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Connected sequences of stable derived functors
and their applications

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1. Introduction

In [5] A. Dold and D. Puppe defined left derived functors

$$L_q T(\cdot, n): \mathcal{A} \rightarrow \mathcal{A}', \quad n, q \in \mathbb{Z}, \quad n \geq 0,$$

of any covariant (not necessary additive) functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ such that $T(0) = 0$ (\mathcal{A} and \mathcal{A}' are abelian categories and \mathcal{A} has enough projectives) and suspensions

$$\sigma_q: L_q T(\cdot, n) \rightarrow L_{q+1} T(\cdot, n+1)$$

such that σ_q is a natural equivalence of functors for $q < 2n$. This permits to consider the functors

$$L_q^s T = \varinjlim \{L_{q+n} T(\cdot, n), \sigma_{q+n}\}: \mathcal{A} \rightarrow \mathcal{A}'$$

called by the authors of [5] the left stable derived functors of the functor T . A justification of the name is clear in view of the equivalences $L_q^s T \cong L_{q+n} T(\cdot, n)$ for $q < n$. Starting with right derived functors of T or left (right) derived functors of any contravariant functor $\hat{T}: \mathcal{A} \rightarrow \mathcal{A}'$ and corresponding suspensions (defined in [5] as well) one can define in a similar fashion right stable derived functors of T and left (right) stable derived functors of \hat{T} . All stable derived functors are additive and they coincide with the derived functors of T and \hat{T} in the sense of Cartan and Eilenberg [2] whenever T and \hat{T} are additive functors.

It is well known that the essential role in the theory of the derived functors of an additive functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ is played by the following three theorems:

- (A) *The left (right) derived functors $L_q T(R^q T): \mathcal{A} \rightarrow \mathcal{A}'$, $q \in \mathbb{Z}$, form connected and exact sequence of functors.*
- (B) *If T is a covariant functor then $L_q T(P) = 0 = R^q T(Q)$ for any projective object P , injective object Q and $q \geq 1$.*
- (C) *Any natural transformation of functors $L_0 T \rightarrow L_0 T'$ uniquely extends to a map of connected sequences.*

One can ask whether these theorems can be carried over on to the stable derived functors of any functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ such that $T(0) = 0$. In this paper we give a partial answer to this question. Our main results state that the left (right) stable derived functors of any functor T form

a connected sequence of functors and this sequence is exact provided T is either a direct sum or a direct product of functors of finite degree (see Definition 4.9). We do not know if it is true for an arbitrary functor T .

Since every additive functor is clearly a functor of finite degree then this is a generalization of Theorem (A). For theorems (B) and (C) it is shown below that they fail in the case of stable derived functors of non-additive functors.

The first part of the paper contains definitions of stable derived functors, constructions of corresponding connected sequences and the proof of their exactness (for T as above). We also prove that if $(\mathcal{A}, \mathcal{A}')$ denotes the "category" of all covariant functors from \mathcal{A} to \mathcal{A}' , then the functors

$$L_q^s: (\mathcal{A}, \mathcal{A}') \rightarrow (\mathcal{A}, \mathcal{A}'), \quad q \in \mathbb{Z},$$

form a connected and exact sequence of functors.

In Sections 7–11 some applications of the above mentioned results are given. In particular it is proved that Eilenberg–MacLane's stable homology and cohomology functors

$$H_q^s(\cdot, G) = H_{q+n}(\cdot, n, G): \mathcal{A}b \rightarrow \mathcal{A}b, \quad n > q,$$

$$H_q^c(\cdot, G) = H^{q+n}(\cdot, n, G): \mathcal{A}b \rightarrow \mathcal{A}b, \quad n > q,$$

($\mathcal{A}b$ denotes the category of abelian groups) form connected and exact sequences of functors. Furthermore, the left stable derived functors of the 2nd symmetric power functor $SP^2: \mathcal{M}_R \rightarrow \mathcal{M}_R$, 2nd exterior power functor $\Lambda^2: \mathcal{M}_R \rightarrow \mathcal{M}_R$ and J. H. C. Whitehead's functor $\Gamma: \mathcal{M}_R \rightarrow \mathcal{M}_R$ are calculated for some categories of modules \mathcal{M}_R . For example, if $R = \mathbb{Z}$ is the ring of rational integers then we have the following natural equivalences of functors

$$L_q^s SP^2 = L_{q-1}^s \Lambda^2 = L_{q-2}^s \Gamma = \begin{cases} 0 & \text{for } q \leq 1, \\ \cdot \otimes_{\mathbb{Z}} \mathbb{Z}_2 & \text{for } q = 2t, t > 0, \\ \text{Tor}_1^{\mathbb{Z}}(\cdot, \mathbb{Z}_2) & \text{for } q = 2t+1, t > 0, \end{cases}$$

where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Observe that this implies that theorems (B) and (C) fail in the case of stable derived functors. In fact, $L_{2q}^s SP^2(\mathbb{Z}) = \mathbb{Z}_2 \neq 0$ for $q > 0$ and hence the trivial natural equivalence of functors $0 = L_0^s 0 \rightarrow L_0^s SP^2 = 0$ does not extend to an isomorphism of corresponding connected sequences. Thus, if we regard the theory of derived functors of an additive functor as an algebraic analogue of Eilenberg–Steenrod's homology and cohomology theories of triangulated pairs of topological spaces [8] (Theorem (B) corresponds to the dimension axiom, of course), then the theory of stable derived functors is an analogue of the generalized homology and cohomology theories [16].

Throughout this paper $T, \hat{T}: \mathcal{A} \rightarrow \mathcal{A}'$ denote a covariant and contravariant functor, respectively, such that $T(0) = \hat{T}(0) = 0$, \mathcal{A} and \mathcal{A}' are abelian categories, $\mathcal{A}b$ denotes the category of abelian groups, \mathcal{M}_R is the category of modules over a commutative ring R with identity element, and Z is the ring of rational integers.

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Some results of the present paper were announced in [13] and [14].

2. Category of complexes

Let \mathcal{A} be an abelian category. A chain complex X in the category \mathcal{A} is a sequence

$$\rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \dots$$

in \mathcal{A} such that $d_n d_{n+1} = 0$ for all n . A complex map $f: X \rightarrow X'$ is a collection of maps $f_n: X_n \rightarrow X'_n$ such that $f_n d_{n+1} = d'_{n+1} f_{n+1}$. A complex X is said to be *left (right)* if $X_n = 0$ for $n < 0$ ($n > 0$).

Let $\mathcal{K}(\mathcal{A})$ denote the category of all complexes in \mathcal{A} together with complex maps. Moreover, we denote by $\mathcal{K}^-(\mathcal{A})$ and $\mathcal{K}^+(\mathcal{A})$ full subcategories of $\mathcal{K}(\mathcal{A})$ consisting of all left and right complexes, respectively. Clearly, $\mathcal{K}(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$ and $\mathcal{K}^+(\mathcal{A})$ are abelian categories.

Two complex maps $f, g: X \rightarrow Y$ are homotopic (we write $f \sim g$) if there exists a collection of maps $s = \{s_n\}$, $s_n: X_n \rightarrow Y_{n+1}$ such that $f_n - g_n = s_{n-1} d_n^X + d_{n+1}^Y s_n$ for all n .

For any complex map $f: X \rightarrow Y$ the cone of f is the complex C_f with $(C_f)_n = Y_n \oplus X_{n-1}$ and the differential given by the matrix

$$\begin{bmatrix} d^Y & f \\ 0 & -d^X \end{bmatrix}.$$

The cone and suspension functors $C, S: \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ are defined as follows (see [12]). CX is the cone of the identity map on X and

$$(2.1) \quad (SX)_n = X_{n-1}, \quad d_n^{SX} = -d_{n-1}^X.$$

If $f: X \rightarrow Y$ is a complex map, then

$$(Cf)_{n+1} = f_{n+1} \oplus f_n, \quad (Sf)_n = f_{n-1}.$$

The natural injections $j_n: X_n \rightarrow (CX)_n$ and natural projections $\pi_n: (CX)_n \rightarrow (SX)_n$ define an exact sequence of complexes

$$(2.2) \quad 0 \rightarrow X \xrightarrow{j(X)} CX \xrightarrow{\pi(X)} SX \rightarrow 0.$$

Moreover, the complex maps $j(X), \pi(X)$ determine natural transformations of functors

$$j: id \rightarrow C, \quad \pi: C \rightarrow S.$$

Note also that $S: \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ is an isomorphism with a functorial inverse

$$S^{-1}: \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$$

defined by $S^{-1}(X)_n = X_{n+1}$, $S^{-1}(f)_n = f_{n+1}$. Obviously, C, S and S^{-1} are exact, chain homotopy preserving functors such that $CX, SX \in \mathcal{K}^-(\mathcal{A})$ ($S^{-1}X \in \mathcal{K}^+(\mathcal{A})$) whenever $X \in \mathcal{K}^-(\mathcal{A})$ ($X \in \mathcal{K}^+(\mathcal{A})$). It follows that (2.2) is a sequence of left complexes for any left complex X and the sequence

$$(2.3) \quad 0 \rightarrow S^{-1}X \rightarrow \bar{C}X \rightarrow X \rightarrow 0$$

(with $\bar{C}X = S^{-1}CX$) is a sequence of right complexes for any right complex X .

It is easy to check that CX is contractible (i.e. $1_{CX} \sim 0$) for any complex X and that two complex maps $f, g: X \rightarrow X'$ are chain homotopic if and only if $f + g$ can be factored by $X \xrightarrow{j(X)} CX$.

In order to give another (equivalent, of course) definition of the homotopy of complex maps recall that if F is a finitely generated free abelian group with a base x_1, \dots, x_n and A is an object of \mathcal{A} , then the tensor product $F \otimes A$ is defined as $\bigoplus_{i=1}^n A_i$, where $A_i = A$. If, further, F' is another finitely generated free abelian group with a basis y_1, \dots, y_m and $f: F \rightarrow F'$ is a group homomorphism such that $f(x_i) = \sum_j a_{ij} y_j$, then clearly the matrix $(a_{ij} 1_A)_{i,j}$ defines a map $f \otimes A: F \otimes A \rightarrow F' \otimes A$. The map $f \otimes A$ does not depend on the choice of free bases of F, F' (see [5]; 3.32) and setting

$$f \otimes g = (f \otimes A) \left(\bigoplus_{i=1}^n g_i \right), \quad g_i = g,$$

we get a covariant functor of two variables \otimes . It permits to define (see e.g. [5]) the tensor product of complexes $K \otimes X$ for any complex K of finitely generated free abelian groups and $X \in \text{ob } \mathcal{K}(\mathcal{A})$. Observe that if $S^n X (n \in \mathbb{Z})$ is the n th suspension of X and the complex S^n is given by

$$(2.4) \quad (S^n)_i = \begin{cases} 0 & \text{for } i \neq n, \\ Z & \text{for } i = n, \end{cases}$$

then $S^n X = S^n \otimes X$.

Let $N(I)$ be the complex

$$(2.5) \quad \dots \rightarrow 0 \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow N(I)_1 = Ze \xrightarrow{d_1} Ze_0 \oplus Ze_1 = N(I)_0 \rightarrow 0 \rightarrow \dots$$

where Ze_0, Ze_1, Ze denote the free abelian groups, generated by symbols e_0, e_1, e , respectively, and $d_1(e) = e_1 - e_0$. Furthermore, let $j_k: S^0 \rightarrow N(I)$, $k = 0, 1$, be complex maps defined by $j_k(1) = e_k$. Then we have the following

2.6. PROPOSITION [3]. *Complex maps $f_0, f_1: X \rightarrow X'$ are homotopic if and only if there exists a complex map $F: N(I) \otimes X \rightarrow X'$ such that $f_k, k = 0, 1$, is the composition*

$$X = S^0 \otimes X \xrightarrow{j_k \otimes 1} N(I) \otimes X \xrightarrow{F} X' \quad (k = 0, 1).$$

For completeness we sketch the proof. If $s = \{s_n: X_n \rightarrow X'_{n+1}\}$ is a chain homotopy joining f_0 with f_1 then the complex map $F: N(I) \otimes X \rightarrow X'$ with

$$F_n: e_0 \otimes X_n \oplus e_1 \otimes X_n \oplus e \otimes X_{n-1} \rightarrow X'_n$$

given by $F_n|_{e_i \otimes X_n} = f_i$ for $i = 0, 1$, and $F_n|_{e \otimes X_{n-1}} = s_{n-1}$ is the required complex map. Conversely, any such F uniquely determines the chain homotopy s .

2.7. DEFINITION (see [2]). An exact sequence of complexes

$$0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0$$

is said to be *normal* if the sequence

$$0 \rightarrow X'_n \xrightarrow{i_n} X_n \xrightarrow{p_n} X''_n \rightarrow 0$$

splits for any $n \in Z$.

Let $\mathcal{N}\mathcal{K}^-(\mathcal{A})$ and $\mathcal{N}\mathcal{K}^+(\mathcal{A})$ denote full subcategories of the category of all short exact sequences in $\mathcal{K}(\mathcal{A})$ whose objects are normal sequences of left and right complexes, respectively.

In what follows, when dealing with a normal sequence of complexes

$$X = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0),$$

we assume that $X_n = X'_n \oplus X''_n$ and i_n, p_n are natural injections and natural projections, respectively.

Let X be a normal sequence of complexes and let θ_n be a composed map

$$X''_n \rightarrow X'_n \oplus X''_n \xrightarrow{d_n} X'_{n-1} \oplus X''_{n-1} \rightarrow X'_{n-1},$$

where the first map is the natural injection and the last one is the natural projection. Then the maps θ_n define a complex map

$$(2.8) \quad \theta(X): X'' \rightarrow SX'$$

and it is easy to verify that the following two lemmas hold:

2.9. LEMMA. *If a commutative diagram*

$$\begin{array}{ccccccc} \mathbf{X} = (0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' \longrightarrow 0) \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ \mathbf{Y} = (0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' \longrightarrow 0) \end{array}$$

has normal rows, then the diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\theta(\mathbf{X})} & SX' \\ \downarrow f'' & & \downarrow Sf' \\ Y'' & \xrightarrow{\theta(\mathbf{Y})} & SY' \end{array}$$

is homotopy commutative.

2.10. LEMMA. *If $f: X \rightarrow Y$ is a complex map, then the sequence*

$$C_f = (0 \rightarrow Y \rightarrow C_f \rightarrow SX \rightarrow 0)$$

with obvious maps is normal and $\theta(C_f) = Sf$.

Let $\mathcal{K}^-(\mathcal{A})^\square$ be the category whose objects are all left complexes and morphisms are homotopy classes of complex maps. A sequence of the form

$$A' \rightarrow A \rightarrow A'' \rightarrow SA'$$

in $\mathcal{K}^-(\mathcal{A})^\square$ is called a *triangle* [9]. Let \mathcal{T}^- denote the category of all triangles in $\mathcal{K}^-(\mathcal{A})^\square$ isomorphic with triangles of the form

$$X' \xrightarrow{i} X \xrightarrow{p} X'' \xrightarrow{\theta(\mathbf{X})} SX'$$

where $\mathbf{X} = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0)$ is a normal sequence in $\mathcal{K}^-(\mathcal{A})$.

