1: f_{j+1}=f_{i(f)}\}\,.$$

If $a_m\ge m-2$ (resp. $a_m<m-2$) then the formula (2.2.10) (*resp.* (2.2.8)) *defines for $i=m(f)$ the coordinates of the vector d corresponding to a minimal projection P^0.*

Remark 2.2.15. In the complex case the formula for the minimal projection given in Theorem 2.13 holds true.

Proof. By Remark 2.2, we can assume without loss of generality that $1=f_1\ge f_2\ge\ldots\ge f_n$. If $P=P_d\in\mathcal{P}$, then $P_{\mathrm{re}(d)}\in\mathcal{P}$, where $\mathrm{re}(d)=(\mathrm{re}(d_1),$

$\ldots, \mathrm{re}(d_n))$. Moreover, by Remark 2.2, $\|P_d\| \geq \|P_{\mathrm{re}(d)}\|$ for any $d \in f^{-1}(1)$, which completes the proof.

II.3. Strongly unique minimal projections onto hyperplanes of $l_\infty^{(n)}$ and $l_1^{(n)}$. The problems considered in this section may be treated as a development of the results obtained in Sections II.1 and II.2. Hence we restrict ourselves to the case $B = l_\infty^{(n)}$ (resp. $B = l_1^{(n)}$), $\mathcal{V} = \mathcal{P}_{\mathrm{Id}}(B, D) \subset \mathcal{K}(B)$, where $D \subset B$ is a hyperplane in $l_\infty^{(n)}$ (resp. in $l_1^{(n)}$). In other words, we will examine when a projection $P^0 \in \mathcal{P}^0_{\mathrm{Id}}(B, D)$ is a strongly unique minimal projection (we will write a SUM projection for brevity) in $\mathcal{P}_{\mathrm{Id}}(B, D)$, i.e. when $P^0 \in \mathcal{P}^0_{\mathrm{Id}}(B, D)$ satisfies

$$\|P\| \geq \|P^0\| + r\|P - P^0\| \tag{2.3.1}$$

for every $P \in \mathcal{P}_{\mathrm{Id}}(B, D)$ with a constant $r > 0$ independent of P.

First we deal with the case $B = l_\infty^{(n)}$. Applying Theorems 1.2.2 and 1.2.5 we prove the following

THEOREM 2.3.1. *Let $D \subset B$ be a hyperplane, i.e. $D = \ker(f)$ for some $f = (f_1, \ldots, f_n) \in S_{B^*}$. Assume $P^0 \in \mathcal{P}_{\mathrm{Id}}(B, D)$ is a minimal projection. Then we have*:

(a) *If $\|P^0\| = 1$, then P^0 satisfies* (2.3.1) *if and only if $|f_i| \geq 1/2$ for exactly one $i \in \{1, \ldots, n\}$. The constant $r = \min\{1 - 2|f_j| : j \neq i\}$ is the best possible.*

(b) *In the real case, if $\|P^0\| > 1$, then P^0 satisfies* (2.3.1) *if and only if $0 < |f_i| < 1/2$ for $i = 1, \ldots, n$.*

Moreover, *the constant*

$$r = \min\{\max\{(1 - 2|f_i|)y_i : i = 1, \ldots, n\} : y \in S_D\} \tag{2.3.2}$$

is the best possible and the following estimate holds:

$$r \geq (1 - 2|f_j|)|f_i|/(1 - |f_i|)\,, \tag{2.3.3}$$

where

$$|f_j| = \max\{|f_k| : k = 1, \ldots, n\} \quad \text{and} \quad |f_i| = \min\{|f_k| : k = 1, \ldots, n\}\,.$$

Proof. (a) Assume that $|f_i| \geq 1/2$ for exactly one index $i \in \{1, \ldots, n\}$. Fix $P \in \mathcal{P}_{\mathrm{Id}}(B, D)$. Denote by d^P (resp. d^0) the vector from B corresponding to P (resp. to P^0). It is clear that $P - P^0 = f(\cdot)(d^0 - d^P)$ and consequently $\|P - P^0\| = \|d^0 - d^P\|_\infty$. Since $|f_i| \geq 1/2$, $\|d^0 - d^P\|_\infty = |d^P_j - d^0_j|$ for some $j \neq i$. By [Bl, Th. 1] $d^0_i = 1/f_i$ and $d^0_j = 0$ for $j \neq i$. Consequently, $\|P - P^0\| = |d^P_j|$ for some $j \neq i$. By (1.2.5), we note that

$$\begin{aligned}\|P\| &\geq \|(x \to x_j) \circ P\| = |1 - f_j d^P| + |d^P_j|(1 - |f_j|)\\ &\geq 1 + |d^P_j|(1 - 2|f_j|) \geq \|P^0\| + \min\{1 - 2|f_k| : k \neq i\}\|P - P^0\|\,,\end{aligned}$$

which gives the result.

Now we shall show that the constant $r = \min\{1 - 2|f_j| : j \neq i\}$ is the best possible. To do this, for $f \in S_{B^*}$ and $d \in f^{-1}(1)$ set

$$P_{f,d} = \mathrm{Id} - f(\cdot)d\,.$$

Since $\|P_{f,d}\| = \|P_{|f|,\overline{d}}\|$ for every $f \in B^*$ and $d \in f^{-1}(1)$ ($\overline{d}_i = d_i$ if $f_i = 0$ and $\overline{d}_i = f_i/|f_i|d_i$ otherwise) we may assume $f \geq 0$. Set $d_k = 0$ if $k \neq i$ and $k \neq j$, $d_i = -f_j/f_i$, $d_j = 1$ and let $d = (d_1, \ldots, d_n)$ (the index j is so chosen that $f_j = \max\{f_k : k \neq i\}$). Let $P = P^0 - f(\cdot)d$. By Theorem 1.2.2 and Remark 1.2.3, it is enough to show that

$$\|(x \to x_k) \circ P\| < 1 + r_1\|P - P^0\|$$

for every $r_1 > r$ and $k = 1, \ldots, n$. First we note that $\|P - P^0\| = \|d\|_\infty = 1$. By (1.2.5), if $k = i$ then

$$\begin{aligned}\|(x \to x_k) \circ P\| &= |1 - f_i(d_i + 1/f_i)| + |d_i + 1/f_i| \cdot |1 - f_i| \\ &= 1/f_i - 1 + d_i(1 - 2f_i) = 1/f_i - 1 + f_j(2f_i - 1)/f_i \\ &\leq 1/f_i - 1 + (1 - f_i)(2f_i - 1)/f_i \\ &= 2(1 - f_i) \leq 1 < 1 + r_1\|P - P^0\|\,.\end{aligned}$$

If $k \neq i$ and $k \neq j$, then $d_k^P = d_k = 0$. Hence

$$\|(x \to x_k) \circ P\| = 1 < 1 + r_1\|P - P^0\|\,.$$

If $k = j$, then

$$\|(x \to x_k) \circ P\| = 2 - 2f_j = 1 + r\|P - P^0\| < 1 + r_1\|P - P^0\|\,.$$

Applying Theorem 1.2.2(b) , we complete the proof of part (a).

(b) As in the previous case we can assume $f_i \geq 0$ for $i = 1, \ldots, n$. Define a function $\phi : S_D \to \mathbb{R}$ by the formula

$$\phi(y) = \min\{(2f_i - 1)y_i : i = 1, \ldots, n\}\,. \tag{2.3.4}$$

Since $f_i > 0$ for $i = 1, \ldots, n$, $\phi(y) < 0$ for every $y \in S_D$. Hence, by the argument of compactness and continuity of ϕ, the constant $\gamma = \max\{\phi(y) : y \in S_D)\}$ is negative. We show that P^0 is a SUM projection with $r = -\gamma$. To do this, by Theorem 1.2.5, Remark 1.2.6 and Theorem 2.1.10, it is enough to prove that for every $P \in \mathcal{P}_{\mathrm{Id}}(B, D)$ there exists $i \in \{1, \ldots, n\}$ with

$$\inf\{((P - P^0)x)_i : x \in A_i(P^0)\} \leq -r\|P - P^0\| \tag{2.3.5}$$

(we write $A_i(P^0)$ instead of $A_{x \to x_i}(P^0)$ (see (1.2.2)). It is clear that $\|P^0 - P\| = \|d^P - d^0\|_\infty$. Set $d = (d^P - d^0)/\|d^P - d^0\|_\infty$ (if $d^P = d^0$ the inequality (2.3.1) is satisfied). Select $i \in \{1, \ldots, n\}$ with $\phi(d) = (2f_i - 1)d_i$. By (1.2.5), $x \in A_i(P^0)$ iff

$$x_j = -\operatorname{sgn}(f_j) = -1 \quad \text{for } j \neq i \quad \text{and} \quad x_i = \operatorname{sgn}(1 - f_i d_i^0) = 1\,.$$

Hence for $x \in A_i(P^0)$ we get

$$((P - P^0)x)_i = f(x)\|d^P - d^0\|_\infty d_i = (f_i - 1)d_i\|d^P - d^0\|_\infty \leq -r\|d^P - d^0\|_\infty\,.$$

By Remark 1.2.6, we have proved our claim.

Now we will show that $r \geq (1-2f_j)f_i/(1-f_i)$, where $f_j = \max\{f_k : k = 1,\dots,n\}$ and $f_i = \min\{f_k : k = 1,\dots,n\}$. To do this, take $y \in S_D$. If $y_k = 1$ for some $k \in \{1,\dots,n\}$, then

$$\phi(y) \leq 2f_k - 1 \leq 2f_j - 1 \leq (2f_j - 1)f_i/(1-f_i)\,,$$

since $f_j < 1/2$ and $f_i < 1/2$.

In the opposite case $y_k = -1$ for some $k \in \{1,\dots,n\}$ and an easy calculation shows that $y_l \geq f_i/(1-f_i)$ for some $l \in \{1,\dots,n\}$. We note that

$$\phi(y) \leq (2f_l - 1)y_l \leq (2f_l - 1)f_i/(1-f_i) \leq (2f_j - 1)f_i/(1-f_i)\,,$$

since $f_l < 1/2$ and $f_j \geq f_l$. Hence $\gamma \leq (2f_j - 1)f_i/(1-f_i)$ and consequently, $r \geq (1-2f_j)f_i/(1-f_i)$.

To prove that the constant r is the best possible, take $r_1 > r$, choose $d \in S_D$ with $\phi(d) > -r_1$, and define $P \in \mathcal{P}_{\mathrm{Id}}(B,D)$ by

$$P = P^0 + f(\cdot)d\,.$$

For $l \in \{1,\dots,n\}$ and $x \in A_l(P^0)$ we have

$$((P-P^0)x)_l = f(x)d_l = (2f_l - 1)d_l \geq \phi(d) > -r_1 = -r_1\|P - P^0\|\,.$$

By Theorem 1.2.5 and Remark 1.2.6, the proof of (b) is complete.

Remark 2.3.2. Suppose that $f \in S_{B^*}$ and an operator $L \in \mathcal{L}(B)$ satisfy the assumptions of Theorem 2.1.15. Assume additionally that $d_{ij} \leq 0$ for $i \neq j$, $d_{ii} \geq 1$ (d_{ij} are the same as in Theorem 2.1.15). Then by Theorem 2.1.11 the operator L has a SUBA element in $\mathcal{L}^1_D$. Moreover, if $\mathrm{dist}(L, \mathcal{L}^1_D) > \|L|_D\|$, then the constant r satisfies the estimate from Theorem 2.3.2(b).

Remark 2.3.3. In the complex case Theorem 3.1(b) does not hold.

Proof. As in the proof of Theorem 3.1(b) we may assume $f \geq 0$. It is easy to show that the projection P^0 considered in Theorem 3.1(b) is also minimal in the complex case. By Remark 1.2.4, $A_i(P^0) = \alpha A_{x\to\alpha\cdot x_i}(P^0)$ for every $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Hence we may restrict ourselves to the case $\alpha = 1$. Take $w \in \mathbb{R}^n \cap S_D$ and let $y = 0 + iw$. For $L = f(\cdot)y$, $j = 1,\dots,n$ and $x \in A_j(P^0)$ we have

$$\mathrm{re}(Lx)_j = \mathrm{re}(f(x)y_j) = (2f_j - 1)\,\mathrm{re}(y_j) = 0 > -r\|y\|$$

for every $r > 0$. Hence, by Theorem 1.2.5(b) and Remark 1.2.6, P^0 does not satisfy (2.3.1) with any constant $r > 0$. However, adopting the reasoning from [6, Th. 1], we can show that the conditions given in Theorem 3.1(b) are equivalent to the uniqueness of a minimal projection in the complex case.

By a similar method to that of Theorem 3.1(a) we get

Corollary 2.3.4. *Let $D \subseteq c_0$ be a hyperplane, $D = \ker(f)$ for some $f \in S_{l_1}$ (we consider the real and complex case). Then $P^0 \in \mathcal{P}^0_{\mathrm{Id}}(c_0, D)$ is a SUM projection if and only if $|f_i| \geq 1/2$ for exactly one index i.*

Now we consider the much more difficult case $B = l_1^{(n)}$.

PROPOSITION 2.3.5. *Assume* $f = (f_1, \ldots, f_n) \in S_B$ *and let* $P^0 \in \mathcal{P}^0_{\mathrm{Id}}(B, D)$, $\|P^0\| = 1$. *Then* P^0 *is a unique minimal projection if and only if* P^0 *is a SUM projection.*

Proof. In view of [Bl, Th. 3] we can assume $1 = f_1 \ge \ldots \ge f_n \ge 0$. By [Bl, Th. 3] card $\mathcal{P}^0_{\mathrm{Id}}(B, \ker(f)) = 1$ and $\lambda_{\mathrm{Id}}(B, \ker(f)) = 1$ if and only if $f_2 > 0$ and $f_3 = f_4 = \ldots = f_n = 0$. So assume $1 = f_1 \ge f_2 > 0 = f_3 = \ldots = f_n$. It is easy to verify that if we put $y = (1/(1+f_2), 1/(1+f_2), 0, \ldots, 0)$ then the operator

$$P_y x = x - f(x) y \quad (x \in B) \tag{2.3.6}$$

belongs to $\mathcal{P}^0_{\mathrm{Id}}(B, \ker(f))$. We show that P_y is a SUM projection with constant $r = f_2$. So take an arbitrary $P \in \mathcal{P}_{\mathrm{Id}}(B, \ker(f))$ and write P in the form $P = \mathrm{Id} - f(\cdot) y^P$. Since $\|f\|_\infty = 1$, $\|P - P^0\| = \|y^P - y\|$. If $y_1^P < 0$ then

$$\begin{aligned}\|y^P - y\| &= \|(y_1^P - 1/(1+f_2), y_2^P - 1/(1+f_2), y_3^P, \ldots, y_n^P)\| \\ &= \sum_{i=1}^n |y_i^P| = \|y^P\|.\end{aligned}$$

Hence, by Remark 2.2.2,

$$\begin{aligned}\|P\| \ge \|Pe_1\| &= |1 - y_1^P| + \|y^P\| - |y_1^P| = 1 - y_1^P + \|y^P\| + y_1^P \\ &= 1 + \|y_P\| = 1 + \|P_y - P\| \ge \|P_y\| + f_2 \|P - P_y\|.\end{aligned}$$

If $y_2^P < 0$, by the same reasoning we have

$$\|P\| \ge \|P_y\| + f_2 \|P_y - P\|.$$

Now suppose that $y_1^P > y_2^P > 0$. It is easy to verify that in this case $\|y - y^P\| = \|y^P\| - 2y_2^P$, since $y_1^P + f_2 y_2^P = 1$. Note that

$$\begin{aligned}\|P\| \ge \|Pe_2\| &= |1 - f_2 y_2^P| + f_2(\|y^P\| - y_2^P) = 1 + f_2(\|y^P\| - 2y_2^P) \\ &= \|P_y\| + f_2 \|y - y^P\| = \|P_y\| + f_2 \|P - P_y\|.\end{aligned} \tag{2.3.7}$$

If $y_2^P > y_1^P > 0$, we have $\|y^P - y\| = \|y^P\| - 2y_1^P$. Hence

$$\|P\| \ge \|Pe_1\| = 1 - y_1^P + \|y^P\| - y_1^P \ge \|P_y\| + f_2 \|P - P_y\|.$$

Since the strong unicity of P_y implies that P_y is a unique minimal projection, the proof of Proposition 2.3.5 is complete.

Remark 2.3.6. Since Remark 2.2.2 holds true in the complex case, Proposition 2.3.5 is also valid in the complex case.

Remark 2.3.7. The constant f_2 obtained in the proof of Proposition 3.5 is the best possible.

Proof. Let $y^P = (y_1^P, y_2^P, 0, \ldots, 0)$, $y_1^P > y_2^P > 0$ and $y_1^P + f_2 y_2^P = 1$. Since $\|Pe_i\| = 1$ for $i > 2$ and $\|Pe_1\| = 1 + \|y^P\| - 2y_1^P < 1$, $\|P\| = \|Pe_2\|$. According to (2.3.7), $\|P\| = \|P_y\| + f_2 \|P_y - P\|$, which proves our claim.

Now we shall investigate a much more difficult case, when the norm of a minimal projection is greater than one. By Remark 2.2.2 and [Bl, Th. 3], in the sequel we can assume that $f = (f_1, \ldots, f_n), n \geq 3$ and $1 = f_1 \geq \ldots \geq f_n, f_3 > 0$. First we prove some preliminary results.

LEMMA 2.3.8. *Let $f \in S_{B^*}$ and let $f_n > 0$. Set $a_m = \sum_{j=1}^m f_j$ for $m \in \{3, \ldots, n\}$. If $a_m > m-2$ and there exists $y \in \ker(f) \setminus \{0\}$ satisfying the system of inequalities*

$$y_j \geq \sum_{i=1, i\neq j}^{m} y_i + \sum_{i=1}^{n-m} |y_{i+m}| \quad \textit{for } j = 1, \ldots, m, \tag{2.3.8}$$

then we can find $y^1 \in \ker(f)\setminus\{0\}$ with

$$y_j^1 > \sum_{i=1, i\neq j}^{m} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1| \quad \textit{for } j = 1, \ldots, m \tag{2.3.9}$$

($\sum_{i=1}^{n-n} |y_{i+n}| = 0$ *by definition*).

Proof. Take $y \in \ker(f)\setminus\{0\}$ satisfying (2.3.8) and consider two cases.

Case I: *There exists $j \in \{1, \ldots, m\}$ with $y_j > \sum_{i=1, i\neq j}^m y_i + \sum_{i=1}^{n-m} |y_{i+m}|$.* Then we can find $\vartheta > 0$ such that

$$y_j - \vartheta > \sum_{i=1, i\neq j}^{m} y_i + \sum_{i=1}^{n-m} |y_{i+m}| + (m-1)\vartheta f_j/(a_m - f_j).$$

Define $y_j^1 = y_j - \vartheta$, $y_i^1 = y_i + \vartheta f_j/(a_m - f_j)$ for $i \in \{1, \ldots, m\}\setminus\{j\}$, $y_i^1 = y_i$ for $i = m+1, \ldots, n$ and put $y^1 = (y_1^1, \ldots, y_n^1)$. Note that

$$\begin{aligned}\sum_{i=1}^{n} f_i y_i^1 &= \sum_{i=1}^{m} f_i y_i^1 + \sum_{i+m+1}^{n} f_i y_i \\ &= f_j y_j - f_j \vartheta + \sum_{i=1, i\neq j}^{m} f_i(y_i + f_j \vartheta/(a_m - f_j)) + \sum_{i=m+1}^{n} f_i y_i \\ &= \sum_{i=1}^{n} f_i y_i = 0.\end{aligned}$$

To finish the proof, fix $i \in \{1, \ldots, m\}$, $i \neq j$. Since $a_m > m-2$, we have $a_m - f_j > f_j(m-3)$, which gives

$$f_j\vartheta/(a_m - f_j) > (m-2) f_j \vartheta/(a_m - f_j) - \vartheta. \tag{2.3.10}$$

Adding (2.3.8) to (2.3.10) we obtain

$$y_i^1 > \sum_{k=1, k\neq i}^{m} y_k^1 + \sum_{k=1}^{n-m} |y_{k+m}^1|,$$

which establishes formula (2.3.9).

