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ON HOMOTOPICALLY REGULAR MAPPINGS OF MANIFOLDS*

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§ 1. On the raising of dimension

Does there exist a homotopically n-regular mapping of one manifold onto another? Since -1-regular mappings one naturally identifies with open (see Definition 3.2) and infinitely regular[1] mappings of compact manifolds coinciding with Serre (as well as Hurewicz) fibrations (see [McA–Tu]), there is a great deal of information on this question in both the extreme situations. However, the problem is far from being exhaustively investigated if $0 \leqslant n < \infty$ (the only exception are 0-regular mappings of 2-manifolds, see [Pu]). In particular, no examples of "wild" n-regular mappings of manifolds were known at the moment of the writing of this paper for $n > -1$. The present problem is to create the machinery for constructing "wild" n-regular mappings of manifolds. First of all it concerns the dimension raising mappings. The principal fact that it is possible to raise the dimension by n-regular mappings for each finite $n \geqslant 0$ has recently been discovered by A. Dranishnikov.

1.1. THEOREM (Dranishnikov [Dr]). *For every integer $n \geqslant -1$ there exists an n-regular mapping D_n: $\mathrm{UMC}^{n+2} \to I^\infty$ which maps the universal Menger compactum of dimension $n+2$ onto the Hilbert cube.*

On the other hand, Dranishnikov proved in [Dr] that an n-regular mapping cannot map an $n+1$-dimensional compactum onto a cube of dimension $> n+1$. The following theorem is also due to Dranishnikov (in [Dr] its weaker version, Theorem 2.5, is proved but the proof of Theorem 1.2 is essentially the same).

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* This paper is in final form and no version of it will be submitted for publication elsewhere.

[1] Throughout this article, "n-regular" means "homotopically n-regular". Mappings which are n-regular for each n are called *infinitely regular.*

The definition of n-regular mappings is given in § 3. All spaces considered are metric, all mappings are continuous.

1.2. THEOREM. *If a mapping of a compactum of dimension $\leqslant n+1$ onto a continuum is n-regular, then it is infinitely regular* $(n \geqslant 0)$.

The following theorem, essentially due to E. Michael, is the basic tool in the theory of homotopically regular mappings.

1.3. THEOREM. *If $f\colon X \to Y$ is an n-regular mapping of a complete metric space X onto a space Y of dimension* $\dim Y \leqslant n+1$, *then each of its cross-sections $g\colon F \to X$ defined over some closed $F \subset Y$ admits an extension to a cross-section defined over some neighbourhood of F.*

Mappings which satisfy the conclusion of above theorem are called in [Sh 1] *locally softly invertible.* Thus one can say shortly that an n-regular mapping of a complete metric space onto a space of dimension $n+1$ is locally softly invertible.

Converselly one has

1.4. THEOREM (see [Sh 1]). *Every locally softly invertible mapping of a compactum onto an n-manifold is $n-1$-regular* $(n > 0)$.

Theorem 1.3 is valid for all $n \geqslant -1$. The case $n = -1$ follows from the main theorem of [Mi 1]. The case $n > -1$ follows from the main theorem of [Mi 2]. The latter theorem is the chief argument in support of our agreement "-1-regularity is openness".

Theorem 1.2 and Theorem 1.3 combined imply that it is impossible to map any $n+1$-dimensional compactum onto a finite-dimensional one of dimension greater than $n+1$ by an n-regular mapping.

However, the following question is open.

1.5. PROBLEM. *Does there exist an infinitely regular mapping which raises the dimension of some compactum?*

A theorem proved by P. Whyte [Whyt] claims that if $f\colon X \to Y$ is a homologically r-regular mapping of compacta, then all sufficiently small Vietoris cycles of dimension $\leqslant r+1$ from Y are covered by small cycles from X. This theorem immediately implies that a homologically ∞-regular (and consequently homotopically ∞-regular) mapping cannot raise the homological dimension. Therefore, invoking R. Edwards theorem (see [Wa 4]), one concludes that the positive solution of 1.5 implies the positive solution of the famous CE-problem.