2.11. LEMMA. *If $(X' \xrightarrow{i} X \xrightarrow{p} X'' \xrightarrow{\theta} SX') \in \text{ob } \mathcal{T}^-$ then $(X \xrightarrow{p} X'' \xrightarrow{u} SX' \xrightarrow{Si} SX) \in \text{ob } \mathcal{T}^-$, too.*

Proof. We assume that \mathcal{A} is the category of abelian groups. The proof in the general case is similar. Moreover, without loss of generality we may assume that $\mathbf{X} = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0)$ is a normal sequence in $\mathcal{K}^-(\mathcal{A})$ and that $\theta = \theta(\mathbf{X})$. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & C_i & \longrightarrow & SX' & \xrightarrow{Si} & SX \\ \downarrow id & & \downarrow t & & \downarrow id & & \downarrow id \\ X & \xrightarrow{p} & X'' & \longrightarrow & SX' & \xrightarrow{Si} & SX \end{array}$$

in $\mathcal{K}^-(\mathcal{A})$, where t is the natural projection. If $h: X'' \rightarrow C_i$ is a complex map defined by $h_n(x''_n) = ((0, x''_n), -\theta_n(x''_n))$ then $th \sim id$, and $ht \sim id$. Then the lemma follows from Lemma 2.10.

A functor $T: \mathcal{K}^-(\mathcal{A}) \rightarrow \mathcal{A}'$ is said to be an *h-functor* if $T(f) = T(g)$ whenever f and g are homotopic maps in $\mathcal{K}^-(\mathcal{A})$.

Using Lemma 2.11 and arguments from the proof of Proposition 1.1, Chapter I in [9], one can easily obtain:

2.12. THEOREM. *Let $L_q: \mathcal{K}^-(\mathcal{A}) \rightarrow \mathcal{A}'$ be a sequence of covariant functors satisfying the following conditions:*

- (i) L_q are *h-functors* and $L_q(0) = 0$.
- (ii) *There are natural equivalences of functors*

$$s_q: L_q \rightarrow L_{q+1} S, \quad q \in \mathbb{Z}.$$

Then the statements (a)–(c) below hold true:

(a) *If $X = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0)$ is a normal sequence in $\mathcal{K}^-(\mathcal{A})$ and $d_q(X)$ is the composition*

$$L_q X'' \xrightarrow{L_q d(X)} L_q S X' \xrightarrow{(s_{q-1} X)^{-1}} L_{q-1} X'$$

then the sequence

$$(2.13) \quad \dots \rightarrow L_q X' \xrightarrow{L_q i} L_q X \xrightarrow{L_q p} L_q X'' \xrightarrow{d_q(X)} L_{q-1} X' \rightarrow \dots$$

is a complex in \mathcal{A}' .

(b) *If $(f', f, f''): X \rightarrow Y$ is a map of normal sequences then the diagrams*

$$\begin{array}{ccc} L_q X'' & \xrightarrow{d_q(X)} & L_{q-1} X' \\ \downarrow L_q f'' & & \downarrow L_{q-1} f' \\ L_q Y'' & \xrightarrow{d_q(Y)} & L_{q-1} Y' \end{array}$$

are commutative for all $q \in \mathbb{Z}$.

(c) *If for any normal sequence Y and any $q \in \mathbb{Z}$ the induced sequence*

$$L_q Y' \rightarrow L_q Y \rightarrow L_q Y''$$

is exact then the sequence (2.13) is exact.

3. Left stable derived functors of covariant functors

Let Δ denote the category whose objects are the sets $[n] = \{0, 1, \dots, n\}$, $n \in \mathbb{Z}$, $n \geq 0$, and whose maps are non-decreasing functions.

Recall that a simplicial object in the category \mathcal{A} is a contravariant functor $X: \Delta \rightarrow \mathcal{A}$ and a simplicial map is a natural transformation of such functors. Write

$$(3.1) \quad \begin{cases} X_n = X([n]), \\ \tilde{\alpha} = X(\alpha): X_m \rightarrow X_n, \end{cases}$$

where $\alpha: [n] \rightarrow [m]$ is a non-decreasing map. Simplicial objects in \mathcal{A} , together with simplicial maps, form a category $s\mathcal{A}$.

Let X be a simplicial object in the category \mathcal{A} . The associated chain complex kX is the following left complex

$$(3.2) \quad \rightarrow X_n \xrightarrow{d_n} X_{n-1} \rightarrow \rightarrow X_1 \xrightarrow{d_1} X_0 \rightarrow 0$$

with $d_n = \sum_{i=0}^n (-1)^i \tilde{\varepsilon}_i$, where

$$(3.3) \quad \varepsilon_i: [n] \rightarrow [n+1] \quad (i = 0, 1, \dots, n)$$

are non-decreasing maps defined by

$$\varepsilon_i(j) = \begin{cases} j & \text{for } j < i, \\ j+1 & \text{for } j \geq i. \end{cases}$$

If $f: X \rightarrow X'$ is a simplicial map then it is clear that the maps $f_n = f([n]): X_n \rightarrow X'_n$ define a complex map $k(f): kX \rightarrow kX'$. Thus we have a functor

$$k: s\mathcal{A} \rightarrow \mathcal{K}^-(\mathcal{A}).$$

Recall that the *normalization functor* (see [5]; 3.1)

$$N: s\mathcal{A} \rightarrow \mathcal{K}^-(\mathcal{A})$$

is defined in the following way:

$$(3.4) \quad \begin{cases} (NX)_n = \bigcap_{i=1}^n \text{Ker}(\tilde{\varepsilon}_i: X_n \rightarrow X_{n-1}) & \text{for } n \geq 0, \\ (NX)_n = 0 & \text{for } n < 0, \\ d_n^{NX} = \tilde{\varepsilon}_0|_{(NX)_n}: (NX)_n \rightarrow (NX)_{n-1} \end{cases}$$

for any simplicial object X in \mathcal{A} . If $f: X \rightarrow X'$ is a simplicial map then $N(f)_n: (NX)_n \rightarrow (NX')_n$ is the restriction of the map f_n to $(NX)_n$.

It is easy to see that NX is a subcomplex of the complex kX and it follows from 3.22 in [5] that the natural inclusion $NX \hookrightarrow kX$ is a homotopy equivalence. It turns out, that the functor N establishes an equivalence of the categories $s\mathcal{A}$ and $\mathcal{K}^-(\mathcal{A})$; it means that there exists an (additive and exact) functor

$$K: \mathcal{K}^-(\mathcal{A}) \rightarrow s\mathcal{A}$$

(see [5]; 3.2) such that the compositions NK and KN are naturally equivalent to identity functors ([5]; 3.6).

Let $T: A^m = \overbrace{A \times A \times \dots \times A}^m \rightarrow A'$ be a covariant functor (in all variables). The functor T induces in a natural way a covariant functor (which will be denoted by the same letter)

$$T: (s\mathcal{A})^m \rightarrow s\mathcal{A}'$$

defined as follows. If X^1, \dots, X^m are simplicial objects in \mathcal{A} , then

$$(3.5) \quad \begin{cases} T(X^1, \dots, X^m)_n = T(X_n^1, \dots, X_n^m), \\ T(X^1, \dots, X^m)(\alpha) = T(X^1(\alpha), \dots, X^m(\alpha)), \end{cases}$$

where $\alpha: [q] \rightarrow [n]$ is non-decreasing map. If $f^i: X^i \rightarrow Y^i$, $i = 0, 1, \dots, m$, are simplicial maps, then

$$T(f^1, \dots, f^m)_n = T(f_n^1, \dots, f_n^m).$$

Let $I(k)$, for $k = 0, 1$, denote simplicial objects in the category of abelian groups such that $I(k)_n$ is a free abelian group generated by the set of all non-decreasing maps $\beta: [n] \rightarrow [k]$ and $I(k)(\alpha): I(k)_n \rightarrow I(k)_m$ is given by

$$I(k)(\alpha)\beta = \beta\alpha$$

for a non-decreasing map $\alpha: [m] \rightarrow [n]$ and a free generator $\beta \in I(k)_n$.

If $\varepsilon_k: [0] \rightarrow [1]$ ($k = 0, 1$) are the simplicial maps (3.3), then it is easy to see that the group homomorphisms

$$e_n^k: I(0)_n \rightarrow I(1)_n,$$

with $e_n^k(\beta) = \varepsilon_k\beta$ for a free generator $\beta \in I(0)_n$, define simplicial maps

$$e^k: I(0) \rightarrow I(1), \quad k = 0, 1.$$

One can also check that the normalization $NI(1)$ of the simplicial object $I(1)$ and the left complex $N(I)$ described in (2.5) coincide.

Recall that in Section 2 the tensor product $F \otimes A$ was defined for any object A of \mathcal{A} and any finitely generated free abelian group F . Then the tensor product $G \otimes X$ is defined for any simplicial object X in \mathcal{A} and any simplicial object G in the category of finitely generated free abelian groups. $((G \otimes X)_n = G_n \otimes X_n, (G \otimes X)(\alpha) = G(\alpha) \otimes X(\alpha))$. Clearly, $I(0) \otimes X \simeq X$.

3.6. DEFINITION [5]. Simplicial maps $f_0, f_1: X \rightarrow X'$ are *homotopic* ($f_0 \sim f_1$), if there exists a simplicial map $F: I(1) \otimes X \rightarrow X'$ such that f_k , $k = 0, 1$, is the composition

$$X \simeq I(0) \otimes X \xrightarrow{e^k \otimes 1} I(1) \otimes X \xrightarrow{F} X'.$$

The simplicial map F is called a *simplicial homotopy* joining f_0 with f_1 .

If $\sigma \in I(1)_n$ is a free generator, then $F_\sigma: X_n \rightarrow X'_n$ denotes the map defined by $F_\sigma(x) = F(\sigma \otimes x)$ for $x \in X_n$.

From 3.31 and 1.15 in [5] we obtain the following

3.7. PROPOSITION. *If $T: \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor, then the functor $T: s\mathcal{A} \rightarrow s\mathcal{A}'$ defined in (3.5) form $\varepsilon = 1$, and the functors N, K are homotopy preserving functors.*

Now, for any covariant functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ and $q \in \mathbb{Z}$, we define the functors

$$D_q T: \mathcal{K}^-(\mathcal{A}) \rightarrow \mathcal{A}'$$

setting $D_q T = H_q NTK$.

It is clear that any natural transformation of functors $\gamma: T \rightarrow T'$ induces a natural transformation

$$(3.8) \quad D_q(\gamma): D_q T \rightarrow D_q T'$$

Immediately from Proposition 3.7 follows the following

3.9. COROLLARY. $D_q T$ are h -functors.

According to [5]; 3.22, the injection $NX \hookrightarrow kX$ is a homotopy equivalence for any simplicial object X . Moreover, it defines a natural transformation of functors $N \rightarrow k$. Hence we get

3.10. COROLLARY. The functors $D_q T$ and $H_q kTK$ are naturally equivalent (in what follows they will be identified).

3.11. COROLLARY. If T is an additive functor, then $D_q T \simeq H_q T$, where $H_q T(X)$ ($X \in \text{ob } \mathcal{K}^-(\mathcal{A})$) is q -th homology object of the complex

$$\dots \rightarrow T(X_n) \xrightarrow{T(d_n)} T(X_{n-1}) \rightarrow \dots \rightarrow T(X_0) \rightarrow 0.$$

Proof. An obvious consequence of the definition of the functor k is that it commutes with additive functors. Hence

$$D_q T \simeq H_q kTK = H_q T kK \simeq H_q TNK \simeq H_q T$$

because, as we mentioned, the complexes kKX and NKX are homotopy equivalent and the composition NK is naturally equivalent to the identity functor.

Let now $T(0) = 0$. Following [5] we define natural transformations of functors

$$(3.12) \quad \sigma_q: D_q T \rightarrow (D_{q+1} T) S$$

where $S: \mathcal{K}^-(\mathcal{A}) \rightarrow \mathcal{K}^-(\mathcal{A})$ is the suspension functor. If X is a left complex, then the normal sequence of left complexes

$$0 \rightarrow X \xrightarrow{j} CX \xrightarrow{\pi} SX \rightarrow 0,$$

with $j = j(X)$, $\pi = \pi(X)$ and the contractible complex CX (see (2.2)), induce the sequence

$$(3.13) \quad NTK(X) \xrightarrow{j_*} NTK(CX) \xrightarrow{\pi_*} NTK(SX)$$

with $\pi_* j_* = 0$ and the contractible complex $NTK(CX)$ (see Proposition 3.7). Let t be a chain homotopy joining zero with the identity map on $NTK(CX)$. Then

$$h = \pi_* t j_*: NTK(X) \rightarrow NTK(SX)$$

is a map of degree 1 and $dh = -hd$ (see 5.6 in [5]), so it produces the maps

$$\sigma_q(X): H_q NTK(X) = D_q TX \rightarrow D_{q+1} TSX = H_{q+1} NTK(SX).$$

It is shown in [5], 5.7, 5.10, that these maps do not depend on the choice of the chain homotopy t and that they define the natural transformation (3.12).

3.14. LEMMA. *If $T^0 \xrightarrow{\alpha} T^1 \xrightarrow{\beta} T^2$ is a sequence of covariant functors from \mathcal{A} to \mathcal{A}' such that $T^i(0) = 0$ for $i = 0, 1, 2$, then there exist chain homotopies t_0, t_1, t_2 joining zero with the identity maps on $NT^0K(CX)$, $NT^1K(CX)$, $NT^2K(CX)$, respectively, such that the diagram*

$$(3.15) \quad \begin{array}{ccccc} NT^0K(X) & \xrightarrow{Na(KX)} & NT^1K(X) & \xrightarrow{N\beta(KX)} & NT^2K(X) \\ \downarrow h_0 = \pi_* t_0 j_* & & \downarrow h_1 = \pi_* t_1 j_* & & \downarrow h_2 = \pi_* t_2 j_* \\ NT^0K(SX) & \xrightarrow{Na(KSX)} & NT^1K(SX) & \xrightarrow{N\beta(KSX)} & NT^2K(SX) \end{array}$$

is commutative (for j_* , π_* see (3.13)).

Proof. If X is a left complex, then in view of Proposition 3.7, $1_{K CX} \sim 0$ (as $1_{CX} \sim 0$). Let $F: I(1) \otimes KCX \rightarrow KCX$ be a simplicial homotopy joining the zero map with $1_{K CX}$. Then the simplicial map

$$T^i(F): I(1) \otimes T^iK(CX) \rightarrow T^iK(CX), \quad i = 0, 1, 2,$$

defined by

$$T^i(F)_\sigma = T^i(F_\sigma)$$

for a free generator $\sigma \in I(1)$, is simplicial homotopy joining zero with the identity map of $T^iK(CX)$ (see 1.13 in [5]). It follows that the diagram

$$\begin{array}{ccccc} N(I(1) \otimes T^0KCX) & \xrightarrow{\bar{\alpha}} & N(I(1) \otimes T^1KCX) & \xrightarrow{\bar{\beta}} & N(I(1) \otimes T^2KCX) \\ \downarrow NT^0(F) & & \downarrow NT^1(F) & & \downarrow NT^2(F) \\ NT^0K(CX) & \xrightarrow{Na(KCX)} & NT^1K(CX) & \xrightarrow{N\beta(KCX)} & NT^2K(CX) \end{array}$$

with $\bar{\alpha} = N(1 \otimes \alpha(KCX))$, $\bar{\beta} = N(1 \otimes \beta(KCX))$, is commutative. Furthermore, the complex map

$$NI(1) \otimes NT^iK(CX) \xrightarrow{\mathcal{V}} N(I(1) \otimes T^iK(CX)) \xrightarrow{NT^i(F)} NT^iK(CX),$$

$i = 0, 1, 2$, with Eilenberg–MacLane’s map \mathcal{V} (see [6]), is a homotopy joining zero with the identity map of $NT^iK(CX)$ (see the proof of Corollary 2.7 in [3]). Consequently, using the naturality of \mathcal{V} we get from

the above diagram the following commutative diagram

$$\begin{array}{ccccc}
NI(1) \otimes NT^0 K O X & \xrightarrow{1 \otimes N\alpha(K O X)} & NI(1) \otimes NT^1 K O X & \xrightarrow{1 \otimes N\beta(K O X)} & NI(1) \otimes NT^2 K O X \\
\downarrow NT^0(F) \cdot \nabla & & \downarrow NT^1(F) \cdot \nabla & & \downarrow NT^2(F) \cdot \nabla \\
NT^0 K(O X) & \xrightarrow{N\alpha(K O X)} & NT^1 K(O X) & \xrightarrow{N\beta(K O X)} & NT^2 K(O X).
\end{array}$$

The homotopies $NT^i(F) \cdot \nabla$ ($i = 0, 1, 2$) yield chain homotopies $t^i = \{t_n^i\}$ joining zero with the identity map of $NT^i K(O X)$ (see the proof of Proposition 2.6) such that

$$N\alpha(K O X)_{n+1} t_n^0 = t_n^1 N\alpha(K O X)_n, \quad N\beta(K O X)_{n+1} t_n^1 = t_n^2 N\beta(K O X)_n$$

for any integer n . This implies that the diagram (3.15) is commutative and the lemma is proved.

