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AB PERCOLATION: A BRIEF SURVEY

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The AB percolation model is a variant of the classical percolation model, which was motivated by applications to polymerization and gelation processes and anti-ferromagnetism. This paper discusses recent developments in the theory, and provides two new results. The main technique used to obtain results is to relate the AB percolation model to an appropriate classical percolation model. New results include a proof that AB percolation occurs on any graph with site percolation critical probability less than 1/2, and a characterization of the AB percolation critical probability of the triangular lattice in terms of a site percolation critical probability on a related lattice.

1. Introduction

In this paper, we consider a variant of the classical percolation model which was introduced independently by several authors in the early 1980's. We will describe the model in terms of atomic or molecular bonding considerations: There are two types of atoms, A and B, which occupy the sites of an infinite lattice graph G, with probabilities p and $1-p$ respectively. Unlike atoms which are connected by an edge of G become bonded together, while like atoms do not bond to each other. The object of study in the model is the probability distribution of the size of the clusters of atoms that are bonded together. As in classical percolation theory, while little can be said about the explicit form of the probability distribution, one is most interested in determining if infinite bonded clusters exist for various values of the parameter p, and progress has been made on this question.

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The model was first studied by Mai and Halley (1980), who named it "AB percolation", motivated by problems in polymerization and gelation. Their Monte Carlo simulation suggested that infinite AB clusters exist for an interval of values of p when G is the triangular lattice. Halley (1983) discussed a more general class of models which he termed "polychromatic percolation", and provided an argument which proved nonexistence of infinite AB percolation clusters on bipartite lattices when $p = 1/2$, which will be discussed in Section 2. Halley (1983) also treated the case of AB percolation on Bethe trees. Turban (1983) and Sevšek, Debierre, and Turban (1983) introduced the model as "anti-percolation", motivated by the study of anti-ferromagnetism. They provided an argument for the existence of infinite AB clusters on the triangular lattice, which, although incorrect, suggested the approach presented in Section 4. Wilkinson (1987) considered two-parameter percolation on bipartite graphs as a model for gelation. AB percolation on a bipartite graph is a special case of his model.

This paper concentrates on rigorous mathematical results concerning AB percolation. Other than the work of Halley (1983), rigorous results have been obtained only recently. Scheinerman and Wierman (1987) provided the first example of a two-dimensional periodic graph on which infinite AB clusters exist with positive probability. Appel and Wierman (1987) proved that AB percolation is impossible on a class of bipartite graphs, partially verifying a conjecture of Halley (1983). Wierman and Appel (1987) proved that infinite AB percolation clusters exist on the triangular lattice, as was claimed by Sevšek, Debierre, and Turban (1983) and suggested by the Monte Carlo study of Mai and Halley (1980). The new results surveyed here stem from a joint research project of the author and Martin Appel, a doctoral student at Johns Hopkins University.

One feature common to all the arguments concerning the model is that they establish a relationship between AB percolation and a classical percolation model. Several of the arguments, while giving some useful information, essentially choose an incorrect classical model, in the sense that they do not give the most complete description of the AB percolation behavior possible.

2. Definitions and background

A graph G consists of a countable set $V(G)$ of vertices and a countable set $E(G)$ of pairs of vertices, called edges. An assignment of a label, A or B, to each vertex of G is a *configuration* on G, i.e. a configuration is an element $\omega \in \{A, B\}^{V(G)}$, or equivalently a function $\omega: V(G) \to \{A, B\}$. The AB percolation model on G is a probability model with sample space $\{A, B\}^{V(G)}$ and probability measure P_p such that the labels of the vertices of G are independent random variables with probability p of labeling each vertex A.

An edge of G is an AB *bond* if the endpoints of the edge have different labels. An AB *path* is an alternating sequence of vertices and edges v_0, e_1, $v_1, \ldots, e_n, v_n$ such that e_i, $1 \leqslant i \leqslant n$, are all AB bonds. [We will use the terms "A path" and "B path" to refer to paths with all vertices labeled A or B respectively. Also, at times we will refer to the labels as "colors", and refer to a path as "monochromatic" if all its vertices have a common label.] The AB *cluster* containing a vertex v, denoted by W_v^{AB}, is the set of vertices which may be reached from v through an AB path. The number of vertices in W_v^{AB} is denoted by $\# W_v^{AB}$. Denote the AB *percolation probability* by

$$\theta_v^{AB}(p) = P_p(\# W_v^{AB} = +\infty).$$

Note that AB paths and AB clusters are unchanged if the label of every vertex is changed, but the parameter of the model is changed from p to $1-p$. Thus,

$$\theta_v^{AB}(p) = \theta_v^{AB}(1-p)$$

for all $p \in [0, 1]$, so the AB percolation probability function is symmetric about 1/2.

