

EXISTENCE THEOREMS FOR CONVOLUTION OF ULTRADISTRIBUTIONS

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Abstract. In the paper we give sufficient conditions for the existence of convolution of two ultradistributions of Beurling type. We distinguish two types of sufficient conditions: a) conditions in terms of the supports of ultradistributions; b) conditions in terms of subspaces of ultradistributions on which convolution can be defined as a bilinear mapping.

1. Introduction. The equivalence of several definitions of convolution of Beurling ultradistributions and, on the other hand, of convolution of tempered ultradistributions is proved in [18] and [7]. Our aim in this paper is to give various sufficient conditions for the existence of convolution of two ultradistributions. Two types of sufficient conditions are considered: a) conditions expressed in terms of the supports of ultradistributions involved; b) conditions which rely on distinguishing subspaces of ultradistributions on which convolution can be defined as a bilinear mapping.

In the first part of the paper (section 3), we prove that under appropriate compatibility conditions on the supports of two ultradistributions their convolution and ultratemppered convolution exist. In the general case of the space of all ultradistributions, the compatibility condition coincides with the well known compatibility condition for distributions. On the other hand, we define the notion of M -compatible supports of tempered ultradistributions which corresponds to

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the concepts of polynomial compatibility introduced for the supports of tempered distributions in [6] and of M_p -compatibility introduced for the supports of generalized functions of Gelfand and Shilov in [21]. We prove that the condition of M -compatibility of the supports implies the existence of ultratemppered convolution (denoted by $*_t$), i.e. the convolution of ultradistributions introduced in many equivalent ways in [8]. These results are obtained under the assumption that the given numerical sequence (M_p) , defining the respective spaces of test functions and of ultradistributions, satisfies the conditions (M.1) and (M.3') (see section 2).

In the second part of the paper (section 4), we examine the weighted \mathcal{D}'_{L^q} and $\mathcal{D}'^{(M_p)}_{L^q}$ spaces (distributional and ultradistributional), denoted by $\mathcal{D}'_{L^q, \mu}$ and $\mathcal{D}'^{(M_p)}_{L^q, \mu}$ (for $q \in [1, \infty]$), respectively. Let us mention here two of the results obtained for convolution in those spaces and to this end fix $\mu, \nu \in \mathbf{R}$ with $\mu + \nu \geq 0$ and $q, r \in [1, \infty]$ with $1/q + 1/r \in [1, 2]$ and put $\rho := \min(\mu, \nu)$; $s := (1/q + 1/r - 1)^{-1}$:

1° if the given sequence (M_p) satisfies conditions (M.1), (M.2') and (M.3') (see section 2), then $f \in \mathcal{D}'_{L^q, \mu}$ and $g \in \mathcal{D}'^{(M_p)}_{L^r, \nu}$ imply $f * g = f *_t g \in \mathcal{D}'^{(M_p)}_{L^s, \rho}$ and the mapping

$$\mathcal{D}'_{L^q, \mu} \times \mathcal{D}'^{(M_p)}_{L^r, \nu} \ni (f, g) \mapsto f * g \in \mathcal{D}'^{(M_p)}_{L^s, \rho}$$

is continuous;

2° if the sequence (M_p) satisfies conditions (M.1), (M.2) and (M.3') (see section 2), then $f \in \mathcal{D}'^{(M_p)}_{L^q, \mu}$ and $g \in \mathcal{D}'^{(M_p)}_{L^r, \nu}$ imply $f * g = f *_t g \in \mathcal{D}'^{(M_p)}_{L^s, \rho}$ and the mapping

$$\mathcal{D}'^{(M_p)}_{L^q, \mu} \times \mathcal{D}'^{(M_p)}_{L^r, \nu} \ni (f, g) \mapsto f * g \in \mathcal{D}'^{(M_p)}_{L^s, \rho}$$

is continuous.

2. Preliminaries. For simplicity we shall consider in this paper only the one-dimensional case of functions, distributions and ultradistributions defined on the real line, but all the results can be easily transferred to the multi-dimensional case. The notation is similar to that introduced in [8]. The sets of all nonnegative integers, positive integers and real numbers are denoted by \mathbf{N}_0, \mathbf{N} and \mathbf{R} , respectively. If f is a function, a distribution or an ultradistribution of one real variable x or of two real variables x, y , then its derivatives will be denoted in the standard way:

$$f^{(\alpha)}(x) := \frac{d^\alpha}{dx^\alpha} f(x), \quad f^{(\alpha, \beta)}(x, y) := \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} f(x, y)$$

for $\alpha, \beta \in \mathbf{N}_0$. We also adopt the notation: $\phi^\Delta(x, y) := \phi(x + y)$ for a given function ϕ on \mathbf{R} and $x, y \in \mathbf{R}$, and

$$(1) \quad \langle x \rangle := (1 + x^2)^{1/2}, \quad x \in \mathbf{R},$$

i.e. the symbol $\langle \cdot \rangle$ will denote the function

$$\mathbf{R} \ni x \mapsto (1 + x^2)^{1/2} \in [1, \infty).$$

The norm in $L^r(X)$ (for $r \in [1, \infty]$), where X is a fixed (measurable) subset of \mathbf{R} is denoted by $\|\cdot\|_{L^r(X)}$, but in case $X = \mathbf{R}$ simply by $\|\cdot\|_r$.

Let (M_p) be a sequence of positive numbers with $M_0 = 1$. The following conditions are frequently imposed on this sequence (cf. [15], [2] and [10]):

$$(M.1) \quad M_p^2 \leq M_{p-1}M_{p+1}, \quad p \in \mathbf{N};$$

$$(M.2) \quad M_p \leq AH^p M_q M_{p-q}, \quad p, q \in \mathbf{N}, \quad 0 \leq q \leq p;$$

$$(M.2') \quad M_p \leq AH^p M_{p-1}, \quad p \in \mathbf{N}$$

and

$$(M.3') \quad \sum_{q \in \mathbf{N}} M_{q-1}/M_q < \infty$$

for some $A > 0$, $H > 0$. We may assume, and we shall do it from now on, that the constants A , H in conditions (M.2) and (M.2') satisfy the inequalities $A \geq 1$, $H \geq 1$.

It is easy to see that conditions (M.1), (M.2) and (M.2') imply the following conditions:

$$(M.a) \quad M_\alpha M_\beta \leq M_{\alpha+\beta}, \quad \alpha, \beta \in \mathbf{N}_0;$$

$$(M.b) \quad M_{\alpha+\beta} \leq AH^{\alpha+\beta} M_\alpha M_\beta, \quad \alpha, \beta \in \mathbf{N}_0, \quad \text{for some } A \geq 1, H \geq 1;$$

$$(M.b') \quad M_{\alpha+\beta} \leq A^\alpha H^{\alpha(\alpha+\beta)} M_\beta, \quad \alpha, \beta \in \mathbf{N}_0, \quad \text{for some } A \geq 1, H \geq 1,$$

respectively. Throughout the paper we assume that the sequence (M_p) satisfies only conditions (M.1) and (M.3'). In sections 3 and 4, however, we shall also additionally impose either condition (M.2) or condition (M.2') on the sequence (M_p) . The letter H will always denote the constant mentioned in the above conditions. The so-called *associated function* for the sequence (M_p) is defined by

$$M(t) := \sup_{p \in \mathbf{N}_0} \log_+(t^p/M_p), \quad t > 0,$$

where $\log_+ t := \max\{\log t, 0\}$.

Since condition (M.1) implies $M_p^{1/p} \leq M_{p+1}^{1/p+1}$ for $p \in \mathbf{N}$ and condition (M.3') is equivalent to

$$\sum_{p=1}^{\infty} M_p^{-1/p} < \infty$$

(cf. [15]), we have

$$\log \frac{t^p}{M_p} \leq \sum_{i=1}^p \log(tM_i^{1/i}) \leq t \sum_{i=1}^{\infty} M_i^{1/i} < \infty$$

for arbitrary $p \in \mathbf{N}$ and $t > 0$, which shows that the associated function is well defined for all $t > 0$. It is easy to see that M is nondecreasing function on $[0, \infty)$, $M = 0$ in a right neighbourhood of 0 and the function e_1 defined by $e_1(t) := \exp(-M(t))$ is integrable on $[0, \infty)$.

Given the sequence (M_p) satisfying conditions (M.1) and (M.3'), several basic spaces of smooth functions in (an open subset of) \mathbf{R}^d are defined, e.g.: $\mathcal{D}_K^{(M_p)}$ (where K is a regular compact set in \mathbf{R}^d - see [10]), $\mathcal{E}^{(M_p)}$, $\mathcal{D}^{(M_p)}$, $\mathcal{D}_{L^s}^{(M_p)}$ ($s \in [1, \infty]$), $\mathcal{B}^{(M_p)}$ and $\mathcal{S}^{(M_p)}$ (see [10] and [8]). The basic spaces are dense in \mathcal{D} (in case of the spaces $\mathcal{D}_K^{(M_p)}$ and $\mathcal{D}^{(M_p)}$, this is a consequence of conditions (M.1) and (M.3') and the known Denjoy-Carleman-Mandelbrojt theorem). The corresponding spaces of ultradistributions are defined as the strong duals of the basic spaces mentioned above: $\mathcal{D}'_K^{(M_p)}$, $\mathcal{E}'^{(M_p)}$, $\mathcal{D}'^{(M_p)}$, $\mathcal{D}'_{L^t}^{(M_p)}$ (where $t = s/(s-1)$), $\mathcal{D}'_{L^1}^{(M_p)}$ and $\mathcal{S}'^{(M_p)}$. The spaces $\mathcal{D}'^{(M_p)}$, $\mathcal{S}'^{(M_p)}$ and $\mathcal{D}'_{L^1}^{(M_p)}$ are called the spaces of all ultradistributions, tempered ultradistributions and integrable ultradistributions of Beurling type, respectively. We shall consider these spaces only in case of functions and ultradistributions defined on \mathbf{R}^d for $d = 1$ or $d = 2$. The definitions of the mentioned spaces can be found in [10] (see also [8]).