We deduce from the above lemma the following

3.16. COROLLARY ([5]; 5.11). *If $\gamma: T \rightarrow T'$ is a natural transformation of functors and $T(0) = T'(0) = 0$, then the diagram*

$$\begin{array}{ccc}
D_q T & \xrightarrow{\sigma_q} & (D_{q+1} T) S \\
\downarrow D_q(\gamma) & & \downarrow D_{q+1}(\gamma) S \\
D_q T' & \xrightarrow{\sigma'_q} & (D_{q+1} T') S
\end{array}$$

is commutative.

3.17. DEFINITION. Let $n \in \mathbb{Z}$. A chain complex X is n -trivial (n -cotrivial), if there exists a complex X' homotopy equivalent with X such that $X'_j = 0$ for $j < n$ (for $j > n$). A simplicial object Q is n -trivial, if NQ is the n -trivial chain complex.

The following Dold–Puppe's result is basic for further considerations.

3.18. THEOREM (see 5.11 in [5]). *If X is an n -trivial left chain complex, then the map $\sigma_q(X): D_q T(X) \rightarrow D_{q+1} T(SX)$ is an isomorphism for $q < 2n$.*

For any integer q and any chain complex X we have a direct system

$$\{D_{q+n} T(S^n X), \sigma_{q+n}(S^n X)\}_{n \in \mathbb{Z}},$$

so we can define

$$\begin{aligned}
D_q^s T(X) &= \varinjlim_n \{D_{q+n} T(S^n X), \sigma_{q+n}(S^n X)\}, \\
D_q^s T(f) &= \varinjlim_n \{D_{q+n} T(S^n f)\},
\end{aligned}$$

for a complex map $f: X \rightarrow X'$. It follows from (3.18) that these limits exist and that

$$(3.19) \quad D_q^s T \approx (D_{q+n} T) S^n \quad \text{for } n > q,$$

where \approx denotes the natural equivalence of functors. Hence and from (3.9) and (3.11) we get

3.20. COROLLARY. $D_q^s T$ are h -functors.

3.21. COROLLARY. If T is an additive functor, then $D_q^s T \approx H_q T$.

3.22. COROLLARY. If there exists a covariant functor $U: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ such that $T(A) = U(A, A)$, $T(f) = U(f, f)$ for any $A \in \text{ob } \mathcal{A}$ and any map f in \mathcal{A} , then $D_q^s T = 0$ for all q .

Proof. In this case the natural transformations σ_q are zero by 5.2 in [5].

3.23. PROPOSITION. (i) $D_q^s T = D_q T = 0$ for $q < 0$.

(ii) $D_q^s T$ are additive functors.

Proof. Taking $n = 0$ in (3.19) we obtain $D_q^s T(X) = H_q(NTKX) = 0$ for $q < 0$, because $(NTKX)_j = 0$ for $j < 0$. To prove (ii) it is sufficient to show that for any complexes X_1, X_2 and natural injections $i_k: X_k \rightarrow X_1 \oplus X_2$, $k = 0, 1$, the maps $D_q^s T(i_k)$ induce an isomorphism

$$D_q^s T(X_1 \oplus X_2) \simeq D_q^s T(X_1) \oplus D_q^s T(X_2).$$

Using (3.19) and the additivity of the functor K , we have for $n > q$

$$D_q^s T(X_1 \oplus X_2) \simeq H_{q+n} NTK(S^n(X_1 \oplus X_2)) \simeq H_{q+n} NT(KS^n X_1 \oplus KS^n X_2).$$

In Section 4 we shall define a functor $T_2: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$, called the 2-nd cross-effect functor of the functor T , such that

$$T(A_1 \oplus A_2) \simeq T(A_1) \oplus T(A_2) \oplus T_2(A_1, A_2)$$

for any objects A_1, A_2 of \mathcal{A} . Hence, by the additivity of the functor N , we get

$$\begin{aligned} & H_{q+n} NT(KS^n X_1 \oplus KS^n X_2) \\ & \simeq H_{q+n} N(TKS^n X_1 \oplus TK S^n X_2 \oplus T_2(KS^n X_1, KS^n X_2)) \\ & \simeq H_{q+n} NTKS^n X_1 \oplus H_{q+n} NTKS^n X_2 \oplus H_{q+n} NT_2(KS^n X_1, KS^n X_2) \\ & \simeq D_q^s T(X_1) \oplus D_q^s T(X_2) \oplus H_{q+n} NT_2(KS^n X_1, KS^n X_2). \end{aligned}$$

Since $S^n X_1, S^n X_2$ are n -trivial complexes and $T_2(0, A) = T_2(A, 0) = 0$, the theorem follows from the following

3.24. LEMMA. If $J: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor such that $J(0, A) = J(A', 0) = 0$ for $A, A' \in \text{ob } \mathcal{A}$, and X_1, X_2 are n -trivial complexes, then

$$H_j N J(KX_1, KX_2) = 0 \quad \text{for } j < 2n.$$

Proof. The lemma is an obvious consequence of (6.10) in [5].

Let us assume that in the category \mathcal{A} there exist arbitrary direct sums and direct limits. It is clear that the categories $\mathcal{K}^-(\mathcal{A})$ and $\text{s}\mathcal{A}$ have the same property.



3.25. PROPOSITION. *If a covariant functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ satisfies the condition $T(0) = 0$ and commutes with direct limits, then*

- (i) *The functors $D_q T$ and $D_q^s T$ commute with direct limits.*
- (ii) *The functors $D_q^s T$ commute with arbitrary direct sums.*

Proof. Observe that (ii) follows from (i) and the well-known fact that any direct sum $\bigoplus_{i \in I} X_i$ of complexes is a direct limit of its subcomplexes of the form $\bigoplus_{i \in L} X_i$, where L is a finite subset of I . Now we prove (i).

By (3.19) we need only show that the functors $D_q T = H_q NTK$ commute with direct limits. Since the functors H_q commute with direct limits and N is left and right adjoint to K (see [3]), then the required result is a consequence of the following well-known

3.26. LEMMA. *If $U: \mathcal{C} \rightarrow \mathcal{C}'$ is a covariant functor left adjoint to $V: \mathcal{C}' \rightarrow \mathcal{C}$ (i.e. $\text{Hom}_{\mathcal{C}'}(U(\cdot), \cdot)$ is naturally equivalent to $\text{Hom}_{\mathcal{C}}(\cdot, V(\cdot))$), then U commutes with direct sums and direct limits, V commutes with direct product.*

We record for a further use the following consequence of the fact that N commutes with direct limits and direct products.

3.27. COROLLARY. *Let $\{T_i\}$ be a direct system of covariant functors such that $T_i(0) = 0$ for all i . Then*

$$D_q^s(\varinjlim T_i) \approx \varinjlim D_q^s T_i, \quad D_q^s\left(\prod_i T_i\right) \simeq \prod_i D_q^s T_i^{(1)}.$$

Let $T: \mathcal{A} \rightarrow \mathcal{A}'$ be as before a covariant functor such that $T(0) = 0$. We define natural equivalences of functors

$$(3.28) \quad s_q: D_q^s T \rightarrow (D_{q+1}^s T)S, \quad q \in \mathbb{Z}.$$

Observe for this purpose that for any left complex X and $q \in \mathbb{Z}$ the sequence of maps

$$\sigma_{q+n}(S^n X): D_{q+n} T(S^n X) \rightarrow D_{q+n+1} T(S^n(SX))$$

is a map of direct systems

$$\{D_{q+n} T(S^n X), \sigma_{q+n}(S^n X)\} \rightarrow \{D_{q+1+n} T(S^n(SX)), \sigma_{q+1+n}(S^{n+1} X)\}$$

and thus it induces a map of limits

$$s_q(X): D_q^s T(X) \rightarrow D_{q+1}^s T(SX).$$

$S^n X$ is the n -trivial complex, so by Theorem 3.18 the maps $\sigma_{q+n}(S^n X)$ are isomorphisms for $n > q$. Hence $s_q(X)$ are isomorphisms and it is easy

⁽¹⁾ $(\prod_i T_i)(A) = \prod_i T_i(A)$, $(\prod_i T_i)f = \prod_i T_i(f)$ for $f: A \rightarrow B$, and similarly for $\varinjlim T_i$.

to verify that they define the natural equivalence of functors (3.28). Moreover, in view of Corollary 3.16, the diagram

$$(3.29) \quad \begin{array}{ccc} D_q^s T & \xrightarrow{s_q} & (D_{q+1}^s T) S \\ \downarrow D_q^s(\gamma) & & \downarrow D_{q+1}^s(\gamma) S \\ D_q^s T' & \xrightarrow{s'_q} & (D_{q+1}^s T') S \end{array}$$

is commutative for any natural transformation of functors $\gamma: T \rightarrow T'$, where $D_q^s(\gamma) = \varinjlim \{D_{q+n}^s(\gamma) S^n\}$ (see 3.8).

The main purpose of the next section is an investigation of a sequence (3.32) described in the following theorem.

3.30. THEOREM. *Let*

$$(3.31) \quad X = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0)$$

be a normal sequence of left complexes and let $\theta(X): X'' \rightarrow SX'$ be the complex map (2.8). If $T: \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor such that $T(0) = 0$, then the sequence

$$(3.32) \quad \dots \rightarrow D_q^s T(X') \xrightarrow{i_*} D_q^s T(X) \xrightarrow{p_*} D_q^s T(X'') \xrightarrow{d_q(X)} D_{q-1}^s T(X')$$

with $d_q(X) = (s_{q-1}(X'))^{-1} D_q^s T(\theta(X))$, is a complex. If the diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & X' & \rightarrow & X & \rightarrow & X'' \rightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ 0 & \rightarrow & Y' & \rightarrow & Y & \rightarrow & Y'' \rightarrow 0 \end{array}$$

is commutative, then the diagram

$$\begin{array}{ccc} D_q^s T(X'') & \xrightarrow{d_q} & D_{q-1}^s T(X') \\ \downarrow D_q^s T(f'') & & \downarrow D_{q-1}^s T(f') \\ D_q^s T(Y'') & \xrightarrow{d_q} & D_{q-1}^s T(Y') \end{array}$$

is commutative.

The theorem immediately follows from Theorem 2.12.

In [2] the notion of connected sequences of functors was defined. We will use this notion in a more general meaning.

3.33. DEFINITION. Let \mathcal{C} be an arbitrary category and let \mathcal{N} be a full subcategory of the category of diagrams of the form $A' \rightarrow A \rightarrow A''$ in \mathcal{C} . A sequence of covariant functors $L_q: \mathcal{C} \rightarrow \mathcal{A}'$, $q \in \mathbb{Z}$ (\mathcal{A}' is an abelian category) is called an \mathcal{N} -connected sequence of covariant functors, if the following conditions are satisfied:

(a) For each $A = (A' \xrightarrow{i} A \xrightarrow{p} A'') \in \text{ob } \mathcal{N}$ there are connecting maps $d_q: L_q A'' \rightarrow L_{q-1} A'$ such that the sequence

$$\dots \rightarrow L_q A' \xrightarrow{L_q i} L_q A \xrightarrow{L_q p} L_q A'' \xrightarrow{d_q} L_{q-1} A' \rightarrow \dots$$

is a complex.

(b) If $(f', f, f''): A \rightarrow B$ is a map in \mathcal{N} , then the diagram

$$\begin{array}{ccc} L_q A'' & \xrightarrow{d_q} & L_{q-1} A' \\ \downarrow L_q f'' & & \downarrow L_{q-1} f' \\ L_q B'' & \xrightarrow{d_q} & L_{q-1} B' \end{array}$$

is commutative.

Similarly one can define an \mathcal{N} -connected sequence of contravariant functors. If \mathcal{C} is an abelian category and \mathcal{N} is a category of all short exact sequences in \mathcal{C} , then any \mathcal{N} -connected sequence of functors is called simply a *connected sequence of functors*.

Recall (see Section 2) that $\mathcal{N}\mathcal{K}^-(\mathcal{A})$ denotes the category of all normal sequences of left complexes in \mathcal{A} . Then Theorem 3.30 and the commutativity of the diagram (3.29) imply

3.34. COROLLARY. *The functors $D_q^s T$, $q \in \mathbb{Z}$, form $\mathcal{N}\mathcal{K}^-(\mathcal{A})$ -connected sequence of covariant functors. Moreover, if $\gamma: T \rightarrow T'$ is a natural transformation of functors, then the natural transformations $D_q^s(\gamma): D_q^s T \rightarrow D_q^s T'$ define a map of connected sequences $D^s(\gamma): \{D_q^s T\} \rightarrow \{D_q^s T'\}$.*

Let \mathcal{A} be an abelian category with enough projectives.

3.35. DEFINITION. Let $q \in \mathbb{Z}$. The q th left stable derived functor of the functor T is the covariant functor $L_q^s T: \mathcal{A} \rightarrow \mathcal{A}'$ defined as follows. If $A \in \text{ob } \mathcal{A}$, then

$$L_q^s T(A) = D_q^s T(P)$$

where P is a projective resolution of A . If $f: \mathcal{A} \rightarrow \mathcal{A}'$ is a map in \mathcal{A} and $F: P \rightarrow P'$ is a resolution of f , then

$$L_q^s T(f) = D_q^s T(F).$$

Since $D_q^s T$ are h -functors, the correctness of this definition can be proved in the same way as in the case of additive functors ([2]; V).

Remark. The functor $L_q^s T$ and the q th left stable derived functor of the functor T defined in [5], 6.3, coincide. In fact, if P is a projective resolution of an object A , then it is clear that the simplicial object $KS^n P$ is a semisimplicial resolution of the pair (A, n) in the sense of [5], 4.1. Hence

$$D_{q+n} T(S^n P) \simeq L_{q+n} T(A, n),$$

what implies the required result.

As an immediate consequence of 3.23 and 3.21 we have:

3.36. COROLLARY. $L_q^s T$ are additive functors and $L_q^s T = 0$ for $q < 0$. If T is an additive functor, then the functors $L_q^s T$ and the left derived functors of the functor T (in the sense of Cartan-Eilenberg) coincide.

3.37. COROLLARY. If $\{T_i\}$ is a direct system of covariant functors, such that $T_i(0) = 0$ for all i , then

$$L_q^s(\varinjlim T_i) \approx \varinjlim L_q^s T_i, \quad L_q^s(\prod_i T_i) \approx \prod_i L_q^s T_i.$$

Proof. This is a consequence of Corollary 3.27.

