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K-UNIRATIONALITY OF CONIC BUNDLES, THE KNESER-TITS CONJECTURE FOR SPINOR GROUPS AND CENTRAL SIMPLE ALGEBRAS

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The aim of the paper is to show the usefulness of finite-dimensional simple algebras for applications to some problems of algebraic geometry and the theory of algebraic groups. In the first part of the paper we review the problem of k-unirationality of conic bundles over curves and discuss related topics. In the second part we discuss the well-known Kneser–Tits conjecture in the spinor case.

1. K-unirationality of conic bundles and central simple algebras over rational functional fields

In the first part of the present paper certain questions on central simple finite-dimensional algebras over rational function fields are considered.

Our interest in algebras over rational function fields is inspired, on the one hand, by the general problem of the study of finite-dimensional simple algebras over an arbitrary field. The well-known Wedderburn theorem on finite skew fields completed in fact the investigation of central simple algebras over finite fields. Next, the theory of simple algebras over number fields was developed. It is mostly due to Hasse, Brauer, Noether and Albert. Thus, one can say that the theory of algebras over "small" fields is quite complete. When we deal with algebras over arbitrary fields it becomes natural to consider central simple algebras over algebraic function fields and in particular over rational function fields.

On the other hand, the consideration of central simple algebras over rational function fields reveals very interesting connections with algebraic geometry, namely with the geometry of conic bundles.

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Recall that a complete absolutely irreducible algebraic surface X over k is called *rational* if $\bar{X} = X \otimes_k \bar{k}$ is birationally equivalent to the projective plane $\mathbf{P}^2_{\bar{k}}$.

Now let k be a field of characteristic $\neq 2$.

DEFINITION 1. A rational surface V over k is called a *conic bundle* if there exists a k-morphism $f: V \to C$ such that C (called a *base*) is a smooth curve of genus zero and the generic fibre V_t of f is an absolutely irreducible curve of genus zero too.

The study of conic bundles is of interest in view of the important role they play in the general classification of rational surfaces over algebraically nonclosed fields (see, e.g., [5]). At present a number of authors (Colliot-Thélène, Coray, Tsfasman, Skorobogatov, Kunyavskii, Salberger, etc.) develop the arithmetic theory of these surfaces.

It is well known that there is a natural correspondence (first established by Witt) between conics over some field K and quaternion algebras over K. In the case where K is the rational function field $k(x)$ a conic over K may be considered as a conic bundle over k. We thus obtain a correspondence between conic bundles over k and quaternion algebras over $k(x)$. A conic bundle has a k-point iff the corresponding quaternion algebra has a k-point.

DEFINITION 2. Let A be a central simple algebra over $k(x)$. One says that a place v of degree one of $k(x)$ (trivial on k) is a *k-point* of A iff $k(x)_v$ (the completion of $k(x)$ at v) is a splitting field of A.

DEFINITION 3. An algebraic variety X of dimension n is said to be *k-unirational* if there exists a rational k-defined map $\mathbf{P}^n \to X$ of finite degree.

Iskovskikh [4] established a remarkable connection between the k-unirationality of conic bundles and splitting fields of quaternion algebras: A conic bundle is k-unirational iff the corresponding quaternion algebra has a k-rational splitting field. Here "k-rational field" means some rational function field $k(z)$.

Iskovskikh also put the question whether a conic bundle is k-unirational if it possesses a k-point (in the case of k infinite). Note that if a conic bundle V over an infinite k is k-unirational then it has a k-point. We call this question here "Problem 1".

In [4], Iskovskikh gave a positive answer to this question in the case when k is the field $\mathbf{R}$ of real numbers (this is also true if k is a really closed field). Thus any quaternion algebra (and therefore any algëbra) over $\mathbf{R}(x)$ with an $\mathbf{R}$-point has a rational splitting field.

In this connection the following question naturally arises. It generalizes Problem 1.

PROBLEM 2. Let A be a finite-dimensional central simple algebra over $k(x)$ with a k-point. Does A have a k-rational splitting field?

As one can see, there is a natural relation between Problem 1 and a particular case of Problem 2, and a positive solution of the latter implies a positive solution of the former.

As noted above Problems 1 and 2 have a positive solution when k is a field of real numbers. Another important result is the following [16].

THEOREM 1. *Let k be a Henselian field and let A be a finite-dimensional central simple algebra over $k(x)$ with a k-point. Then A has a k-rational splitting field.*

In particular, any conic bundle over a local field k with a k-point is k-unirational. There is one more class of fields for which Problems 1,2 have a positive solution.