Case II: *For every* $j \in \{1, \ldots, m\}$,

$$y_j = \sum_{i=1, i\neq j}^{m} y_i + \sum_{i=1}^{n-m} |y_{i+m}| \,. \tag{2.3.11}$$

Then for each $j \in \{1, \ldots, m\}$, $y_1 = \ldots = y_m$ (to show this we subtract for fixed $j, u \in \{1, \ldots, m\}$ equalities (2.3.11)). Consequently, by (2.3.11),

$$-(m-2)y_1 = \sum_{i=1}^{n-m} |y_{i+m}| \,. \tag{2.3.12}$$

Hence if $n = m$, then $y = 0$; a contradiction.

In the opposite case

$$\begin{aligned} 0 &= \sum_{i=1}^{n} f_i y_i = \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{n-m} f_{i+m} y_{i+m} = y_1 \Big(\sum_{i=1}^{m} f_i \Big) + \sum_{i=1}^{n-m} f_{i+m} y_{i+m} \\ &\le y_1 a_m + \sum_{i=1}^{n-m} |f_{i+m}| \cdot |y_{i+m}| \le y_1 a_m + \sum_{i=1}^{n-m} |y_{i+m}| \\ &< y_1(m-2) + \sum_{i=1}^{n-m} |y_{i+m}| = 0 \,, \end{aligned}$$

since according to (2.3.12), $y_1 < 0$ and $a_m > m - 2$. So we may exclude case II. The lemma is proved.

LEMMA 2.3.9. *Let* $f \in S_{B^*}$, $f = (f_1, \ldots, f_n)$, $n \ge 3$, $f_3 > 0$, $f_2 < 1$. *Suppose* $m \in \{3, \ldots, n\}$ *satisfies* $a_m < m-2$, $a_{m-1} > m-3$. *If there exists* $y \in \ker(f)\setminus\{0\}$ *satisfying*

$$\begin{aligned} y_j &\ge \sum_{i=1, i\neq j}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}| \quad \textit{for } j = 2, \ldots, m-1 \,, \\ y_m &\ge \sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}| \,, \end{aligned} \tag{2.3.13}$$

then there exists $y^1 \in \ker(f)\setminus\{0\}$ *with*

$$\begin{aligned} y^1_j &> \sum_{i=1, i\neq j}^{m-1} y^1_i + \sum_{i=1}^{n-m+1} |y^1_{i+m-1}| \quad \textit{for } j = 2, \ldots, m-1 \,, \\ y^1_m &> \sum_{i=1}^{m-1} y^1_i + \sum_{i=1}^{n-m} |y^1_{i+m}| \,. \end{aligned} \tag{2.3.14}$$

Proof. Take $y \in \ker(f)\setminus\{0\}$ satisfying (2.3.13) and consider three cases.

C a s e I: $y_m > \sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}|$. Then we can select $\vartheta > 0$ with

$$y_m > \sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}| - \vartheta + (m-2)\vartheta/(a_{m-1}-1)\,.$$

Define $y_1^1 = y_1 - \vartheta$, $y_j^1 = y_j + \vartheta/(a_{m-1}-1)$ $(j = 2,\dots,m-1)$, $y_j^1 = y_j$ for $j = m,\dots,n$ and set $y^1 = (y_1^1,\dots,y_n^1)$. Note that

$$f(y^1) = \sum_{i=1}^{n} f_i y_i^1 = y_1 - \vartheta + \sum_{i=2}^{m-1} f_i(y_i + \vartheta/(a_{m-1}-1)) + \sum_{i=m}^{n} f_i y_i = \sum_{i=1}^{n} f_i y_i = 0\,.$$

Since $a_{m-1} > m-3$, $\vartheta/(a_{m-1}-1) > -\vartheta + (m-3)\vartheta/(a_{m-1}-1)$. Combining this inequality with (2.3.13) we get

$$y_j^1 > \sum_{i=1, i\neq j}^{m-1} y_i^1 + \sum_{i=1}^{n-m+1} |y_{i+m+1}^1| \quad \text{for } j = 2,\dots,m-1\,,$$

$$y_m^1 > \sum_{i=1}^{m-1} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1|\,,$$

which proves our claim.

C a s e II. *There exists* $j \in \{2,\dots,m-1\}$ *with*

$$y_j > \sum_{i=1, i\neq j}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}|\,.$$

Then $y_j - f_j^{-1}\vartheta > \sum_{i=1,i\neq j}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}| + \vartheta$ for $\vartheta > 0$ sufficiently small. Since $f_2 < 1$,

$$\vartheta - f_j^{-1}\vartheta < 0\,. \tag{2.3.15}$$

Define $y^1 = (y_1^1,\dots,y_n^1)$, where $y_1^1 = y_1 + \vartheta$, $y_j^1 = y_j - f_j^{-1}\vartheta$, $y_i^1 = y_i$ for $i \neq 1, j$. It is clear that $y^1 \in \ker(f)$. Adding (2.3.13) to (2.3.15) we get for each $k \in \{2,\dots,m-1\}\backslash\{j\}$,

$$y_k^1 > \sum_{i=1, i\neq k}^{m-1} y_i^1 + \sum_{i=1}^{n-m+1} |y_{i+m-1}^1|$$

and

$$y_m^1 > \sum_{i=1}^{m-1} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1|\,,$$

which completes the proof of this case.

C a s e III:

$$y_j = \sum_{i=1, i\neq j}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}| \quad \text{for } j = 2,\dots,m-1\,, \tag{2.3.16}$$

$$(2.3.17)\qquad y_m = \sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}|\,.$$

First we show that $y_m > 0$. If not, then by (2.3.16), for every $j \in \{2,\ldots,m-1\}$,

$$(2.3.18)\qquad y_j = \sum_{i=1, i\neq j}^{m-1} y_i + \sum_{i=2}^{n-m+1} |y_{i+m-1}| - y_m\,.$$

Subtracting equalities (2.3.16) for fixed $j,k \in \{2,\ldots,m-1\}$ we get $y_2 = \ldots = y_{m-1}$. By (2.3.17) and (2.3.18), $y_{m-1} = 0$, which gives $0 = \sum_{i=1}^{n-m+1} |y_{i+m-1}| + y_1$. Since $f_2 < 1$ and $y \in \ker(f)$, $y_{i+m-1} = 0$ for $i = 1,\ldots,n-m+1$ and consequently $y = 0$; a contradiction. Hence $y_m > 0$ and reasoning as above we get $y_2 = y_3 = \ldots = y_{m-1}$. Subtracting (2.3.16) from (2.3.17) we get $y_2 - y_m = y_m - y_2$, which gives $y_2 = y_3 = \ldots = y_m > 0$. By (2.3.17), $y_1 = -(m-3)y_m - \sum_{i=1}^{n-m} |y_{i+m}|$. Hence

$$\begin{aligned}
0 = \sum_{i=1}^{n} f_i y_i &= \sum_{i=1}^{m} f_i y_i + \sum_{i=m+1}^{n} f_i y_i \\
&= -(m-3)y_m - \sum_{i=m+1}^{n} |y_i| + (a_m - 1)y_m + \sum_{i=m+1}^{n} f_i y_i \\
&= (a_m - (m-2))y_m + \sum_{i=m+1}^{n} f_i y_i - \sum_{i=m+1}^{n} |y_i| < 0\,,
\end{aligned}$$

since $a_m < m-2$ and $y_m > 0$. Thus we can exclude case III and the proof of the lemma is complete.

Remark 2.3.10. Assume $P \in \mathcal{P}_{\mathrm{Id}}(B,D)$, $D = \ker(f)$, $f = (1, f_2, \ldots, f_n)$, $n \geq 3$ and $f_3 > 0$. Put $C_i = \{g \in \mathrm{ext}_{B^*} : \pm g(Pe_i) = \|Pe_i\|\}$ for $i = 1,\ldots,n$. Then $g \in \mathrm{crit}^*(P)$ if and only if $g \in \bigcup_{i\in A} C_i$, where

$$(2.3.19)\qquad A = \{i \in \{1,\ldots,n\} : \|Pe_i\| = \|P\|\}\,.$$

Proof. Assume $g \in \mathrm{crit}^*(P)$. Since $\mathrm{ext}_{B^*} = \{\pm e_i\}_{i=1}^{n}$, we have $(g\circ P)e_i = \|P\|$ for some $i \in \{1,\ldots,n\}$. It is clear that $i \in A$. The converse is obvious.

LEMMA 2.3.11. *Let $f \in S_{B^*}$ and let $m = m(f)$ be chosen as in Corollary 2.2.14. Assume that $P^0 \in \mathcal{P}_{\mathrm{Id}}(B, \ker(f))$ is a unique minimal projection. Then*

(a) *If $a_m > m-2$ and $m = i(f)$* (see Cor. 2.2.14) *then $A = \{1,\ldots,m\}$.*

If $a_m > m-2$ and $m < i(f)$ then $A = \{1,\ldots,l\}$, where $l = \max\{i \geq m+1 : f_i^{-1} = \beta_m\}$.

(b) *If $f_2 < 1$, $a_{m-1} > m-3$ and $a_m < m-2$, then $A = \{2,\ldots,l\}$, where $l = \max\{i \geq m : f_i = f_m\}$.*

Proof. (a) By Theorem 2.2.13 and Corollary 2.2.14, the vector d^0 corresponding to P^0 has coordinates

$$d_1^0 = x(\beta_m - f_1^{-1})/2,\ \ldots,\ d_m^0 = x(\beta_m - f_m^{-1})/2\,,$$
$$d_i^0 = 0 \quad \text{for } i = m+1, \ldots, n\,,$$

where x is given by (2.2.10). Hence it is easy to verify that

$$\|d^0\| = x\beta_m \tag{2.3.20}$$

and that the system of inequalities

$$\begin{aligned} 1 + f_j(\|d^0\| - 2d_j^0) = 1 + x = \|P^0\| \quad &\text{for } j = 1, \ldots, m\,,\\ 1 + f_j\|d^0\| < 1 + x = \|P^0\| \quad &\text{for } j \ge m+1 \end{aligned} \tag{2.3.21}$$

is consistent. In view of Remark 2.2.2 and (2.3.20) we get our claim.

(b) According to (2.2.8) and Corollary 2.2.14, the vector y^0 corresponding to P^0 has coordinates

$$\begin{aligned} &y_1^0 = x((m-2)(\beta_m - f_m^{-1}) - f_m^{-1} - 1)/2\,,\\ &y_j^0 = x(f_m^{-1} - f_j^{-1})/2 \quad \text{for } j = 2, \ldots, m\,,\\ &y_j^0 = 0 \quad \text{for } j \ge m+1\,, \end{aligned}$$

where x is given by (2.2.8). It is easy to verify that

$$\|y^0\| = f_m^{-1}x \tag{2.3.22}$$

and that the system of inequalities

$$\begin{aligned} 1 + f_j(\|y^0\| - 2y_j^0) = 1 + x = \|P^0\| \quad &\text{for } j = 2, \ldots, m\,,\\ 1 + f_j\|y^0\| \le 1 + x = \|P^0\| \quad &\text{for } j \ge m+1 \end{aligned} \tag{2.3.23}$$

is consistent. By Remark 2.2.2 we get the desired result.

Now we are able to prove the main result of this section.

THEOREM 2.3.12. *Let $f \in S_{B^*}$ (we consider the real case), $f = (1, f_2, \ldots, f_n)$, $f_3 > 0$ and let $m = m(f)$ be as in Corollary 2.2.14. Then the projection P^0 defined in Corollary 2.2.14 is a SUM projection if and only if either*

$$a_m > m - 2 \tag{2.3.24}$$

or

$$f_2 < 1, \quad a_{m-1} > m - 3 \quad \textit{and} \quad a_m < m - 2\,. \tag{2.3.25}$$

Proof. Assume that f satisfies (2.3.24) and (2.3.25). Consider a function $\phi : S_D \to \mathbb{R}$ given by

$$\phi(y) = \min\{f_{k(g)} \cdot g(y) : g \in C\} \tag{2.3.26}$$

where

$$C = \{g \in \mathrm{crit}^*(P^0) : g(P^0 e_i) = \|P^0\| \text{ for some } i \in \{1, \ldots, n\}\} \tag{2.3.27}$$

and

(2.3.28) $$k(g)=\min\{i\in\{1,\dots,n\}: g(P^0e_i)=\|P^0\|\}\,.$$

Assume we have proved that $\phi(y)<0$ for every $y\in S_D$. Hence, by the compactness of S_D and the continuity of ϕ, the constant $\gamma=\sup\{\phi(y): y\in S_D\}$ is strictly negative. We will prove that P^0 is a SUM projection with $r=-\gamma$. To do this, by Theorem 1.2.5(b), it is enough to show that for every $P\in\mathcal{P}_{\mathrm{Id}}(B,D)$ there exists $g\in C$ (it is clear that $C\cup -C=\mathrm{crit}^*(P^0)$ and $C\cap -C=\emptyset$) with

$$\inf\{g(P-P^0)e_i: e_i\in A_g\}\le -r\|P-P^0\|\,.$$

So fix $P\in\mathcal{P}_{\mathrm{Id}}(B,D)$ and let $P-P^0=f(\cdot)d$ for some $d\in D$ (we can assume $d\ne 0$). Select $g\in C$ with $f_{k(g)}g(d/\|d\|)=\phi(d/\|d\|)$. Note that for every $e_i\in A_g$,

$$g(P-P^0)e_i=f_ig(d/\|d\|)\|d\|\ge f_{k(g)}g(d/\|d\|)\|d\|\,,$$

since $\phi(d/\|d\|)<0$. Hence

$$\inf\{g(P-P^0)e_i: e_i\in A_g\}=f_{k(g)}g(d)=\phi(d/\|d\|)\|d\|\le\gamma\|d\|=-r\|P-P^0\|\,,$$

which, by Theorem 1.2.5, gives our assertion.

By the same reasoning as in Theorem 3.1, we can show that the constant r is the best possible.

So to end the proof, it suffices to show that $\phi(y)<0$ for each $y\in S_D$. By (2.3.26) and (2.3.28), $k(g)\in A$ (see (2.3.19)). Hence in view of Lemma 3.11 and Remark 2.2.2, $f_{k(g)}>0$. According to (2.3.26), it is enough to verify that $\inf\{g(y): g\in C\}<0$ for every $y\in S_D$. Suppose, on the contrary, that there exists $y\in S_D$ with $g(y)\ge 0$ for every $g\in C$ and consider two cases.

Case I: $a_m>m-2$. If $m=i(f)$ (see Cor. 2.2.14) then by Lemma 3.11 the set A corresponding to P^0 is $\{1,\dots,m\}$. Consequently, in view of Remark 3.10 and (2.3.25),

$$C=\bigcup_{i=1}^{m}D_i,\quad\text{where}\quad D_i=\{g\in\mathrm{ext}_{B^*}: g(P^0e_i)=\|P^0\|\}\,.$$

By Remark 2.2.2,

$$D_i=\{(-1,\dots,1_i,-1,\dots,-1_m,\varepsilon_1,\dots,\varepsilon_{n-m}):\ \varepsilon=(\varepsilon_1,\dots,\varepsilon_{n-m})\in\{-1,1\}^{n-m}\}\,.$$

Hence the inequalities $g(y)\ge 0$ for every $g\in C$ form the system (2.3.8). By Lemma 3.8, we can find $y^1\in S_D$ with $g(y^1)>0$ for every $g\in C$. Hence for every $g\in C$ and $e_i\in A_g$,

(2.3.29) $$f(e_i)g(y^1)>0$$

since $i\le m$ and $f_m>0$. Now define $P=P^0+f(\cdot)y^1$ and note that (2.3.29) yields for every $g\in C$,

(2.3.30) $$\inf\{g(P-P^0)e_i: e_i\in A_g\}>0\,.$$

By Theorem 1.2.5(b), P^0 is not a minimal projection; a contradiction.

If $m < i(f)$ then $A = \{1, \dots, l\}$, where l is given in Lemma 3.11. Hence $C = \bigcup_{i=1}^{l} D_i$, where the sets D_i are as above for $i = 1, \dots, m$ and

$$D_i = \{(-1, \dots, -1_m, \varepsilon_1, \dots, 1_i, \varepsilon_i, \dots, \varepsilon_{n-m-1}) : \\ \varepsilon \in \{-1, 1\}^{n-m-1}\} \quad \text{for } i = m+1, \dots l\,.$$

So we must add to (2.3.8) the system

$$y_j \ge \sum_{i=1}^{m} y_i + \sum_{i=1, i\ne j}^{n-m} |y_{i+m}| \quad \text{for } j = m+1, \dots, l\,.$$

In view of Lemma 3.8, there exists $y^1 \in \ker(f)$ with

$$y_j^1 > \sum_{i=1, i\ne j}^{m} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1| \quad \text{for } j = 1, \dots, m\,.$$

Now replace f by $f^1 = (1, f_2, \dots, f_m, f_{m+1}^1, \dots, f_n^1)$ where $f_{m+1} > f_{m+1}^1 \ge \dots \ge f_n^1$. Note that, by Corollary 2.2.14, the operator P_1^0 defined by

$$P_1^0 x = x - f^1(x) y^0 \quad \text{for } x \in B \tag{2.3.31}$$

is a minimal projection from B onto $\ker(f^1)$. If the change of f_{m+1} is minor, then modifying slightly the $n-m$ last coordinates of the vector y^1 we get $y^2 = (y_1^1, \dots, y_m^1, y_{m+1}^2, \dots, y_n^2) \in \ker(f^1)$ satisfying (2.3.9). Since $\beta_m < 1/f_{m+1}^1$, applying Theorem 1.2.5, we see that P_1^0 is not a minimal projection from B onto $\ker(f^1)$; a contradiction.

Case II: $a_m < m-2$, $a_{m-1} > m-3$, $f_2 < 1$. If $m < i(f)$ or $m = i(f)$ and $f_{m+1} < f_m$ then by Lemma 3.11, $A = \{2, \dots, m\}$ and $C = \bigcup_{i=2}^{m} D_i$, where the sets D_i are defined as in Case I. By Remark 2.2.2,

$$D_i = \{(-1, \dots, -1, 1_i, -1, \dots, -1_{m-1}, \varepsilon_1, \dots, \varepsilon_{n-m+1}) : \varepsilon \in \{-1, 1\}^{n-m+1}\}$$

for $i = 2, \dots, m-1$ and

$$D_m = \{(-1, \dots, -1, 1_m, \varepsilon_1, \dots, \varepsilon_{n-m}) : \varepsilon \in \{-1, 1\}^{n-m}\}\,.$$

Hence the inequalities $g(y) \ge 0$ for every $g \in C$ form system (2.3.13). By Lemma 3.9 there exists $y^1 \in D$ with $g(y^1) > 0$ for every $g \in C$. Reasoning as in Case I we get a contradiction with the minimality of P^0.

