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On foliations with leaves satisfying some  
geometrical conditions

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## CONTENTS

Introduction . . . . .	5
1. Preliminaries . . . . .	6
1. A. Foliations . . . . .	7
1. B. Geometry of submanifolds . . . . .	9
2. The characteristic form . . . . .	11
3. Stability of minimal foliations . . . . .	18
4. A metric on the space of foliations . . . . .	24
5. Jacobi fields on leaves . . . . .	27
6. The Gauss mapping of a foliation . . . . .	37
References. . . . .	45



## Introduction

Recent years have brought intense progress in the study of foliations. The theory of foliations developed toward two main directions: the quantitative and qualitative theories. The first one concerns the following question: How many geometrically distinct foliations can be constructed on a given manifold? The characteristic classes of foliations are a very useful tool in the investigation of this problem. Several kinds of such classes have been discovered (see, for example, [6], [29], [33]). Many authors ([43], [46] and others) examined relationship of the Riemannian geometry of the normal bundle of a foliation and its characteristic classes. For a comprehensive exposition of the quantitative theory we refer to [39].

Properties of foliations are the object of qualitative theory studies. The first remarkable results in this area have been obtained by Reeb [45]. The Stability Theorems (compare Section 1) are the main results of [45]. Reeb formulated also the following question: Given a foliation  $F$  of a manifold  $M$  with each leaf compact and a compact subset  $K$  of  $M$ , are the volumes of leaves meeting  $K$  bounded? Reeb showed that the answer is "yes" if  $\text{codim } F = 1$ . Vogt [56] and Edwards, Millett and Sullivan [14] showed that if  $\text{codim } F = 2$  and  $M$  is compact, then the answer is also affirmative. However, examples constructed by Reeb [45] and Epstein and Vogt [21] show that if either  $\text{codim } F = 2$  and  $M$  is not compact or  $\text{codim } F > 2$ , then the affirmative answer is not possible in general. The similar results for holomorphic foliations have been obtained by Duchamp and Kalka [12], Holmann [31], [32], Kaup [34] and Müller [41]. Epstein [18] found several conditions equivalent to the local boundness of volumes of leaves of a foliation with all leaves compact. Another conditions of this kind have been found by Wadsley [57] and Rummler [50]. These last conditions are expressed in terms of geometry of leaves. Namely, they proved (Wadsley in the case of one-dimensional foliations and Rummler in general) that volumes of leaves of an orientable foliation  $F$  with all leaves compact of an orientable manifold  $M$  are locally bounded if and only if  $M$  admits a Riemannian structure  $g$  such that each leaf of  $F$  is a minimal submanifold of a Riemannian manifold  $(M, g)$ . This result caused our interest in the geometry of leaves of a foliation of a Riemannian manifold.

The present paper is devoted to the study of foliations of Riemannian

manifolds with leaves being minimal, totally geodesic or totally umbilical submanifolds. Section 1 contains basic definitions and facts concerning foliations and geometry of submanifolds. In Section 2, the characteristic form of a foliation is studied. This form was defined and applied by Rummier [48] (see also [3] and [51]). Here, we get some other applications of the characteristic form. For instance, we calculate its Laplacian and prove some facts about cohomology of flat manifolds foliated by totally geodesic submanifolds. Section 3 is devoted to the question mentioned above. We generalize it asking: Given a subset  $X$  of a foliated Riemannian manifold  $M$  saturated by compact minimal leaves, is the space of leaves meeting  $X$  Hausdorff? The general answer is "no" even if all leaves are totally geodesic. We get the affirmative answer under a stronger assumption: We suppose that there exists a positive number  $\eta$  such that balls  $B_M(x, r)$  and  $B_L(x, r)$  satisfy the condition  $B_L(x, r) = B_M(x, r) \cap L$  for any  $r < \eta$ , leaf  $L \subset X$  and  $x$  of  $L$ . The proof of this result consists in construction of a separating points family of continuous functions on the considered space of leaves. In the same section, we construct a continuous (with respect to  $C^r$ -topologies) function  $F \mapsto g_F$  which assigns a Riemannian metric  $g_F$  such that leaves of  $F$  are minimal with respect to  $g_F$  to any foliation  $F$  of some neighbourhood of a foliation satisfying some conditions. In Section 4, we endow the set  $\text{Fol}_p^c(M)$  of all  $p$ -dimensional foliations with compact leaves of a given Riemannian manifold  $M$  with a metric  $d_H$  and show that stable foliations form a closed subset of the metric space  $(\text{Fol}_p^c(M), d_H)$ . In Section 5, Jacobi fields on minimal leaves of foliations are investigated. Some results concerning the existence of minimal or totally geodesic foliations of some Riemannian manifolds (such as  $S^n$ ,  $P^nC$ , manifolds of either non-positive or non-negative curvature) are obtained. In Section 6, the Gauss mapping of a foliation of a Riemannian manifold is defined. The class of Riemannian metrics with holonomy groups contained in a given group is introduced. We show that the Gauss mapping of a foliation with all leaves totally geodesic factors through the space of leaves. The Gauss mappings of foliations with all leaves minimal or totally umbilical are investigated.

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## 1. Preliminaries

All the manifolds, the mappings, the Riemannian metrics and the foliations considered in this paper are assumed to be differentiable of the class  $C^\infty$ . However, some results remain valid under weaker assumptions of differentiability.

### 1. A. Foliations

(1.1) DEFINITION. A  $p$ -dimensional *foliation* of an  $m$ -dimensional manifold  $M$  is a family  $F$  of disjoint connected subsets of  $M$  such that  $\bigcup F = M$  and for any point  $x$  of  $M$  there exists a chart  $\varphi = (\varphi_1, \varphi_2): U \rightarrow \mathbb{R}^p \times \mathbb{R}^{m-p}$  defined on an open neighbourhood  $U$  of  $x$  and such that for each  $L$  of  $F$  the connected components of  $U \cap L$  are described by the equation  $\varphi_2 = \text{const}$ . Sets of the family  $F$  are called *leaves* of the foliation, the number  $q = m - p$  is the *codimension* of  $F$ , and charts referred to in the definition are said to be *distinguished* by  $F$ . The subset  $X$  of  $M$  is said to be *saturated* if it is the union of a subfamily of  $F$ .

It follows from the definition that leaves are submanifolds immersed in  $M$ . These immersions are not necessarily embeddings; the natural manifold topology of leaves is not necessarily the one induced from  $M$ . Leaves with induced topology are called *proper*. Every compact leaf is proper.

Define the relation " $\sim$ " on  $M$  by

$$x \sim y \quad \text{if and only if} \quad y \in L_x,$$

where  $L_x$  is the leaf of  $F$  passing through  $x$ . Clearly, " $\sim$ " is an equivalence. The quotient space  $M/\sim$  will be called the *leaf space* and denoted by  $M/F$ . The canonical projection  $M \rightarrow M/F$  is always open ([45], p. 106) but it need not be closed.

The field of tangent spaces of leaves of  $F$  forms a  $p$ -dimensional subbundle  $TF$  of the tangent bundle  $TM$  of  $M$  called the *tangent bundle* of  $F$ . This bundle is involutive, i.e. the Lie bracket  $[X, Y]$  of two sections  $X$  and  $Y$  of  $TF$  is a section of  $TF$  again. The classical Frobenius Theorem states that any involutive subbundle of  $TM$  is the tangent bundle of a foliation of  $M$  (see [54], Chapter III, §5). The quotient  $NF = TM/TF$  is called the *normal bundle* of  $F$ . If  $M$  carries a Riemannian structure, then  $NF$  can be identified with the field of orthogonal complements of fibres of  $TF$ .

(1.2) DEFINITION. A foliation  $F$  is *orientable* (resp., *transversely orientable*) if and only if the bundle  $TF$  (resp.,  $NF$ ) is orientable.

Note that leaves of an orientable foliation are orientable and that a foliation of an orientable manifold is orientable if and only if it is transversely orientable.

A mapping  $f: N \rightarrow M$  is *transverse* to a foliation  $F$  of  $M$  if

$$f_*(T_x N) + T_{f(x)} F = T_{f(x)} M$$

for any  $x$  of  $N$ . In this case, the connected components of sets of the form  $f^{-1}(L)$ , where  $L$  is a leaf of  $F$ , form a foliation of  $N$ . This foliation will be denoted by  $f^*F$ . Note that

$$\text{codim } f^*F = \text{codim } F.$$

Now, we are going to define the holonomy of a leaf of a foliation. We follow Lawson [39]. Other equivalent definitions can be found in [14], [18], [45].

Let  $i: L \rightarrow M$  be the inclusion map of a leaf  $L$  of a foliation  $F$  of a manifold  $M$ . There exists an immersion  $I: NL \rightarrow M$ , where  $NL = i^*NF$  is the normal bundle of  $L$ , satisfying the following conditions:

- (i)  $I(O_x) = x$  for each  $x$  of  $L$ , where  $O_x$  denotes the origin of  $N_xL$ ,
- (ii)  $I$  maps the fibres of  $NL$  transversely to  $F$ .

The foliation  $I^*F$  of  $NL$  is transverse to fibres of  $NL$  and the zero section of  $NL$  appears as one of its leaves. It follows that for any curve  $c: [0, 1] \rightarrow L$  there exists a neighbourhood  $U$  of the origin of  $N_{c(0)}L$  such that for any point  $v$  of  $U$  there exists a unique curve  $c_v$  on the leaf of  $I^*F$  passing through  $v$  with the initial point  $v$  and such that  $\pi \circ c_v = c$ , where  $\pi: NL \rightarrow L$  is the projection. The mapping

$$U \ni v \mapsto c_v(1)$$

is a diffeomorphism of  $U$  onto a neighbourhood of the origin of  $N_{c(1)}L$ . This diffeomorphism depends only on the homotopy class of  $c$ . Considering germs of such diffeomorphisms corresponding to all loops based at a fixed point  $x$  of  $L$  we can construct a homomorphism  $h_L$  of the fundamental group  $\pi_1(L)$  into the group  $\Gamma^q$  of germs at 0 of local diffeomorphisms of  $\mathbb{R}^q$  which map the origin to itself. The conjugacy class of  $h_L$  does not depend on the choice of the immersion  $I$ , of the base point in the definition of  $\pi_1(L)$  and other choices involved in the above construction.

(1.3) DEFINITION.  $h_L$  is called the *holonomy homomorphism* of  $L$ . We say that  $L$  has *finite holonomy* (resp.,  $L$  is *generic*) if and only if the group  $\text{im } h_L$  is a finite (resp., trivial) subgroup of  $\Gamma^q$ .

Of course, all simply connected leaves are generic.

Let us recall the following result due to Epstein, Millett and Tischler [20].

(1.4) LEMMA. *The union of all generic leaves of a foliation of a paracompact manifold  $M$  is a dense  $G_\delta$  subset of  $M$ .*

The next theorem establishes the importance of the notion of holonomy.

(1.5) THEOREM (Reeb stability). *Let  $L$  be a compact leaf of a codimension  $q$  foliation  $F$  of a manifold  $M$  and suppose that  $L$  has finite holonomy. Let  $p: \tilde{L} \rightarrow L$  be the finite covering corresponding to the subgroup  $\ker h_L$  of  $\pi_1(L)$ . Then there exists a saturated neighbourhood  $U$  of  $L$  in  $M$  and a finite covering  $f: \tilde{L} \times D \rightarrow U$ , where  $D$  is an open neighbourhood of the origin of  $\mathbb{R}^q$ , such that  $f|\tilde{L} \times \{0\} = p$  and  $f^*(F|U) = \{\tilde{L} \times \{u\}; u \in D\}$ . If  $L$  is, moreover, generic, then  $p = \text{id}$  and  $f$  is a diffeomorphism.*

For the proof see [39] or [45].

From this theorem and Lemma (1.4), it follows that the union of all generic leaves of a foliation  $F$  of  $M$  is an open dense subset of  $M$  if only all leaves of  $F$  are compact.

For examples and more detailed exposition of the theory of foliations we refer to [6], [38] and [39].

**1. B. Geometry of submanifolds.** Let  $(M, g)$  be a Riemannian manifold. We denote the Levi-Civita connection on  $(M, g)$  by  $\nabla$ . The same symbol  $\nabla$  will be used for connections determined by  $\nabla$  on tensor bundles. The curvature tensor of the connection  $\nabla$  will be denoted by  $R$  and the sectional curvature of the plane spanned by vectors  $v, w$  by  $K(v \wedge w)$ . The Ricci tensor on  $(M, g)$  will be denoted by  $\text{Ric}$ . Recall that

$$\text{Ric}(v, w) = \text{Trace}(u \mapsto R(u, v)w)$$

for any  $v, w$  of  $T_x M$ ,  $x$  of  $M$ . The trace of the bilinear form  $\text{Ric}$  is the scalar curvature of  $(M, g)$ .

Throughout the paper the symbol  $\langle v, w \rangle$  will be used instead of  $g(v, w)$ .

Let  $L$  be a submanifold of  $M$ . Then the inclusion map  $i: L \rightarrow M$  is an immersion and induces the Riemannian metric  $\bar{g} = i^*g$  on  $L$ . The Levi-Civita connection, curvature tensor, sectional curvature and Ricci tensor on  $(L, \bar{g})$  will be denoted by  $\bar{\nabla}$ ,  $\bar{R}$ ,  $\bar{K}$  and  $\bar{\text{Ric}}$ , respectively.  $NL$  is the normal bundle of  $L$ . If  $x \in L$  and  $v \in T_x M$ , then  $v^\top$  and  $v^\perp$  denote the tangent and normal components of  $v$ :

$$v = v^\top + v^\perp, \quad v^\top \in T_x L, \quad v^\perp \in N_x L.$$

The second fundamental form of  $L$  can be considered either as a section of the bundle  $\text{Hom}(NL, \text{Hom}(TL, TL))$  or as a section of the bundle  $\text{Hom}(TL \otimes TL, NL)$ . In the first case, it will be denoted by  $A$  and defined by

$$A^w(v) = -(\nabla_v W)^\top,$$

where  $v \in T_x L$ ,  $w \in N_x L$  and  $W$  is a section of  $NL$  such that  $W(x) = w$ . In the second case, the second fundamental form of  $L$  will be denoted by  $B$  and defined by

$$B(u, v) = (\nabla_u V)^\perp,$$

where  $u, v \in T_x L$  and  $V$  is a section of  $TL$  such that  $V(x) = v$ . The following formulae follow from definitions of  $A$  and  $B$ :

$$\langle A^w u, v \rangle = \langle u, A^w v \rangle = \langle B(u, v), w \rangle,$$

$$\nabla_u V = \bar{\nabla}_u V + B(u, v),$$

where  $u, v \in T_x L$ ,  $w \in N_x L$ ,  $x \in L$ , and  $V$  is a vector field on  $L$  such that  $V(x) = v$ .

The curvature tensors  $R$  and  $\bar{R}$  are related by

$$(1) \quad \bar{R}(u, v)w = (R(u, v)w)^\top + \sum_j (\langle B(v, w), w_j \rangle A^{w_j} u - \langle B(u, w), w_j \rangle A^{w_j} v),$$

where  $u, v, w \in T_x L$ ,  $x \in L$ ,  $w_1, \dots, w_q$  is an orthonormal frame of  $N_x L$ ,  $q = \text{codim } F$ . It follows that

$$(2) \quad \bar{K}(u \wedge v) = K(u \wedge v) + \sum_j (\langle B(u, u), w_j \rangle \langle B(v, v), w_j \rangle - \langle B(u, v), w_j \rangle^2),$$

where  $u, v \in T_x L$ ,  $x \in L$ ,  $\|u\| = \|v\| = 1$ ,  $\langle u, v \rangle = 0$ . The equalities (1) and (2) are known as the Gauss equations. Recall also the Codazzi equation

$$(3) \quad (R(u, v)w)^\perp = (\nabla'_u B)(v, w) - (\nabla'_v B)(u, w),$$

where  $u, v, w \in T_x L$ ,  $x \in L$ , and  $\nabla'$  is the connection on the bundle  $\text{Hom}(TL \otimes TL, NL)$  determined by the Levi-Civita connection on  $L$  and the connection  $\nabla^\perp$  on  $NL$  given by

$$\nabla_v^\perp W = (\nabla_v W)^\perp$$

for  $v$  of  $TL$  and sections  $W$  of  $NL$ .

The trace of the second fundamental form  $B$  is called the mean curvature vector of  $L$  and denoted by  $H$ .  $H$  is a section of  $NL$ . At any point  $x$  of  $L$  we have

$$H(x) = \sum_i B(e_i, e_i),$$

where  $e_1, \dots, e_p$ ,  $p = \dim L$ , is an orthonormal frame of  $T_x L$ .