As is shown in [14], the space $\mathcal{S}^{(M_p)}$ can be equivalently defined as

$$\mathcal{S}^{(M_p)} := \text{proj} \lim_{m \rightarrow \infty} \mathcal{S}_2^{M_p, m}$$

or

$$\mathcal{S}^{(M_p)} := \text{proj} \lim_{m \rightarrow \infty} \mathcal{S}_\infty^{M_p, m},$$

where $\mathcal{S}_2^{(M_p), m}$ and $\mathcal{S}_\infty^{(M_p), m}$ are the spaces of all smooth functions ϕ such that

$$\sigma_{m,2}(\phi) := \left(\sum_{\alpha, \beta \in \mathbf{N}_0} \int_{\mathbf{R}} \left| \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \langle x \rangle^\beta \phi^{(\alpha)}(x) \right|^2 dx \right)^{1/2} < \infty$$

and

$$\sigma_{m,\infty}(\phi) := \sup_{\alpha, \beta \in \mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle \cdot \rangle^\beta \phi^{(\alpha)}\|_\infty < \infty$$

with the topology defined by the above norms $\sigma_{m,2}(\phi)$ and $\sigma_{m,\infty}(\phi)$, respectively. Consequently, each $\phi \in \mathcal{S}^{(M_p)}$ satisfies for every $m > 0$ the estimate

$$(2) \quad \sup_{\alpha, \beta \in \mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle \cdot \rangle^\beta \phi^{(\alpha)}\|_\infty < \infty.$$

Observe that if $\mathcal{D}_{L^\infty}^{(M_p)}$ is equipped with the topology induced by $\mathcal{E}^{(M_p)}$, the corresponding topological space is denoted by $\mathcal{B}_C^{(M_p)}$. The space $\mathcal{B}^{(M_p)}$ is dense in $\mathcal{B}_C^{(M_p)}$ and $(\mathcal{B}_C^{(M_p)})' = \mathcal{D}'_{L^1}^{(M_p)}$, because every bounded set in $\mathcal{D}_{L^\infty}^{(M_p)}$ is relatively compact. Since $(\mathcal{D}_{L^1}^{(M_p)})' = \mathcal{B}^{(M_p)}$, the space $\mathcal{B}_C^{(M_p)}$ is semireflexive.

Among several kinds of approximate units introduced in [8] we shall need in this paper only strong $\mathcal{D}^{(M_p)}$ -approximate units in \mathbf{R} and in \mathbf{R}^2 , called briefly *strong approximate units*.

Recall that a sequence (η_n) of elements of $\mathcal{D}^{(M_p)}(\mathbf{R}^d)$ ($d = 1$ or 2) is called a *strong approximate unit* if for every compact set $K \subset \mathbf{R}^d$ there exists an index $n_0 \in \mathbf{N}$ such that $\eta_n(x) = 1$ for $x \in K$ and $n \geq n_0$ and, moreover,

$$(3) \quad \sup_{n \in \mathbf{N}} \sup_{\alpha \in \mathbf{N}_0^d} \left(\frac{m^{|\alpha|}}{M_{|\alpha|}} \|\eta_n^{(\alpha)}\|_\infty \right) < \infty$$

for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}_0^d$, where $|\alpha| := \alpha_1 + \dots + \alpha_d$ (hence (η_n) converges to 1 in $\mathcal{E}^{(M_p)}(\mathbf{R}^d)$).

We shall need the following two propositions about representations of convergent sequences of ultradistributions and tempered ultradistributions:

PROPOSITION 1. *Let $f \in \mathcal{D}'^{(M_p)}$, $f_n \in \mathcal{D}'^{(M_p)}$ for $n \in \mathbf{N}$ and suppose that $f_n \rightarrow f$ in $\mathcal{D}'^{(M_p)}$ as $n \rightarrow \infty$. Then, for each open, relatively compact set G in \mathbf{R} , there are measures $f_\alpha, f_{n,\alpha} \in \mathcal{C}'(\bar{G})$ for $n \in \mathbf{N}$, $\alpha \in \mathbf{N}_0$ and positive constants L and B such that*

$$(4) \quad f|_G = \sum_{\alpha \in \mathbf{N}_0} f_\alpha^{(\alpha)}, \quad f_n|_G = \sum_{\alpha \in \mathbf{N}_0} f_{n,\alpha}^{(\alpha)},$$

$$(5) \quad \|f_\alpha\|_{\mathcal{C}'(\bar{G})} \leq BL^\alpha/M_\alpha, \quad \|f_{n,\alpha}\|_{\mathcal{C}'(\bar{G})} \leq BL^\alpha/M_\alpha,$$

for arbitrary $\alpha, n \in \mathbf{N}_0$ and

$$(6) \quad \lim_{n \rightarrow \infty} \|f_{n,\alpha} - f_\alpha\|_{\mathcal{C}'(\bar{G})} = 0$$

for every $\alpha \in \mathbf{N}_0$.

Proof. The above proposition is an extension of Theorem 8.1 of [10] to the case of a sequence of ultradistributions. Note that the existence of the representations (4) follows directly from that theorem, but we cannot deduce from it that the constants B and L in the second inequality of (5) do not depend on n . This and (6) follow, however, from the proof of the mentioned theorem with small modifications. For completeness, we present here the whole proof.

Let K be the closure of G . We shall prove that the restrictions of f and f_n to $\mathcal{D}_K^{(M_p)}$ have representations (4) and the series in (4) are convergent in the strong topology of $(\mathcal{D}_K^{(M_p)})'$. Since the inclusion mapping $\mathcal{D}^{(M_p)}(G) \rightarrow \mathcal{D}_K^{(M_p)}$ is continuous, both series in (4) converge also in the strong topology of the space $\mathcal{D}^{(M_p)}(G)$. Note that

$$\mathcal{D}_K^{(M_p)} := \text{proj} \lim_{j \rightarrow \infty} X_j,$$

where X_j is the Banach space of all $\phi \in \mathcal{D}_K$ such that

$$\lim_{\alpha \rightarrow \infty} (M_\alpha^{-1} j^\alpha \|\phi^{(\alpha)}\|_{\mathcal{C}(K)}) = 0$$

with the norm

$$\|\phi\|_{X_j} := M_\alpha^{-1} \sup_{\alpha \in \mathbf{N}_0} j^\alpha \|\phi^{(\alpha)}\|_{\mathcal{C}(K)}.$$

Since $\mathcal{D}_K^{(M_p)}$ is a strict projective limit of the spaces X_j , it follows from a theorem in [3] (2.6 Satz, p. 147) that the restrictions of f and f_n to $\mathcal{D}_K^{(M_p)}$ can be extended to elements g and g_n of the same space X'_j for $n \in \mathbf{N}$, so that $g_n \rightarrow g$ in X'_j as $n \rightarrow \infty$.

On the other hand, using an appropriate mapping, one can identify X_j with a closed subspace of the space

$$Y_j := \{\phi = (\phi_\alpha) : \phi_\alpha \in \mathcal{C}(K), \lim_{\alpha \rightarrow \infty} (M_\alpha^{-1} j^\alpha \|\phi_\alpha\|_{\mathcal{C}(K)}) = 0\}$$

and Y_j with a closed subspace of $\mathcal{C}_0(K^{\mathbf{N}})$, where $\mathcal{C}(K)$ is the space of all continuous functions with supports in K , the set $K^{\mathbf{N}}$ is the disjoint union of countably many copies of K regarded as a locally compact space, and $\mathcal{C}_0(K^{\mathbf{N}})$ is the space of all continuous functions vanishing on the boundary of $K^{\mathbf{N}}$.

By the Hahn–Banach theorem, we extend g and g_n to measures \tilde{g} and \tilde{g}_n on $K^{\mathbf{N}}$ (for $n \in \mathbf{N}$), so that $\tilde{g}_n \rightarrow \tilde{g}$ in $\mathcal{C}'_0(K^{\mathbf{N}})$ as $n \rightarrow \infty$. This implies the existence of measures $f_\alpha, f_{n,\alpha} \in \mathcal{C}'(K)$ for $n \in \mathbf{N}, \alpha \in \mathbf{N}_0$, which satisfy (5) and (6). For $\phi \in X_j$, we have

$$\langle \tilde{g}_n, \phi \rangle = \langle f_n, \phi \rangle = \sum_{\alpha \in \mathbf{N}_0} (-1)^\alpha \langle f_{n,\alpha}, \phi^{(\alpha)} \rangle = \sum_{\alpha \in \mathbf{N}_0} \langle f_{n,\alpha}^{(\alpha)}, \phi \rangle$$

and, similarly, $\langle \tilde{g}, \phi \rangle = \langle f, \phi \rangle = \sum_{\alpha \in \mathbf{N}_0} \langle f_\alpha^{(\alpha)}, \phi \rangle$. ■

PROPOSITION 2. *Suppose that $f \in \mathcal{S}'^{(M_p)}$ and $f_n \in \mathcal{S}'^{(M_p)}$ for $n \in \mathbf{N}_0$, and $f_n \rightarrow f$ in $\mathcal{S}'^{(M_p)}$ as $n \rightarrow \infty$. Then there are functions $F_\alpha, F_{n,\alpha} \in L^2$ for $n, \alpha \in \mathbf{N}_0$ and positive constants λ, L and B such that*

$$\begin{aligned} f|_G &= \sum_{\alpha \in \mathbf{N}_0} (E_\lambda F_\alpha)^{(\alpha)}, & f_n|_G &= \sum_{\alpha \in \mathbf{N}_0} (E_\lambda F_{n,\alpha})^{(\alpha)} \\ \|F_\alpha\|_2 &\leq BL^\alpha/M_\alpha, & \|F_{n,\alpha}\|_2 &\leq BL^\alpha/M_\alpha, \end{aligned}$$

for arbitrary $\alpha, n \in \mathbf{N}_0$, and

$$\lim_{n \rightarrow \infty} \|F_{n,\alpha} - F_\alpha\|_2 = 0$$

for every $\alpha \in \mathbf{N}_0$, where $E_\lambda(x) = e^{M(\lambda|x|)}$ for $x \in \mathbf{R}$.

