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TAQDIR HUSAIN

Multipliers of topological algebras

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WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

1. Introduction

All algebras considered in the sequal are associative. An algebra A over the field of real or complex numbers, endowed with a Hausdorff topology is called a *topological algebra* if the maps: $(x, y) \to x+y$, $(\lambda, x) \to \lambda x$ and $(x, y) \to xy$ are jointly continuous in both variables together, where $x, y \in A$ and λ is a real or complex scalar. If the topology of a topological algebra is given by a metric, then it is called *metrizable*. A topological algebra is called a *locally convex (LC) algebra* if its topology is described by a family $\{\varrho_\alpha\}$ of seminorms. If each member of a defining family $\{\varrho_\alpha\}$ of seminorms of a LC algebra A is submultiplicative, i.e., $\varrho_\alpha(xy) \leqslant \varrho_\alpha(x)\varrho_\alpha(y)$ for all $x, y \in A$, then it is called *locally m-convex (LMC)* ([15]). A complete metrizable LC algebra is called a *B_0-algebra* ([16]) and a complete metrizable LMC algebra is called a *Fréchet algebra* ([2]).

Let A be a topological algebra over the complex field $\boldsymbol{C}$. A sequence $\{x_i\}$ of A is said to be a *basis* of A if for each $x \in A$ there is a unique sequence $\{\lambda_i(x)\}$ of complex numbers, depending upon x, such that $x = \sum \lambda_i(x) x_i$. Each $\lambda_i = \lambda_i(x)$ is a linear functional on A. If each λ_i is continuous, the basis $\{x_i\}$ is called a *Schauder basis*. A basis $\{x_i\}$ is called *orthogonal* if $x_i x_j = \delta_{ij} x_i$. If $\{x_i\}$ is an orthogonal basis, then each λ_i is multiplicative ([7]).

A self-map T of an algebra A is called a *multiplier* of A if $Tx \cdot y = x \cdot Ty$ for all $x, y \in A$. The set of all multipliers of A will be denoted by $M(A)$. The purpose of this paper is to study the set $M(A)$ when A is a topological algebra, in particular, a Banach algebra with orthogonal basis.

For instance, it is shown that each element $T \in M(A)$ is determined by a complex function on the maximal ideal space of A (Theorem 2.5) which is bounded if A is a Banach algebra (Theorem 2.7). It is known ([7]) that each Banach space with an unconditional normalized basis can be given an equivalent norm and multiplication making it a Banach algebra with the given basis becoming orthogonal. Here we show (Theorem 3.10) that on the other hand, each Banach algebra with an orthogonal basis can be endowed with an equivalent norm in which the basis is not orthogonal. We construct an example

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of a Banach algebra A possessing an orthogonal basis for which $M(A)$ is the smallest (Example 3.12 and Proposition 3.13). The author is thankful to K. Jarosz for this example and other results in Sections 7 and 8.

As is the case in scalar sequence spaces, we define π-algebras for the topological algebras with orthogonal bases and study some of their properties. See Sections 4 and 5. Further, as in the case of Banach spaces or more generally in locally convex vector spaces Schauder decompositions have been defined and studied, similarly here we define orthogonal decomposition of topological algebras and show that an orthogonal decomposition of a B_0-algebra is very well behaved (Theorem 6.2). Moreover, π-algebras associated with Schauder decompositions of a Banach algebra are subalgebras of the algebra l_∞ of all bounded complex sequences and they are representable by operators on appropriate Banach spaces (Theorem 6.5). In Section 7, we show that under certain conditions, a Schauder basis of a commutative Banach algebra can be perturbed to obtain an orthogonal basis (Theorem 7.5). In Section 8, we show that each orthogonal basis of a Banach algebra is an unconditional orthogonal basis of a subalgebra. Moreover, we introduce a new method of defining topological algebras which have an orthogonal basis obtained by taking the tensor product of bases.

For a background on multipliers of a Banach algebra, the reader is referred to Larsen [9] and for a general study of Banach algebras, see Żelazko [13]. The theory of orthogonal bases in topological algebras was first introduced by this author and has been extensively studied by him in collaboration with Liang and Watson among others. See, for example, the references [3], [4], [5], [6], [7], [8]. Thanks to the referee for pointing out some obscurities and omissions.

2. Multipliers of topological algebras

For the time being, A will stand for a complex algebra without a topology and then later it will be a topological algebra with appropriate conditions.

To study some properties of multipliers we assume that A is proper: A is called *proper* if for any $x \in A$, $xA = Ax = \{0\}$ implies $x = 0$. If A has identity, then A is proper. Moreover, a topological algebra with an approximate identity is proper. We also known ([5], Proposition 3.1) that each topological algebra with an orthogonal basis is proper.

A map $T\colon A \to A$ is called a *multiplier* if $Tx \cdot y = x \cdot Ty$ (also written as $Txy = xTy$ if no confusion arises). If A is proper, then a multiplier T on A is linear because for $x, y \in A$ and $\alpha, \beta \in \boldsymbol{C}$, $[T(\alpha x + \beta y) - \alpha Tx - \beta Ty]z = \alpha x Tz + \beta y Tz - \alpha x Tz - \beta y Tz = 0$ for all $z \in A$ ([12]) and satisfies another important equation: $T(xy) = Tx \cdot y (= x \cdot Ty)$ because $[T(xy) - Tx \cdot y]z = T(xy) \cdot z - (Tx \cdot y)z = xyTz - xyTz$ (using the associative law) $= 0$ for all $z \in A$ ([12]). Recalling that $M(A)$ denotes the set of all multipliers of A, we see that if A is proper, then $M(A)$ is an algebra with the identity map of A as its identity.

PROPOSITION 2.1. *Let A be a commutative algebra with identity. Then $M(A)$ is algebraically isomorphic with A.*

Proof. For each $x \in A$, if we put $L_x(y) = xy$ for all $y \in A$, then it is easy to verify that $L_x \in M(A)$ and the map: $x \to L_x$ into $M(A)$ is an injective (because A is proper) algebra homomorphism. Suppose $T \in M(A)$. Put $x_0 = T(e)$. Then for all $x \in A$, $Tx = T(ex) = x_0 x = L_{x_0}(x)$ shows that A and $M(A)$ are isomorphic. ∎

Proposition 2.1 shows that the study of multipliers for a commutative algebra with identity is somewhat uninteresting.

Now suppose A stands for a Hausdorff topological algebra. Let $L_c(A)$ denote the set of all continuous linear self-maps of A. Clearly $L_c(A)$ is an algebra under composition and has identity. We topologize $L_c(A)$ as follows: Let $\{U\}$ denote the family of all closed circled neighbourhoods of $0 \in A$. Let β denote the family of all bounded (resp. finite) subsets of A. For each $B \in \beta$ and $U \in \{U\}$, let

$$T(B, U) = \{T \in L_c(A):\ T(B) \subset U\}.$$

As B runs over β and U on $\{U\}$, the family $\{T(B, U)\}$ defines a topology on $L_c(A)$, called the *strong* (resp. *simple*)*-operator topology*. It is easily seen that $L_c(A)$ is a Hausdorff topological vector space. Moreover, for any given $T(B, U)$, a neighbourhood of 0 in $L_c(A)$ and for any fixed $T_0 \in L_c(A)$, if we choose $V \in \{U\}$ such that $T_0(V) \subset U$, then for all $S \in T(B, V)$

$$T_0 S(B) \subset T_0(V) \subset U,$$

shows that the map: $(T, S) \to TS$ is separately continuous. In other words, $L_c(A)$ endowed with the strong topology is a semitopological algebra. If A is a complete metrizable topological vector space (in particular, algebra), then $L_c(A)$ is complete in the strong topology ([12], p. 112). Thus if A is a B_0-algebra (complete metrizable LC algebra) or a Fréchet algebra (complete metrizable LMC algebra), then $L_c(A)$ is a complete semitopological algebra. If a B_0-algebra A has a countable fundamental system of bounded sets, then $L_c(A)$ is a complete metrizable semitopological algebra, hence by Arens' theorem (see [27]) $L_c(A)$ is also a complete metrizable topological algebra.

THEOREM 2.2. *If A is a proper topological algebra, then each multiplier of A has closed graph.*

Proof. Let T be a multiplier of A. Let $\{x_\alpha\}$ be a net in A such that $x_\alpha \to x$ and $Tx_\alpha \to y$ in A. Let $z \in A$ be an arbitrary element. Then by the continuity of multiplication, $x_\alpha z \to xz$ and $Tz \cdot x_\alpha \to Tz \cdot x$. Since T is a multiplier, $T(zx_\alpha) = zTx_\alpha = Tz \cdot x_\alpha$. Hence

$$\lim_\alpha zTx_\alpha = \lim_\alpha Tz \cdot x_\alpha = zy = Tz \cdot x = zTx.$$

Thus $z(y - Tx) = 0$ for all $z \in A$. Since A is proper, we have $y = Tx$, proving that the graph of T is closed. ∎

COROLLARY 2.3. *Each multiplier on a proper complete metrizable algebra, in particular B_0-algebra or Fréchet or Banach algebra is continuous.*

Proof. It follows from the Closed Graph Theorem ([1]) in view of Theorem 2.2. ∎

THEOREM 2.4. *Let A be a complete metrizable proper topological algebra. Then $M(A)$ is a closed subalgebra of $L_c(A)$ endowed with the strong topology.*

Proof. Since A is proper, each $T \in M(A)$ is linear. By Corollary 2.3, $M(A) \subset L_c(A)$ and $M(A)$ is an algebra (Proposition 2.1). To prove that $M(A)$ is closed, let $\{T_\alpha\}$ be a net in $M(A)$ such that $T_\alpha \to T \in \overline{M(A)} \subset L_c(A)$. (Note that $L_c(A)$ is complete). Thus for all $x, y \in A$, $T_\alpha x \cdot y = x \cdot T_\alpha y$ implies $Tx \cdot y = x \cdot Ty$, i.e. $T \in M(A)$. ∎

Now let A be a semisimple commutative topological algebra and $M(A)$ the set of multipliers on A. We show that each linear $T \in M(A)$ can be represented by a continuous map on $\Delta(A)$, provided $\Delta(A)$ is nonempty. (Note $\Delta(A)$ denotes the set of all nonzero continuous linear multiplicative functionals on A endowed with the w^*-topology which may be empty in general.) Let $M_l(A)$ denote the subset of $M(A)$ consisting of linear multipliers. $M(A) = M_l(A)$ is an algebra if A is proper or in particular has identity. For each $x \in A$, we put

$$\hat{x}(f) = f(x), \quad f \in \Delta(A),$$

the Gelfand transform of x. We have:

THEOREM 2.5. *To each $T \in M_l(A)$ there corresponds a continuous map:*

$$\mu^T\colon \Delta(A) \to \boldsymbol{C}$$

such that

$$T\hat{x} = \mu^T \cdot \hat{x}$$

for all $x \in A$.