(1.6) DEFINITION. A submanifold  $L$  of a Riemannian manifold  $(M, g)$  is *minimal* (resp., *totally geodesic*, resp., *totally umbilical*) if and only if  $H = 0$  (resp.,  $B = 0$ , resp.,  $B = \bar{g} \otimes W$  for some section  $W$  of  $NL$ ). A foliation  $F$  of  $(M, g)$  is *minimal* (resp., *totally geodesic*, resp., *totally umbilical*) if and only if all leaves of  $F$  are minimal (resp., totally geodesic, resp., totally umbilical) submanifolds of  $(M, g)$ .

Totally geodesic submanifolds can be characterized as follows:  $L$  is totally geodesic if and only if geodesics on  $M$  tangent to  $L$  at one point lie entirely on  $L$ .

Minimal submanifolds appear as critical points of the volume function. Namely, if  $f: L \times \mathbb{R} \rightarrow M$  is an immersion,  $f_0 = \text{id}_L$ ,  $L$  is compact and  $V(t)$  is the volume of  $L$  with respect to the Riemannian metric  $f_t^*g$ , then

$$V'(0) = - \int_L \langle H, X \rangle, \quad \text{**}$$

where  $X$  is the section of  $NL$  given by  $X(x) = ((t \mapsto f_t(x))'(0))^\perp$ . It follows that  $H = 0$  if and only if  $V'(0) = 0$  for all such  $f$ . The second derivative of the volume function is given by (see [13], [53])

$$V''(0) = \int_L \langle -\nabla^2 X + \text{Ric}'(X) - A'(X), X \rangle,$$

where  $\nabla^2 X(x)$  is the trace of the bilinear form

$$T_x L \times T_x L \ni (v, w) \mapsto \nabla_v (\nabla X)(w),$$

$$\text{Ric}'(u) = \sum_i (R(e_i, u) e_i)^\perp$$

for  $u$  of  $N_x L$ , and

$$(4) \quad \langle A'(w), w' \rangle = \langle A^w, A^{w'} \rangle = \sum_i \langle A^w e_i, A^{w'} e_i \rangle$$

for  $w, w'$  of  $N_x L$ ; here  $e_1, \dots, e_p$  is an orthonormal frame of  $T_x L$ . Note that ([53], p. 65)

$$(5) \quad \int_L \langle \nabla^2 X, Y \rangle = - \int_L \langle \nabla X, \nabla Y \rangle$$

for any sections  $X$  and  $Y$  of  $NL$ .

(1.7) DEFINITION. A section  $X$  of the normal bundle  $NL$  of a minimal submanifold  $L$  of a Riemannian manifold  $(M, g)$  is said to be a *Jacobi field* if and only if there exists a mapping  $f: L \times \mathbf{R} \rightarrow M$  such that

- (i) each  $f_t$  is a minimal immersion of  $L$  into  $M$ ,
- (ii)  $X(x) = ((t \mapsto f_t(x))'(0))^\perp$  for all  $x$  of  $L$ .

It is known that Jacobi fields satisfy the equation

$$(6) \quad \nabla^2 X - \text{Ric}'(X) + A'(X) = 0.$$

In the case of geodesics (= one-dimensional minimal submanifolds) the converse is true: every section of  $NL$  satisfying (6) is a Jacobi field. Recently, Burns [9] found an example of a field  $X$  satisfying (6) which is not associated with any one-parameter family of minimal submanifolds.

For more information about submanifolds of Riemannian manifolds we refer to [11], [36], [53].

## 2. The characteristic form

Let  $F$  be an oriented foliation of a Riemannian manifold  $(M, g)$ ,  $p = \dim F$ . Let us take a positive oriented, orthonormal frame  $e_1, \dots, e_p$  of  $T_x F$ ,  $x \in M$ . Put

$$(7) \quad \omega_F(v_1, \dots, v_p) = \det [\langle v_i, e_j \rangle; i, j = 1, \dots, p]$$

for all  $v_1, \dots, v_p$  of  $T_x M$ . The formula (7) defines properly a  $p$ -form  $\omega_F$  on  $M$ .

(2.1) DEFINITION. The form  $\omega_F$  is called the *characteristic form* of  $F$ .

The form  $\omega_F$  was defined and investigated by Andrzejczak [3] and Rummier [48] who proved the following

(2.2) LEMMA. Let  $e_1, \dots, e_p$  be a positive oriented, orthonormal frame of  $T_x F$ ,  $v, w \in N_x F$ . Then

$$d\omega_F(e_1, \dots, e_p, v) = (-1)^{p+1} \langle v, H(x) \rangle$$

and

$$d\omega_F(e_1, \dots, e_{p-1}, v, w) = (-1)^p \langle e_p, [V, W](x) \rangle,$$

where  $H$  is the mean curvature vector of the leaf  $L_x$  passing through  $x$  and  $V, W$  are sections of the bundle  $NF$  such that  $V(x) = v, W(x) = w$ . Thus the form  $\omega_F$  is closed if and only if the foliation  $F$  is minimal and the bundle  $NF$  is involutive.

As an immediate application of this lemma we obtain the following result.

(2.3) PROPOSITION. *Let  $F$  be an orientable minimal foliation of a Riemannian manifold  $(M, g)$ ,  $\text{codim } F = 1$ . If  $F$  has a compact leaf, then  $H^{m-1}(M, \mathbf{R}) \neq 0$  ( $m = \dim M$ ).*

PROOF. Let  $L$  be a compact leaf of  $F$ . The characteristic form  $\omega_F$  of  $F$  is closed and  $i^*\omega_F$ , where  $i: L \rightarrow M$  is the inclusion, does not vanish at any point of a compact orientable manifold  $L$ . Therefore, the cohomology class  $[i^*\omega_F] = i^*[\omega_F] \neq 0$ .

It follows that the sphere  $S^m, m > 1$ , can not be equipped with a codimension-one foliation having a compact leaf and minimal with respect to any Riemannian metric on  $S^m$ . In particular, the Reeb foliation (see [39], [45]) may not be minimal with respect to any Riemannian metric on  $S^3$ . Note that the only compact leaf of the Reeb foliation is embedded into  $S^3$  as a minimal hypersurface called the Clifford torus. More general, Novikov [42] proved that any codimension-one foliation of a compact three-dimensional manifold  $M$  has a compact leaf provided that the fundamental group  $\pi_1(M)$  is finite. Novikov's theorem and the above proposition yield the following result.

(2.4) COROLLARY. *Let  $(M, g)$  be a compact Riemannian manifold,  $\dim M = 3$ . If  $\pi_1(M)$  is finite, then there are no codimension-one minimal foliations of  $M$ .*

The next theorem and proposition contain other applications of Lemma (2.2).

(2.5) THEOREM. *Suppose that  $L$  is a compact leaf of a codimension-one minimal foliation  $F$  of a compact connected Riemannian manifold  $(M, g)$  of dimension  $m$ . If the Ricci curvature of  $(M, g)$  is non-negative, then  $L$  is totally geodesic.*

PROOF. Passing to a suitable covering space we may suppose that both  $M$  and  $F$  are orientable.

Using the cohomology sequence of the pair  $(M, L)$  and the Alexander–Pontrjagin duality we get the following exact sequence

$$\dots \rightarrow H^{m-1}(M) \xrightarrow{\alpha} H^{m-1}(L) \xrightarrow{\beta} H_0(M-L) \xrightarrow{\gamma} H^m(M) \rightarrow 0.$$

Lemma (2.2) shows that  $\alpha$  is surjective. Therefore,  $\text{im } \beta = 0 = \ker \gamma$  and  $H_0(M-L) \approx \text{im } \gamma = H^m(M) \approx \mathbf{R}$ . On the other hand, the homology sequence of the pair  $(M, M-L)$  and the Alexander–Pontrjagin duality yield the exact sequence

$$\dots \rightarrow H_1(M-L) \xrightarrow{\alpha} H_1(M) \xrightarrow{\beta} H^{m-1}(L) \xrightarrow{\gamma} H_0(M-L) \xrightarrow{\delta} H_0(M) \rightarrow 0.$$

Assume that  $L$  is not totally geodesic. Then, according to Kelly [35], the natural homomorphism  $H_1(L) \rightarrow H_1(M)$  is surjective. Considering a tubular

neighbourhood of  $L$  one can easily show that the homomorphism  $\alpha$  is also surjective. Therefore,  $\text{im } \beta = 0 = \ker \gamma$  and  $H_0(M-L) \simeq \text{im } \gamma \oplus \text{im } \delta \simeq H^{m-1}(L) \oplus H_0(M) \simeq \mathbf{R} \oplus \mathbf{R}$ . Contradiction.

(2.6) PROPOSITION. *Let  $(M, g, J)$  be a nearly Kähler manifold [27], i.e. an almost Hermitian manifold satisfying*

$$(\nabla_X J)X = 0$$

for any vector field  $X$  on  $M$ . Suppose that  $G$  is a connected group of isometries of  $(M, g)$  with almost complex orbits of codimension two. Then either  $H^{\dim M - 2}(M, \mathbf{R}) \neq 0$  or  $G$  has no compact orbits.

Proof.  $G$ -orbits form an orientable foliation  $F$  of  $M$ . Leaves of  $F$  are minimal according to [26]. The normal bundle  $NF$  is involutive. In fact, if  $X$  is a section of  $NF$ ,  $\|X\| = 1$ ,  $v \in T_x F$ , and  $Z$  is a Killing vector field on  $M$  such that  $Z(x) = J(v)$ , then

$$\begin{aligned} \langle (\nabla_X JX)(x), v \rangle &= -\langle \nabla_X JX, JZ \rangle(x) = -\langle J(\nabla_X X), JZ \rangle(x) \\ &= \langle X, \nabla_X Z \rangle(x) = \langle X, [X, Z] \rangle(x) \\ &= \frac{1}{2} L_Z g(X, X)(x) = 0; \end{aligned}$$

similarly,

$$\langle \nabla_{JX} X(x), v \rangle = 0,$$

i.e.  $[X, JX](x) \in N_x F$ . Consequently, the characteristic form  $\omega_F$  is closed and  $H^{\dim M - 2}(M, \mathbf{R}) \neq 0$  if only compact orbits exist.

Now, we are going to calculate the Laplacian  $\Delta \omega_F$  of the characteristic form  $\omega_F$  of an oriented foliation  $F$  of a Riemannian manifold  $(M, g)$ . We will apply the following Weitzenböck formula for the Laplacian of any  $p$ -form  $\omega$  on a Riemannian manifold  $(M, g)$ :

$$(8) \quad \Delta \omega = \nabla^2 \omega + S(\omega),$$

where  $\nabla^2 \omega$  is the trace of the bilinear form

$$(X, Y) \mapsto \nabla_X (\nabla_Y \omega)(Y),$$

and  $S(\omega)$  is a  $p$ -form given by

$$(9) \quad \begin{aligned} S(\omega)(v_1, \dots, v_p) &= \sum_{i < k} \sum_s \omega(v_1, \dots, v_{i-1}, R(e_s, v_k) v_i - \\ &\quad - R(e_s, v_i) v_k, v_{i+1}, \dots, v_{k-1}, e_s, v_{k+1}, \dots, v_p) + \\ &\quad + \sum_{s,k} \omega(v_1, \dots, v_{k-1}, R(e_s, v_k) e_s, v_{k+1}, \dots, v_p) \end{aligned}$$

for all  $v_1, \dots, v_p$  of  $T_x M$ ,  $x$  of  $M$ ; here  $e_1, \dots, e_m$  ( $m = \dim M$ ) is an orthonormal frame of  $T_x M$ . The formula (8) can be found in [15], [16], [40].

At first, let us consider the form  $S(\omega_F)$ . We can choose the frame  $e_1, \dots, e_m$  in (9) such that  $e_1, \dots, e_p$  form a positive oriented frame of  $T_x F$ . If at least three of  $v_1, \dots, v_p$  belong to  $N_x F$ , then  $S(\omega_F)(v_1, \dots, v_p) = 0$ . Thus we have to consider the following three cases:

$$(i) (v_1, \dots, v_p) = (e_1, \dots, e_{p-2}, v, w),$$

$$(ii) (v_1, \dots, v_p) = (e_1, \dots, e_{p-1}, v),$$

$$(iii) (v_1, \dots, v_p) = (e_1, \dots, e_p),$$

where  $v, w \in N_x F$ ,  $\|v\| = \|w\| = 1$ ,  $\langle v, w \rangle = 0$ .

In the case (i), we have

$$\begin{aligned} S(\omega_F)(e_1, \dots, e_{p-2}, v, w) &= \sum_{i < k < p-1} \sum_s \omega_F(e_1, \dots, e_{i-1}, R(e_s, e_k) e_i - \\ &\quad - R(e_s, e_i) e_k, e_{i+1}, \dots, e_{k-1}, e_s, e_{k+1}, \dots, e_{p-2}, v, w) + \\ &\quad + \sum_{i < p-1} \sum_s \omega_F(e_1, \dots, e_{i-1}, R(e_s, v) e_i - \\ &\quad - R(e_s, e_i) v, e_{i+1}, \dots, e_{p-2}, e_s, w) + \\ &\quad + \sum_{i < p-1} \sum_s \omega_F(e_1, \dots, e_{i-1}, R(e_s, w) e_i - \\ &\quad - R(e_s, e_i) w, e_{i+1}, \dots, e_{p-2}, v, e_s) + \\ &\quad + \sum_s \omega_F(e_1, \dots, e_{p-2}, R(e_s, w) v - R(e_s, v) w, e_s) + \\ &\quad + \sum_{k < p-1} \sum_s \omega_F(e_1, \dots, e_{k-1}, R(e_s, e_k) e_s, e_{k+1}, \dots, e_{p-2}, v, w) + \\ &\quad + \sum_s \omega_F(e_1, \dots, e_{p-2}, R(e_s, v) e_s, w) + \\ &\quad + \sum_s \omega_F(e_1, \dots, e_{p-2}, v, R(e_s, w) e_s). \end{aligned}$$

The first three terms of the above sum vanish as well as the last three. We get

$$(10) \quad S(\omega_F)(e_1, \dots, e_{p-2}, v, w) = -2\langle R(e_p, v) w, e_{p-1} \rangle + 2\langle R(e_p, w) v, e_{p-1} \rangle.$$

The similar calculation shows that

$$(11) \quad S(\omega_F)(e_1, \dots, e_{p-1}, v) = \sum_{s > p} \langle R(e_s, v) e_s, e_p \rangle$$

and

$$(12) \quad S(\omega_F)(e_1, \dots, e_p) = - \sum_{k \leq p} \sum_{i > p} K(e_i \wedge e_k).$$

Setting about the computation of  $\nabla^2\omega_F$  let us extend vectors  $e_1, \dots, e_m$  of  $T_xM$  to orthonormal vector fields  $E_1, \dots, E_m$  in a neighbourhood of  $x$  satisfying  $\nabla_{E_i}E_j(x) = 0$  for all  $i, j = 1, \dots, m$ . Then

$$\nabla^2\omega_F(x) = \sum_i \nabla_{E_i}\nabla_{E_i}\omega_F(x),$$

i.e.