The probability that any particular edge of G is an AB bond is $2p(1-p)$, which has its maximum at 1/2 and is monotone on each side of 1/2. Intuitively, one expects the AB percolation probability to have these properties also, which would imply that there is a single interval of values of p for which the AB percolation probability is positive. However, there is no general proof of the claim of monotonicity, and it has not been proven that there cannot be multiple intervals of positivity.

While the value of $\theta_v^{AB}(p)$ may depend on the vertex v, the set of values of p for which $\theta_v^{AB}(p) > 0$ is independent of the choice of vertex if G is a connected graph, as for classical percolation models. Thus, for a connected graph G and an arbitrary site v, we define the AB critical probability by

$$p_H^{AB}(G) = \inf\{p \colon \theta_v^{AB}(p) > 0\}.$$

[By convention, if the set on the right side is empty, $p_H^{AB}(G) = \infty$.] We use the notation p_H rather than p_C for the critical probability because equality of several definitions of critical probability has not yet been proved for AB percolation. We will denote the classical site percolation critical probability by p_C, since we assume that the results of Menshikov, Molchanov, and Sidorenko (1986) or Aizenman and Barsky (1987) apply to G. Note that since we do not know that the percolation probability is monotone for $p < 1/2$, it may be possible that $\theta_v^{AB}(p) = 0$ for a value $p \in (p_H^{AB}, 1/2]$.

An inclusion principle holds as for classical percolation models: If $G \subset H$, then $p_H^{AB}(G) \geqslant p_H^{AB}(H)$. To see this, note that inserting additional edges in G may create additional AB edges for any configuration but destroys no AB edges that were already present, so the AB cluster in H containing v is at least as large as the AB cluster in G containing v, for every configuration ω.

For any graph G, we define the graph G_2 as follows: Let G_2 have the same vertex set as G. The edge set of G_2 contains all edges of G as well as an edge between each pair of vertices which may be connected by a path of 2 edges in G.

Since successive vertices on an AB path have alternating labels, if an infinite AB path exists on G, then there exists an infinite A path on G_2 and an infinite B path on G_2. The existence of such paths requires that $p > p_C(G_2)$ and $1-p > p_C(G_2)$ respectively. Therefore, we have the lower bound:

$$p_C(G_2) \leqslant p_H^{AB}(G).$$

This lower bound implies that $p_H^{AB}(G) > 0$ for any periodic graph G (as defined by Kesten (1982)), since one may apply a Peierl's argument to the graph G_2 to show that $p_C(G_2) > 0$. We may also observe that if $p_C(G_2) > 1/2$, then $\theta_v^{AB}(p) = 0$ for all $p \in [0, 1]$, i.e. AB percolation cannot occur on G.

A graph is *bipartite* if there exists a partition of the vertex set into two sets V_1 and V_2 such that every edge in G has one endpoint in V_1 and the other endpoint in V_2. We will call the sets V_1 and V_2 a *bipartition*. Note that any path on a bipartite graph passes through vertices of V_1 and V_2 alternately.

Halley (1983) proved that if G is a bipartite graph with site percolation critical probability greater than 1/2, then $\theta_v^{AB}(1/2) = 0$. Using symmetry when $p = 1/2$, one sees that reversing the labels on one set of the bipartition of G preserves the probability measure. However, the label reversal converts each AB cluster into a monochromatic cluster. Thus, when $p = 1/2$, the probabilities of existence of infinite AB, A and B clusters on G are all equal, so if the classical site percolation critical probability is strictly greater than 1/2, they must all be zero. If it were proved that the AB percolation probability assumes its maximum at 1/2, AB percolation could not occur on any bipartite graph. While the claim is intuitive, its proof remains an open problem.