Proof. One can prove the proposition following the idea of the proof of the structural theorem for $\mathcal{S}'^{(M_p)}$, first noticing that $\mathcal{S}^{(M_p)}$ is a strict projective limit of the spaces $\mathcal{S}_2^{M_p, m}$ (see [8]) and then, similarly to the proof of Proposition 1, applying a theorem proved in [3] (2.6 Satz, p. 147). ■

3. Compatibility conditions on supports. Let us recall the notion of compatibility of supports used in the theory of distributions: two subsets X and Y of \mathbf{R} are called *compatible* if one of the following equivalent conditions holds (cf. [1], [4] and [9]):

- (a) for every bounded interval I in \mathbf{R} the set $(X \times Y) \cap I^\Delta$ is bounded in \mathbf{R}^2 , where $I^\Delta := \{(x, y) : x + y \in I\}$;
- (b) for every bounded interval I in \mathbf{R} the set $X \cap (I - Y)$ is bounded in \mathbf{R} ;
- (c) for every bounded interval I in \mathbf{R} the set $(X - I) \cap Y$ is bounded in \mathbf{R} ;
- (d) $x_n \in X, y_n \in Y$ ($n \in \mathbf{N}$), $|x_n| + |y_n| \rightarrow \infty$ implies $|x_n + y_n| \rightarrow \infty$;
- (e) for every $R > 0$ the set $T_R := \{(x, y) : x \in X, y \in Y, |x + y| \leq R\}$ is bounded in \mathbf{R}^2 .

In [7] (see also [8]), it is proved that several general conditions for the existence of the convolution $f * g$ of given ultradistributions f and g in $\mathcal{D}'^{(M_p)}$ are equivalent. In the theorem below we shall prove that if the supports of ultradistributions f and g are compatible, then these conditions are fulfilled.

THEOREM 1. *Suppose that a given sequence (M_p) satisfies conditions (M.1) and (M.3').*

- (i) *If $f, g \in \mathcal{D}'^{(M_p)}$ are ultradistributions whose supports $\text{supp } f$ and $\text{supp } g$ are compatible, then the convolution $f * g$ exists.*
- (ii) *If $f_n, g_n \in \mathcal{D}'^{(M_p)}$, $\text{supp } f_n \subset X$, $\text{supp } g_n \subset Y$ for $n \in \mathbf{N}$, where X, Y are compatible sets in \mathbf{R} , and if $f_n \rightarrow f$, $g_n \rightarrow g$ in $\mathcal{D}'^{(M_p)}$, then $f_n * g_n \rightarrow f * g$ in $\mathcal{D}'^{(M_p)}$ as $n \rightarrow \infty$.*

Proof. It suffices to show that one of the equivalent conditions for the existence of convolution in $\mathcal{D}'^{(M_p)}$ is satisfied. We shall prove that the limit $\lim_{k \rightarrow \infty} \langle f \otimes g, \eta_k \phi^\Delta \rangle$ exists for every $\phi \in \mathcal{D}^{(M_p)}(\mathbf{R})$ and for every strong approximate unit (η_k) in $\mathcal{D}^{(M_p)}(\mathbf{R}^2)$ (cf. [8]).

Assume that $\text{supp } \phi$ is contained in the ball of radius $R > 0$. Since the set T_R is bounded, there exist open bounded sets Ω_1 and Ω_2 in \mathbf{R} such that the closure K of T_R is contained in $\Omega_1 \times \Omega_2$. By Theorem 8.1 of [10], there are measures $f_\alpha \in \mathcal{C}'(\overline{\Omega}_1)$, $g_\beta \in \mathcal{C}'(\overline{\Omega}_2)$ on Ω_1 and Ω_2 for $\alpha, \beta \in \mathbf{N}_0$ and positive constants B and L such that

$$f|_{\Omega_1} = \sum_{\alpha \in \mathbf{N}_0} f_\alpha^{(\alpha)}, \quad g|_{\Omega_2} = \sum_{\beta \in \mathbf{N}_0} g_\beta^{(\beta)},$$

$$\|f_\alpha\|_{\mathcal{C}'(\overline{\Omega}_1)} \leq BL^\alpha / M_\alpha, \quad \|g_\beta\|_{\mathcal{C}'(\overline{\Omega}_2)} \leq BL^\beta / M_\beta$$

for $\alpha, \beta \in \mathbf{N}_0$. This and (M.1) imply that

$$(7) \quad \|f_\alpha \otimes g_\beta\|_{\mathcal{C}'(\overline{\Omega}_1 \times \overline{\Omega}_2)} \leq AB^2 \frac{(HL)^{\alpha+\beta}}{M_{\alpha+\beta}}$$

for $\alpha, \beta \in \mathbf{N}_0$.

Let $\eta \in \mathcal{D}(\Omega_1 \times \Omega_2)$ be equal to 1 on \overline{T}_R . There exists an index $k_0 \in \mathbf{N}$ such that

$$\begin{aligned} \langle f \otimes g, \eta_k \phi^\Delta \rangle &= \langle f \otimes g, \eta_k \phi^\Delta \eta \rangle = \sum_{\alpha \in \mathbf{N}_0} \sum_{\beta \in \mathbf{N}_0} \langle f_\alpha^{(\alpha)} \otimes g_\beta^{(\beta)}, \phi^\Delta \eta \rangle \\ &= \sum_{\alpha \in \mathbf{N}_0} \sum_{\beta \in \mathbf{N}_0} \langle f_\alpha^{(\alpha)} \otimes g_\beta^{(\beta)}, \phi^\Delta \rangle \end{aligned}$$

for $k > k_0$. Put

$$a_{i,j} := \sum_{\alpha \leq i} \sum_{\beta \leq j} \langle f_\alpha^{(\alpha)} \otimes g_\beta^{(\beta)}, \phi^\Delta \eta \rangle$$

for $i, j \in \mathbf{N}_0$ and notice that

$$\begin{aligned} \lim_{j \rightarrow \infty} a_{i,j} &= \sum_{\alpha \leq i} \langle f_\alpha^{(\alpha)} \otimes g, \phi^\Delta \eta \rangle \quad (i \in \mathbf{N}); \\ \lim_{i \rightarrow \infty} a_{i,j} &= \sum_{\beta \leq j} \langle f \otimes g_\beta^{(\beta)}, \phi^\Delta \eta \rangle \quad (j \in \mathbf{N}). \end{aligned}$$

Let (i_p) and (j_p) be arbitrary strictly increasing sequences of positive integers and set $\tilde{a}_p = a_{i_p, j_p}$ for $p \in \mathbf{N}$. It follows from (7) that

$$\begin{aligned} (8) \quad |\tilde{a}_r - \tilde{a}_p| &\leq \sum_{i_p < \alpha \leq i_r} \sum_{j_p < \beta \leq j_r} |\langle f_\alpha \otimes g_\beta, (\phi^\Delta)^{(\alpha+\beta)} \rangle| \\ &\leq C \sum_{i_p < \alpha \leq i_r} \sum_{j_p < \beta \leq j_r} \frac{(LH)^{\alpha+\beta}}{M_{\alpha+\beta}} \sup_{|x+y| \leq R} |\phi^{(\alpha+\beta)}(x+y)| \\ &\leq 2^{-2p} C \sup_{i \in \mathbf{N}_0} \sup_{|t| \leq R} \frac{(4LH)^i}{M_i} |\phi^{(i)}(t)| \sum_{i=1}^{\infty} 2^{-i} \end{aligned}$$

for $r > p$, where $C = \pi R^2 AB$. This implies that (\tilde{a}_p) is a Cauchy sequence, i.e. it converges to a certain number a . It is easy to see that a does not depend on the choice of sequences (i_p) and (j_p) . Consequently, the double limit $\lim_{i,j \rightarrow \infty} a_{i,j}$ exists and

$$(9) \quad \lim_{i,j \rightarrow \infty} a_{i,j} = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} a_{i,j} = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} a_{i,j} = a.$$

Now, by (9), we have

$$\begin{aligned} (10) \quad \langle f * g, \phi \rangle &= \langle f \otimes g, \eta_k \phi^\Delta \eta \rangle = \sum_{\alpha, \beta \in \mathbf{N}_0} \langle f_\alpha \otimes g_\beta, (\phi^\Delta)^{(\alpha+\beta)} \rangle \\ &= \sum_{\alpha, \beta \in \mathbf{N}_0} \langle f_\alpha * g_\beta, \phi^{(\alpha+\beta)} \rangle, \end{aligned}$$

which proves (i).