$$(13) \quad \nabla^2\omega_F(v_1, \dots, v_p) \\ = \sum_i (E_i E_i \omega_F(X_1, \dots, X_p) - 2 \sum_j E_i \omega_F(X_1, \dots, X_{j-1}, \nabla_{E_i} X_j, X_{j+1}, \dots, X_p) + \\ + 2 \sum_{j < k} \omega_F(X_1, \dots, X_{j-1}, \nabla_{E_i} X_j, X_{j+1}, \dots, X_{k-1}, \nabla_{E_i} X_k, X_{k+1}, \dots, X_p) + \\ + \sum_j \omega_F(X_1, \dots, X_{j-1}, \nabla_{E_i} \nabla_{E_i} X_j, X_{j+1}, \dots, X_p))(x),$$

where  $v_1, \dots, v_p \in T_xM$  and  $X_1, \dots, X_p$  are vector fields on a neighbourhood of  $x$  such that  $X_i(x) = v_i$ . Analogously to the calculation of  $S(\omega_F)$ , we have to consider cases (i), (ii), and (iii). We can extend vectors  $e_1, \dots, e_p$  (resp.,  $v$  and  $w$ ) to local orthonormal sections  $X_1, \dots, X_p$  of  $TF$  (resp.,  $Y$  and  $Z$  of  $NF$ ). In all cases, the first term of the sum in (13) vanishes. In the case (i), we have

$$\begin{aligned} & \nabla^2\omega_F(e_1, \dots, e_{p-2}, v, w) \\ & = -2 \sum_{j < p-1} \sum_i e_i \omega_F(X_1, \dots, X_{j-1}, \nabla_{E_i} X_j, \dots, X_{p-2}, Y, Z) - \\ & \quad - 2 \sum_i e_i \omega_F(X_1, \dots, X_{p-2}, \nabla_{E_i} Y, Z) - \\ & \quad - 2 \sum_i e_i \omega_F(X_1, \dots, X_{p-2}, Y, \nabla_{E_i} Z) + \\ & \quad + 2 \sum_{j < k < p-1} \sum_i \omega_F(e_1, \dots, e_{j-1}, \nabla_{e_i} X_j, e_{j+1}, \dots, e_{k-1}, \\ & \quad \quad \quad \nabla_{e_i} X_k, e_{k+1}, \dots, e_{p-2}, v, w) + \\ & \quad + 2 \sum_{j < p-1} \sum_i \omega_F(e_1, \dots, e_{j-1}, \nabla_{e_i} X_j, e_{j+1}, \dots, e_{p-2}, \nabla_{e_i} Y, w) + \\ & \quad + 2 \sum_{j < p-1} \sum_i \omega_F(e_1, \dots, e_{j-1}, \nabla_{e_i} X_j, e_{j+1}, \dots, e_{p-2}, v, \nabla_{e_i} Z) + \\ & \quad + \sum_{j < p-1} \sum_i \omega_F(e_1, \dots, e_{j-1}, \nabla_{e_i} \nabla_{E_i} X_j, e_{j+1}, \dots, e_{p-2}, v, w) + \\ & \quad + \sum_i \omega_F(e_1, \dots, e_{p-2}, \nabla_{e_i} \nabla_{E_i} Y, w) + \sum_i \omega_F(e_1, \dots, e_{p-2}, v, \nabla_{e_i} \nabla_{E_i} Z) + \\ & \quad + 2 \sum_i \omega_F(e_1, \dots, e_{p-2}, \nabla_{e_i} Y, \nabla_{e_i} Z). \end{aligned}$$

All terms of the above sum but the last one vanish. Therefore,

$$\begin{aligned} \nabla^2 \omega_F(e_1, \dots, e_{p-2}, v, w) \\ = 2 \sum_i (\langle \nabla_{e_i} Y, e_{p-1} \rangle \langle \nabla_{e_i} Z, e_p \rangle - \langle \nabla_{e_i} Z, e_{p-1} \rangle \langle \nabla_{e_i} Y, e_p \rangle). \end{aligned}$$

Denoting by  $B$  the second fundamental form of the leaf  $L_x$  passing through  $x$  we obtain

$$\begin{aligned} (14) \quad \nabla^2 \omega_F(e_1, \dots, e_{p-2}, v, w) \\ = 2 \sum_{i \leq p} (\langle B(e_i, e_{p-1}), v \rangle \langle B(e_i, e_p), w \rangle - \langle B(e_i, e_{p-1}), w \rangle \langle B(e_i, e_p), v \rangle) + \\ + 2 \sum_{i > p} (\langle \nabla_{e_i} Y, e_{p-1} \rangle \langle \nabla_{e_i} Z, e_p \rangle - \langle \nabla_{e_i} Z, e_{p-1} \rangle \langle \nabla_{e_i} Y, e_p \rangle). \end{aligned}$$

Similarly,

$$\begin{aligned} (15) \quad \nabla^2 \omega_F(e_1, \dots, e_{p-1}, v) \\ = 2 \sum_{i \leq p} \sum_{j < p} \langle B(e_i, e_j), v \rangle \langle \nabla_{e_i} X_j, e_p \rangle + 2 \sum_{i > p} \sum_{j < p} \langle \nabla_{e_i} X_j, v \rangle \langle \nabla_{e_i} X_j, e_p \rangle - \\ - 2 \sum_i \langle \nabla_{e_i} Y, \nabla_{e_i} X_p \rangle - \sum_i \langle \nabla_{e_i} \nabla_{E_i} Y, e_p \rangle \end{aligned}$$

and

$$(16) \quad \nabla^2 \omega_F(e_1, \dots, e_p) = - \sum_{i, j \leq p} \sum_{k > p} \langle B(e_i, e_j), e_k \rangle^2 - \sum_{j \leq p} \sum_{i, k > p} \langle \nabla_{e_i} X_j, e_k \rangle^2.$$

(2.7) PROPOSITION. *Let  $F$  be an oriented foliation of a Riemannian manifold  $(M, g)$ . If the sectional curvature of  $(M, g)$  is non-negative, then the characteristic form  $\omega_F$  is harmonic if and only if  $F$  is totally geodesic, the bundle  $NF$  is involutive and the foliation  $F^\perp$  determined by  $NF$  is also totally geodesic.*

*Proof.* Suppose that  $\omega_F$  is harmonic, i.e. that  $\Delta \omega_F = \nabla^2 \omega_F + S(\omega_F) = 0$ . From the formulae (12) and (16), it follows that  $B = 0$  and

$$(17) \quad \langle \nabla_X V, X' \rangle = - \langle V, \nabla_X X' \rangle = 0$$

for all sections  $X, X'$  of  $NF$  and  $V$  of  $TF$ . The equality  $B = 0$  shows that leaves of  $F$  are totally geodesic while (17) implies  $\langle V, [X, X'] \rangle = \langle V, \nabla_X X' \rangle - \langle V, \nabla_{X'} X \rangle = 0$ . This proves that  $NF$  is involutive. The equality (17) shows also that leaves of  $F^\perp$  are totally geodesic.

Conversely, if  $NF$  is involutive and both foliations  $F$  and  $F^\perp$  are totally geodesic, then formulae (14), (15), and (16) show that  $\nabla^2 \omega_F = 0$ . Moreover, in this situation  $M$  is locally isometric to the product of leaves of  $F$  and  $F^\perp$ . It follows that  $R(u, v) = 0$  whenever  $u \in TF$  and  $v \in NF$ . From (10), (11), and (12), we obtain  $S(\omega_F) = 0$ .

Applying the above proposition to foliations of flat manifolds we get the following result.

(2.8) THEOREM. *Let  $(M, g)$  be a compact flat Riemannian manifold,  $m = \dim M$ . If  $M$  admits a  $p$ -dimensional orientable parallel foliation, then  $H^p(M, \mathbb{R}) \neq 0$ .*

*Proof.* Let  $F$  be a parallel oriented foliation of  $M$ ,  $p = \dim F$ . Let us take a point  $x$  of  $M$  and an orthonormal frame  $e_1, \dots, e_m$  of  $T_x M$  such that  $e_1, \dots, e_p$  form a positive oriented frame of  $T_x F$ . We can extend vectors  $e_i$  to vector fields  $X_i$  on a neighbourhood of  $x$  satisfying  $\nabla X_i = 0$ . Vector fields  $X_1, \dots, X_p$  are local sections of  $TF$  because  $F$  is totally geodesic. From the equalities  $\langle X_i, X_j \rangle = \delta_{ij}$ ,  $i, j = 1, \dots, m$ , it follows that  $X_{p+1}, \dots, X_m$  are local sections of  $NF$ . Consequently,  $NF$  is involutive and  $F^\perp$  (compare (2.7)) totally geodesic. Therefore,  $\omega_F$  is a harmonic  $p$ -form on  $M$ . Of course,  $\omega_F \neq 0$ . From the well known Hodge theorem (see, for example, [25], p. 76), it follows that  $H^p(M, \mathbb{R}) \neq 0$ .

(2.9) THEOREM. *Let  $(M, g)$  be a compact Riemannian manifold of non-negative Ricci curvature. If  $\dim H^1(M, \mathbb{R}) \geq p$ , then  $M$  admits an orientable parallel foliation of dimension  $p$ .*

*Proof.* If  $\dim H^1(M, \mathbb{R}) \geq p$ , then there are harmonic 1-forms  $\eta_1, \dots, \eta_p$  on  $M$  linearly independent over  $\mathbb{R}$ . If the curvature of  $M$  is non-negative, then forms  $\eta_1(x), \dots, \eta_p(x)$  are linearly independent at any point  $x$  of  $M$ . Therefore, putting

$$E_x = \text{the orthogonal complement of } \{v \in T_x M; \iota_v \eta = 0\},$$

where  $\eta = \eta_1 \wedge \dots \wedge \eta_p$ ,  $x \in M$ , we define a  $p$ -dimensional subbundle  $E$  of  $TM$ . This bundle is involutive and orientable. The foliation  $F$  determined by  $E$  is parallel. In fact, the parallel form  $\eta$  is the characteristic form of  $F$  (with respect to a suitable orientation).

(2.10) COROLLARY. *If  $(M, g)$  is a compact orientable flat Riemannian manifold,  $m = \dim M$ , and  $\dim H^1(M, \mathbb{R}) \geq \min(p, m-p)$ , then  $M$  admits an orientable parallel foliation of dimension  $p$ .*

(2.11) EXAMPLE. There are just 10 affine diffeomorphism classes of compact flat Riemannian manifolds of dimension 3 (compare [60], Chapter 3). Six of them (denoted by  $G_i$ ,  $i \leq 6$ ) consist of orientable and four (denoted by  $B_i$ ,  $i \leq 4$ ) of non-orientable manifolds. The cohomology group  $H^1(M, \mathbb{R})$  is isomorphic to  $\mathbb{R}^3$  if  $M$  is of the class  $G_1$ , to  $\mathbb{R}^2$  if  $M$  is of the class  $B_i$  with  $i \leq 2$ , to  $\mathbb{R}$  if  $M$  is either of the class  $G_i$  with  $2 \leq i \leq 5$  or of the class  $B_i$  with  $i \geq 3$ . If  $M$  is of the class  $G_6$ , then  $H^1(M, \mathbb{R}) = 0$ . Therefore, all compact connected flat manifolds of dimension 3 but these of the class  $G_6$  admit an orientable totally geodesic foliation of dimension 1 and a (not necessarily orientable) totally geodesic foliation of dimension 2. Manifolds of the class  $G_i$  with  $i \leq 5$



admit an orientable totally geodesic foliation of dimension 2 as well as manifolds of the class  $B_i$  with  $i \leq 2$ . If  $M$  is of the class  $B_i$  with  $i \geq 3$ , then  $H^2(M, \mathbf{R}) = 0$  and  $M$  does not admit any orientable totally geodesic foliation of dimension 2.

(2.12) EXAMPLE.  $Z_k$ -manifolds are compact connected Riemannian manifolds of the cyclic holonomy group of order  $k$ , where  $k$  is a prime number. For any integers  $a, b, c$  such that  $a, b > 0, c \geq 0, b > c$ , and  $a \geq c$  there exists a  $Z_k$ -manifold  $M$  of dimension  $(k-1)a + b$  and the first integral homology group isomorphic to  $Z^b \oplus Z_k^{a-c}$  (see [10]). It follows that for any natural number  $m \geq k$  there exists a  $Z_k$ -manifold  $M, \dim M = m$ , such that  $M$  admits a totally geodesic orientable foliation of dimension  $p$  for any  $p \leq m + 1 - k$ . In particular, for any  $m \geq 2$  there are  $m$ -dimensional  $Z_2$ -manifolds which admit orientable totally geodesic foliations of arbitrary dimension. One can also observe that if  $m \geq 2(k-1)$ , then there are  $m$ -dimensional  $Z_k$ -manifolds which admit (not necessarily orientable) totally geodesic foliations of arbitrary dimension.

Remark. If  $M$  is a compact connected flat Riemannian manifold, then Theorems (2.8) and (2.9) give the following implications:

$$\dim H^1(M, \mathbf{R}) \geq p \stackrel{(c)}{\iff} H^p(M, \mathbf{R}) \neq 0$$

(a)  $\Downarrow$ 
 $\Uparrow$  (b)

$M$  admits an parallel geodesic foliation of dimension  $p$ .

Clearly, the converse of (a) is not true as well as the converse of (c). The analysis of the implication converse to (b) seems to be reasonable.

### 3. Stability of minimal foliations

Let  $F$  be a foliation of a manifold  $M$  such that all leaves of  $F$  are compact. Such foliations will be called compact. If either  $\text{codim } F = 1$  or  $\text{codim } F = 2$  and  $M$  is compact, then, according to Reeb [45] and Edwards, Millett and Sullivan [14], all leaves of  $F$  have finite holonomy groups. If either  $\text{codim } F \geq 3$  or  $\text{codim } F = 2$  but  $M$  is not compact, then this need not be true. For suitable examples we refer to [17], [21], [39], [45], and [55]. Epstein [18] collected several conditions equivalent to the statement "the leaf  $L$  has finite holonomy". Namely, he proved the following

(3.1) THEOREM. *Let  $F$  be a compact foliation of a manifold  $M$ . Take an arbitrary Riemannian metric  $g$  on  $M$  and denote by  $V(x)$  the volume of the leaf  $L_x$  of  $F$  passing through a point  $x$  of  $M$ . The following conditions are equivalent:*

- (i)  $L_x$  has finite holonomy,
- (ii) the volume function  $V$  is bounded in a neighbourhood of  $x$ ,
- (iii) the topology of  $M/F$  is Hausdorff in a neighbourhood of  $L_x$ .

The following definition is motivated by Theorems (1.5) and (3.1).

(3.2) DEFINITION. A compact foliation  $F$  of a manifold  $M$  is said to be *stable* if and only if the space  $M/F$  is Hausdorff.

From the above definition and Theorem (3.1), it follows that for any compact foliation  $F$  of an arbitrary manifold  $M$  there exists an open dense saturated set  $U$  such that the restricted foliation  $F|U$  is stable.

The structure theorem of Ehresmann (see [18], Theorem 4.3) asserts that the leaf space  $M/F$  of a stable foliation  $F$  is a generalized manifold in the sense of Satake [52]. Using this fact and the relative de Rham cohomology of foliations [49], Rummmler [50] was able to associate with any oriented stable foliation  $F$  a form  $\eta_F$  such that

- (i)  $\eta_F$  is  $F$ -positive, i.e.  $\eta_F(e_1, \dots, e_p) > 0$  for any positive oriented frame  $e_1, \dots, e_p$  of  $T_x F$ ,  $x \in M$ ,
- (ii)  $\eta_F$  is  $F$ -closed, i.e.  $d\eta_F(v_1, \dots, v_p, v) = 0$  for any  $v_1, \dots, v_p$  of  $T_x F$ ,  $v$  of  $T_x M$ ,  $x$  of  $M$ .

This form can be chosen "pure", i.e. such that  $\eta_F(x) = v_1^* \wedge \dots \wedge v_p^*$  for some  $v_1^*, \dots, v_p^*$  of  $T_x^* M$  at any point  $x$  of  $M$ . For such form  $\eta_F$  one can find a Riemannian metric  $g$  on  $M$  such that the characteristic form  $\omega_F$  of the foliation  $F$  of the Riemannian manifold  $(M, g)$  is equal to  $\eta_F$ . From Lemma (2.2), it follows that leaves of  $F$  are minimal submanifolds of  $(M, g)$ . Conversely, if  $F$  is a minimal foliation of a Riemannian manifold  $(M, g)$ , then the form  $\omega_F$  is  $F$ -positive and  $F$ -closed. In the case of a compact manifold  $M$ , the stability of  $F$  follows immediately from a slight modification of the main result of [14]. Rummmler [50] improved this result and showed the following

(3.3) THEOREM. A compact orientable foliation of an orientable manifold  $M$  is stable if and only if  $F$  is minimal with respect to some Riemannian metric on  $M$ .

Theorem (3.3) generalizes the Wadsley's result [57]: A compact one-dimensional foliation is stable if and only if leaves are geodesics with respect to some Riemannian metric.

Rummmler's results suggest the following problem. Let  $F$  be a foliation of a Riemannian manifold  $(M, g)$ . Let  $M_0$  be a saturated subset of  $M$  such that all leaves of  $M_0$  are compact minimal submanifolds of  $(M, g)$ . We ask the following

(3.4) QUESTION. Is  $M_0/F$  a Hausdorff topological space?