Appel and Wierman (1987) gave a partial confirmation of Halley's claim. Let G be a bipartite graph with bipartition sets V_1 and V_2. For each V_i, construct a graph $G(V_i)$ with vertex set V_i, such that two vertices u and v are adjacent in $G(V_i)$ if and only if u and v are adjacent to a common vertex in G. Let p_1 and p_2 denote the site percolation critical probabilities of $G(V_1)$ and $G(V_2)$ respectively. Since labels on an AB path in G alternate, if there is an infinite AB path in G, there exists an infinite A path on one of the two derived graphs, and an infinite B path on the other. This leads to the following result: If $p_1 + p_2 > 1$, or if $p_1 + p_2 = 1$ and each of $G(V_1)$ and $G(V_2)$ is a member of a matching pair of graphs (in the sense of Sykes and Essam (1964)), is periodic and has one axis of symmetry, then $\theta_v^{AB} = 0$ for all $p \in [0, 1]$. For example, this result shows that AB percolation does not exist on the hexagonal lattice. While the result does not apply to the square lattice, a separate proof shows that AB percolation does not exist on the square lattice either.

Since the beginning of the study of AB percolation, it has been of interest to prove that infinite AB percolation clusters can actually occur on one of the common planar lattices studied in physics. One may see that a natural lattice to study for this problem is the triangular lattice: By the inclusion principle, a more richly connected graph has a higher probability of occurrence of AB percolation. Since edges may be inserted in any planar graph to obtain a fully triangulated graph, one concludes that if AB percolation occurs on any planar graph, it must occur on some fully triangulated graph. With the regularity and periodicity of the triangular lattice, it is perhaps the easiest in the class to work with.

Mai and Halley (1980) performed Monte Carlo simulations of AB percolation on the triangular lattice which suggest that infinite AB clusters exist when $p \in [0.2145, 0.7855]$, but provided no proof that the phenomenon occurred for any value of p.

Sevšek, Debierre, and Turban (1983) gave an incorrect argument which claims that an infinite AB cluster exists on the triangular lattice when $p = 1/2$. They remark that, on the triangular lattice, all the boundary sites of a site percolation cluster (of A's) belong to the same AB cluster. Since the critical probability of the triangular lattice is 1/2 (see Kesten (1982)), they claim that when $p = 1/2$ there is an infinite A cluster, and that following its boundary provides an infinite AB cluster. However, this argument is invalid for two reasons. First, the principal result of Kesten (1982) established that for site percolation on a class of periodic two-dimensional graphs, which includes the triangular lattice, there is almost surely no infinite cluster at the critical probability. Second, an open cluster in a classical site percolation model contains arbitrarily large circuits, in which case the boundary of the cluster consists of a union of finite length circuits, so one must verify that the AB clusters obtained by following the segments of the boundary can be linked together to obtain an infinite AB cluster.

Wierman and Appel (1987) gave the first rigorous proof of existence of AB percolation on the triangular lattice. They examined the boundary of finite AB clusters, finding that they must be blocked by monochromatic "double circuits" (i.e. two circuits of the same color with no vertices in the region between). By showing that there can only exist finitely many such circuits around the origin, it was proved that an infinite AB percolation cluster exists when $p \in (1 - p_T^{AA}, p_T^{AA})$, where p_T^{AA} is a critical probability defined in terms of the size of the cluster of sites which are connected to each other by "double paths" of A's. Since the existence of a double path of A's on the triangular lattice implies the existence of an A path on the square lattice, it can be shown that p_T^{AA} is at least as large as the site percolation critical probability of the square lattice, which is at least .503 (see Tóth (1985)). The interval (.497, .503) is much smaller than that suggested by Mai and Halley (1983). The proof in Section 4 provides a somewhat wider interval.

3. An existence result

In this section, we give a simple argument to show that infinite AB percolation clusters exist with positive probability on a large collection of graphs. The argument presented here is based on a suggestion by Geoffrey Grimmett.

PROPOSITION 3.1. *If G is an infinite connected graph with $p_C(G) < 1/2$, then $\theta_v^{AB}(p) > 0$ for all $p \in (p_C(G), 1-p_C(G))$.*

Proof. For simplicity, begin by assuming that $p = 1/2$. Consider the following inductive procedure for constructing an AB cluster containing a specific vertex v. Let $W_0 = v$ and $C_0 = \emptyset$. For each $n \geqslant 1$ choose a vertex w which is not in $W_{n-1} \cup C_{n-1}$ but is adjacent to a vertex $z \in W_{n-1}$. If w and z have opposite labels, then let $W_n = W_{n-1} \cup \{w\}$ and $C_n = C_{n-1}$. If w and z have the same label, then let $W_n = W_{n-1}$ and $C_n = C_{n-1} \cup \{w\}$. If at some point there is no vertex w as required, the process stops and we have generated a cluster, which we will denote as W. If the process continues indefinitely, we use W to denote $\bigcup W_n$.