If $m = i(f)$ and $f_{m+1} = f_m$, then $C = \bigcup_{i=2}^{l} D_i$, where l is defined in Lemma 3.11 and

$$D_i = \{(-1, \dots, -1_{m-1}, \varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_i, \dots, \varepsilon_{n-m}) : \varepsilon \in \{-1, 1\}^{n-m}\}$$

for $i \ge m$ (for $i = 2, \dots, m$, the D_i are defined as above). Hence we must add the following inequalities to (2.3.13):

$$y_j \ge \sum_{i=1}^{m} y_i + \sum_{i=1, i\ne j}^{n-m} |y_{i+m-1}|\,. \tag{2.3.32}$$

By Lemma 3.9, there exists $y^1 \in D$ with

$$y^1_j > \sum_{i=1, i\neq j} y^1_i + \sum_{i=1}^{n-m+1} |y^1_{i+m-1}| \quad \text{for } i = 2, \ldots, m-1\,,$$
$$y^1_m > \sum_{i=1}^{m-1} y^1_i + \sum_{i=2}^{n-m+1} |y^1_{i+m-1}|\,.$$

Modifying, as in Case I, f to f^1 where $f^1 = (f_1, \ldots, f_m, f^1_{m+1}, \ldots, f^1_n)$, $f^1_{m+1} < f_m$, and y^1 to y^2 belonging to $\ker(f^1)$ we get a contradiction as in Case I.

To prove the converse, first suppose that $a_m = m-2$, where m is defined by (2.2.18). Then

$$(\beta_m - f_m^{-1})(m-2) - a_m f_m^{-1} - m = a_m \beta_m - m\,.$$

Moreover, $\beta_m > f_m^{-1}$ by the definition of m. Consequently, by Corollary 2.2.14, the solutions of the systems (2.2.7) and (2.2.9) define two different minimal projections (see (2.2.8) and (2.2.10)).

If $a_m < m-2$ and $a_{m-1} = m-3$, then by the definition of $i(f)$, $m(f) = i(f)$ and consequently, $\beta_{i(f)} > f^{-1}_{i(f)}$. Note that by (2.2.17),

$$(\beta_m - f_m^{-1})(m-2) + a_m f_m^{-1} - m - ((\beta_{m-1} - f^{-1}_{m-1})(m-3) - a_{m-1} f^{-1}_{m-1} - (m-1)) = (f^{-1}_{m-1} - f^{-1}_m) \cdot (a_{m-1} - (m-3)) = 0\,.$$

Hence the solutions of (2.2.7) and (2.2.9) for $i = m-1$ define two different minimal projections.

If $a_m < m-2$ and $f_2 = 1$ then also $m = i(f)$ and $\beta_i > f_i^{-1}$, by the definition of $i(f)$. Then it is easy to see that if $d^i = (d_1, \ldots, d_i)$, z, x is a solution of (2.2.7) for $i = i(f)$ then $d^{i_1} = (d_2, d_1, \ldots, d_i)$ also defines a minimal projection. Since $\beta_{i(f)} > f^{-1}_{i(f)}$, $d^i \neq d^{i_1}$. The proof of Theorem 2.3.12 is complete.

Reasoning as in Remark 2.3, we can establish the following

Remark 2.3.13. In the complex case Theorem 2.3.12 does not hold. However, the conditions (2.3.24) and (2.3.25) are equivalent to the unicity of a minimal projection in the complex case as well.

Proof. In view of Remark 2.2.2, without loss of generality, we can consider the case $1 = f_1 \geq \ldots \geq f_n$, $f_3 > 0$. It is also clear, by Remark 2.2.2, that $\|P_d\| \geq \|P_{\mathrm{re}(d)}\|$ ($\mathrm{re}(d) = (\mathrm{re}(d_1), \ldots, \mathrm{re}(d_n))$ for any $P = P_d \in \mathcal{P}_{\mathrm{Id}}(B, D)$. Moreover, if $\mathrm{im}(d) = (\mathrm{im}(d_1), \ldots, \mathrm{im}(d_n)) \neq 0$ then $\|P_d\| > \|P_{\mathrm{re}(d)}\|$. Hence the problem of unicity of a minimal projection reduces to the real case.

Remark 2.3.14. The equivalence of conditions (2.3.24) and (2.3.25) to the unicity of a minimal projections was proved, by a different method from that of Theorem 3.12, in [Od] or [OdL, p. 70]. Theorem 3.12 in a slightly different version was proved in [OdL, Th. III.3.11, p. 113].

II.4. Minimal projections onto subspaces of $l_\infty^{(n)}$ of codimension two. In this section, unless otherwise stated, $B = l_\infty^{(n)}$ (we consider only the real case) and $D \subset B$ is a subspace of codimension 2 (i.e. $D = \ker(f_1) \cap \ker(f_2)$) where $f_1, f_2 \in S_{B^*}$ are linearly independent functionals). We present a partial solution of the problem of calculating $\lambda_{\mathrm{Id}}(B, D)$ (see (0.1.18)) as well as the problem of finding a minimal projection. As in Section II.2, for brevity, we will write λ_{Id} instead of $\lambda_{\mathrm{Id}}(B, D)$, $\mathcal{P}$ instead of $\mathcal{P}_{\mathrm{Id}}(B, D)$ and $\mathcal{P}^0$ instead of $\mathcal{P}^0_{\mathrm{Id}}(B, D)$. In the sequel we will need the following well known

LEMMA 2.4.1 (see e.g. [Bl, Lemma 1]). *Assume B is a normed space and let D be a subspace of codimension k, $D = \bigcap_{i=1}^k \ker(f^i)$, where the $f^i \in B^*$ are linearly independent. Then there exist $y^1, \ldots, y^k \in B$ satisfying $f^i(y^j) = \delta_{ij}$ for $i, j = 1, \ldots, k$ such that*

$$Px = x - \sum_{i=1}^k f^i(x) y^i \quad \text{for } x \in B\,. \tag{2.4.1}$$

On the other hand, if $y^1, \ldots, y^k \in B$ satisfy $f^i(y_j) = \delta_{ij}$ then the operator $P = \mathrm{Id} - \sum_{i=1}^k f^i(\cdot) y^i$ belongs to $\mathcal{P}$.

First we prove some preliminary results. We start with

Remark 2.4.2. Let $D \subset B$ be a subspace of codimension two. Then for every $i \in \{1, \ldots, n\}$ with $e_i \notin D$ there exists a unique (up to a constant) $f^i \in B^* \setminus \{0\}$ such that $f^i|_D = 0$ and $f^i_i = 0$.

Proof. Put $D_i = D \oplus [e_i]$. Since $e_i \notin D$, $\dim D_i = n - 1$. Consequently, there exists a unique (up to a constant) $f^i \in B^*$ satisfying $f^i|_{D_i} = 0$, as desired.

LEMMA 2.4.3. *Let $D \subset \ker(f)$ for some $f \in S_{B^*}$. If there exists $i \in \{1, \ldots, n\}$ satisfying*

$$|f_i| \geq \sum_{k \neq i} |f_k| \tag{2.4.2}$$

then for every $L \in \mathcal{L}(B, \ker(f))$,

$$\|e_i \circ L\| \leq \max_{j \neq i} \|e_j \circ L\|\,. \tag{2.4.3}$$

Moreover, if we have strict inequality in (2.4.2) *then the same holds in* (2.4.3).

Proof. Take any $f \in S_{B^*}$ such that we have strict inequality in (2.4.2). Then it is easy to deduce that $\|e_i|_{\ker(f)}\| < 1$. Hence

$$\|e_i \circ L\| \leq \|e_i|_{\ker(f)}\| \|L\| < \|L\|$$

and consequently,

$$\|e_i \circ L\| < \max_{l \neq i} \|e_l \circ L\|\,.$$

If we have equality in (2.4.2), then we can approximate f by a sequence $\{f^n\} \subset B^*$ such that f^n satisfies the strict inequality in (2.4.2). From this, it is easy to derive that (2.4.3) holds true for any $L \in \mathcal{L}(B, \ker(f))$.

LEMMA 2.4.4. *Let $D = \ker(f^1) \cap \ker(f^2)$, where $f^1, f^2 \in B^*$ are linearly independent. Let $P \in \mathcal{P}$, $P = \mathrm{Id} - (f^1(\cdot)y^1 + f^2(\cdot)y^2)$ where $y^1, y^2 \in B$. Then*

$$\|P\| = \max_{i=1,\dots,n} |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{j \neq i} |f_j^1 y_i^1 + f_j^2 y_i^2| . \tag{2.4.4}$$

Proof. Note that $\|P\| = \max_{i=1,\dots,n} \|e_i \circ P\|$. So to finish the proof, it is sufficient to show that for each $i \in \{1, \dots, n\}$,

$$\|e_i \circ P\| = |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{j \neq i} |f_j^1 y_i^1 + f_j^2 y_i^2| . \tag{2.4.5}$$

To do this, take any $x \in S_{B^*}$. Then

$$\begin{aligned}(e_i \circ P)x &= x_i - f^1(x) y_i^1 - f^2(x) y_i^2 \\ &= x_i - \Big(\sum_{j=1}^n f_j^1 x_j\Big) y_i^1 - \Big(\sum_{j=1}^n f_j^2 x_j\Big) y_i^2 \\ &= x_i(1 - f_i^1 y_i^1 - f_i^2 y_i^2) - \sum_{j \neq i} x_j (f_j^1 y_i^1 + f_j^2 y_i^2) \\ &\le |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{j \neq i} |f_j^1 y_i^1 + f_j^2 y_i^2| .\end{aligned}$$

Hence

$$\|e_i \circ P\| \le |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{i \neq j} |f_j^1 y_i^1 + f_j^2 y_i^2| .$$

If we take $x = (x_1, \dots, x_n)$, where $x_i = \operatorname{sgn}(1 - f_i^1 y_i^1 - f_i^2 y_i^2)$ and $x_j = -\operatorname{sgn}(f_j^1 y_i^1 + f_j^2 y_i^2)$ for $j \neq i$, we get (2.4.5), which completes the proof of the lemma.

LEMMA 2.4.5. *Let f^1, f^2 be as in Lemma 4.4. Put*

$$g^1 = (\operatorname{sgn}(f_1^2) f_1^1, \dots, \operatorname{sgn}(f_n^2) f_n^1)$$

and

$$g^2 = (|f_1^2|, \dots, |f_n^2|) . \tag{2.4.6}$$

Then the set $\mathcal{P}_{\mathrm{Id}}(B, D)$ is linearly isometric to $\mathcal{P}_{\mathrm{Id}}(B, D_1)$ where $D_1 = \ker(g^1) \cap \ker(g^2)$.

Proof. It is easily seen that the mapping $L : B \to B$ defined by

$$Lx = (x_1 \operatorname{sgn}(f_1^2), \dots, x_n \operatorname{sgn}(f_n^2))$$

is a linear isometry such that $L(D_1) = D$. Hence the mapping $\Psi(P) = L^{-1} \circ P \circ L$ for $P \in \mathcal{P}_{\mathrm{Id}}(B, D)$ is the required linear isometry between $\mathcal{P}_{\mathrm{Id}}(B, D)$ and $\mathcal{P}_{\mathrm{Id}}(B, D_1)$.

Now we can state the main result of this section.

THEOREM 2.4.6. *Let $D \subset B$ be a subspace of codimension two. Assume furthermore that there are $i_0 \in \{1, \ldots, n\}$ and $f^{i_0} \in B^* \setminus \{0\}$ with $f^{i_0}_{i_0} = 0$ and $f^{i_0}|_D = 0$ such that*

$$|f^{i_0}_j| \geq \sum_{k \neq j} |f^{i_0}_k| \quad \text{for some } j \in \{1, \ldots, n\}\,. \tag{2.4.7}$$

Let $f^j \in S_{B^*}$ satisfy the assumptions of Remark 4.2 for $i = j$. Then if there exists $j_0 \in \{1, \ldots, n\}$ such that

$$|f^j_{j_0}| \geq \sum_{k \neq j_0} |f^j_k| \tag{2.4.8}$$

then $\lambda_{\mathrm{Id}} = 1$. In the opposite case

$$\lambda_{\mathrm{Id}} = 1 + \Big(\sum_{i=1}^{n} |f^j_i| / (1 - 2|f^j_i|) \Big)^{-1}.$$

Proof. First we consider the case described by (2.4.7) and (2.4.8). Define $y^j = (y^j_1, \ldots, y^j_n)$ by

$$y^j_k = \begin{cases} 0 & \text{if } k \neq j_0, j\,, \\ 1/f^j_{j_0} & \text{if } k = j_0\,, \\ -f^{i_0}_{j_0}/(f^j_{j_0} f^{i_0}_j) & \text{if } k = j \end{cases} \tag{2.4.9}$$

and $y^{i_0} = (y^{i_0}_1, \ldots, y^{i_0}_n)$ by

$$y^{i_0}_k = \begin{cases} 0 & \text{if } k \neq j\,, \\ 1/f^{i_0}_j & \text{if } k = j\,. \end{cases} \tag{2.4.10}$$

Consider the operator

$$P = \mathrm{Id} - f^{i_0}(\cdot) y^{i_0} - f^j(\cdot) y^j\,. \tag{2.4.11}$$

Since $f^j(y^{i_0}) = f^{i_0}(y^j) = 0$ and $f^j(y^j) = f^{i_0}(y^{i_0}) = 1$ the operator P belongs to $\mathcal{P}$. In view of Lemma 4.4,

$$\|P\| = \max\{1, \|e_{j_0} \circ P\|, \|e_j \circ P\|\}\,.$$

According to (2.4.7), (2.4.8) and Lemma 4.3, $\|P\| = 1$ and consequently, $\lambda_{\mathrm{Id}} = 1$.

Now suppose f^{i_0} satisfies (2.4.7) and f^j does not satisfy (2.4.8). In view of Lemma 4.5 we can assume that $f^j \geq 0$. Put $w^j = (w^j_1, \ldots, w^j_n)$ where

$$w^j_k = \begin{cases} (\sum)^{-1}/(1 - 2f^j_k) & \text{if } k \neq j\,, \\ -\sum_{l \neq j} (f^{i_0}_l (\sum)^{-1})/((1 - 2f^j_l) f^{i_0}_k) & \text{if } k = j \end{cases} \tag{2.4.12}$$

(we will write

$$\Big(\sum\Big)^{-1} = \Big(\sum_{l=1}^{n} |f^j_l| / (1 - 2|f^j_l|) \Big)^{-1} \tag{2.4.13}$$

for brevity; cf. [Bl]). It is easy to check that $f^j(w^j) = 1$ and $f^{i_0}(w^j) = 0$. Define

$$P^1 = \mathrm{Id} - f^{i_0}(\cdot)y^{i_0} - f^j(\cdot)w^j\,. \tag{2.4.14}$$

In view of Lemma 4.4 one can check that

$$\|P^1\| = \max\Big\{\|e_j \circ P^1\|, 1 + \Big(\sum_{l=1}^{n} |f_l^j|/(1-2|f_l^j|)\Big)^{-1}\Big\}.$$

According to Lemma 4.3, $\|e_j \circ P^1\| \le \max_{l\ne j} \|e_l \circ P^1\|$. Hence

$$\|P^1\| = 1 + \Big(\sum_{k=1}^{n} |f_k^j|/(1-2|f_k^j|)\Big)^{-1}$$

and $\lambda_{\mathrm{Id}} \le 1 + (\sum_{k=1}^{n} |f_k^j|/(1-2|f_k^j|))^{-1}$.

To prove the opposite inequality, we apply Theorem 1.2.5(a). First observe that $\mathrm{crit}^*(P^1) \supset E = \{i : f_i^j > 0\}$. According to the proof of Lemma 4.4 and (2.4.14), for each $i \in \mathrm{crit}^*(P^1) \setminus \{j\}$,

$$\{x^{i,1} = (x_1^{i,1}, , \ldots, x_n^{i,1}), x^{i,2} = (x_1^{i,2}, \ldots, x_n^{i,2})\} \subset \mathcal{A}_i$$

(see Theorem 1.2.5), where

$$x_l^{i,1} = \begin{cases} 1 & \text{if } l = j, i \text{ or } f_l^j = 0\,, \\ -1 & \text{if } f_l^j > 0 \text{ and } l \ne j, i \end{cases} \tag{2.4.15}$$

and

$$x_l^{i,2} = \begin{cases} 1 & \text{if } l = i \text{ or } (f_l^j = 0,\ l \ne j)\,, \\ -1 & \text{if } l = j \text{ or } (f_l^j > 0,\ l \ne i)\,. \end{cases} \tag{2.4.16}$$

Now we will show that for every $L \in \mathcal{L}_D = \{L \in \mathcal{L}(B, D) : L|_D = 0\}$ there exists an index $i \in E$ such that $e_i(Lx^{i,k}) \le 0$ for $k = 1$ or $k = 2$. Fix $L \in \mathcal{L}_D$. According to Lemma 4.1, $L = f^{i_0}(\cdot)z^{i_0} + f^j(\cdot)z^j$ for some $z^{i_0}, z^j \in D$. First observe that, by (2.4.7), for any $i \in E$,

$$f^{i_0}(x^{i,1})z_i^{i_0} \le 0 \quad \text{or} \quad f^{i_0}(x^{i,2})z_i^{i_0} \le 0\,. \tag{2.4.17}$$

Hence if $z_i^j = 0$ for some $i \in E$ then

$$e_i(Lx^{i,1}) \le 0 \quad \text{or} \quad e_i(Lx^{i,2}) \le 0\,.$$

Now let $z_i^j \ne 0$ for any $i \in E$. Since $z^j \in D$, $f_j^j = 0$ and $\|f^j\| = 1$, $z_i^j > 0$ for some $i \in E$. Moreover, according to (2.4.15) and (2.4.16), $f^j(x^{i,1}) = f^j(x^{i,2})$. Hence for $k = 1, 2$,

$$f^j(x^{i,k})z_i^j = \Big(\sum_{l\ne j} f_l^j x_l^{i,k}\Big)z_i^j = z_i^j\Big(|f_i^j| - \sum_{l\ne i,j} |f_l^j|\Big) < 0\,, \tag{2.4.18}$$

since f^j does not satisfy (2.4.8). Consequently, by (2.4.17) and (2.4.18) there exist $i \in E \subset \mathrm{crit}^*(P^1)$ and $k \in \{1, 2\}$ such that $e_i(Lx^{i,k}) \le 0$. Since $x^{i,k} \in \mathcal{A}_i$ for $k = 1, 2$, by Theorem 1.2.5(a), the proof of Theorem 4.6 is complete.

COROLLARY 2.4.7. *Suppose* f^{i_0} *satisfies* (2.4.7) *and* f^j *satisfies* (2.4.8). *Then the projection* P *defined by* (2.4.11) *belongs to* $\mathcal{P}^0$. *If* f^j *does not satisfy* (2.4.8) *the same holds for* P^1 *defined by* (2.4.14).

EXAMPLE 2.4.8. Let $f = (0, 1/3, 0, 1/3, 0, 1/3)$ and $g = (1/3, 0, 1/3, 0, 1/3, 0)$. Then $D = \ker(f) \cap \ker(g)$ does not satisfy the assumptions of Theorem 4.6.

Remark 2.4.9. Theorem 4.6 was proved by the author in [LG4].