Proof. We define $\mu^T\colon \Delta(A) \to \boldsymbol{C}$ by

$$\mu^T(f) = f \circ T(x),$$

where $x \in A$ is such that $f(x) = 1$. To see that μ^T is well defined, suppose $x_1, x_2 \in A$ with $f(x_1) = 1 = f(x_2)$. Then

$$f \circ T(x_1 x_2) = f(x_1) f(Tx_2) = f \circ T(x_2),$$
$$f \circ T(x_2 x_1) = f(x_2) f(Tx_1) = f \circ T(x_1).$$

Moreover, we see that for all $f \in \Delta(A)$ and $x \in A$,

$$f \circ T(x) = \mu^T(f) f(x).$$

For, if $x_0 \in A$ is fixed such that $f(x_0) = 1$, then choose $\alpha, \beta \in C (\beta \neq 0)$ so that $f(\alpha x_0 + \beta x) = 1$. Then by definition of μ^T (using the linearity of T).

$$f \circ T(\alpha x_0 + \beta x) = f(\alpha x_0 + \beta x) \mu^T(f),$$

and thus we have

$$f \circ T(x) = \mu^T(f) f(x)$$

for all $f \in \Delta(A)$ and $x \in A$.

To prove that μ^T is continuous at $f_0 \in \Delta(A)$, let $\varepsilon > 0$ be given. There is $x_0 \in A$ with $f_0(x_0) = 1$. Let U' be the set of all $f \in \Delta(A)$ such that

$$|f(Tx_0) - f_0(Tx_0)| < \varepsilon.$$

Then U' is a w^*-neighbourhood of f_0 in $\Delta(A)$ and for all $f \in U'$,

$$|\mu^T(f) - \mu^T(f_0)| = |f(Tx_0) - f_0(Tx_0)| < \varepsilon.$$

This proves the continuity of μ^T. ∎

PROPOSITION 2.6. *The map ψ: $T \to \mu^T$ of $M(A)$ into $C(\Delta(A))$ is an algebra homomorphism, if A is proper.*

Proof. If $T, S \in M_l(A)$, then clearly $\alpha T + \beta S \in M_l(A)$ for $\alpha, \beta \in C$ and $TS \in M_l(A)$. But then,

$$\psi(\alpha T + \beta S) = \mu^{\alpha T + \beta S} = \alpha \mu^T + \beta \mu^S = \alpha \psi(T) + \beta \psi(S)$$

and similarly

$$\psi(TS) = \psi(T) \psi(S). \ ∎$$

The following is known (see [12]):

THEOREM 2.7. *If A is a semisimple commutative Banach algebra A, then for each $T \in M(A)$, μ^T is a bounded function on $\Delta(A)$.*

Proof. By Corollary 2.3, T is a bounded linear operator on A and so $\|T\| < \infty$. Also for each $f \in \Delta(A)$, $\|f\| \leqslant 1$. For each $f \in \Delta(A)$, let

$$\alpha_f = \sup_{\|x\| = 1} |f(x)| \leqslant 1$$

By definition, $f(x)\mu^T(f) = f \circ T(x)$ for $x \in A$ (Theorem 2.5). Therefore

$$|f(x)|\,|\mu^T(f)| = |f(T(x))| \leqslant \|f\|\,\|T\|\,\|x\| \leqslant \alpha_f \|T\|\,\|x\|.$$

Whence for $f \in \Delta(A)$,

$$\begin{aligned} |\mu^T(f)| &\leqslant \frac{\alpha_f \|T\|\,\|x\|}{|f(x)|} \qquad (\text{if } f(x) \neq 0) \\ &\leqslant \inf_{\|x\|=1} \frac{\alpha_f \|T\|\,\|x\|}{|f(x)|} \\ &\leqslant \frac{\alpha_f \|T\|}{\sup\limits_{\|x\|=1} |f(x)|} = \|T\| < \infty. \end{aligned}$$

This completes the proof. ∎

COROLLARY 2.8. *The map $\psi\colon T \to \mu^T$ of $M(A)$ into $C_b(\Delta(A))$ is norm decreasing, where $C_b((A))$ is the Banach algebra of all bounded continuous complex-valued functions on $\Delta(A)$.*

Proof. By the above theorem

$$\|\mu^T\| = \sup_{f \in \Delta(A)} |\mu^T(f)| \leqslant \|T\|.$$

THEOREM 2.9. *Let A be a semisimple commutative topological algebra such that $\Delta(A)$ is point-separating. Then each linear map $T\colon A \to A$ is a multiplier iff $T(M) \subset M$ for all $M \in \{M_f\}$, where $f^{-1}(0) = M_f$ for $f \in \Delta(A)$.*

Proof. Suppose $T \in M(A)$. Then by Theorem 2.5,

$$\mu^T(f) = f \circ T(x)$$

for some $x \in A$ with $f(x) = 1$. Let M_f be a fixed member of $\{M_f\}$. Suppose $x \notin M_f$ and $y \in M_f$. Then $xy \in M_f$ and so $f(xy) = 0$. From $Tx \cdot y = x \cdot Ty$ we have

$$f(Tx)f(y) = f(x)f(Ty).$$

Since $f(x) \neq 0$, we have $f(Ty) = 0$ for all $y \in M_f$. This proves that $TM_f \subset M_f$.

Conversely, suppose $T(M) \subset M$ for all $f \in \Delta$. Then for $x \in M_f$ (or $y \in M_f$)

$$f(x \cdot Ty) - f(Tx \cdot y) = f(x)f(Ty) - f(Tx)f(y) = 0,$$

because $Tx \in M_f$ (or $Ty \in M_f$). Now if both $x, y \in M_f$, then M_f being a maximal ideal, there are $u, v \in M_f$, $\alpha, \beta \in C$ such that $x = u + \alpha z$, $y = v + \beta z$ for some $z \in A$. But then

$$\begin{aligned} x \cdot Ty - Tx \cdot y &= (u + \alpha z)(Tv + \beta Tz) - (Tu + \alpha Tz)(v + \beta z) \\ &= (uTv - Tuv) + \alpha(zTv - Zzv) + \beta(uTz - Tuz). \end{aligned}$$

From this we derive that

$$f(x\cdot Ty)-f(Tx\cdot y)=0 \qquad \text{for all } f\in\Delta(A).$$

Since $\Delta(A)$ is point-separating, we have $x\cdot Ty=Tx\cdot y$, i.e. $T\in M(A)$. ∎

We restate Corollary 2.8 differently for future use.

COROLLARY 2.10. *Let A be a commutative semisimple Banach algebra. Then for each $T\in M(A)$*

$$\mu^T(f)=f\circ T, \qquad f\in\Delta(A),$$

with

$$\|\mu^T\|\leqslant\|T\|.$$

In the following sections we shall deal with special algebras, which are automatically proper and hence each multiplier on them is linear. Moreover, $\Delta(A)$ is point-separating.

3. Multipliers of topological algebras with orthogonal bases

From now on, we assume A to be a topological algebra possessing an orthogonal basis $\{x_i\}$. Thus A is commutative ([7]) and proper ([5]). We note that the basis $\{x_i\}$ is automatically a Schauder basis [see S. El-Helaly and T. Husain, *Orthogonal bases are Schauder bases and a characterization of Φ-algebras*, Pacific J. Math. 132 (1988), 265–275]. It is known ([5]) that the set $\Delta(A)$ of all closed maximal ideals consists of all kernels of coordinate functionals λ_i's, i.e.

$$M_i=\{x\in A\colon\ \lambda_i(x)=0\}.$$

Thus $\Delta(A)=\{M_i\}$ is a countable discrete space homeomorphic with the discrete space of positive integers. In this section, we study the multipliers of topological algebras with an orthogonal basis, in particular Banach algebras possessing orthogonal bases.

Let A stand for a topological algebra with an orthogonal Schauder basis $\{x_i\}$ in the sequel. Hence A is a commutative semisimple topological algebra such that $\Delta(A)$ is point-separating. We derive the following results from the previous ones as special cases:

PROPOSITION 3.1. *Each multiplier of A is determined by a sequence of complex number.*

Proof. Since $\Delta(A)=\{M_i\}$, we see that it is a particular case of Theorem 2.5. ∎

Remark 3.2. Since each $T \in M(A)$ of a topological algebra with an orthogonal basis $\{x_i\}$ is determined by a sequence $\{\mu_i\}$ of complex numbers, we shall denote $\{\mu_i\}$ by

$$\mu^T = \{\mu_i^T\} \in s,$$

where s is the algebra of all complex sequences with pointwise operations.

If $\hat{x}$ denotes the Gelfand transform of $x \in A$, then as seen above,

$$\hat{x}(i) = \lambda_i(x), \quad i \geqslant 1 \text{ and } x \in A,$$

$$\widehat{Tx}(i) = \mu_i^T \lambda_i(x) \quad \text{for all } i \geqslant 1 \text{ and } x \in A.$$

If ϕ denotes the mapping from A to s, defined by $\phi(x) = \{\lambda_i(x)\}$, then

$$\phi(Tx) = \{\mu_i^T \lambda_i(x)\} \in s.$$

Thus we can define $\psi\colon M(A) \to s$ by $\psi(T) = \{\mu_i^T\} = \mu^T$.

PROPOSITION 3.3. *If A is a topological algebra with an orthogonal Schauder basis $\{x_i\}$, then the map $\psi\colon T \to \mu^T$ of $M(A)$ into s is an injective continuous homomorphism.*

Proof. Such an algebra is semisimple ([7], Corollary 1.5) and so it is a particular case of Proposition 2.6. The only thing that needs a proof is the continuity of ψ. But this follows from the assumption that the basis is Schauder. ∎

THEOREM 3.4. *Let A be a Banach algebra with an orthogonal basis $\{x_i\}$. Then for each $T \in M(A)$, $\mu^T \in l_\infty$, the Banach algebra of all bounded complex sequences.*

Proof. This is a particular case of Theorem 2.7. ∎

Remark 3.5. Setting

$$M^{\#}(A) = \{\mu^T = \{\mu_i^T\}\colon\ T \in M(A)\},$$

we see that $M^{\#}(A)$ is algebraically isomorphic with $M(A)$ by Proposition 3.3.

If A is a Banach algebra with an orthogonal basis, then $M^{\#}(A) \subset l_\infty$ (Theorem 3.4). We can therefore endow $M^{\#}(A)$ with the induced sup-norm topology. On the other hand, we have:

THEOREM 3.6. *Let A be a B_0-algebra with identity e and possessing an orthogonal unconditional basis $\{x_i\}$. Then $l_\infty \subset M^{\#}(A)$. Since $l_\infty \subset M^{\#}(A)$ implies that the basis $\{x_i\}$ is unconditional, one can state this theorem as follows: The basis is unconditional iff $l_\infty \subset M^{\#}(A)$.*

Proof. We know that $e = \sum x_i$. Let $\beta = \{\beta_i\} \in l_\infty$. Since the basis is unconditional, the series $\sum \beta_i x_i$ converges in A. Let $x_0 = \sum \beta_i x_i$ and consider $L_{x_0}(x) = x_0 x$ $(x \in A)$. Then $L_{x_0} \in M(A)$ (Proposition 2.1(b)). If $x = \sum \lambda_i(x) x_i$, then $L_{x_0}(x) = \sum \beta_i \lambda_i(x) x_i$ or equivalently $\lambda_i(L_{x_0}(x)) = \beta_i \lambda_i(x)$ for all $i \geqslant 1$. Thus $\{\beta_i\}$ is defined by a multiplier L_{x_0} of A and so $\beta \in M^{\#}(A)$. ∎

Remark. We note that though there is only one Fréchet algebra satisfying the hypothesis of Theorem 3.6, namely $A = s$, but there are other nontrivial B_0-algebras of this type. For example, $A = H(D)$, the algebra of all holomorphic functions on the open unit disc endowed with the compact-open topology. A is a B_0-algebra under multiplication

$$f * g(x) = \frac{1}{2\pi i} \int\limits_{|z|=r} f(z) g(xz^{-1}) z^{-1} \, dz,$$

where $x \in D$, $|x| < r < 1$ with an orthogonal unconditional basis $\{z^n\}_{n \geqslant 0}$ and identity $z^0 = 1$ (see [7] for details).