Note that $W \subseteq W_v^{AB}$, since (by construction) every vertex in W is reached from v by an AB path. In general, however, $W \neq W_v^{AB}$, because the procedure tests each vertex only once, from just one of its neighbors, and those vertices which fail may actually be in W_v^{AB} if tested from a different neighbor.

Now view the process from a different perspective. Consider W_n to be the set of open vertices and C_n to be the set of closed vertices in the classical percolation model. At each step, a new vertex is added to the open cluster with probability 1/2. Each vertex is tested only once. Thus, the cluster size $\# W$ obtained has exactly the same probability distribution as the open cluster size in a classical percolation model on the graph with parameter 1/2. Hence, if the critical probability $p_C(G) < 1/2$, there is positive probability that $\# W = \infty$, and thus positive probability that $\# W_v^{AB} = \infty$. We may then conclude that infinite AB percolation clusters exist on a graph G when $p = 1/2$ if the site percolation critical probability of G is strictly less than 1/2.

If we now return to the argument in the case where $p \neq 1/2$ we see that the cluster generated by the process is larger than an open cluster in the classical percolation model with parameter equal to $\min\{p, 1-p\}$ and smaller than the AB cluster. From this we may conclude that if $p_H(G) < 1/2$ then AB percolation occurs for all $p \in (p_H(G), 1-p_H(G))$. ∎

The result does not require any symmetry or periodicity properties, nor does it depend on the dimension of the lattice. Note that due to the inequalities between the cluster sizes used in the argument, we should expect that infinite AB clusters exist for values of p outside the interval specified. For example, for the triangular lattice, which has site percolation critical probability equal to 1/2, this argument does not prove that AB percolation exists, a fact which was proved by Wierman and Appel (1987). Nevertheless, this proof shows that the phenomenon does occur on a large class of graphs.

4. Existence on the triangular lattice: direct approach

In this section, we give a proof of the existence of AB percolation on the triangular lattice which differs from that of Wierman and Appel (1987). The idea of the proof is similar to that of Sevšek, Debierre, and Turban (1983), which attempted to construct an infinite AB cluster by following the boundary of a monochromatic cluster. The difference is that rather than considering monochromatic clusters in the original triangular lattice, we look at clusters in the T_2 lattice. The proof shows that this is the "correct" lattice to consider, since the proof actually identifies the critical probability of AB percolation on the triangular lattice as the classical site percolation critical probability on the T_2 lattice.

We begin by establishing an upper bound for the site percolation critical probability of the T_2 lattice. Combined with the following theorem, it also provides an improved upper bound for the AB percolation critical probability of the triangular lattice. The proof is based on a technique suggested by Tomasz Łuczak.

LEMMA 4.1. $p_C(T_2) \leqslant .4031$.

Proof. Represent the triangular lattice with equilateral triangular faces with one set of edges parallel to the x-axis. Consider successive groups of three vertices on each horizontal line to represent sites in a new triangular lattice. [See Fig. 1.] Relative to a site percolation model on the original lattice, each group will be called "open" if either the center vertex is open or both endpoint vertices are open. Thus each group is open with probability $p+(1-p)p^2 = p+p^2-p^3$. It is easy to check that if two groups are open and are diagonal neighbors or one is two lines above the other, then all open sites in the two groups are in a common open cluster in the T_2 lattice. Since the adjacency