Let us prove now the second assertion of the theorem. By Proposition 1, there

exist measures $f_{n,\alpha}$ and $g_{n,\beta}$ and constants $B, L > 0$ such that

$$(11) \quad f_n|_{\Omega_1} = \sum_{\alpha \in \mathbf{N}_0} f_{n,\alpha}^{(\alpha)}, \quad g_n|_{\Omega_2} = \sum_{\alpha \in \mathbf{N}_0} g_{n,\alpha}^{(\alpha)},$$

$$(12) \quad \|f_{n,\alpha}\|_{\mathcal{C}'(\bar{\Omega}_1)} \leq BL^\alpha/M_\alpha, \quad \|g_{n,\alpha}\|_{\mathcal{C}'(\bar{\Omega}_2)} \leq BL^\alpha/M_\alpha$$

for $\alpha, n \in \mathbf{N}_0$ and

$$(13) \quad \|f_{n,\alpha} - f_n\|_{\mathcal{C}'(\bar{\Omega}_1)} \rightarrow 0, \quad \|g_{n,\alpha} - g_n\|_{\mathcal{C}'(\bar{\Omega}_2)} \rightarrow 0,$$

for $\alpha \in \mathbf{N}_0$ as $n \rightarrow \infty$. By the first part of the theorem, the convolutions $f * g$ and $f_n * g_n$ exist for $n \in \mathbf{N}$ and, due to (10) and the estimation used in (8), we have

$$(14) \quad \begin{aligned} |\langle f_n * g_n, \phi \rangle| &\leq \sum_{\alpha, \beta \in \mathbf{N}_0} |\langle f_{n,\alpha} * g_{n,\beta}, \phi^{(\alpha+\beta)} \rangle| \\ &= \sum_{\alpha, \beta \in \mathbf{N}_0} |\langle f_{n,\alpha} \otimes g_{n,\beta}, (\phi^\Delta)^{(\alpha+\beta)} \rangle| \\ &\leq C \sup_{i \in \mathbf{N}_0} \sup_{|t| \leq R} \frac{(2LH)^i}{M_i} |\phi^{(i)}(t)| < \infty \end{aligned}$$

for any fixed $\phi \in \mathcal{D}^{(M_p)}$. Moreover,

$$(15) \quad \begin{aligned} |\langle f_n * g_n - f * g, \phi \rangle| &\leq \sum_{\alpha, \beta \in \mathbf{N}_0} |\langle f_{n,\alpha} * g_{n,\beta} - f_\alpha * g_\beta, \phi^{(\alpha+\beta)} \rangle| \\ &\leq \left(\sum_{\substack{\alpha \leq m \\ \beta \leq m}} + \sum_{\substack{\alpha > m \\ \beta \in \mathbf{N}_0}} + \sum_{\substack{\alpha \in \mathbf{N}_0 \\ \beta > m}} \right) |\langle f_{n,\alpha} * g_{n,\beta} - f_\alpha * g_\beta, \phi^{(\alpha+\beta)} \rangle|, \end{aligned}$$

where $m \in \mathbf{N}_0$.

Let $\epsilon > 0$ be fixed. Notice that the series

$$\sum_{\alpha, \beta \in \mathbf{N}_0} \langle f_\alpha * g_\beta, \phi^{(\alpha+\beta)} \rangle$$

is convergent and all the series

$$\sum_{\alpha, \beta \in \mathbf{N}_0} \langle f_{n,\alpha} * g_{n,\beta}, \phi^{(\alpha+\beta)} \rangle$$

for $n \in \mathbf{N}$ are uniformly bounded by a convergent series (see (14)). Therefore each of the last two sums in (15) is less than $\epsilon/3$ for m large enough. On the other hand, in view of (12), we have

$$\begin{aligned} &|\langle f_{n,\alpha} * g_{n,\beta} - f_\alpha * g_\beta, \phi^{(\alpha+\beta)} \rangle| \\ &\leq \|\phi^{(\alpha+\beta)}\|_{\mathcal{C}(K)} (\|f_{n,\alpha} * (g_{n,\beta} - g_\beta)\|_{\mathcal{C}'(K)} + \|(f_{n,\alpha} - f_\alpha) * g_\beta\|_{\mathcal{C}'(K)}) \\ &\leq C \|\phi^{(\alpha+\beta)}\|_{\mathcal{C}(K)} (L^\alpha M_\alpha^{-1} \|g_{n,\beta} - g_\beta\|_{\mathcal{C}'(K)} + L^\beta M_\beta^{-1} \|f_{n,\alpha} - f_\alpha\|_{\mathcal{C}'(K)}) \end{aligned}$$

for some constant $C > 0$ and all $\alpha, \beta \in \mathbf{N}_0$ and $n \in \mathbf{N}$. Consequently, by (13), it follows that

$$\sum_{\substack{\alpha \leq m \\ \beta \leq m}} |\langle f_{n,\alpha} * g_{n,\beta} - f_\alpha * g_\beta, \phi \rangle| \leq \epsilon/3$$

for n large enough, which completes the proof of (ii) as well as of the whole theorem. ■

The following notion is a modification of polynomial compatibility of supports, introduced for tempered distributions in [5] (see also [6]) and then generalized to the case of the space $K\{M_p\}'$ of Gelfand and Shilov in [21] (see also [22] and [9]). Coincidentally, the sequence of functions defining a Gelfand-Shilov space is denoted also by (M_p) as the sequence defining the space of ultradistributions and the mentioned above condition in the Gelfand-Shilov spaces is called M_p -compatibility. To avoid misunderstanding, we shall call the notion introduced below M -compatibility, which is justified by the use of the associated function M in its definition.

DEFINITION. Sets $X, Y \subset \mathbf{R}$ are said to be M -compatible if

$$(16) \quad M(|x|) + M(|y|) \leq M(d|x+y|), \quad x \in X, y \in Y,$$

for some $d > 0$ (or, equivalently, $d \geq 1$).

The following assertion is an analogue of Theorem 1 for the convolution $f *_t g$ of tempered ultradistributions f and g in the space $\mathcal{S}'^{(M_p)}$ (see [8]).

THEOREM 2. Suppose that the sequence (M_p) satisfies conditions (M.1), (M.2) and (M.3').

(i) If $f, g \in \mathcal{S}'^{(M_p)}$ and $\text{supp } f$ and $\text{supp } g$ are M -compatible, then the convolution $f *_t g$ in $\mathcal{D}'^{(M_p)}$ exists.

(ii) If $f_n, g_n \in \mathcal{S}'^{(M_p)}$ and $\text{supp } f_n \subset X$, $\text{supp } g_n \subset Y$ for $n \in \mathbf{N}$, where X, Y are M -compatible sets in \mathbf{R} , and if $f_n \rightarrow f$, $g_n \rightarrow g$ in $\mathcal{S}'^{(M_p)}$, then $f_n *_t g_n \rightarrow f *_t g$ in $\mathcal{S}'^{(M_p)}$ as $n \rightarrow \infty$.

Proof. By [14], there exist constants $\lambda, L, B > 1$ and functions $f_\alpha, g_\beta \in L^2$ such that

$$(17) \quad f = \sum_{\alpha \in \mathbf{N}_0} (E_\lambda f_\alpha)^{(\alpha)}, \quad g = \sum_{\beta \in \mathbf{N}_0} (E_\lambda g_\beta)^{(\beta)}$$

and

$$(18) \quad \|f_\alpha\|_2 \leq BL^\alpha/M_\alpha, \quad \|g_\beta\|_2 \leq BL^\beta/M_\beta,$$

for $\alpha, \beta \in \mathbf{N}$, where $E_\lambda(x) = e^{M(\lambda|x|)}$ for $x \in \mathbf{R}$.

Let $X := \text{supp } f$ and $Y := \text{supp } g$ and fix $\phi \in \mathcal{S}^{(M_p)}$, a strong approximate unit (η_n) in $\mathcal{D}^{(M_p)}(\mathbf{R}^2)$ and indices $i, j \in \mathbf{N}$, $j > i$. Now let $\eta := \eta_{i,j} := \eta_j - \eta_i$ and $K := K_{i,j} := \text{supp}(\eta_j - \eta_i)$.

By (17), we have

$$(19) \quad a := a_{i,j} := |\langle f \otimes g, \eta \phi^\Delta \rangle| \leq \sum_{\alpha, \beta \in \mathbf{N}_0} | \langle (E_\lambda f_\alpha)^{(\alpha)} \otimes (E_\lambda g_\beta)^{(\beta)}, \eta \phi^\Delta \rangle | \\ \leq \sum_{\alpha, \beta \in \mathbf{N}_0} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} b_{\gamma, \delta}^{\alpha, \beta},$$

where

$$b_{\gamma, \delta}^{\alpha, \beta} := \int_X \int_Y |[(E_\lambda f_\alpha) \otimes (E_\lambda g_\beta)] \eta^{(\alpha-\gamma, \beta-\delta)}(\phi^{(\gamma+\delta)})^\Delta|.$$

In view of (16), we have

$$(20) \quad b_{\gamma, \delta}^{\alpha, \beta} \leq \int_X \int_Y |(f_\alpha \otimes g_\beta) \eta^{(\alpha-\gamma, \beta-\delta)}(E_{d\lambda} \phi^{(\gamma+\delta)})^\Delta|.$$

Notice that (M.2) implies the inequality

$$2M(t) \leq M(at) + a, \quad t > 0,$$

for some constant $a \geq 1$. Hence there exists a constant $b \geq 1$ such that

$$M(d\lambda t) \leq -2M(t) + M(bt) + b, \quad t > 0.$$

Let

$$A_{\alpha, \beta} := \sup_{n \in \mathbf{N}} \sup_{(x, y) \in \mathbf{R}^2} |\eta^{(\alpha, \beta)}(x, y)|, \\ B_\alpha := \sup_{t \in \mathbf{R}} e^{M(b|t|)} |\phi^{(\alpha)}(t)|, \quad C_0 := \int_{\mathbf{R}} e^{-M(|t|)} dt < \infty$$

and

$$\kappa := \kappa_{i,j} := \sup\{e^{-M(|x|)} : x \in K = K_{i,j}\}.$$

By (20), the Schwarz inequality and inequalities (18), (M.b) and (M.a), we obtain

$$b_{\gamma, \delta}^{\alpha, \beta} \leq A_{\alpha-\gamma, \beta-\delta} B_{\gamma+\delta} \int_{\mathbf{R}} e^{-2M(|t|)} \left[\int_{\mathbf{R}} |f_\alpha(x) g_\beta(t-x)| dx \right] dt \\ \leq \kappa C_0 A_{\alpha-\gamma, \beta-\delta} B_{\gamma+\delta} \|f_\alpha\|_2 \|g_\beta\|_2 \\ \leq \kappa A B^2 C_0 (LH)^{\alpha+\beta} \frac{A_{\alpha-\gamma, \beta-\delta}}{M_{\alpha+\beta-\gamma-\delta}} \frac{B_{\gamma+\delta}}{M_{\gamma+\delta}} \\ \leq \frac{\kappa A B^2 C_0}{4^{\alpha+\beta}} \cdot \frac{(4LH)^{\alpha+\beta-\gamma-\delta}}{M_{\alpha+\beta-\gamma-\delta}} A_{\alpha-\gamma, \beta-\delta} \cdot \frac{(4LH)^{\gamma+\delta}}{M_{\gamma+\delta}} B_{\gamma+\delta}.$$

By (19) and (20), this yields

$$a \leq \sum_{\alpha, \beta \in \mathbf{N}_0} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} b_{\gamma, \delta}^{\alpha, \beta} \leq \sum_{\alpha, \beta \in \mathbf{N}_0} \frac{\kappa C}{2^{\alpha+\beta}}$$

for some constant $C > 0$, in view of (2) and (3).