PROPOSITION 3.7. *For any Banach algebra A with an orthogonal basis $\{x_i\}$, the mapping ψ: $M(A) \to M^{\#}(A)$* (see Proposition 3.3) *is norm decreasing.*

Proof. For each $T \in M(A)$, from the proof of Theorem 3.4, we have

$$\|\mu^T\|_\infty = \sup_{i \geqslant 1} |\mu_i^T| \leqslant \|T\|.$$

COROLLARY 3.8. *Let A be a function algebra (i.e. a Banach algebra in which $\|x\|_\infty = \|x\|$) with an orthogonal basis $\{x_i\}$. Then $M(A)$ and $M^{\#}(A)$ are isometric and hence $M^{\#}(A)$ is the Banach algebra l_∞ because the only such an algebra A is c_0.*

As we have seen (Theorem 2.9) above, a multiplier, as a linear operator of a topological algebra with an orthogonal basis can be characterized by its action on those maximal ideals which are kernels of nonzero multiplicative functionals. A particular case is the following:

THEOREM 3.9. *Let A be a topological algebra with an orthogonal Schauder basis $\{x_i\}$. Let $\{\lambda_i\}$ denote the sequence of its coordinate functionals and*

$$M_i = \{x \in A\colon\ \lambda_i(x) = 0\}$$

for each $i \geqslant 1$. A linear map T: $A \to A$ is a multiplier of A iff $T(M_i) \subset M_i$ for all $i \geqslant 1$.

We know ([7], Proposition 4.1) that each Banach space with an unconditional normalized basis can be given a multiplication and an equivalent norm making it a Banach algebra having the given basis as an orthogonal basis. Now we have a result in the reverse direction as follows:

THEOREM 3.10. *Let A be a Banach algebra with an orthogonal basis $\{x_i\}$. Then there exists an equivalent norm and a new multiplication making A a Banach algebra with $\{x_i\}$ as its basis but not an orthogonal basis.*

Proof. Let $T\in M(A)$ such that $Tx_i \neq x_i$ for at least one i. Let $\mu^T = \{\mu_i^T\} \in M^{\#}(A)$ which is determined by T (Proposition 3.1). Thus for each $x = \sum \lambda_i(x)x_i \in A$,

$$\lambda_i(Tx) = \mu_i^T \lambda_i(x)$$

for all $i \geqslant 1$ and $x\in A$. We define a new multiplication in A by

$$x \circ y = xTy$$

for $x, y\in A$. It is easy to check that $x\circ y$ satisfies all the properties that a multiplication in an algebra should. Here the multiplication is associative, even though in general this formula may not define an associative multiplication.

To define a new norm on A, we put

$$\|x\|' = \|T\|\,\|x\|,$$

for all $x\in A$. (Note that by Corollary 2.3 T is a bounded linear operator.) It is clear that $\|x\|'$ is an equivalent norm on A and

$$\|x\circ y\|' = \|T\|\,\|xTy\| \leqslant \|T\|\,\|x\|\,\|T\|\,\|y\| = \|x\|'\,\|y\|',$$

thus proving that A with the new multiplication and norm is a Banach algebra. Since for each $x = \sum \lambda_i(x)x_i \in A$, the series converges in the equivalent norm $\|\cdot\|'$, it follows that $\{x_i\}$ is a basis of the new Banach algebra $(A, \|\cdot\|', \circ)$. But

$$x_i \circ x_j = x_i Tx_j = T(x_i x_j) = T(\delta_{ij} x_i) = \delta_{ij} T(x_i)$$

shows that $x_i \circ x_j \neq \delta_{ij} x_i$ unless $Tx_i = x_i$ for all $i \geqslant 1$, which is ruled out. ■

Corollary 3.11. *Let A be a Banach algebra with an orthogonal basis $\{x_i\}$. Let $M(A)$ and $M^{\#}(A)$ be as before. Then for each $i \geqslant 1$ $\lambda_i \circ T = \mu_i^T$ for all $T\in M(A)$, where $\{\lambda_i\}$ is the sequence of coordinate functionals.*

Example 3.12. We construct an example of a commutative Banach algebra A with an orthogonal (but not unconditional) basis such that

$$M(A) \simeq l_1 \oplus C.$$

That is the set of multipliers of A is the smallest possible. Let l_1 be the usual Banach space. Let

$$f_n = (1, 1, \ldots, 1, 0, 0, \ldots), \qquad n \geqslant 1.$$

Then $\{f_n\}$ is a Schauder basis of l_1 but not an unconditional one. We note that for any scalar sequence $\{\alpha_i\}$,

$$\textstyle\sum \alpha_n f_n \in l_1 \quad \text{iff } \sum \alpha_n \text{ is convergent.}$$

We define multiplication in l_1 by

$$\left(\textstyle\sum \alpha_n f_n\right)\cdot\left(\sum \beta_n f_n\right) = \sum 2^{-n} \alpha_n \beta_n f_n.$$

It is easy to verify that with this multiplication l_1 is a Banach algebra. Moreover,

$$f_n \cdot f_m = 2^{-n}\delta_{nm}$$

and hence, if $g_n = 2^n f_n$, then $\{g_n\}$ is an orthogonal basis of l_1. We denote l_1 with this multiplication and orthogonal basis by A. To complete the construction of the example, we prove:

PROPOSITION 3.13. *A map T: $A \to A$ is a multiplier of A iff for some $\{\mu_n\} \in l_1 \oplus C$*

$$T(\sum \alpha_n f_n) = \sum \mu_n \alpha_n f_n.$$

Proof. $\Rightarrow$ is trivial. For the reverse implication assume T is a multiplier of A given by a sequence $\{\mu_n\}$. We have to show that there exists a sequence $\{\beta_n\} \in l_1$ and $\lambda \in C$ such that for all $n \geqslant 1$.

$$\mu_n = \beta_n + \lambda.$$

First we show that $\{\mu_n\}$ is convergent. Suppose not, then there are subsequences $\{k_n\}$, $\{l_n\}$ of positive integers such that $\mu_{k_n} \to \lambda_0$ and $\mu_{l_n} \to \lambda_1$ with $\lambda_0 \neq \lambda_1$. (Such subsequences exist because $\{\mu_n\}$ is clearly bounded.) Replacing $\alpha T + \beta$ by T with appropriate α and β, without loss of generality we can assume that $\lambda_0 = 1$ and $\lambda_1 = 0$. Taking subsequences, if necessary, we can also assume that

$$k_n < l_n < k_{n+1} \quad \text{for all } n > 1.$$

Now we define $\{\alpha_n\}$ by

$$\alpha_{k_n} = \frac{1}{n}, \quad \alpha_{l_n} = -\frac{1}{n}, \quad \alpha_n = 0 \quad \text{for all other } n.$$

Then the series $\sum \alpha_n$ is convergent and so $\sum \alpha_n f_n \in A$. But $\{\mu_n \alpha_n\}$ is of the form:

$$\mu_{k_n}\alpha_{k_n} = \mu_{k_n}\frac{1}{n}, \quad \text{in which } \mu_{k_n} \to 1,$$

$$\mu_{l_n}\alpha_{l_n} = -\mu_{l_n}\frac{1}{n}, \quad \text{in which } \mu_{l_n} \to 0,$$

$$\mu_n \alpha_n = 0 \qquad \text{otherwise}.$$

Now the series $\sum \alpha_n$ is divergent and so $\sum \alpha_n \mu_n f_n \notin A$, i.e., T is not a multiplier. Hence $\{\mu_n\}$ must be convergent. Let $\lambda = \lim \mu_n$ and put $\beta_n = \mu_n - \lambda$. Then $\lim \beta_n = 0$.

We show that $\{\beta_n\} \in l_1$. Suppose not, then by multiplying T by an appropriate scalar, if necessary, we may assume that $\{\beta_n'\} \notin l_1$, where $\beta_n' = \max(0, \operatorname{Re}\beta_n)$, $\operatorname{Re}\beta_n$ being the real part of β_n. Let $\{k_n\}$ be the subsequence of all positive integers such that $\beta_{k_n}' \geqslant 0$.

We define a sequence $\{r_n\}$ of positive integers as follows: Let r_1 be the smallest integer ($\geqslant 1$) such that

$$\sum_{j=1}^{r_1} \beta'_{k_j} \geqslant 1.$$

(Such an r_1 exists because $\sum \beta'_n$ is divergent.) Let $r_2 > r_1$ be any integer such that

$$\beta'_j \geqslant \frac{1}{2r_1} \quad \text{for all } j \geqslant k_{r_2}$$

(such an r_2 exists because $\lim \beta'_n = 0$). Put $r_3 = r_2 + r_1$. Inductively, we choose r_{3j+1} to be the smallest positive integer such that

$$\sum_{j=r_{3j}+1}^{r_{3j+1}} \beta'_{kj} \geqslant 1,$$

r_{3j+2} is such that $r_{3j+2} > r_{3j+1}$ and such that

$$\beta'_j \leqslant \frac{1}{r_{3j+1} - r_{3j}} \cdot \frac{1}{2} \quad \text{for all } j \geqslant k_{3j+2}.$$

Put $r_{3j+3} = r_{3j+2} + (r_{3j+1} - r_{3j})$. Now we define $\{\alpha_n\}$ by:

$$\begin{aligned}
\alpha_n &= 0 && \text{if } \beta'_n < 0,\\
\alpha_{k_n} &= \frac{1}{j} && \text{if } r_{3j}+1 \leqslant n \leqslant r_{3j+1},\\
\alpha_{l_n} &= 0 && \text{if } r_{3j+1} < n \leqslant r_{3j+2},\\
\alpha_{k_n} &= \frac{-1}{j} && \text{if } r_{3j+2} < n \leqslant r_{3j+3}.
\end{aligned}$$

Then we see that $\sum \alpha_n$ converges to 0 and $\sum \beta_n \alpha_n$ is divergent because

$$\begin{aligned}
\mathrm{Re}\Big(\sum_{n=1}^{k_{3j+3}} \beta_n \alpha_n\Big) &\geqslant 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3} - \frac{1}{2}\cdot\frac{1}{3} + \ldots + \frac{1}{j} - \frac{1}{2}\cdot\frac{1}{j}\\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2j} \to \infty.
\end{aligned}$$

Thus we have shown that $\{\beta_n\} \in l_1$ with

$$\mu_n = \beta_n + \lambda.$$

This concludes the proof. ∎

We close this section with the answer of the following problem: Suppose A is a topological algebra with an orthogonal basis. Does there exist a topological algebra B such that $A \approx M(B)$?