II.5. Uniqueness of minimal projection onto subspaces of $l_\infty^{(n)}$ of codimension two. In this section we present necessary and sufficient conditions for D under which P and P^1 given by (2.4.11) and (2.4.14) are unique minimal projections. The notation used in this section is the same as in Section II.4. We start with

THEOREM 2.5.1. *Suppose* f^{i_0} *satisfies* (2.4.7) *and* f^j *satisfies* (2.4.8). *Then the projection* P *defined by* (2.4.11) *is a unique minimal projection if and only if either*

$$|f^{i_0}_{j_0}| = |f^{i_0}_j| \quad \text{and} \quad |f^j_{j_0}| > |f^j_l| \quad \text{for } l \neq j_0\,, \tag{2.5.1}$$

or

$$|f^{i_0}_j| > |f^{i_0}_l| \quad \text{for } l \neq j \quad \text{and} \quad |f^j_{j_0}| > |f^j_l| \quad \text{for } l \neq j_0 \tag{2.5.2}$$

(*the indices* i_0, j, j_0 *are the same as in Theorem* 2.4.6).

Proof. Assume that (2.5.2) holds. We will show that P is a SUM (strongly unique minimal) projection. According to (2.4.11) and Lemma 2.4.4 the set $\{i : i \neq j, j_0\} \subset \text{crit}^*(P)$. Moreover, for each $i \neq j$, j_0 the set $\mathcal{A}_i$ (see Th. 1.2.5) contains all the vectors from the set ext_B having the ith coordinate equal to 1. Take any $L \in \mathcal{L}_D \setminus \{0\}$. We will show that there exist $i \neq j$, j_0 and $x \in \mathcal{A}_i$ such that $e_i(Lx) < 0$. By Lemma 2.4.1,

$$L = f_{i_0}(\cdot) z^{i_0} + f^j(\cdot) z^j$$

where z_{i_0}, $z^j \in D$. Since $L \neq 0$ either $z^{i_0}_i \neq 0$ or $z^j_i \neq 0$ for some $i \in \{1, \ldots, n\}$. Moreover, since $f^j_j = 0$, $f^{i_0}_j \neq 0$ and $f^j_{j_0} \neq 0$, we can assume that $i \neq j, j_0$. Now we divide the proof into two cases.

Case 1: $z^{i_0}_i \neq 0$ *and* $z^j_i = 0$. By (2.5.2), there exist $x^1, x^2 \in \mathcal{A}_i$ such that $f^{i_0}(x^1) f^{i_0}(x^2) < 0$. Consequently, $e_i(Lx^k) < 0$ for $k = 1$ or 2.

Case 2: $z^j_i \neq 0$. Reasoning as above, we can find $x^1, x^2 \in \mathcal{A}_i$ such that $f^j(x^1) f^j(x^2) < 0$. Modifying the jth coordinate of x^1 and x^2 if necessary, we can assume that $f^{i_0}(x^1) z^{i_0}_i \leq 0$ and $f^{i_0}(x^2) z^{i_0}_i \leq 0$. Consequently, $e_i(Lx^k) < 0$ for $k = 1$ or 2.

Now consider the function

$$F(L) = \min_{i \neq j, j_0} \min_{x \in \mathcal{A}_i} e_i(Lx)$$

for $L \in \mathcal{L}_D$. It is easily seen that F is continuous as the minimum of a finite number of continuous functions. Moreover, by the reasoning presented before, $F(L) < 0$ for every $L \neq 0$. Since $\mathcal{L}_D$ is finite-dimensional, the constant $\gamma = \sup_{L\in S(\mathcal{L}_D)} F(L)$ is strictly negative. We will show that P given by (2.4.11) is a SUM projection with constant $r = -\gamma$. To do this, take any $L \in \mathcal{L}_D \setminus \{0\}$. Then there exist $i \neq j, j_0$ and $x \in \mathcal{A}_i$ with

$$e_i(L/\|L\|)x = F(L/\|L\|)\,.$$

Hence

$$e_i(Lx) \le -r\|L\|\,,$$

which, according to Theorem 1.2.5(b), completes the proof of this part.

Now assume that (2.5.1) holds. Define

$$g^{i_0} = f^{i_0} + (-f^{i_0}_{j_0}/f^{j}_{j_0})f^j\,.$$

Then $D = \ker(g^{i_0}) \cap \ker(f^j)$ and according to (2.5.1), g^{i_0} and f^j satisfy (2.5.2), which completes the proof of this part.

Now suppose neither (2.5.1) nor (2.5.2) holds. First we consider the case $|f^{i_0}_k| = |f^{i_0}_j|$ for some $k \neq i_0, j_0, j$. Define $z = (z_1, \ldots, z_n) \in B$ by

$$z_l = \begin{cases} 1/f^{i_0}_k & \text{if } l = k\,, \\ -f^j_k/(f^{i_0}_k f^j_{j_0}) & \text{if } l = j_0\,, \\ 0 & \text{if } l \neq k, j_0\,. \end{cases}$$

Set

$$Q = \mathrm{Id} - f^{i_0}(\cdot)z - f^j(\cdot)y^j$$

where y^j is given by (2.4.9). Since $|f^j_{j_0}|, |f^{i_0}_j|, |f^{i_0}_k| \ge 1/2$, in view of Lemma 2.4.4, $\|e_l \circ Q\| \le \max_{n\neq l}\|e_n \circ Q\|$ for $l = j, j_0, k$. Observe that for $l \neq j, j_0, k$, $e_l \circ Q = e_l$. Consequently, $\|Q\| = 1$. Since $z \neq y^{i_0}$ (see (2.4.10), Q is a minimal projection different from P, which completes the proof of this case.

Now let $|f^j_k| = |f^j_{j_0}|$ for some $k \neq j, j_0$. Changing the roles of f^{i_0} and f^j and reasoning as before we can construct a projection Q of norm one different from P. The proof of Theorem 5.1 is complete.

LEMMA 2.5.2. *Assume* $D = \ker(f^{i_0}) \cap \ker(f^j)$ *where* f^j *and* f^{i_0} *are as in Theorem* 2.4.6. *Assume furthermore that*

$$f^j_k \neq 0 \quad \textit{for } k \neq j$$

and

$$1/2 = |f^{i_0}_j| > |f^{i_0}_k| \quad \textit{for } k \neq j\,.$$

Then $\|e_j \circ P^1\| < \|P^1\|$ *where* P^1 *is defined by* (2.4.14).

Proof. In view of Lemma 2.4.5, we may assume $f^j_l > 0$ for $l \neq j$. Observe that by Lemma 2.4.4, (2.4.10) and (2.4.12),

$$\|e_j \circ P^1\| = \sum_{k\neq j} \Big| f_k^{i_0} - f_k^j \Big(\sum_{l\neq i_0,j} f_l^{i_0}/(1-2f_l^j)\Big) \cdot \Big(\sum\Big)^{-1} \Big| / |f_j^{i_0}| \,.$$

(The symbol $(\sum)^{-1}$ is introduced in (2.4.13)). If we fix f^j and $f_j^{i_0}$ the above formula may be considered as a convex function (we will denote it by φ) of the variables $f_k^{i_0}$, $k \neq j$, satisfying, $\sum_{k\neq j} |f_k^{i_0}| = 1/2$. Hence to finish the proof, it is sufficient to show that φ is strictly convex. To do this, we prove that for α, $\beta \neq 0$, $|\alpha| + |\beta| = 1/2$, and two different indices $k, l \neq i_0, j$,

$$\varphi(\alpha e_k + \beta e_l) < \|P^1\| = 1 + \Big(\sum\Big)^{-1}.$$

Observe that

$$\begin{aligned}\varphi(\alpha e_k + \beta e_l) = (1/f_j^{i_0})\Big(&\Big|\alpha - f_k^j(\alpha/(1-2f_k^j) + \beta/(1-2f_l^j))\Big(\sum\Big)^{-1}\Big| \\ &+ \Big|\beta - f_l^j(\alpha/(1-2f_k^j) + \beta/(1-2f_l^j))\Big(\sum\Big)^{-1}\Big| \\ &+ \sum_{m\notin\{j,l,k\}} |f_m^j| \Big|(\alpha/(1-2f_k^j) + \beta/(1-2f_l^j))\Big(\sum\Big)^{-1}\Big| \,.\end{aligned}$$

Hence if $\varphi(\alpha e_k + \beta e_l) = \|P^1\|$ then

$$\begin{aligned}\text{(2.5.3)}\quad &\Big|\alpha\Big(1 - (f_k^j/(1-2f_k^j))\Big(\sum\Big)^{-1}\Big) - (\beta f_k^j/(1-2f_l^j))\Big(\sum\Big)^{-1}\Big| \\ &= \Big|\alpha\Big(1 - (f_k^j/(1-2f_k^j))\Big(\sum\Big)^{-1}\Big)\Big| + \Big|(\beta f_k^j/(1-2f_l^j))\Big(\sum\Big)^{-1}\Big|\end{aligned}$$

and

$$\text{(2.5.4)}\qquad |\alpha/(1-2f_k^j) + \beta/(1-2f_l^j)| = |\alpha/(1-2f_k^j)| + |\beta(1-2f_l^j)| \,.$$

Consequently, according to (2.5.4), $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\beta)$. But, by (2.5.3), $\operatorname{sgn}(\alpha) = -\operatorname{sgn}(\beta)$; a contradiction ($\alpha, \beta \neq 0$). The proof of Lemma 5.2 is complete.

THEOREM 2.5.3. *Let f^{i_0}, f^j be as in Theorem* 2.4.6. *Assume furthermore that f^j does not satisfy* (2.4.8) *and either*

$$\text{(2.5.5)}\qquad |f_k^{i_0}| = |f_j^{i_0}| = 1/2 \quad \textit{for some } k \neq j, i_0$$

or

$$\text{(2.5.6)}\qquad f_l^j = 0 \quad \textit{for some } l \neq j \,.$$

Then the projection P^1 defined by (2.4.14) *is not a unique minimal projection.*

Proof. In view of Lemma 2.4.5 we can assume that $f^j \geq 0$. Let f^{i_0} satisfy (2.5.5). For $\alpha \in \mathbb{R}$ put

$$u_\alpha^j = w^j - (\alpha e_{i_0} + \beta e_j + \gamma e_k)\,,$$

where w^j is given by (2.4.12) and β, γ are so chosen that $f^j(u^j_\alpha) = 1$, $f^{i_0}(u^j_\alpha) = 0$. For $x \in B$ define

$$Q^\alpha x = x - f^{i_0}(x)y^{i_0} - f^j(x)u^j_\alpha\,,$$

where y^{i_0} satisfies (2.4.10). By Lemma 2.4.1, Q^α belongs to $\mathcal{P}$. We will show that $\|Q^\alpha\| = \lambda_{\rm Id}$ for α sufficiently small. According to (2.4.5) and (2.4.12), $\|e_{i_0} \circ Q^\alpha\| < \lambda_{\rm Id}$ for $\alpha \geq 0$ sufficiently small. By Lemma 2.4.3,

$$\|e_l \circ Q^\alpha\| \leq \max_{u \neq l} \|e_u \circ Q^\alpha\|$$

for $l = j, k$. For $l \neq i_0, j, k$, in view of (2.4.4) and (2.4.12), $\|e_l \circ Q^\alpha\| = \lambda_{\rm Id}$ (since f^j does not satisfy (2.4.8) such an index exists). Consequently, $\|Q^\alpha\| = \lambda_{\rm Id}$, which proves that P^1 is not a unique minimal projection.

Now let f^j satisfy (2.5.6). For $\alpha \in \mathbb{R}$ define

$$z^j_\alpha = w^j - (\alpha e_l + \beta e_j)\,,$$

where w^j is given by (2.4.12) and β is so chosen that $f^j(z^j_\alpha) = 1$ and $f^{i_0}(z^j_\alpha) = 0$. For $x \in l^{(n)}_\infty$ define

$$Z^\alpha x = x - f^{i_0}(x)y^{i_0} - f^j(x)z^j_\alpha\,.$$

By Lemma 2.4.1, Z^α belongs to $\mathcal{P}$. Reasoning in a similar way to the case of Q^α we can show that $\|Z^\alpha\| = \lambda_{\rm Id}$ for α sufficiently small. The proof of Theorem 5.3 is complete.

To present the next result of this section set for $i \neq j$,

$$E_1 = \{i : f^{i_0}(x^{i,1}) = 0\} \tag{2.5.7}$$

and

$$E_2 = \{i : f^{i_0}(x^{i,2}) = 0\} \tag{2.5.8}$$

where $x^{i,1}, x^{i,2}$ are defined by (2.4.15) and (2.4.16).

THEOREM 2.5.4. *Let* f^{i_0}, f^j *be as in Theorem* 2.4.6. *Assume furthermore that* f^{i_0} *does not satisfy* (2.5.5) *and* f^j *does not satisfy* (2.5.6). *Then* P^1 *is not a unique minimal projection if and only if* E_1 *and* E_2 *are nonempty sets.*

Proof. In view of Lemma 2.4.5, we can assume that $f^j \geq 0$ and $f^{i_0}_j > 0$. Suppose E_1, E_2 are nonempty sets. By Theorem 0.2.7, $\operatorname{card} \operatorname{ext}_{\mathcal{L}^*(B)}$ is finite. Hence, according to Proposition 0.2.17, it is sufficient to show that P^1 is not a SUM projection. To do this, take $y \in S_D$ such that $y_k < 0$ for $k \in E_1$, $y_k > 0$ for $k \in E_2$ and $y_k = 0$ for $k \notin E_1 \cup E_2 \cup \{j\}$. Put $L = f^{i_0}(\cdot)y$. We will show that for any $i \in \operatorname{crit}^*(P^1)$,

$$\inf_{x \in \mathcal{A}_i} e_i(Lx) \geq 0\,. \tag{2.5.9}$$

By Lemma 5.2 and (2.4.14), $\operatorname{crit}^*(P^1) = \{1, \ldots, n\} \setminus \{j\}$. Since $f^{i_0}_j > 0$, either for any $k \in E_1$,

$$f^{i_0}(x^{k,l}) \leq 0 \qquad (l = 1, 2)\,,$$

or for any $k \in E_2$,

$$f^{i_0}(x^{k,l}) \geq 0 \quad (l = 1, 2)\,.$$

Consequently, for any $i \neq j$ and $l = 1, 2$, $f^{i_0}(x^{i,l})y_i \geq 0$. Since f^j does not satisfy (2.5.6), $\mathcal{A}_i = \{x^{i,1}, x^{i,2}\}$ for $i \neq j$. Hence, by the above reasoning, (2.5.9) holds true. In view of Theorem 1.2.5(b), P^1 is not a SUM projection as desired.

To prove the converse, first suppose that $E_1 \neq \emptyset$ and $E_2 = \emptyset$. Take any $L \in \mathcal{L}_D \setminus \{0\}$. We will show that there is an $i \in \mathrm{crit}^*(P^1)$ such that

$$\inf_{x \in \mathcal{A}_i} e_i(Lx) < 0\,.$$

According to Lemma 2.4.1, $L = f^{i_0}(\cdot)z^{i_0} + f^j(\cdot)z^j$. Since $L \neq 0$, $z_i^{i_0} \neq 0$ or $z_i^j \neq 0$ for some $i \in \{1, \ldots, n\}$. Moreover, since $f_j^{i_0} \neq 0$, we may assume that $i \neq j$. If $z^j \neq 0$, reasoning as in Theorem 2.4.6, we can show that $e_i(Lx^{i,1}) < 0$ or $e_i(Lx^{i,2}) < 0$. In the opposite case, since $f_i^j > 0$ for $i \neq j$, either $z_i^{i_0} > 0$ for some $i \in E_1$ or $z_i^{i_0} \neq 0$ for some $i \notin E_1 \cup \{j\}$. Since E_2 is an empty set, it is easy to check that $\inf_{x \in \mathcal{A}_i} e_i(Lx) < 0$, where the index i is defined as above. To finish the proof of this part, consider the function F defined by (2.5.3). Applying Theorem 1.2.5(b) and reasoning as in Theorem 5.1, we find that P^1 is a SUM projection.

If $E_2 \neq \emptyset$ and $E_1 = \emptyset$ or $E_1, E_2 = \emptyset$, reasoning in the same manner as in the previous part of the proof, we conclude that P^1 is a SUM projection. The proof of Theorem 5.4 is complete.

Example 2.5.5 (nonuniqueness). Let

$$f^{i_0} = (1/2, 0, 1/4, -1/4), \qquad f^j = (0, 1/3, 1/3, 1/3)\,.$$

Then it is easy to check that $E_1 = \{4\}$, $E_2 = \{3\}$. Hence, in view of Theorem 5.4, P^1 is not a unique minimal projection.

Example 2.5.6 (uniqueness). Let

$$f^{i_0} = (1/2, 0, 1/4, 1/4), \qquad f^j = (0, 1/3, 1/3, 1/3)\,.$$

Then $E_1 = \{2\}$, $E_2 = \emptyset$. By Theorem 3.4, P^1 is a SUM projection.

Remark 2.5.7. Theorems 5.1 and 5.3 were proved by the author in [LG4].

II.6. Strong unicity criterion in some space of operators. Throughout this section, unless otherwise stated, B will stand for a finite-dimensional real Banach space and f will be a functional from S_{B^*}. For given $L \in \mathcal{L}(B)$ we will write for brevity $\mathcal{P}_D(L) = \{L_0 \in \mathcal{L}_D : \|L - L_0\| = \mathrm{dist}(L, \mathcal{L}_D)\}$. If $D \subset B$ is a linear subspace and $A \subset B^*$ then $A|_D$ stands for the set of all restrictions of functionals from A. We start with two preliminary results.

Remark 2.6.1. For $L \in \mathcal{L}(B)$ set

$$\mathrm{crit}(L) = \{x \in S_B : \|Lx\| = \|L\|\}\,. \tag{2.6.1}$$

Assume $L_0 \in \mathcal{P}_D(L)$, $D = \ker(f)$, $\|f\| = 1$ and $\|L - L_0\| > \|L|_D\|$. Put

$$C_{L-L_0} = \{x \in \mathrm{crit}(L - L_0) : f(x) > 0\}\,. \tag{2.6.2}$$

Then C_{L-L_0} is a nonempty closed set, $C_{L-L_0} \cap -C_{L-L_0} = \emptyset$ and

$$C_{L-L_0} \cup -C_{L-L_0} = \mathrm{crit}(L - L_0)\,.$$

P r o o f. It is clear that the set $A = \{x \in \mathrm{crit}(L - L_0) : f(x) \geq 0\}$ is closed. Since $\mathrm{dist}(L, \mathcal{L}_D) > \|L|_D\|$ and D is a hyperplane,

$$A = \{x \in \mathrm{crit}(L - L_0) : f(x) > 0\}\,,$$

which proves that C_{L-L_0} is closed. The fact that $\mathrm{crit}(L - L_0) \cap D = \emptyset$ implies immediately that $C_{L-L_0} \cup -C_{L-L_0} = \mathrm{crit}(L)$. By (2.6.2), $C_{L-L_0} \cap -C_{L-L_0} = \emptyset$.

R e m a r k 2.6.2. Let $L \in \mathcal{L}(B)$, $\mathrm{dist}(L, \mathcal{L}_D) > \|L|_D\|$, $L_0 \in \mathcal{L}_D$. Define

$$D_{L-L_0} = \{h \in \mathrm{crit}^*(L - L_0) : C_{L-L_0} \cap A_h \neq \emptyset\} \tag{2.6.3}$$

(see (1.2.1)) and if $L \in \mathcal{L}(B, D)$,

$$D^D_{L-L_0} = \{h \in \mathrm{crit}^*(L - L_0) : C_{L-L_0} \cap A_h \neq \emptyset\}\,. \tag{2.6.4}$$

Then D_{L-L_0} (resp. $D^D_{L-L_0}$) is a compact set, and $D_{L-L_0} \cap -D_{L-L_0} = \emptyset$ (resp. $D^D_{L-L_0} \cap -D^D_{L-L_0} = \emptyset$).

