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COBORDISM OF IMMERSIONS AND SINGULAR MAPS, LOOP SPACES AND MULTIPLE POINTS*

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§ 1. Some motivating questions

1. In 1974 Banchoff proved that, given a closed surface F^2, the only restriction for the possible values of the number t of triple points of an immersion $F^2 \to R^3$ is that $t \equiv \chi(F^2)$ (modulo 2), where $\chi(F^2)$ is the Euler characteristic of F^2. What is the relation between the number of triple points and singular points if we consider arbitrary (generic) maps $F^2 \to R^3$?

What is the right generalization of this question to higher dimensions?

2. Given a closed manifold M^n and a bordism class $x \in \Omega_i(M^n)$ of M^n, how can we decide whether x is realizable by an immersion or not? If not, then what is the simplest singularity which we are forced to admit to realize x?

3. Which n-dimensional manifolds can be immersed into R^{n+k} up to cobordisms?

What follows can be considered as developing tools to handle these sorts of problems. But the main motivation is the hope that generalizing the Pontryagin–Thom construction to the cobordism of *singular maps* one can obtain a model of the loop spaces of Thom spaces. Let me now explain how we do arrive at this model.

DEFINITION 0. We say that a map $f\colon M^n \to R^{n+k}$ of an n-manifold into R^{n+k} is a C_q-*map* $(1 \leqslant q \leqslant \infty)$ if the composition of f and the standard inclusion $R^{n+k} \hookrightarrow R^{n+k+q}$ becomes an embedding after an arbitrarily small alteration.

The usual Pontryagin–Thom construction gives an isomorphism $\mathrm{Emb}(n, k) \approx \pi_{n+k}(MO(k))$, where $MO(k)$ is the Thom space and $\mathrm{Emb}(n, k)$ is the cobordism group of the embeddings of n-manifolds into R^{n+k}.

On the other hand, the cobordism group of C_q-maps of n-manifolds into R^{n+k} is isomorphic to $\pi_{n+k}(\Omega^q MO(k+q))$.

* This survey paper is in final form and will not be submitted for publication elsewhere.

Now, our hope lies in the following. Consider a class $\mathscr{P}$ of maps of manifolds which contains the class of embeddings and is contained in the class of all C_q-maps. One can define

a) the cobordism group of maps $M^n \to R^{n+k}$ of class $\mathscr{P}$, denoting it by $\mathscr{P}(n, k)$, and

b) a "classifying" space $P(k)$ such that $\mathscr{P}(n, k) \approx \pi_{n+k}(P(k))$.

Enlarging smartly class $\mathscr{P}$ (for example allowing more numerous and more complicated singularities) we can describe how the corresponding space $P(k)$ changes. The resulting spaces $P(k)$ will give better and better approximations to $\Omega^q MO(k+q)$.

§ 2. Immersions with restricted selfintersections

Usually two main types of maps are considered in differential topology: embeddings and immersions. Embeddings have no selfintersections at all while an immersion may have selfintersections of arbitrary multiplicity. Here we consider a notion lying in a sense between the two types. Namely, we fix a natural number l and allow selfintersections of multiplicity l but forbid selfintersections of higher multiplicity. Immersions of this type will be called l-immersions. This notion is due to F. Uchida [U1] and to M. Gromov (unpublished). The cobordism groups of l-immersions can be defined in two different ways.

Definition 1 (see [Sz1]). The objects are l-immersions of arbitrary n-manifolds into the sphere S^{n+k}. The cobordisms are l-immersions of $(n+1)$-manifolds (with boundary) into the cylinder $S^{n+k} \times I$. Notation: $\mathrm{Imm}_l(n, k)$. For $l = 1$ and $l = \infty$ we also use the notation: $\mathrm{Emb}(n, k)$ and $\mathrm{Imm}(n, k)$, respectively. If the underlying manifolds are all oriented then we obtain the definition of another group, which we denote by $\mathrm{Imm}_l^{SO}(n, k)$

Definition 2 (see [U1] and [Sz6]). The objects are l-immersions of arbitrary n-manifolds into arbitrary $(n+k)$-manifolds. The cobordisms are l-immersions of $(n+1)$-manifolds into $(n+k+1)$-manifolds (both with boundaries). Notation: $C(n, k; l)$. The oriented version of this group is denoted by $C^o(n, k; l)$.

For $l = 1$ and $l = \infty$, i.e. for embeddings and immersions these groups have been interpreted in terms of algebraic topology and investigated by several authors, as is shown below.

	$l = 1$ (embeddings)	$1 < l < \infty$	$l = \infty$ (immersions)
$\mathrm{Imm}_l(n, k)$	$\pi_{n+k}(MO(k)) \approx \mathrm{Emb}(n, k)$ Pontryagin, Thom	?	$\pi^S_{n+k}(MO(k)) \approx \mathrm{Imm}(n, k)$ Wells
$C(n, k; l)$	$\mathfrak{N}_{n+k}(MO(k)) \approx C(n, k; 1)$ Wall	?	$\mathfrak{N}_{n+k}(\Omega^\infty S^\infty MO(k)) \approx C(n, k; \infty)$ Schweitzer

The groups $C(n, k; l)$ (see Definition 2) have been introduced and investigated by F. Uchida (mainly for $l = 2$). He constructed some interesting exact sequences joining these groups with some other groups, which are more or less computable. However, these exact sequences do not allow one to compute these groups completely. The construction in the next section makes it possible to interpret groups $C(n, k; l)$ in algebraic topological terms and compute them completely.