Since $M(B)$ always contains identity, we assume that A has identity to answer the above question. Moreover, since $M(B)$ is often complete, we assume that A is complete. With these remarks we have:

THEOREM 3.14. *Let A be a complete topological algebra with identity and possessing an unconditional orthogonal basis $\{x_i\}$. Then there is a topological algebra B such that $A \approx M(B)$.*

Proof. For B we take A', the dual of A endowed with the weak*-topology. We show that B is a topological algebra. For this, let $f \in B$ and put $\alpha_i = f(x_i)$. If $x = \sum_{i=1}^{\infty} \lambda_i(x) x_i \in A$, by the continuity of f we have:

$$f(x) = f\Big(\sum_{i=1}^{\infty} \lambda_i(x)\, x_i\Big) = \sum_{i=1}^{\infty} \lambda_i(x)\, \alpha_i.$$

Since f is continuous, it is spectrally continuous which means that f, restricted to l_∞ (because the basis $\{x_i\}$ being unconditional implies l_∞ is embedded in A) with the sup-norm, is continuous. Thus $\{\alpha_i\} \in l_1$.

Now if $f \in A'$ is fixed and $x = \sum \lambda_i(x) x_i \in A$, then $y = \sum \lambda_i(y) x_i \in A$, where $\lambda_i(y) = \operatorname{sgn}[\alpha_i \lambda_i(x)]$. Thus we have

$$|\alpha_i \lambda_i(x)| = \alpha_i \lambda_i(x) \lambda_i(y) = \alpha_i \lambda_i(xy).$$

Whence

$$f(xy) = \sum \alpha_i \lambda_i(x) \lambda_i(y) = \sum |\alpha_i \lambda_i(x)| < \infty.$$

This proves that $\{\alpha_i \lambda_i(x)\} \in l_1$. (We note that A' cannot have identity. But this is not needed in the proof.)

To show that A' is an algebra, let $f, g \in A'$ and define

$$f \cdot g(x) = \sum_{i=1}^{\infty} f(x_i)\, g(x_i)\, \lambda_i(x) \qquad \text{for } x = \sum \lambda_i(x) x_i \in A.$$

Since $\{f(x_i)\} \in l_1$ and $\{f(x_i)\lambda_i(x)\} \in l_1$ as shown above, $f \cdot g$ is a well-defined continuous functional on A. Moreover, the map $(f, g) \to f \cdot g$ is jointly continuous, i.e. B is a commutative topological algebra. Furthermore, $f \cdot g$ can be represented as a composition of the following maps:

$$x \to \{\lambda_i \alpha_i\} \to \sum_{i=1}^{\infty} \alpha_i \beta_i \lambda_i,$$

where $\alpha_i = f(x_i)$, $\beta = g(x_i)$, $\lambda_i = \lambda_i(x)$.

To complete the proof, we must show that $A \simeq M(B)$. Clearly $A \subset M(B)$ by the preceeding composition map. Let $\alpha = \{\alpha_i\} \in M(B)$ and we show that $\alpha \in A$. If $x \in A$, the map $f \to \sum \alpha_i f(x_i) \lambda_i(x)$ of A' into C is a continuous linear functional. This means that $\alpha \in A''$ Since A' is equipped with the weak*-topology and A is complete, we have $A'' = A$ and so $\alpha \in A$ ([15]).

It is easy to see that $\{\lambda_i\}$ is an orthogonal Schauder basis of A' and thus the proof is completed. ∎

4. π-Algebras

As before, let A denote a complete LC algebra with an unconditional orthogonal Schauder basis $\{x_i\}$ and identity e. We set

$$m(A) = \{x = \sum \lambda_i(x) x_i \in A\colon \sup_i |\lambda_i(x)| < \infty\}.$$

It is not hard to verify that $m(A)$ is a Banach algebra with the norm

$$\|x\|_s = \sup_{i \geq 1} |\lambda_i(x)|$$

and that $\{x_i\} \subset m(A)$. In this section, we show that $m(A)$ is the largest Banach algebra in a certain sense. But first we note that each subalgebra B of A containing $\{x_i\}$ (and identity e) is semisimple: Recall A is semisimple ([7]). Let $N_i = \{x - xx_i\colon x \in B\}$. Then each N_i is a maximal ideal of B and $\bigcap N_i = \{0\}$. In particular, $m(A)$ is a semisimple subalgebra of A.

PROPOSITION 4.1. *Every Banach algebra B contained in A and containing $\{x_i\}$, is a subset of $m(A)$.*

Proof. Let $\|\cdot\|$ denote the norm of B. Since $x_i \neq 0$ for all $i \geq 1$, $\|x_i\| > 0$ for all $i \geq 1$. Now for each $x \in B$, $x = \sum \lambda_i(x) x_i$ in A and so $xx_i = \lambda_i(x) x_i$ for all $i \geq 1$. But then

$$|\lambda_i(x)| \, \|x_i\| = \|\lambda_i(x) x_i\| = \|xx_i\| \leq \|x\| \, \|x_i\|$$

implies that

$$|\lambda_i(x)| \leq \|x\| \quad \text{for all } i \geq 1.$$

This shows that $x \in m(A)$. ∎

DEFINITION 4.2. A Banach algebra $(B, \|\cdot\|)$ contained in A and containing $\{x_i\}$ is called a *π-algebra* if there exists $K > 0$ such that

$$\sup_n \Big\| \sum_{i=1}^{n} xx_i \Big\| \leq K \|x\|$$

for all $x \in B$. (See also [13].)

PROPOSITION 4.3. *$m(A)$ is a π-algebra.*

Proof. If $x = \sum_i \lambda_i(x) x_i$ in A, then $xx_i = \lambda_i(x) x_i$ for all $i \geq 1$ and so

$$|\lambda_i(x)| \, \|x_i\|_s = \|\lambda_i(x)(x_i)\|_s \leq \|x_i\|_s \, \|x\|_s.$$

Since $\|x_i\|_s = 1$, where $\|\cdot\|_s$ is the norm of $m(A)$, it follows that

$$\sup_n \|\sum_{i=1}^{n} xx_i\|_s = \sup_n \|\sum_{i=1}^{n} \lambda_i(x)x_i\|_s \leqslant \|x\|_s.$$

Thus $K = 1$ verifies the definition of π-algebra. ∎

THEOREM 4.4. *Every π-algebra contained in A is continuously embedded in $m(A)$ and is semisimple. Moreover all norms are equivalent.*

Proof. The first part follows from Propositions 4.1, 4.2 and 4.3. The second part follows from the Open Mapping Theorem, since all Banach topologies on semisimple Banach algebras are equivalent (e.g. cf. [16]). ∎

DEFINITION 4.5. A π-algebra $B \subset m(A)$ is called of *second category* if $\mathrm{cl}_s B = B$, where cl_s denotes the closure of B in the sup-norm topology of $m(A)$, otherwise B is called of *first category.*

It is clear that A is of second category iff A is uniform.

PROPOSITION 4.6. *Every π-algebra $B \subset m(A)$ which is of second category contains $c(A) = \{x \in A\colon x = \sum \lambda_i(x)x_i$ with that $\lim \lambda_i(x)$ exists$\}$.*

Proof. Let $x \in c(A)$. Then $x = \sum \lambda_i(x)x_i$ (in A) for which $\lim_i \lambda_i(xy)$ exists for all $y \in m(A)$. Since $e \in m(A)$, we have $\lim_i \lambda_i(x) = \mu$ for $x \in c(A)$. But then

$$\sum \mu x_i = \mu \sum x_i = \mu e \in c(A).$$

Put

$$y_n = \mu e + \sum_{i=1}^{n} (\lambda_i(x) - \mu)x_i \quad \text{for } n = 1, \ldots$$

Then $y_n \in c(A)$ and so $y_n - \mu e \in c(A)$ for all $n \geqslant 1$. Clearly

$$\|y_n - \mu e\|_s = \sup_{1 \leqslant i \leqslant n} |\lambda_i(x) - \mu| \to 0 \quad \text{as } n \to \infty.$$

Since B is a π-algebra, $\{x_i\} \subset B$ and so $y_n - \mu e \in B$ for all $n \geqslant 1$. Thus

$$x = \lim_{n}{}_s \sum_{i=1}^{n} \lambda_i(x)x_i = \mu e + \lim_{n}{}_s (y_n - \mu e) \in \mathrm{cl}_s B = B,$$

because B is of second category and $B = \bar{B} \subset \mathrm{cl}_s B = B$ in which $\bar{B}$ is the closure of B in the norm topology of B which is coarser than the $\|\cdot\|_s$-norm topology in $m(A)$ ∎

EXAMPLE 4.7. (π-algebra of first category). Set

$$b(A) = \{x \in A\colon \sum_i |\lambda_i(xy) - \lambda_{i+1}(xy)| < \infty \quad \text{for all } y \in m(A)\}.$$

Since $e = \sum x_i \in m(A)$ for all $x \in b(A)$, we have $\sum_i |\lambda_i(x) - \lambda_{i+1}(x)| < \infty$. But then from

$$\lambda_{n+1}(x) = -\sum_{i=1}^{n} [\lambda_i(x) - \lambda_{i+1}(x)] + \lambda_1(x),$$

we see that $\{\lambda_i(x)\}$ is bounded for all $x \in b(A)$, i.e. $b(A) \subset m(A)$. It is clear that $\{x_i\} \subset b(A)$ and so $\phi_A \subset b(A) \subset m(A)$, where ϕ_A is the set of all finite linear combinations of $\{x_i\}$.

To show that actually $b(A) \subset c(A)$, let $x = \sum \lambda_i(x) x_i \in b(A)$. Then $\lim_i \lambda_i(x) = \mu$ and so $x = \mu e + x'$ in which $x' \in c_0(A) = \{x \in m(A)$ with $\lim \lambda_i(x) = 0\}$. It is easy to see that $b(A)$ is a vector space and

$$\|x\|_b = \sup_i |\lambda_i(x)| + \sum_i |\lambda_i(x) - \lambda_{i+1}(x)|$$

is a norm on $b(A)$. If $y \in m(A)$ and $n > m$, then

$$(4.1) \qquad |\lambda_n(xy) - \lambda_m(xy)| = \Big|\sum_{i=m}^{n-1} [\lambda_i(xy) - \lambda_{i+1}(xy)]\Big|$$

$$\leqslant \sum_{i=m}^{n-1} |\lambda_i(xy) - \lambda_{i+1}(xy)| \to 0 \quad \text{as } n, m \to \infty.$$

This shows that $\lim_i \lambda_i(xy)$ exists for each $y \in m(A)$, proving that $b(A) \subset c(A)$.