3.38. COROLLARY. If \mathcal{A} is a category of modules and the functor T commutes with direct limits, then the functors $L_q^s T$ commute with direct limits and direct sums.

Proof. This is a consequence of Lemma 9.5, Chapter V in [2] and Proposition 3.25.

Now let

$$A = (0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0)$$

be an exact sequence in \mathcal{A} . It is easy to check (as in [2], V, 2.2) that there exists a normal sequence of complexes

$$(3.39) \quad P = (0 \rightarrow P' \xrightarrow{\bar{i}} P \xrightarrow{\bar{p}} P'' \rightarrow 0)$$

such that the complexes P', P, P'' are projective resolutions of A', A, A'' and \bar{i}, \bar{p} are resolutions of the maps i, p , respectively. Applying Theorem 3.31 to the sequence P (called a projective resolution of A), we obtain a complex

$$(3.40) \quad \dots \rightarrow L_q^s T(A') \xrightarrow{i_*} L_q^s T(A) \xrightarrow{p_*} L_q^s T(A'') \xrightarrow{d_q} L_{q-1}^s T(A') \rightarrow \dots$$

with $d_q = d_q(P) = (s_{q-1}(P'))^{-1} D_q^s T(\theta(P))$. If

$$\begin{array}{ccccccc} A & = & (0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0) \\ & & & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ B & = & (0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0) \end{array}$$

is a commutative diagram with exact rows, then as in [2], V, 2.3, one can prove that there exists a commutative diagram

$$\begin{array}{ccccccc} P & = & (0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0) \\ & & & & \downarrow \bar{f}' & & \downarrow \bar{f} & & \downarrow \bar{f}'' & & \\ \tilde{P} & = & (0 & \longrightarrow & \tilde{P}' & \longrightarrow & \tilde{P} & \longrightarrow & \tilde{P}'' & \longrightarrow & 0) \end{array}$$

such that \tilde{P} is a projective resolution of the sequence B and $\bar{f}', \bar{f}, \bar{f}''$ are resolutions of f', f, f'' , respectively. In view of Lemma 2.9 it follows that the diagram

$$\begin{array}{ccc} P'' & \xrightarrow{\theta(P)} & SP' \\ \downarrow f'' & & \downarrow sf' \\ \tilde{P}'' & \xrightarrow{\theta(\tilde{P})} & S\tilde{P}' \end{array}$$

is homotopy commutative. But $D_q^s T$ are h -functors and s_q are natural transformations of functors, so it is clear that the diagram

$$\begin{array}{ccc} L_q^s T(A'') & \xrightarrow{d_q} & L_{q-1}^s T(A') \\ \downarrow f_*'' & & \downarrow f_*' \\ L_q^s T(B'') & \xrightarrow{d_q} & L_{q-1}^s T(B') \end{array}$$

is commutative. It follows that the maps d_q do not depend on the choice of the resolution P and that the following corollary holds.

3.41. COROLLARY. *The functors $L_q^s T$ form a connected sequence of covariant functors. If $\gamma: T \rightarrow T'$ is a natural transformation of functors, then the map $D^s(\gamma)$ from Corollary 3.34 induces a map of connected sequences $d^s(\gamma): \{L_q^s T\} \rightarrow \{L_q^s T'\}$.*

3.42. THEOREM. *If T is an additive functor, then the connected sequence $\{L_q T\}$ of left derived functors of the functor T and the connected sequence $\{L_q^s T\}$ of stable derived functors of the functor T are isomorphic.*

Proof. In this case we deduce from Corollary 3.36 and the assertion (c) of Theorem 2.12 that the sequence (3.40) is exact for any exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$. Moreover, there is a natural equivalence of functors $\tau_0: L_0 T \rightarrow L_0^s T$ and $L_q^s T(A) = L_q T(A) = 0$ for $q > 0$ and any projective object A . Consequently, by the well-known theorem [2] τ_0 extends to an isomorphism of connected sequences.

3.43. DEFINITION. A sequence of functors from \mathcal{A} to \mathcal{A}'

$$T' \rightarrow T \rightarrow T''$$

is called an *exact (p -exact) sequence*, if the sequence

$$T'(A) \rightarrow T(A) \rightarrow T''(A)$$

is exact for any (any projective) object A of \mathcal{A} .

3.44. THEOREM. *If $0 \rightarrow T' \xrightarrow{\alpha} T \xrightarrow{\beta} T'' \rightarrow 0$ is an exact sequence of covariant functors from \mathcal{A} to \mathcal{A}' and $T(0) = 0$, then there exist natural transformations of functors*

$$d_q: D_q^s T'' \rightarrow D_{q-1}^s T', \quad d_q: L_q^s T'' \rightarrow L_{q-1}^s T'$$

such that the sequences

$$(3.45) \quad \dots \rightarrow D_q^s T' \xrightarrow{D_q^s(\alpha)} D_q^s T \xrightarrow{D_q^s(\beta)} D_q^s T'' \xrightarrow{d_q} D_{q-1}^s T' \rightarrow \dots,$$

$$(3.46) \quad \dots \rightarrow L_q^s T' \xrightarrow{L_q^s(\alpha)} L_q^s T \xrightarrow{L_q^s(\beta)} L_q^s T'' \xrightarrow{d_q} L_{q-1}^s T' \rightarrow \dots$$

are exact. If, moreover, a commutative diagram of covariant functors

$$\begin{array}{ccccccc} 0 & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & T'' \longrightarrow 0 \\ & & \downarrow \gamma' & & \downarrow \gamma & & \downarrow \gamma'' \\ 0 & \longrightarrow & \tilde{T}' & \longrightarrow & \tilde{T} & \longrightarrow & \tilde{T}'' \longrightarrow 0 \end{array}$$

has exact rows, then the diagrams

$$\begin{array}{ccc} D_q^s T'' \xrightarrow{d_q} D_{q-1}^s T' & & L_q^s T'' \xrightarrow{d_q} L_{q-1}^s T' \\ \downarrow D_q^s(\gamma'') & & \downarrow L_q^s(\gamma'') \\ D_q^s \tilde{T}'' \xrightarrow{d_q} D_{q-1}^s \tilde{T}' & & L_q^s \tilde{T}'' \xrightarrow{d_q} L_{q-1}^s \tilde{T}' \end{array}$$

are commutative.

Proof. Clearly, we can confine ourselves to the case of the functors D_q^s . If X is a left complex, then we conclude from the exactness of the normalization functor N and the assumption that the diagram (3.15) (for $T^0 = T'$, $T^1 = T$, $T^2 = T''$) has exact rows. Then applying the Homology Sequence Theorem to this diagram we get the following commutative diagram with exact columns

$$\begin{array}{ccc} \downarrow & & \downarrow \\ D_{q+n} T' (S^n X) & \xrightarrow{\sigma'_{q+n}(S^n X)} & D_{q+n+1} T' (S^{n+1} X) \\ \downarrow D_{q+n}(\alpha)(S^n X) & & \downarrow D_{q+n+1}(\alpha)(S^{n+1} X) \\ D_{q+n} T (S^n X) & \xrightarrow{\sigma_{q+n}(S^n X)} & D_{q+n+1} T (S^{n+1} X) \\ \downarrow D_{q+n}(\beta)(S^n X) & & \downarrow D_{q+n+1}(\beta)(S^{n+1} X) \\ D_{q+n} T'' (S^n X) & \xrightarrow{\sigma''_{q+n}(S^n X)} & D_{q+n+1} T'' (S^{n+1} X) \\ \downarrow d_{q+n}(S^n X) & & \downarrow d_{q+n+1}(S^{n+1} X) \\ D_{q-1+n} T' (S^n X) & \xrightarrow{\sigma'_{q-1+n}(S^n X)} & D_{q+n} T' (S^{n+1} X) \\ \downarrow & & \downarrow \end{array}$$

Hence, if we put $d_q(X) = \varinjlim_n d_{q+n}(S^n X)$, then the sequence (3.45) is exact. Making use of Lemma 3.14 and the naturality of σ_q it is easy to see that d_q define a natural transformation of functors $d_q: D_q^s T'' \rightarrow D_{q-1}^s T'$ satisfying the second part of the theorem.

3.47. COROLLARY. *The sequence (3.46) is exact for any p -exact sequence $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$.*

4. Functors with extensions

This section contains the main steps of the proof of the exactness of connected sequences constructed in Section 3 (for functors of finite degree).

Let $T: \mathcal{A} \rightarrow \mathcal{A}'$ be a covariant functor (such that $T(0) = 0$) between abelian categories $\mathcal{A}, \mathcal{A}'$. We recall the definition of the cross-effect functors $T_k: \mathcal{A}^k \rightarrow \mathcal{A}'$, $k = 0, 2, \dots$, due to Eilenberg and MacLane [7]. Let $T_1 = T$. If A_1, A_2 are objects of \mathcal{A} and $i_{A_j}: A_j \rightarrow A_1 \oplus A_2$, $p_{A_j}: A_1 \oplus A_2 \rightarrow A_j$, $j = 1, 2$, denote the natural injections and natural projections, respectively, then we put

$$(4.1) \quad T_2(A_1, A_2) = \text{Im}(1_{T(A_1 \oplus A_2)} - T(i_{A_1} p_{A_1}) - T(i_{A_2} p_{A_2})).$$

It is clear that there are maps

$$\begin{aligned} \lambda_2(A_1, A_2): T_2(A_1, A_2) &\rightarrow T(A_1 \oplus A_2), \\ \varrho_2(A_1, A_2): T(A_1 \oplus A_2) &\rightarrow T_2(A_1, A_2) \end{aligned}$$

such that

$$(4.2) \quad 1_{T(A_1 \oplus A_2)} = T(i_{A_1})T(p_{A_1}) + T(i_{A_2})T(p_{A_2}) + \lambda_2(A_1, A_2)\varrho_2(A_1, A_2).$$

If $f_j: A_j \rightarrow A'_j$, $j = 1, 2$, are any maps in \mathcal{A} then, by definition, $T_2(f_1, f_2)$ is the unique map for which the diagram

$$(4.3) \quad \begin{array}{ccccc} T_2(A_1, A_2) & \xrightarrow{\lambda_2(A_1, A_2)} & T(A_1 \oplus A_2) & \xrightarrow{\varrho_2(A_1, A_2)} & T_2(A_1, A_2) \\ \downarrow T_2(f_1, f_2) & & \downarrow T(f_1 \oplus f_2) & & \downarrow T_2(f_1, f_2) \\ T_2(A'_1, A'_2) & \xrightarrow{\lambda_2(A'_1, A'_2)} & T(A'_1 \oplus A'_2) & \xrightarrow{\varrho_2(A'_1, A'_2)} & T_2(A'_1, A'_2) \end{array}$$

is commutative (the existence of such maps follows from Definition 4.1). Thus the functor $T_2: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ is defined; it is called the *second cross-effect functor* of the functor T .

The commutativity of the diagram (4.3) and the equality (4.2) yield

$$(4.4) \quad T(f_1 \oplus f_2) = T(i_{A'_1})T(f_1)T(p_{A_1}) + T(i_{A'_2})T(f_2)T(p_{A_2}) + \lambda_2(A'_1, A'_2)T_2(f_1, f_2)\varrho_2(A_1, A_2).$$

Furthermore, it follows from the above-mentioned equality (4.2) that the maps $T(i_{A_1}), T(i_{A_2}), \lambda_2(A_1, A_2)$ induce an isomorphism

$$(4.5) \quad T(A_1 \oplus A_2) \simeq T(A_1) \oplus T(A_2) \oplus T_2(A_1, A_2).$$

Higher cross-effect functors of the functor T are defined by induction. Namely

$$(4.6) \quad T_k(A_1, \dots, A_k) = (T_{k-1}(A_1, \dots, A_{k-2}, \cdot))_2(A_{k-1}, A_k)$$

for $A_1, \dots, A_k \in \text{ob } \mathcal{A}$. If $f_j: A_j \rightarrow A'_j$ are maps in $\mathcal{A}, j = 1, \dots, k$, then $T_k(f_1, \dots, f_k)$ is the unique map for which the diagram

$$\begin{array}{ccc} T_k(A_1, \dots, A_k) & \xleftarrow[\varrho_2]{\lambda_2} & T_{k-1}(A_1, \dots, A_{k-2}, A_{k-1} \oplus A_k) \\ \downarrow T_k(f_1, \dots, f_k) & & \downarrow T_{k-1}(f_1, \dots, f_{k-2}, f_{k-1} \oplus f_k) \\ T_k(A'_1, \dots, A'_k) & \xleftarrow[\varrho_2]{\lambda_2} & T_{k-1}(A'_1, \dots, A'_{k-2}, A'_{k-1} \oplus A'_k) \end{array}$$

is commutative.

Let $\lambda_k(A_1, \dots, A_k)$ and $\varrho_k(A_1, \dots, A_k)$ denote the compositions

$$(4.7) \quad \begin{cases} T_k(A_1, \dots, A_k) \xrightarrow{\lambda_2} T_{k-1}(A_1, \dots, A_{k-2}, A_{k-1} \oplus A_k) \xrightarrow{\lambda_2} \dots \\ \dots \xrightarrow{\lambda_2} T(A_1 \oplus \dots \oplus A_k), \\ T(A_1 \oplus \dots \oplus A_k) \xrightarrow{\varrho_2} T_2(A_1, A_2 \oplus \dots \oplus A_k) \xrightarrow{\varrho_2} \dots \xrightarrow{\varrho_2} T_k(A_1, \dots, A_k), \end{cases}$$

respectively. Then, it is clear that the diagram

$$(4.8) \quad \begin{array}{ccccc} T_k(A_1, \dots, A_k) & \xrightarrow{\lambda_k(A_1, \dots, A_k)} & T(A'_1 \oplus \dots \oplus A'_k) & \xrightarrow{\varrho_k(A_1, \dots, A_k)} & T_k(A_1, \dots, A_k) \\ \downarrow T_k(f_1, \dots, f_k) & & \downarrow T(f_1 \oplus \dots \oplus f_k) & & \downarrow T_k(f_1, \dots, f_k) \\ T_k(A'_1, \dots, A'_k) & \xrightarrow{\lambda_k(A'_1, \dots, A'_k)} & T(A'_1 \oplus \dots \oplus A'_k) & \xrightarrow{\varrho_k(A'_1, \dots, A'_k)} & T_k(A'_1, \dots, A'_k) \end{array}$$

is commutative for any maps $f_j: A_j \rightarrow A'_j$ in \mathcal{A} and $j = 1, \dots, k$.

4.9. DEFINITION. A covariant functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ such that $\hat{T}(0) = 0$ is said to be a *functor of degree n* (we write $\text{deg } T = n$) if $T_k = 0$ for $k > n$ and $T_n \neq 0$. If $T_k \neq 0$ for all $k \geq 1$, then we put $\text{deg } T = \infty$. A functor of degree ≤ 2 is called a *quadratic functor*.

Similarly one can define cross-effect functors and the degree of any contravariant functor $\hat{T}: \mathcal{A} \rightarrow \mathcal{A}'$ such that $\hat{T}(0) = 0$.