If  $M_0$  is a manifold, then the positive answer follows from Theorem (3.3). In general case, Rummmler's proof breaks down for several reasons. The general answer to (3.4) is "no" even if leaves passing through  $M_0$  are totally geodesic. In fact, if  $M = T^2 \times R$ ,  $F$  is the foliation induced on  $M$  by the foliation of  $R^3$  determined by the vector field  $X$  given by

$$X(x) = \frac{\partial}{\partial x_1}(x) + x_3 \frac{\partial}{\partial x_2}(x), \quad x = (x_1, x_2, x_3),$$

and  $M_0 = T^2 \times A$ ,  $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ , then all leaves of  $F$  are totally geodesic with respect to the natural Riemannian metric on  $M$ , leaves passing through  $M_0$  are compact and the space  $M_0/F$  is not Hausdorff. We shall prove below that the answer to (3.4) is "yes" if leaves of  $M_0$  are totally geodesic and satisfy some additional condition. We begin with proof of the following

(3.5) PROPOSITION. *Let  $(M, g)$  be a Riemannian manifold and denote by  $d(x, y)$  the distance of points  $x$  and  $y$  determined by  $g$ . A compact foliation  $F$  of  $M$  is stable if and only if the function  $d_L: M/F \rightarrow \mathbb{R}$  given by*

$$d_L(L') = d(L, L') = \inf \{d(x, x'); x \in L, x' \in L'\}$$

is continuous for every leaf  $L$  of  $F$ .

PROOF. If  $d_L$  is continuous for each  $L$  and  $L_1, L_2$  are two distinct leaves, then  $d_{L_1}(L_1) = 0$  and  $d_{L_1}(L_2) = a > 0$ . The sets

$$d_{L_1}^{-1}([0, a/2]), \quad d_{L_1}^{-1}((a/2, +\infty))$$

are disjoint open neighbourhoods of  $L_1$  and  $L_2$ , respectively. Thus  $M/F$  is Hausdorff.

In the proof of the converse implication, we will use the following elementary fact:

(3.6) LEMMA. *Let  $A, B, C$  be compact subsets of a metric space  $(X, d)$ ,  $\varepsilon > 0$ . If  $d(b, C) < \varepsilon$  and  $d(c, B) < \varepsilon$  for all points  $b$  of  $B$  and  $c$  of  $C$ , then*

$$|d(A, B) - d(A, C)| < \varepsilon.$$

In fact, the inequalities  $d(b, C) < \varepsilon$  for all  $b$  yield

$$\begin{aligned} d(A, C) &= \inf_c d(c, A) \leq \inf_c d(c, a_0) \\ &\leq \inf_c d(c, b_0) + d(a_0, b_0) = d(b_0, C) + d(A, B) < d(A, B) + \varepsilon, \end{aligned}$$

where  $a_0 \in A$  and  $b_0 \in B$  are points satisfying  $d(A, B) = d(a_0, b_0)$ . Similarly, inequalities  $d(c, B) < \varepsilon$  for all  $c$  yield

$$d(A, B) < d(A, C) + \varepsilon.$$

Coming back to the proof of the proposition let us suppose that  $F$  is stable and take any two leaves  $L$  and  $L'$  of  $F$ . Considering (3.6) we see that the function  $d_L$  is continuous at  $L'$  if and only if for any positive number  $\varepsilon$  there exists an open neighbourhood  $U$  of  $L'$  in  $M/F$  such that if  $L''$  belongs to  $U$ , then  $d(x', L'') < \varepsilon$  and  $d(x'', L') < \varepsilon$  for all  $x'$  of  $L'$  and  $x''$  of  $L''$ . Put

$$W = \{x \in M; d(x, L') < \varepsilon\}.$$

$W$  is an open neighbourhood of  $L'$  in  $M$  and  $F$  is stable, therefore (compare [18], Theorem 4.1) there exists a saturated open neighbourhood  $V$  of  $L'$  contained in  $W$ . Put  $U_1 = \pi(V)$ , where  $\pi: M \rightarrow M/F$  is the canonical projec-

tion. Let us choose points  $x_1, \dots, x_n$  of  $L'$  such that the balls  $B(x_i, \frac{1}{2}\varepsilon)$ ,  $i = 1, \dots, n$ , cover  $L'$  and put  $U_2 = \{L'' \in M/F; L'' \cap B(x_i, \frac{1}{2}\varepsilon) \neq \emptyset \text{ for any } i \leq n\}$ . The set  $U = U_1 \cap U_2$  is open and has the required property. This guaranties continuity of  $d_L$  and ends the proof of the proposition.

From the above proof, one can see that if  $X$  is an arbitrary saturated subset of  $M$  and all restricted functions  $d_L|X/F$ , where  $L \subset X$ , are continuous, then the space  $X/F$  is Hausdorff. Conversely, if  $X$  is locally compact and  $X/F$  is Hausdorff, then the function  $d_L|X/F$  is continuous for any leaf  $L$  of  $F$ .

(3.7) DEFINITION. We say that a submanifold  $L$  of a Riemannian manifold  $(M, g)$  is  $\eta$ -regular if and only if the ball  $B_L(x, r)$  is equal to the intersection  $B_M(x, r) \cap L$  for every  $x$  of  $L$ ,  $r < \eta$ .

Clearly, if  $L$  is  $\eta$ -regular for some  $\eta > 0$ , then  $L$  is totally geodesic. Conversely, every compact totally geodesic submanifold is  $\eta$ -regular for some  $\eta > 0$ . Some properties of  $\eta$ -regular submanifolds are found and applied in [59].

(3.8) THEOREM. Let  $\eta > 0$ ,  $F$  be a foliation of a Riemannian manifold  $(M, g)$  and  $M_0$  a saturated subset of  $M$  such that all leaves of  $M_0$  are compact  $\eta$ -regular submanifolds of  $(M, g)$ . Then the quotient  $M_0/F$  is a Hausdorff topological space.

Proof. According to (3.5) and (3.6), we have to prove that if  $L \subset M_0$  and  $\varepsilon$  is a positive number, then there exists an open neighbourhood  $U$  of  $L$  in  $M_0/F$  such that if  $L' \in U$ , then

$$d(x, L') < \varepsilon \quad \text{for each } x \text{ of } L$$

and

$$d(x', L) < \varepsilon \quad \text{for each } x' \text{ of } L'.$$

We claim that there exists a positive number  $\lambda$  such that for any  $x$  of  $M_0$  and any  $v$  of  $T_x F$  such that  $d(x, L) < \lambda$  and  $\|v\| < 2\varepsilon$  the inequality  $d(\exp v, L) < \varepsilon$  holds. In fact, the set

$$E_\varepsilon = \{v \in TF; d(\exp v, L) < \varepsilon\}$$

is open and contains the tangent bundle  $TL$  of  $L$ . Therefore, for any point  $x$  of  $L$  there exists a number  $\lambda(x) > 0$  such that if  $y \in B(x, \lambda(x))$ ,  $v \in T_y F$  and  $\|v\| < 2\varepsilon$ , then  $v \in E_\varepsilon$ . We can choose points  $x_1, \dots, x_s$  of  $L$  such that the balls  $B(x_i, \frac{1}{2}\lambda(x_i))$ ,  $i = 1, \dots, s$ , cover  $L$ . The number  $\lambda = \frac{1}{2} \min \{\lambda(x_1), \dots, \lambda(x_s)\}$  has the required property.

Now, let us take positive numbers  $\delta$  and  $\varepsilon'$  such that  $\delta + \varepsilon' < \min(\varepsilon, \lambda)$  and cover  $L$  by a finite number of balls  $B(y_i, \delta)$ , where  $y_1, \dots, y_k \in L$ . Put

$$U = \{L' \in M_0/F; L' \cap B(y_i, \varepsilon') \neq \emptyset \text{ for all } i \leq k\}.$$

$U$  is an open subset of  $M_0/F$  and  $L \subset U$ . If  $L \subset U$  and  $x \in L$ , then

$$d(x, L) \leq \min \{d(x, y_i) + d(y_i, L); i \leq k\} < \delta + \varepsilon' < \varepsilon.$$

Let us take any leaf  $L' \in U$  and put  $L'_\varepsilon = \{x' \in L'; d(x', L) < \varepsilon\}$ .  $L'_\varepsilon$  is non-empty and open in  $L'$ . In order to complete the proof of the theorem we have to show that it is closed.

Let  $(x'_n)$  be a sequence of points of  $L'_\varepsilon$  such that  $x' = \varinjlim x'_n$ . For any  $n$  we can find a point  $x_n$  of  $L$  such that  $d(x_n, x'_n) < \varepsilon$ . Passing to a subsequence, if necessary, we can suppose that the sequence  $(x_n)$  is convergent to a point  $x$  of  $L$ . There exists a point  $x''$  of  $L$  such that  $d(x, x'') < \delta + \varepsilon'$ . For any  $n$  we have

$$d(x', x'') \leq d(x', x'_n) + d(x'_n, x_n) + d(x_n, x) + d(x, x'').$$

Passing with  $n$  to infinity we get

$$d(x', x'') \leq \varepsilon + \delta + \varepsilon' < 2\varepsilon.$$

Taking  $\varepsilon$  small enough ( $\varepsilon < \frac{1}{2}\eta$ ) we can express  $x'$  in the form  $\exp v$  for some  $v$  of  $T_x L$  such that  $\|v\| < 2\varepsilon$ . Moreover,

$$d(x'', L) \leq d(x, x'') < \delta + \varepsilon' < \lambda.$$

Therefore,  $d(x', L) = d(\exp v, L) < \varepsilon$ , i.e.  $x' \in L'_\varepsilon$ .

This ends the proof.

One can observe that we proved more:  $M_0/F$  is a Tychonoff space. On the other hand, it satisfies the second axiom of countability. Therefore,  $M_0/F$  is metrizable.

Another problem motivated by Theorem (3.3) can be considered. Let us endow the set  $\text{Fol}_{c,s}^p(M)$  of all compact stable  $p$ -dimensional foliations of  $M$  and the set  $\mathcal{R}(M)$  of all Riemannian metrics on  $M$  with some topologies. Construct a continuous function

$$\text{Fol}_{c,s}^p(M) \ni F \mapsto g_F \in \mathcal{R}(M)$$

such that any leaf of any foliation  $F \in \text{Fol}_{c,s}^p(M)$  is a minimal submanifold of the Riemannian manifold  $(M, g_F)$ .

In the case of  $C^r$ -topologies, we can define such a function locally, in neighbourhoods of foliations satisfying some additional conditions. Namely, we have the following result.

(3.9) THEOREM. *Let  $F_0$  be a foliation of a compact manifold  $M$ ,  $m = \dim M$ ,  $p = \dim F_0$ . Assume that there exists a closed  $p$ -form  $\omega$  on  $M$  such that  $\omega(v_1, \dots, v_p) \neq 0$  for all linearly independent vectors  $v_1, \dots, v_p$  of  $T_x F_0$ ,  $x \in M$ . Then there exists a neighbourhood  $A$  of  $F_0$  open in the  $C^0$ -topology for the set  $\text{Fol}^p(M)$  of all  $p$ -dimensional foliations of  $M$  and a function  $A \ni F \mapsto g_F \in \mathcal{R}(M)$  continuous with respect to  $C^r$ -topologies,  $r = 0, 1, \dots, +\infty$ ,*

and such that any leaf  $L$  of any foliation  $F \in A$  is minimal with respect to the Riemannian metric  $g_F$ .

**Proof.** Put

$$A = \{F \in \text{Fo}^p(M); \omega(v_1, \dots, v_p) \neq 0 \text{ for all linearly independent vectors of } T_x F, x \in M\}.$$

Clearly,  $A$  is open in the  $C^0$ -topology and  $F_0 \in A$ . Put

$$\omega_F(v_1, \dots, v_p) = \omega(\pi_F v_1, \dots, \pi_F v_p),$$

where  $\pi_F: TM \rightarrow TF$  is the projection given by

$$\pi_F v = w \quad \text{iff} \quad \iota_w(\omega(x)|A^p T_x F) = \iota_v \omega(x)|A^{p-1} T_x F,$$

$v, w \in T_x M, x \in M, F \in A$ .  $\omega_F$  is a  $p$ -form on  $M$  such that (see [50], pp. 59–68)

- (i)  $\dim \{v \in T_x M; \iota_v \omega_F = 0\} = m - p$  for any  $x$  of  $M$ ,
- (ii)  $\omega_F(v_1, \dots, v_p) = \omega(v_1, \dots, v_p)$  for any  $v_1, \dots, v_p$  of  $T_x F, x \in M$ ,
- (iii)  $d\omega_F(v_1, \dots, v_p, w) = 0$  for any  $v_1, \dots, v_p$  of  $T_x F, w \in T_x M, x \in M$ .

Let us fix a Riemannian metric  $g$  on  $M$ . From (i) and (ii), it follows that the formulae

$$\begin{aligned} g'_F(v, w) &= g(v, w) && \text{when either } v, w \in TF \text{ or } \iota_v \omega_F = \iota_w \omega_F = 0, \\ g'_F(v, w) &= 0 && \text{when } v \in TF \text{ and } \iota_w \omega_F = 0 \end{aligned}$$

define properly a Riemannian metric  $g'_F$  on  $M$ . We can define an orientation of any  $F$  of  $A$  saying that a basis  $v_1, \dots, v_p$  of  $T_x F, x \in M$ , is positive oriented if and only if  $\omega(v_1, \dots, v_p) > 0$ . One can see that the characteristic form of such oriented foliation  $F$  of the Riemannian manifold  $(M, g'_F)$  is of the form  $\lambda_F \omega_F$ , where  $\lambda_F$  is a positive smooth function on  $M$  given by

$$\lambda_F(x) = 1/\omega_F(v_1, \dots, v_p),$$

where  $x \in M$  and  $v_1, \dots, v_p$  is a positive oriented  $g$ -orthonormal frame of  $T_x F$ . Put

$$g_F = \lambda_F^{-2/p} g'_F.$$

Clearly, the function  $F \mapsto g_F$  is continuous with respect to  $C^r$ -topologies. Let us take a positive oriented basis  $w_1, \dots, w_p$  of  $T_x F, x \in M, F \in A$ , such that  $g_F(w_i, w_j) = \delta_{ij}, i, j = 1, \dots, p$ . For any vectors  $v_1, \dots, v_p$  of  $T_x M$  we have

$$\begin{aligned} \det [g_F(v_i, w_j); i, j \leq p] &= \det [\lambda_F^{-2/p}(x) g'_F(v_i, w_j); i, j \leq p] \\ &= \det [\lambda_F^{-1/p}(x) g'_F(v_i, \lambda_F^{-1/p}(x) w_j); i, j \leq p] \\ &= \lambda_F^{-1}(x) \det [g'_F(v_i, \lambda_F^{-1/p}(x) w_j); i, j \leq p] \\ &= \omega_F(v_1, \dots, v_p), \end{aligned}$$

because

$$g'_F(\lambda_F^{-1/p}(x)w_i, \lambda_F^{-1/p}(x)w_j) = g_F(w_i, w_j) = \delta_{ij}.$$

Therefore, the characteristic form of the foliation  $F$  of  $(M, g_F)$  equals  $\omega_F$ . From (iii) and Lemma (2.2), it follows that leaves of  $F$  are minimal submanifolds of  $(M, g_F)$ .

**Remark.** Note that all compact orientable codimension-one foliations of any orientable manifold satisfy conditions of Theorem (3.9).

#### 4. A metric on the space of foliations

Let us denote by  $\text{Fol}_p^c(M)$  the set of all  $p$ -dimensional compact foliations of a manifold  $M$ . Let  $g$  be a Riemannian metric on  $M$  and denote by  $d(x, y)$  the distance of points  $x$  and  $y$  of  $M$  determined by  $g$ . For any compact subsets  $A$  and  $B$  of  $M$  we denote the Hausdorff distance of sets  $A, B$  by  $d_H(A, B)$ , i.e.

$$d_H(A, B) = \sup_{x \in A} d(x, B) + \sup_{x \in B} d(x, A).$$

Let  $F$  and  $F'$  be compact foliations of  $M$ ,  $\dim F = \dim F' = p$ . Put

$$(18) \quad d_H(F, F') = \sup_{x \in M} \min(1, d_H(L_x, L'_x)),$$

where  $L_x$  and  $L'_x$  denote the leaves of  $F$  and  $F'$  passing through  $x$ . Clearly, the pair  $(\text{Fol}_p^c(M), d_H)$  is a metric space.

If  $M$  is compact, then the topology of the metric space  $(\text{Fol}_p^c(M), d_H)$  does not depend on the choice of  $g$ . In fact, if  $g'$  is another Riemannian metric on  $M$  and  $T^1M$  is the unit tangent bundle of the Riemannian manifold  $(M, g)$ , then there are positive numbers  $\mu_0, \mu_1$  such that

$$\mu_0 \leq \|v\|' \leq \mu_1$$

for any  $v$  of  $T^1M$ . Consequently,

$$\mu_0 L(c) \leq L'(c) \leq \mu_1 L(c)$$

and

$$(19) \quad \mu_0 d(x, y) \leq d'(x, y) \leq \mu_1 d(x, y)$$

for any smooth curve  $c$  on  $M$  and any points  $x$  and  $y$  of  $M$ . ( $L(c)$  and  $L'(c)$  denote the length of  $c$  on  $(M, g)$  and  $(M, g')$ , respectively;  $d'(x, y)$  is the distance of  $x$  and  $y$  determined by  $g'$ .) Defining the distance  $d'_H(F, F')$  of foliations  $F, F' \in \text{Fol}_p^c(M)$  by the formula analogous to (18) we get from (19)

$$\mu_0 d_H(F, F') \leq d'_H(F, F') \leq \mu_1 d_H(F, F').$$

Clearly, the analogous result for a non-compact manifold need not be true.