To see that $b(A)$ is a Banach space in the norm $\|\cdot\|_b$, let $\{y_n\}$ be a Cauchy sequence in $b(A)$. Let $\varepsilon > 0$ be given, there is n_0 such that for $m, n \geqslant n_0$, $\|y_n - y_m\|_b < \varepsilon$. As in the previous paragraph, it follows that $\{\lambda_i(y_n)\}_{n \geqslant 1}$ is a Cauchy sequence of complex numbers for each $i \geqslant 1$ and therefore $\lim_n \lambda_i(y_n) = \mu_i$ for all $i \geqslant 1$. Since for all $i \geqslant 1$

$$|\mu_i| = \lim_n |\lambda_i(y_n)| \leqslant \sup_n \|y_n\|_b < \infty,$$

(because $\{y_n\}$ being a Cauchy sequence of $b(A)$ is bounded) it follows that $\{\mu_i\} \in l_\infty$. Therefore there is $y \in A$ with $y = \sum_i \mu_i x_i$. Since $\{\mu_i\} \in l_\infty$, $y \in m(A)$. Now for any $z \in m(A)$, from (4.1)

$$\sum_{i=p}^{q} |\lambda_i(y_n z) - \lambda_{i+1}(y_n z)| \to 0 \quad \text{as } p, q \to \infty$$

for each $n = 1, 2,$ we derive that

$$\sum_{i=p}^{q} |\lambda_i(yz) - \lambda_{i+1}(yz)| \to 0 \quad \text{as } p, q \to \infty,$$

thus showing that $y \in b(A)$. It is easy to see that

$$\|y_n - y\|_b \to 0 \quad \text{as } n \to \infty.$$

Now to complete the example, we establish that $b(A)$ is of first category. We note that if $a_n = (-1)^{n-1}/n$, $n = 1, 2, \ldots$, then $\{a_n\} \in c_0$ and $x = \sum a_n x_n \in c(A)$, but $x \notin b(A)$. Hence $b(A)$ cannot be of second category by Proposition 4.6.

5. Orthogonal decomposition of topological algebras

In this section, we wish to study orthogonal decomposition of topological algebras similar to (Schauder) decomposition of topological vector spaces, in particular, Banach spaces.

DEFINITION 5.1. A commutative topological algebra A is said to have an *orthogonal* (*Schauder*) *decomposition* if there exists a countable family $\{M_i\}$ of ideals (respectively, closed ideals) of A such that $M_iM_j = 0$ for $i \neq j$ and for each $x \in A$ there exists a unique sequence $\{x_i\}$, $x_i \in M_i$ for all $i \geqslant 1$ such that

$$x = \sum x_i. \tag{5.1}$$

From the definition it is clear that for each $x \in A$, $xx_i = x_i^2$ for all $i \geqslant 1$, thus if x_i is idempotent, then $xx_i = x_i^2 = x_i$. It is easy to see that each orthogonal decomposition is a Schauder decomposition.

If a topological algebra A has an orthogonal basis $\{x_i\}$, then $x = \sum \lambda_i(x) x_i$ (uniquely). Since it is clear that $Ax_i = M_i$ is an ideal of A and $xx_i = \lambda_i x_i$ for all $i \geqslant 1$, we obtain an orthogonal decomposition of A.

The unique representation of x in (5.1) insures the existence of a projection map

$$P_i\colon A \to M_i$$

defined by $P_i(x) = x_i$ for all $i = 1, 2, \ldots$ Moreover we have

$$P_jP_i(x) = P_jx_i = 0 \quad \text{for } i \neq j,$$
$$P_i^2(x) \quad P_ix_i = x_i \quad \text{for all } i \geqslant 1.$$

Thus in the algebra of all self-maps of A, the sequence $\{P_i\}$ of projections forms an orthogonal family, viz.:

$$P_iP_j = \delta_{ij}P_i.$$

Furthermore, since $x_N = \sum_{i=1}^N x_i$, $y_M = \sum_{i=1}^M y_i$ imply by the continuity of multiplication in A that

$$xy = \lim_{N,M\to\infty} x_N y_M = \lim_{N\to\infty} \sum_{i=1}^N x_i y_i = \sum_{i=1}^\infty x_i y_i,$$

we have

$$P_i(xy) = x_i y_i = P_i(x)P_i(y) \quad \text{for all } i \geqslant 1.$$

In other words, each P_i is an algebra homomorphism of A onto M_i.

THEOREM 5.2. *Let $\{M_i\}$ define an orthogonal Schauder decomposition of a B_0-algebra A. Then for each $i \geqslant 1$ here exists a closed ideal N_i of A such that*

$$A = M_i \oplus N_i$$

(*direct sum*).

Proof. For each i, we define

$$N_i = \{x \in A\colon\ x = \sum_{j \neq i} x_j, \quad x_j \in M_j\}.$$

By the uniqueness of representation of x in (5.1), we have

$$x = x_i + \sum_{j \neq i} x_j.$$

Hence $A = M_i + N_i$, $M_i \cap N_i = \{0\}$ and thus

$$A = M_i \oplus N_i \quad \text{for each } i \geqslant 1.$$

To show that N_i is an ideal of A, let $x = \sum_{j \neq i} x_j \in N_i$ and $y = \sum y_i \in A$. Then

$$xy = \sum_{i \neq j} x_j y = \lim_n \sum_{j \neq i}^{n} x_j y_j = \sum_{j \neq i} x_j y_j \in N_i.$$

To prove N_i is closed, consider

$$L = \{\{x_i\} \in \prod M_i\colon\ \sum x_i \text{ exists in } A\}.$$

Since each M_i is a closed ideal of a B_0-algebra, it is itself a B_0-algebra and so is $\prod M_i$. We endow L with the topology induced from $\prod M_i$. More explicitly, if $\{\varrho_n\}$ is a countable family of continuous seminorms defining the B_0-topology of $\prod M_i$, then for each $x \in \prod M_i$,

$$|x| = d(x, 0) = \sum_n \frac{1}{2^n} \frac{\varrho_n(x)}{1 + \varrho_n(x)}$$

defines a quasinorm on $\prod M_i$ and for each $\{x_i\} \in L$, we set

$$\|\{x_i\}\|_d = \sup_{n \geqslant 1} d\Big(\sum_{i=1}^{n} x_i, 0\Big) = \sup_n \Big|\sum_{i=1}^{n} x_i\Big|.$$

Note that $\|\{x_i\}\|_d$ is a finite number, since a convergent sequence in a metric space is bounded. Thus it is routine to verify that $\|\{x_i\}|_d$ is a quasinorm on L. For completeness of L, let $\{x_{nj}\}_{j \geqslant 1}$ be a Cauchy sequence in L. For any $\varepsilon > 0$ there is n_0 such that for all $p, q \geqslant n_0$,

$$\|\{x_{pj}\} - \{x_{qj}\}\|_d = \sup_n \Big|\sum_{j=1}^{n} (x_{pj} - x_{qj})\Big| < \frac{\varepsilon}{2}.$$

Thus

$$\sup_n |x_{pn} - x_{qn}| \leqslant 2 \sup_n \Big|\sum_{j=1}^{n} (x_{pj} - x_{qj})\Big| = 2\, \|\{x_{pj}\} - \{x_{qj}\}\|_d < \varepsilon.$$

Since each M_i is complete, $\lim_n x_{ni} = x_{0i}$ for all $i \geqslant 1$. For each n there is $N_n \geqslant n_0$ such that $|\sum_{i=1}^{n}(x_{0i}-x_{N_n i})| < \varepsilon$. Hence

$$\|\{x_{0j}-x_{pj}\}\|_d = \sup_n \Big|\sum_{j=1}^{n}(x_{0j}-x_{pj})\Big|$$

$$\leqslant \Big\{\sup_n \Big|\sum_{j=1}^{n}(x_{N_n j}-x_{0j})\Big| + \sup_n \Big|\sum_{j=1}^{n}(x_{N_n j}-x_{pj})\Big|\Big\}$$

for all $p \geqslant n_0$. Since $\{x_{n_0 j}\} \in L$, there is $m_0 > n_0$ for which

$$\Big|\sum_{j=n}^{m} x_{n_0 j}\Big| < \varepsilon$$

whenever $n, m \geqslant m_0$. Thus

$$\Big\|\sum_{j=n}^{m} x_{0j}\Big\|_d \leqslant \Big\|\sum_{j=n}^{m}(x_{0j}-x_{n_0 j})\Big\|_d + \Big\|\sum_{j=n}^{m} x_{n_0 j}\Big\|_d$$

$$< 2\sup_n \Big\|\sum_{j=i}^{n}(x_{0j}-x_{mj})\Big\|_d + \varepsilon < 2\varepsilon$$

for all $n, m \geqslant m_0$ and so the limit exists in A, i.e. $\sum_{j=1}^{\infty} x_{0j}$ is convergent, proving the completeness of L.

Now let T: $L \to A$ be defined as follows: $T(\{x_i\}) = \sum_{i=1}^{\infty} x_i$. By the uniqueness of representation of elements of A, $\sum x_i = 0$ iff $x_i = 0$ for all $i \geqslant 1$. This proves that T is injective and by hypothesis $\{M_i\}$ being a decomposition of A, T is surjective. T is clearly linear and the inequality

$$\|T(\{x_j\})\|_d = \Big|\sum x_j\Big| \leqslant \sup_n \Big|\sum_{j=1}^{n} x_j\Big| = \|\{x_j\}\|_d$$

shows that T is continuous. By the open mapping theorem, T is open or equivalently T^{-1} is continuous.

Now to prove the closedness of N_i, let $y_0 \in \bar{N}_i$ such that $\lim_n y_n = y_0$ with $y_n \in N_i$. Since $y_n = \sum_{j=1, j\neq i}^{\infty} y_{nj}$, $y_0 = \sum_{j=1}^{\infty} y_{0j}$ $(y_{nj}, y_{0j} \in M_j)$ and $y_{ni} = 0$ for all $n \geqslant 1$, by the isomorphism of T^{-1}, we have

$$\{y_{0j}\} = T^{-1}(y_0) = T^{-1}(\lim_n y_n) = (\lim_n y_{nj}\}.$$

By definition of the metric on L, it follows that

$$y_{0i} = \lim_n y_{ni} = 0.$$

Hence $y_0 \in N_i$ and so each N_i is closed.

THEOREM 5.3. *Let A be a B_0-algebra. The following statements are equivalent*:

(i) *There exists a countable family* $\{M_i\}$ *of closed ideals in A forming an orthogonal Schauder decomposition.*

(ii) *There exists a sequence* $\{P_i\}$ *of continuous projections of A such that* $P_iP_j = \delta_{ij}P_i$, *each* P_i *is multiplicative and for each* $x \in A$, $x = \lim_n \sum_{i=1}^n P_i x$.

Proof. (i) $\Rightarrow$ (ii). As seen above, each Schauder orthogonal decomposition $\{M_i\}$ gives rise to a family of desired projections $\{P_i\}$. To show that each P_i is continuous, we note that for each M_i there exists (Theorem 5.2) a closed ideal N_i such that $A = M_i \oplus N_i$ for each $i \geqslant 1$. If $\{y_i\}$ is a sequence in A converging to $y \in A$ such that $\{P_i(y_j)\}_j$ converges to $z \in M_i$, then $\lim_j (y_j - P_i y_j) = y - z$. Since $P_i y_j \in M_i$ and $y_j - P_i y_j \in N_i$, it follows that $z \in M_i$ and $y - z \in N_i$. Hence $P_i y = P_i z = z$, showing that the graph of P_i is closed. Since both A and M_i are complete metrizable topological vector spaces, by the Closed Graph Theorem, P_i is continuous.