CE-PROBLEM. *Does there exist a cell-like mapping which raises the dimension of some compactum?*

The replacement of the word "compactum" by "compact manifolds" poses a problem which is known to be equivalent to the CE-problem. On the contrary, Problem 1.5 concerning mappings of compacta certainly cannot easily be reduced to a corresponding problem for manifolds. In connection with this let us remark that one of E. Dyer's theorems ([Dy], Theorem 10) claims that homologically ∞-regular mappings with acyclic fibers of a closed manifold are homeomorphisms. Hence, infinitely soft [2] mappings cannot raise dimension of closed manifolds. At the same time it is not known whether ∞-soft mappings can raise dimension of compacta or not (Dranishnikov's problem, see [Dr]). The above-mentioned theorem of Dyer gives some grounds for making the following conjecture.

1.6. CONJECTURE. *Infinitely regular mappings cannot raise dimension of manifolds.*

§ 2. On n-fibrations

Another kind of "wildness" is represented by n-regular closed mappings which are not $n+1$-fibrations (an n-fibration is a surjective mapping having the covering homotopy property for n-dimensional polyhedra). Let us recall the theorems which clarify the relationships between these notions.

The following theorem is a consequence of Theorem 3.4 from Michael [Mi 3].

2.1. THEOREM. *Every closed n-regular mapping of complete metric spaces is an n-fibration.*

The coverse theorem is due to McAuley (see [McA–Tu]).

2.2. THEOREM. *If an n-fibration between metric spaces has the domain* LC^{n-1} *and the range* LC^n, *then it is $n-1$-regular.*

Thus the condition of n-regularity for closed mappings of manifolds, being stronger than n-fibration, is weaker than $n+1$-fibration. The following two theorems demonstrate that an n-regular mapping is closer to an $n+1$-fibration than to an n-fibration.

The first theorem is a consequence of Lemma 5.1 from E. Michael [Mi 3].

2.3. THEOREM. *A closed n-regular mapping of complete metric spaces is an approximate $n+1$-fibration (in the sense of Coram–Duvall* [30]).

The second theorem is due to Dranishnikov.

2.4. THEOREM (see [Dr], Theorem [3] 2.1). Let $f\colon X \to Y$ be a closed $n-1$-*regular mapping of complete metric spaces and let $y \in Y$ be an arbitrary point.*

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[2] n-soft mappings of complete metric spaces are characterized as $n-1$-regular mappings with $n-1$-connected fibres (see [Dr]). Hence an inifnitely soft mapping of a compactum is the same as an infinitely regular and cell-like one.

[3] This theorem is formulated only for n-soft mappings; however, its proof works equally well for locally n-soft, that is, $n-1$-regular mappings.

Then it is possible to define in a natural way an exact sequence of homotopical groups

$$\pi_n f^{-1} y \to \pi_n X \to \pi_n Y \to \pi_{n-1} f^{-1} y \to \ldots \to \pi_0 Y$$

Thus $n-1$-regular mappings generate the same exact sequence as n-fibrations.

Examples of open -1-regular mappings which are not 0-fibrations are contained in R. D. Anderson [A] and they are indeed "wild". Namely, he constructs in [A]

(1) a monotone open mapping of a 2-sphere onto itself which is not a homeomorphism;

(2) a monotone open mapping of the Sierpiński universal curve onto I^2.

The proof that these mappings are not 0-fibrations is based upon the following theorem.

2.5. THEOREM. *A 0-fibration of a plane continuum onto the square I^2 is necessarily a light open mapping.*

The above theorem allows us to formulate the famous Stoilov theorem [St] as follows:

2.6. THEOREM. *For every mapping f: $E^2 \to E^2$ (E^2 is the plane) the following two conditions are equivalent:*

(1) f is a 0-fibration,

(2) $f = g \circ h$, where h is an autohomeomorphism of E^2 and g is a complex analytic function.

The topological part of the proof presented in [Pu] is naturally divided into two parts. The first part is the proof that a light open mapping of the plane is a 0-fibration. It was generalized by E. Floyd.

2.7. THEOREM (see [Fl]). *Every light open mapping of metric compacta is a 0-fibration.*

The second part of Stoilov's proof demonstrates in fact that every 0-fibration f: $E^2 \to E^2$ has discrete fibres. One easily modifies this proof so as to obtain an analogous result for mappings of plane continua onto the square. Thus Theorem 2.5 can be proved on the basis of Stoilov's arguments.

The above mentioned examples of Anderson seem to have n-dimensional analogues.