P r o o f. Assume $h \in \mathrm{cl}(D_{L-L_0})$ and let $\{h_n\} \subset D_{L-L_0}$, $h_n \to h$. By (2.6.3), for every $n \in \mathbb{N}$ there exists $x_n \in C_{L-L_0} \cap A_{h_n}$, i.e. $h_n(L - L_0)x_n = \|L - L_0\|$. Passing to a subsequence if necessary, we can assume $x_n \to x$. By Remark 6.1, $x \in C(L - L_0)$. Note that

$$\begin{aligned} h(L - L_0)x &= h_n(L - L_0)x + (h - h_n)(L - L_0)x \\ &= h_n(L - L_0)x_n + h_n(L - L_0)(x - x_n) + (h - h_n)(L - L_0)x\,. \end{aligned}$$

Since the last two terms tend to 0 as $n \to \infty$, $h(L - L_0)x = \|L - L_0\|$ and consequently, $x \in A_h$. Since $x \in C_{L-L_0}$, $h \in D_{L-L_0}$ by (2.6.3). Note that $\mathrm{dist}(L, \mathcal{L}_D) > \|L|_Y\|$ implies $D_{L-L_0} \cap -D_{L-L_0} = \emptyset$.

The proof for the set $D^D_{L-L_0}$ goes in the same manner, so we omit it.

Now we state the main result of this section.

THEOREM 2.6.3. *Assume* $L \in \mathcal{L}(B)$ *and let* $D = \ker(f)$, $\|f\| = 1$. *Assume furthermore that* $\mathrm{dist}(L, \mathcal{L}_D) > \|L|_D\|$ *and let* $L_0 \in \mathcal{L}_D$. *Then the following conditions are equivalent*:

(a) L_0 *is a SUBA to* L *in* $\mathcal{L}_D$ (*resp.* $L_0 \in \mathcal{P}_D$).
(b) $0 \in \mathrm{int\,conv}\, D_{L-L_0}|_D$ (*resp.* $0 \in \mathrm{conv}\, D_{L-L_0}|_D$).

P r o o f. Assume L_0 is a SUBA to L in $\mathcal{L}_D$ and let $0 \notin$ int conv $D_{L-L_0}|_D$. This means that there exists $\psi \in D^{**}$ with $\psi(h) \geq 0$ for every $h \in D_{L-L_0}|_D$ (we can assume $\|\psi\| = 1$). Since D is finite-dimensional, $\psi = d$ for some $d \in S_D$.

Define $L_1 = f(\cdot)d$ and note that $L_1 \in \mathcal{L}_D$. By (2.6.3) and Remark 6.2, for every $h \in D_{L-L_0}$ we have

$$\begin{aligned}\inf\{h(L_1x) : x \in A_h\} &= \inf\{f(x)h(d) : x \in A_h\}\\ &= h(d)\inf\{f(x) : x \in A_h\} \geq 0 > -r\|L_1\|\end{aligned}$$

for every $r > 0$. By Theorem 1.2.5(b), L_0 is not a SUBA to L in $\mathcal{L}_D$; a contradiction. Since by Remark 6.2 the set D_{L-L_0}, and consequently $\operatorname{conv} D_{L-L_0}$, is compact, the same reasoning applies to the second case.

To prove the converse, define a function $g : S|_D \to \mathbb{R}$ by

$$g(d) = \inf\{g_h(d) : h \in D_{L-L_0}\} \quad \text{for } d \in S_D\,, \tag{2.6.5}$$

where $g_h(d) = \inf\{f(x)h(d) : x \in A_h\}$. Note that the function $S_D \ni d \to f(x)h(d)$ is continuous and consequently the functions g_h and g are upper-semicontinuous.

Now assume $0 \in \operatorname{int}\operatorname{conv} D_{L-L_0}|_D$. This means that for every $d \in S_D$ there exsists $h \in D_{L-L_0}$ with $h(d) < 0$. (If not, then $D_{L-L_0}|_D \subset \{h \in D^* : h(d) \geq 0\}$ for some $d \in S_D$ and consequently $\operatorname{int}\operatorname{conv} D_{L-L_0}|_D \subset \{h \in D^* : h(d) > 0\}$. But $0 \in \operatorname{int}\operatorname{conv} D_{L-L_0}$; a contradiction.) Since $\operatorname{dist}(L, \mathcal{L}_D) > \|L|_D\|$ and D is a hyperplane, $g(d) < 0$ for every $d \in S_D$. Since g is upper-semicontinuous, the value $\gamma = \max\{g(d) : d \in S_D\}$ is attained at some point $d_0 \in S_D$ and consequently $\gamma < 0$. We show that L_0 is a SUBA to L in $\mathcal{L}_D$ with $r = -\gamma$. To do this, fix $L_1 \in \mathcal{L}_D \backslash \{0\}$. It is clear that $L_1 = f(\cdot)d_1$ for some $d \in D\backslash\{0\}$. Put $d_2 = d_1/\|d_1\|$, fix $\varepsilon > 0$ and take $h \in D_{L-L_0}$ with $g_h(d_2) < g(d_2) + \varepsilon$. Note that

$$\begin{aligned}g_h(d_2) &= \inf\{f(x)h(d_2) : x \in A_h\} = \inf\{h(L_1x)/\|d_1\| : x \in A_h\}\\ &\leq g(d_2) + \varepsilon \leq -r + \varepsilon\,,\end{aligned}$$

which gives

$$\inf\{h(L_1x) : x \in A_h\} \leq -(r - \varepsilon)\|L_1\|\,.$$

By Theorem 1.2.5(b), L_0 is a SUBA to L in $\mathcal{L}_D$ with constant $r - \varepsilon$ for every $\varepsilon > 0$ and consequently with constant r. The proof is complete.

Remark 2.6.4. If $L \in \mathcal{L}(B, D)$ then the set D_{L-L_0} in Theorem 6.3 can be replaced by $D^D_{L-L_0}$ (see (2.6.4)).

As an immediate consequence of Theorem 6.3 we get

Corollary 2.6.5. *Assume $L \in \mathcal{L}(B)$, $L_0 \in \mathcal{L}_D$, $\|L - L_0\| > \|L|_D\|$. Then the set $D_{L-L_0}|_D$ is linearly dependent. If $L \in \mathcal{L}(B, D)$ the same holds for $D^D_{L-L_0}$.*

Reasoning as in Theorem 2.3.1 we can show

Remark 2.6.6. The constant r from Theorem 6.3 is the best possible.

Now we will point out when the assumption $\operatorname{dist}(L, \mathcal{L}_D) > \|L|_D\|$ is satisfied.

Remark 2.6.7. Assume B is a Banach space and let $D \subset B$ be its complemented subspace. Take $P^0 \in \mathcal{P}_{\mathrm{Id}}(B, D)$ and note that

$$\operatorname{dist}(P^0, \mathcal{L}_D) = \inf\{\|P\| : P \in \mathcal{P}_{\mathrm{Id}}(B, D)\} = \lambda_{\mathrm{Id}}(B, D)\,.$$

In many cases of hyperplanes $\operatorname{dist}(P^0, \mathcal{L}_D) > \|P^0|_D\| = 1$ (see e.g. [Ba2, Bl]). It is well known that if B is not a Hilbert space then there exists a hyperplane D in B satisfying $\lambda_{\mathrm{Id}}(B, D) > 1$. If $\operatorname{dist}(P^0, \mathcal{L}_D) > 1$ then it is easy to show that

$$\operatorname{dist}(L, \mathcal{L}_D) > \|L|_D\| \quad \text{if} \quad \|L - P^0\| < \operatorname{dist}(P^0, \mathcal{L}_D) - 1\,.$$

Now we show an estimate from above of the number $\operatorname{dist}(L, \mathcal{L}_D)$.

PROPOSITION 2.6.8. *Assume B is a Banach space and let D be its complemented subspace. Then for every $L \in \mathcal{L}(B, D)$,*

$$\|L|_D\| \le \operatorname{dist}(L, \mathcal{L}_D) \le \lambda_{\mathrm{Id}}(B, D)\|L|_D\|\,.$$

Proof. Fix $L \in \mathcal{L}(B, D)$ and $\varepsilon > 0$. Take $P_\varepsilon \in \mathcal{P}_{\mathrm{Id}}(B, D)$ with $\|P_\varepsilon\| < \lambda(B, D) + \varepsilon$ and put $L_\varepsilon = L \circ (I - P_\varepsilon)$. It is clear that $L_\varepsilon \in \mathcal{L}_D$. Note that

$$\|L - L_\varepsilon\| = \|L - L \circ (I - P_\varepsilon)\| = \|L \circ P_\varepsilon\| \le \|L|_D\| \cdot \|P_\varepsilon\|\,,$$

which gives the desired result.

COROLLARY 2.6.9. *Assume that $\lambda(B, D) = 1$. Then*

$$\operatorname{dist}(L, \mathcal{L}_D) = \|L|_D\| \quad \textit{for every } L \in \mathcal{L}(B, D)\,.$$

In particular, if there exists $P^0 \in \mathcal{P}_{\mathrm{Id}}(B, D)$ with $\|P^0\| = 1$, then the operator $L_0 = L \circ (I - P^0) \in \mathcal{P}_D(L)$.

Since for D being a hyperplane we have $\lambda_{\mathrm{Id}}(B, D) \le 2$ (for more precise results see [Bl], [OdL, p. 84], [Ro]) we immediately get

COROLLARY 2.6.10. *Assume $D \subset B$ is a hyperplane. Then*

$$\|L|_D\| \le \operatorname{dist}(L, \mathcal{L}_D) \le 2\|L|_D\| \quad \textit{for every } L \in \mathcal{L}(B, D)\,.$$

Now we apply Theorem 6.3 to generalize Theorem I.1.3 of [OdL].

THEOREM 2.6.11. *Assume B is a three-dimensional Banach space and let $D \subset B$ be a hyperplane. Assume furthermore that $L \in \mathcal{L}(B, D)$ and* $\operatorname{dist}(L, \mathcal{L}_D) > \|L|_D\|$. *Then there exists $L_0 \in \mathcal{L}_D$ which is a SUBA to L in $\mathcal{L}_D$.*

Proof. Since $\mathcal{L}_D$ is a finite-dimensional linear space, the set $\mathcal{P}_D(L)$ is nonempty. Take an arbitrary $L_0 \in \mathcal{P}_D(L)$. By Theorem 6.3 and Remark 6.4 it is sufficient to show that $0 \in \operatorname{int} \operatorname{conv} D^D_{L-L_0}$ (see (2.6.4)). Suppose, on the contrary, that this is not true. By Theorem 6.3, $0 \in \operatorname{conv} D^D_{L-L_0}$. Since $\dim D = 2$, we have $0 = \alpha h_1 + (1-\alpha)h_2$ where $h_1, h_2 \in D^D_{L-L_0}$ and $\alpha \in (0, 1)$. Since $\|h_1\| = \|h_2\| = 1$, we easily get $\alpha = 1/2$. Consequently, $h_1 = -h_2$, which gives $h_1 \in D^D_{L-L_0} \cap -D^D_{L-L_0}$; a contradiction with Remark 6.2.

Remark 2.6.12. The assumption $\operatorname{dist}(L, \mathcal{L}_D) > \|L|_D\|$ in Theorem 6.11 is essential. Take e.g. $B = l^3_\infty$, $D = \ker(f)$, $f = (1/2, 1/2, 0)$. It is easy to check that the operators $P_1 = \mathrm{Id} - f(\cdot)(2, 0, 0)$ and $P_2 = \mathrm{Id} - f(\cdot)(0, 2, 0) \in \mathcal{P}_{\mathrm{Id}}(B, D)$, $P_1 \ne P_2$, $\|P_1\| = \|P_2\| = 1$. Consequently, $\mathcal{P}_D(P_1) \supset \{0, P_1 - P_2\}$ and strong unicity does not hold.

Remark 2.6.13. The assumption $\dim D = 3$ in Theorem 6.11 is essential. Take e.g. $B = l^4_\infty$, $D = \ker(f)$, $f = (1/3, 1/3, 1/3, 0)$. It is well known (see e.g. [Bl]) that $\lambda_{\mathrm{Id}}(B, D) = 4/3$. Take $P^0 \in \mathcal{P}_{\mathrm{Id}}(B, D)$ with $\|P^0\| = 4/3$ (the formula for such a projection is given in Cor. 2.1.16). Then $\mathrm{dist}(P^0, \mathcal{L}_D) = \lambda_{\mathrm{Id}}(B, D) = 4/3 > 1 = \|P^0|_D\|$. By Theorem 2.3.1(b), 0 is not a SUBA to P^0 in $\mathcal{L}_D$.

Remark 2.6.14. Theorems 6.3 and 6.11 were proved by the author in [LG2]. For the case $L \in \mathcal{P}_{\mathrm{Id}}(B, D)$, Theorem 6.11 was proved by W. Odyniec (see e.g. [OdL], Th. I.1.3).

Chapter III

III.1. Extensions of linear operators from finite-dimensional subspaces I. First we present terminology and notation which will be frequently used in this section. The symbol D will stand for an n-dimensional subspace of a normed space B (we consider the real and complex case unless otherwise stated). Given $P^0 \in \mathcal{P}(B, D)$ (we will write $\mathcal{P}(B, D)$ instead of $\mathcal{P}_{\mathrm{Id}}(B, D)$) and $A_1, A_2 \in \mathcal{L}(D)$ note that

$$\lambda_{A_1}(B, D) = \mathrm{dist}(A_1 \circ P^0, \mathcal{L}_D(B, D)),$$
$$\lambda_{A_2}(B, D) = \mathrm{dist}(A_2 \circ P^0, \mathcal{L}_D(B, D)),$$

where

(3.1.1) $$\mathcal{L}_D(B, D) = \{L \in \mathcal{L}(B, D) : L|_D = 0\}.$$

Hence

(3.1.2) $$|\lambda_{A_1}(B, D) - \lambda_{A_2}(B, D)| \le \|P^0 \circ (A_1 - A_2)\|.$$

Now we present some preliminary results. Let $\{L_\beta\}$ be a net in $\mathcal{L}(B, D)$ (here we assume $D = Z^*$ for some Banach space Z). Then we say that $L_\beta \to_\tau L$ iff

(3.1.3) $$(L_\beta x)z \to (Lx)z$$

for every $x \in B$ and $z \in Z$. The topology defined by (3.1.3) will be called the *τ-topology*.

THEOREM 3.1.1 [Is]. *Let $\mathcal{V} \subset \mathcal{L}(B, D)$ be a τ-closed set. Then for every $L \in \mathcal{L}(B, D)$ the set*

$$\{U \in \mathcal{V} : \|U - L\| = \mathrm{dist}(L, \mathcal{V})\}$$

is nonvoid.

If we put $\mathcal{V} = \mathcal{L}_D(B, D)$ then we get immediately

COROLLARY 3.1.2. *For every $A \in \mathcal{L}(D)$ there exists $L_0 \in \mathcal{L}(B, D)$ with*

$$\|L_0\| = \lambda_A(B, D)$$

(*in other words*, *there exists an extension of minimal norm*).

DEFINITION 3.1.3. Let (T, Σ, μ) be a measure space. A family $\{U_i\}_{i\in I}$ of nonvoid measurable sets is called a *partition* of T iff $\bigcup_{i\in I} U_i = T$ and $U_i \cap U_j = \emptyset$ for $i \neq j$.

For two partitions $\{U_i\}_{i\in I}$, $\{V_j\}_{j\in J}$ we write $\{U_i\} \le \{V_j\}$ iff for every $i \in I$ there exists $J_i \subset J$ with $U_i = \bigcup_{j\in J_i} V_j$.

In the sequel we will need the following

Remark 3.1.4. Assume D is an n-dimensional subspace of a normed space B with a basis $y_1, \ldots, y_n$. Assume furthermore that we have n sequences $\{y_1^m\}, \ldots, \{y_n^m\} \subset B$ satisfying $\lim_{m\to\infty} y_i^m = y_i$ for $i = 1, \ldots, n$. Set $D_m = \mathrm{Span}\{y_1^m, \ldots, y_n^m\}$ and put

$$K = \sup_m \Big\{ \max_{i=1,\ldots,n} |\alpha_i^m| : y \in S_{D_m},\ y = \sum_{i=1}^n \alpha_i^m y_i^m \Big\} \tag{3.1.4}$$

Then

(a) $K < \infty$,
(b) $\sup_m \{\mathrm{dist}(y, D) : y \in S_{D_m}\} \to 0$.

Proof. Suppose that $K = \infty$. Then, passing to a subsequence if necessary, we can choose for each $m \in \mathbb{N}$, $y_m \in S_{D_m}$, $y_m = \sum_{i=1}^n \alpha_i^m y_i^m$, and $i_0 \in 1, \ldots, n$ independent of m with the following properties:

$$\lim_{m\to\infty} |\alpha_{i_0}^m| = \infty, \qquad \sup_m \max_{i=1,\ldots,n} |\alpha_i^m|/|\alpha_{i_0}^m| < \infty\,.$$

Put $\gamma_i^m = \alpha_i^m/\alpha_{i_0}^m$. We can assume without loss of generality that $\gamma_i^m \to \gamma_i$ for $i = 1, \ldots, n$. Hence $\sum_{i=1}^m \gamma_i^m y_i^m$ tends to $y = \sum_{i=1}^m \gamma_i y_i$. Since $\gamma_{i_0} = 1$, $\|y\| > 0$. But $y_m = \sum_{i=1}^m \alpha_i^m y_i^m = \alpha_{i_0}^m \sum_{i=1}^m \gamma_i^m y_i^m$. From this we derive $\|y_m\| \to \infty$; a contradiction.

To prove (b), put $a_m = \sup_m \{\mathrm{dist}(y, D) : y \in S_{D_m}\}$ and suppose $a_m \ge d > 0$ (we pass to a subsequence if necessary). For $m \in \mathbb{N}$ take $y_m \in S_{D_m}$ satisfying $a_m = \mathrm{dist}(y_m, D)$ and let $y_m = \sum_{i=1}^m \alpha_i^m y_i^m$. By (a), we can assume $\alpha_i^m \to \alpha_i$ for $i = 1, \ldots, n$. Put $y = \sum_{i=1}^n \alpha_i y_i$. Then

$$\mathrm{dist}(y_m, D) \le \|y - y_m\| = \Big\| \sum_{i=1}^n \alpha_i^m y_i^m - \sum_{i=1}^n \alpha_i y_i \Big\| \to 0 \quad \text{as } m \to \infty\,;$$

a contradiction.

Remark 3.1.5. Assume D is an n-dimensional normed space. Let $A \in \mathcal{L}(D)$ be a singular mapping. Then for every $\varepsilon > 0$ there exists a nonsingular mapping $A_\varepsilon \in \mathcal{L}(D)$ with $\|A - A_\varepsilon\| < \varepsilon$.

Proof. Fix a singular mapping $A \in \mathcal{L}(D)$. Let $y_1, \ldots, y_n$ be a basis of D such that $y_1, \ldots, y_k (k < n)$ is a basis of $\ker(A)$. Let $w_1, \ldots, w_n$ be a basis of D such that $w_{k+1} = Ay_{k+1}, \ldots, w_n = Ay_n$ is a basis of $\mathrm{Im}(A)$. For each $l \in \mathbb{N}$ define a linear mapping $A_l : D \to D$ by

$$A_l y_i = \begin{cases} 0 & \text{for } i > k\,, \\ w_i/l & \text{for } i \le k\,. \end{cases}$$

Note that $A+A_l$ is a nonsingular mapping. Indeed, $(A+A_l)y = 0$ iff $\sum_{i=1}^{k} \alpha_i w_i/l + \sum_{i=k+1}^{n} \alpha_i w_i = 0$, which is equivalent to $\alpha_i = 0$ for $i = 1, \ldots, n$ ($y = \sum_{i=1}^{k} \alpha_i y_i$). It is clear that $\|A_l\| < \varepsilon$ for l sufficiently large, which completes the proof.