(ii) $\Rightarrow$ (i). Put $P_i(A) = M_i$. Then by assumptions on P_i, each M_i is a closed ideal and for each $x \in A$, $x = \sum_i x_i$ where $P_i x = x_i$ for all $i \geqslant 1$. To prove the uniqueness, suppose there is $\{y_j\}$ such that $\lim_n \sum_{j=1}^n y_j = 0$, $y_i \in M_i$. Since each P_i is continuous, we have

$$0 = \lim_n P_i\Big(\sum_{j=1}^{n} y_j\Big) = \lim_n \sum_{j=1}^{n} P_i y_j = P_i y_i = y_i.$$

Further, if $z_i = P_i(x) \in M_i$ and $z_j = P_j(y) \in M_j$, then $z_i z_j = P_i(x) P_j(y)$ implies

$$P_i(z_i z_j) = P_i(x)\,[P_i P_j(y)] = 0 \qquad \text{for } i \neq j.$$

This proves that $\{M_i\}$ is an orthogonal family satisfying (i). ∎

Remark. The Banach algebra l_∞ does not admit an orthogonal decomposition. Actually, there is no Banach algebra with unit which admits an orthogonal decomposition.

6. Π-Algebras associated with orthogonal decompositions

In Section 4, we studied π-algebras associated with topological algebras with orthogonal bases. Now we consider π-algebras associated with orthogonal decompositions.

DEFINITION 6.1. Let $\{M_i\}$ be a Schauder orthogonal decomposition of a topological algebra A. Let $\{P_i\}$ be the associated sequence of projections. Define

$$A_s = \Big\{\{\alpha_i\} \in s\colon \sum_i \alpha_i P_i x \text{ exists for all } x \text{ in } A\Big\}.$$

A_s will be called the *Π-algebra* associated with the orthogonal decomposition $\{M_i\}$.

It is easy to verify that A_s with pointwise operations is an algebra, containing the basic vectors $e_i = \{\delta_{ij}\}_j$, $i = 1, 2, \ldots$ Indeed, $\{e_i\}$ forms an orthogonal basis, called the *canonical basis*, of the Fréchet algebra s.

We first start considering the Π-algebras associated with orthogonal decompositions of Banach algebras, since this special case is neater. We let A stand for a Banach algebra with a Schauder decomposition $\{M_i\}$ until further notice.

PROPOSITION 6.2. $A_s \subset l_\infty$.

Proof. Let $\alpha = \{\alpha_i\} \in A_s$. Then for $x \in A$, the series $\sum \alpha_i P_i(x)$ is convergent in A. Since A is a Banach algebra with Schauder decomposition, each P_i being continuous, it follows that $\sum_{i=1}^{n} \alpha_i P_i$ is a bounded linear operator on A for each n and so

$$T_\alpha(x) = \lim_n \Big(\sum_{i=1}^{n} \alpha_i P_i\Big) x = \lim_n \sum_{i=1}^{n} \alpha_i P_i(x) = \sum_{i=1}^{\infty} \alpha_i P_i(x)$$

is defined for all $x \in A$. Being the pointwise limit of a sequence $\{\sum_{i=1}^{n} \alpha_i P_i\}$ of bounded linear operators, by the Banach–Steinhaus theorem T_α is a bounded linear operator. Clearly $T_\alpha(P_i(x)) = \alpha_i P_i(x)$ and so

$$|\alpha_i| \, \|P_i(x)\| \leqslant \|T_\alpha\| \, \|P_i(x)\|.$$

We choose i such that $P_i(x) \neq 0$ if $x \neq 0$ and so

$$\sup_i |\alpha_i| \leqslant \|T_\alpha\|,$$

proving that $\{\alpha_i\} \in l_\infty$. ∎

DEFINITION 6.3. Let A be a Banach algebra with an orthogonal Schauder decomposition $\{M_i\}$. Set

$$\Pi(A) = \{T_\alpha \colon \alpha = \{\alpha_i\} \in A_s\}.$$

We endow $\Pi(A)$ with the norm induced from $B(A)$, the algebra of all bounded linear operators on A. Clearly $\Pi(A)$ is a normed algebra.

THEOREM 6.4. *The map $\alpha \to T_\alpha$ of A_s into $\Pi(A)$ is injective and $\|\alpha\|_\infty \leqslant \|T_\alpha\|$.*

Proof. Suppose $T_\alpha = 0$. $T_\alpha(x) = \sum \alpha_i P_i(x) = 0$ for all $x \in A$ and so $T_\alpha(P_i x) = \alpha_i P_i(x) = 0$, because $P_i P_j = 0$ for $i \neq j$. Since M_i is nontrivial, i.e. $M_i \neq 0$, we have $P_i(x) \neq 0$ and so $\alpha_i = 0$ for all $i \geqslant 1$, i.e. $\alpha = \{0\}$, proving the injectivity. The other part follows from Proposition 6.2 ∎

THEOREM 6.5. *There exists a norm A_s such that A_s is a π-algebra and the map $\alpha \to T_\alpha$ of A_s onto $\Pi(A)$ is an unital isometry.*

Proof. If $\alpha = \{\alpha_i\}$, $\beta = \{\beta_i\}$ are in A_s, then $T_\alpha x = \sum \alpha_i P_i x$ and $T_\beta x = \sum \beta_i P_i x$ for $x \in A$ imply

$$T_\alpha(T_\beta x) = \sum \alpha_i P_i(T_\beta x) = \sum \alpha_j \beta_i P_i x = T_{\alpha\beta} x.$$

Thus $T_\alpha T_\beta \in \Pi(A)$. Clearly the map $\alpha \to T_\alpha$ is linear, actually an algebra homomorphism. In view of Theorem 6.4, this map is an isomorphism. To prove that this map is unital, let $T_\alpha = I$ be the identity of $\Pi(A)$. Then $x = \sum P_i x \in A$ implies that $T_\alpha x = \sum P_i x = x$. Thus $\alpha = \{1\}$, the identity of l_∞. By Theorem 6.4, the mapping $T_\alpha \to \alpha$ is continuous.

To show that $\Pi(A)$ is a Banach algebra, it is enough to show that $\Pi(A)$ is closed in $B(A)$. Let $\{T_{\alpha_n}\}$ be a sequence in $\Pi(A)$ such that $\|T_{\alpha_n} - T\| \to 0$ as $n \to \infty$ for some $T \in \operatorname{cl} \Pi(A)$. Let $\alpha_n = \{\alpha_{ni}\}_{i \geqslant 1} \in A_s$ such that

$$T_{\alpha_n} x = \sum_{i=1}^{\infty} \alpha_{ni} P_i x^-$$

for all $n \geqslant 1$ and $x \in A$. Then $T_{\alpha_n}(P_i x) = \alpha_{ni} P_i x$ and so by Theorem 6.4

$$\|\alpha_n - \alpha_m\|_\infty \leqslant \|T_{\alpha_n} - T_{\alpha_m}\| \to 0$$

as $n, m \to \infty$. Hence for each $i \geqslant 1$, $\{\alpha_{ni}\}_{n \geqslant 1}$ being a Cauchy sequence is convergent. Let $\lim_n \alpha_{ni} = \alpha_{0i}$ for $i = 1, 2, \ldots$ and set $\alpha_0 = \{\alpha_{0i}\}_{i \geqslant 1}$. Then $\|\alpha_n - \alpha_0\|_\infty \to 0$ as $n \to \infty$, i.e., $\lim_n \alpha_n = \alpha_0$ in l_∞. Since

$$\lim_n T_{\alpha_n} P_i x = \lim_n \alpha_{ni} P_i x = \alpha_{0i} P_i x,$$

we have

$$\sum_{i=1}^{m} \alpha_{0i} P_i x = \lim_n \sum_{i=1}^{m} \alpha_{ni} P_i x = T_{\alpha_0}\left(\sum_{i=1}^{m} P_i x\right).$$

This proves that $T = T_{\alpha_0}$ and so $\Pi(A)$ is closed. Thus by the open mapping theorem we have an isomorphic homeomorphism of A_s onto $\Pi(A)$.

To obtain isometry, we endow A_s with a new norm: $\|\alpha\|' = \|T_\alpha\|$ for $\alpha \in A_s$. It is clear that A_s with the new norm is a π-algebra for the following reasons: If $e_i = \{\delta_{ij}\}_{j \geqslant 1} \in A_s$, then $T_{e_i} = P_i \in \Pi(A)$ for all $i = 1, 2, \ldots$ and

$$\|\alpha\beta\|' = \|T_{\alpha\beta}\| = \|T_\alpha T_\beta\| \leqslant \|T_\alpha\| \, \|T_\beta\| = \|\alpha\|' \, \|\beta\|'$$

for $\alpha, \beta \in A_s$. Thus A_s is a commutative Banach algebra in the new norm isometric with $\Pi(A)$. Moreover,

$$\|e\|' = \|T_e\| = \|I\| = 1 = \|\{1\}\|_\infty$$

shows the unital isometry of $(A_s, \|\cdot\|')$ onto $\Pi(A)$. ∎

Let A be a Banach algebra with an orthogonal decomposition $\{M_i\}$. Then each M_i is also a Banach algebra. Assume each M_i is proper. Then A is also proper. If T is a multiplier on A, then for each $x = \sum x_i \in A$, $x_i \in M_i$, we have $Tx = \sum x_i^T$, say, where $x_i^T \in M_i$ for all $i \geqslant 1$. Thus for $x = \sum x_i$, $y = \sum y_i$

$$Tx \cdot y = x \cdot Ty$$

implies

$$x_i^T y_i = x_i y_i^T \quad \text{for all } i \geqslant 1.$$

Clearly each projection P_i: $A \to M_i$ gives

$$P_i Tx = x_i^T$$

and so

$$P_i(Tx) y_i = x_i P_i(Ty).$$

Hence it is clear that $T_i = P_i T$ is a multiplier on M_i and

$$Tx = \sum_{i=1}^{\infty} T_i x \quad \text{for all } x \in A.$$

Thus we have shown:

THEOREM 6.6. *Each $T \in M(A)$ can be expressed as $Tx = \sum_{i=1}^{\infty} T_i x$ $(x \in A)$ where $T_i \in M(M_i)$ for all $i \geqslant 1$.*

7. Perturbation of orthogonal bases

In this section we are concerned with the problem of obtaining an orthogonal Schauder basis by perturbation from a given Schauder basis of a commutative Banach algebra.

LEMMA 7.1. *Let A be a Banach algebra and $\tilde{A} = A \oplus C$ its unitization. Let e denote the identity of $\tilde{A}$ and $a \in A$ such that $\|a\| \leqslant \varepsilon < 1$. Then*

(i) $e-(e+a)^{-1/2} \in A$,

(ii)
$$\|e-(e+a)^{-1/2}\| \leqslant \frac{1}{2} \frac{\|a\|}{1-\|a\|}.$$

Proof. (i) We recall that if for any element a in a Banach algebra A, the spectrum $\sigma(a) \subset P$ and f is analytic on P with $0 \in P$ such that $f(0) = 0$, then $f(a) \in A$ ([16], Chap. III). Since the spectral radius $r(a) \leqslant \|a\|$, if we put $f(\lambda) = 1-(1+\lambda)^{-1/2}$, $\lambda \in C$, $|\lambda| \leqslant \|a\|$, then $f(a) = e-(e+a)^{-1/2} \in A$.