2.8. CONJECTURE. *For every $n \geqslant 0$ there exists an n-regular mapping of the sphere S^{2n+4} onto itself which is not a homeomorphism.*

2.9. CONJECTURE. *For every $n \geqslant 0$ there exists an n-regular mapping which maps an $n+2$-dimensional compact subset of E^{2n+4} onto the cube I^{2n+4}.*

On the other hand, as follows from Theorem 3.1, every n-fibration between closed manifolds of the same dimension $2n+2$ is a homeomorphism. And I hope that the following is true (cf. Problem 5 from [Sh 1]).

2.10. CONJECTURE. *It is impossible to raise the dimension of a compact subset of* E^{2n+2} *by an* n*-fibration.*

Openness is a very strong condition for mappings of 1-manifolds. For example, all open mappings of $S^1 \to S^1$ are infinitely regular. Openness is a very weak condition for mappings of n-manifolds if $n > 2$. Thus J. Walsh has proved in [Wa 1] that every mapping of a triangulable n-manifold onto a simply connected ANR can be approximated by open mappings in $n > 2$. The crucial dimension is 2. Here one sees the phenomenon of "restricted misbehaviour". Thus, an open mapping can raise the dimension of a plane continuum, as shown by Anderson second example. However, it cannot map it onto a 3-dimensional compactum. On the contrary, as shown by Stoilov's theorem, the behaviour of 0-fibrations in dimension 2 is quite "tame". One expects an analoguous picture in higher dimensions. The crucial dimension for $n-1$-regular mappings of manifolds in $2n+2$.

2.11. QUESTION. *Does there exist an* $n-1$*-regular mapping of a compact subset of* E^{2n+1} *which raises the dimension?*

0-fibrations of n-manifolds for $n > 2$ can raise the dimension, because light open mappings can do so. J. Walsh has proved in [Wa 2] that every open mapping $f\colon M^m \to N^n$ of compact triangulable manifolds may be approximated by light open mappings if $n \geqslant m > 2$.

An open mapping $f\colon S^2 \to S^2$ admits a natural representation in the form $f = g \circ h$, where h is monotone and acyclic and g is light and open. Then $\operatorname{im}(h) \cong S^2$ (Moore's theorem) and h is a limit of a sequence of homeomorphism; therefore h is approximable by 0-fibrations.

2.12. QUESTION. *Is every open mapping between compact manifolds approximable by 0-fibrations?*

More generally, one asks:

2.13. QUESTION (approximation problem). *Is every* n*-regular mapping of compact manifolds approximable by* $n+1$*-fibrations?*

§ 3. The main theorem and its corollaries

The following theorem is my principal result:

3.1. THEOREM. *If* $f\colon M^m \to N^n$ *is an* r*-regular* $(r \geqslant 0)$ *mapping of manifolds, then* f *is infinitely regular at a point* $x \in \operatorname{Int} M$ *if one of the following conditions is satisfied*:

(1) $m \leqslant 2r+2$ *and* $m-n \leqslant r$,

(2) $m = 2r+3$, $n > r+2$ *and* f *is an approximate* $r+1$*-fibration*,

(3) $m = 2r+3$, $n = r+2$, $f(x) \in \operatorname{Int} N$ *and* f *is an approximate* $r+1$-*fibration*,

(4) $m = 2r+4$, $n > r+3$ *and* f *is an* $r+1$-*fibration*,

(5) $m = 2r+4$, $n = r+3$, $f(x) \in \operatorname{Int} N$ *and* f *is an* $r+1$-*fibration.*

Let us explain the condition of regularity at a point.

3.2. DEFINITION. A mapping $f\colon X \to Y$ is said to be *n-regular at a point* $x \ni X$ if for every open $U \ni x$ there exist open $V \ni x$ and open $W \in f(x)$ such that, for every mapping $g\colon B^k \to V \cap f^{-1}(y)$ where $y \in W$ and B^k is the unit ball in E^k and $0 \leqslant k \leqslant n+1$, there exists a mapping $G\colon B^k \to U \cap f^{-1}(y)$ such that $G|_{\partial B^k} = g$.

In this definition we allow k to take the value 0, assuming that B^0 is a one point space and $\partial B^0 = \emptyset$. This assumption implies $f(U) \supset W$ and therefore f is open at x for all $n \geqslant -1$.