DEFINITION 4. A perfect field k is called *pseudo-algebraically closed* (PAC) if any nonvoid absolutely irreducible algebraic variety over k has a k-point. For example, as follows from the Riemann–Weyl inequality, an infinite algebraic extension of a finite field F_q is PAC.

THEOREM 2 (see [13]). *Let k be a* PAC *field. Then any central simple algebra over $k(x)$ has a rational splitting field, and any conic bundle over k is k-unirational.*

So in particular for any infinite algebraic extension k of F_q Problems 1, 2 have a positive solution. Note that Problem 1 in this case has been solved earlier in [12] by other methods.

Now consider some related results in this theory of central simple algebras. These results concern with those ones of local-global principle type.

DEFINITION 5. A central simple algebra over $k(x)$ is said to *belong to the separable part* $\mathrm{Br}_s\, k(x)$ of $\mathrm{Br}\, k(x)$, where $\mathrm{Br}\, k(x)$ is the Brauer group of $k(x)$, if it splits by $k^{\mathrm{sep}}(x)$, where k^{sep} is a separable closure of k. (As follows from Tsen's theorem, any algebra over $k(x)$ splits by $k^{\mathrm{alg}}(x)$)

THEOREM 3 [17]. *Let k be a global field whose central simple algebra A over k belongs to* $\mathrm{Br}_s\, k(x)$. *If for any place v of k, $k_v(x)$ is a splitting field of A, then A is trivial.*

In terms of Brauer groups this result is formulated as follows: the natural homomorphism $\mathrm{Br}_s\, k(x) \to \prod_{v\in V_k} \mathrm{Br}\, k_v(x)$ is injective.

Remark. The analogous result for Witt rings was obtained by Colliot-Thélène, Coray and Sansuc in 1980 [2].

THEOREM 4 [14]. *Let A be as in Theorem 3. Assume that there exists a subset $\mathfrak{a} \subseteq k$ (which is an arithmetic progression in case* $\mathrm{char}\, k = 0$ *and an infinite subring $\mathfrak{a}$ of k with $\mathfrak{a} \not\subseteq k^{\mathrm{char}\, k}$ otherwise) such that $k(x)_{x-a}$, the completion of $k(x)$ at the place corresponding to the polynomial $x-a$ is a splitting field of A for all $a \in \mathfrak{a}$. Then A is trivial.*

COROLLARY. *Let k be a global field of characteristic $\neq 2$, and V a k-defined conic bundle with k-rational base C. Assume that for each k-point $t \in C$ the fibre V_t is k-rational. Then V is k-rational. Furthermore, the generic fibre V_τ is rational over the residue field of the generic point.*

Finally, a few words on open questions.

At present we have no answers to Problems 1, 2 except the cases of really closed, Henselian or PAC field k. The classical cases of finite fields and algebraic number fields are of great importance.

Further, it is easy to see that if a conic bundle over an infinite field k is k-unirational then it contains infinitely many k-points. Thus, we can pose the following two problems which seem to be slightly simpler than Problems 1, 2.

PROBLEM 3. Let X be a conic bundle over infinite k with a k-point. Does X contain infinitely many k-points?

PROBLEM 4. Let A be a central simple algebra over $k(x)$ with a k-point. Does A possess an infinite set of k-points?

It follows from the above theorems that Problems 3, 4 have positive answers in the cases of really closed, Henselian or PAC field k.

At present we know nothing about these problems in other cases.

2. Skew fields, classical groups and the Kneser–Tits conjecture

The second theme which we shall discuss is the Kneser–Tits conjecture. We shall suppose that the fundamental field k is infinite of characteristic not two.

Let G be an algebraic group defined over k, quasi-simple and isotropic over k, let G_k be the group of all elements of G rational over k, and let G_k^+ be the subgroup of G_k generated by all the unipotent elements of G_k which are contained in the unipotent radical of some parabolic subgroup of G, defined over k. The group $W(G_k) = G_k/G_k^+$ is called the *Whitehead group* of G. It was proved by J. Tits [10], that every subgroup of G_k which is normalized by G_k^+ either is central in G, or contains G_k^+. In particular, the quotient of G_k^+ by its center is a simple group. And the well-known Kneser–Tits conjecture was formulated in connection with the above result as follows: If G is simply connected then $W(G_k)$ is trivial.

Later on deep connections were established between the Kneser–Tits conjecture and problems of approximation and rationality in the theory of algebraic groups, problems of algebraic K-theory, etc. [7, 8].