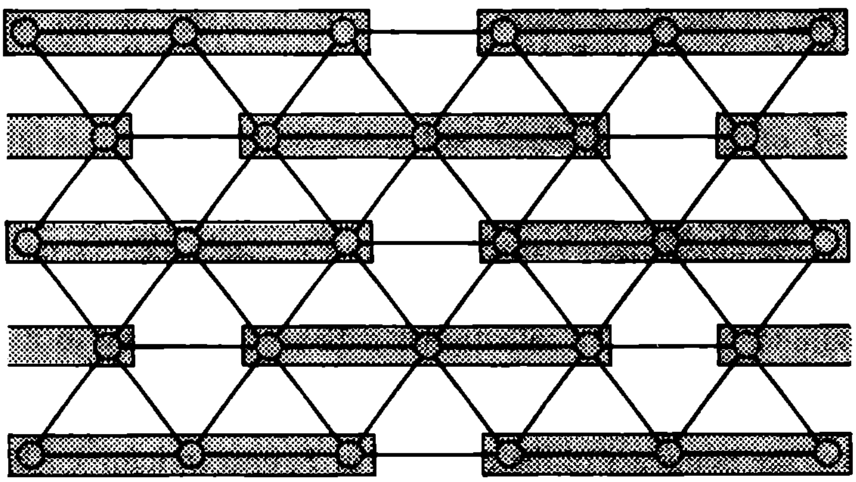

Fig. 1. A portion of the triangular lattice, showing groups of three vertices used in finding an upper bound for the critical probability of the T_2 lattice. Each group is open if either the center vertex is open or both end vertices are open

relationship between the groups provides the triangular lattice, there is an infinite open cluster of such groups if $p+p^2-p^3 > p_C(T) = 1/2$, and thus if $p \geqslant .4031$. Since the infinite cluster of groups implies an infinite cluster on the T_2 lattice, $p \geqslant .4031$ implies $p > p_C(T_2)$. ∎

THEOREM 4.2. $p_H^{AB}(T) = p_C(T_2)$.

Proof. Clearly, if $p < p_C(T_2)$ an infinite AB cluster cannot occur on T, so $p < p_H^{AB}(T)$, and thus $p_C(T_2) \leqslant p_H^{AB}(T)$.

On the other hand, if $p > p_C(T_2)$, then an infinite cluster of A's occurs in T_2 with probability one. If in addition, $p < 1-p_C(T_2)$, then $1-p > p_C(T_2)$, so there exists an infinite cluster of B's on T_2 containing a specific site with positive probability. Fix a vertex b_0 and consider a configuration ω such that b_0 is in an infinite cluster of B's in T_2. Let $b = (b_0, b_1, b_2, \ldots)$ denote an infinite path of B vertices contained in this cluster.

We may represent the infinite A cluster on T_2 as a connected set in the plane as follows: If three vertices of the cluster are pairwise adjacent, we will say that the triangle with those vertices is in the cluster. [Here we consider the triangles to include their interiors.] The cluster is represented in the plane as the union of its triangles and the line segments representing its edges which are not in such triangles. Note that this region representing the cluster may contain vertices of T which are labeled B.

Consider the topological boundary of the region representing the cluster. It consists only of edges of the graph T_2; it is not possible to have a portion of the boundary made up of parts of two edges of T_2 which cross, since then the four A vertices which are endpoints of the edges cause the region to be contained in triangles in the cluster.

The boundary of the A cluster in T_2 consists of a collection of simple polygonal curves. I claim that for each of these polygons, all vertices of T on the polygon are in a common AB cluster on T. Consider three cases, corresponding to the three types of edges of T_2. Let e be an edge of the boundary. If e is an edge of T, then there is a vertex labeled B which is adjacent to both endpoints of e. If e crosses an edge of T perpendicularly, then one of the endpoints of the crossed edge is not in the cluster and thus is labeled B. If e covers two edges of T, we must consider two subcases: If the vertex at its midpoint is labeled B, the endpoints are in a common AB cluster. If it is labeled A, then there are two vertices labeled B outside the cluster which link the three A vertices on e into a common AB cluster. [See Fig. 2.] If one of the polygonal curves is infinite, we have an infinite AB path corresponding to it. Thus, we assume all the polygonal curves have finite length.