The set  $\text{Fol}_c^p(M)$  can be endowed with other reasonable topologies. For example, for any  $p$ -dimensional foliation  $F$  of  $M$  the tangent bundle  $TF$  of  $F$  can be considered as a section of the Grassmann bundle  $G^p(M)$  of all  $p$ -planes of tangent spaces  $T_x M$ ,  $x \in M$ . The set of all differentiable mappings of  $M$  into  $G^p(M)$  carries the weak and strong  $C^r$ -topologies,  $r = 0, 1, \dots, +\infty$  (see [30], pp. 34–36). These topologies and the mapping

$$\text{Fol}_c^p(M) \ni F \mapsto TF$$

induce topologies on the set  $\text{Fol}_c^p(M)$  which are also called weak and strong  $C^r$ -topologies, respectively. Weak  $C^r$ -topologies are metrizable. If  $M$  is compact, then weak and strong topologies are the same. Other topologies for the space of foliations are considered in [19].

The following examples show that  $C^r$ -topologies and the topology of the metric space  $(\text{Fol}_c^p(M), d_H)$  are not comparable. More precisely, the first example shows that the topology of  $(\text{Fol}_c^p(M), d_H)$  need not be weaker than the strong  $C^\infty$ -topology while the second one demonstrates that the  $C^0$ -topology need not be weaker than the topology of  $(\text{Fol}_c^p(M), d_H)$ .

(4.1) EXAMPLE. Let us consider the plane  $\mathbb{R}^2$  with Euclidean coordinates  $(u, v)$ . Put  $X_1 = \partial/\partial u$ ,  $X_2 = \partial/\partial v$  and

$$Y_n = X_1 + \frac{1}{n} X_2, \quad n = 1, 2, \dots$$

Foliations determined by the vector fields  $X_1$  and  $Y_n$  induce in a natural way compact foliations  $F$  and  $F_n$  of the torus  $S^1 \times S^1$ . Clearly,  $F_n \rightarrow F$  in the  $C^\infty$ -topology but  $d_H(F_n, F) = 1$  for any  $n$ ;  $d_H$  being the metric on  $\text{Fol}_c^1(S^1 \times S^1)$  determined by the standard Riemannian metric of the flat torus.

(4.2) EXAMPLE. Let us keep notations of the previous example. Put  $Z_n = X_1 + (\varphi_n \circ \text{pr}_2) \cdot X_2$ ,  $n = 1, 2, \dots$ , where  $\text{pr}_2: \mathbb{R}^2 \rightarrow \mathbb{R}$  is the canonical projection on the second factor and  $\varphi_n$  is a differentiable function with the following properties (see Fig. 1):

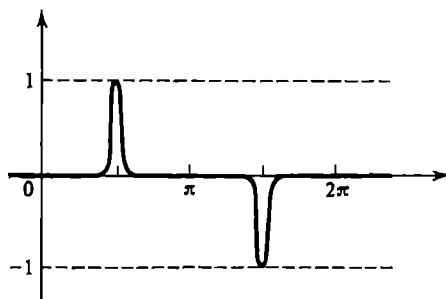


Fig. 1

- (i)  $\varphi_n: \mathbf{R} \rightarrow [-1, 1]$ ,
- (ii)  $\varphi_n(\pi/2) = 1$  and  $\varphi_n \equiv 0$  on the set  $[0, \pi/2 - 1/n] \cup [\pi/2 + 1/n, \pi]$ ,
- (iii)  $\varphi_n(\pi + t) = -\varphi_n(t)$  for  $t \in [0, \pi]$ ,
- (iv)  $\varphi_n(2\pi + t) = \varphi_n(t)$  for all  $t$  of  $\mathbf{R}$ .

Foliations determined by the vector fields  $X_1$  and  $Z_n$  induce in a natural way compact foliations  $F$  and  $\bar{F}_n$  of the torus  $S^1 \times S^1$ . One can see that  $d_H(\bar{F}_n, F) \rightarrow 0$  but the sequence  $(\bar{F}_n)$  does not converge to  $F$  in the  $C^0$ -topology.

The next theorem shows that, in spite of the above examples, the metric  $d_H$  can be of some interest.

(4.3) THEOREM. *The set of all stable compact  $p$ -dimensional foliations of a manifold  $M$  is closed in the metric space  $(\text{Fol}_p^c(M), d_H)$ .*

Proof. According to Proposition (3.5), it is sufficient to show that if  $F^n, F \in \text{Fol}_p^c(M)$ ,  $d(F^n, F) \rightarrow 0$  and  $x \in M$ , then the sequence  $(d_x^n)$  converges uniformly to  $d_x$ , where the functions  $d_x^n, d_x: M \rightarrow \mathbf{R}$  are given by

$$d_x^n(y) = d(L_x^n, L_y^n), \quad d_x(y) = d(L_x, L_y),$$

$L_x^n, L_x, L_y^n, L_y$  being leaves of  $F^n$  and  $F$  passing through  $x$  and  $y$ , respectively.

Let us take a number  $\varepsilon \in (0, 1)$  and find a natural number  $n_0$  such that  $d_H(F^n, F) < \frac{1}{2}\varepsilon$  for  $n > n_0$ . Then for any  $n > n_0$ ,  $z \in M$ ,  $z' \in L_z$  and  $z'' \in L_z^n$  we have

$$d(z', L_z^n) < \frac{1}{2}\varepsilon \quad \text{and} \quad d(z'', L_z) < \frac{1}{2}\varepsilon.$$

Consequently,

$$|d(A, L_z^n) - d(A, L_z)| < \frac{1}{2}\varepsilon$$

for any compact set  $A \subset M$ ,  $n > n_0$  and  $z$  of  $M$  (see Lemma (3.6)). From this inequality, it follows that

$$\begin{aligned} |d_x^n(y) - d_x(y)| &= |d(L_x^n, L_y^n) - d(L_x, L_y)| \\ &\leq |d(L_x^n, L_y^n) - d(L_x^n, L_y)| + |d(L_x^n, L_y) - d(L_x, L_y)| < \varepsilon \end{aligned}$$

for any  $x, y$  of  $M$ ,  $n > n_0$ . This proves the theorem.

The analogous result for  $C^r$ -topologies can not be obtained. Clearly, every unstable compact foliation of an arbitrary manifold  $M$  is a point of accumulation of the set of unstable foliations of  $M$ . However, there are unstable compact foliations which are points of accumulation of the set of stable foliations. The foliation  $F$  considered below has this property.

(4.4) EXAMPLE. Let  $M = (\mathbf{R}^2 - \{(0, 0), (1, 0), (-1, 0)\}) \times S^1$  and introduce coordinates  $(u, v, t)$  on  $M$ , where  $(u, v)$  are linear coordinates on  $\mathbf{R}^2$  and  $t$  is a parameter for  $S^1$ . The vector field

$$X = -v(r^2 - 1)^2 \frac{\partial}{\partial u} + u(r^2 - 1)^2 \frac{\partial}{\partial v} + v \frac{\partial}{\partial t},$$

where  $r^2 = u^2 + v^2$ , does not vanish at any point of  $M$ , and, therefore, generates a 1-dimensional foliation  $F$  of  $M$ . The foliation of  $N = (\mathbb{R}^2 - \{(0, 0), (1, 0), (-1, 0)\}) \times \mathbb{R}$  induced from  $F$  by the canonical projection  $f: N \rightarrow M$  is sketched on Fig. 2. Leaves of  $f^*F$  lie on cylinders  $r = \text{const}$ .

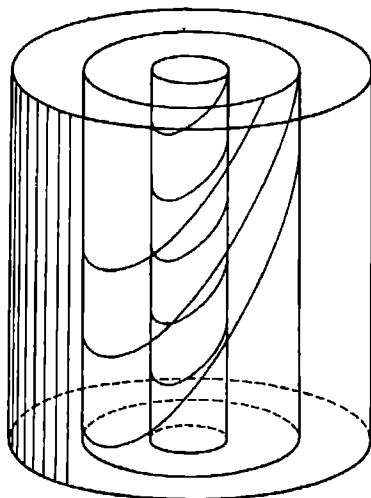


Fig. 2

For  $r = 1$  leaves are vertical lines. For  $r \neq 1$  leaves are ellipses obtained by slicing the cylinder with a family of planes transverse to the  $t$ -axis. The foliation  $F$  is compact but it is not stable. (It is a slight modification of the foliation considered in [17] and [39].)

Put

$$X_n = -v\varphi_n(r^2) \frac{\partial}{\partial u} + u\varphi_n(r^2) \frac{\partial}{\partial v} + v \frac{\partial}{\partial t},$$

where  $(\varphi_n)$ ,  $\varphi_n: \mathbb{R} \rightarrow \mathbb{R}$  is a sequence of smooth positive functions which converges to  $\varphi$ ,  $\varphi(t) = (t-1)^2$ , with respect to the strong  $C^r$ -topology for the set  $C^\infty(\mathbb{R}; \mathbb{R})$ . Clearly, foliations  $F_n$  of  $M$  generated by vector fields  $X_n$  have all leaves compact, are stable and  $F_n \rightarrow F$  with respect to the strong  $C^\infty$ -topology for the set  $\text{Fol}_c^1(M)$ . Note that  $d_H(F_n, F) \geq 2$  for any  $n$ , where  $d_H$  is the metric on  $\text{Fol}_c^1(M)$  determined by the standard Riemannian metric on  $M$ .

## 5. Jacobi fields on leaves

Let  $L$  be a compact generic leaf of a minimal foliation  $F$  of a Riemannian manifold  $(M, g)$ . Theorem (1.5) asserts that there exists a diffeomorphism  $f$  of a product  $L \times D$ , where  $D$  is an open subset of  $\mathbb{R}^q$ ,

$q = \text{codim } F$ , onto a saturated open neighbourhood  $U$  of  $L$  such that  $f^*F = \{L \times \{y\}; y \in D\}$ . The foliation  $f^*F$  is minimal with respect to the Riemannian metric  $f^*g$ . The vector fields  $V_i$ ,  $i = 1, \dots, q$ , given by  $V_i(x, 0) = (0, v_i)$ , where  $x \in L$  and  $v_1, \dots, v_q$  is a basis of the tangent space  $T_0D$ , satisfy the equalities

$$V_i(x, 0) = (t \mapsto f_i(x, t))'(0),$$

where  $f_i$  is a mapping of  $L \times R$  into  $L \times D$  defined by

$$f_i(x, t) = (x, c_i(t))$$

with  $c_i: R \rightarrow D$  satisfying  $c_i(0) = 0$  and  $c_i'(0) = v_i$ . The immersions  $f_i(\cdot, t)$  are minimal. Therefore, the normal components  $X_i$  of fields  $V_i$  are Jacobi fields on  $L \times \{0\}$ . Coming back to the manifold  $M$  we get the following

(5.1) THEOREM. *The normal bundle of a compact generic leaf of a minimal foliation is spanned by pointwise linearly independent Jacobi fields.*

The remaining part of this section is devoted to some applications of the above theorem. At first, foliations of manifolds of non-positive and non-negative sectional curvature will be considered.

(5.2) THEOREM. *Let  $F$  be a totally geodesic foliation of a non-positively curved Riemannian manifold  $(M, g)$ . If  $L$  is a compact leaf of  $F$ , then any Jacobi field  $X$  on  $L$  satisfies  $\nabla X = 0$  and  $\langle \text{Ric}' X, X \rangle = 0$ . If all leaves of  $F$  are compact, then  $\langle \text{Ric}' v, v \rangle = 0$  for any  $v$  of  $NF$ . Consequently, if the sectional curvature of  $(M, g)$  is strictly negative at one point of  $M$ , then  $M$  does not admit a compact totally geodesic foliation.*

Proof. The first statement of the theorem follows from (5) and (6) in the following way:

$$0 \leq \int_L \langle \nabla X, \nabla X \rangle = - \int_L \langle \nabla^2 X, X \rangle = - \int_L \langle \text{Ric}' X, X \rangle \leq 0.$$

The second statement follows from the first one, Theorem (5.1) and Lemma (1.4). The last part of the theorem is clear now.

(5.3) THEOREM. *Let  $F$  be a codimension-one foliation of a non-negatively curved Riemannian manifold  $(M, g)$ . If  $F$  is minimal and  $L$  is a compact generic leaf of  $F$ , then  $L$  is a totally geodesic submanifold of  $(M, g)$  and  $\langle \text{Ric}' v, v \rangle = 0$  for any  $v$  of  $NL$ . Consequently, if the sectional curvature of  $(M, g)$  is positive at one point of  $M$ , then  $M$  does not admit a compact minimal foliation of codimension one. Conversely, if  $M$  is compact,  $L$  is a totally geodesic leaf of  $F$  and  $\dim H^1(M, \mathbb{R}) \leq 1$ , then  $F$  has a compact totally geodesic leaf.*

Proof. Theorem (5.1) asserts that if  $F$  is a minimal foliation, then there exists a Jacobi field  $X$  on  $L$  which does not vanish at any point of  $L$ . Let  $V$

be a unit section of the normal bundle of  $L$ . Then  $X = f \cdot V$ , where  $f$  is a strictly positive (or: negative) function. According to (6), we get

$$\begin{aligned} 0 &= \int_L \langle -\nabla^2 X + \text{Ric}' X - A'(X), V \rangle \\ &= \int_L \langle \nabla X, \nabla V \rangle + \int_L f \langle \text{Ric}' V, V \rangle - \int_L f \langle A^V, A^V \rangle, \end{aligned}$$

where  $A$  is the second fundamental form of  $L$ . The first term vanishes because  $\nabla V = 0$  (codim  $F = 1$ !). The remaining two terms are non-positive and their sum is zero. Therefore,  $A^V = 0$  and  $\langle \text{Ric}' V, V \rangle = 0$ .

Assume now that  $L$  is totally geodesic. Then the sectional curvature of  $L$  is non-negative. Let us take a point  $x$  of  $L$  and denote by  $V_L(r)$  the volume of the ball  $B(x, r) \subset L$  with the radius  $r$  and the centre  $x$ . Then, according to Bishop [4] (see also [5] and [58]),

$$V_L(r) \leq c \cdot r^p, \quad p = \dim F,$$

for some constant  $c$ . It follows that

$$\lim_{r \rightarrow \infty} \inf \frac{1}{r} \log V_L(r) = 0,$$

i.e. that  $L$  has a non-exponential growth. Plante [44] proved that the closure of a leaf  $L$  with non-exponential growth contains a leaf  $L'$  of polynomial growth of degree  $k \leq \max(0, \dim H^1(M, \mathbf{R}) - 1)$ , i.e. that there exists a polynomial  $P$  of degree  $k$  such that

$$V_{L'}(r) \leq P(r)$$

for all  $r > 0$ . If  $\dim H^1(M, \mathbf{R}) \leq 1$ , then  $k = 0$ , i.e. the volume of  $L$  is finite and  $L$  is compact.

The last part of the above theorem can be generalized as follows. From the Gauss equation (1), it follows that the Ricci tensor  $\overline{\text{Ric}}$  of a  $p$ -dimensional submanifold  $L$  of a Riemannian manifold  $(M, g)$  satisfies

$$\begin{aligned} \overline{\text{Ric}}(u, v) &= \sum_{i \leq p} \langle R(e_i, u)v, e_i \rangle + \langle H(x), B(u, v) \rangle - \\ &\quad - \sum_{i \leq p} \langle B(u, e_i), B(v, e_i) \rangle, \end{aligned}$$

where  $u, v \in T_x L$ ,  $e_1, \dots, e_p$  is an orthonormal frame of  $T_x L$ ,  $H$  and  $B$  are the mean curvature vector and the second fundamental form of  $L$ , respectively. From this formula, it follows that if the sectional curvature of  $(M, g)$  is non-negative, codim  $L = 1$ , and the second fundamental form of  $L$  is semi-definite at any point of  $L$ , then the Ricci curvature of  $L$  is non-negative. Applying once again Bishop's and Plante's results we get

(5.4) PROPOSITION. Let  $(M, g)$  be a compact Riemannian manifold of non-negative sectional curvature,  $H^1(M, \mathbf{R}) \approx \mathbf{R}$ . If  $F$  is a codimension-one foliation of  $M$ , then any leaf of  $F$  with the semi-definite second fundamental form contains a compact leaf in its closure.