Now an n-regular mapping can be defined as a mapping which is n-regular at every point of its domain. And this definitions in the case of $n = -1$ changes into a definition of an open mapping.

Theorem 3.1 implies most of the results of my paper [Sh 1], in particular the following three theorems:

3.3. COROLLARY. *It is impossible to map a manifold of dimension* $m \leqslant 2r+ +2(2r+3)$ *onto a manifold of a greater dimension by an r-regular* (*resp. closed r-regular*) *mapping.*

3.4. COROLLARY. *Every closed r-regular mapping of n-manifolds without boundary is a local homeomorphism if* $n \leqslant 2r+3$.

3.5. COROLLARY (theorem on spheres). *An r-regular mapping* $f\colon S^{2r+3} \to S^{r+2}$ *exists iff* $r = -1, 0, 2, 6$.

Consider now the question of the necessity of the assumptions of Theorem 3.1. The projection $p\colon S^m \to B^n$, being $m-n-1$-regular and being an $m-n$-fibration (see Theorem 3.11 below), shows the necessity of all dimension restrictions in this theorem. In particular, the projection p, being ∞-regular at each x with $p(x) \in \operatorname{Int} B^n$, is not $m-n$-regular at any others points. But it seems to me that the restrictions on dim N are inessential in cases 1–3 if one considers manifolds without boundary.

3.6. CONJECTURE. *Every closed r-regular mapping of manifolds without boundary is infinitely regular if its domain has dimension* $\leqslant 2r+3$.

For n-fibrations Theorem 3.1 implies

3.7. COROLLARY. *It is impossible to map a manifold of dimension* $m \leqslant 2r+4$ *onto a manifold of a greater dimension by an* $r+1$-*fibration.*

3.8. COROLLARY. *Every n-fibration between manifolds without boundary of the same dimension* $2n+2$ *is a local homeomorphism.*

Theorems 3.7 and 3.8 are analogous to 3.3 and 3.4. Let us consider which theorem is analogous to 3.5.

3.9. QUESTION. *For which n does there exist an n-fibration $p: E^{2n+2} \to E^{n+1}$ which is not a Serre fibration (E^n is a Euclidean space)?*

For $n = 0, 1, 3, 7$ the corresponding multiplications look like n-fibrations. It is not difficult to prove that all these multiplications are $n-1$-regular. I have no proof that they are n-fibrations if $n = 1, 3, 7$, but I hope that the following more general conjecture is true.

3.10. CONJECTURE. *Every closed simplicial n-regular mapping is an $n+1$-fibration.*

This conjecture explains the word "wild" applied above to characterize n-regular mappings which are not $n+1$-fibrations.

An important result which confirms this conejcture has been proved recently by Dranishnikov (unpublished).

3.11. THEOREM. *An orthogonal projection of the unit sphere S^n onto a k-dimensional unit ball B^k lying in some k-plane is an $n-k$-fibration.*

The proof of the above theorem involves considerations analogous to those of [Dr], where it is proved that the Dranishnikov mapping D_n: $\mathrm{UMC}^{n+2} \to I^{\infty}$ see Theorem 1.1 is an $n+1$-fibration. It is a promissing problem to investigate the topology of n-fibrations of E^{2n+2} onto E^k. The first non-trivial case is $n = 1$, $k = 1, 2$. First of all, I am interested in 1-fibrations of E^4 onto E^2. The following conjecture is weaker than 3.10.

3.12. CONJECTURE. *Every analytic function of two complex variables realizes a 1-fibration of E^4 onto E^2 if it has connected fibres.*

It might be asked if the converse of this conjecture is also true and every 1-fibration of E^4 onto E^2 is topologically equivalent to some analytic function. Unfortunately it is not so. I have a counterexample. Nevertheless, it does not seem improbable that the following question has a positive answer.