4.10. DEFINITION. If we have a diagram

$$\begin{array}{ccc} A' & & A'' \\ \downarrow h' & \swarrow s & \downarrow h'' \\ B' & & B'' \end{array}$$

in the category \mathcal{A} , then the triple $(h', h''; s)$ is called a *skew map* and we write

$$(4.11) \quad (h', h''; s): (A', A'') \rightarrow (B', B'').$$

If moreover $(f', f''; t): (B', B'') \rightarrow (C', C'')$ is another skew map, then we have the diagram

$$(4.12) \quad \begin{array}{ccc} A' & & A'' \\ \downarrow h' & \swarrow s & \downarrow h'' \\ B' & & B'' \\ \downarrow f' & \swarrow t & \downarrow f'' \\ C' & & C'' \end{array}$$

and the composition of skew maps is defined by the formula

$$(4.13) \quad (h', h''; s)(f', f''; t) = (f' h', f'' h''; f' s + t h'').$$

Let $\mathcal{A} \wr \mathcal{A}$ denote the category, whose objects are the pairs (A', A'') of objects of \mathcal{A} and whose maps are skew maps. In what follows the map $(h', h''): (A', A'') \rightarrow (B', B'')$ from the category $\mathcal{A} \times \mathcal{A}$ and the skew map $(h', h''; 0)$ will be identified. Thus the category $\mathcal{A} \times \mathcal{A}$ is considered as a subcategory of the category $\mathcal{A} \wr \mathcal{A}$.

4.14. DEFINITION. If $(h', h''; s): (A', A'') \rightarrow (B', B'')$ is a skew map, then the map

$$h = i_{B''} h'' p_{A''} + i_{B'} h' p_{A'} + i_{B'} s p_{A''}: A' \oplus A'' \rightarrow B' \oplus B''$$

is called a *map induced by the skew map* $(h', h''; s)$. If \mathcal{A} and \mathcal{A}' are categories of modules, then $h(a', a'') = (h' a' + s a'', h'' a'')$.

Then it is easy to check that the following diagram is commutative

$$(4.15) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & A' \oplus A'' & \xrightarrow{p} & A'' \longrightarrow 0 \\ & & \downarrow h' & & \downarrow h & & \downarrow h'' \\ 0 & \longrightarrow & B' & \xrightarrow{i} & B' \oplus B'' & \xrightarrow{p} & B'' \longrightarrow 0. \end{array}$$

4.16. DEFINITION. If $\mathbf{h} = (h', h, h'')$ is a triple of maps from \mathcal{A} such that the diagram (4.15) is commutative, then the skew map

$$(4.17) \quad (h', h''; \theta(\mathbf{h})): (A', A'') \rightarrow (B', B'')$$

with

$$(4.18) \quad \theta(\mathbf{h}) = p_{B'} h i_{A''}$$

is called a *map associated with* \mathbf{h} .

Remark. Let $\text{tr } \mathcal{A}$ denotes the category, whose objects are sequences in \mathcal{A} of the form

$$A = (0 \rightarrow A' \xrightarrow{i_{A'}} A' \oplus A'' \xrightarrow{p_{A''}} A'' \rightarrow 0)$$

and maps are triples $h = (h', h, h'')$ of such maps from \mathcal{A} that the diagram (4.15) is commutative. It is clear that the correspondence

$$\begin{aligned} A &\rightarrow (A', A'') \in \text{ob } \mathcal{A} \swarrow \mathcal{A}, \\ h &\rightarrow (h', h''; \theta(h)) \end{aligned}$$

has a functorial character and establishes an isomorphism of the category $\text{tr } \mathcal{A}$ with the category $\mathcal{A} \swarrow \mathcal{A}$.

4.19. DEFINITION. A covariant functor $\tilde{F}: \mathcal{A} \swarrow \mathcal{A} \rightarrow \mathcal{A}'$ is an *extension of the functor* $F: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$, if the diagram

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ \downarrow j & \searrow \tilde{F} & \\ \mathcal{A} \swarrow \mathcal{A} & & \end{array}$$

is commutative, where the vertical arrow denotes the natural injection of the category $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A} \swarrow \mathcal{A}$.

4.20. DEFINITION. A functor $\tilde{F}: \mathcal{A} \swarrow \mathcal{A} \rightarrow \mathcal{A}'$ is called a *trivial extension of the functor* F , if

$$\tilde{F}(h', h''; s) = \tilde{F}(h', h''; 0) = F(h', h'')$$

for any skew map $(h', h''; s)$.

Obviously any functor of two variables admits the trivial extension. An important example is the following:

4.21. EXAMPLE. Let $T: \mathcal{A} \rightarrow \mathcal{A}'$ be a covariant functor and let $T_2: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ be the second cross-effect functor of the functor T . We define an extension $\tilde{T}_2: \mathcal{A} \swarrow \mathcal{A} \rightarrow \mathcal{A}'$ of the functor T_2 as follows:

- (i) $\tilde{T}_2(A', A'') = T_2(A', A'')$ for $A', A'' \in \text{ob } \mathcal{A}$,
- (ii) if $(h', h''; s): (A', A'') \rightarrow (B', B'')$ is a skew map and $h: A' \oplus A'' \rightarrow B' \oplus B''$ is the map induced by $(h', h''; s)$, in the sense of Definition 4.14, then $\tilde{T}_2(h', h''; s)$ is the composed map

$$T_2(A', A'') \xrightarrow{\lambda_2(A', A'')} T(A' \oplus A'') \xrightarrow{T(h)} T(B' \oplus B'') \xrightarrow{\varrho_2(B', B'')} T_2(B', B''),$$

where λ_2 and ϱ_2 are natural transformations of functors from (4.3).

Applying equality (4.2) and definition (4.13) one can prove that \tilde{T}_2 is a functor.

Let now $F: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ be a covariant functor such that $F(A, 0) = 0$ for $A \in \text{ob } \mathcal{A}$. We will denote by

$$F_k(A, \cdot, \dots, \cdot): \mathcal{A}^k \rightarrow \mathcal{A}'$$

the k th cross-effect functor of the functor $F(A, \cdot): \mathcal{A} \rightarrow \mathcal{A}'$, and by

$$\begin{aligned} \lambda_k(A, B_1, \dots, B_k): F_k(A, B_1, \dots, B_k) &\rightarrow F(A, B_1 \oplus \dots \oplus B_k) \\ \varrho_k(A, B_1, \dots, B_k): F(A, B_1 \oplus \dots \oplus B_k) &\rightarrow F_k(A, B_1, \dots, B_k) \end{aligned}$$

the maps (4.7) for the functor $T = F(A, \cdot)$. Note that for $k = 2$ and any maps $f: A \rightarrow A'$, $f_j: B_j \rightarrow B'_j$ ($j = 1, 2$) there exists a unique map

$$F_2(f, f_1, f_2): F_2(A, B_1, B_2) \rightarrow F_2(A', B'_1, B'_2)$$

such that the diagram

$$(4.22) \quad \begin{array}{ccccc} F_2(A, B_1, B_2) & \xrightarrow{\lambda_2(A, B_1, B_2)} & F(A, B_1 \oplus B_2) & \xrightarrow{\varrho_2(A, B_1, B_2)} & F_2(A, B_1, B_2) \\ \downarrow F_2(f, f_1, f_2) & & \downarrow F(f, f_1 \oplus f_2) & & \downarrow F_2(f, f_1, f_2) \\ F_2(A', B'_1, B'_2) & \xrightarrow{\lambda_2(A', B'_1, B'_2)} & F(A', B'_1 \oplus B'_2) & \xrightarrow{\varrho_2(A', B'_1, B'_2)} & F_2(A', B'_1, B'_2) \end{array}$$

is commutative. It is easy to see that we can consider F_2 as a functor of three variables.

We will often use the skew map

$$(4.23) \quad p(A, B) = (1_A, p_B; p_A): (A, B \oplus A) \rightarrow (A, B),$$

which, for simplicity, will be identified with the map induced by $p(A, B)$ (in the sense of Definition 4.17).

4.24. LEMMA. *If $F: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor such that $F(A, 0) = 0$ for $A \in \text{ob } \mathcal{A}$ and $\tilde{F}: \mathcal{A} \wr \mathcal{A} \rightarrow \mathcal{A}'$ is an extension of the functor F , then for any skew map $(h', h''; s): (A', A'') \rightarrow (B', B'')$ the following equality holds:*

$$(4.25) \quad \begin{aligned} \tilde{F}(h', h''; s) &= F(h', h'') + \\ &+ \tilde{F}(p(B', B'')) \lambda_2(B', B'', B') \tilde{F}_2(h', h''; s) \varrho_2(A', A'', A'') F(1_{A'}, d_{A''}), \end{aligned}$$

where $d_{A''}: A'' \rightarrow A'' \oplus A''$ is the diagonal map.

Proof. Let $(h', h''; s): (A', A'') \rightarrow (B', B'')$ be a skew map. Then the diagram

$$\begin{array}{ccc} (A', A'' \oplus A'') & \xleftarrow{(1_{A'}, d_{A''}, 0)} & (A', A'') \\ \downarrow (h', h'' \oplus s, 0) & & \downarrow (h', h''; s) \\ (B', B'' \oplus B') & \xrightarrow{p(B', B'')} & (B', B'') \end{array}$$

is commutative. Furthermore, $\tilde{F}(h', h'' \oplus s; 0) = F(h', h'' \oplus s)$ and $\tilde{F}(1_{A'}, d_{A''}; 0) = F(1_{A'}, d_{A''})$. Hence the above diagram induces the diagram

(4.26)

$$\begin{array}{ccccc}
 F(A', A'') & & & & \\
 \downarrow F(h', s) & \swarrow F(1_{A'}, p_{A''}) & & & \\
 F(A', A'') & \xrightarrow{F(1_{A'}, p_{A''})} & F(A', A'' \oplus A'') & \xleftarrow{F(1_{A'}, d_{A''})} & F(A', A'') \\
 \downarrow F(h', h'') & \searrow \varrho_2(A', A'', A'') & \downarrow F(h', h'' \oplus s) & & \downarrow \tilde{F}(h', h''; s) \\
 F_2(A', A'', A'') & & & & \\
 \downarrow F_2(h', h'', s) & & & & \\
 F_2(B', B'', B') & \xrightarrow{\lambda_2(B', B'', B')} & & & \\
 \downarrow F(1_{B'}, i_{B''}) & & & & \\
 F(B', B'') & \xrightarrow{F(1_{B'}, i_{B''})} & F(B', B'' \oplus B') & \xrightarrow{\tilde{F}(p(B', B''))} & F(B', B'') \\
 \downarrow F(h', s) & \swarrow F(1_{B'}, i_{B'}) & & & \\
 F(B', B') & & & &
 \end{array}$$

Using (4.4) for $T = F(B', \cdot)$, $f_1 = s$, $f_2 = h''$, and the commutativity of the diagram (4.22), we get

$$\begin{aligned}
 (4.27) \quad F(h', h'' \oplus s) &= F(1_{B'}, h'' \oplus s) F(h', 1_{A''} \oplus 1_{A'}) \\
 &= [F(1_{B'}, i_{B'}) F(1_{B'}, s) F(1_{B'}, p_{A''}) + F(1_{B'}, i_{B''}) F(1_{B'}, h'') F(1_{B'}, p_{A''}) \\
 &\quad + \lambda_2(B', B'', B') \tilde{F}_2(1_{B'}, h''; s) \varrho_2(B', A'', A'')] F(h', 1_{A''} \oplus 1_{A'}) \\
 &= F(1_{B'}, i_{B'}) F(h', s) F(1_{A'}, p_{A''}) + F(1_{B'}, i_{B''}) F(h', h'') F(1_{A'}, p_{A''}) + \\
 &\quad + \lambda_2(B', B'', B') \tilde{F}_2(h', h'', s) \varrho_2(A', A'', A'').
 \end{aligned}$$

Furthermore, the following equalities hold

$$(4.28) \quad F(1_{A'}, p_{A''}) F(1_{A'}, d_{A''}) = 1_{F(A', A'')},$$

$$(4.29) \quad \tilde{F}(p(B', B'')) F(1_{B'}, i_{B''}) = 1_{F(B', B'')},$$

$$(4.30) \quad \tilde{F}(p(B', B'')) F(1_{B'}, i_{B'}) = 0.$$

In fact, the first equality follows from the definition of the diagonal map, the second from (4.13) and the third from the commutativity of the diagram

$$\begin{array}{ccc}
 (B', B'' \oplus B') & \xrightarrow{p(B', B'')} & (B', B'') \\
 \uparrow (1_{B'}, i_{B''}; 0) & & \uparrow (1_{B'}, 0; 0) \\
 (B', B') & \xrightarrow{(1_{B'}, 0; 1_{B'})} & (B', 0)
 \end{array}$$

In virtue of the commutativity of the right square of the diagram (4.26)

$$\tilde{F}(h', h''; s) = \tilde{F}(p(B', B''))F(h', h'' \oplus s)F(1_{A'}, d_{A''}).$$

Hence, using (4.27)–(4.30), we get the required formula (4.25).

If $F: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ is an additive functor with respect to the second variable, then clearly $F_2 = 0$ and from Lemma 4.24 we obtain

4.31. COROLLARY. *If $F: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ is an additive functor with respect to the second variable, then it admits only the trivial extension.*

For the proof of the main theorem of this section we need the following

4.32. LEMMA. *Under the assumptions of Lemma 4.24 there exists a map φ such that the following diagram is commutative*

$$(4.33) \quad \begin{array}{ccccc} F_2(A', A'', A') & \xrightarrow{\lambda_2(A', A'', A')} & F(A', A'' \oplus A') & \xrightarrow{\tilde{F}(p(A', A''))} & F(A', A'') \\ \downarrow \varphi & & & & \downarrow \tilde{F}(h', h''; s) \\ F_2(B', B'', B') & \xrightarrow{\lambda_2(B', B'', B')} & F(B', B'' \oplus B') & \xrightarrow{\tilde{F}(p(B', B''))} & F(B', B''). \end{array}$$

Proof. In view of Lemma 4.24 we have

$$\begin{aligned} & \tilde{F}(h', h''; s)\tilde{F}(p(A', A''))\lambda_2(A', A'', A') \\ &= F(h', h'')\tilde{F}(p(A', A''))\lambda_2(A', A'', A') + \tilde{F}(p(B', B''))\lambda_2(B', B'', B')\psi_1 \end{aligned}$$

for some map $\psi_1: F_2(A', A'', A') \rightarrow F_2(B', B'', B')$. Moreover, by the definition of λ_2 , $F(1_{A'}, p_{A''})\lambda_2(A', A'', A') = 0$. Hence the first summand in the above sum has the following form

$$\begin{aligned} & F(h', h'')\tilde{F}(p(A', A''))\lambda_2(A', A'', A') \\ & \stackrel{(4.23)}{=} \tilde{F}(h', h''p_{A''}; h'p_{A'})\lambda_2(A', A'', A') \\ & \stackrel{(4.24)}{=} F(h', h''p_{A''})\lambda_2(A', A'', A') + \tilde{F}(p(B', B''))\lambda_2(B', B'', B')\psi_2 \\ & = \tilde{F}(p(B', B''))\lambda_2(B', B'', B')\psi_2, \end{aligned}$$

where ψ_2 is some map from $F_2(A', A'', A')$ to $F_2(B', B'', B')$. Then the diagram (4.33) is commutative, if we put $\varphi = \psi_1 + \psi_2$. The lemma is proved.

The following theorem plays a key role in the further considerations.

4.34. THEOREM. *If $F: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor with an extension $\tilde{F}: \mathcal{A} \not\sim \mathcal{A} \rightarrow \mathcal{A}'$ satisfying the condition $\tilde{F}(A, 0) = F(A, 0) = 0$ for any object A , then there exist covariant functors F', F'' with extensions \tilde{F}', \tilde{F}'' , respectively, and natural transformations of functors $\alpha: \tilde{F}' \rightarrow \tilde{F}$, $\beta: \tilde{F} \rightarrow \tilde{F}''$ such that*

- (i) $0 \rightarrow \tilde{F}' \xrightarrow{\alpha} \tilde{F} \xrightarrow{\beta} \tilde{F}'' \rightarrow 0$ is an exact sequence of functors.
- (ii) $F'(A, 0) = 0$ for any object A of \mathcal{A} .