We now show that the infinite B cluster on T_2 links these AB clusters in T that follow the boundary of the A cluster in T_2 into an infinite AB cluster in T. To reduce the number of cases in the following analysis, we introduce the following convention: Whenever an edge of b covers two edges of T and the

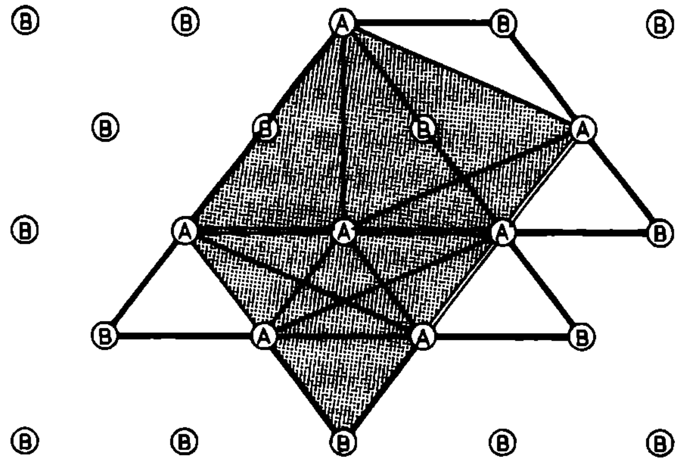

Fig. 2. Thin lines represent bonds in an A cluster on the T_2 lattice. The dotted region indicates the A cluster on T_2. Thick lines show the AB path which follows the boundary of the region representing the cluster

vertex at its midpoint is labeled B, we insert the midpoint into the path b. Thus, in the following, whenever such an edge occurs in b, its midpoint is labeled A. Consider a segment of the path b, say $b_{i_1}, \ldots, b_{i_n}$, such that b_{i_1} and b_{i_n} are in (or on the boundary of) separate components of the complement of the A cluster, and b_{i_j} are in the interior of the A cluster for all $j = 2, \ldots, n-1$.

Consider a vertex v which is contained in the interior of the A cluster in T_2. v has six neighbors in T, and one of the following holds: (a) Three of the neighbors are labeled A, no two of which are adjacent in T. (b) There are four neighbors labeled A, in two opposite pairs. In case (a), the three neighbors are vertices of an equilateral triangle which contains v in the interior. In case (b), the union of four triangles (with edges in T_2) contains v in the interior. In either case, more of the neighbors may be labeled A. [See Fig. 3.]

Suppose the edge $b_{i_j}b_{i_{j+1}}$ is an edge of T. If either one (or both) of the vertices is not in the interior of the A cluster, then both their common neighbors are labeled A, so they are in a common AB cluster on T. If either endpoint is in the interior of the A cluster, by the previous paragraph, at least

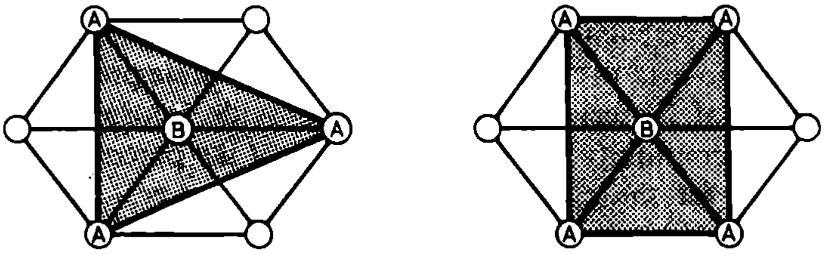

Fig. 3. Two portions of the triangular lattice, illustrating the two possible ways (including rotations) that a B site may be in the interior of an A cluster in T_2. Unlabeled sites may take either label

one of the common neighbors is labeled A. If both endpoints are on the boundary of the A cluster, then at least one of their common neighbors is labeled A, so they are in a common AB cluster.

Suppose the edge $b_{i_j}b_{i_{j+1}}$ crosses an edge of T perpendicularly. If one of the endpoints is in the interior of the A cluster, then in either case (a) or (b) above, one of the common neighbors is labeled A. If both are not in the interior of the A cluster, then if neither common neighbor was labeled A, the endpoints would be in or on the boundary of the same component of the complement of the A cluster, which is a contradiction. Otherwise, there is an A vertex which links the endpoints into a common AB cluster in T.

Suppose the edge $b_{i_j}b_{i_{j+1}}$ covers two edges of T. By our convention, the site at the midpoint of the edge is labeled A, so the endpoints are in the same AB cluster.