In 1971, when I was a student of M. Gromov, he posed to me the problem: How to compute the first type cobordism groups of l-immersions, i.e. the groups $\mathrm{Imm}_l(n, k)$? As a matter of fact he himself suggested the main idea of the solution. The idea was that one should construct a "classifying space" (an analogue of the Thom space) attaching to each other the total spaces of the universal bundles of normal bundles of simple, double, triple etc. points.

§ 3. The classifying space for 2-immersions

Consider an immersion $f: M^n \to R^{n+k}$ which has no triple points and such that the selfintersection at double points is transversal. Then the double set $\{y \in R^{n+k} | \; x_1, x_2 \in M^n; \; x_1 \neq x_2 \text{ and } f(x_1) = f(x_2) = y\}$ forms an $(n-k)$-dimensional submanifold in R^{n+k}, which we denote by V^{n-k}. The $2k$-dimensional normal bundle of V^{n-k} in R^{n+k} admits the group $G = O(k) \wr Z_2$ (the wreath product of the orthogonal group $O(k)$ by the group Z_2) consisting of all matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ where A and B are arbitrary orthogonal $k \times k$-matrices.

(*Proof.* In every normal fibre R_x^{2k} $(x \in V^{n-k})$ there is a "cross" $C = R^k \times 0 \cup 0 \times R^k$, which is the union of the tangent vectors of the two branches of $f(M)$ at x. The structure group of the normal bundle of V leaves invariant these "crosses". $O(k) \wr Z_2$ is the maximal subgroup of $O(2k)$ which acts on R^{2k} leaving the cross $C = R^k \times 0 \cup 0 \times R^k \subset R^{2k}$ invariant.)

Let us denote by $\eta: EG \xrightarrow{R^{2k}} BG$ the universal vector bundle with structure group $G = O(k) \wr Z_2$. We can choose a cross $C_b \in R_b^{2k}$ in the fibre over $b \in BG$ continuously with respect to b. Let SG and DG be the unit sphere and unit disc bundle, respectively, associated with η. If $\hat{C}$ denotes the intersection $SG \cap \{\bigcup C_b | \; b \in BG\}$, then $\hat{C}$ is an (infinite-dimensional) submanifold in SG of codimension k. By the Pontryagin–Thom construction this submanifold determines a map $\varrho: SG \to MO(k)$. Now, the classifying space for 2-immersions is $DG \cup_{\varrho} MO(k)$. This symbol stands for the disjoint union $DG \cup MO(k)$ in which we identify points $x \in SG \subset DG$ with $\varrho(x) \in MO(k)$. We denote this space by $\Gamma_2 MO(k)$.

THEOREM. $\mathrm{Imm}_2(n, k) \approx \pi_{n+k}(\Gamma_2 MO(k))$.

The proof is straightforward (see [Sz1], [Sz4]).

We leave to the reader to guess how to continue this construction, i.e. how to construct a space $\Gamma_l MO(k)$ for any natural l such that $\mathrm{Imm}_l(n, k) \approx \pi_{n+k}(\Gamma MO(k))$. (This construction can be found in [Sz6]). The same construction can be repeated for any closed subgroup H of the orthogonal group. For example, if $H = SO(k)$, we obtain a space $\Gamma_l MSO(k)$ such that $\mathrm{Imm}_l^{SO}(n, k) \approx \pi_{n+k}(\Gamma_l MSO(k))$. Analogously, a classifying space can be defined for the cobordism group of C_1 l-immersions. It will be denoted by $\bar{\Gamma}_l MO(k)$ (when the underlying manifolds are not necessarily orientable) or by $\bar{\Gamma}_l MSO(k)$ (when the underlying manifolds are oriented).

§ 4. First applications: homotopy groups of spheres

Let us consider the case when H is the trivial group. It is well-known that the first successes in computing the homotopy groups of spheres were achieved by Pontryagin with use of *geometrical* methods (cobordism groups of framed embeddings). Since then the methods of computation became purely algebraic. Now the question is whether we can return to geometrical methods if we consider the cobordism groups of framed 2-immersions (or l-immersions) and not only embeddings. My answer is yes and no. No, because I am not able to prove anything new by this method (and probably algebra always remains more powerful in algebraic topology than geometry). Yes, because there are some (old) theorems, which can be proved in this way.