If now $i, j \in \mathbf{N}$, $j > i$ so that $i \rightarrow \infty$, then $\kappa = \kappa_{i,j} \rightarrow 0$ and, since the above estimate can be expressed as

$$(21) \quad a_{i,j} = |\langle f \otimes g, (\eta_j - \eta_i)\phi^\Delta \rangle| \leq \kappa_{i,j}C,$$

we conclude that the sequence (a_n) , where

$$a_n := |\langle f \otimes g, \eta_n\phi^\Delta \rangle|,$$

is a Cauchy sequence. It is easy to see that its limit does not depend on the choice of a strong approximate unit (η_n) and this proves the first assertion (cf. [8]).

The second assertion can be proved in a similar way as the second statement of Theorem 1. ■

4. Convolution in weighted spaces. The weighted ultradistributional spaces $\mathcal{D}'_{L^s, \mu}^{(M_p)}$ are defined in [12], similarly to the distributional spaces $\mathcal{D}'_{L^s, \mu}$ introduced in [16]. We recall some definitions and assertions from [12].

Let $s \in [1, \infty]$, $h > 0$, $\mu \in \mathbf{R}$. Then $\mathcal{D}_{L^s, \mu}^{M_p, h}$ is defined to be the space of all functions $\phi \in \mathcal{E}$ such that $(\langle \cdot \rangle^\mu \phi)^{(\alpha)} \in L^s$ (see (1)) for each $\alpha \in \mathbf{N}_0$ and

$$\|\phi\|_{\mathcal{D}_{L^s, \mu}^{M_p, h}} = \sum_{\alpha \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|(\langle \cdot \rangle^\mu \phi)^{(\alpha)}\|_s < \infty,$$

equipped with the topology induced by the norm $\|\cdot\|_{\mathcal{D}_{L^s, \mu}^{M_p, h}}$. Further, we define

$$\mathcal{D}_{L^s, \mu}^{(M_p)} := \text{proj} \lim_{h \rightarrow \infty} \mathcal{D}_{L^s, \mu}^{M_p, h}.$$

The space $\dot{\mathcal{B}}_\mu^{(M_p)}$ is defined to be the subspace of $\mathcal{D}_{L^\infty, \mu}^{(M_p)}$ which is the completion of the space $\mathcal{D}^{(M_p)}$ in the topology of the family of the norms $\|\cdot\|_{\mathcal{D}_{L^s, \mu}^{M_p, h}}$,

One can easily prove that the mappings:

$$\begin{aligned} \mathcal{D}_{L^s}^{(M_p)} &\ni \phi \mapsto \langle \cdot \rangle^{-\mu} \phi \in \mathcal{D}_{L^s, \mu}^{(M_p)}, \\ \dot{\mathcal{B}}_\mu^{(M_p)} &\ni \phi \mapsto \langle \cdot \rangle^{-\mu} \phi \in \dot{\mathcal{B}}_\mu^{(M_p)} \end{aligned}$$

are homeomorphisms. The spaces $\mathcal{D}_{L^s, \mu}^{(M_p)}$ and $\dot{\mathcal{B}}_\mu^{(M_p)}$ are (FG) -spaces and the spaces $\mathcal{D}_{L^s, \mu}^{(M_p)}$ are (FS) -spaces for $s > 1$ (for the definitions see [3]). The proof of this assertion may be done in a way similar to the proof given in [16] for $\mathcal{D}_{L^s}^{(M_p)}$, $s > 1$.

Let us recall some properties of the spaces defined above which will be needed in the sequel; for their proofs we refer to [12]. Let $\mu, \nu \in \mathbf{R}$ and $q, r, s \in [1, \infty]$. If

$1/q + 1/r \geq 1/s$, then the pointwise multiplications

$$\begin{aligned} \mathcal{D}_{L^q, \mu}^{(M_p)} \times \mathcal{D}_{L^r, \nu}^{(M_p)} &\ni (\phi, \psi) \mapsto \phi\psi \in \mathcal{D}_{L^s, \mu+\nu}^{(M_p)}, \\ \mathcal{D}_{L^s, \mu}^{(M_p)} \times \dot{\mathcal{B}}_{\nu}^{(M_p)} &\ni (\phi, \psi) \mapsto \phi\psi \in \mathcal{D}_{L^s, \mu+\nu}^{(M_p)}, \\ \mathcal{D}_{L^\infty, \mu}^{(M_p)} \times \dot{\mathcal{B}}_{\nu}^{(M_p)} &\ni (\phi, \psi) \mapsto \phi\psi \in \dot{\mathcal{B}}_{\mu+\nu}^{(M_p)}, \\ \dot{\mathcal{B}}_{\mu}^{(M_p)} \times \dot{\mathcal{B}}_{\nu}^{(M_p)} &\ni (\phi, \psi) \mapsto \phi\psi \in \dot{\mathcal{B}}_{\mu+\nu}^{(M_p)} \end{aligned}$$

are continuous mappings. If (a) $q < r$ and $\nu < \mu$ or (b) $q > r$ and $\nu < \mu + 1/q - 1/r$, then $\mathcal{D}_{L^q, \mu}^{(M_p)} \hookrightarrow \mathcal{D}_{L^r, \nu}^{(M_p)}$, where the symbol $E \hookrightarrow F$ means that the space E is continuously embedded into the space F and E is dense in F .

Given a $q \in (1, \infty]$, define r to satisfy the equation $1/q + 1/r = 1$. We denote by $\mathcal{D}'_{L^q, \mu}^{(M_p)}$ and $\mathcal{D}'_{L^1, \mu}^{(M_p)}$ the strong duals of the spaces $\mathcal{D}_{L^r, -\mu}^{(M_p)}$ and $\dot{\mathcal{B}}_{-\mu}^{(M_p)}$, respectively.

An ultradistribution $g \in \mathcal{D}'^{(M_p)}$ belongs to $\mathcal{D}'_{L^q, \mu}^{(M_p)}$ if there exist $h > 0$ and a sequence $(G_\alpha)_{\alpha \in \mathbf{N}}$ of elements of L^q such that

$$(22) \quad g = \sum_{\alpha \in \mathbf{N}_0} G_\alpha^{(\alpha)}$$

and

$$(23) \quad \sum_{\alpha \in \mathbf{N}_0} \frac{M_\alpha}{h^\alpha} \|\langle \cdot \rangle^\mu G_\alpha\|_q < \infty.$$

Conversely, if (23) holds for a sequence $(G_\alpha)_{\alpha \in \mathbf{N}_0}$ in L^q , then g defined by (22) belongs to $\mathcal{D}'_{L^q, \mu}^{(M_p)}$.

Let $q, r \in [1, \infty]$ and $\mu, \nu \in \mathbf{R}$. If (a) $q \leq r$ and $\nu \leq \mu$; or (b) $q > r$ and $\nu < \mu + 1/q - 1/r$, then we have the following embeddings:

$$\begin{array}{ccccccccc} \mathcal{E}' & \hookrightarrow & \mathcal{D}'_{L^r, \nu} & \hookrightarrow & \mathcal{D}'_{L^q, \mu} & \hookrightarrow & \mathcal{S}' & \hookrightarrow & \mathcal{D}' \\ \hookrightarrow & & \hookrightarrow & & \hookrightarrow & & \hookrightarrow & & \hookrightarrow \\ \mathcal{E}'^{(M_p)} & \hookrightarrow & \mathcal{D}'_{L^r, \nu}^{(M_p)} & \hookrightarrow & \mathcal{D}'_{L^q, \mu}^{(M_p)} & \hookrightarrow & \mathcal{S}'^{(M_p)} & \hookrightarrow & \mathcal{D}'^{(M_p)}. \end{array}$$

The space $\mathcal{D}'_{L^q, \mu}$ is a proper subset of $\mathcal{D}'_{L^q, \mu}^{(M_p)} \cap \mathcal{D}'$.

Now we shall prove some results (Theorems 3 and 4) on convolution in the weighted distributional and ultradistributional spaces.

THEOREM 3. *Fix $\phi \in \mathcal{S}^{(M_p)}$ and $q \in (1, \infty]$. If the sequence (M_p) satisfies conditions (M.1) and (M.3'), then*

- (i) $f \in \mathcal{D}'_{L^q, \mu}$ implies $f * \phi \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$;
- (ii) $f \in \mathcal{D}'_{L^1, \mu}$ implies $f * \phi \in \dot{\mathcal{B}}_{\mu}^{(M_p)}$.

If (M_p) satisfies conditions (M.1), (M.2') and (M.3'), then

- (iii) $f \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$ implies $f * \phi \in \mathcal{D}_{L^q, \mu}$;

(iv) $f \in \mathcal{D}'_{L^1, \mu}^{(M_p)}$ implies $f * \phi \in \dot{\mathcal{B}}_\mu$.