(ii) For $\alpha \in C$, $|\alpha| < 1$ and $a \in A$ with $\|a\| < 1$, we have

$$(e+a)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{n!} a^n.$$

Thus

$$(e+a)^{-1/2} - e = \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1) \dots (-\frac{1}{2}-n+1)}{n!} a^n,$$

in which

$$\left|\frac{(-\frac{1}{2})(-\frac{1}{2}-1)\dots(-\frac{1}{2}-n+1)}{n!}\right| \leqslant \frac{1}{2}$$

for all $n \geqslant 1$. Whence we have

$$\|(e+a)^{-1/2}-e\| \leqslant \frac{1}{2}\frac{\|a\|}{1-\|a\|}. \blacksquare$$

LEMMA 7.2. *Let A be a Banach algebra and $a \in A$ such that*

$$\|a^2-a\| \leqslant \varepsilon < \tfrac{1}{4}.$$

Then there exists $b \in A$ such that $b^2 = b$ and

$$\|a-b\| \leqslant \frac{2\varepsilon}{1-4\varepsilon}(\|a\|+\tfrac{1}{2}).$$

Proof. By hypothesis

$$\|4a^2-4a\| \leqslant 4\varepsilon < 1.$$

Hence by Lemma 7.1, there is $z \in \tilde{A}$ such that

$$\|z-e\| \leqslant \frac{2\varepsilon}{1-4\varepsilon}.$$

Put

$$b = \tfrac{1}{2}(e-z)+az.$$

Then it is easy to verify that $b^2 = b$. Furthermore,

$$\|a-b\| = \|a-\tfrac{1}{2}(e-z)-az\| = \|a(e-z)-\tfrac{1}{2}(e-z)\| = \|(e-z)(a-\tfrac{1}{2})\|$$

$$\leqslant \|e-z\|\,\|a-\tfrac{1}{2}\| \leqslant \frac{2\varepsilon}{1-4\varepsilon}(\|a\|+\tfrac{1}{2}).$$

Now to prove that $b \in A$ first we observe that $b \in \tilde{A}$. In view of the statement in the proof of (i) in Lemma 1, we have

$$b-a = (e-z)(\tfrac{1}{2}-a) = f(a) \in A,$$

where

$$f(\lambda) = [1-(1+4\lambda^2-4\lambda)^{1/2}](\tfrac{1}{2}-\lambda).$$

Thus

$$b = a+f(a) \in A. \blacksquare$$

DEFINITION 7.3. A sequence $\{y_n\}$ in a topological vector space is said to be *strongly linearly independent* if for any scalar sequence $\{\alpha_n\}$, $\sum_{n=1}^{\infty}\alpha_n y_n = 0$ implies $\alpha_n = 0$ for all $n \geqslant 1$.

PROPOSITION 7.4. *Let E be a Banach space with a Schauder basis $\{x_n\}$* such that

$$\inf_n \|x_n\| > 0.$$

If $\{y_n\}$ in a sequence of strongly linearly independent elements in E such that $\sum \|x_n - y_n\| < \infty$, then $\{y_n\}$ is also a Schauder basis.

Proof. Since for each $x \in E$,

$$x = \sum \lambda_n(x)\, x_n,$$

we define

$$F_n(x) = \lambda_n(x)\, x_n.$$

Clearly each F_n is a bounded linear operator on E and since the series $\sum \lambda_n x_n$ is convergent, by the Banach–Steinhaus theorem, $\sup_n \|F_n\| < \infty$. Since $\inf_n \|x_n\| > 0$ by hypothesis, it follows that

$$\sup_n |\lambda_n(x)| \leqslant K\,\|x\|,$$

for some $K > 0$. If we define a map Φ: $E \to E$ by

$$\Phi\left(\sum \lambda_n x_n\right) = \sum \lambda_n y_n,$$

then we show that Φ is well defined by proving that $\sum \lambda_n y_n$ is convergent whenever $\sum \lambda_n x_n$ is convergent. For this, it is enough to show that $\{\sum_{i=1}^n \lambda_i y_i\}$ is a Cauchy sequence whenever $\{\sum_{i=1}^n \lambda_i x_i\}$ is a Cauchy sequence. From the inequalities,

$$\Big\|\sum_{i=m}^n \lambda_i y_i\Big\| \leqslant \Big\|\sum_{i=m}^n \lambda_i (y_i - x_i)\Big\| + \Big\|\sum_{i=m}^n \lambda_i x_i\Big\| \leqslant K\,\|x\| \sum_{i=m}^n \|y_i - x_i\| + \Big\|\sum_{i=m}^n \lambda_i x_i\Big\|,$$

we conclude that $\{\sum_{i=1}^n \lambda_i y_i\}$ is a Cauchy sequence because the series $\sum_{i=1}^\infty \|y_i - x_i\|$, $\sum_{i=1}^\infty \lambda_i x_i$ being convergent imply $\{\sum_{i=1}^n \|y_i - x_i\|\}$ and $\{\|\sum_{i=1}^n \lambda_i x_i\|\}$ are Cauchy sequences

Now choose an integer n_0 such that

$$\sum_{n=n_0}^\infty \|x_n - y_n\| < \frac{1}{2K}.$$

Put

$$\begin{aligned} X_1 &= \operatorname{lin\,span}\{x_1, \ldots, x_{n_0}\}, \\ X_2 &= \operatorname{Cl\,lin\,span}\{x_n\colon\ n \geqslant n_0 + 1\}, \\ Y_1 &= \operatorname{lin\,span}\{y_1, \ldots, y_{n_0}\}, \\ Y_2 &= \operatorname{Cl\,lin\,span}\{y_n\colon\ n \geqslant n_0 + 1\}. \end{aligned}$$

Let $\Phi_1 = \Phi | X_1$, $\Phi_2 = \Phi | X_2$. Then Φ_1 maps X_1 into Y_1 and Φ_2 maps X_2 into Y_2. For any $x \in X_2$, $\|x\| = 1$, $x = \sum_{n=n_0}^{\infty} \lambda_n x_n$ we have:

$$\|\Phi_2(x) - x\| \leqslant \Big\| \sum_{n=n_0}^{\infty} \lambda_n (x_n - y_n) \Big\| \leqslant K \frac{\|x\|}{2K} = \frac{1}{2} \|x\|.$$

Thus for all $x \in X_2$, we have

$$\tfrac{1}{2} \|x\| \leqslant \|\Phi_2(x)\| \leqslant \tfrac{3}{2} \|x\|.$$

This shows that Φ_2 is a bounded linear injective map from X_2 onto a closed linear subspace Y_0 of Y_2. We show that $Y_0 = Y_2$. We have shown that for all $x \in X_2$, $\|x\| = 1$, there is $y \in Y_0$ ($y = \Phi(x)$) such that $\|x - y\| \leqslant \frac{1}{2}$. Thus by Proposition 15.4 ([10], p. 80)

$$n_0 = \operatorname{Codim}_X X_2 = \operatorname{Codim}_X Y_0.$$

On the other hand, since $\{y_n\}$ is strongly linearly independent, $\{y_1, \ldots, y_{n_0}\}$ are linearly independent elements of Y_0/Y_2. Thus

$$\operatorname{Codim}_X Y_2 = n_0.$$

This proves that $Y_0 = Y_2$ and so for each $x \in Y_1 \oplus Y_2 = E$ there is a sequence $\{\beta_n\}$ of scalars such that

$$x = \sum_{n=1}^{\infty} \beta_n y_n,$$

proving that $\{y_n\}$ is a Schauder basis. ∎

Now we prove the main result of this section:

THEOREM 7.5. *Let A be a commutative Banach algebra with a Schauder basis $\{x_n\}$ such that*

(i) $\|x_n\| \geqslant 2\varepsilon_n$ *for all* $n \geqslant 1$,

(ii) $\|x_n x_m - \delta_{nm} x_n\| \leqslant \varepsilon_n < \frac{1}{10}$ *for all* $n, m \geqslant 1$.

Then there is a Schauder basis $\{y_n\}$ in A such that $\{y_n\}$ is orthogonal (i.e. $y_n y_m = \delta_{nm}$) and

$$\|x_n - y_n\| \leqslant \frac{2\varepsilon_n}{1 - 4\varepsilon_n} (\|x_n\| + \tfrac{1}{2}) < 10\varepsilon_n \|x_n\|.$$

Remark. The assumption (i) is to avoid the situation when $\|x_n^2 - x_n\|$ is small not because x_m is "almost idempotent" but because $\|x_n\|$ is very small.)

Proof. By putting $a = x_n$ in Lemma 7.2, we find a sequence $\{y_n\}$ in A such that $y_n^2 = y$ and

$$\|x_n - y_n\| \leqslant \frac{2\varepsilon_n}{1 - 4\varepsilon_n} (\|x_n\| + \tfrac{1}{2}).$$

In order to appeal to Proposition 7.4, we first show that $\inf \|x_n\| > 0$. We have

$$\|x_n\| - \|x_n\|^2 \leqslant \|x_n^2 - x_n\| \leqslant \varepsilon_n.$$

Hence

$$\|x_n\| \geqslant \tfrac{1}{2}(1+\sqrt{1-4\varepsilon_n}) > 1-2\varepsilon_n > \tfrac{1}{2}$$

or

$$\|x_n\| \leqslant \tfrac{1}{2}(1-\sqrt{1-4\varepsilon_n}) < 2\varepsilon_n,$$

which is contrary to our assumption (i). Thus

$$\inf \|x_n\| \geqslant \tfrac{1}{2} > 0.$$

Moreover,

$$\frac{2\varepsilon_n}{1-4\varepsilon_n}(\|x_n\| + \tfrac{1}{2}) \leqslant \|x_n\|.$$

Hence by the inequality in Lemma 7.2, we have $y_n \neq 0$ for all $n \geqslant 1$. Since $y_n^2 = y_n$, we get $\|y_n\| \geqslant 1$. Now for all $n \neq m$ ($n, m > 0$ integers) we have

$$(y_n y_m)(y_n y_m) = (y_n y_n)(y_m y_m) = y_n y_m,$$

which shows that $\|y_n y_m\| \leqslant \varepsilon_n < 1$. Thus $y_n y_m$ is an idempotent of norm less than 1, which is not possible. Therefore $y_n y_m = 0$ for $n \neq m$.

Now if $\sum \alpha_n y_n = 0$, then

$$\alpha_m y_m = (\textstyle\sum \alpha_n y_n) y_m = 0$$

for all $m \geqslant 1$ which implies that $\alpha_n = 0$ for all $n \geqslant 1$, proving the strong linear independence of $\{y_n\}$. Thus the hypotheses of Proposition 7.4 are satisfied and the proof is completed. ∎

Remark. 7.6. Note we have shown that "if $\sum \varepsilon_n < \infty$ then the perturbation is valid". Johnson [13] has shown that for L_p ($1 < p < \infty$) our theorem cannot be strengthened to the effect that "if $\sup |\varepsilon_n| < \varepsilon_0$ for some $\varepsilon_0 > 0$ then the perturbation is valid".