For example:

1) the exact sequence of Toda;
2) the EHP sequence;
3) the theorem of Freudenthal;
4) the double suspension E^2: $\pi_{2k}(S^k) \to \pi_{2k+2}(S^{k+2})$ is an epimorphism if k is even;
5) $\pi_{n+k+1}(S^{k+1}) \approx \pi_{n+k}(S^{2k}) \oplus \pi_{n+k}(S^k)$ if $k = 1, 3, 7$ and $n < 2k$;
6) if we add to the tools the generalized Hurewicz theorem (which is proved by the algebraic method of spectral sequences) then we can prove geometrically the following theorem of Serre: E^2: $\pi_{n+k}(S^k) \to \pi_{n+k+2}(S^{k+2})$ induces isomorphisms of p-components for k odd and $p > n/k$ (see [Sz3]). As an example *I* sketch the proof of 4).[1]

Denote by $\mathrm{Imm}_2^{fr}(n, k)$ the cobordism group of 2-immersions of n-manifolds into R^{n+k} with trivial (and trivialized) normal bundle. Further denote by $\overline{\mathrm{Imm}}_2^{fr}(n, k)$ the cobordism group of 2-immersions of n-manifolds into R^{n+k} with trivialized normal bundle and with the extra condition C_1 (see Definition 0). From Hirsch's theory of immersions it follows easily that if $n < 2k-1$ then

[1] In the proof the integer k will be replaced by $k+1$.

a) $\mathrm{Imm}_2^{fr}(n, k) \approx \pi^S(n)$ (= the nth stable homotopy group of spheres);

b) $\overline{\mathrm{Imm}_2^{fr}}(n, k) \approx \pi_{n+k+1}(S^{k+1})$.

It is easy to see that an immersion satisfies condition C_1 iff no path lying in the double set $V^{n-k} \subset R^{n+k}$ can interchange the two branches of $f(M)$.

(Moving a point P along a closed path in V we have a cross $R^k \times 0 \cup 0 \times R^k$ in the normal bundle at every position of P. It may happen that at the return of P to its starting point the two linear subspaces ($R^k \times 0$ and $0 \times R^k$) of the fibre interchange their positions. This does not happen for any closed curve in V if and only if the immersion satisfies condition C_1).

The infinite suspension homeomorphism E^∞: $\pi_{n+k+1}(S^{k+1}) \to \pi^S(n)$ corresponds to the forgetting map $\overline{\mathrm{Imm}_2^{fr}}(n, k) \to \mathrm{Imm}_2^{fr}(n, k)$. When $n = k + 1$ and k is odd then this homomorphism is obviously an epimorphism. Indeed, in this case the double set is a 1-dimensional manifold and thus it is orientable. We have two framing sets of vectors along every double circle, both consisting of k vectors. Should they interchange their roles when a point P is moving along this circle, its normal bundle would not be orientable, which is impossible.

§ 5. Classifying spaces of l-immersions and models of loop spaces of suspensions

It is interesting that if we continue the construction of the classifying spaces of l-immersions for $l = 3, 4$, etc. to the infinity, we obtain nothing else but the Barratt–Eccles model of the space $\Omega^\infty S^\infty X$ (see [B–E]) for $X = MO(k)$ (see [Sz6]). The fact that we have to obtain a space homotopically equivalent to $\Omega^\infty S^\infty MO(k)$ follows from the Hirsch theory and was noticed by R. Wells ([We]).

Remark 1. If we consider only C_1-maps then, having constructed the classifying space of C_1 l-immersions for $l = 1, 2, \ldots, \infty$, we find that

1) from one hand, by the Hirsch theory, we must obtain the space $\Omega SMO(k)$ and ,

2) on the other hand, for $l = \infty$ this space is nothing else but the James product of the space $MO(k)$.

Hence, for the special case of Thom spaces we obtain a new proof of the fact that the James product gives a model for ΩSX (see [Sz1]).

Remark 2. Analogously, considering only C_q maps we obtain P. May's model for $\Omega^q S^q X$ when X is the Thom space $MO(k)$.

The constructed spaces make it possible to compute completely Uchida's groups $C(n, k; l)$ and obtain some information on the first type cobordism groups of l-immersions (on $\mathrm{Imm}_l(n, k)$).

§ 6. Groups $C(n, k; l)$

The groups $C(n, k; l)$ are direct sums of finitely many copies of Z_2. The number of terms to sum up can be expressed explicitly by using the number-theoretical properties of n, k and l (see [Sz2]).

The computation relies on the following facts:

1) $C(n, k; l) \approx \mathfrak{N}_{n+k}(\Gamma_l MO(k))$, where $\mathfrak{N}_i(X)$ denotes the ith bordism group of the space X (in the sense of Conner and Floyd);

2) $\mathfrak{N}_*(X) \approx \mathfrak{N}_* \otimes H_*(X; Z_2)$ where $\mathfrak{N}_*$ is the cobordism ring of unoriented manifolds;

3) $H_*(\Gamma_l MO(k); Z_2) = \bigoplus_{i=1}^{l} H_*(\Gamma_i MO(k)/\Gamma_{i-1} MO(k); Z_2)$ (see [B–E]);

4) $\Gamma_i MO(k)/\Gamma_{i-1} MO(k) = \underbrace{(MO(k) \wedge MO(k) \wedge \ldots \wedge MO(k))}_{i\text{-factors}} \times_{S(i)} WS(i)$,

where $WS(i)$ is a contractible space with free $S(i)$ action; $S(i)$ is the ith symmetric group. (This follows from the construction of spaces $\Gamma_i MO(k)$.)

5) The homology groups of the space $(X \wedge \ldots \wedge X) \times_{S(i)} WS(i)$ have been computed by Vogel.

The ranks of the groups $C^o(n, k; l)$ can be computed quite similarly (see [Sz6]).

§ 7. Groups $\mathrm{Imm}_l(n, k)$

The results concerning these groups are less satisfactory than those on the groups $C(n, k; l)$ (see [Sz1]). (These groups are isomorphic to the homotopy groups of spaces $\Gamma_l MO(k)$ and the computation of homotopy groups is always more difficult than that of the homologies.)