If (M_p) satisfies conditions (M.1), (M.2) and (M.3'), then

(v) $f \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$ implies $f * \phi \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$;

(vi) $f \in \mathcal{D}'_{L^1, \mu}$ implies $f * \phi \in \dot{\mathcal{B}}_\mu^{(M_p)}$.

Proof. To prove (i) suppose that $f \in \mathcal{D}'_{L^q, \mu}$. In view of a result of [16], it follows that $f * \phi \in \mathcal{D}'_{L^q, \mu}$ for each $\phi \in \mathcal{S}$. Therefore

$$\langle \cdot \rangle^{-\mu} (f * \phi)^{(\alpha)} = \langle \cdot \rangle^{-\mu} (f * \phi^{(\alpha)}) \in L^q$$

for $\alpha \in \mathbf{N}$. Since $f \in \mathcal{D}'_{L^q, \mu}$, there exist $\gamma \in \mathbf{N}_0$ and functions F_β in L^q such that

$$f = \sum_{\beta \leq \gamma} (\langle \cdot \rangle^{-\mu} F_\beta)^{(\beta)}$$

in $\mathcal{D}'^{(M_p)}$. Applying suitable properties of convolution (see [7]) and the following elementary inequality:

$$\langle a + b \rangle^\mu \leq 4^{|\mu|} \langle a \rangle^\mu \langle b \rangle^{|\mu|}, \quad a, b, \mu \in \mathbf{R}$$

(called *Peetre's inequality*), we get

$$\begin{aligned} & \sum_{\alpha \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|\langle \cdot \rangle^\mu (f * \phi)^{(\alpha)}\|_q \\ &= \sum_{\alpha \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|\langle \cdot \rangle^\mu \sum_{\beta \leq \gamma} (\langle \cdot \rangle^{-\mu} F_\beta) * \phi^{(\alpha+\beta)}\|_q \\ &\leq \sum_{\alpha \in \mathbf{N}_0} \sum_{\beta \leq \gamma} \frac{h^\alpha}{M_\alpha} \left(\int_{\mathbf{R}} \langle x \rangle^{\mu q} \left(\int_{\mathbf{R}} \langle x-t \rangle^{-\mu} |\phi^{(\alpha+\beta)}(x-t) F_\beta(t)| dt \right)^q dx \right)^{1/q} \\ &\leq 4^{|\mu|} \sum_{\alpha \in \mathbf{N}_0} \sum_{\beta \leq \gamma} \frac{h^\alpha}{M_\alpha} \|\langle \cdot \rangle^{|\mu|} |\phi^{(\alpha+\beta)}| * |F_\beta|\|_q. \end{aligned}$$

Hence, by Young's inequality,

$$\begin{aligned} \sum_{\alpha \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|\langle \cdot \rangle^\mu (f * \phi)^{(\alpha)}\|_q &\leq 4^{|\mu|} \sum_{\beta \leq \gamma} \sum_{\alpha \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|F_\beta\|_q \|\langle \cdot \rangle^{|\mu|} \phi^{(\alpha+\beta)}\|_1 \\ &\leq 4^{|\mu|} \max_{\beta \leq \gamma} \|F_\beta\|_q \max_{\beta \leq \gamma} (M_{\alpha+\beta} M_\alpha^{-1} h^{-\beta}) \sum_{\gamma \in \mathbf{N}_0} \frac{h^\gamma}{M_\gamma} \|\langle \cdot \rangle^{|\mu|} \phi^{(\gamma)}\|_1 < \infty. \end{aligned}$$

This yields $f * \phi \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$, as desired.

The proof of (ii) is similar.

To prove (iii) suppose that $f \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$. There are functions F_β in L^q and $h > 0$ such that

$$(24) \quad f = \sum_{\beta \in \mathbf{N}_0} (\langle \cdot \rangle^{-\mu} F_\beta)^{(\beta)}; \quad \sum_{\beta \in \mathbf{N}_0} \frac{M_\beta}{h^\beta} \|F_\beta\|_q < \infty$$

(cf. (22) and (23)). From the proof of Theorem 6.10 in [16] (p. 71), it follows that $f * \phi \in \mathcal{E}$. Similarly to the previous case, using (M.b') and Young's inequality, we get

$$\begin{aligned} \|\langle \cdot \rangle^\mu (f * \phi)^{(\alpha)}\|_q &= \|\langle \cdot \rangle^\mu \sum_{\beta \in \mathbf{N}_0} (\langle \cdot \rangle^{-\mu} F_\beta) * \phi^{(\alpha+\beta)}\|_q \\ &\leq 4^{|\mu|} \sum_{\beta \in \mathbf{N}_0} \|F_\beta * (\langle \cdot \rangle^{|\mu|} \phi^{(\alpha+\beta)})\|_q \leq C \sum_{\beta \in \mathbf{N}_0} \frac{h^\beta}{M_\beta} \|\langle \cdot \rangle^{|\mu|} \phi^{(\alpha+\beta)}\|_1 \\ &\leq C_\alpha \sum_{\beta \in \mathbf{N}_0} \frac{(H^\alpha h)^{\alpha+\beta}}{M_{\alpha+\beta}} \|\langle \cdot \rangle^{|\mu|} \phi^{(\alpha+\beta)}\|_1 \leq C_\alpha \sum_{\beta \in \mathbf{N}_0} \frac{(H^\alpha h)^\gamma}{M_\gamma} \|\langle \cdot \rangle^{|\mu|} \phi^{(\gamma)}\|_1 < \infty \end{aligned}$$

for each $\alpha \in \mathbf{N}_0$, where

$$(25) \quad C := 4^{|\mu|} \sup_{\beta \in \mathbf{N}_0} \frac{M_\beta}{k^{\beta}} \|F_\beta\|; \quad C_\alpha := \frac{CA^\alpha}{M_\alpha}$$

(C and C_α are finite, in view of (24)). Hence $f * \phi \in \mathcal{D}_{L^q, \mu}$.

The proof of (iv) is similar.

To prove (v) suppose as before that $f \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$. By (24), condition (M.b) and Young's inequality, we obtain

$$\begin{aligned} \sum_{\alpha \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|\langle \cdot \rangle^\mu (f * \phi)^{(\alpha)}\|_q &\leq \sum_{\beta \in \mathbf{N}_0} \sum_{\alpha \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|\langle \cdot \rangle^\mu |(\langle \cdot \rangle^{-\mu} F_\beta) * \phi^{(\alpha+\beta)}|\|_q \\ &\leq 4^{|\mu|} \sum_{\alpha, \beta \in \mathbf{N}_0} \frac{h^\alpha}{M_\alpha} \|F_\beta * (\langle \cdot \rangle^{|\mu|} \phi^{(\alpha+\beta)})\|_q \leq C \sum_{\alpha, \beta \in \mathbf{N}_0} \frac{h^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle \cdot \rangle^{|\mu|} \phi^{(\alpha+\beta)}\|_1 \\ &\leq AC \sum_{\alpha, \beta \in \mathbf{N}_0} \frac{(Hh)^{\alpha+\beta}}{M_{\alpha+\beta}} \|\langle \cdot \rangle^{|\mu|} \phi^{(\alpha+\beta)}\|_1 \leq AC \sum_{\gamma \in \mathbf{N}_0} \frac{(Hh)^\gamma}{M_\gamma} \|\langle \cdot \rangle^{|\mu|} \phi^\gamma\|_1 < \infty, \end{aligned}$$

where C is the constant defined in (25). This means that $f * \phi \in \mathcal{D}_{L^q, \mu}$.

The proof of (vi) is similar. ■

THEOREM 4. *Suppose that the sequence (M_p) satisfies conditions (M.1) and (M.3'). Fix $\mu, \nu \in \mathbf{R}$ such that $\mu + \nu \geq 0$ and $q, r \in [1, \infty]$ such that $1 \leq 1/q + 1/r \leq 2$.*

(i) *If (M_p) satisfies additionally condition (M.2') and if $f \in \mathcal{D}'_{L^q, \mu}$ and $g \in \mathcal{D}'_{L^r, \nu}^{(M_p)}$, then $f * g = f *_t g \in \mathcal{D}'_{L^s, \rho}^{(M_p)}$ and the mapping*

$$(26) \quad \mathcal{D}'_{L^q, \mu} \times \mathcal{D}'_{L^r, \nu}^{(M_p)} \ni (f, g) \mapsto f * g \in \mathcal{D}'_{L^s, \rho}^{(M_p)}$$

is continuous, where $\rho := \min(\mu, \nu)$ and $s := (1/q + 1/r - 1)^{-1}$ (then $s \in [1, \infty]$).

(ii) *If (M_p) satisfies additionally condition (M.2) and if $f \in \mathcal{D}'_{L^q, \mu}^{(M_p)}$ and $g \in$*

$\mathcal{D}'_{L^r, \nu}^{(M_p)}$, then $f * g = f *_t g \in \mathcal{D}'_{L^s, \rho}^{(M_p)}$ and the mapping

$$(27) \quad \mathcal{D}'_{L^q, \mu}^{(M_p)} \times \mathcal{D}'_{L^r, \nu}^{(M_p)} \ni (f, g) \mapsto f * g \in \mathcal{D}'_{L^s, \rho}^{(M_p)},$$

is continuous, where s and ρ are defined above.

Proof. The proof is divided into five steps.