To present the main result of this section we introduce some notations. Let $\{D_1, \ldots, D_m, \ldots\}$ be a sequence of n-dimensional subspaces of a normed linear space B. Let $D \subset B$ be a fixed n-dimensional subspace. We write

$$D_m \to D \tag{3.1.5}$$

iff for each $m \in \mathbb{N}$ there exists a basis $y_1^m, \ldots, y_n^m$ of D_m such that $y_i^m \to y_i$ for $i = 1, \ldots, n$, where $y_1, \ldots, y_n$ is a fixed basis of D.

Now assume that $A \in \mathcal{L}(D)$. Then we can define $A_m \in \mathcal{L}(D_m)$ by

$$A_m = \varphi_m^{-1} \circ A \circ \varphi_m \tag{3.1.6}$$

where $\varphi_m \in \mathcal{L}(D_m, D)$ is given by

$$\varphi_m(y_i^m) = y_i \tag{3.1.7}$$

($\{y_i^m\}$ and $\{y_i\}$ are fixed bases of D_m which satisfy (3.1.5)).

It is easy to see that

$$\|\varphi_m - \mathrm{Id}|_{D_m}\| \to 0 \quad \text{and} \quad \|\varphi_m^{-1} - \mathrm{Id}|_D\| \to 0\,. \tag{3.1.8}$$

Now we can state the main result of this section.

THEOREM 3.1.6. *Let B be a separable Banach space*, $B = \mathrm{cl}(\bigcup_{m=1}^{\infty} B_m)$, *where $\{B_m\}$ is an increasing sequence of finite-dimensional vector subspaces of B. Assume $D \subset B$ is an n-dimensional subspace of B and let $D_m \subset B_m$ be so chosen that $D_m \to D$* (see (3.1.5))*. Then for $A \in \mathcal{L}(D)$ the following holds*:

(a) *if A is a nonsingular mapping then for every $P \in \mathcal{P}_A(B, D)$ there exists a sequence $R_k \in \mathcal{P}_{A_k}(B, D_k)$ (A_k is given by* (3.1.6)*) such that*

$$\|P - R_k\| \to 0\,, \tag{3.1.9}$$

$$\text{(b)}\quad \lambda_A(B, D) = \lim_{k\to\infty} \lambda_{A_k}(B_k, D_k)\,. \tag{3.1.10}$$

Proof. We divide the proof into three cases.

Case I: $A = \mathrm{Id}|_D$. Fix $P \in \mathcal{P}(B, D)$, $P = \sum_{i=1}^{n} \varphi_i y_i$. For $k \in \mathbb{N}$ define $P_k = \sum_{i=1}^{n} \varphi_i y_i^k$. Since $D_k \to D$, $\|P - P_k\| \to 0$. Now, we will show that

$$\lim_{k\to\infty} \|P_k|_{D_k} - \mathrm{Id}|_{D_k}\| = 0\,.$$

To do this, take $w_k \in S_k$ and compute

$$\begin{aligned} \|P_k w_k - w_k\| &\le \|P_k w_k - P w_k\| + \|P w_k - w_k\| \\ &\le \|P - P_k\| + (1 + \|P\|)\,\mathrm{dist}(w_k, D)\,. \end{aligned}$$

By Remark 1.4(b) we get our assertion. Hence for $k \ge k_0$,

$$\|P_k y_k|_{D_k} - \mathrm{Id}|_{D_k}\| < 1\,,$$

which shows that $P_k|_{D_k}$ is invertible. Put $R_k = (P_k|_{D_k}^{-1} \circ P_k)$. It is clear that $R_k \in \mathcal{P}(B, D_k)$. Note that

$$\begin{aligned}\|P_k - R_k\| &= \|P_k|_{D_k} \circ R_k - R_k\| \le \|R_k\| \cdot \|P_k|_{D_k} - \mathrm{Id}|_{D_k}\| \\ &\le \|P_k|_{D_k}^{-1}\| \cdot \|P_k\| \cdot \|P_k|_{D_k} - \mathrm{Id}|_{D_k}\| \\ &\le \|P_k\| \cdot \|P_k|_{D_k} - \mathrm{Id}|_{D_k}\|/(1 - \|P_k|_{D_k} - \mathrm{Id}|_{D_k}\|)\end{aligned}$$

and consequently, $\|P_k - R_k\| \to 0$, which finally gives $\|P - R_k\| \to 0$.

To prove (b), take $P_0 \in \mathcal{P}(B, D)$ with $\|P_0\| = \lambda(B, D)$ (by Corollary 1.2 such a P_0 exists). By the above reasoning, $\|P_0 - R_k\| \to 0$ for some $R_k \in \mathcal{P}(B, D_k)$. Now fix $\varepsilon > 0$. Since $\mathrm{cl}(\bigcup_{k=1}^\infty B_k) = B$, $|\,\|P_0\| - \|P_0|_{B_k}\|\,| < \varepsilon$ for $k \ge k_0$ and consequently,

$$\begin{aligned}|\,\|P_0\| - \|R_k|_{B_k}\|\,| &\le |\,\|P_0\| - \|P_0|_{B_k}\|\,| + |\,\|P_0|_{B_k}\| - \|R_k|_{B_k}\|\,| \\ &\le |\,\|P_0\| - \|P_0|_{B_k}\|\,| + \|P_0 - R_k\| \le 2\varepsilon\end{aligned}$$

for k sufficiently large. Since $R_k|_{B_k} \in \mathcal{P}(B_k, D_k)$, $\lambda(B, D) \ge \limsup \lambda(B_k, D_k)$.

To finish this part of the proof, it is sufficient to show that $\liminf \lambda(B_k, D_k) \ge \lambda(B, D)$. Assume that this is not true and for $k = 1, 2, \ldots$ choose $P_k \in \mathcal{P}(B_k, D_k)$ with $\|P_k\| = \lambda(B_k, D_k)$. Passing to a subsequence if necessary, we can assume $\lim_k \|P_k\| < \|P_0\| - \varepsilon$ for some $\varepsilon > 0$. Fix $f \in \bigcup_{k=1}^\infty B_k, \|f\| = 1$. Note that $\sup \|P_k f\| \le \|P_0\| - \varepsilon$ By Remark 1.4(a), we can choose a subsequence (m_k) depending on f such that $P_{m_k} f \to d_f \in D$. Now let $(f_1, f_2, \ldots)$ be a Hamel basis of $\bigcup_{k=1}^\infty B_k$ with $\|f_i\| = 1$ for $i = 1, 2, \ldots$ Applying the diagonal argument, we can select a subsequence (m_k) in such a way that $P_{m_k} f_i \to d_i$ for $i = 1, 2, \ldots$ Now we can define a linear mapping $\overline{P}_0 : \bigcup_{k=1}^\infty B_k \to D$ by $\overline{P}_0 f_i = d_i$ for $i = 1, 2, \ldots$

We show that $\overline{P}_0$ is continuous and $\|\overline{P}_0\| < \|P_0\| - \varepsilon$. To do this, take $w_0 \in \bigcup_{k=1}^\infty B_k$ with $\|w_0\| = 1$ and write $w_0 = \sum_{i=1}^l \alpha_i f_i$. Then

$$\begin{aligned}\overline{P}_0 w_0 &= \sum_{i=1}^l \alpha_i d_i = \sum_{i=1}^l \alpha_i \lim_{k\to\infty} P_{m_k} f_i \\ &= \lim_{k\to\infty} \Big(\sum_{i=1}^l \alpha_i P_{m_k} f_i\Big) = \lim_{k\to\infty} P_{m_k} w_0\,.\end{aligned}$$

Hence, $\|\overline{P}_0 w_0\| \le \limsup_{k\to\infty} \|P_{m_k}\| < \|P_0\| - \varepsilon$. Since $B = \mathrm{cl}(\bigcup_{k=1}^\infty B_k)$, we can extend $\overline{P}_0$ onto the whole space B to a linear continuous mapping Q_0 such that $\|Q_0\| < \|P_0\| - \varepsilon$.

It is sufficient to show that $Q_0 \in \mathcal{P}(B, D)$. To do this, we check that $Q_0 y_i = y_i$ for $i = 1, \ldots, n$ ($\{y_i\}$ is a basis of D). Now fix $\varepsilon > 0$ and $i \in \{1, \ldots, n\}$. Since $D_k \to D$, we can choose $k_0 \in \mathbb{N}$ such that for $k, m \ge k_0$, $\|y - y_k\| < \varepsilon/(3(\|P_0\| + 1))$ and $\|y_m - y_k\| < \varepsilon/(3(\|P_0\| + 1))$. Next select $m_0 \ge k_0$ such that

$\|Q_0 y_i^{k_0} - P_{m_0} y_i^{k_0}\| < \varepsilon/3$. Note that

$$\begin{aligned}\|Q_0 y_i - y_i\| &\le \|Q_0 y_i - y_i^{k_0}\| + \|y_i^{k_0} - y_i\| \\ &\le \|Q_0(y_i^{k_0} - y_i)\| + \|Q_0 y_i^{k_0} - y_i^{k_0}\| + \|y_i - y_i^{k_0}\| \\ &\le (\|Q_0\| + 1)\|y_i^{k_0} - y_i\| + \|Q_0 y_i^{k_0} - P_{m_0} y_i^{k_0}\| + \|P_{m_0} y_i^{k_0} - y_i^{k_0}\| \\ &\le (\|Q_0\| + 1)\|y_i - y_i^{k_0}\| + \|Q_0 y_i^{k_0} - P_{m_0} y_i^{k_0}\| \\ &\quad + \|P_{m_0}(y_i^{k_0} - y_i^{m_0})\| + \|y_i^{m_0} - y_i^{k_0}\| \\ &\le (\|Q_0\| + 1)\|y_i^{k_0} - y_i\| + \|Q_0 y_i^{k_0} - P_{m_0} y_i^{k_0}\| \\ &\quad + (\|P_{m_0}\| + 1)\|y_i^{k_0} - y_i^{m_0}\| \\ &\le (\|P_0\| + 1)(\|y_i^{k_0} - y_i^{m_0}\| + \|y_i^{k_0} - y_i\|) + \|Q_0 y_i^{k_0} - P_{m_0} y_i^{k_0}\|\,.\end{aligned}$$

Hence finally we get $\|Q_0 y_i - y_i\| < \varepsilon$, which gives $Q_0 \in \mathcal{P}(B,D)$. But $\|Q_0\| < \|P_0\| - \varepsilon$ and $\|P_0\| = \lambda(B,D)$; a contradiction.

Case II: $A \in \mathcal{L}(D)$, A *nonsingular*. Fix $P_0 \in \mathcal{P}_A(B,D)$. Then $P_0 = A \circ Q_0$ where $Q_0 \in \mathcal{P}(B,D)$. By Case I, there exists a sequence $\{R_k\} \in \mathcal{P}(B,D_k)$ with $\|Q_0 - R_k\| \to 0$. Put $P_k = A_k \circ R_k$, where A_k is given by (3.1.6). It is clear that $P_k \in \mathcal{P}_{A_k}(B,D_k)$. Moreover,

$$\begin{aligned}\|P_0 - P_k\| &= \|A \circ Q_0 - A_k \circ R_k\| \\ &\le \|A \circ Q_0\| \cdot \|\varphi_k^{-1} - \mathrm{Id}|_D\| + \|\varphi_k^{-1} \circ A\| \cdot \|Q_0 - \varphi_k \circ R_k\| \\ &\le \|A \circ Q_0\| \cdot \|\varphi_k^{-1} - \mathrm{Id}|_D\| \\ &\quad + (\|Q_0 - R_k\| + \|R_k\| \cdot \|\varphi_k - \mathrm{Id}|_{D_k}\|) \cdot \|\varphi_k^{-1} \circ A\|\,.\end{aligned}$$

By (3.1.8), $\|P_0 - P_k\| \to 0$. Reasoning as in Case I, we get

$$\lambda_A(B,D) \ge \limsup \lambda_{A_k}(B_k,D_k)\,.$$

To finish the proof of Case II it is sufficient to show that $\liminf \lambda_{A_k}(B_k,D_k) \ge \lambda_A(B,D)$. Suppose, on the contrary, that this is not true and choose $P_k \in \mathcal{P}_{A_k}(B_k,D_k)$ with $\lambda_{A_k}(B_k,D_k) = \|P_k\|$. Reasoning as in Case I we can construct $Q_0 \in \mathcal{L}(B,D)$ such that $\|Q_0\| < \lambda_A(B,D) - \varepsilon$ for some $\varepsilon > 0$. To show that $Q_0 \in \mathcal{P}_A(B,D)$ it is sufficient to check that $A^{-1} \circ Q_0 \in \mathcal{P}(B,D)$. Since for each k, $A_k^{-1} \circ P_k \in \mathcal{P}(B,D)$ and

$$\|A_k^{-1} \circ P_k\| \le 2\|A\| \cdot \|P_0\| \quad \text{for } k \ge k_0\,,$$

this can be proved as in Case I.

Case III: $A \in \mathcal{L}(D)$. By Remark 1.5, we can select a sequence $\{A^l\} \subset \mathcal{L}(D)$ of nonsingular mappings such that $A^l \to A$. Fix any $P_0 \in \mathcal{P}(B,D)$ and put $P_A = A \circ P_0$, $P_{A^l} = A^l \circ P_0$. Let $\{A_k\}$ (resp. $\{A_k^l\}$) denote the sequence defined by (3.1.6) for A (for A^l resp.). By Case I, take a sequence $\{P_k\} \in \mathcal{P}(B_k,D_k)$ with $\|P_k - P_0\| \to 0$ and define

$$P_{A_k} = A_k \circ P_k, \quad P_{A_k^l} = A_k^l \circ P_k\,.$$

According to (3.1.2) and (3.1.8) we get immediately

$$|\lambda_A(B,D)-\lambda_{A^l}(B,D)| \le \|P_A-P_{A^l}\| \le \|A-A^l\|\cdot\|P_0\| \\ \le |\lambda_{A_k}(B_k,D_k)-\lambda_{A^l_k}(B_k,D_k)| \le \|P_{A_k}-P_{A^l_k}\| \le 2\|A-A^l\|\cdot\|P_0\|\,.$$

Since the right side of the last inequality does not depend on k, applying Case II for A_l we easily get $\lambda_{A_k}(B_k,D_k)\to\lambda_A(B,D)$. The proof of Theorem 1.6 is complete.

Note that if we fix a basis $(x_1,\dots,x_{m_k})$ in each B_k, we can identify each $x\in B_k$ with the sequence of its coefficients with respect to this basis. Hence

$$\lambda_{A_k}(B_k,D_k)=\lambda_{A^1_k}(\mathbb{R}^{m_k},D^1_k)$$

where $D^1_k\subset\mathbb{R}^{m_k}$ is an n-dimensional subspace and $A^1_k\in\mathcal{L}(D^1_k)$ is defined by $A^1_k w=A_kEw$ for $w\in\mathbb{R}^{m_k}$ (E denotes the identifying mapping and the norm in $\mathbb{R}^{m_k}$ is induced by E from B). For this reason it is important to know how this norm in $\mathbb{R}^{m_k}$ looks like. Now we present some examples concerning this problem.

Example 3.1.7. Let (T,Σ,μ) be a measure space. Assume μ is σ-finite, atomless and separable. Fix a sequence $\{T_n\}$ of measurable sets with $T_n\subset T_{n+1}$ and $\mu(T_n)$ finite and set $T=\bigcup_{n=1}^\infty T_n$. For each $n\in\mathbb{N}$ fix a partition $\{U^n_i\}$ of T_n (see Def. 1.5) such that $\{U^n_i\}\le\{U^{n+1}_i\}$ and $\mu(U^n_i)\le 1/n$. Let $L^\varphi(T,\Sigma,\mu)$, where $\varphi(t,\cdot)$ is a convex function for every $t\in T$, denote the Orlicz–Musielak space equipped with the Luxemburg norm (see [Mu, p. 33] for basic information about these spaces). Assume furthermore that φ is locally integrable and satisfies condition Δ_2 [Mu, p. 52]. Put

$$B_n=\Big\{\sum_{i=1}^{l_n}\lambda_i\chi_{U^n_i}\Big\}$$

where $l_n=\operatorname{card}\{U^n_i\}$. In view of [Mu, pp. 36, 53], $\operatorname{cl}(\bigcup_{n=1}^\infty B_n)=L^\varphi$. Note that

$$\varrho_\varphi\Big(\sum_{i=1}^{l_n}\lambda_i\chi_{U^n_i}\Big)=\sum_{i=1}^{l_n}\int\limits_{U^n_i}\varphi(t,|\lambda_i|)\,d\mu\,.$$

Hence ϱ_φ induces on K^{l_n} ($K=\mathbb{R}$ or $K=\mathbb{C}$) a modular $\overline{\varrho}_\varphi$ given by

$$\overline{\varrho}_\varphi(\lambda_1,\dots,\lambda_{l_n})=\sum_{i=1}^{l_n}\overline{\varphi}_i(\lambda_i)$$

where $\overline{\varphi}_i=\int_{U^n_i}\varphi(t,|\lambda_i|)\,d\mu(t)$. Consequently, in this case for every $n\in\mathbb{N}$ we consider the space K^{l_n} equipped with the Luxemburg norm $\|\ \|_{\overline{\varphi}}$.

Example 3.1.8. Let (T,Σ,μ) and B_n be as in Example 1.7. Consider the space $L_p(T,\Sigma,\mu)$ for $1\le p<\infty$. It is well known that $\operatorname{cl}(\bigcup_{n=1}^\infty B_n)=L_p(T,\Sigma,\mu)$.

Moreover, for $n \in \mathbb{N}$ and $x \in B_k$ we have

$$\|x\| = \Big(\int_T \Big| \sum_{i=1}^{l_n} \lambda_i \chi_{U_i^n} |^p \, d\mu \Big)^{1/p} = \Big(\int_{U_n} |\sum_{i=1}^{l_n} \lambda_i \chi_{U_i^n} |^p \, d\mu \Big)^{1/p}$$

$$= \Big(\int_{U_n} \sum_{i=1}^{l_n} |\lambda_i|^p |\chi_{U_i^n}| d\mu \Big)^{1/p} = \Big(\sum_{i=1}^{n} \mu(U_i^n) |\lambda_i|^p \Big)^{1/p}.$$

Now fix $n \in \mathbb{N}$ and put $a_i = \mu(U_i^n)^{1/p}$ for $i = 1, \ldots, l_n$. Consider the linear mapping $T_n : K^{l_n} \to K^{l_n}$ defined by

$$T_n e_i = a_i e_i \quad \text{for } i = 1, \ldots, l_n .$$

To calculate the constant $\lambda_A(L_p, D)$ it is sufficient (by Theorem 1.6) to calculate the constants $\lambda_{A_n}(l_p^{(l_n)}, T_n(D_n))$, where $D_n \to D$ is a sequence of subspaces of B.

Now we show how to construct subspaces D_n for the space of polynomials of degree $\leq n$.