The preceding analysis shows that the AB clusters formed by following the boundary of the A cluster inside each component of its complement may be connected by AB paths along the parts of the path b which cross the A cluster. Thus, the AB cluster containing the origin is infinite. ∎

5. Complements and open problems

The proof in Section 4 identifies the critical probability of AB percolation on the triangular lattice, but required much case-by-case analysis. A different approach, based on duality methods from classical percolation theory, is also being developed by Appel and Wierman, in a forthcoming paper. The approach appears to be more easily generalizable to other graphs, but does not make as direct a connection between the AB cluster on T and the A cluster on T_2, so it provides some different information. For example, the duality approach shows that two critical probabilities are equal for AB percolation: Let $p_T^{AB} = \inf\{p\colon E|\# W_v^{AB}| < +\infty\}$ denote the threshold at which the expected AB cluster size becomes infinite. The method shows that $p_T^{AB} = p_H^{AB}$. The duality method also shows that $\theta_v^{AB}(p_H^{AB}) = 0$ for the triangular lattice, i.e. there is almost surely no infinite AB cluster at the AB critical probability.

One goal of the research on AB percolation is to characterize the class of graphs on which AB percolation can occur, and determine the critical probability of AB percolation on graphs for which it does. We expect the duality approach to lead to considerable progress on these problems in the case of two-dimensional lattices. It should also provide information (as in the classical case) on continuity and differentiability of the percolation probability function, bounds for the cluster size distribution, power estimates, and the nature of the singularity of the clusters-per-site function at the critical probability.

An interesting question concerning critical exponents for AB percolation arises from the Monte Carlo simulation of Mai and Halley (1980). They fitted an equation of the form $\theta^{AB}(p) = K(p-p_H^{AB})^{\beta_{AB}}$ for $p > p_H^{AB}$ and estimated the critical exponent $\beta_{AB} = .121 \pm .004$, compared to estimates of .14 for the site percolation critical exponent on the triangular lattice. While Mai and Halley's results suggest that the critical exponent of AB percolation on the triangular lattice is different than the critical exponent of classical site percolation on the triangular lattice, Wilkinson (1987) presents arguments that bipartite percolation and classical percolation are in the same universality class, and thus have the same critical exponents. The duality approach of Appel and Wierman also suggests that the behavior of AB percolation is similar to that of classical percolation. Either a proof of equality of critical exponents or a counterexample would be an interesting development.

References

[1] M. Aizenman and D. Barsky, *Sharpness of the phase transition in percolation models*, Comm. Math. Phys. 108 (1987), 489–526.

[2] M. J. Appel and J. C. Wierman, *On the absence of infinite AB percolation clusters in bipartite graphs*, J. Phys. A 20 (1987), 2527–2531.

[3] J. W. Halley, *Polychromatic percolation*, in: G. Deutscher, R. Zallen, and J. Adler (eds.), Percolation Structures and Processes, Adam Hilger, Bristol 1983, 323–351.

[4] H. Kesten, *Percolation Theory for Mathematicians*, Birkhaüser, Boston 1982.

[5] T. Mai and J. W. Halley, *AB percolation on a triangular lattice*, in: S. Sinha (ed.), Ordering in Two Dimensions, North-Holland, Amsterdam 1980, 369–371.

[6] M. V. Men'shikov, S. A. Molchanov, and A. F. Sidorenko, *Percolation theory and some applications*, Itogi Nauki i Tekhniki (Series of Probability Theory, Mathematical Statistics, Theoretical Cybernetics) 24 (1986), 53–110.

[7] E. R. Scheinerman and J. C. Wierman, *Infinite AB percolation clusters exist*, J. Phys. A 20 (1987), 1305–1307.

[8] F. Sevšek, J.-M. Debierre, and L. Turban, *Antipercolation on Bethe and triangular lattices*, ibid. 16 (1983), 801–810.

[9] M. F. Sykes and J. W. Essam, *Exact critical percolation probabilities for bond and site problems in two dimensions*, J. Math. Phys. 5 (1964), 1117–1127.

[10] B. Tóth, *A lower bound for the critical probability of the square lattice site percolation*, Z. Wahrsch. Verw. Gebiete 69 (1985), 19–22.

[11] L. Turban, *Antipercolation and site-bond Potts model in the $q = 1$ limit*, J. Phys. A 16 (1983), L15–L17.

[12] J. C. Wierman and M. J. Appel, *Infinite AB percolation clusters exist on the triangular lattice*, ibid. 20 (1987), 2533–2537.

[13] M. K. Wilkinson, *Bi-partite percolation and gelation*, ibid. 20 (1987), 3011–3018.

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