Comparing Corollary (2.10) and Theorem (5.3) we obtain the following

(5.5) COROLLARY. Any compact orientable flat Riemannian manifold  $(M, g)$  with  $H^1(M, \mathbf{R}) \approx \mathbf{R}$  admits an orientable compact totally geodesic foliation of codimension one.

We start with a series of results providing estimations of dimensions of minimal foliations of some Riemannian manifolds.

(5.6) PROPOSITION. If  $S^p$  is a generic leaf of a minimal foliation of  $S^m$ , then either  $p = m$  or  $p \leq \frac{1}{2}(m-1)$ .

PROOF. Put  $q = m - p$ . According to Theorem (5.1), it suffices to prove that if  $p+1 > q > 0$ , then any Jacobi fields  $X_1, \dots, X_q$  on  $S^p$  are linearly dependent at some points of  $S^p$ . Let  $V_1, \dots, V_q$  be sections of the normal bundle of  $S^p$  in  $S^m$  such that  $\nabla V_i = 0$  and  $\langle V_i, V_j \rangle = \delta_{ij}$  for all  $i, j \leq q$ . The linear combination  $\sum_i g_i V_i$  is a Jacobi field if and only if  $g_i$  are restrictions to  $S^p$  of linear functions on  $\mathbf{R}^{p+1}$ . Jacobi fields  $X_i = \sum_j g_i^j V_j$ ,  $i = 1, \dots, q$ , where  $g_i^j = f_i^j|_{S^p}$  and  $f_i^j: \mathbf{R}^{p+1} \rightarrow \mathbf{R}$  are linear functions, are linearly independent at a point  $x$  of  $S^p$  if and only if

$$\det [f_i^j(x); i, j \leq q] \neq 0.$$

However, if  $p+1 > q > 0$ , then the intersection of the kernels  $\ker f_i^j$ , where  $i$  is fixed and  $j = 1, \dots, q$ , contains non-zero elements of  $\mathbf{R}^{p+1}$ . Clearly,  $\det [f_i^j(x); i, j \leq q] = 0$  at any point  $x$  of this intersection. This proves the proposition.

The inequality in (5.6) can be improved by use of some advanced results due to Allard and Ferus. If  $F$  is a  $p$ -dimensional minimal foliation of  $S^m$  admitting a totally geodesic generic leaf  $L$ , then there exists an open saturated subset  $U$  of  $S^m$ ,  $L \subset U$ , such that the restricted foliation  $F|U$  is of the form  $f^*F'$ , where  $F'$  is the foliation of the product  $L \times D$ ,  $D \subset \mathbf{R}^{m-p}$ , by the leaves  $L \times \{u\}$ ,  $u \in D$ , and  $f: U \rightarrow L \times D$  is a diffeomorphism. Therefore, the volume function is constant on  $U$  and, according to Allard ([1], Sec. 8), all leaves of  $F|U$  are totally geodesic. Applying results of Ferus [24] we obtain the following

(5.7) THEOREM. If  $S^p$  is a generic leaf of a minimal foliation of  $S^m$ , then either  $p = m$  or  $p \leq k_m$ , where  $k_m$  is the largest integer  $k$  satisfying the inequality

$$k \leq a(m-k)$$

with  $a((2j+1) \cdot 2^{4l+h}) = 2^h + 8l - 1$  for  $h = 0, 1, 2, 3, j, l = 0, 1, \dots$

Reading the proof of Proposition (5.6) one can see that the arguments are the following: Any Jacobi field  $X$  on  $S^p$  is the normal component of a Killing vector field  $V$  on  $S^m$  and if  $p+1 > q > 0$  and  $V_1, \dots, V_q$  are Killing fields on  $S^{p+q}$ , then some linear combination  $\sum_i a_i V_i$ ,  $a_i \in \mathbb{R}$ ,  $\sum_i a_i^2 > 0$ , is tangent to  $S^p$  at some points. For the geometric interpretation of the last fact in the case  $p = q = 1$  see Fig. 3: The Killing vector field marked with the aid of arrows is tangent to the equator  $S^1$  of  $S^2$  at the point  $x_0$ .

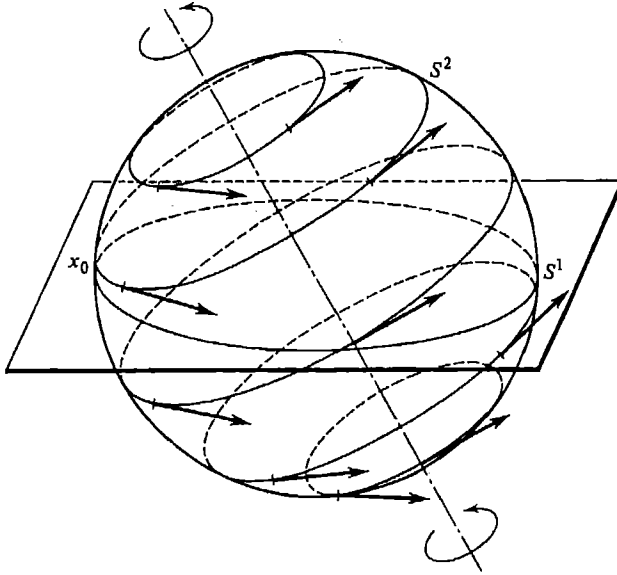


Fig. 3

It seems to be reasonable to ask about the analogous facts for foliations of other symmetric spaces. Up to now, we have obtained such results for complex and quaternionic projective space, i.e. for symmetric spaces of the rank one. We formulate and prove our result in the case of the complex projective space  $P^m\mathbb{C}$  with the natural Riemannian structure of a symmetric space. The result and argumentation for the quaternionic projective space  $P^m\mathbb{Q}$  is similar.

(5.8) THEOREM. *If  $P^p\mathbb{C}$  is a leaf of a minimal foliation of  $P^m\mathbb{C}$ , then either  $p = m$  or  $p \leq \frac{1}{2}(m-1)$ .*

The proof of the theorem is broken into several lemmas.

(5.9) LEMMA. *If  $f: P^p\mathbb{C} \times \mathbb{R} \rightarrow P^m\mathbb{C}$  is a mapping such that  $f_0$  is a totally geodesic immersion and all  $f_t$  are minimal immersions of  $P^p\mathbb{C}$  into  $P^m\mathbb{C}$ , then there exists a number  $\varepsilon > 0$  such that for any  $t$  satisfying  $|t| < \varepsilon$  the immersion  $f_t$  is totally geodesic.*

Proof. Let  $s_t$  be the scalar curvature of  $f_t(P^pC)$ . There exists a number  $\varepsilon > 0$  such that

$$(20) \quad s_0 - s_t \leq \frac{2(2p+1)(m-p)}{4(m-p)-1} \quad \text{for } |t| < \varepsilon.$$

Lawson [37] proved that the inequality (20) implies the total geodesy of  $f_t$ .

Remark. If all  $f_t$  are holomorphic immersions, then the statement of Lemma (5.9) can be obtained in another way. Namely, we can show that there exists a positive number  $\delta$  such that any compact complex submanifold  $L$  of  $P^mC$  with the second fundamental form  $A$  satisfying  $\|A\|^2 < \delta$  is totally geodesic and then we can proceed as in the above proof of the lemma.

In order to prove the result announced above let us remind the Simons expression of  $\nabla^2 A$  (see [53]). Let  $L$  be any minimal submanifold of an arbitrary Riemannian manifold  $(M, g)$ . Let  $e_1, \dots, e_p$ ,  $p = \dim L$ , be an orthonormal frame of  $T_x L$ ,  $x \in L$ . Take vectors  $u, v$  of  $T_x L$ ,  $w$  of  $N_x L$ , and put

$$\langle R^w(u, v) \rangle = \sum_i \langle (\nabla_u R)(e_i, v)e_i, w \rangle + \sum_i \langle (\nabla_{e_i} R)(e_i, u)v, w \rangle$$

and

$$\begin{aligned} \langle R(A)^w u, v \rangle = & \sum_i \{ 2 \langle R(e_i, v)B(u, e_i), w \rangle + 2 \langle R(e_i, u)B(v, e_i), w \rangle - \\ & - \langle A^w u, R(e_i, v)e_i \rangle - \langle A^w v, R(e_i, u)e_i \rangle + \\ & + \langle R(e_i, B(u, v))e_i, w \rangle - 2 \langle A^w e_i, R(e_i, u)v \rangle \}. \end{aligned}$$

Then

$$\nabla^2 A = -A \circ A' - A'' \circ A + R(A) + R',$$

where  $A'$  is defined by (4) and  $A''(s) = \text{Trace}((w, w') \mapsto [A^w [A^{w'}, s]])$  for any linear transformation  $s$  of  $T_x L$ ,  $x \in L$ . In the case of a complex submanifold  $L$  of the complex projective space,  $R' = 0$  because of the symmetry of  $P^mC$  and

$$\langle R(A), A \rangle = \left(\frac{3}{2}p - \frac{1}{2}\right) \|A\|^2,$$

where  $p$  denotes the complex dimension of  $L$ . The last equality can be obtained by the direct but long calculation involving the definition of  $R(A)$  and the following formula for the curvature tensor of a Kähler manifold  $(M, g, J)$  of the constant holomorphic curvature  $c$  (see [36], Volume II, p. 167):

$$(21) \quad \langle R(u, v)w, z \rangle = \frac{c}{4} \{ \langle u, z \rangle \langle v, w \rangle - \langle u, w \rangle \langle v, z \rangle + \langle u, Jz \rangle \langle v, Jw \rangle - \\ - \langle u, Jw \rangle \langle v, Jz \rangle - 2 \langle u, Jv \rangle \langle w, Jz \rangle \}.$$

Using these facts and the inequality

$$\langle A \circ A' + A'' \circ A, A \rangle \leq \frac{1}{2} q^{-1} (4q - 1) \cdot \|A\|^4,$$

where  $q$  is the complex codimension of  $L$  (see [53], p. 93), we get

$$0 \leq - \int_L \langle \nabla^2 A, A \rangle \leq \int_L \|A\|^2 \left( -\frac{3}{2} p + \frac{1}{2} + \frac{1}{2} q^{-1} (4q - 1) \|A\|^2 \right).$$

It follows that if  $\|A\|^2 < \delta$ , where  $\delta = q(3p - 1)/(4q - 1)$ , then  $A = 0$ .

(5.10) LEMMA. *Every Jacobi field on  $P^p C$  is a normal component of a Killing vector field on  $P^m C$ .*

Proof. Let  $X$  be a Jacobi field on  $P^p C$ . Then there exists a mapping  $f: P^p C \times \mathbb{R} \rightarrow P^m C$  satisfying the conditions of Definition (1.7). From Lemma (5.9), it follows that  $f_t$  is a totally geodesic immersion for  $t$  sufficiently close to zero. Therefore, there exists a differentiable curve  $(-\varepsilon, \varepsilon) \ni t \mapsto F_t$  on the group  $G$  of isometries of  $P^m C$  such that  $F_0 = \text{id}$  and  $f_t = F_t|_{P^p C}$  for  $|t| < \varepsilon$ .

Such curve can be constructed as follows: Take a point  $x_0$  of  $P^p C$  and an orthonormal frame  $w_1, \dots, w_q, Jw_1, \dots, Jw_q$ ,  $q = m - p$ , of the orthogonal complement of the tangent space of  $P^p C$  at  $x_0$ . Extend vectors  $w_i$  to orthonormal vector fields  $W_i$  on a neighbourhood of  $x_0$  in such a manner that vectors  $W_i(f_t(x_0))$  belong to the orthogonal complement of the tangent space of the submanifold  $f_t(P^p C)$  at  $f_t(x_0)$ . Define linear transformations

$$A_t: T_{x_0} P^m C \rightarrow T_{f_t(x_0)} P^m C$$

putting  $A_t = f_{t*}$  on  $T_{x_0} P^p C$ ,  $A_t(w_i) = W_i(f_t(x_0))$ , and  $A_t(Jw_i) = JW_i(f_t(x_0))$ .  $A_t$  are orthogonal transformations such that  $A_t \circ J = J \circ A_t$ . From (21) and the well known facts about isometries of Riemannian manifolds, it follows that  $A_t = F_{t*}$  for a unique isometry  $F_t$  of  $P^m C$  (see [36], Volume I, Chapter VI).

Put  $u = (t \mapsto F_t)'(0)$ ;  $u$  is an element of the tangent space  $T_e G$ , where  $G$  is the group of isometries of  $P^m C$ . Let  $h_t = \exp(tu)$  and denote by  $V$  the Killing vector field on  $P^m C$  generated by the subgroup  $(h_t)$  of  $G$ . Then we have at any point  $x$  of  $P^p C$

$$\begin{aligned} V(x) &= (t \rightarrow h_t(x))'(0) = \tau_{x*}((t \rightarrow h_t)'(0)) \\ &= \tau_{x*}(u) = \tau_{x*}((t \rightarrow F_t)'(0)) \\ &= (t \rightarrow F_t(x))'(0) = (t \rightarrow f_t(x))'(0), \end{aligned}$$

where  $\tau_x$  is the mapping of  $G$  into  $P^m C$  given by  $\tau_x(h) = h(x)$  for  $h$  of  $G$ . Therefore,  $V^\perp = X$ .

Remark. The similar result in the case of Jacobi fields along geodesics on arbitrary symmetric spaces can be found in [7], where the authors proved the following. Let  $M = G/K$  be a symmetric space,  $c: [0, 1] \rightarrow M$  a geodesic such that the vector  $\dot{c}(0)$  is orthogonal to the orbit  $K \cdot c(0)$ . Any Jacobi field

$X$  along  $c$  obtained by a variation of  $c$  through geodesics  $c_t$ , orthogonal to the orbits  $K \cdot c_t(0)$  is of the form  $V \circ c$  for some Killing vector field  $V$  on  $M$ , if only  $X(0)$  and  $X(1)$  are tangent to the orbits  $K \cdot c(0)$  and  $K \cdot c(1)$ , respectively.

(5.11) LEMMA. *Let  $M = G/K$  be a homogeneous space, where  $G$  is a Lie group acting on  $M$  as a group of isometries and  $K$  is a closed subgroup of  $G$ . Denote by  $\pi$  the canonical projection of  $G$  onto  $M$ . Let  $H$  be another closed subgroup of  $G$ . The Killing vector field  $X$  corresponding to an element  $X$  of  $\mathfrak{g}$  the Lie algebra of  $G$  is tangent to the orbit  $H \cdot \pi(e)$  at the point  $h \cdot \pi(e)$ ,  $h \in H$ , if and only if  $X \in \mathfrak{h} + \text{ad}(h^{-1})\mathfrak{k}$ , where  $\mathfrak{h}$  and  $\mathfrak{k}$  are the Lie algebras of  $H$  and  $K$ , respectively.*

Proof. Put  $L = H \cdot \pi(e)$ ,  $x = h \cdot \pi(e)$ . The tangent space  $T_x L$  consists of vectors of the form  $Y^*(x)$ , where  $Y$  belongs to  $\mathfrak{h}$ . The field  $X^*$  is tangent to  $L$  at  $x$  if and only if the field  $(Y - X)^*$  vanishes at  $x$  for some  $Y$  of  $\mathfrak{h}$ . The equality  $(X - Y)^*(x) = 0$  holds if and only if the curve

$$t \mapsto h^{-1} \exp(t(X - Y))h$$

lies on  $K$ , i.e. if  $X - Y \in \text{ad}(h^{-1})\mathfrak{k}$ .