Of course, this general conjecture was known to be true in all known partial cases, for example, for classical groups over a k. In that case, when G_k is a classical group over a skew field over k, the first results pertaining to this problem were obtained by Nakayama and Matsushima in 1943 and by Wang in 1950 in the context of the so-called Tannaka–Artin problem. They proved that $W(G_k) = \{1\}$ if $G_k = \mathrm{SL}_n(D)$, $n \geqslant 2$, where $\mathrm{SL}_n(D) = \{a \in \mathrm{GL}_n(D) | \mathrm{Nrd}(a) = 1\}$, Nrd is the reduced norm homomorphism and D is a finite-dimensional division algebra over a p-adic or an algebraic number field. For a long time, nothing was known in other nontrivial cases. But in 1974–1984 in the works of V. P. Platonov and V. I. Yanchevskii this problem was solved for all groups G of classical type except the case where G is a simply connected group of type D_l, i.e. a spinor group over a skew field. And new methods were developed for the computation of the Whitehead group when it is not trivial [7, 8, 11, 15].

More precisely, here is a table which illustrates the recent situation.

Type of G		Realization of G_k	Answer to the Kneser–Tits conjecture in general
A_l	${}^1A_l$	$\mathrm{SL}_n(D)$, $n \geqslant 2$, D a finite-dimensional skew field over k	negative (V. P. Platonov, 1975).
	${}^2A_l$	$\mathrm{SU}_n(M, D)$, $n \geqslant 2$, D a finite-dimensional skew field over k with involution of the second kind, M a skew-hermitian isotropic form	negative (V. P. Platonov, V. I. Yanchevskii, 1975)
B_l		$\mathrm{Spin}_{2l+1}(f)$, f an isotropic quadratic form on a vector space over k	positive (see e.g. [3])
C_l		$\mathrm{Sp}_{2l}(f)$, f a skew form on a vector space over k	positive (see e.g. [1])
		$\mathrm{SU}_n(M, D)$, $n \geqslant 2$, D a finite-dimensional skew field over k with involution of the first kind and first type, M a skew-hermitian isotropic form	positive (see e.g. [3])
D_l		$\mathrm{Spin}_n(M, D)$, $n \geqslant 2$, D a finite-dimensional skew field over k with involution of the first kind and second type, M a skew-hermitian isotropic form	?

The last case was an open problem for a long time. But recently it has been solved by A. P. Monastyrnyi and V. I. Yanchevskii and the answer is negative in general [6].

Now we shall state the results obtained in this case.

Let D be a finite-dimensional central division algebra over k with involution α such that $2 \dim_k S_\alpha < \dim_k D$ and $k \subset S_\alpha$, where $S_\alpha = \{a \in D | a^\alpha = a\}$, V an n-dimensional right vector space over D, and M a nondegenerate skew-hermitian form on $V \times V$ relative to α. In the sequel we shall assume that the index of M is > 0. Let $\mathrm{U}_n(M, D)$ be the corresponding unitary group. It is known that $\mathrm{U}_n(M, D) = \mathrm{SU}_n(M, D)$, where $\mathrm{SU}_n(M, D) = \mathrm{U}_n(M, D) \cap \mathrm{SL}_n(D)$. A simply connected group of type D_l will be viewed as a universal covering of $\mathrm{U}_n(M, D)$ and denoted by $\mathrm{Spin}_n(M, D)$.

The first problem is to describe the group $W(\mathrm{Spin}_n(M, D))$ in terms of the skew field D only.

There exists the well-known Wall spinor norm homomorphism θ_{W}: $\mathrm{U}_n(M, D) \to D^*/\Sigma_\alpha(D)[D^*, D^*]$ $(n \geqslant 3)$ [3], where $\Sigma_\alpha(D)$ is the multiplicative group generated by the nonzero elements of S_α, and $[D^*, D^*]$ is the commutator subgroup of the multiplicative group D^* of nonzero elements of D.

In [6] the reduced spinor norm homomorphism

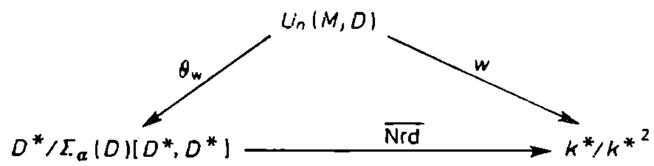
$$w\colon\ \mathrm{U}_n(M, D) \to k^*/k^{*2},$$

is defined, which is a generalization of the classical Eichler spinor norm. It is also proved that the following sequence is exact:

$$\{-1, 1\} \to \mathrm{W}(\mathrm{Spin}_n(M, D)) \to \mathrm{Ker}\, w/\mathrm{Ker}\, \theta_{\mathrm{w}} \to 1.$$

The following result is important:

The diagram



is commutative.