R. Wells has obtained some interesting results on the p-components of these groups for $l = \infty$. Namely, he proved that $\mathrm{Imm}(n, k)$ has no p-components if $n < k + 2p^2 - 2p - 1$. Combining this theorem with a theorem of Barratt and Eccles which states that the inclusion $\Gamma_l MO(k) \subset \Gamma_{l+1} MO(k)$ induces a monomorphism of the mod p homology groups, and using the generalized Hurewicz theorem we obtain that $\mathrm{Imm}_l(n, k) \otimes Z_p = 0$ if $n < k + 2p^2 - 2p - 1$ for any l.

If k is odd then the groups $\mathrm{Imm}_l(n, k)$ have no p-components for $p > l$.

For k even, $l \geqslant 2$, $n < 3k - 2$, these groups may have nontrivial torsion-free part, namely rank $\mathrm{Imm}_l(n, k) = \pi_k(n - k/4)$, where

$$\pi_k(x) = \begin{cases} 0 \text{ if } x \text{ is not an integer} \\ \text{the number of partitions of } x \text{ into the sum of numbers } \leqslant k. \end{cases}$$

If we consider the oriented version of Definition 1 then we obtain groups $\mathrm{Imm}_l^{SO}(n, k)$. Hardly anything is known about these groups.

The group $\mathrm{Imm}_2^{SO}(2, 1)$ (i.e. the cobordism group of immersions without triple points of oriented surfaces in R^3) is isomorphic to the direct sum of infinitely many copies of the group Z_2 ([Sz1]).

§ 8. Classyfying space for Σ^1 singular maps

When investigating the cobordism groups of immersions we arrive at a natural and interesting question, which cobordism classes of n-manifolds contain a manifold immersible into R^{n+k}. Thus the problem is to compute the natural homomorphism $\mathrm{Imm}^{SO}(n, k) \to \Omega_n$ which sends a class of immersions into the cobordism class of the underlying manifold (i.e. the domain of an immersion from the given class). This problem has been considered and completely solved modulo finite groups by Burlet [Bu]. His method was algebraic. I would like to sketch a more geometrical method, which allows one to compute the odd components of this map as well, at least in the metastable case (i.e. when $n < 2k-1$).

One can think of Ω_n as the cobordism group of arbitrary smooth maps of n-manifolds into R^{n+k}. On the other hand, if $n < 2k-1$ then an arbitrary "generic" map has only Σ^1 type singularities. So the idea is to construct the analogue of the Thom space for the cobordism group of Σ^1 singular maps and compare this space with the classifying space constructed for immersions. This comparison turns out to be fairly easy in this case, since the classifying space for the Σ^1 singular cobordisms can be obtained from the classifying space of immersions by attaching to it a well-describable space.

More precisely, we define the cobordism group of so-called S-maps (see [H], [Sz4]).

Definition. A map $f\colon M^n \to N^{n+k}$ is called an *S-map* if

1) rank $df(x) \geqslant n-1$ for $x \in M^n$;

2) singular points are not double;

3) f has no triple points;

4) If $(x_1, \ldots, x_n)$ are local coordinates at a singular point and the x_1-coordinate line is the direction of $\mathrm{Ker}\, df$, then the $(2n-1)$ vectors

$$\frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 f}{\partial x_1 \partial x_n}$$

are linearly independent.

Remark. S-maps $M^n \to N^{n+k}$ are dense in $C^\infty(M^n, N^{n+k})$ if $n < 2k-1$. Now, the cobordism group of S-maps of oriented n-manifolds into R^{n+k} can be defined by replacing in the definition of $\mathrm{Imm}_l^{SO}(n, k)$ the word "l-immersion" by the word "S-map". The group obtained thus will be denoted by $S(n, k)$. We are looking for a space $X(k)$ such that $S(n, k) \approx \pi_{n+k}(X(k))$. The space $X(k)$ can be constructed from two blocks. The first one is the classifying space for oriented 2-immersions $\Gamma_2 MSO(k)$. The second block is the unit disc bundle of the universal $(2k+1)$-dimensional vector bundle ξ with structure group consisting of all matrices

$$\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & A \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 0 & A \\ \hline 0 & A & 0 \\ \hline \end{array}, \quad \text{where } A \in O(k).$$

Denote this disc bundle by D and the sphere bundle associated to it by ∂D. An attaching map ϱ: $\partial D \to \Gamma_2 MSO(k)$ can be defined. (For a precise definition of ϱ see [Sz4] or [Sz5].)

Now let $X(k) = D \cup_{\varrho} \Gamma_2 MSO(k)$, meaning that in the disjoint union $D \cup \Gamma_2 MSO(k)$ we identify every point $x \in \partial D \subset D$ with its image. The proof of the isomorphism $S(n, k) \approx \pi_{n+k}(X(k))$ can be found in [Sz4] and [Sz5]. The main step in the proof is a lemma which says that the image of an S-map near the image of the singular *submanifold* has in some sense a canonical form. This is a globalization of the local results of Whitney and Haefliger on the canonical form of Σ^1 singular map near a singular *point*. This lemma is due to Haefliger.