We will apply Lemma (5.11) to our situation. We have  $M = P^m C$ ,  $L = P^p C$ ,  $G = \text{SU}(m+1)$ ,  $\mathfrak{g} = \mathfrak{su}(m+1)$ ,

$$H = \left\{ \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \in \text{SU}(m+1); A \in \text{SU}(p+1) \right\},$$

$$\mathfrak{h} = \left\{ \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{su}(m+1); Y \in \mathfrak{su}(p+1) \right\},$$

$$K = \left\{ \begin{bmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{bmatrix}; A \in U(m) \right\},$$

$$\mathfrak{k} = \left\{ \begin{bmatrix} -\text{Tr } Y & 0 \\ 0 & Y \end{bmatrix}; Y \in \mathfrak{u}(m) \right\}.$$

(5.12) LEMMA. *If  $Z_1, Z_2$  are arbitrary elements of  $\mathfrak{su}(m+1)$  and  $p \geq \frac{1}{2}m$ , then some non-trivial linear combination of  $Z_1$  and  $Z_2$  belongs to the union  $\bigcup_{h \in H} (\mathfrak{h} + \text{ad}(h)\mathfrak{k})$ .*

Proof. If  $Z_1, Z_2 \in \mathfrak{su}(m+1)$ , then there are real numbers  $\beta_1, \beta_2$ ,  $|\beta_1| + |\beta_2| > 0$ , such that the matrix  $Z = \beta_1 Z_1 + \beta_2 Z_2$  is of the form

$$Z = \begin{bmatrix} Y & U \\ -{}^t \bar{U} & X \end{bmatrix},$$

where  $Y \in \mathfrak{su}(p+1)$  and  $X \in \mathfrak{su}(m-p)$ . Let

$$U = [u_{jk}; j = 1, \dots, p+1, k = 1, \dots, m-p].$$

The condition  $p \geq \frac{1}{2}m$  yields the inequality  $\text{rank } U \leq p$  which asserts that there exists a matrix  $A = [a_{ij}] \in \text{SU}(p+1)$  such that

$$\text{rank} \begin{bmatrix} a_{12} & \cdots & a_{1,p+1} & u_{11} & \cdots & u_{1,m-p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p+1,2} & \cdots & a_{p+1,p+1} & u_{p+1,1} & \cdots & u_{p+1,m-p} \end{bmatrix} = p.$$

Therefore, there exists a matrix

$$X' = [x_{ij}; i = 2, \dots, p+1, j = 1, \dots, m-p]$$

such that

$$a_{12}x_{2j} + \cdots + a_{i,p+1}x_{p+1,j} = u_{ij}$$

for all  $i \leq p+1, j \leq m-p$ . The matrix

$$W = \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 | -'X' & X \end{array} \right]$$

belongs to  $\mathfrak{f}$  and

$$Z = \left[ \begin{array}{cc} Y & 0 \\ 0 & 0 \end{array} \right] + \text{ad}(h)W,$$

where

$$h = \left[ \begin{array}{cc} A & 0 \\ 0 & I \end{array} \right] \in H.$$

This proves the lemma.

Theorem (5.8) follows immediately from the above lemmas and Theorem (5.1).

Applying once again Theorem (5.1) and known facts concerning Jacobi fields on some minimal submanifolds of spheres we obtain the following results.

(5.13) THEOREM. *If  $S^p$  is a generic leaf of a minimal foliation of  $S^m$ , then the number  $m-p$  is even.*

PROOF. According to Theorem (5.1), it is sufficient to prove that if  $m-p$  is odd, then every  $m-p$  Jacobi fields on  $S^p$  are linearly dependent at some points.

Let us take such fields  $V_1, \dots, V_{m-p}$ . There are Killing vector fields  $X_1, \dots, X_{m-p}$  such that  $V_i$  are the normal components of  $X_i$ . Vectors  $V_1(x), \dots, V_{m-p}(x)$ , where  $x = (x_0, \dots, x_p, 0, \dots, 0) \in S^p \subset S^m$ , are linearly dependent if and only if

$$(22) \quad f(x) = \det [\langle X_i(x), e_j \rangle; i = 1, \dots, m-p, j = p+1, \dots, m] = 0,$$

where  $e_j = (\delta_j^0, \dots, \delta_j^m)$ . One can see that  $f$  is a homogeneous polynomial of degree  $m-p$ . If  $m-p$  is odd, then  $f(-x) = -f(x)$  and, therefore, the equation (22) has a solution on  $S^p$ . This proves the theorem.

(5.14) THEOREM. *If  $S^k \times S^l$  embedded into  $S^m$  by the mapping*

$$S^k \times S^l \ni (x_0, \dots, x_k, y_0, \dots, y_l) \\ \mapsto \left( \frac{k}{k+l} x_0, \dots, \frac{k}{k+l} x_k, \underset{k \text{ times}}{0}, \dots, 0, \frac{l}{k+l} y_0, \dots, \frac{l}{k+l} y_l, \underset{l \text{ times}}{0}, \dots, 0 \right)$$

is a generic leaf of a minimal foliation of  $S^m$ , then the number  $q = m - (k+l)$  is odd.

Proof. Let us take Jacobi fields  $V_1, \dots, V_q$  on  $S^k \times S^l$ . From the recent results due to Allard and Almgren [2], it follows that we can find Killing vector fields  $X_1, \dots, X_q$  on  $S^m$  such that  $V_i$  are the normal components of  $X_i$ . Vectors  $V_1(x), \dots, V_q(x)$ , where  $x = (x_0, \dots, x_k, 0, \dots, 0, y_0, \dots, y_l, 0, \dots, 0) \in S^k \times S^l$  are linearly dependent if and only if

$$(23) \quad f(x) = \det [\langle X_i(x), N_j(x) \rangle; i, j = 1, \dots, q] = 0,$$

where  $N_1, \dots, N_q$  are sections of the normal bundle of  $S^k \times S^l$  in  $S^m$  given by

$$N_1 = \frac{\partial}{\partial x_{k+1}}, \dots, N_{k'} = \frac{\partial}{\partial x_{k+k'}}, N_{k'+1} = \frac{\partial}{\partial y_{l+1}}, \dots, N_{k'+l'} = \frac{\partial}{\partial y_{l+l'}}$$

and

$$N_q(x) = \frac{l}{k+l} \left( x_0 \frac{\partial}{\partial x_0} + \dots + x_k \frac{\partial}{\partial x_k} \right) - \frac{k}{k+l} \left( y_0 \frac{\partial}{\partial y_0} + \dots + y_l \frac{\partial}{\partial y_l} \right).$$

It is easy to see that  $f$  is a homogeneous polynomial of degree  $q+1$ . If  $q$  is even, then the equation (23) has a solution on  $S^k \times S^l$ . The theorem is proved.

It would be interesting to get the estimation of the codimension  $q$  assuming that  $S^k \times S^l$  is a generic leaf of a minimal foliation of  $S^m$ . Methods used in the proof of Proposition (5.6) can not be applied. Reading the proof of the last theorem one can see that the problem can be reduced to the following: Prove that if  $q$  is sufficiently small, then the system of equations

$$(24) \quad \begin{cases} f(x_0, \dots, x_k, y_0, \dots, y_l) = 0, \\ x_0^2 + \dots + x_k^2 = k/(k+l), \\ y_0^2 + \dots + y_l^2 = l/(k+l), \end{cases}$$

where  $f$  is a homogeneous polynomial of degree  $q+1$ , has a solution. Of course, there are homogeneous polynomials such that the system (24) has no

solutions. However, we can get more informations about  $f$ . For example, if  $q = 1$ , then  $f$  is of the form

$$\sum_{i=0}^k \sum_{j=0}^l a_{ij} x_i y_j$$

and the system (24) has a solution. Therefore, the product  $S^k \times S^l$  can not become a leaf of a minimal foliation of  $S^{k+l+1}$ . This fact follows also directly from Proposition (2.3). If  $q > 1$ , then the structure of  $f$  is much more complicated and we are not able to get any information concerning the existence of solutions of (24).

**Remark.** Known examples of Riemannian submersions with totally geodesic fibres (see [22] and [23]) show that the estimations of dimensions of minimal foliations of  $S^m$  and  $P^m C$  obtained in Proposition (5.6) and Theorem (5.8) are best possible for some values of  $m$ . For example, if  $m = 3, 7$  or  $15$ , then  $S^p$ ,  $p = \frac{1}{2}(m-1)$ , is a leaf of a totally geodesic foliation of  $S^m$ . Similarly,  $P^1 C$  is a leaf of a totally geodesic foliation of  $P^3 C$ .

## 6. The Gauss mapping of a foliation

Let  $F$  be a  $p$ -dimensional foliation of a Riemannian manifold  $(M, g)$ . For any differentiable curve  $c: [0, 1] \rightarrow M$  the parallel displacement  $\tau_c$  of tangent vectors is a linear isomorphism of  $T_x M$  onto  $T_y M$ , where  $x = c(0)$  and  $y = c(1)$ . This isomorphism generates a diffeomorphism, denoted still by  $\tau_c$ , of the Grassmann manifold  $G_x^p$  of  $p$ -dimensional linear subspaces of  $T_x M$  onto the corresponding  $G_y^p$ . The holonomy group  $H(x)$  of  $(M, g)$  at a point  $x$  of  $M$  consists of all isomorphisms of the form  $\tau_c$ , where  $c: [0, 1] \rightarrow M$ ,  $c(0) = c(1) = x$ . Therefore,  $H(x)$  acts on the Grassmann manifold  $G_x^p$ . Put  $G_x^p = G_x^p / H(x)$ .  $G_x^p$  carries always the natural structure of a topological space and in some cases the structure of a differentiable manifold. Let  $\text{pr}: G_x^p \rightarrow G_x^p$  be the canonical projection.

(6.1) **DEFINITION.** The *Gauss mapping* of  $F$  is a mapping  $\gamma: M \rightarrow G_x^p$  defined by

$$\gamma(y) = \text{pr}(\tau_c(T_y F)),$$

where  $c$  is an arbitrary differentiable curve on  $M$  joining  $y$  to  $x$ .

The above definition is correct, i.e. if  $c_1$  and  $c_2$  are two curves joining  $y$  to  $x$ , then  $\text{pr}(\tau_{c_1}(T_y F)) = \text{pr}(\tau_{c_2}(T_y F))$ . In fact,

$$\tau_{c_1}(T_y F) = h(\tau_{c_2}(T_y F)),$$

where  $h = \tau_{c_1 c_2^{-1}} \in H(x)$ .

The mapping  $\gamma$  is continuous and if  $G_x^p$  carries the structure of a differentiable manifold, then it is smooth.

The Gauss mapping corresponds to a point  $x$ . However, if we take another point  $y$  of  $M$  and denote by  $\gamma'$  the Gauss mapping of  $F$  corresponding to  $y$ , then

$$(25) \quad \gamma' = \tau_{c_0} \circ \gamma,$$

where  $c_0$  is a differentiable curve joining  $x$  to  $y$  and  $\tau_{c_0}: G_x^p \rightarrow G_y^p$  is a homeomorphism given by  $\tau_{c_0}(H(x) \cdot A) = H(y) \cdot \tau_{c_0}(A)$  for all  $A$  of  $G_x^p$ . In fact, if  $\text{pr}': G_y^p \rightarrow G_y^p$  is the projection,  $z \in M$ , and  $c$  is a curve joining  $z$  to  $y$ , then

$$\begin{aligned} \gamma'(z) &= \text{pr}'(\tau_{c_0}(\tau_{c_0}^{-1}c(T_z F))) \\ &= \tau_{c_0}(\text{pr}(\tau_{c_0}^{-1}c(T_z F))) = \tau_{c_0}(\gamma(z)). \end{aligned}$$

(6.2) PROPOSITION. *The Gauss mapping  $\gamma$  of a totally geodesic foliation splits into the composition*

$$M \xrightarrow{\pi} M/F \rightarrow G_x^p,$$

where  $\pi$  is the canonical projection.

PROOF. It is sufficient to prove that  $\gamma$  is constant on leaves of  $F$ . If  $z$  is a point of the leaf  $L_x$  passing through  $x$ , then  $\gamma(z) = H(x) \cdot T_x F$  because the parallel displacement along any curve  $c$  on  $L_x$  maps the space  $T_{c(0)}L$  onto  $T_{c(1)}L$ . Thus,  $\gamma$  is constant on  $L_x$ . If  $L_y$ ,  $y \in L_y$ , is another leaf of  $F$ , then the Gauss mapping  $\gamma'$  corresponding to  $y$  is constant on  $L_y$ . From (25), it follows that  $\gamma$  is also constant on  $L_y$ .

The Gauss mapping of  $F$  depends on the Riemannian metric  $g$  on  $M$ . We will deform  $g$  and consider Gauss mappings corresponding to different Riemannian metrics.

(6.3) DEFINITION. Let  $\mathcal{R} = \mathcal{R}(M)$  be the space of all Riemannian metrics on  $M$  with the topology of uniform convergence with all derivatives on compact sets. Choose a point  $x$  of  $M$  and a subgroup  $H$  of the group  $\text{Gl}(T_x M)$  of all linear isomorphisms of the tangent space of  $M$  at  $x$ . We will denote by  $\mathcal{R}_H$  the set of all Riemannian metrics  $g$  on  $M$  such that the holonomy group  $H(x)$  of  $g$  at  $x$  is contained in  $H$ . An  $H$ -deformation of a Riemannian metric  $g$  of  $\mathcal{R}_H$  is a continuous curve  $[0, 1] \ni t \mapsto g_t \in \mathcal{R}_H$  such that  $g_0 = g$ . Two metrics  $g$  and  $g'$  of  $\mathcal{R}_H$  are  $H$ -equivalent if and only if there exists an  $H$ -deformation  $(g_t)$  such that  $g_0 = g$  and  $g_1 = g'$ .

If  $F$  is a foliation of a Riemannian manifold  $(M, g)$ ,  $x \in M$ , and the holonomy group  $H(x)$  is a subgroup of  $H$ , then the Gauss mapping  $\gamma: M \rightarrow G_x^p$  induces a mapping  $\gamma^H: M \rightarrow G_H^p = G_x^p/H$ .

(6.4) PROPOSITION. *If Riemannian metrics  $g$  and  $g'$  are  $H$ -equivalent, then the mappings  $\gamma^H$  and  $\gamma'^H$  corresponding to  $g$  and  $g'$ , respectively, are homotopic.*

Proof. Let  $(g_t)$  be an  $H$ -deformation joining  $g$  to  $g'$ . Denote by  $\gamma_t$  the Gauss mapping of a foliation  $F$  corresponding to  $g_t$ . The mapping  $\Gamma: [0, 1] \times M \rightarrow \mathcal{G}_H^p$  given by  $\Gamma(t, z) = \gamma_t^H(z)$  provides the required homotopy.

From Propositions (6.2) and (6.4), we get the following result.

(6.5) THEOREM. *Let  $F$  be a foliation of a Riemannian manifold  $(M, g)$ ,  $x \in M$ . If  $H$  is a subgroup of  $\text{Gl}(T_x M)$  and the metric  $g$  is  $H$ -equivalent to a metric  $g'$  such that leaves of  $F$  are totally geodesic submanifolds of  $(M, g')$ , then the mapping  $\gamma^H: M \rightarrow \mathcal{G}_H^p$  is homotopic to the composition*

$$M \xrightarrow{\pi} M/F \rightarrow \mathcal{G}_H^p.$$

The following example shows that the space  $\mathcal{R}_H$  need not be arcwise connected.

(6.6) EXAMPLE. Let  $M = \mathbb{R}^2 - \{0\}$  and  $F$  be the foliation of  $M$  by circles centred at the origin. Denote by  $(u^1, u^2)$  the Euclidean coordinates on  $M$  and put

$$g = (du^1)^2 + (du^2)^2, \quad g' = ((u^1)^2 + (u^2)^2)^{-1} \cdot g.$$

Leaves of  $F$  are geodesics of the Riemannian metric  $g'$ . Both metrics  $g$  and  $g'$  have trivial holonomy, i.e. belong to  $\mathcal{R}_H$ , where  $H$  is the trivial subgroup of  $\text{Gl}(T_x M)$ ,  $x \in M$ . The Gauss mapping  $\gamma'$  of  $F$  corresponding to  $g'$  splits into the composition  $M \rightarrow M/F \rightarrow \mathcal{G}_x^1$ . The space  $M/F \approx \mathbb{R}$  is contractible. Therefore,  $\gamma'$  is homotopic to a constant mapping. (In fact, it is constant according to the next proposition.) On the other hand, if  $L$  is a leaf of  $F$ , then the restriction  $\gamma|_L$  of the Gauss mapping  $\gamma$  of  $F$  corresponding to  $g$  is a two-fold covering of  $\mathcal{G}_x^1 \approx S^1$ . Therefore,  $\gamma$  is not homotopic to a constant mapping and, consequently, the metrics  $g$  and  $g'$  are not  $H$ -equivalent.