3.13. TRIANGULATION PROBLEM. *Is every 1-fibration f: $M^4 \to N^2$ of a triangulable manifold without boundary topologically equivalent to some simplicial mapping?*

In asking an analogous question in higher dimensions more care is needed. Indeed, a Serre fibration between manifolds may have the Bing's dogbon space as a fibre and therefore not be triangulable. However, F. Raymond [Ra] proves that all fibers of an arbitrary Serre fibration[4] between manifolds without boundary are generalized manifolds. Now, invoking the

[4] The notions of a Serre fibration and a Hurewicz fibration for mappings of finite-dimensional ANR's are known to be equivalent (see [U]).

famous Cannon–Edwards–Quinn theorem on the topological characterization of manifolds, we conclude that if $p\colon E \to B$ is a Serre fibration of closed manifolds, then the superposition $p \circ \mathrm{proj}\colon E \times S^2 \to E \to B$ has only closed manifolds as fibres. Hence $p \circ \mathrm{proj}$ is a bundle by virtue of the Chapman–Ferry Theorem [Cha–F]. Thus every Serre fibration of closed triangulable manifolds is stably triangulable.

3.14. QUESTION. *Is every n-fibration $f\colon M^{2n+2} \to N^k$ of a closed triangulable manifold stably triangulable?*

§ 4. Regular and UV^k-mappings

The theories of open mappings and of monotone mappings are closely connected. The theory of open mappings is the background of the theory of homotopically n-regular mappings and the theory of monotone mappings is in turn the background for the theory of UV^k-mappings. Open as well as monotone mappings cannot raise the dimension of 2-manifolds and can raise the dimension of 3-manifolds. They can both raise the dimension of 1-dimensional compacta but not of 0-dimensional ones. Thus open mappings and monotone mappings have the same crucial dimensions in the sense of [Sh 1]. Open mappings are -1-regular-monotone mappings are UV^0. In general I suppose an n-regular mapping to have the same crucial dimensions as UV^{n+1}. Just now A. V. Chernavskii is about to publish (see [Che]) his construction of a UV^k-mapping of I^{2k+3} onto I^{2k+4} (a result announced twenty years ago, see [Ke–Che]). Hence I expect that there exists a $k-1$-regular mapping which maps the same cubes.

Closed $k-1$-regular mappings as well as closed UV^k-mappings are approximate k-fibrations. And a natural class lying at the intersection of these two is the class of polyhedrally $k+1$-soft mappings (see [Sh 1], [Dr]). Polyhedrally n-soft mappings may be defined as $n-1$-fibrations with $n-1$-connected fibers. It may be asked whether polyhedrally n-soft mappings have the same crucial dimensions as approximate $n-1$-fibrations.

The same proof as that used in [Sh 1] for Corollary 3 of Theorem 4 gives a more general result.

4.1. THEOREM. *An approximate n-fibration cannot map a closed subset of a $2n+1$-manifold onto a polyhedron of dimension $> 2n+1$.*

I am sure that following is true:

4.2. CONJECTURE. *For every natural n there exists an n-fibration moreover, a polyhedrally $n+1$-soft mapping which maps the cube I^{2n+3} onto the Hilbert cube.*

As regards the crucial dimension I have a question and a theorem.

4.3. QUESTION. *Does there exist an approximate n-fibration which maps a subset of* I^{2n+2} *onto* I^{2n+3}?

4.4. THEOREM. *If there exists an n-fibration of a compactum X onto the cube* I^{2n+3} *then X is not embeddable in* E^{2n+2}.

Proof. Every n-fibration is invertible over every combinatorialy contractible polyhedron. We denote by T^n the n-fold product of the triod T. Since I^{2n+3} contains the product $T^{n+1}\times I$, the theorem may be deduced from the following lemma:

4.5. LEMMA. *The cube* I^{2n} *does not contain an uncountable family of disjoint closed sets homeomorphic to* T^n.

Proof of the lemma. Suppose on the contrary that $\{T_\alpha\colon \alpha\in A\}$ is an uncountable family of disjoint closed subsets of I^{2n} homeomorphic to T^n. According to [Sh 2] there exists a mapping $f\colon S^{2n-1}\to T^n$ which does not identify any pair of antipodes. Let us choose for each $\alpha\in A$ a mapping f_α of this type. Thus for every $\alpha\in A$ the number $h(f_\alpha)=\inf\limits_x\|f(x)-f(-x)\|$ is positive. The set A, being uncountable, provides the uncountability of the set $A_\varepsilon=\{\alpha\in A\colon h(f_\alpha)>\varepsilon\}$ for a certain $\varepsilon>0$. The space of mappings of S^{2n-1} to I^{2n} is separable, and hence it cannot contain a discrete uncountable subset. Therefore one finds such $\alpha,\beta\in A$ that $\sup\limits_x\|f_\alpha(x)-f_\beta(x)\|<\varepsilon$. If x, y are distinct points from E^{2n}, one defines a point $D(x,y)\in S^{2n-1}$ by the formula $D(x,y)=(x-y)/\|x-y\|$. Let us now consider the following two mappings of S^{2n-1} onto itself. The first is $D_\alpha(x)=D(f_\alpha(x),f_\alpha(-x))$, the second is $D_{\alpha,\beta}(x)=D(f_\alpha(x),f_\beta(-x))$. As is easy to see, the mapping D_α sends antipodes to antipodes and therefore has an odd degree. The choice of α, β ensures that the points $D_\alpha(x)$ and $D_{\alpha,\beta}(x)$ are never a pair of antipodes. Therefore the mappings D_α and $D_{\alpha,\beta}$ are homotopic. But, on the other hand, one sees that the mapping $D_{\alpha,\beta}$ is a composition of two mappings of which one maps S^{2n+1} into $T_\alpha\times T_\beta$, and the other is the restriction of D over $T_\alpha\times T_\beta$. The space $T_\alpha\times T_\beta$ being contractible, makes $D_{\alpha,\beta}$ homotopic to a constant mapping and therefore of degree 0. A contradiction.

For a mapping $f\colon X\to Y$ we denote by $M(f)$ its mapping cylinder. The natural projection of $M(f)$ onto the interval $I=[0,1]$ is denoted by $d(f)$.

4.6. THEOREM. *If* $f\colon X\to Y$ *is a mapping of an* LC^n *compactum, then f is* UV^n *if and only if* $d(f)$ *is n-regular.*

The part "if" is an immediate consequence of G. Whyburn's theorem proved in [Whyb]. The part "only if" is a consequence of the following more general theorem:

4.7. THEOREM (on factorization). *If the superposition* $f\circ g$ *of mappings of compacta is n-regular and g is* UV^n, *then f is n-regular.*

Indeed, the mapping $d(f)$ is by definition a left factor of the projection

$X \times I \to X$, and its right cofactor has, as is easy to see, the same UV-properties as f.

The proof of Theorem 4.7 is based on E. Michael [Mi 2]. The main technical tool is Theorem 4.1 from [Mi 2]. The following theorem has almost the same proof:

4.8. THEOREM. *If the superposition of mappings of compacts $f \circ g$ is an n-fibration and g is an approximate n-fibration, then f is an n-fibration.*

4.9. DEFINITION. A mapping of compacts $f\colon X \to Y$ will be called *n-resolvable* (with the fiber F) if there exist such an LC^n-compactum F and such a UV^n-mapping $g\colon Y \times F \to X$ that the superposition $f \circ g$ coincides with the natural projection pr: $Y \times F \to Y$.

Theorem 4.7 shows that n-resolvable mappings are n-regular. What about the converse?

An obvious property of resolvable mappings is that they have global cross-sections. However, one hardly expects an arbitrary n-regular mapping to have more than local cross-sections over at most $n+1$-dimensional subsets (see Theorem 1.3).

4.10. DEFINITION. A mapping $f\colon X \to Y$ is called *locally n-resolvable* if every point $y \in Y$ has a neighborhood U such that $f|_U$ is n-resolvable.

4.11. THE RESOLUTION PROBLEM. *Is every n-regular mapping of a compactum onto a compactum of dimension $\leqslant n+1$ locally n-resolvable?*

I can prove that the answer is "yes" if $\dim Y = 0$.

§ 5. The bundle problem

Find the conditions under which a mapping is a bundle – this general problem is one of the interesting and fruitful problems of in topology. The obvious necessary condition is to have a constant fiber.

5.1. DEFINITION. A mapping is said to have a *constant fiber* if all its point-inverses are homeomorphic to each other.

The first result relating to this problem that I know is

5.2. THEOREM (Dyer–Hamstrom [Dy–Ha]). *If $p\colon E \to B$ is a 0-regular mapping of finite-dimensional compacta with a constant fiber, which is a manifold of dimension $\leqslant 2$, then p is a bundle.*

In high dimensions one has

5.3. THEOREM (Chapman–Ferry [Cha–F]). *If $p\colon E \to B$ is an infinitely regular mapping of finite-dimensional compacta with a constant fiber which is a closed manifold of dimension $\geqslant 5$, then p is a bundle.*

More precisely, in [Cha–F] a slightly weaker result is formulated.