(iii) \tilde{F}'' is the trivial extension of the functor F'' .

(iv) The n -th cross-effect functor $F'_n(A, \cdot, \dots, \cdot)$ of the functor $F'(A, \cdot)$ is given by

$$F'_n(A, B_1, \dots, B_n) = \text{Im} [\tilde{F}(p(A, B_1 \oplus \dots \oplus B_n)) \lambda_{n+1}(A, B_1, \dots, B_n, A)].$$

Proof. Let $A', A'' \in \text{ob } \mathcal{A}$. Consider the sequence

$$F_2(A', A'', A') \xrightarrow{\lambda_2(A', A'', A')} F(A', A'' \oplus A') \xrightarrow{\tilde{F}(p(A', A''))} F(A', A''),$$

where $p(A', A'')$ is the skew map (4.23) and put

$$\tilde{F}'(A', A'') = F'(A', A'') = \text{Im} \tilde{F}(p(A', A'')) \lambda_2(A', A'', A').$$

Let $\alpha(A', A''): \tilde{F}'(A', A'') \rightarrow \tilde{F}(A', A'')$ be the natural monomorphism. Furthermore, let

$$\tilde{F}''(A', A'') = F''(A', A'') = \text{Coker } \alpha(A', A'')$$

and let $\beta(A', A''): \tilde{F}(A', A'') \rightarrow \tilde{F}''(A', A'')$ be the natural epimorphism. If $(h', h''; s): (A', A'') \rightarrow (B', B'')$ is a skew map, then it follows from Lemma 4.32 that there exist the unique maps $\tilde{F}'(h', h''; s)$, $\tilde{F}''(h', h''; s)$ such that the diagram

$$(4.35) \quad \begin{array}{ccccccc} 0 & \rightarrow & \tilde{F}'(A', A'') & \xrightarrow{\alpha(A', A'')} & \tilde{F}(A', A'') & \xrightarrow{\beta(A', A'')} & \tilde{F}''(A', A'') \rightarrow 0 \\ & & \downarrow \tilde{F}'(h', h''; s) & & \downarrow \tilde{F}(h', h''; s) & & \downarrow \tilde{F}''(h', h''; s) \\ 0 & \rightarrow & \tilde{F}'(B', B'') & \xrightarrow{\alpha(B', B'')} & \tilde{F}(B', B'') & \xrightarrow{\beta(B', B'')} & \tilde{F}''(B', B'') \rightarrow 0 \end{array}$$

is commutative. Of course, we must put $F'(h', h'') = \tilde{F}'(h', h''; 0)$ and $F''(h', h'') = \tilde{F}''(h', h''; 0)$. It is clear that $\alpha(A', A'')$, $\beta(A', A'')$ define natural transformations α, β such that the sequence $0 \rightarrow \tilde{F}' \xrightarrow{\alpha} \tilde{F} \xrightarrow{\beta} \tilde{F}'' \rightarrow 0$ is exact.

The assertion (ii) follows from (i). To prove (iii), it is sufficient to show that the diagram (4.35) is commutative, if we put $s = 0$ in the map $\tilde{F}''(h', h''; s)$. By Lemma 4.24 and the definition of the functor \tilde{F}' , there exists a map $\psi: F(A', A'') \rightarrow F(B', B'')$ such that

$$\tilde{F}(h', h''; s) = F(h', h'') + \alpha(B', B'')\psi.$$

Hence, using the commutativity of the diagram (4.35) for $s = 0$ and the definition of F'' and β , we get

$$\begin{aligned} \beta(B', B'')\tilde{F}(h', h''; s) &= \beta(B', B'')F(h', h'') + \beta(B', B'')\alpha(B', B'')\psi \\ &= \beta(B', B'')F(h', h'') = F''(h', h'')\beta(A', A'') \end{aligned}$$

what means that \tilde{F}'' is a trivial extension of F'' .

The condition (iv) is proved by induction on n . For $n = 1$ it is the definition of the functor F' . Let $n \geq 1$. Then it follows from the description of the $(n+1)$ th cross-effect functor and Definition 4.1 that

$$(4.36) \quad \begin{aligned} F'_{n+1}(A, B_1, \dots, B_{n+1}) &= [F'_n(A, B_1, \dots; B_{n-1}, \cdot)]_2(B_n, B_{n+1}) \\ &= \text{Im} [1_{F'_n(A, B_1, \dots, B_{n-1}, B_n \oplus B_{n+1})} - F'(1_A, 1_{B_1}, \dots, 1_{B_{n-1}}, i_{B_n} p_{B_n}) - \\ &\quad - F'(1_A, 1_{B_1}, \dots, 1_{B_{n-1}}, i_{B_{n+1}} p_{B_{n+1}})]. \end{aligned}$$

By the induction assumption $F'_n(A, C_1, \dots, C_n)$ is the image of the composed map

$$F_{n+1}(A, B_1, \dots, B_n, A) \xrightarrow{\lambda_{n+1}} F(A, B_1 \oplus \dots \oplus B_n \oplus A) \xrightarrow{\tilde{F}(p(A, B_1 \oplus \dots \oplus B_n))} F(A, B_1 \oplus \dots \oplus B_n).$$

Moreover, if

$$F_{n+1}(A, C_1, \dots, C_n, A) \xrightarrow{\gamma_n} F'_n(A, C_1, \dots, C_n) \xrightarrow{\alpha_n} F(A, C_1 \oplus \dots \oplus C_n)$$

is the appropriate factorization of this map (i.e. $\gamma_n = \gamma_n(A, C_1, \dots, C_n)$ is an epimorphism and $\alpha_n = \alpha_n(A, C_1, \dots, C_n)$ is a monomorphism) then, in view of the definition of $F'_n(A, f_1, \dots, f_n)$, the following diagram is commutative for any $f_j: C_j \rightarrow C'_j$ in \mathcal{A} ($j = 1, \dots, n$):

(4.37)

$$\begin{array}{ccccc} F_{n+1}(A, C_1, \dots, C_n, A) & \xrightarrow{\quad} & F(A, C_1 \oplus \dots \oplus C_n) & & \\ \downarrow F_{n+1}(1, f_1, \dots, f_n, 1) & \searrow \gamma_n & \downarrow F'_n(1, f_1, \dots, f_n) & \nearrow \alpha_n & \downarrow F(1, f_1 \oplus \dots \oplus f_n) \\ & & F'_n(A, C_1, \dots, C_n) & & \\ & & \downarrow F'_n(1, f_1, \dots, f_n) & & \\ & & F'_n(A, C'_1, \dots, C'_n) & \nearrow \alpha'_n & \\ \downarrow F_{n+1}(1, f_1, \dots, f_n, 1) & \swarrow \gamma'_n & \downarrow F'_n(1, f_1, \dots, f_n) & \searrow \alpha'_n & \\ F_{n+1}(A, C'_1, \dots, C'_n, A) & \xrightarrow{\quad} & F(A, C'_1 \oplus \dots \oplus C'_n) & & \end{array}$$

Since $\text{Im}(fg) = \text{Im}f$, provided g is an epimorphism, then the equality (4.36) and the commutativity of this diagram yield

$$\begin{aligned} F'_{n+1}(A, B_1, \dots, B_{n+1}) &= \text{Im} [1_{F'_n(A, B_1, \dots, B_{n-1}, B_n \oplus B_{n+1})} - \\ &\quad - F'_n(1_A, 1_{B_1}, \dots, 1_{B_{n-1}}, i_{B_n} p_{B_n}) - F'_n(1_A, 1_{B_1}, \dots, 1_{B_{n-1}}, i_{B_{n+1}} p_{B_{n+1}})] \\ &= \text{Im} [\gamma_n [F_{n+1}(1_A, 1_{B_1}, \dots, 1_{B_{n-1}}, 1_{B_n \oplus B_{n+1}}, 1_A) - \\ &\quad - F_{n+1}(1_A, 1_{B_1}, \dots, 1_{B_{n-1}}, i_{B_n} p_{B_n}, 1_A) - \\ &\quad - F_{n+1}(1_A, 1_{B_1}, \dots, 1_{B_{n-1}}, i_{B_{n+1}} p_{B_{n+1}}, 1_A)]] = \text{Im} \gamma_n \lambda_2 \rho_2, \end{aligned}$$

where

$$\lambda_2: F_{n+2}(A, A, B_1, \dots, B_{n+1}) \rightarrow F_{n+1}(A, A, B_1, \dots, B_{n-1}, B_n \oplus B_{n+1}),$$

$$\varrho_2: F_{n+1}(A, A, B_1, \dots, B_{n-1}, B_n \oplus B_{n+1}) \rightarrow F_{n+2}(A, A, B_1, \dots, B_{n+1})$$

are maps from (4.3) for the functor $T = F_{n+1}(A, A, B_1, \dots, B_{n-1}, \cdot)$. Since ϱ_2 is an epimorphism and α_n is a monomorphism,

$$\text{Im}(\gamma_n \lambda_2 \varrho_2) = \text{Im}(\gamma_n \lambda_2) = \text{Im}(\alpha_n \gamma_n \lambda_2).$$

But

$$\begin{aligned} \alpha_n \gamma_n \lambda_2 &= \tilde{F}(p(A, B_1 \oplus \dots \oplus B_{n+1})) \lambda_{n+1}(A, B_1, \dots, B_{n-1}, B_n \oplus B_{n+1}, A) \\ &\stackrel{(4.7)}{=} \tilde{F}(p(A, B_1 \oplus \dots \oplus B_{n+1})) \lambda_{n+2}(A, B_1, \dots, B_{n+1}, A). \end{aligned}$$

Consequently,

$$\begin{aligned} F'_{n+1}(A, B_1, \dots, B_{n+1}) \\ = \text{Im} \tilde{F}(p(A, B_1 \oplus \dots \oplus B_{n+1})) \lambda_{n+2}(A, B, \dots, B_{n+1}, A) \end{aligned}$$

and the theorem is proved.

If $\deg F(A, \cdot) \leq n$, then obviously $F_{n+1}(A, \cdot, \dots, \cdot) = 0$ and thus $\lambda_{n+1}(A, \cdot, \dots, \cdot) = 0$. In view of the assertion (iv) of the above theorem, it follows $F'_n(A, \cdot, \dots, \cdot) = 0$. Hence, we obtain

4.38. COROLLARY. $\deg F'(A, \cdot) \leq \deg F(A, \cdot) - 1$ for any object A in \mathcal{A} .

4.39. EXAMPLE. Let \mathcal{M}_R denote the category of modules over a commutative ring R and let $SP^n: \mathcal{M}_R \rightarrow \mathcal{M}_R$ be the n th symmetric power functor (see Section 7). We will find the functors $\tilde{F}', \tilde{F}'': \mathcal{M}_R \vee \mathcal{M}_R \rightarrow \mathcal{M}_R$ for the functor $\tilde{F} = SP_2^n: \mathcal{M}_R \vee \mathcal{M}_R \rightarrow \mathcal{M}_R$ defined in Example 4.21. In virtue of [5], 10.3,

$$F(A, B) = SP_2^n(A, B) = \bigoplus_{j=1}^{n-1} SP^{n-j}(A) \otimes_R SP^j(B)$$

and the natural injection

$$SP_2^n(A, B) \rightarrow SP^n(A \oplus B)$$

is given by

$$(a_1 \vee \dots \vee a_{n-j}) \otimes (b_1 \vee \dots \vee b_j) \rightarrow (a_1, 0) \vee \dots \vee (a_{n-j}, 0) \vee (0, b_1) \vee \dots \vee (0, b_j)$$

where $(a_1 \vee \dots \vee a_{n-j}) \otimes (b_1 \vee \dots \vee b_j) \in SP^{n-j}(A) \otimes SP^j(B)$ and $(a_i, 0), (0, b_i) \in A \oplus B$, $a_i \in A$, $b_i \in B$. Hence the second cross-effect functor $F_2(A, \cdot, \cdot)$ of the functor $F(A, \cdot) = SP_2^n(A, \cdot)$ sends B_1, B_2 into

$$F_2(A, B_1, B_2) = \bigoplus_{j=1}^{n-1} \bigoplus_{i=1}^{j-1} SP^{n-j}(A) \otimes SP^i(B_1) \otimes SP^{j-i}(B_2)$$

and the straightforward computation shows that the image $\tilde{F}'(A, B)$ of the composed map

$$F_2(A, B, A) \xrightarrow{i_2(A, B, A)} F(A, B \oplus A) \xrightarrow{\tilde{F}'(A, B)} F(A, B)$$

is

$$\tilde{F}'(A, B) = \bigoplus_{i=1}^{n-2} SF^{n-i}(A) \otimes SP^i(B).$$

If $(h', h''; s): (A, B) \rightarrow (A', B')$ is a skew map, then

$$\tilde{F}''(h', h''; s) = h' \otimes SF^{n-1}(h'')$$

and $\tilde{F}''(h', h''; s)((a_1 \vee \dots \vee a_{n-j}) \otimes (b_1 \vee \dots \vee b_j))$ is a sum of elements of the form $(h'(a_1) \vee \dots \vee h'(a_{n-j}) \vee s(b_{i_1}) \vee \dots \vee s(b_{i_k}) \otimes h''(b_{i_{k+1}}) \vee \dots \vee h''(b_{i_j}))$.

Let

$$(4.40) \quad \mathbf{Y} = (0 \rightarrow Y' \xrightarrow{i} Y \xrightarrow{p} Y'' \rightarrow 0)$$

be an exact sequence of simplicial objects in the category \mathcal{A} . We say that \mathbf{Y} is a *normal sequence* if it splits in all dimensions. The full category of diagrams whose objects are all normal sequences of simplicial objects in the category \mathcal{A} is denoted by $\text{Ns}\mathcal{A}$.

If $\alpha: [m] \rightarrow [n]$ is a non-decreasing map and \mathbf{Y} is a normal sequence (4.40) of simplicial objects, then clearly the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y'_n & \xrightarrow{i_n} & Y_n & \xrightarrow{p_n} & Y''_n \longrightarrow 0 \\ & & \downarrow Y''(\alpha) & & \downarrow Y(\alpha) & & \downarrow Y'(\alpha) \\ 0 & \longrightarrow & Y'_m & \xrightarrow{i_m} & Y_m & \xrightarrow{p_m} & Y''_m \longrightarrow 0 \end{array}$$

is commutative (we assume $Y_j = Y'_j \oplus Y''_j$ and i_j, p_j are the natural injections and natural projections, respectively). Therefore there exists a skew map

$$(Y'(\alpha), Y''(\alpha); \theta(Y(\alpha))): (Y'_n, Y''_n) \rightarrow (Y'_m, Y''_m)$$

associated with the map $Y(\alpha)$ in the sense of Definition 4.16.