Now, from the exact homotopy sequence of the pair $(X(k), \Gamma_2 MSO(k))$ we obtain easily the answer to the question: What are, respectively, the image and the kernel of the natural map φ: $\mathrm{Imm}^{SO}(n, k) \to \Omega_n$?

THEOREM. *Suppose $n < 2k-1$. Then* Coker φ *is a finite 2-group.*
Ker φ *is also a finite 2-group if k is odd, otherwise it is isomorphic to the group Ω_{n-k} modulo finite 2-groups.*

Proof. The pair $(X(k), \Gamma_2 MSO(k))$ is $2k$-connected. Thus by the homotopy excision theorem, in the exact sequence

$$(\ast) \quad \ldots \to \pi_{n+k}(\Gamma_2 MSO(k)) \to \pi_{n+k}(X(k)) \to \pi_{n+k}(X(k), \Gamma_2 MSO(k)) \to \ldots$$

for $n < 2k$ the relative homotopy groups can be replaced by the homotopy groups of the factor-space $X(k)/\Gamma_2 MSO(k)$, which coincides with the Thom space $T\xi$ of the vector bundle ξ. A standard computation shows that for $n < 2k-1$ the groups $\pi_{n+k}(T\xi)$ are isomorphic modulo the class of finite 2-groups to Ω_{n-k-1} (if k is even) or 0 (if k is odd). Using the isomorphisms

$$\pi_{n+k}(X(k)) \approx \Omega_n \quad \text{and} \quad \pi_{n+k}(\Gamma_2 MSO(k)) \approx \mathrm{Imm}_2^{SO}(n, k) \approx \mathrm{Imm}^{SO}(n, k)$$

the exact sequence $(\ast)$ can be rewritten as follows,

$$\to \pi_{n+k+1}(T\xi) \to \mathrm{Imm}(n, k) \to \Omega_n \to \pi_{n+k}(T\xi) \to$$

which modulo finite 2-groups is the same as

$$\to \Omega_{n-k} \overset{\leftarrow - - -}{\rightarrow} \mathrm{Imm}^{SO}(n, k) \to \Omega_n \to \Omega_{n-k-1} \quad (\text{if } k \text{ is even})$$

or

$$0 \to \mathrm{Imm}^{SO}(n, k) \to \Omega_n \to 0 \quad (\text{if } k \text{ is odd}).$$

In the first case a splitting map $\mathrm{Imm}^{SO}(n, k) \to \Omega_{n-k}$ (dotted arrow) can be defined which assigns to a class of immersions the cobordism class of the manifold formed by the double points of an arbitrary immersion from the given class. Q.E.D.

As regards the third motivating question, we formulate the following

THEOREM (see [Sz4] Corollary II). *Let us denote by $\mathscr{C}$ an arbitrary class of Abelian groups containing the class of all finite 2 primary groups. Let M^n be a manifold such that $H^i(M^n, Z) \in \mathscr{C}$ if $i \equiv 1 \bmod 4$. Let us denote by $\alpha_j(M^n)$ the subgroup of the bordism group $\Omega_j(M^n)$ consisting of the elements which can be represented by immersions. Then for $j < \frac{2}{3}n$ the factor groups $\Omega_j(M^n)/\alpha_j(M^n)$ belong to the class $\mathscr{C}$.*

Proof. Consider the sequence of spaces $\Gamma_2(k) \subset X(k) \xrightarrow{\mathrm{pr}} X(k)/\Gamma_2(k)$. For any CW complex K it gives an exact sequence of homotopy sets

$$(*_*^*) \qquad [K, \Gamma_2(k)] \xrightarrow{\varkappa} [K, X(k)] \to [K, X(k)/\Gamma_2(k)].$$

If $\dim K \leqslant n$ then these homotopy sets are stable (and so they are groups) because all the spaces $\Gamma_2(k)$, $X(k)$, $X(k)/\Gamma_2(k)$ are $(k-1)$-connected and $n < 2k-1$. For $K = M^n$ the group $[M^n, X(k)]$ can be identified with $\Omega_{n-k}(M^n)$ the image of $\varkappa$ can be identified with $\alpha_{n-k}(M^n)$. Hence the factorgroup $\Omega_{n-k}(M^n)/\alpha_{n-k}(M^n)$ is isomorphic to a subgroup of $[M^n, X(k)/\Gamma_2(k)]$.

It is well-known that there exists a spectral sequence, its second member being $E_2^{-p,q} = H^{q-2p}(M, \pi_{q-p}(X(k)/\Gamma_2(k)))$, which converges to the graduated set of stable homotopy classes $\bigoplus_p \{M^n, X(k)/\Gamma_2(k)\}_p$ (see Mosher–Tangora's book, Ch. 14.).

Now, from $(*_*^*)$ and from the conditions it follows that in this spectral sequence all the groups $E_2^{-p,q}$ belong to $\mathscr{C}$. ∎

§ 9. Classifying spaces for higher singularities

The classifying spaces for the cobordism groups of maps with Σ^{11}, Σ^{111}, Σ^{1111},... type singularities can also be constructed. (Here we use the normal forms of these types of singularities given by Morin [Mo].) This way we get a model of the (first) loop space of the Thom space. The classifying space of Σ^2 singularities has been constructed by Mirnij (see [M]). What about the classifying spaces of maps with higher singularities? Probably they can be constructed also from some simple blocks, but I am not able to prove this. However, the existence of these classifying spaces can be deduced from E. Brown's representability theorem (we explain this later on). For some applications it is enough to know that these spaces exist.