Instead of ∞-regularity the authors require f to be a Hurewicz fibration, which implies an additional condition on B, namely that it should be locally connected. However, the proof from [Cha–F] yields Theorem 5.3 if one pays attention to the following theorem, essentially due to S. Ungar [U].

5.4. THEOREM. *A mapping of finite-dimensional compacta is infinitely regular iff it is strongly regular and has* ANR *fibres.*

Theorem 5.2 supports the following

5.5. CONJECTURE. *If $f\colon X \to Y$ is an n-regular mapping between compacta with a constant fiber which is a closed manifold of dimension $\leqslant 2n+2$, then f is infinitely regular.*

Let us show that the conjecture fails if the dimension of the fibers is allowed to take the value $2n+3$. For every n there exists a UV^n-mapping $f\colon S^{2n+3} \to S^{2n+3}$ whose degree is 0. Such mappings were constructed by D. Wilson for $n = 0, 1$ [Wil] and by J. Walsh [Wa 3] for all n. As is easy to see, for the mapping f in question, the corresponding $d(f)$, being n-regular by virtue of 4.6, fails to be a Serre fibration, and hence is not ∞-regular. But there are no counterexemples to the Conjecture 5.5 in the form $d(g)$, because of the following theorem:

5.6. THEOREM (R. Lacher [La]). *Every UV^n-mapping of a closed manifold M^m onto itself is cell-like if $m \leqslant 2n+2$.*

An example constructed in [To-We] shows that a Hurewicz fibration with I^∞ as a fiber is not necessarily a bundle. To ensure that a fibration is a bundle one needs some General Position Conditions in the case of infinite-dimensional fibers [To–We].

But what is the situation if the fiber is finite-dimensional?

5.7. QUESTION. *Let $p\colon E \to B$ be a ∞-regular mapping of compacta. Suppose it has a closed n-manifold as a constant fiber. Is p a bundle?*

As mentioned above, this problem is known to have a positive solution if $n \neq 3, 4$ and $\dim B < \infty$.

I can only prove the following

5.8. THEOREM. *If $p\colon E \to B$ is a 0-regular mapping of compacta with a 1-dimensional polyhedron as a constant fiber, then p is a bundle.*

Thus Question 5.7 has a positive answer if $n = 1$. But a positive solution to this problem for all n seems to be too strong a result for us to hope for. Namely, it would imply a negative solution of the CE-problem and the following strong result in the direction of a positive solution of the Homeomorphism Group problem: the group $\mathrm{Auth}(M)$ of autohomeomorphisms of an arbitrary closed manifold M is an absolute neighbourhood extensor for compacta. Indeed, if $\mathrm{Auth}(M) \notin \mathrm{ANE}$ (compacta) then there exists a mapping $H\colon F \to \mathrm{Auth}(M)$ of a compact subset of I^∞

which admits no local extension. Let I_1 and I_2 be two copies of I^∞. Denote by B the space obtained by sewing together I_1 and I_2 along F. And denote by E the space obtained by sewing together $I_1 \times M$ and $I_2 \times M$ by a mapping $h\colon F \times M \to F \times M$ where $h(x, y) = (x, H(x)(y))$. Defined in a natural way the mapping of E onto B, being ∞-regular, fails, as is easy to see, to be a bundle. The construction described in [Ko–Mi–Wa] transforms a hypothetical dimension-raising cell-like mapping into a ∞-regular mapping with a sphere as a fiber which fails to be an ANR-fibration in the sense of [To–We] or a locally absolutely soft mapping in the sense of [Sh 1]. Problem 5.7 is naturally divided in two parts.

5.8. QUESTION. *Let $p\colon E \to B$ be a ∞-regular mapping of compacta with a closed n-manifold as a fiber. Is p a Hurewicz fibration?*

5.9. QUESTION. *Suppose that $p\colon E \to B$ is a Hurewicz fibration of* LC^0-*compacta with a closed n-manifold as a fiber. Is it a bundle?*

Question 5.9 seems to be independent of the CE-problem. And both questions seems to have good chances to have a positive solution in the case $n = 2$ since $\mathrm{Auth}(M^2)$ is known to be ANR.