4.41 DEFINITION. Let $\tilde{F}: \mathcal{A} \vee \mathcal{A} \rightarrow \mathcal{A}'$ be a covariant functor such that $\tilde{F}(A, 0) = 0$ for $A \in \text{ob}\mathcal{A}$. The functor

$$\tilde{F}^*: \text{Ns}\mathcal{A} \rightarrow \mathcal{A}'$$

is defined as follows:

(i) if $\mathbf{Y} \in \text{ob}\text{Ns}\mathcal{A}$ has the form (4.40) and $\alpha: [m] \rightarrow [n]$ is a non-decreasing map, then

$$\begin{aligned} \tilde{F}^*(\mathbf{Y})_n &= \tilde{F}(Y'_n, Y''_n), \\ \tilde{F}^*(\mathbf{Y})(\alpha) &= \tilde{F}(Y'(\alpha), Y''(\alpha); \theta(Y(\alpha))); \end{aligned}$$

(ii) if $h: \mathbf{Y} \rightarrow \mathbf{Z}$ is a map in $\text{Ns}\mathcal{A}$, with $h_n = (h'_n, h_n, h''_n)$, then

$$\tilde{F}^*(h)_n = \tilde{F}(h'_n, h''_n; \theta(h_n)) \quad (\text{see 4.18}).$$

Immediately from the definition follows

4.42. COROLLARY. If $\tilde{F}: \mathcal{A} \wr \mathcal{A} \rightarrow \mathcal{A}'$ is a trivial extension of the functor $F: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$, then

$$\tilde{F}^*(\mathbf{Y}) = F(Y', Y'')$$

for any normal sequence of simplicial objects (4.40) (see 3.5).

Corollaries 4.31 and 4.42 yield

4.43. COROLLARY. If $T: \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant quadratic functor (see Definition 4.9) and \tilde{T}_2 is the extension of the functor T_2 defined in the Example 4.21, then

$$\tilde{T}_2^*(\mathbf{Y}) = T_2(Y', Y'')$$

for any normal sequence of simplicial objects (4.40).

4.44. PROPOSITION. If $J: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor with an extension $\tilde{J}: \mathcal{A} \wr \mathcal{A} \rightarrow \mathcal{A}'$ satisfying the condition $J(A, 0) = \tilde{J}(A, 0) = 0$, then there exist covariant functors $\tilde{J}_i: \mathcal{A} \wr \mathcal{A} \rightarrow \mathcal{A}'$, $i = 1, 2, 3, \dots$, such that

$$(i) \quad \tilde{J}_1 = \tilde{J},$$

(ii) if $\mathbf{Y} = (0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0)$ is a normal sequence of simplicial objects in \mathcal{A} , then

$$\tilde{J}_1^*(\mathbf{Y}) \leftarrow \tilde{J}_2^*(\mathbf{Y}) \leftarrow \dots \leftarrow \tilde{J}_n^*(\mathbf{Y}) \leftarrow \tilde{J}_{n+1}^*(\mathbf{Y}) \leftarrow \dots$$

and the inclusions $\tilde{J}_n^*(\mathbf{Y}) \leftarrow \tilde{J}_{n+1}^*(\mathbf{Y})$, $n = 1, 2, 3, \dots$, define natural transformations of functors $\tilde{J}_n^* \leftarrow \tilde{J}_{n+1}^*$,

(iii) if the sequence \mathbf{Y} is n -trivial (it means that simplicial objects Y' , Y , Y'' are n -trivial), then

$$H_j(\tilde{J}_i^*(\mathbf{Y})/\tilde{J}_{i+1}^*(\mathbf{Y})) = 0$$

for $j < 2n$ and $i = 1, 2, 3, \dots$,

(iv) if $\deg J(A, \cdot) \leq m$ for all A , then $0 = \tilde{J}_{m+1} = \tilde{J}_{m+2} = \dots$

Proof. We define the functors \tilde{J}_i by induction. Let $\tilde{J}_1 = \tilde{J}$. If $\tilde{J}_1, \dots, \tilde{J}_n$ are defined and satisfy conditions (i)–(iv), then taking $\tilde{F} = \tilde{J}_n$ in Theorem 4.34 we get an exact sequence of functors

$$0 \rightarrow \tilde{F}' \rightarrow \tilde{F} \rightarrow \tilde{F}'' \rightarrow 0$$

such that \tilde{F}'' is a trivial extension of the functor F'' . We put $\tilde{J}_{n+1} = \tilde{F}'$. It is clear that conditions (i) and (ii) hold. To prove (iii), it is sufficient

to show that $H_j \tilde{F}''^*(Y) = 0$ for any n -trivial sequence Y and $j < 2n$. But $\tilde{F}''^*(Y) = F''(Y', Y'')$, by Corollary 4.42, so the required result follows from Lemma 3.24. Condition (iv) is a consequence of Corollary 4.38 and the construction of the functors \tilde{J}_i . The theorem is proved.

4.45. Remark. If $T = SP^n: \mathcal{M}_R \rightarrow \mathcal{M}_R$ (see Example 4.39) and \tilde{T}_2 is the extension of the functor SP_2^n defined in Example 4.21, then one can prove that the functors \tilde{J}_i given by

$$(a) \tilde{J}_i(A, B) = \bigoplus_{k=1}^{n-k} SP^{n-k}(A) \otimes SP^k(B) \subset SP_2^n(A, B),$$

(b) $\tilde{J}_i(h', h''; s)$ is the restriction of the map $\overline{SP}_2^n(h', h''; s)$ to $\tilde{J}_i(A, B)$, satisfy conditions (i)–(iv) of Proposition 4.44 for $\tilde{J} = \tilde{T}_2$.

The main result of this section is the following

4.46. THEOREM. *If $T: \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor such that $T(0) = 0$ and $\deg T < \infty$, then*

$$H_j \tilde{T}_2^*(Y) = 0$$

for any normal sequence $Y = (0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0)$ of n -trivial simplicial objects and $j < 2n$, where \tilde{T}_2 is the extension of the functor T_2 defined in Example 4.21.

Proof. Let $\deg T = m < \infty$ and let Y be a normal and n -trivial sequence of simplicial objects in \mathcal{A} . Applying Proposition 4.44 (for $\tilde{J} = \tilde{T}_2$), we get a filtration

$$\tilde{T}_2^*(Y) = \tilde{J}_1^*(Y) \leftarrow \tilde{J}_2^*(Y) \leftarrow \dots \leftarrow \tilde{J}_{m-1}^*(Y) \leftarrow \tilde{J}_m^*(Y) = 0.$$

We prove that $H_j \tilde{J}_i^*(Y) = 0$ for $j < 2n$ and $i = m, m-1, \dots, 2, 1$. Since trivially $H_j \tilde{J}_m^*(Y) = 0$ for $j < 2n$, it is sufficient to show that the assumption $H_j \tilde{J}_k^*(Y) = 0$ for $j < 2n$ implies $H_j \tilde{J}_{k-1}^*(Y) = 0$ for $j < 2n$. For this purpose note that the exact sequence of simplicial objects

$$0 \rightarrow \tilde{J}_k^*(Y) \hookrightarrow \tilde{J}_{k-1}^*(Y) \rightarrow \tilde{J}_{k-1}^*(Y)/\tilde{J}_k^*(Y) \rightarrow 0$$

induces an exact homology sequence

$$\dots \rightarrow H_j \tilde{J}_k^*(Y) \rightarrow H_j \tilde{J}_{k-1}^*(Y) \rightarrow H_j (\tilde{J}_{k-1}^*(Y)/\tilde{J}_k^*(Y)) \rightarrow \dots$$

But $H_j \tilde{J}_k^*(Y) = H_j (\tilde{J}_{k-1}^*(Y)/\tilde{J}_k^*(Y)) = 0$ for $j < 2n$ by our assumptions and the condition (iii) of Proposition 4.44. Thus $H_j \tilde{J}_{k-1}^*(Y) = 0$ for $j < 2n$ and the theorem is proved.

5. On the exactness of connected sequences

In this section we prove that left stable derived functors of any covariant functors of a finite type (see Definition 5.10) form connected and exact sequence of functors.

Let $T: \mathcal{A} \rightarrow \mathcal{A}'$ be a covariant functor such that $T(0) = 0$ and let \tilde{T}_2 be the extension of the functor T_2 defined in Example 4.21.

5.1. LEMMA. *If $Y = (0 \rightarrow Y' \xrightarrow{i} Y \xrightarrow{p} Y'' \rightarrow 0)$ is a normal sequence of simplicial objects in the category \mathcal{A} , then the simplicial objects $\text{Ker}T(p)/\text{Im}T(i)$ and $\tilde{T}_2^*(Y)$ (see 4.41) are isomorphic.*

Proof. From the sequence Y we get the sequence

$$T(Y') \xrightarrow{T(i)} T(Y) \xrightarrow{T(p)} T(Y'')$$

with $T(p)T(i) = 0$. Hence, $\text{Im}T(i) \subset \text{Ker}T(p)$. Without loss of generality we may assume that

$$T(Y_n) = T(Y'_n) \oplus T(Y''_n) \oplus T_2(Y'_n, Y''_n)$$

and that the maps $T(i_n)$, $T(p_n)$ are natural injections and projections, respectively (see 4.5). Then

$$\text{Ker}T(p_n) = T(Y'_n) \oplus T_2(Y'_n, Y''_n), \quad \text{Ker}T(p_n)/\text{Im}T(i_n) = T_2(Y'_n, Y''_n).$$

Moreover, it follows from Example 4.21 and Definition 4.41 that

$$\begin{aligned} \tilde{T}_2^*(Y)(\alpha) &= \tilde{T}_2(Y'(\alpha), Y''(\alpha); \theta(Y(\alpha))) \\ &= \varrho_2(Y'_m, Y''_m)T(Y(\alpha))\lambda_2(Y'_n, Y''_n) = (\text{Ker}T(p)/\text{Im}T(i))(\alpha) \end{aligned}$$

for any non-decreasing map $\alpha: [m] \rightarrow [n]$. Thus the lemma is proved.

5.2. THEOREM. *If $X = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0)$ is a normal sequence of left complexes and*

$$H_j \tilde{T}_2^*(KX) = 0 \quad \text{for } j < 2n,$$

then the sequence

$$(5.3) \quad D_{q+n}T(X') \xrightarrow{i_*} D_{q+n}T(X) \xrightarrow{p_*} D_{q+n}T(X'')$$

is exact for $n > q + 1$, where $D_i T: \mathcal{K}^-(\mathcal{A}) \rightarrow \mathcal{A}'$ are the functors described in Section 3.

Proof. The functor $K: \mathcal{K}^-(\mathcal{A}) \rightarrow s\mathcal{A}$ carries normal sequences of complexes into normal sequences of simplicial objects. Therefore, from the normal sequence X we get a normal sequence of simplicial objects

$$KX = (0 \rightarrow KX' \xrightarrow{K(i)} KX \xrightarrow{K(p)} KX'' \rightarrow 0)$$

and now, using Lemma 5.1, we have a commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & & \tilde{T}_2^*(KX) & & \\
 & & & & \uparrow & & \\
 0 & \longrightarrow & \text{Ker } TK(p) & \xrightarrow{i''} & T(KX) & \xrightarrow{TK(p)} & T(KX'') \longrightarrow 0 \\
 & & \uparrow & \nearrow^{TK(i)} & & & \\
 & & T(KX') & & & & \\
 & & \uparrow & & & & \\
 & & 0 & & & &
 \end{array}$$

in which vertical and horizontal rows are exact sequences of simplicial objects. This diagram induces a commutative diagram

$$\begin{array}{ccccc}
 H_{q+n} \tilde{T}_2^*(KX) & & & & \\
 \uparrow & & & & \\
 H_{q+n} \text{Ker } T(p) & \xrightarrow{i''} & H_{q+n} T(KX) & \xrightarrow{p_*} & H_{q+n} T(KX'') \\
 \uparrow & \nearrow^{i_*} & & & \\
 H_{q+n} T(KX') & & & & \\
 \uparrow & & & & \\
 H_{q+n+1} \tilde{T}_2^*(KX) & & & &
 \end{array}$$

such that the vertical and horizontal rows are exact. By our assumptions $H_{q+n+1} \tilde{T}_2^*(KX) = H_{q+n} \tilde{T}_2^*(KX) = 0$ for $n > q+1$, so the map i'_* is an isomorphism. Hence the sequence (5.3) is exact and the theorem is proved.

An immediate consequence of Theorem 4.46 is the following

5.4. COROLLARY. *If X (in 5.2) is a normal sequence of n -trivial left complexes and $T: \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor of finite degree, then the sequence (5.3) is exact for $n > q+1$.*

The main result of this section is the following

5.5. THEOREM. *Let $T: \mathcal{A} \rightarrow \mathcal{A}'$ be a covariant functor of finite degree (between abelian categories $\mathcal{A}, \mathcal{A}'$) such that $T(0) = 0$. If $X = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0)$ is a normal sequence of left complexes in \mathcal{A} , then the sequence*

$$(5.6) \quad \dots \rightarrow D_q^s T(X') \xrightarrow{i_*} D_q^s T(X) \xrightarrow{p_*} D_q^s T(X'') \xrightarrow{d_q} D_{q-1}^s T(X') \rightarrow \dots$$

is exact.

Proof. Recall that we get the sequence (5.6) applying Theorem 2.12 to the functors $D_q^s T$ and the natural equivalences s_q (see 3.28). Hence, in virtue of Theorem 2.12(c), it is sufficient to show that the sequence

$$(5.7) \quad D_q^s T(X') \xrightarrow{i_*} D_q^s T(X) \xrightarrow{p_*} D_q^s T(X'')$$

is exact for any normal sequence \mathbf{X} and $q \in \mathbb{Z}$. If $n > q + 1$ then, applying Theorem 5.2 to the normal sequence

$$0 \rightarrow S^n X' \xrightarrow{S^n(i)} S^n X \xrightarrow{S^n(p)} S^n X'' \rightarrow 0,$$

we obtain the exact sequence

$$D_{q+n} T(X') \rightarrow D_{q+n} T(X) \rightarrow D_{q+n} T(X'').$$

But the sequence (5.7) is a direct limit of the above sequences, so it is exact and the theorem is proved.

5.8. COROLLARY. *Let \mathcal{A} be an abelian category with enough projectives. If $T: \mathcal{A} \rightarrow \mathcal{A}'$ is a covariant functor of finite degree and*

$$0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0$$

is an exact sequence in \mathcal{A} , then the sequence

$$(5.9) \quad \dots \rightarrow L_q^s T(A') \xrightarrow{i_*} L_q^s T(A) \xrightarrow{p_*} L_q^s T(A'') \xrightarrow{d_q} L_{q-1}^s T(A') \rightarrow \dots$$

is exact.

5.10. DEFINITION. A covariant (contravariant) functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ is called a *functor of finite type*, if T is either a direct sum or a direct product of functors of finite degree.

5.11. COROLLARY. *If T is a covariant functor of finite type, then the sequences (5.7) and (5.9) are exact, i.e. $\{D_q^s T\}_{q \in \mathbb{Z}}$ form an $\mathcal{N}\mathcal{K}^-(\mathcal{A})$ -connected and exact sequence (see 3.33) of functors, $\{L_q^s T\}_{q \in \mathbb{Z}}$ form a connected and exact sequence of functors.*

Proof. As before we must only prove that the sequence (5.7) is exact. But, in view of Corollary 3.27 and Theorem 5.5, it is a direct sum (or a direct product or direct limit) of exact sequences. It follows that it is exact.

6. Right and left stable derived functors of contravariant functors.

Right stable derived functors of covariant functors

Let Δ be the category described in Section 3. Recall that we denote by $\mathcal{K}^-(\mathcal{A})$, $\mathcal{K}^+(\mathcal{A})$ and $s\mathcal{A}$ the categories of all left, right chain complexes and all simplicial objects in \mathcal{A} , respectively.

Following [5] we define a cosimplicial object in the category \mathcal{A} as a covariant functor $Y: \Delta \rightarrow \mathcal{A}$ and a cosimplicial map as a natural transformation of such functors. Clearly, cosimplicial objects together with cosimplicial maps form a category $\text{co-}s\mathcal{A}$.