1° We shall show that, for every strong approximate unit (η_n) in $\mathcal{D}^{(M_p)}(\mathbf{R}^2)$ and $\psi \in \mathcal{D}'_{L^{s/(s-1)}, -\rho}^{(M_p)}$, the limit

$$\lim_{n \rightarrow \infty} \langle f \otimes g, \eta_n \psi^\Delta \rangle$$

exists and the mapping

$$\mathcal{D}'_{L^{s/(s-1)}, -\rho}^{(M_p)} \ni \psi \mapsto \lim_{n \rightarrow \infty} \langle f \otimes g, \eta_n \psi^\Delta \rangle \in \mathbf{R}$$

defines an element of $\mathcal{D}'_{L^s, \rho}^{(M_p)}$. This will imply that

$$f * g = f *_t g \in \mathcal{D}'_{L^s, \rho}^{(M_p)}.$$

By (22) and the representation

$$f = \sum_{\alpha \leq \alpha_0} (\langle \cdot \rangle^{-\mu} F_\alpha)^{(\alpha)}, \quad (F_\alpha)_\alpha \in L^q, \quad \alpha \leq \alpha_0,$$

we have

$$f \otimes g = \sum_{\alpha \leq \alpha_0} \sum_{\beta \in \mathbf{N}_0} F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)},$$

where, for simplicity of notation, we set

$$F_{\mu, \alpha} := \langle \cdot \rangle^{-\mu} F_\alpha, \quad G_{\nu, \beta} := \langle \cdot \rangle^{-\nu} G_\beta;$$

the sum on the right hand side converges weakly in $\mathcal{D}'^{(M_p)}(\mathbf{R}^2)$.

Consider an arbitrary subsequence of a given strong approximate unit (η_i) and select a subsequence (η_{m_i}) for which there exist increasing sequences of positive numbers (a_i) and (b_i) divergent to ∞ such that, for all $i \in \mathbf{N}$, we have $a_i < b_i < a_{i+1}$, $\eta_{m_i} \equiv 1$ on $[-a_i, a_i] \times [-a_i, a_i]$ and $\text{supp } \eta_{m_i} \subset [-b_i, b_i] \times [-b_i, b_i]$.

Let (n_i) be another increasing sequence of positive integers. We shall prove that $((A(m_i, n_i))_{i \in \mathbf{N}}$ is a Cauchy sequence, where

$$A(m_i, n_i) := \sum_{\alpha \leq \alpha_0} \sum_{\beta \leq n_i} \langle F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)}, \bar{\eta}_{m_i} \psi^\Delta \rangle.$$

It will also be seen that the limit does not depend on the sequences (m_i) and (n_i) and this will imply that the limit

$$\lim_{i \rightarrow \infty} \sum_{\alpha \leq \alpha_0} \sum_{\beta \in \mathbf{N}_0} \langle F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)}, \eta_i \psi^\Delta \rangle,$$

exists and is equal to

$$\sum_{\alpha \leq \alpha_0} \sum_{\beta \in \mathbf{N}_0} \lim_{i \rightarrow \infty} \langle F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)}, \eta_i \psi^\Delta \rangle.$$

First note that, for each $\varepsilon > 0$, there exists a $\beta_0 \in \mathbf{N}$ such that

$$(28) \quad \sum_{\beta \geq \beta_0} \frac{M_\beta}{h^\beta} \|G_\beta\|_r < \varepsilon.$$

Next, notice that conditions (M.b') and (M.a) imply that, for each $d \geq 1$ and $\alpha, \beta, \gamma, \delta \in \mathbf{N}$ such that $\gamma \leq \alpha$ and $\delta \leq \beta$, we have

$$(29) \quad \frac{1}{M_\beta} \leq \frac{A^\beta H^{(\alpha+\beta)\beta}}{M_{\alpha+\beta}} \leq \frac{(dAH^\beta)^{\alpha+\beta}}{d^\beta M_{\alpha+\beta}} \leq \frac{(dAH^\beta)^{\alpha+\beta}}{d^\beta M_{\alpha-\gamma+\beta-\delta} M_{\gamma+\delta}}.$$

Fix now the sequence (m_i) and denote $\bar{\eta}_i = \eta_{m_i}$. Next, fix indices $i, k \in \mathbf{N}$ so that $i < k$ and denote for short $a := a_i, b := b_k$ (hence we have $0 < a < b$). It is easy to see that

$$\text{supp}(\bar{\eta}_k - \bar{\eta}_i) \subset K_1 \cup K_2,$$

where

$$K_1 := (J_1 \cup J_2) \times J; \quad K_2 := J \times (J_1 \cup J_2)$$

and

$$J_1 := [-2b, -a]; \quad J_2 := [a, 2b]; \quad J := [-2b, 2b].$$

Putting

$$c_{\gamma, \delta}^{\alpha, \beta} := \binom{\alpha}{\gamma} \binom{\beta}{\delta}, \quad d_{\gamma, \delta} := \|(\bar{\eta}_k - \bar{\eta}_i)^{(\gamma, \delta)}\|_\infty$$

for $0 \leq \gamma \leq \alpha, 0 \leq \delta \leq \beta$, we have

$$(30) \quad \sum_{\substack{\alpha \leq \alpha_0 \\ n_i \leq \beta \leq n_k}} |\langle F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)}, (\bar{\eta}_k - \bar{\eta}_i) \psi^\Delta \rangle| = \sum_{\substack{\alpha \leq \alpha_0 \\ n_i \leq \beta \leq n_k}} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} c_{\gamma, \delta}^{\alpha, \beta} d_{\gamma, \delta} (I_1 + I_2),$$

where

$$(31) \quad I_i = \int_{K_i} |(F_{\mu, \alpha} \otimes G_{\nu, \beta})(\psi^{(\alpha-\gamma, \beta-\delta)})^\Delta|$$

for $i = 1, 2$.

Assume first that $\rho = \nu$, i.e. $\nu \leq \mu, \mu \geq 0$. In this case, Peetre's inequality yields

$$\langle x \rangle^{-\mu} \langle t-x \rangle^{-\nu} \leq 4^{|\nu|} \langle x \rangle^{-\mu+|\nu|} \langle t \rangle^{-\nu} \leq 4^{|\nu|} \langle t \rangle^{-\nu}$$

for any $x, t \in \mathbf{R}$ and thus

$$\begin{aligned} I_1 &= \int_{J_1 \cup J_2} \left(\int_{J_x} |F_{\mu, \alpha}(x) G_{\nu, \beta}(t-x) \psi^{(\alpha+\beta-\gamma-\delta)}(t)| dt \right) dx \\ &\leq 4^{|\nu|} \int_{J_1 \cup J_2} \int_{J'} |F_\alpha(x) G_\beta(t-x) \langle t \rangle^{-\nu} \psi^{(\alpha+\beta-\gamma-\delta)}(t)| dx dt, \end{aligned}$$

where

$$J_x := \{t \in \mathbf{R} : t - x \in J\}$$

and

$$J' := (J_1 \cup J_2) + J = [-4b, 2b - a] \cup [-2b + a, 4b].$$

Now, assume that $\rho = \mu$, i.e. $\mu \leq \nu$, $\nu \geq 0$. Then, by Peetre's inequality,

$$\langle y - t \rangle^{-\mu} \langle y \rangle^{-\nu} \leq 4^{|\mu|} \langle y \rangle^{|\mu| - \nu} \langle t \rangle^\mu \leq 4^{|\mu|} \langle t \rangle^{-\mu}$$

for any $y, t \in \mathbf{R}$ and, since $J \times J^y = (J_1 \cup J_2) \times J_x$, we have

$$\begin{aligned} I_1 &= \int_J \left(\int_{J^y} |F_{\mu, \alpha}(y - t) G_{\nu, \beta}(y) \psi^{(\alpha + \beta - \gamma - \delta)}(t)| dt \right) dy \\ &\leq 4^{|\mu|} \int_{J_1 \cup J_2} \left(\int_{J_x} |F_\alpha(x) G_\beta(t - x) \langle t \rangle^{-\mu} \psi^{(\alpha + \beta - \gamma - \delta)}(t)| dt \right) dx \\ &\leq 4^{|\mu|} \int_{J_1 \cup J_2} \int_{J'} |F_\alpha(x) G_\beta(t - x) \langle t \rangle^{-\mu} \psi^{(\alpha + \beta - \gamma - \delta)}(t)| dx dt, \end{aligned}$$

where $J^y := \{t \in \mathbf{R} : y - t \in J_1 \cup J_2\}$.

Hence, putting $C := 4^{|\rho|}$, $D_{\gamma, \delta}^{\alpha, \beta} := \|\langle \cdot \rangle^{-\rho} \psi^{(\alpha + \beta - \gamma - \delta)}\|_{s/(s-1)}$ and denoting by F'_α the function which equals to 0 on the interval $[-a, a]$ and coincides with F_α otherwise, we get in both cases

$$\begin{aligned} (32) \quad I_1 &\leq C \int_{-\infty}^{\infty} |\langle t \rangle^{-\rho} \psi^{(\alpha + \beta - \gamma - \delta)}(t)| \left(\int_{\mathbf{R} \setminus [-a, a]} |F_\alpha(x) G_\beta(t - x)| dx \right) dt \\ &\leq C \int_{-\infty}^{\infty} |\langle t \rangle^{-\rho} \psi^{(\alpha + \beta - \gamma - \delta)}(t)| (|F'_\alpha| * |G_\beta|)(t) dt \\ &\leq C D_{\gamma, \delta}^{\alpha, \beta} \| |F'_\alpha| * |G_\beta| \|_s \leq C D_{\gamma, \delta}^{\alpha, \beta} \|F'_\alpha\|_q \|G_\beta\|_r, \end{aligned}$$

in view of Hölder's and Young's inequalities. Similarly, we prove that

$$(33) \quad I_2 \leq C D_{\gamma, \delta}^{\alpha, \beta} \|F_\alpha\|_q \|G'_\beta\|_r,$$

where G'_β are functions equal to 0 on $[-a, a]$ and to G_β otherwise. Combining (30)–(33) and taking into account that

$$\sup_{\beta, \gamma \in \mathbf{N}_0} \frac{(dAH^\beta)^\gamma}{M_\gamma} < \infty, \quad \sup_{\alpha, \beta \in \mathbf{N}_0} \frac{(dAH^\beta)^\alpha}{M_\alpha} < \infty,$$

we get

$$\begin{aligned} \sum_{\substack{\alpha \leq \alpha_0 \\ n_i \leq \beta \leq n_k}} |\langle F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)}, (\bar{\eta}_k - \bar{\eta}_i) \psi^\Delta \rangle| &\leq C \sum_{\substack{\alpha \leq \alpha_0 \\ n_i \leq \beta \leq n_k}} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} c_{\gamma, \delta}^{\alpha, \beta} d_{\gamma, \delta} D_{\gamma, \delta}^{\alpha, \beta} \\ &\quad \times \frac{M_\beta}{h^\beta} (\|F_\alpha\|_{L^q(\mathbf{R} \setminus [-a, a])} \|G_\beta\|_r + \|F_\alpha\|_q \|G_\beta\|_{L^r(\mathbf{R} \setminus [-a, a])}). \end{aligned}$$