§ 10. An application: Eccles' theorem

In 1979 at the Siegen conference Peter Eccles posed, and in most cases answered the question, for what values of n there exists an immersion of an n-manifold into R^{n+1} which has exactly one $(n+1)$-tuple point. In particular he proved that there is no such immersion if the manifold is orientable and n is even. Using the classifying spaces for singular maps one can prove the following generalization of this fact.

DEFINITION. Given a map $f\colon M^n \to R^{n+k}$, we say that the image $f(Q)$ of a singular point Q *has multiplicity greater than or equal to* k if in any neighbourhood of $f(Q)$ there is a point which has k (or more) preimages.

THEOREM. *If the multiplicities of all singular points of a map* $f\colon M^n \to R^{n+1}$ *do not exceed* $n-1$, *where* M^n *is orientable and* n *is even, then the number of* $(n+1)$-*tuple points cannot be* 1.

We sketch the proof only for C_1-maps (see Definition 0). First we give a proof for the nonsingular case. Let me remind that the space $\Gamma_l MSO(k)$ has been already mentioned in this paper. Denote this space for $k=1$ by Γ_l. Recall that $\Gamma_1 = MSO(1) = S^1$, and Γ_l can be obtained by attaching an l-dimensional cube to Γ_{l-1} using an attaching map from the boundary of the l-dimensional cube into Γ_{l-1} (see [Sz1]). The points of Γ_{l-1} can be considered as finite sequences $(x_1, \dots, x_j)$ of length $j \leqslant l-1$ each element of which lies in the unit interval $[0, 1]$. Two sequences are identified if, by omitting elements which are 0 or 1, they become the same. The attaching map $\varrho_l\colon \partial I^l \to \Gamma_{l-1}$ is the following. Any point $P \in \partial I^l$ can be represented as a sequence $\underline{x} = (x_1, \dots, x_l)$ where $\forall x_i \in [0, 1]$ and $\exists x_i = 0$ or 1. Omitting any element of x which is equal to 0 or 1 we obtain $\varrho_l(x)$. Recall also that there is an isomorphism $\pi_{n+1}(\Gamma_l) \approx \overline{\mathrm{Imm}}_l^{SO}(n, 1)$, where $\overline{\mathrm{Imm}}_l^{SO}(n, 1)$ denotes the cobordism group of C_1 l-immersions of oriented n manifolds into R^{n+1}.

Now suppose, on the contrary, that there exists a C_1-immersion $f\colon M^{2k} \to R^{2k+1}$ with a unique point $P \in R^{2k+1}$ such that the number of its preimages equals to $2k+1$ (i.e. $|f^{-1}(P)| = 2k+1$).

LEMMA. *If there exists such a map f then the natural "forgetting" map* $\overline{\mathrm{Imm}}_{2k}^{SO}(2k-1, 1) \to \overline{\mathrm{Imm}}_{2k+1}^{SO}(2k-1, 1)$ *is a monomorphism.*

Proof. Suppose there exists an immersion of a $(2k-1)$-dimensional manifold into R^{2k} which bounds an immersion $\varphi\colon N^{2k} \to R^{2k} \times I$. We have to show that there exists an immersion $\tilde{\varphi}$: $\tilde{N}^{2k} \to R^{2k} \times I$ with the same boundary ($\partial\varphi = \partial\tilde{\varphi}$) and having no $(2k+1)$-tuple selfintersection points.

Denote the composition $M^{2k} \xrightarrow{f} R^{2k+1} \hookrightarrow R^{2k+1} \cup \text{point} = S^{2k+1}$ also by f. Consider a small disc U centered around $P \in S^{2k+1}$, where P has $(2k+1)$ preimages. The $(2k+1)$-tuple selfintersection points of φ form a finite set. Let Q be such a point. Consider a small disc V centered around Q. Replace V by the disc $S^{2k+1} \setminus U$ in such a way that the set $\{\varphi(N^{2k}) \setminus V\} \cup \{f(M) \cap (S^{2k+1} \setminus U)\}$ forms the image of an immersion which has one less $(2k+1)$-tuple point. Having repeated this procedure for all other $(2k+1)$-tuple selfintersection points of φ we obtain an immersion $\tilde{\varphi}$ which has no $(2k+1)$-tuple points at all. The lemma is proved.