(6.7) PROPOSITION. *The Gauss mapping of a parallel foliation of a flat Riemannian manifold is constant.*

Proof. Let  $F$  be a parallel foliation of a flat Riemannian manifold  $(M, g)$ . The normal bundle of  $F$  is involutive. Denote by  $F^\perp$  the foliation of  $M$  with  $TF^\perp = NF$ . Covering  $M$  by a family of charts distinguished by  $F$  and  $F^\perp$  simultaneously it is easy to show that an arbitrary curve  $c: [0, 1] \rightarrow M$  is homotopic to the composition  $c_1 c_2 \dots c_{2k-1} c_{2k}$ , where  $c_i$  are curves on  $M$  such that

$$\dot{c}_{2j-1}(t) \in TF, \quad \dot{c}_{2j}(t) \in NF$$

for any  $t, j = 1, \dots, k$ . Clearly,  $\tau_{c_i}(T_{x_i} F) = T_{y_i} F$ , where  $x_i$  and  $y_i$  are the start and end points of  $c_i$ ,  $i = 1, \dots, 2k$ . On the other hand,

$$\tau_c = \tau_{c_1} \circ \tau_{c_2} \circ \dots \circ \tau_{c_{2k}}$$

because  $(M, g)$  is flat. Therefore,  $\tau_c(T_{c(0)} F) = T_{c(1)} F$ . This ends the proof.

The Gauss mapping of a minimal foliation of a flat manifold can be complicated. For example, if  $F$  and  $F^\perp$  are mutually orthogonal foliations of the torus  $T^n$ ,  $\dim F + \dim F^\perp = n$ , and  $F$  is minimal, then the Gauss mapping  $\gamma$  of  $F$  need not be harmonic.

In the above situation,  $\gamma$  maps  $M$  into  $G_x^p$ ,  $p = \dim F$ ,  $x \in T^n$ .  $G_x^p$  carries the natural Riemannian structure induced from  $G_n^p$ , the Grassmann manifold of  $p$ -dimensional linear subspaces of  $\mathbf{R}^n$ , by an isometry  $T_x T^n \rightarrow \mathbf{R}^n$ .

The differential  $d\gamma$  is a section of the bundle  $T^*T^n \otimes \gamma^*TG_n^p$ . The bundle  $TG_n^p$  splits into the tensor product  $K^* \otimes K^\perp$ , where  $K$  is the canonical bundle of  $G_n^p$  (i.e. the fibre of  $K$  at a point  $\alpha$  of  $G_n^p$  is  $\alpha$  itself) and  $K^\perp$  is the orthogonal complement of  $K$ . It follows that

$$\gamma^*TG_n^p \approx T^*F \otimes NF.$$

The above isomorphism of bundles is an affine transformation: it maps the natural connection on  $\gamma^*TG_n^p$  onto the connection on  $T^*F \otimes NF$  induced by the Levi-Civita connection on  $T^n$ . Using this isomorphism we can consider  $d\gamma$  as a section of  $T^*T^n \otimes T^*F \otimes NF$ . Define a section  $\beta$  of the bundle  $T^*F \otimes T^*F \otimes NF$  as follows:  $\beta(y) = B(y)$ , where  $B$  is the second fundamental form of the leaf  $L_y$  of  $F$  passing through  $y$ . Then, analogously to [47],

$$(26) \quad d\gamma(X, Y) = \beta(X, Y)$$

for any sections  $X, Y$  of  $TF$ .  $\nabla d\gamma$  is a section of the bundle  $T^*T^n \otimes T^*T^n \otimes T^*F \otimes NF \approx \text{Hom}(TT^n \otimes TT^n, T^*F \otimes NF)$ . Using the decomposition  $TM \approx TF \oplus NF$  we can express  $\tau(\gamma) = \text{Trace } \nabla d\gamma$  in the form  $\tau_F(\gamma) + \tau_F^\perp(\gamma)$ , where  $\tau_F(\gamma) = \text{Trace}(\nabla d\gamma|TF \otimes TF)$  and  $\tau_F^\perp(\gamma) = \text{Trace}(\nabla d\gamma|NF \otimes NF)$ . Clearly,  $\tau_F(\gamma)$  and  $\tau_F^\perp(\gamma)$  are sections of the bundle  $T^*F \otimes NF$ . From (26), the Codazzi equation (3), and the equality  $\text{Tr} \circ \nabla = \nabla \circ \text{Tr}$ , it follows that

$$(27) \quad \tau_F(\gamma)(y) = \nabla H(y) - d\gamma(H(y)),$$

where  $H$  is the mean curvature vector of  $L_y$ . Similarly, considering the Gauss mapping  $\gamma^\perp$  of the foliation  $F^\perp$  determined by  $NF$  we obtain

$$\tau_F^\perp(\gamma^\perp)(y) = \nabla H^\perp(y) - d\gamma^\perp(H^\perp(y)),$$

where  $H^\perp$  is the mean curvature vector of the leaf of  $F^\perp$  passing through a point  $y$  of  $M$ . But,  $\gamma^\perp = I \circ \gamma$ , where  $I$  is the isometry of  $G_n^p$  onto  $G_n^{n-p}$  which sends any  $p$ -dimensional subspace of  $\mathbf{R}^n$  on its orthogonal complement. It follows that

$$(28) \quad \tau_F^\perp(\gamma)(y) = dI(\nabla H^\perp(y)) - dI(d\gamma(H^\perp(y))).$$

The Gauss mapping  $\gamma$  is harmonic if and only if  $\tau(\gamma) = 0$  (see, for example, [15] or [16]). From (27) and (28), it follows that if  $F$  is minimal, then  $\tau_F(\gamma) = 0$ . The equality  $\tau_{F^\perp}(\gamma) = 0$  does not hold in general.

Now, let  $F$  be a  $p$ -dimensional foliation of an arbitrary Riemannian manifold  $(M, g)$ ,  $x \in M$ ,  $\gamma: M \rightarrow G_x^p$  the Gauss mapping of  $F$ . If  $U_x$  is an open neighbourhood of the origin of the tangent space  $T_x M$  such that the exponential mapping  $\exp_x$  of  $(M, g)$  at  $x$  is defined on  $U_x$  and maps  $U_x$  diffeomorphically onto a neighbourhood  $U$  of  $x$ , then we can define a mapping  $\Gamma: U_x \rightarrow G_x^p$  by

$$\Gamma(v) = \tau_{c_v}^{-1}(T_{\exp_x v} F),$$

where  $c_v(t) = \exp(tv)$  is the geodesic joining  $x$  to  $\exp_x v$ . The mapping  $\Gamma$  is differentiable and is related to  $\gamma$  as follows:

$$\gamma|U = \text{pr} \circ \Gamma \circ \exp_x^{-1}|U,$$

where  $\text{pr}: G_x^p \rightarrow G_x^p$  is the canonical projection.

Let us take an orthonormal frame  $X_1, \dots, X_m$  of the module of vector fields on  $U$  such that  $X_i$  are sections of  $TF$  for  $i = 1, \dots, p$ . Put  $e = (X_1(x), \dots, X_m(x))$ . The orthonormal frame  $e$  of  $T_x M$  determines a diffeomorphism (denoted also by  $e$ ) of  $G_x^p$  onto  $G_m^p = O(m)/O(p) \times O(m-p)$ . The differential  $d(e \circ \Gamma \circ \exp_x^{-1})$  maps  $T_x M$  into  $T_0 G_m^p$ , where  $o$  is the orbit of the unit matrix. The tangent space  $T_0 G_m^p$  is canonically isomorphic to the quotient  $\mathfrak{o}(m)/\mathfrak{o}(p) \times \mathfrak{o}(m-p)$ , where  $\mathfrak{o}(k)$  denotes the Lie algebra of the orthogonal group  $O(k)$ . Using this isomorphism we get

$$\begin{aligned} d(e \circ \Gamma \circ \exp_x^{-1})(v) &= (e \circ \Gamma \circ \exp_x^{-1} \circ c_v)'(0) \\ &= \left[ \lim_{t \rightarrow 0} \frac{1}{t} \left( \langle X_i(x), \tau_{c_{tv}}^{-1}(X_j(\exp_x tv)) \rangle - \langle X_i(x), X_j(x) \rangle \right); i, j \leq m \right]_{\mathfrak{o}(p) \times \mathfrak{o}(m-p)} \\ &= \left[ \langle X_i(x), \lim_{t \rightarrow 0} \frac{1}{t} (\tau_{c_{tv}}^{-1}(X_j(\exp_x tv)) - X_j(x)) \rangle; i, j \leq m \right]_{\mathfrak{o}(p) \times \mathfrak{o}(m-p)} \\ &= [\langle X_i(x), \nabla_v X_j \rangle; i, j \leq m]_{\mathfrak{o}(p) \times \mathfrak{o}(m-p)} \end{aligned}$$

for every  $v$  of  $T_x M$ .

Let us suppose that the foliation  $F$  is totally umbilical. Then for any  $w$  of  $N_x F$  there exists a number  $\alpha(w)$  such that

$$\langle w, \nabla_X Y(x) \rangle = \alpha(w) \langle X, Y \rangle(x)$$

for all sections  $X$  and  $Y$  of  $TF$ . In this case the matrix representing the value of  $d(e \circ \Gamma \circ \exp_x^{-1})$  at  $X_i(x)$ ,  $i \leq p$ , has the form

$$\left[ \begin{array}{ccc|ccc} & & & 0 & \dots & 0 \\ & & & \vdots & & \vdots \\ & & & 0 & \dots & 0 \\ & & \mathbf{A} & \alpha_1 & \dots & \alpha_{m-p} \\ & & & 0 & \dots & 0 \\ & & & \vdots & & \vdots \\ & & & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & -\alpha_1 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & -\alpha_{m-p} & 0 & \dots & 0 \end{array} \right] \quad \mathbf{B}$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are skew-symmetric matrices of order  $p$  and  $m-p$ , respectively, and  $\alpha_j = \alpha(X_{p+j}(x))$ .

Let us assume that the foliation  $F$  is not geodesic at  $x$ , i.e. that the second fundamental form of the leaf  $L_x$  passing through  $x$  does not vanish at  $x$ . Then  $\alpha_j \neq 0$  for some  $j \leq m-p$  and, consequently,  $d(e \circ \Gamma \circ \exp_x^{-1})(v) \neq 0$  for all  $v$  of  $T_x F$ ,  $v \neq 0$ . In this situation,

(29)  $\text{rank } \Gamma \geq p \quad \text{at } x.$

The most interesting results can be obtained when  $p = m-1$ . Then  $\dim G_x^p = p$  and  $\Gamma$  is of maximal rank at  $x$ .

(6.8) THEOREM. *If  $F$  is a totally umbilical foliation of a Riemannian manifold  $(M, g)$  which is not geodesic at any point of  $M$ , then the Gauss mapping  $\gamma$  of  $F$  is open provided  $\text{codim } F = 1$ . If, in addition,  $M$  is compact, then  $\gamma$  maps  $M$  onto  $G_x^{m-1}$ ,  $m = \dim M$ .*

Proof. From (29), it follows that if  $V$  is an open neighbourhood of the origin  $O_x$  of  $T_x M$ , then  $\Gamma(V)$  contains an open neighbourhood of  $\Gamma(O_x)$ . The projection  $\text{pr}$  is open. Consequently, if  $U$  is an open neighbourhood of  $x$ , then  $\gamma(U)$  contains an open neighbourhood of  $\gamma(x)$ .

Let us take an arbitrary open subset  $U$  of  $M$  and a point  $\zeta = \gamma(y)$ , where  $y \in U$ . We have to show that  $\gamma(U)$  contains an open neighbourhood of  $\zeta$ . Let  $\gamma': M \rightarrow G_y^{m-1}$  be the Gauss mapping of  $F$  corresponding to  $y$ . Then (compare (25))  $\gamma = \tau_c \circ \gamma'$ , where  $\tau_c$  is the homeomorphism of  $G_y^{m-1}$  onto  $G_x^{m-1}$  induced by the parallel displacement along a curve  $c$  joining  $y$  to  $x$ . The set  $\gamma'(U)$  contains an open neighbourhood  $V$  of  $\gamma'(y)$ . Put  $W = \tau_c(V)$ .  $W$  is open,  $\zeta \in W$ , and  $W \subset \gamma(U)$ . This ends the proof of the first statement. The second one follows immediately from the first.

Remark. It is easy to see that complete flat Riemannian manifolds do

not admit foliations satisfying conditions of (6.8). In fact, any such foliation gives rise to a foliation of  $R^m$  by  $(m-1)$ -dimensional spheres which does not exist. The problem of existence of  $p$ -dimensional,  $p < m-1$ , totally umbilical foliations of  $m$ -dimensional complete flat Riemannian manifolds is open. Such foliations produce foliations of  $R^m$  by  $p$ -dimensional spheres. Recently, I have been informed of the following result due to Millett:  $R^3$  does not admit foliations by circles.

Note that the non-existence of satisfying conditions of (6.8) foliations of some flat manifolds can be obtained by use of the last theorem. In fact, if  $(M, g)$  is compact and flat, then the holonomy group  $H(x)$  of  $(M, g)$  at a point  $x$  of  $M$  is finite. If, in addition, the action of  $H(x)$  on  $G_x^{m-1}$  is free, then  $G_x^{m-1} = G_x^{m-1}/H(x)$  carries the natural structure of a differentiable manifold. If  $F$  satisfies conditions of Theorem (6.8), then  $\gamma$  is a local diffeomorphism of  $M$  onto  $G_x^{m-1}$ . Therefore,  $\gamma$  is a covering (see [36], Volume I, p. 178) and the fundamental group of  $M$  is isomorphic to a subgroup of  $\pi_1(G_x^{m-1})$ . The fundamental group  $\pi_1(G_x^{m-1})$  is finite because the projection  $G_x^{m-1} \rightarrow G_x^{m-1}$  is a finite covering and the fundamental group of  $G_x^{m-1} \approx P^{m-1}R$  is finite. Therefore,  $\pi_1(M)$  must be finite. On the other hand, the Bieberbach Theorem states that  $M$  can be covered by the torus  $T^m$ . Therefore,  $\pi_1(M)$  must be infinite.

Recent results due to Brito, Langevin and Rosenberg [8] allow to improve the result mentioned in the above remark. They proved that if  $F$  is an orientable codimension-one foliation of a connected compact orientable  $(m+1)$ -dimensional Riemannian manifold  $(M, g)$  of constant sectional curvature  $c$ , then

$$\int_M \eta_k = \begin{cases} c^{k/2} \cdot \binom{m/2}{k/2} \cdot \text{vol}(M), & \text{when } k \text{ and } m \text{ are even,} \\ 0, & \text{when either } k \text{ or } m \text{ is odd,} \end{cases}$$

where  $\eta_k(x) = \sigma_k(\lambda_1(x), \dots, \lambda_m(x))$ ,  $\sigma_k$  is the  $k$ th elementary symmetric function of  $m$  variables, and  $\lambda_1(x), \dots, \lambda_m(x)$  are the principal curvatures at  $x$  of the leaf  $L_x$  with respect to a fixed unit section  $V$  of the bundle  $NF$ . It follows that if either  $c \leq 0$  or  $m$  is odd, then

$$\int_M \eta_2 \leq 0.$$

If leaves of  $F$  are umbilical, then  $\eta_2 = \binom{m}{2} \lambda^2$ , where  $\lambda$  is a differentiable function on  $M$  defined at any point  $x$  as the unique eigenvalue of the second fundamental form of  $L_x$  at  $x$ . These facts yield the following

(6.9) PROPOSITION. *Let  $(M, g)$  be a compact Riemannian manifold of constant sectional curvature  $c$ . If either  $c \leq 0$  or  $\dim M$  is even, then any*

*totally umbilical codimension-one foliation of  $(M, g)$  is totally geodesic. If  $\dim M$  is odd and  $(M, g)$  admits a totally geodesic foliation of codimension one, then  $(M, g)$  is flat.*

The similar argumentation shows that compact Riemannian manifolds of constant sectional curvature can not be foliated by codimension-one submanifolds with definite second fundamental forms. (Such submanifolds are called strictly convex because of the following: If the second fundamental form of a hypersurface  $N$  of  $R^{n+1}$  is definite at a point  $x$  of  $N$ , then  $x$  is the only point of some its neighbourhood on  $N$  which lies on the tangent hyperplane  $T_x N$ . See [36], Volume II, p. 40.) In fact, if  $F$  is an orientable foliation of such manifold  $(M, g)$ ,  $x \in M$ , and the second fundamental form of the leaf  $L_x$  is definite at  $x$ , then  $\eta_k(x) \neq 0$  for any  $k$ . If  $\eta_k(x) \neq 0$  at every point  $x$  of  $M$ , then

$$\int_M \eta_k \neq 0.$$

Taking  $k$  odd we get the contradiction.

**Remark.** If  $(M, g)$  is a compact Riemannian manifold of even dimension and constant sectional curvature  $c \neq 0$ , then the Euler–Poincaré characteristic of  $M$  does not vanish and, consequently,  $M$  admits no codimension-one foliations. Comparing this fact with the second part of Proposition (6.9) we can see that compact space forms of non-vanishing sectional curvature do not admit totally geodesic foliations of codimension one.

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