If now i, j with $j > i$ vary so that $i \rightarrow \infty$, then $a = a_i \rightarrow \infty$ and the above estimate, in view of (28), shows that $((A(m_i, n_i))_{i \in \mathbf{N}})$ is a Cauchy sequence. It is easy to see that its limit does not depend on subsequences $(\bar{\eta}_i)$. Consequently,

$$\begin{aligned} \langle f * g, \psi \rangle &= \lim_{i \rightarrow \infty} \langle f \otimes g, \eta_i \psi^\Delta \rangle \\ &= \lim_{i \rightarrow \infty} \sum_{\substack{\alpha \leq \alpha_0 \\ \beta \in \mathbf{N}_0}} \langle F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)}, \eta_i \psi^\Delta \rangle = \sum_{\substack{\alpha \leq \alpha_0 \\ \beta \in \mathbf{N}_0}} \langle F_{\mu, \alpha}^{(\alpha)} * G_{\nu, \beta}^{(\beta)}, \psi \rangle. \end{aligned}$$

Hence (see [18]) f and g are convolvable, $f * g = f *_t g$ and

$$\langle f *_t g, \psi \rangle = \sum_{\substack{\alpha \in \mathbf{N}_0 \\ \beta \in \mathbf{N}_0}} \langle F_{\mu, \alpha} * G_{\nu, \beta}^{(\alpha + \beta)}, \psi \rangle$$

for each $\psi \in \mathcal{D}_{L^s/(s-1), -\rho}^{(M_p)}$, which finishes the proof of the first step.

2° Let us prove that, under the assumptions of (ii), $f * g = f *_t g$. According to (22), there exist a sequence $(F_\alpha)_\alpha$ of elements of L^q , a sequence $(G_\beta)_\beta$ of elements of $L^r(\mathbf{R})$ and $h > 0$ such that

$$f = \sum_{\alpha \in \mathbf{N}_0} F_{\mu, \alpha}^{(\alpha)}, \quad g = \sum_{\beta \in \mathbf{N}_0} G_{\nu, \beta}^{(\beta)},$$

where the two series converge in the weak sense, and that

$$\sum_{\alpha \in \mathbf{N}_0} \frac{M_\alpha}{h^\alpha} \|F_\alpha\|_r < \infty, \quad \sum_{\beta \in \mathbf{N}_0} \frac{M_\beta}{h^\beta} \|G_\beta\|_q < \infty.$$

It follows that

$$f \otimes g = \sum_{\alpha \in \mathbf{N}_0} \sum_{\beta \in \mathbf{N}_0} F_{\mu, \alpha}^{(\alpha)} \otimes G_{\nu, \beta}^{(\beta)},$$

where the series on the right hand side converges weakly in $\mathcal{D}'^{(M_p)}(\mathbf{R}^2)$.

Conditions (M.b) and (M.a) imply that

$$1 \leq \frac{AH^{\alpha+\beta} M_\alpha M_\beta}{M_{\alpha+\beta}} \leq \frac{A(dH)^{\alpha+\beta}}{M_{\alpha-\gamma+\beta-\delta} M_{\gamma+\delta}} \frac{M_\alpha M_\beta}{d^\alpha d^\beta}$$

for each $d > 0$, $\alpha, \beta \in \mathbf{N}$ and $\gamma, \delta \in \mathbf{N}$ such that $\gamma \leq \alpha$, $\delta \leq \beta$. Applying the above estimate instead of (29) and repeating the arguments used in 1°, one can conclude that f and g are convolvable and

$$\langle f *_t g, \psi \rangle = \sum_{\alpha \in \mathbf{N}_0} \sum_{\beta \in \mathbf{N}_0} \langle F_{\mu, \alpha} * G_{\nu, \beta}^{(\alpha + \beta)}, \psi \rangle$$

for $\psi \in \mathcal{D}_{L^s/(s-1), -\rho}^{(M_p)}$.

The next part of the proof is similar to that of Proposition 9 in [16].

3° Suppose that condition (M.2') holds. If $f \in \mathcal{D}'_{L^q, \mu}$ is fixed, then the mapping

$$\mathcal{D}'_{L^q, \nu} \ni g \mapsto f * g \in \mathcal{D}'_{L^s, \rho},$$

has the closed graph. Indeed, if (g_i) converges to zero in $\mathcal{D}'_{L^r, \nu}$ and $(f * g_i)$ converges to h in $\mathcal{D}'_{L^s, \rho}$, then

$$\langle h, \phi \rangle = \lim_{i \rightarrow \infty} \langle f * g_i, \phi \rangle = \lim_{i \rightarrow \infty} \langle g_i(\check{f} * \phi), 1 \rangle$$

for each $\phi \in \mathcal{D}^{(M_p)}$ (see [16], Proposition 6). Note that $\check{f} * \phi \in \mathcal{D}_{L^q, \mu}^{(M_p)}$. Moreover, the multiplication, as a mapping from $\mathcal{D}_{L^q, \mu}^{(M_p)} \times \mathcal{D}'_{L^r, \nu}$ into $\mathcal{D}'_{L^1, \mu+\nu}$, is separately continuous, and $\mathcal{D}'_{L^1, \mu+\nu}$ is continuously embedded into \mathcal{D}'_{L^1} . This implies that $g_i(\check{f} * \phi)$ converges to zero in \mathcal{D}'_{L^1} as $i \rightarrow \infty$. Therefore $\langle h, \phi \rangle = 0$ for each $\phi \in \mathcal{D}^{(M_p)}$. Consequently, the linear mapping in question has the closed graph.

If $g \in \mathcal{D}'_{L^q, \nu}$ is fixed, then the mapping

$$\mathcal{D}'_{L^q, \mu} \ni f \mapsto g * f \in \mathcal{D}'_{L^s, \rho}$$

has the closed graph. Indeed, if (f_i) converges to zero in $\mathcal{D}'_{L^q, \mu}$ and $(f_i * g)$ converges to h in $\mathcal{D}'_{L^s, \rho}$, then

$$\langle h, \phi \rangle = \lim_{i \rightarrow \infty} \langle f_i * g, \phi \rangle = \lim_{i \rightarrow \infty} \langle f_i(\check{g} * \phi), 1 \rangle$$

for each $\phi \in \mathcal{D}^{(M_p)}$ (see [18], Proposition 6, or [7]). We have $\check{g} * \phi \in \mathcal{D}_{L^r, \nu}$ and, moreover, the multiplication, as a mapping from $\mathcal{D}_{L^q, \mu} \times \mathcal{D}'_{L^r, \nu}$ to $\mathcal{D}'_{L^1, \mu+\nu}$, is separately continuous, and $\mathcal{D}'_{L^1, \mu+\nu}$ is continuously embedded into \mathcal{D}'_{L^1} . Therefore $f_i(\check{g} * \phi)$ converges to zero in \mathcal{D}'_{L^1} as $i \rightarrow \infty$ and thus $\langle h, \phi \rangle = 0$ for each $\phi \in \mathcal{D}^{(M_p)}$. Consequently, the linear mapping in question has a closed graph.

4° Similarly, one can prove that if condition (M.2) is fulfilled and $g \in \mathcal{D}'_{L^r, \nu}$ is fixed, then the mapping

$$\mathcal{D}'_{L^q, \mu} \ni f \mapsto f * g \in \mathcal{D}'_{L^s, \rho},$$

has the closed graph.

5° Note that

- (a) the space $\mathcal{D}'_{L^q, \rho}^{(M_p)}$ is the strong dual of a Fréchet space;
- (b) the space $\mathcal{D}'_{L^q, \nu}$, $q \in [1, \infty]$, is an inductive limit of Banach spaces (see [16], Theorem 9).

As we have noted, since $\mathcal{D}'_{L^1}^{(M_p)} = (\mathcal{B}_C^{(M_p)})'$ and the space $\mathcal{B}_C^{(M_p)}$ is semireflexive (see [18]), it follows that $\mathcal{D}'_{L^1}^{(M_p)}$ is barrelled. Therefore the space $\mathcal{D}'_{L^1, \nu}^{(M_p)}$, which is isomorphic to $\mathcal{D}'_{L^1}^{(M_p)}$, is barrelled. It follows that the spaces $\mathcal{D}'_{L^1, \nu}^{(M_p)}$ are distinguished Fréchet spaces (see [4], p. 228, Proposition 1). Thus $\mathcal{D}'_{L^1, \nu}^{(M_p)}$ is a

bornological space ([4], p. 289, Theorem 1) and hence, being complete, is an inductive limit of Banach spaces.

It follows from statements given in [20] (Appendix, p. 159, Theorem 3, and p. 164) that if E is an inductive limit of Banach spaces (which implies that E is ultrabornological), F is a strong dual of a Fréchet space and $T : E \rightarrow F$ is a sequentially closed linear mapping, then T is continuous. Consequently, the convolution mappings (26) and (27) are partially continuous by the closed graph theorem. But then they are continuous (see [4], p. 364, Exercise 10). ■

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