Thus it is enough to prove that the map $\overline{\mathrm{Imm}}_{2k}^{SO}(2k-1, 1) \to \overline{\mathrm{Imm}}_{2k+1}^{SO}(2k-1, 1)$ is not a monomorphism, i.e. that the map α: $\pi_{2k}(\bar{\Gamma}_{2k}) \to \pi_{2k}(\bar{\Gamma}_{2k+1})$ induced by the inclusion $\bar{\Gamma}_{2k} \subset \bar{\Gamma}_{2k+1}$ is not a monomorphism. Since $\bar{\Gamma}_{2k+1} = \bar{\Gamma}_{2k} \cup_{\varrho} D^{2k+1}$, the map α is a monomorphism iff the attaching map ϱ: $\partial D^{2k+1} \to \bar{\Gamma}_{2k}$ is null-homotopic. Denote by $\tilde{\Gamma}_l$ the universal covering of $\bar{\Gamma}_l$. It is easy to see that the factor-space $\tilde{\Gamma}_{2k}/\tilde{\Gamma}_{2k-1}$ is an infinite wedge product of spheres S^{2k}. The map ϱ can be lifted into a map $\tilde{\varrho}$: $S^{2k} \to \tilde{\Gamma}_{2k}$ and composing $\tilde{\varrho}$ with the projection $\tilde{\Gamma}_{2k} \to \tilde{\Gamma}_{2k}/\tilde{\Gamma}_{2k-1}$ we obtain a map $S^{2k} \to \bigvee_{i=1}^{\infty} S_{(i)}^{2k}$. The degree of such a map is an infinite sequence of integers. In our case some of these integers will be odd, so the map is not homotopic to zero. This completes the proof of the Theorem for C_1-immersions. The proof for singular maps only slightly differs from this.

In that case we have to consider those singular C_1-maps of oriented n-manifolds into R^{n+1} which satisfy the following conditions:

a) the multiplicities of their singular points do not exceed $2k-1$;

b) the multiplicity of their selfintersections does not exceed l.

The cobordism group of these maps will be denoted by $\Sigma_l^k(n, 1)$. These groups can also be represented homotopically, i.e. there exists a space $X_l = X_l(k)$ such that

$$\pi_{n+1}(X_l(k)) \approx \Sigma_l^k(n, 1).$$

The existence of these spaces X_l can be deduced from Brown's representability theorem. Define a functor from the category of simplicial complexes into the category of sets as follows. Consider in a simplicial space K a subset β such that for any open simplex $\Delta^S \subset K$ the intersection $\beta \cap \Delta^S$ is the image of a proper map of an $(s-1)$-dimensional manifold which satisfies conditions a) and b). A cobordism joining two subsets of this type is a subset of this type in $K \times I$. The set of cobordism classes will be denoted by $\Sigma_l(K)$. Notice that if K is the $(n+1)$-sphere, then $\Sigma_l(K)$ is the same as $\Sigma_l(n, 1)$. If f: $L \to K$ is a p.l. map and β is a subset in K as above, then

$f^{-1}(\beta)$ will define an element in $\Sigma_l(L)$. So we obtain a map f^*: $\Sigma_l(K) \to \Sigma_l(L)$. Hence $\Sigma_l(\ )$ is a functor indeed. It is easy to see that this functor satisfies the condition of Brown's theorem. (If the elements $\beta_i \in \Sigma_l(K_i)$ $(i = 1, 2)$ "restricted" to $K_1 \cap K_2$ give the same element then there exists a unique $\beta \in \Sigma_l(K_1 \cup K_2)$ such that $\beta|_{K_i} = \beta_i$.) So X_l does exist.

A map $\bar{\Gamma}_l \to X_l$, which may be supposed to be an embedding, corresponds to the natural forgetting map $\overline{\mathrm{Imm}_l^{SO}}(\cdot, 1) \to \Sigma_l(\cdot)$. Hence $\bar{\Gamma}_l \subset X_l$. We will be interested mainly in X_l for $l = 2k-1$, $2k$ and $2k+1$. It is not hard to see that X_l for $l \geqslant 2k-1$ can be obtained from the disjoint union $X_{2k-1} \cup \bar{\Gamma}_l$ by identifying the subspaces $\bar{\Gamma}_{2k-1} \subset X_{2k-1}$ and $\bar{\Gamma}_{2k-1} \subset \bar{\Gamma}_l$.

Now the proof of the Theorem for the singular case is exactly the same as for the case of immersions. We simply have to replace in the proof the group $\overline{\mathrm{Imm}_l^{SO}}(n, 1)$ by $\Sigma_l(n, 1)$ and spaces $\bar{\Gamma}_l$ by X_l.

At the final step of the proof we considered the map $S^{2k} \to \tilde{\Gamma}_l/\tilde{\Gamma}_{l-1}$ for $l = 2k$. ($\tilde{\Gamma}_l$ denoted the universal covering of $\bar{\Gamma}_l$.) The inclusion $\bar{\Gamma}_l \subset X_l$ induces an isomorphism of the fundamental groups and so the universal covering of X_l—denote it by $\tilde{X}_l$—contains the universal covering of $\bar{\Gamma}_l$ and the factor space $\tilde{X}_l/\tilde{X}_{l-1}$ will be same as $\tilde{\Gamma}_l/\tilde{\Gamma}_{l-1}$.

It turns out that we have to consider the same map $S^{2k} \to \tilde{X}_{2k}/\tilde{X}_{2k-1} = \tilde{\Gamma}_{2k}/\tilde{\Gamma}_{2k-1}$ as before. And as in the case of immersions, it is enough to show that this map is not homotopic to zero. But this has already been proved. ■

§ 11. Application: Points which are both double and singular

Definition. Given a generic map f: $M^n \to R^{n+k}$, we say that a point $P \in R^{n+k}$ is an *α-point* if $f^{-1}(P)$ consists of two points one of which is singular, while the other is nonsingular.

Remark. The α-points form an $(n-(2k+1))$-dimensional manifold. Hence, when $n = 2k+1$ (and M^n is compact), the number of α-points is finite.