References

[A] R. D. Anderson, *On monotone interior mappings in the plane*, Trans. Amer. Math. Soc. 73 (1952), 211–222.

[Cha F] T. A. Chapman and S. Ferry, *Approximating homotopy equivalences by homeomorphism*, Amer. J. Math. 101, 583–607.

[C Du] D. Coram and P. F. Duvall, *Approximate fibrations*, Rocky Mountain J. Math. 7 (1977), 257–288.

[Che] A. V. Chernavskii, *A generalization of L. V. Keldysh's construction of monotone maps of cube onto cube of higher dimension* (in Russian), Uspekhi Mat. Nauk 40 (4) (1985), 209–211.

[Dr] A. N. Dranishnikov, *Absolute extensors in dimension n and n-soft maps raising dimension* (in Russian), Uspekhi Mat. Nauk 39 (5) (1984), 55–95.

[Dy] E. Dyer, *Regular mappings and dimension*, Ann. of Math. 67 (1958), 119–149.

[Dy Ha] E. Dyer and M. E. Hamstrom, *Completely regular mappings*, Fund. Math. 45 (1958), 103–118.

[Fl] E. E. Floyd, *Some characterizations of interior maps*, Ann. of Math. 51 (1950), 571–575.

[Ke Che] L. V. Keldysh and A. V. Chernavskii, *Topological embeddings in euclidean space*, Proc. of the second Prague Top. Symp. 1966.

[Ko Mi Wa] G. Kozlowski, J. van Mill and J. Walsh, AR-*maps obtained from cell-like*, Proc. Amer. Math. Soc. 82 (1981), 299–302.

[La] R. C. Lacher, *Cellularity criteria for maps*, Michigan Math. J. 17 (1970), 385–396.

[McA–Tu] F. McAuley and P. A. Tulley, *Fiber spaces and n-regularity*, Topology Seminar Wisconsin, 1965, Ann. of Math. Studies 60.

[Mi1] E. A. Michael, *Continuous selections*, I, Ann. of Math. 63 (1956), 361–382.

[Mi2] –, *Continuous selections*, II, Ann. of Math. 64 (1956), 562–580.

[Mi3] –, *Continuous selections*, III, Ann. of Math. 65 (1957), 375–390.

[Pu] W. T. Puchett, *On 0-regular surface transformations*, Trans. Amer. Math. Soc. 47 (1940), 95–113.

[Ra] F. Raymond, *Local triviality for Hurewicz fiberings of manifolds*, Topology 3 (1965), 43–57.

[Sh1] E. V. Shchepin, *Soft maps of manifolds*, Uspekhi Mat. Nauk 39 (5) (1984), 209–224.

[Sh2] —, *On a problem of L. A. Tumarkin*, Soviet Math. Dokl. 15 (1974) 1024–1026.

[St] S. Stoilov, *Lecons sur les Principes Topologiques*...., Paris Gauthier–Villars, 1956.

[To We] H. Toruńczyk and J. West, *Fibrations and Bundles with Hilbert Cube Manifold Fibers*, preprint.

[U] G. S. Ungar, *Conditions for a map to have the slicing structure property*, Pacific J. Math. 30 (1969), 549–553.

[Wa1] J. J. Walsh, *Monotone and open mappings onto* ANR's, Proc. Amer. Math. Soc. 60 (1976), 286–288.

[Wa2] --, *Light open and open mappings on manifolds*, II, Trans. Amer. Math. Soc. 217 (1976), 271–284.

[Wa3] —, *A general method for constructing* UV^k*-mappings on manifolds with applications to spheres*, preprint.

[Wa4] —, *Dimension, cohomological dimension and cell-like mappings*, in *Shape Theory and Geometric Topology*, Dubrovnik 1981, Lecture Notes in Math. 870.

[Whyt] P. Whyte, *Regular transformations*, Duke Math. J. 12 (1945), 101–106.

[Whyb] G. T. Whyburn, *Regular convergence and monotone transformations*, Amer. J. Math. 57 (1935), 902–906.

[Wil] D. C. Wilson, *On constructing monotone and* UV^1*-mappings of arbitrary degree*, Duke Math. J. 41 (1974), 103–109.

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