Example 3.1.9. Let $B = L_p[0,1]$, $1 \leq p < \infty$, with the Lebesgue measure. Let D be the space of polynomials of degree $\leq m$ restricted to $[0,1]$ (we consider the real case) with the basis $(1, t, \ldots, t^m)$. For $n \in \mathbb{N}$ consider a partition $\{U_i^n\}$ of $[0,1]$ such that $\mu(U_i^n) = 1/2^n$. Let B_n be as in Example 1.8. Put

$$y_0^n = 1, \quad y_j^n = \sum_{i=1}^{2^n} ((i-1)/2^n)^j \chi_{U_i^n}$$

for $j = 1, \ldots, m$. It is obvious that $y_i^n \to t^i$ for $i = 0, \ldots, m$ in L_p norm. So to calculate the constant $\lambda_A(B, D)$ it is sufficient to know the constants $\lambda_{A_n}(l_p^{(2^n)}, D_n)$, where $D_n = \operatorname{span}(y_0^n, \ldots, y_n^m)$.

It is clear that a similar construction can be done for T being a Lebesgue measurable subset of positive measure in K^n ($K = \mathbb{R}$ or $K = \mathbb{C}$) and D being an arbitrary finite-dimensional subspace of $L_p(T)$, $1 \leq p < \infty$. So the question of major importance is to calculate or estimate the constant $\lambda_A(l_p^{(n)}, D)$ where D is a linear subspace of $l_p^{(n)}$. Unfortunately, we know the exact formulas only for the case $p = 1, \infty$, $A = \mathrm{Id}$ and D being a hyperplane in $l_p^{(n)}$ [Bl].

Example 3.1.10. Let $B = W_s^p[0,1]$ (the Sobolev space) equipped with the norm

$$\|f\| = \sum_{i=0}^{s-1} |f^{(i)}(0)| + \Big(\int_0^1 |f^{(s)}(t)|^p \Big)^{1/p}.$$

Since $f^{(s-1)}$ is absolutely continuous, the mapping $\| \ \|$ is in fact the norm in $W_s^p[0,1]$. Hence $W_s^p[0,1]$ is linearly isometric to $K^{s-1} \times L_p[0,1]$ equipped with the norm

$$\|x\| = \sum_{i=0}^{s-1} |x_i| + \Big(\int_0^1 |x_s(t)|^p \, dt \Big)^{1/p}$$

where $x = (x_0, \ldots, x_s)$. Let $\{U_i^n\}$ be as in Example 2.6. Put $B_n = K^{s-1} \times \{\sum_{i=1}^{2^n} \lambda_i \chi_{U_i^n}\}$ for $n \in \mathbb{N}$. Then B_n can be identified with $\mathbb{R}^{s-1+2^n}$ with the norm defined by

$$\|x\| = \sum_{i=1}^{s-1} |x_i| + \Big(\sum_{i=1}^{2^n} |x_{i+s-1}|^p \Big)^{1/p}.$$

III.2. Extensions of linear operators from finite-dimensional subspaces II. We start with the following

THEOREM 3.2.1. *Assume B is a normed space with B^* separable and let $D \subset B$ be a linear n-dimensional subspace. Assume that $\{\varphi_i\}_{i=1}^{\infty} \subset S_{B^*}$ is so chosen that* $\mathrm{cl}(\mathrm{Span}\{\varphi_i\}) = B^*$ *(the closure is taken with respect to the norm topology in B^*). Let $A : D \to D$ be a nonsingular linear mapping. Put*

$$\mathcal{D}_A^m(B, D) = \Big\{ L \in \mathcal{P}_A : L = \sum_{i=1}^m \varphi_i(\cdot) y_i \Big\}. \tag{3.2.1}$$

Then for every $P \in \mathcal{P}_A(B, D)$ there exists a sequence $P_m \in \mathcal{D}_A^m(B, D)$ such that

(a) $\|P_m x - Px\| \to 0$ *for every* $x \in B$,

(b) $\|P_m\| \to \|P\|$.

Proof. Let $P \in \mathcal{P}_A(B, D)$, $P = \sum_{i=1}^n f_i(\cdot) y_i$. Since $\mathrm{cl}(\mathrm{Span}\{\varphi_i\}) = B^*$, for $i = 1, \ldots, n$ there exists a sequence $\{\psi_i^m\}$, $\psi_i^m \in \mathrm{Span}[\varphi_1, \ldots, \varphi_m]$, with $\psi_i^m \to f_i$. Put $U_m = \sum_{i=1}^n \psi_i^m(\cdot) y_i$. It is clear that $U_m x \to Px$ for every $x \in B$. Since D is finite-dimensional, $\|U_m|_D - P|_D\| \to 0$. Since $P|_D = A$ and A is invertible, $U_m|_D$ is invertible for $m \ge m_0$. Put $P_m = A \circ (U_m|_D)^{-1} \circ U_m$ and $y_i^m = A \circ (U_m|_D)^{-1} y_i$. Note that

$$P_m x = \sum_{i=1}^n \psi_i^m(\cdot) y_i^m,$$

which gives $P_m \in \mathcal{D}_A^m(B, D)$. Since A is invertible, $(U_m|_D)^{-1} \to A^{-1}$. From this we derive that $\|P_m x - Px\| \to 0$ for every $x \in B$.

To show that $\|P_m\| \to \|P\|$, fix $x \in S_x$ and $\varepsilon > 0$ with $\|Px\| \ge \|P\| - 2\varepsilon$. Hence

$$\|P_m\| \ge \|P_m x\| \ge \|P\| - 2\varepsilon$$

for $m \ge m_0$ and consequently, $\liminf \|P_m\| \ge \|P\|$. To prove that $\limsup \|P_m\| \le \|P\|$ assume, on the contrary, that $\|P\| + \varepsilon < \|P_m x_m\|$ for some $x_m \in S_B$ and $\varepsilon > 0$. Since $P_m = A \circ (U_m|_D)^{-1}) \circ U_m$ and $U_m \to P$ and $(U_m|_D)^{-1} \to A^{-1}$ it follows that $\sup_m \|P_m\| \le M < \infty$. Hence, passing to a subsequence if necessary, we can assume $P_m x_m \to y \in D$. Since $\{x_m\} \subset S_B$, by the Banach–Alaoglu Theorem,

$\{x_m\}$ has a cluster point γ in $S_{B^{**}}$ with respect to the weak* topology in $S_{B^{**}}$. Since B^* is separable, $S_{B^{**}}$ with the weak* topology is metrizable. Compute, passing to a subsequence if necessary,

$$\Big\|P_m x_m - \sum_{i=1}^n \gamma(f_i)y_i\Big\| = \Big\|\sum_{i=1}^n \psi_i^m(x_m)y_i^m - \sum_{i=1}^n \gamma(f_i)y_i\Big\|$$
$$\le \Big\|\sum_{i=1}^n \psi_i^m(x_m)(y_i^m - y_i)\Big\| + \Big\|\sum_{i=1}^n (\psi_i^m(x_m) - \gamma(f_i))y_i\Big\|.$$

Note that the right side of the above inequality tends to 0, since $y_i^m \to y_i$ for $i = 1, \ldots, n$ and $\psi_i^m \to f_i$ in the norm of B^*. Hence $\|\sum_{i=1}^n \gamma(f_i)y_i\| \ge \|P\| + \varepsilon$, which, by Goldstine's Theorem, leads to a contradiction.

Remark 3.2.2. To show that $\liminf \|P_m\| \ge \|P\|$ it is sufficient to demonstrate that $\mathrm{cl}(\mathrm{Span}\{\varphi_i\}) = B^*$, where the closure is taken with respect to the weak* topology in B^*. The proof of this part of Theorem 2.1 is in fact the same as in [Che5, Theorem 1.1].

Now we state a result concerning the space $C^{(k)}[a,b]$ equipped with the norm

$$\|f\|_1 = \sum_{i=1}^{k-1} |f^{(i)}(a)| + \|f^{(k)}\|_{\sup}. \tag{3.2.2}$$

The method of proof is similar to that of [Che5].

THEOREM 3.2.3. *Let $E = \{t_i\}$ be a dense countable set in $[a,b]$. For $i = 1, 2, \ldots$ and $x \in C^{(k)}[a,b]$, define*

$$\widehat{t_i}(x) = \sum_{j=1}^{k-1} x^j(a) + x^{(k)}(t_i). \tag{3.2.3}$$

Assume $D \subset C^{(k)}[a,b]$ is an n-dimensional subspace and let $A \in \mathcal{L}(D)$ be a nonsingular mapping. Then for every $P \in \mathcal{P}_A(B,D)$ there exists a sequence $P_m \in \mathcal{D}_A^m(B,D)$ such that $\|P_m\| \to \|P\|$.

Proof. Note that $C^{(k)}[a,b]$ is linearly isometric to $\mathbb{R}^{k-1} \times C[a,b]$ with the norm

$$\|(z_1, \ldots, z_{k-1}, f)\| = \sum_{i=1}^{k-1} |z_i| + \|f\|_{\sup}.$$

Hence $(C^{(k)}[a,b])^* = \mathbb{R}^{k-1} \times (C[a,b])^*$ (the norm in $\mathbb{R}^{k-1} \times (C[a,b])^*$ is given by

$$\|(z_1, \ldots, z_{k-1}, \nu)\| = \max\{\max_{i=1,\ldots,k-1} |z_i|, |\nu|\}, \tag{3.2.4}$$

the symbol $|\nu|$ denoting the variation of the Radon measure ν). Now take $P \in \mathcal{P}_A(B,D)$, $P = \sum_{i=1}^n f_i(\cdot)y_i$ ($f_i(x) = (\alpha_1, \ldots, \alpha_{k-1}, u_i)$ where u_i is a Radon measure). For each $m \in \mathbb{N}$ let $T_1^m, \ldots, T_{k_m}^m$ denote a partition of $[a,b]$ (see Def. 3.1.3) into disjoint intervals such that $\mu(T_j^m) \le 1/m$ (μ denotes the Lebesgue measure). Put

$$\psi_i^m(x) = \sum_{i=1}^{k-1} \alpha_i x^{(i)}(a) + \sum_{j=1}^{k_m} u_i(T_j^m) x^{(k)}(t_j^m) \tag{3.2.5}$$

(t_j^m are fixed points from the sets $T_j^m \cap E$). It is easy to verify that $\psi_i^m \to f_i$ weak* in B^* for $i = 1, \dots, n$. Hence, using the functionals ψ_i^m, we can define the operators U_m and P_m as in Theorem 2.1. By the proof of Theorem 2.1 and Remark 2.2, $\|P_m x - Px\| \to 0$ for every $x \in B$ and $\liminf \|P_m\| \ge \|P\|$. To show that $\limsup \|P_m\| \le \|P\|$, we prove that $\|\widehat{t} \circ U_m\| \le \|\widehat{t} \circ P\|$ for every $t \in E$. Note that

$$\begin{aligned}(\widehat{t} \circ P)x &= \sum_{j=1}^n f_j(x)\widehat{t}(y_j) = \sum_{j=1}^n \Big(\sum_{l=0}^{k-1} \alpha_l x^{(l)}(a) + u_j \circ x^k\Big)\widehat{t}(y_j) \\ &= \Big(\sum_{j=1}^n \widehat{t}(y_j) u_j\Big) \circ x^{(k)} + \sum_{l=0}^{k-1} \alpha_l \Big(\sum_{j=1}^n \widehat{t}(y_j)\Big) x_l(a)\,.\end{aligned}$$

Hence, by (3.2.4),

$$\|\widehat{t} \circ P\| = \max\Big\{|\nu_t|(T), \max_l \Big|\alpha_l \sum_{j=1}^n \widehat{t}(y_j)\Big|\Big\},$$

where $\nu_t = \sum_{j=1}^n \widehat{t}(y_j) u_j$. Note that

$$\begin{aligned}|\widehat{t} \circ U_m(x)| &= \Big|\sum_{j=1}^n \psi_j^m(x)\widehat{t}(y_j)\Big| \\ &= \Big|\sum_{j=1}^n \Big(\sum_{z=0}^{k-1} \alpha_z x^{(z)}(a) + \sum_{l=1}^{k_m} u_j(T_l^m) x^{(k)}(t_l^m)\Big)\widehat{t}(y_j)\Big| \\ &= \Big|\sum_{z=0}^{k-1} \alpha_z x^{(z)}(a)\Big(\sum_{j=1}^n \widehat{t}(y_j)\Big) + \sum_{l=1}^{k_m} x^{(k)}(t_l^m)\Big(\sum_{j=1}^n \widehat{t}(y_j) u_j(T_l^m)\Big)\Big| \\ &\le \Big(\sum_{z=0}^{k-1} |x^{(z)}(a)| + \|x^{(k)}\|_{\sup}\Big) \max\Big\{\sum_{l=1}^{k_m} |\nu_t|(T_l^m), \max_z \Big|\alpha_z \sum_{j=1}^n \widehat{t}(y_j)\Big|\Big\} \\ &\le \|x\| \max\Big\{|\nu_t|(T), \max_z \Big|\alpha_z \sum_{j=1}^n \widehat{t}(y_j)\Big|\Big\} = \|\widehat{t} \circ P\| \cdot \|x\|\,.\end{aligned}$$

Hence $\|U_m\| \le \|P\|$ for $m = 1, 2, \dots$ Since $P_m = A \circ (U_m|_D)^{-1} \circ U_m$, we have $\|U_m - P_m\| \to 0$ and consequently, $\limsup \|P_m\| \le \|P\|$, which completes the proof.

To present the next result we introduce some notations. Let B be a normed space. Assume $\{\varphi_i\}_{i=1}^{\infty} \subset S_{B^*}$. Let $D \subset B$ be an n-dimensional subspace. Assume furthermore that $\text{Span}\{\varphi_1|_D, \dots, \varphi_n|_D\} = D^*$ and let $A \in \mathcal{L}(D)$. We say that a sequence $\{\mathcal{D}_A^m(B,D)\}$ has *property* $(*)$ if and only if for every $P \in \mathcal{P}_A(B,D)$ there exists a sequence $P_m \in \mathcal{D}_A^m(B,D)$ for $m \geq n$ such that

$$P_m x \to Px \quad \text{for every } x \in B$$

and

$$\|P_m\| \to \|P\|. \tag{3.2.6}$$

Theorems 0.1.2, 2.1 and 2.3 present examples of sequences $\mathcal{D}_A^m(B,D)$ having property $(*)$. For $P \in \mathcal{D}_A^m(B,D)$ put

$$N_m(P) = \sup_{\|f\| \leq 1} \max_{i=1,\dots,m} |\varphi_i(Pf)| \tag{3.2.7}$$

and

$$N_m = \inf\{N_m(P) : P \in \mathcal{D}_A^m(B,D)\}. \tag{3.2.8}$$

Now we can state the following

PROPOSITION 3.2.4. *Let B be a normed space and let $D \subset B$ be an n-dimensional subspace. Suppose $\{\varphi_i\}_{i=1}^{\infty}$ is a weak*-dense subset of S_{B^*}. Let $A \in \mathcal{L}(D)$ be so chosen that $\mathcal{D}_A^m(B,D)$ satisfies condition $(*)$ with respect to the sequence $\{\varphi_i\}$. Then*

$$\lambda_A(B,D) = \lim_{m\to\infty} N_m.$$

Proof. Fix $\varepsilon > 0$ and let $P_0 \in \mathcal{P}_A(B,D)$ be a minimal extension. Choose $m_0 \in \mathbb{N}$ and $Q_{m_0} \in \mathcal{D}_A^{m_0}(B,D)$ with $|\,\|Q_{m_0}\| - \|P_0\|\,| < \varepsilon$. Note that for $m \geq m_0$,

$$N_m \leq N_m(Q_{m_0}) \leq \|Q_{m_0}\| \leq \|P_0\| + \varepsilon,$$

which gives $\limsup_{m\to\infty} N_m \leq \|P_0\|$. To show that $\liminf N_m \geq \|P_0\|$ suppose, on the contrary, that $\lim N_m < \|P_0\| - \varepsilon$ for some $\varepsilon > 0$ (pass to a subsequence if necessary). Define a function $\|\ \|_1$ by

$$\|y\|_1 = \max_{i=1,\dots,n} |\varphi_i(y)| \quad \text{for } y \in D.$$

Since $\text{Span}\{\varphi_1|_D, \dots, \varphi_n|_D\} = D^*$, $\|\ \|_1$ is a norm in D. Since D is finite-dimensional, there exists $w > 0$ with $\|y\| \leq w\|y\|_1$ for every $y \in D$. Hence for every $x \in B$ and $m \geq n = \dim D$,

$$\begin{aligned}\|P_m x\| &\leq w\|P_m x\|_1 = w \max_{i=1,\dots,n} |\varphi_i(Px)| \\ &\leq w \max_{i=1,\dots,m} |\varphi_i(Px)| \leq \|x\| N_m(P_m) w < \|P_0\| w.\end{aligned}$$

($P_m \in \mathcal{D}_A^m(B,D)$ is so chosen that $N_m(P_m) < N_m + 1/m$). Consequently, $\|P_m\| \leq w\|P_0\|$ for $m \geq m_0$, which gives that $\{P_m\}$ has a cluster point $P_1 \in \mathcal{P}_A(B,D)$ with respect to the τ-topology (see (3.1.3)). Since $\{\varphi_i\}$ is a norming set, there exist $m_0 \in \mathbb{N}$, $x \in S_B$ and $\delta_m \to 0$ with

$$\|P_1\| - \varepsilon/3 < |\varphi_{m_0}(P_1 x)| \leq N_m(P_m) + \delta_m \leq \|P_0\| - \varepsilon/2$$

for $m \geq m_0$. Finally, we get $\|P_1\| < \|P_0\| - \varepsilon/6$; a contradiction with the minimality of P_0.