Theorem. *The number of α-points cannot be equal to* 1.

Sketch of the proof. Consider the cobordism group of maps of $2k$-dimensional manifolds into R^{3k}, where the cobordism maps have no α-points. Denote this group by $A(k)$. Again, there exists a space $Y(k)$ such that $A(k) \approx \pi_{3k}(Y(k))$. Denote by $B(k)$ the cobordism group of arbitrary maps of $2k$-dimensional manifolds into R^{3k}. There exists a space $Z(k)$ such that $B(k) \approx \pi_{3k}(Z(k))$ and $Z(k)$ can be obtained from $Y(k)$ by attaching a $(3k+1)$-dimensional disc to it by the use of a map $\partial D^{3k+1} \to Y(k)$. The analogue of the Lemma in § 10 is true also in this case; so, if the Theorem is false, then the natural map $A(k) \to B(k)$ is a monomorphism. Hence the map $\pi_{3k}(Y(k)) \to \pi_{3k}(Z(k))$ induced by the inclusion $Y(k) \subset Z(k)$ is a monomorphism. This means that the attaching map $\partial D^{3k+1} \to Y(k)$ is null-homotopic and hence $Z(k)$ is (homotopically equivalent to) the wedge product $Y(k) \vee S^{3k+1}$. Then there is a retraction r: $Z(k) \to Y(k)$, $r|_{Y(k)} =$ identity. Now we are going to lead the last statement to a contradiction. Consider two maps f, g: $M^{2k} \to R^{3k}$ of a $2k$-dimensional manifold into R^{3k}, such that f has an even number of triple points and g has an odd number of triple points. These maps are cobordant in the class of all maps. Let H: $S^{3k} \times I \to Z(k)$ be the homotopy corresponding to the cobordism h joining f and g. It is not hard to see that the number of α-points of h modulo 2 equals the difference of the numbers of triple points of f and g. (Proof: the triple lines of h may end at the α-points of h or at the triple points of f and g.) Thus for any cobordism joining f and g, the number of its α-points is odd. But the number of α-points of a cobordism can be expressed in terms of the homotopy H corresponding to this cobordism as follows. Let φ_H be the (uniquely defined) map for which the following diagram commutes:

The left vertical arrow is the natural factorizing map, which maps the two bases $S^{3k} \times 0$ and $S^{3k} \times 1$ into one point.

$$\begin{array}{ccc} S^{3k} \times I & \xrightarrow{H} & Z(k) \\ \downarrow & & \downarrow \\ S^{3k+1} & \xrightarrow{\varphi_H} & (Z(k)/Y(k)) = S^{3k+1} \end{array}$$

The number of α-points of the cobordism map h corresponding to H equals the degree of the map φ_H.

But if $Y(k)$ is a retract of $Z(k)$ then the map H can be changed (keeping fixed on the bases $S^{3k} \times 0$ and $S^{3k} \times 1$) so that φ_H have zero degree. (It suffices to replace H by $r \circ H$, where r is the retraction $Z(k) \to Y(k)$.)

The contradiction thus obtained proves the Theorem.

§ 12. The analogue of Banchoff's theorem

Finally, we answer the very first motivating question. There are no restrictions on the number of triple points and singular points (if the latter is positive) of a generic map of a closed surface into R^3 except that the number of singular points must be even. (Proof: multiple points form a graph whose vertices of odd degree are at singular points.) So this is not the right question. To formulate the right question we need a

DEFINITION. Let f: $M^{2k} \to R^{3k}$ be a generic map. Such a map has no singular points except for Σ^1-type points and these points form a $(k-1)$-dimensional submanifold V^{k-1} in M^{2k}. The image of this submanifold under f is the boundary of the image of the submanifold Δ formed by double points. Consider the outward normal vector field v of $f(V)$ in $f(\Delta)$. If the vectorfield v is small enough then its endpoints form a submanifold $\tilde{V}$ in R^{3k}. Let $l(f)$ be the linking number of $\tilde{V}$ and $f(M)$ in R^{3k}. It is called the *linking number of map f*. (If M is oriented and k is even then $l(f)$ is an integer, otherwise it is an element of the group Z_2.)

Now, the right analogue of Banchoff's theorem for singular maps is the following.

THEOREM ([Sz7]). *For any generic map f: $F^2 \to R^3$ of a closed surface into R^3 we have $\chi(F^2)+t(f)+l(f) \equiv 0 \bmod 2$ (where $t(f)$ is the number of triple points) and this is the only restriction.*

Remark. F. Ronga proved a formula for the homology class of double and triple points of immersions. When the immersion is a map from an n-manifold into R^{n+k} then these classes are $\bar{W}_k$ and $\bar{W}_k^2$, respectively. He also showed that the formula for the homology class of double points holds for singular maps as well. And what about the formula for triple points? It does not hold in general. The "error", i.e. the difference between the two sides for a singular map f: $M^{2k} \to R^{3k}$ equals $l(f)$ ([Sz8]).

Added in proof. 1. F. Ronga showed me that § 11. can be replaced by the following short remark: "The set of α-points forms a 0-homological cycle (the boundary of triple points)".

2. In the theorem on the page 251 the homological conditions on M^n can be omitted. This will be shown in a subsequent paper.

3. The results discussed in this paper have been or will be published with full proofs elsewhere.

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