8. Banach algebras with unconditional bases and multipliers

Let A be a Banach algebra with an orthogonal basis $\{x_i\}$. We show that there exists a subalgebra B of A and another norm on B such that $\{x_i\}$ becomes an unconditional orthogonal basis of B. Furthermore, we introduce a method of introducing new Banach algebras with orthogonal bases from old and find their multipliers.

In the sequel, A stands for a Banach algebra with an orthogonal basis.

DEFINITION 8.1. For each $x = \sum_{n=1}^{\infty} \lambda_n(x) x_n \in A$, we put

$$\|x\|' = \begin{cases} \sup \|\sum_{n=1}^{\infty} \beta_n x_n\|, & \text{if } \sum \beta_n x_n \in A \text{ for all } \{\beta_n\} \text{ with } |\beta_n| \leqslant |\lambda_n(x)|, \\ \infty, & \text{otherwise.} \end{cases}$$

Let $B = \{x \in A\colon \|x\|' < \infty\}$.

PROPOSITION 8.2. *$(B, \|\cdot\|')$ is a normed algebra.*

Proof. We break up the proof in two parts: (a) to show that $\|\cdot\|'$ is a norm on B, trivially for each $x \in B$ and $\lambda \in C$, $\lambda x \in B$. If $x, y \in B$, then $x = \sum \alpha_n x_n$, $y = \sum \beta_n x_n$ and so

$$\|x+y\|' = \sup\{\|\sum \gamma_n x_n\|\colon |\gamma_n| \leqslant |\alpha_n + \beta_n| \text{ for all } n \geqslant 1\}.$$

Put

$$\gamma_n' = \frac{\gamma_n \alpha_n}{\alpha_n + \beta_n}, \qquad \gamma_n'' = \frac{\gamma_n \beta_n}{\alpha_n + \beta_n}.$$

Then

$$\begin{aligned} \|x+y\|' &\leqslant \sup\{\|\sum \gamma_n' x_n\| + \|\sum \gamma_n'' x_n\|\colon |\gamma_n'| \leqslant |\alpha_n| \text{ and } |\gamma_n'| \leqslant |\beta_n|\} \\ &\leqslant \|x\|' + \|y\|'. \end{aligned}$$

Since $\|x\|' = 0 \Leftrightarrow x = 0$ is clear, it follows that $(B, \|\cdot\|')$ is a normed vector space. (b) $(B, \|\cdot\|')$ is a normed algebra. Observe that if E is any Banach space with a Schauder basis $\{x_i\}$, then for $x = \sum \lambda_i x_i$ we define

$$F_n(x) = F_n\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) = \lambda_n.$$

By the Banach–Steinhaus theorem,

$$\sup_n \|F_n\| \, \|x_n\| < \infty.$$

Hence $\sup_n \|F_n\| < \infty$. We let

$$K = \sup_n \|F_n\| < \infty.$$

Now for $x, y \in B$, $x = \sum \alpha_n x_n$, $y = \sum \beta_n x_n$, we have

$$\|xy\|' = \sup\{\|\sum \gamma_n x_n\|\colon |\gamma_n| \leqslant |\alpha_n \beta_n| \text{ for all } n \geqslant 1\}.$$

Since for all $n \geqslant 1$,

$$|\alpha_n| \leqslant K\|x\| \leqslant K\|x\|'$$

we have

$$\|xy\|' \leqslant K\,\|x\|' \sup\{\|\textstyle\sum \gamma_n x_n\|: |\gamma_n| \leqslant |\beta_n| \text{ for all } n \geqslant 1\}$$
$$= K\,\|x\|'\,\|y\|',$$

proving that the multiplication in $(B, \|\cdot\|')$ is continuous and so $(B, \|\cdot\|')$ is a normed algebra.

We remark that if we want the new norm to be submultiplicative, then we choose the new norm to be $\|x\|'' = K\,\|x\|'$. ∎

PROPOSITION 8.3. *$(B, \|\cdot\|')$ is a Banach algebra.*

Proof. We only have to prove that $(B, \|\cdot\|')$ is complete. Let

$$y^{(j)} = \sum \lambda_n^j x_n \in B, \quad j = 1, 2, \ldots$$

Suppose that $\{y^{(j)}\}$ is a Cauchy sequence in $(B, \|\cdot\|')$. Clearly $\|x\| \leqslant \|x\|'$ for all $x \in B$ and so $\{y^{(j)}\}$ is a Cauchy sequence in the original norm $\|\cdot\|$. Therefore, it is convergent in A. Let $y^{(j)} \to y_0 \in A$, where $y_0 = \sum \lambda_n^0 x_n$. It is clear that $y_0 \in B$. Now put

$$A_{j,i} = \{\textstyle\sum \gamma_n x_n \in B: |\gamma_n| \leqslant |\lambda_n^j - \lambda_n^i| \text{ for all } n \geqslant 1\}.$$

Then we have

$$A_{j,0} \subset \mathrm{Cl}\Big(\bigcup_{i=j+1}^{\infty} A_{j,i}\Big) \quad \text{(closure in } \|\cdot\|\text{-norm)}.$$

Thus

$$\begin{aligned}
\|y^{(j)} - y_0\|' &= \sup\{\|x\|: x \in A_{j,0}\} \\
&\leqslant \sup\{\|x\|: x \in \bigcup_{i=j+1}^{\infty} A_{j,i}\} \\
&\leqslant \sup_{i=j+1} \sup\{\|x\|: x \in A_{j,i}\} \\
&= \sup_{i \geqslant j+1} \|y^{(j)} - y^{(i)}\|' \to 0. \ ∎
\end{aligned}$$

THEOREM 8.4. *$\{x_n\}$ is an unconditional orthogonal basis of the Banach algebra $(B, \|\cdot\|')$.*

Proof. In view of the implications: (i) ⇔ (vii) of Proposition 1.1 ([9]) and the definition of the norm $\|\cdot\|'$, it is sufficient to prove only that $\{x_n\}$ is a Schauder basis of $(B, \|\cdot\|')$.

Suppose $\{x_n\}$ is not a basis of $(B, \|\cdot\|')$. Then there is $a = \sum \alpha_n x_n \in B$ such that $\{\sum_{i=1}^n \alpha_i x_i\}$ does not converge to a in the norm $\|\cdot\|'$. Hence

$$\Big\|\sum_{i=n+1}^{\infty} \alpha_i x_i\Big\|' = \sup\Big\{\Big\|\sum_{i=n+1}^{\infty} \gamma_i x_i\Big\|: |\gamma_i| \leqslant |\alpha_i|\Big\}$$

does not converge to zero. By induction we define a sequence $\{n_k\}$ of positive integers such that $n_{k+1} > n_k$ and such that there is $\varepsilon > 0$ with

$$\|\sum_{i=n_k+1}^{n_{k+1}} \alpha_i x_i\|' > \varepsilon \quad \text{for all } k \geqslant 1. \tag{8.1}$$

From this we see that $\|\sum_{i=n}^{\infty} \alpha_i x_i\| \not\to 0$ implies that for some ε ($\varepsilon \in \boldsymbol{R} \cup \{\infty\}$) which is a cluster point of $\|\sum_{i=n}^{\infty} \alpha_i x_i\|'$. Let n_1 be such that

$$\|\sum_{n=n_1}^{\infty} \alpha_i x_i\|' > \frac{\varepsilon}{2}.$$

By definition of $\|x\|'$ there is γ_n, $|\gamma_n| \leqslant |\alpha_n|$, such that

$$\|\sum_{n=n_1}^{\infty} \alpha_i x_i\| > \frac{\varepsilon}{2}.$$

Hence there is n_2 such that

$$\|\sum_{n=n_1}^{n_2} \gamma_n x_n\| > \frac{\varepsilon}{2}, \qquad \|\sum_{i=n_2}^{\infty} \alpha_i x_i\|' > \frac{\varepsilon}{2}.$$

Assume we have chosen n_k such that (8.1) holds. Let $\{\gamma_n\}$ be given by the definition of $\|\cdot\|'$ such that

$$\|\sum_{n_k+1}^{n_{k+1}} \gamma_n x_n\| > \varepsilon \qquad \text{and} \qquad |\gamma_n| \leqslant |\alpha_n|.$$

Thus the sequence $\{\|\sum_{i=n_1}^{n_k} \gamma_i x_i\|\}$ is not convergent in the norm $\|\cdot\|$ because it is not $\|\cdot\|$-Cauchy. But by our assumption $\sum \alpha_n x_n \in B$ and so by the defintion of $\|\cdot\|'$ and $|\gamma_n| \leqslant |\alpha_n|$ imply that $\sum \gamma_n x_n \in A$. This contradicts the assumption that $\{x_n\}$ is a Schauder basis of $(A, \|\cdot\|)$. This completes the proof. ∎

Remark 8.5. If $\{x_n\}$ is an unconditional orthogonal basis of A, then $A = B$.

Definition 8.6. Let X be a topological algebra. We define

$$X^k = \{a^k = a \ldots a \ (k \text{ factors})\colon a \in X\} \qquad \text{and} \qquad X^\infty = \bigcap_{k=1}^{\infty} X^k.$$

If A is a Banach algebra with an orthogonal basis $\{x_i\}$, then we define

$$A(X) = \{\{a_n\} \in X^\infty\colon \{\|a_n\|\} \in B\},$$

where B is the subalgebra of A as defined in Definition 8.1.

Proposition 8.7. (a) *$A(X)$ is a Banach algebra.*

(b) *$A(X)$ has an orthogonal basis $\{x_n * y_n\}$, where $\{y_n\}$ is an orthogonal basis of X.*

Proof. (a) The proof is similar to Propositions 8.1–8.3.

(b) Note that $A(X)$ can be thought of as the tensor product of A and X and thus the proof is immediate. ■

THEOREM 8.8.

$$M(A(X)) = \{\{T_n\}:\ T_n \in M(A) \text{ for all } n \geqslant 1 \text{ with } \sup_n \|T_n\| < \infty\}.$$

Proof. If $T \in M(A(X))$, then for all $a = \{a_n\} \in A^\infty$, and $b = \{b_n\} \in A^\infty$,

$$Ta \cdot b = a \cdot Tb.$$

If we put $(Ta)_n = T_n a \in A$, then $T_n a \cdot b = a \cdot T_n b$ and so $T_n \in M(A)$ for all $n \geqslant 1$. Since $\{\|a_n\|\} \in B$, we conclude that $\sup_n \|T_n\| < \infty$.

On the other hand, if $\{T_n\}$ is a sequence such that $T_n \in M(A)$ with $\sup_n \|T_n\| < \infty$, then we see that for $T = \{T_n\}$, $a \in A(X)$, $b \in A(X)$, with $a = \{a_n\} \in A^\infty$, $b = \{b_n\} \in A^\infty$ $(a_n, b_n \in A)$ where $\{\|a_n\|\} \in B$ and $\{\|b_n\|\} \in B$, we have $Ta \cdot b = a \cdot Tb$, because $T_n a_n \cdot b_n = a_n \cdot T_n b_n$ for all $n \geqslant 1$. This proves that $T \in M(A(X))$. ■

COROLLARY 8.9. *For* $X = l_p$ $(1 \leqslant p < \infty)$,

$$M(A(l_p)) \cong l_\infty(M(A)) = l_\infty \otimes M(A),$$

(injective tensor product).

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DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO
CANADA L8S 4K1