Now we present a result concerning the case $B = C(T)$. Let D and $\{\varphi_i\}_{i=1}^{\infty}$ be as in Proposition 2.4. For $m \geq n$ define

$$D_m = \{x \in \mathbb{K}^m : \varphi_i(y) = x_i,\ i = 1, \ldots, m,\ y \in D\}, \tag{3.2.9}$$

and define $A_m \in \mathcal{L}(D_m)$ by

$$A_m(w) = (\varphi_1(Ay), \ldots, \varphi_m(Ay)) \tag{3.2.10}$$

where $y \in D$ denotes an element satisfying (3.2.9). Since $\operatorname{Span}\{\varphi_1|_D, \ldots, \varphi_n|_D\} = D^*$, for every $w \in D_m$ there exists exactly one $y \in D$ satisfying (3.2.9). Hence A_m is well defined. Now we can state

THEOREM 3.2.5. *Assume $B = C(T)$ (we consider the complex and real case) where T is a compact metrizable set. For $i = 1, 2, \ldots$ let $\varphi_i = t_i$ where $\{t_i\}$ is a dense, countable subset of T. Let D be an n-dimensional subspace of B and let $A \in \mathcal{L}(D)$. Then*

$$\lambda_A(B, D) = \lim_{m\to\infty} \lambda_{A_m}(l_\infty^{(m)}, D_m).$$

Proof. First we assume that A is a nonsingular mapping. By the same reasoning as in [Che5, Th. 1.1], the sequence $\{D_A^m(B, D)\}$ has property $(*)$. Hence, by Proposition 2.4, $\lambda_A(B, D) = \lim_{m\to\infty} N_m$. For every $m \geq n$ select $P_m \in \mathcal{D}_A^m(B, D)$ with $N_m(P_m) \leq N_m + 1/m$. Note that by the Tietze–Urysohn Theorem,

$$N_m(P_m) = \sup\{\max_{i=1,\ldots,m} |\varphi_i(P_m f)| : f \in B,\ \max_{i=1,\ldots,m} |\varphi_i(f)| \leq 1\}. \tag{3.2.11}$$

(We denote by $W_m(P_m)$ the right side of (3.2.11).) For each $m \geq n$ define $Q_m : l_\infty^{(m)} \to l_\infty^{(m)}$ by

$$Q_m x = (\varphi_1(P_m f_x), \ldots, \varphi_m(P_m f_x)) \tag{3.2.12}$$

where $f_x \in C(T)$ is so chosen that $f_x(t_i) = x_i$ for $i = 1, \ldots, m$ and $\|f_x\| = \|x\|$. Since the value of $P_m f_x$ only depends on the values of f_x at the points t_i, the operator Q_m is well defined. Note that by (3.2.10), for $x \in D_m$,

$$Q_m x = (\varphi_1(P_m y), \ldots, \varphi_m(P_m y)) = (\varphi_1(Ay), \ldots, \varphi_m(Ay)) = A_m x.$$

Hence $Q_m \in \mathcal{P}_{A_m}(l_\infty^{(m)}, D_m)$. Moreover, by (3.2.12), $\|Q_m\| = W_m(P_m) = N_m(P_m)$. Hence

$$\lambda_{A_m}(l_\infty^{(m)}, D_m) \leq \|Q_m\| \leq N_m(P_m) \leq N_m + 1/m,$$

which gives, in view of Proposition 2.4,

$$\limsup \lambda_{A_m}(l_\infty^{(m)}, D_m) \leq \lambda_A(B, D).$$

To show that $\liminf \lambda_{A_m}(l_\infty^{(m)}, D_m) \geq \lambda_A(B, D)$, take $Q_m \in \mathcal{P}_{A_m}(l_\infty^{(m)}, D_m)$ of minimal norm. Define $L_m \in \mathcal{L}(B, D)$ by

$$L_m f = (Q_m(\varphi_1(f), \ldots, \varphi_m(f))^{**}$$

(** denotes the unique extension of $Q_m(\varphi_1(f),\ldots,\varphi_m(f))$ onto the space D; see (3.2.9)). We show that $L_m \in \mathcal{D}_A^m(B,D)$. Since for $f \in D$,

$$L_m f = [(Q_m(\varphi_1(f),\ldots,\varphi_m(f))]^{**} = (A_m(\varphi_1(f),\ldots,\varphi_m(f))^{**} = (\varphi_1(Af),\ldots,\varphi_m(Af))^{**} = Af\,,$$

we have $L_m \in \mathcal{P}_A(B,D)$. Moreover, if $\varphi_1(f) = \ldots = \varphi_m(f) = 0$ then $L_m(f) = 0$, which finally gives $L_m \in \mathcal{D}_A^m(B,D)$. It is clear that

$$\lambda_{A_m}(l_\infty^{(m)}, D_m) = \|Q_m\| = W_m(L_m) = N_m(L_m) \ge N_m$$

and consequently,

$$\liminf \lambda_{A_m}(l_\infty^{(m)}, D_m) \ge \liminf N_m = \lambda_A(B,D)\,,$$

which completes the proof in the case of nonsingular operators.

Now suppose $A \in \mathcal{L}(D)$ is a singular mapping. Fix $\varepsilon > 0$. By Remark 3.2.2, there exists a nonsingular mapping $A^\varepsilon \in \mathcal{L}(D)$ with $\|A - A^\varepsilon\| < \varepsilon$. Let $y_1,\ldots,y_n$ be a basis of D and let $\psi_1,\ldots,\psi_n \in B^*$, $\|\psi_i\| = 1$ for $i = 1,\ldots,n$, be so chosen that $\psi_i(y_j) = \delta_{ij}$. Define $L = \sum_{i=1}^n \psi_i(\cdot)Ay_i$ and$L^\varepsilon = \sum_{i=1}^n \psi_i(\cdot)A^\varepsilon y_i$. By (3.1.2),

$$|\lambda_{A^\varepsilon}(B,D) - \lambda_A(B,D)| \le \|L^\varepsilon - L\| \le n \max_{i=1,\ldots,n} \|y_i\| \cdot \|A - A_\varepsilon\|\,.$$

Moreover, for each $m \ge n$,

$$|\lambda_{A_m^\varepsilon}(l_\infty^{(m)}, D_m) - \lambda_{A_m}(l_\infty^{(m)}, D_m)| \le \|L_m^\varepsilon - L_m\|\,,$$

where A_m (resp. A_m^ε) is defined by (3.2.10) and L_m (resp. L_m^ε) is given by (3.2.12). Observe that

$$\|L_m^\varepsilon - L_m\| = W_m(L^\varepsilon - L) = N_m(L^\varepsilon - L) = \sup_{\|f\|\le 1} \max_{i=1,\ldots,m} |\varphi_i(L - L^\varepsilon)f| \le \|L - L^\varepsilon\| \le n \max_{i=1,\ldots,n} \|y_i\| \cdot \|A - A^\varepsilon\|\,.$$

Note that

$$|\lambda_A(B,D) - \lambda_{A_m}(l_\infty^{(m)}, D_m)| \le |\lambda_A(B,D) - \lambda_{A^\varepsilon}(B,D)| + |\lambda_{A^\varepsilon}(B,D) - \lambda_{A_m^\varepsilon}(l_\infty^{(m)}, D_m)| + |\lambda_{A_m^\varepsilon}(l_\infty^{(m)}, D_m) - \lambda_{A_m}(l_\infty^{(m)}, D_m)|\,.$$

By the previous part of the proof we can choose $m_0 \in \mathbb{N}$ such that for $m \ge m_0$,

$$|\lambda_{A^\varepsilon}(B,D) - \lambda_{A_m^\varepsilon}(l_\infty^{(m)}, D_m)| \le \varepsilon$$

(A^ε is a nonsingular mapping). Applying the above estimates we get for $m \ge m_0$,

$$|\lambda_A(B,D) - \lambda_{A_m}(l_\infty^{(m)}, D_m)| \le \varepsilon + 2n \max_{i=1,\ldots,n} \|y_i\| \cdot \|A - A^\varepsilon\| \le (1 + 2n \max_{i=1,\ldots,n} \|y_i\|)\varepsilon\,,$$

which completes the proof of the theorem.

Note that to show a theorem similar to Theorem 2.5 for any normed linear space B it is necessary to know that the sequence $\{\mathcal{D}_A^m(B,D)\}$ has property $(*)$ (see (3.2.6)) and satisfies the following condition:

(3.2.13) for every $m \geq n = \dim D$ and every $x \in l_\infty^{(m)}$ with $\|x\| = 1$ there exists $\psi \in B^{**}$, $\|\psi\| = 1$, with $\psi(f_i) = x_i$.

For this reason it is possible to prove

THEOREM 3.2.6. *Assume $B = C^{(k)}[a,b]$ and let $\varphi_i = \widehat{t_i}$ for $i = 1, 2, \ldots$ where $\{t_i\}$ is a dense, countable subset of $[a,b]$ and $\widehat{t_i}$ is defined by* (3.2.3). *If D is an n-dimensional subspace of B and $A \in \mathcal{L}(D)$ then*

$$\lim_{m\to\infty} \lambda_{A_m}(l_\infty^{(m)}, D_m) = \lambda_A(B, D).$$

Proof. By Theorem 2.4 the sequence $\{\mathcal{D}_A^m(B, D)\}$ has property $(*)$. To finish the proof it is sufficient to show that $\{\mathcal{D}_A^m(B, D)\}$ satisfies (3.2.13). Fix $x \in l_\infty^{(m)}$ with $\|x\| = 1$. By the Tietze–Urysohn Theorem there exists $f \in C[a,b]$ with $\|f\| = 1$ and $f(x_i) = x_i$ for $i = 1, \ldots, m$. Let $g_1(t) = \int_a^t f(s)\,ds$ and $g_i(t) = \int_a^t g_{i-1}(s)\,ds$ for $i = 2, \ldots, k$. Note that $g_k^{(j)} = g_{k-j}$ for $j = 0, \ldots, k$ $(g_0 = f)$. Since $g_j(a) = 0$ for $j = 0, \ldots, k-1$, by (3.2.3) we have $\widehat{t_i}(g_k) = f(t_i) = x_i$ for $i = 1, 2, \ldots$ Moreover,

$$\|g_k\| = \sum_{j=0}^{k-1} |g_k^{(j)}(a)| + \|f\|_{\sup} = \|f\|_{\sup} = 1.$$

Since g_k may be treated as a functional from B^{**} of norm 1, the proof is complete.

THEOREM 3.2.7. *Assume B is a normed space and let $\{\varphi_i\}$ be a countable weak*-dense subset of S_{B^*}. Let $D \subset B$ be an n-dimensional subspace and let $A \in \mathcal{L}(D)$. Then*

$$\lambda_A(B, D) \leq \lim_{m\to\infty} \lambda_{A_m}(l_\infty^{(m)}, D_m),$$

where D_m is defined by (3.2.9) *and A_m by* (3.2.10).

Proof. It is well known that B can be linearly isometrically embedded in $C(S_{B^*})$ (in S_{B^*} we have the weak* topology). Then, by Theorem 2.5,

$$\lambda_A(B, D) \leq \lambda_A(C(S_{B^*}), D) = \lim_{m\to\infty} \lambda_{A_m}(l_\infty^{(m)}, D_m)$$

as required.

The last result of this section will be devoted to the case of sequence Orlicz–Musielak spaces [Mu, p. 33].

THEOREM 3.2.8. *Assume l_ψ is a sequence Orlicz–Musielak space (ψ is a locally integrable function) equipped with the Luxemburg norm satisfying the δ_2 condition* [Mu, p. 52]. *Let $A \in \mathcal{L}(D)$. For $i = 1, 2, \ldots$ and $x \in l_\psi$ put $\varphi_i(x) = x_i$. Then*

$$\lambda_A(B, D) = \lim_{m\to\infty} \lambda_{A_m}(l_\psi^{(m)}, D_m),$$

where D_m is defined by (3.2.9) *and $l_\psi^{(m)}$ denotes the space K^m ($K = \mathbb{R}$ or $K = \mathbb{C}$) equipped with the norm induced from l_ψ.*

Proof. For $m \in \mathbb{N}$ and $P \in \mathcal{D}_A^m(l_\psi, D)$ put

$$U_m(P) = \sup\{\|(\varphi_1(Pf), \ldots, \varphi_m(Pf))\|_\psi : \|(\varphi_1(f), \ldots, \varphi_m(f))\|_\psi \le 1\}$$

and

$$U_m = \inf\{U_m(P) : P \in \mathcal{D}_A^m(B, D)\}\,.$$

Note that for every $P \in \mathcal{D}_A^m(B, D)$,

$$U_m(P) = \sup\{\|(\varphi_1(Pf), \ldots, \varphi_m(Pf))\|_\psi : \|f\|_\psi \le 1\} \tag{3.2.14}$$

(for each $x \in l_\psi^{(m)}$, $\|x\|_\psi = \|(x, 0, \ldots)\|_\psi$ and since $P \in \mathcal{D}_A^m(B, D)$ the value of Pf only depends on the first m coordinates). By the δ_2 condition [Mu, pp. 36, 53], the sequence $\{\mathcal{D}_A^m(B, D)\}$ has property $(*)$. By (3.2.14), $U_m \le U_m(P) \le \|P\|$ for every $P \in \mathcal{D}_A^m(B, D)$. Hence reasoning as in Proposition 2.4 we get $\lim_{m\to\infty} U_m = \lambda_A(l_\psi, D)$.

To finish the proof it is sufficient to show that $\lambda_{A_m}(l_\psi^{(m)}, D_m) = U_m$, which is an immediate consequence of (3.2.14). The proof is complete.

Remark 3.2.9. Assume B is a finite-dimensional real or complex Banach space. Then the number N_m in Proposition 2.4 can be replaced by $W_m = \inf\{W_m(P) : P \in \mathcal{D}_A^m(B, D)\}$ ($W_m(P)$ is defined by (3.2.11)). In the real case W_m can be calculated by the Remez algorithm (see e.g. [Che1, p. 54]). Therefore, by Proposition 2.4, we can calculate $\lambda_A(B, D)$ numerically. Moreover, repeating this procedure for the sequence (B_m, D_m) described in Theorem 3.1.6 (see also Examples 3.1.7–3.1.9) we can estimate the constant $\lambda_A(B, D)$ not only in the finite-dimensional case (see also Theorem 3.8). Theorem 2.5 extends Theorem 0.1.3 of [Che6].

III.3. Algorithms for seeking the constant W_m

First form of the extremum problem. In this section $D \subset B$ is an n-dimensional subspace (we consider the real case). Let $f_1, \ldots, f_m \in S_{B^*}$, $m \ge n$, satisfy (3.2.13). Assume furthermore that $f_1|_D, \ldots, f_n|_D$ is a basis of D^*. For given $F \in \mathcal{L}(D)$, we seek to determine $P \in \mathcal{D}_F^m$ (see (3.2.1)) satisfying $W_m(P) = W_m$ (W_m is considered with respect to F and $f_1, \ldots, f_m$). Since $f_1|_D, \ldots, f_n|_D$ is a basis of D^* there exist $d_1, \ldots, d_n \in D$ with $f_i(d_j) = \delta_{ij}$. Put $Q_0 = \sum_{j=1}^n f_j(\cdot)d_i$ and $P_0 = F \circ Q_0$. It is clear that $P_0 \in \mathcal{D}_F^m$. Moreover, if $n = m$ then $\mathcal{D}_F^m = \{P_0\}$. If $m > n$ then $P \in \mathcal{D}_F^m$ satisfying $W_m(P) = W_m$ must be of the form $P = P_0 - L$ where $L = \sum_{i=1}^m f_i(\cdot)u_i$ for some $u_1, \ldots, u_n \in D$. So in order to determine P it is necessary to determine $u_1, \ldots, u_n$. Note that $u_i = \sum_{j=1}^n f_j(u_i)d_j$ for $i = 1, \ldots, m$ and one sees that the following equations are equivalent:

$$L(D) = 0\,, \tag{3.3.1}$$

$$L(d_j) = 0 \quad \text{for } j = 1, \ldots, n\,, \tag{3.3.2}$$

$$\sum_{i=1}^m f_i(d_j)u_i = 0 \quad \text{for } j = 1, \ldots, n\,, \tag{3.3.3}$$

$$\sum_{i=1}^{m} f_i(d_j)\sum_{k=1}^{n} f_k(u_i)d_k = 0 \quad \text{for } j=1,\dots,n\,, \tag{3.3.4}$$

$$\sum_{i=1}^{m} f_i(d_j)f_k(u_i) = 0 \quad \text{for } j=1,\dots,n,\ k=1,\dots,n\,, \tag{3.3.5}$$

$$B\circ A = 0, \qquad A_{ij} = f_i(d_j), \qquad B_{ji} = f_j(u_i)\,. \tag{3.3.6}$$

The $m\times n$ matrix A is determined by the choice of the functionals $f_1,\dots,f_m$. The $n\times m$ matrix B is arbitrary except for the condition $B\circ A=0$. Each choice of B satisfying this condition leads to a set of functions $u_1,\dots,u_m$ via the equation

$$u_i = \sum_{j=1}^{n} f_j(u_i)d_j = \sum_{j=1}^{n} B_{ji}d_j\,.$$

From these we obtain the operators L and P by the above equations. For convenience, define $d_{n+1}=d_{n+2}=\ldots=d_m=0$. Then P can be written as

$$P = \sum_{j=1}^{m} f_j(\cdot)(Fd_j - u_j) = \sum_{j=1}^{m} f_j(\cdot)\Big(Fd_j - \sum_{k=1}^{n} B_{kj}d_k\Big)\,.$$

Since the functionals $f_1,\dots,f_n$ satisfy (3.2.13) the expression to be minimized is

$$W_m(P) = \max\Big\{\sum_{j=1}^{m}\Big|f_i(Fd_j) - \sum_{k=1}^{n} B_{kj}f_i(d_k)\Big| : i=1,\dots,m\Big\}$$

and this is to be done by choosing B freely, except for the constraint $B\circ A=0$.

Second form of the extremum problem. In the reasoning presented above, the problem of calculating the constant W_m was formulated as a constrained minimization problem. Here the problem will be given in unconstrained form. The constraint $B\circ A=0$ states that the rows of B lie in the orthogonal complement of the columns of A. This orthogonal complement has as a basis the rows of the matrix

$$D_{ij} = \begin{cases} -f_{n+i}(d_j) & \text{if } i\le j\le n \text{ and } 1\le i\le m-n\,,\\ \delta_{j,n+i} & \text{if } n+1\le j\le m \text{ and } 1\le i\le m-n\,. \end{cases}$$

The asserted orthogonality is proved by observing that A and D are partitioned matrices of the form $A=(I_n,K)^T$ and $D=(-K,I_{m-n})$, whence $D\circ A = -K\circ I_n + I_{m-n}\circ K = 0$. Now each row of B can be an arbitrary linear combination of the rows of D. Hence we can write $B=C\circ D$ where C is now a free (unconstrained) $n\times(m-n)$ matrix. The expression to be minimized by the free choice of C is

$$W_m(P) = \max\Big\{\sum_{j=1}^{m}\Big|f_i(Fd_j) - \sum_{k=1}^{n}\sum_{l=1}^{m-n} C_{kl}D_{lj}f_i(d_k)\Big| : i=1,\dots,m\Big\}\,.$$

If we extend A to an $m\times m$ matrix by setting $A_{ij}=f_i(d_j)$ and $d_j=0$ for $j>n$, then we have a new interpretation of the problem:

$$W_m(P) = \|A \circ F - A \circ C \circ D\|_\infty = \min . \tag{3.3.10}$$

Here the matrix norm is given by $\|A\|_\infty = \max_i \sum_j |A_{ij}|$, and we put $C_{ij} = 0$ when $i > n$.

Third form of the extremum problem. In order to turn (3.3.10) into a Chebyshev approximation problem, let $S = \{-1, 1\}^m$. Then

$$\begin{aligned} W_m(P) &= \max_i \sum_{j=1}^m \Big| f_i(Fd_j) - \sum_{k=1}^n \sum_{l=1}^{m-n} C_{kl} D_{lj} f_i(d_k) \Big| \\ &= \max_{s\in S} \max_i \Big| \sum_{j=1}^m s_j \Big(f_i(Fd_j) - \sum_{k=1}^n \sum_{l=1}^{m-n} C_{kl} D_{lj} f_i(d_k) \Big) \Big| \\ &= \max_i \max_{s\in S} \Big| h(i,s) - \sum_{k=1}^n \sum_{l=1}^{m-n} C_{kl} g_{kl}(i,s) \Big| = \Big\| h - \sum_{k,l} C_{kl} g_{kl} \Big\|_\infty . \end{aligned}$$

Here

$$h(i,s) = \sum_{j=1}^m s_j f_i(Fd_j), \qquad g_{kl}(i,s) = f_i(d_k) \sum_{j=1}^m s_j D_{lj}$$

for $(i,s) \in \{1,\ldots,m\} \times S$. Therefore the minimization of $W_m(P)$ is a problem to which the Remez algorithm can be applied. The problem is nondegenerate, in the sense that the set of functions g_{kl} is linearly independent (this was checked in [Che6]). The convergence of this algorithm is established, for example, in [Che1].

Remark 3.3.1. In view of Theorems 3.2.3, 3.2.5 and 3.2.6 the methods of calculation of the constant W_m presented above can be applied for seeking the constant $\lambda_A(B,D)$ for $B = C_{\mathbb{R}}(T)$ (T compact, metrizable) and $B = C^{(k)}[a,b]$. Moreover, by Theorem 3.2.7 we can use these methods for estimation from above of the constant $\lambda_A(B,D)$ if B has weak*-separable dual space.

Remark 3.3.2. The results presented in this section are an adaptation of the results proved in [Che6] for the case $F = \mathrm{Id}$.

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