And then we have

THEOREM 1. *There exists an exact sequence*

$$\{-1, 1\} \to \mathrm{W}(\mathrm{Spin}_n(M, D)) \to R(D)/\Sigma_\alpha(D)[D^*, D^*] \to 1,$$

where $R(D) = \{a \in D^* \mid \mathrm{Nrd}(a) \in k^{*2}\}$.

In view of this theorem the group $K_1\,\mathrm{Spin}(D) = R(D)/\Sigma_\alpha(D)[D^*, D^*]$, which we call the *spinor Whitehead group* of D, is of great importance for the computation of $\mathrm{W}(\mathrm{Spin}_n(M, D))$. It is well known [11] that if $\dim_k D = 4$ then $\mathrm{W}(\mathrm{Spin}_n(M, D))$ is trivial. Therefore we are interested in the case where $\dim_k D = 2^{2S}$ and $S \geqslant 2$. In the case where $S = 2$ we have

THEOREM 2. *Let* $S = 2$. *Then* $\mathrm{W}(\mathrm{Spin}_n(M, D)) = \{1\}$.

It is evident that if the group $K_1\,\mathrm{U}(D) = D^*/\Sigma_\alpha(D)[D^*, D^*]$ is trivial then $K_1\,\mathrm{Spin}(D)$ is trivial. The group $K_1\,\mathrm{U}(D)$, where D is a Henselian skew field, was computed in [9] by V. P. Platonov and V. I. Yanchevskii. It follows from this computation that $K_1 \operatorname{Spin}(D)$ is trivial for a wide class of Henselian skew fields.

Nevertheless the answer in general to the Kneser–Tits conjecture for spinor groups has turned out to be negative. And we want to emphasize the central role of Henselian division algebras for the construction of a counterexample.

THEOREM 3. *Let $k = K\langle x_1\rangle\langle x_2\rangle\langle x_3\rangle$, where $E\langle y\rangle$ denotes the field of formal power series in y over the field E. Let $(a, b)_k$ $(a, b \in k^*)$ be the corresponding quaternion algebra and suppose $D = (a, x_1)\otimes_k (b, x_2)\otimes_k (c, x_3)$ $(a, b, c \in K)$ is a division algebra. Let*

$$R_k = \{d \in K(\sqrt{a}, \sqrt{b}, \sqrt{c})^* \mid N_{K(\sqrt{a},\sqrt{b},\sqrt{c})|K}(d) \in K^{*2}\},$$

$$R_{K(\sqrt{a})} = \{d \in K(\sqrt{a}, \sqrt{b}, \sqrt{c})^* \mid N_{K(\sqrt{a},\sqrt{b},\sqrt{c})|K(\sqrt{a})}(d) \in K(\sqrt{a})^{*2}\},$$

$$\Sigma'_K = \{d \in K(\sqrt{a}, \sqrt{b}, \sqrt{c})^* \mid N_{K(\sqrt{a},\sqrt{b},\sqrt{c})|K(\sqrt{a})}(d) \in K^*\}.$$

Then $|K_1 \operatorname{Spin}(D)| \leqslant |R_k/R_{K(\sqrt{a})}\Sigma'_K|$.

Let $K = \mathbf{Q}(x, y)$, $a = x$, $b = y$, $c = (1+xy)/(1+y)$. Then $R_K/R_{K(\sqrt{a})}\Sigma'_K \neq \{1\}$, and $K_1 \operatorname{Spin}(D) \neq \{1\}$.

Finally, we shall present a useful formula for computing $K_1 \operatorname{Spin}(D)$ in a special but important case.

Let $k = K\langle x_1\rangle \ldots \langle x_t\rangle$, $a_1, \ldots, a_t \in K$. Then $D = (a_1, x_1)\otimes_k \ldots \otimes_k (a_t, x_t)$ is a division algebra iff $(K(\sqrt{a_1}, \ldots, \sqrt{a_t}) : K) = 2^t$.

THEOREM 4. *Suppose $D = (a_1, x_1)\otimes_k \ldots \otimes_k (a_t, x_t)$ is a division algebra, set*

$$R_K = \{d \in K(\sqrt{a_1}, \ldots, \sqrt{a_t})^* \mid N_{K(\sqrt{a_1},\ldots,\sqrt{a_t})|K}(d) \in K^{*2}\},$$

and let $T_1, \ldots, T_l$ be all subfields of $K(\sqrt{a_1}, \ldots, \sqrt{a_t})$ of codimension two. Then $K_1 \operatorname{Spin}(D) \cong R_K/T_1^* \ldots T_l^*$.

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