Let Y be a cosimplicial object in \mathcal{A} . The associated chain complex $\bar{k}Y$ is the following right complex

$$0 \rightarrow Y_0 \xrightarrow{d_0} Y_{-1} \rightarrow \dots \rightarrow Y_{-n} \xrightarrow{d_{-n}} Y_{-n-1} \rightarrow \dots,$$

where $Y_{-n} = Y([n])$, $d_{-n} = \sum_{i=0}^n (-1)^i Y(\varepsilon_i)$ and $\varepsilon_i: [n] \rightarrow [n+1]$ ($i = 0, 1, \dots, n$) are the non-decreasing maps (3.3). If $f: Y \rightarrow Y'$ is a cosimplicial map, then it is easy to see that the maps $f([n]): Y_{-n} \rightarrow Y'_{-n}$ define a complex map $\bar{k}f: \bar{k}Y \rightarrow \bar{k}Y'$. Thus the covariant functor

$$\bar{k}: \text{co-s } \mathcal{A} \rightarrow \mathcal{K}^+(\mathcal{A})$$

is determined.

Now let $\hat{T}: \mathcal{A} \rightarrow \mathcal{A}'$ be a fixed contravariant functor such that $\hat{T}(0) = 0$. Similarly as in Section 3, the functor \hat{T} induces in the natural way a covariant functor

$$\hat{T}: \text{s } \mathcal{A} \rightarrow \text{co-s } \mathcal{A}'$$

and then we can define a contravariant functors

$$D^q \hat{T}: \mathcal{K}^-(\mathcal{A}) \rightarrow \mathcal{A}' \quad (q \in \mathbb{Z})$$

as $H_{-q} \bar{k} \hat{T} K$, where $K: \mathcal{K}^-(\mathcal{A}) \rightarrow \text{s } \mathcal{A}$ is the functor from ([5]; 3.2). If \mathcal{A}^* denotes the opposite category to the category \mathcal{A} and $\delta_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A}^*$ is the dualization functor, then it is easy to deduce that

$$(6.1) \quad D^q \hat{T} = \delta_{\mathcal{A}^*} D_q(\delta_{\mathcal{A}'} \hat{T}).$$

This fact, together with (3.9) and (3.11), implies the following

6.2. COROLLARY. (i) *The functors $D^q \hat{T}$ are h-functors.*

(ii) *If \hat{T} is an additive functor then, for any complex X , $D^q \hat{T}(X)$ is the q -th homology object of the complex*

$$\dots \leftarrow \hat{T}(X_n) \xleftarrow{\hat{T}(d_n)} \hat{T}(X_{n-1}) \leftarrow \dots \leftarrow \hat{T}(X_0) \leftarrow 0$$

The action of the functor $\delta_{\mathcal{A}^*}: \mathcal{A}'^* \rightarrow \mathcal{A}'^{**} = \mathcal{A}'$ on the natural transformations (3.12) (for $T = \delta_{\mathcal{A}'} \hat{T}$) provides natural transformations

$$\sigma^q: (D^{q+1} \hat{T}) S \rightarrow D^q \hat{T}$$

such that the following corollary holds:

6.3. COROLLARY. *If X is an n -trivial left complex, then $\sigma^q(X)$ is an isomorphism for $q < 2n$ (see 3.18).*

Now we define functors

$$D_s^q \hat{T}: \mathcal{K}^-(\mathcal{A}) \rightarrow \mathcal{A}' \quad (q \in \mathbb{Z})$$

by formulas

$$D_s^q \hat{T}(X) = \varprojlim_n \{D^{q+n} \hat{T}(S^n X), \sigma^{q+n}(S^n X)\},$$

$$D_s^q \hat{T}(f) = \varprojlim_n \{D^{q+n} \hat{T}(S^n f)\},$$

where X, X' are left complexes and $f: X \rightarrow X'$ is a complex map.

If $n > q$, then in view of (6.1) and (6.3) we have

$$(6.4) \quad D_s^q \hat{T} \approx (D^{q+n} \hat{T}) S^n \approx (\delta_{\mathcal{A}'} D_{q+n}(\delta_{\mathcal{A}'} \hat{T})) S^n \approx (H_{-q-n} \bar{k} \hat{T} K) S^n$$

what implies (see 3.19) that

$$(6.5) \quad D_s^q \hat{T} \approx \delta_{\mathcal{A}'} D_q^s(\delta_{\mathcal{A}'} \hat{T}).$$

Finally, from (3.20) and (3.23), we get

6.6. COROLLARY. (i) $D_s^q \hat{T}$ are additive h -functors.

(ii) $D_s^q \hat{T} = 0$ for $q < 0$.

The action of the functor $\delta_{\mathcal{A}'}$ on the natural equivalences

$$s_q: D_q^s(\delta_{\mathcal{A}'} \hat{T}) \rightarrow D_{q+1}^s(\delta_{\mathcal{A}'} \hat{T}) S$$

gives as the result natural equivalences

$$s^q: (D_s^{q+1} \hat{T}) S \rightarrow D_s^q \hat{T}.$$

Now assume that \mathcal{A} is a category with enough projectives.

6.7. DEFINITION. Let $q \in \mathbb{Z}$. A q -th right stable derived functor of the functor \hat{T} is the functor

$$R_s^q \hat{T}: \mathcal{A} \rightarrow \mathcal{A}'$$

defined as follows:

$$R_s^q \hat{T}(A) = D_s^q \hat{T}(P)$$

where P is a projective resolution of the object A of \mathcal{A} . If $f: A' \rightarrow A$ is a map in \mathcal{A} , and $\bar{f}: P' \rightarrow P$ is a resolution of f , then

$$R_s^q \hat{T}(f) = D_s^q \hat{T}(\bar{f}).$$

Equality (6.5) and Definition 6.7 yield

$$6.8. \text{ COROLLARY. } R_s^q \hat{T} = \delta_{\mathcal{A}'} L_q^s(\delta_{\mathcal{A}'} \hat{T}).$$

6.9. COROLLARY. If \hat{T} is an additive functor, then the functors $R_s^q \hat{T}$ are naturally equivalent to the right derived functors of the functor \hat{T} in the sense of [2].

Proof. By equality (6.4) we have

$$R_s^q \hat{T}(A) = D_s^q \hat{T}(P) \approx D^{q+n} T(S^n P)$$

which, in view of Corollary 6.2, gives the required result.

6.10. THEOREM. If $X = (0 \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow 0)$ is a normal sequence of left complexes and $A = (0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0)$ is an exact sequence in \mathcal{A} , then there exist connecting maps

$$\begin{aligned} d^q(\mathbf{X}) &: D_s^q \hat{T}(X') \rightarrow D_s^{q+1} \hat{T}(X''), \\ d^q(\mathbf{A}) &: R_s^q \hat{T}(A') \rightarrow R_s^{q+1} \hat{T}(A'') \end{aligned}$$

such that the sequences $\{D_s^q \hat{T}\}_{q \in \mathbb{Z}}$ and $\{R_s^q \hat{T}\}_{q \in \mathbb{Z}}$, together with these maps, form $\mathcal{N}\mathcal{K}^-(\mathcal{A})$ -connected and connected sequence of contravariant functors. Moreover, if \hat{T} is a functor of finite type (see Definition 5.10), then the sequences

$$(6.11) \quad \dots \rightarrow D_s^q \hat{T}(X'') \xrightarrow{p^*} D_s^q \hat{T}(X) \xrightarrow{i^*} D_s^q \hat{T}(X') \xrightarrow{d^q} D_s^{q+1} \hat{T}(X'') \rightarrow \dots,$$

$$(6.12) \quad \dots \rightarrow R_s^q \hat{T}(A'') \xrightarrow{p^*} R_s^q \hat{T}(A) \xrightarrow{i^*} R_s^q \hat{T}(A') \xrightarrow{d^q} R_s^{q+1} \hat{T}(A'') \rightarrow \dots$$

are exact.

Proof. By definition,

$$\begin{aligned} d^q(\mathbf{X}) &= D_s^{q+1} \hat{T}(\theta(\mathbf{X})) (s^q(X'))^{-1}, \\ d^q(\mathbf{A}) &= d^q(\mathbf{P}), \end{aligned}$$

where $\theta(\mathbf{X}): X'' \rightarrow SX'$ is the complex map defined in Lemma 2.8 and \mathbf{P} is a projective resolution of the sequence \mathbf{A} . It is easy to see that the maps $d^q(\mathbf{X})$ and $d^q(\mathbf{A})$ arise from the connecting maps

$$d_{q+1}: D_{q+1}^s(\delta_{\mathcal{A}'} \hat{T})(X'') \rightarrow D_q^s(\delta_{\mathcal{A}'} \hat{T})(X')$$

(in the sequence (3.32) for $T = \delta_{\mathcal{A}'} \hat{T}$) and

$$d_{q+1}: L_{q+1}^s(\delta_{\mathcal{A}'} \hat{T})(A'') \rightarrow L_q^s(\delta_{\mathcal{A}'} \hat{T})(A')$$

(in the sequence (3.40)), respectively, by the action of the functor $\delta_{\mathcal{A}'} \cdot$. It follows that the sequences (6.11) and (6.12) can be obtained by the action of the functor $\delta_{\mathcal{A}'} \cdot$ on the sequences (3.32) and (3.40), respectively. Now the theorem follows from Theorem 3.30 and Corollary 5.11.

Remark. If \hat{T} is an additive functor, then the connected sequence of right derived functors of the functor \hat{T} and the connected sequence of right stable derived functors of the functor \hat{T} are isomorphic. A proof of this statement is a repetition of the proof of Theorem 3.42.

To define left stable derived functors of the functor $\hat{T}: \mathcal{A} \rightarrow \mathcal{A}'$ and right stable derived functors of a covariant functor $T: \mathcal{A} \rightarrow \mathcal{A}'$ such that $\hat{T}(0) = T(0) = 0$, we define functors

$$\alpha_q^s \hat{T}, \alpha_q^s T: \mathcal{K}^+(\mathcal{A}) \rightarrow \mathcal{A}'$$

taking

$$\alpha_q^s \hat{T} = D_q^s(\hat{T} \delta_{\mathcal{A}'} \bar{\delta}), \quad \alpha_q^s T = D_q^s(T \delta_{\mathcal{A}'} \bar{\delta}),$$

where $\bar{\delta}: \mathcal{K}^+(\mathcal{A}) \rightarrow \mathcal{K}^-(\mathcal{A}^*)$ is a contravariant functor defined in the following way. For $X \in \text{ob } \mathcal{K}^+(\mathcal{A})$

$$(6.13) \quad \begin{cases} (\bar{\delta}X)_n = \delta_{\mathcal{A}}(X_{-n}), \\ d_n^{\bar{\delta}(X)} = \delta_{\mathcal{A}}(d_{-n-1}). \end{cases}$$

If $f = \{f_n\}: X \rightarrow X'$ is a complex map, then $\bar{\delta}(f) = \{\delta_{\mathcal{A}}f_n\}$.

Now, let \mathcal{A} be a category with enough injectives.

6.14. DEFINITION. If $q \in \mathbb{Z}$, then the q -th left (right) stable derived functor of the functor $\hat{T}(T)$ is the functor

$$L_q^s \hat{T}(R_s^q T): \mathcal{A} \rightarrow \mathcal{A}'$$

defined by

$$L_q^s \hat{T}(A) = \alpha_q^s \hat{T}(Q) \quad (R_s^q T(A) = \alpha_s^q T(Q)),$$

where Q is an injective resolution of the object \mathcal{A} . If $f: \mathcal{A}' \rightarrow \mathcal{A}$ is a map in \mathcal{A} and $\bar{f}: Q' \rightarrow Q$ is a resolution of f , then

$$L_q^s \hat{T}(f) = \alpha_q^s \hat{T}(\bar{f}) \quad (R_s^q T(f) = \alpha_s^q T(\bar{f})).$$

Using suitable properties of the functors $D_q^s T$, $D_s^q \hat{T}$ and $L_q^s T$, $R_s^q \hat{T}$, we can also in this case prove assertions analogous to 3.20, 3.21, 3.23, 3.34, 3.36, 5.11 and 6.10.

7. Symmetric power functor SP^n and exterior power functor A^n

We recall that the functors SP^n and A^n for $n > 0$ are defined as follows. If M is an R -module and $\otimes^n(M) = M \otimes \dots \otimes M$ ($\otimes = \otimes_R$) is a tensor product of n copies of M , then we put

$$(7.1) \quad \begin{cases} SP^n(M) = \otimes^n(M)/U(M), \\ A^n(M) = \otimes^n(M)/V(M), \end{cases}$$

where $U(M)$ is a submodule of $\otimes^n(M)$ generated by all elements $m_1 \otimes \dots \otimes m_n - m_{i_1} \otimes \dots \otimes m_{i_n}$, $m_j \in M$, i_1, \dots, i_n is a permutation of the integers $1, 2, \dots, n$; and $V(M)$ is a submodule of $\otimes^n(M)$ generated by all elements $m_1 \otimes \dots \otimes m_n$, $m_j \in M$, such that $m_i = m_j$ for some $i \neq j$. If we write $m_1 \vee \dots \vee m_n$, $m_1 \wedge m_2 \wedge \dots \wedge m_n$ for the images of the element $m_1 \otimes \dots \otimes m_n \in \otimes^n(M)$ under the natural epimorphisms $\otimes^n(M) \rightarrow SP^n(M)$, $\otimes^n(M) \rightarrow A^n(M)$, respectively, and if $f: M \rightarrow N$ is a homomorphism of R -modules, then the homomorphisms $SP^n(f)$, $A^n(f)$ are given by

$$(7.2) \quad \begin{cases} SP^n(f)(m_1 \vee m_2 \vee \dots \vee m_n) = f(m_1) \vee f(m_2) \vee \dots \vee f(m_n), \\ A^n(f)(m_1 \wedge m_2 \wedge \dots \wedge m_n) = f(m_1) \wedge f(m_2) \wedge \dots \wedge f(m_n). \end{cases}$$

Finally, we define the functors SP^0 and A^0 by $SP^0(M) = A^0(M) = R$, $SP^0(f) = A^0(f) = 1_R$ and we form the following functors:

$$(7.3) \quad \begin{cases} SP = \bigoplus_{n=0}^{\infty} SP^n, & \widetilde{SP} = \bigoplus_{n=1}^{\infty} SP^n, \\ A = \bigoplus_{n=0}^{\infty} A^n, & \widetilde{A} = \bigoplus_{n=1}^{\infty} A^n. \end{cases}$$

It is well known that the functors SP, A, SP^n, A^n commute with direct limits. Then by Corollary 3.38 we have

7.4. COROLLARY. *If G is an R -module, then the left stable derived functors of the functors $\widetilde{SP} \otimes G, \widetilde{A} \otimes G, SP^n \otimes G, A^n \otimes G$ (2) commute with direct limits and direct sums.*

Let us recall that in Section 4 the cross-effect functors of any covariant functor T , such that $T(0) = 0$, were defined. The r th cross-effect functor of the functor $SP^n, n > 0$, is given by the formula

$$(7.5) \quad SP_r^n = \bigoplus_{\substack{i_1 + \dots + i_r = n \\ i_j > 0}} SP^{i_1} \otimes \dots \otimes SP^{i_r}$$

for $r \leq n$ and $SP_r^n = 0$ for $r > n$ (see [5]; 10.4). Similarly, one can prove that

$$(7.6) \quad A_r^n = \bigoplus_{\substack{i_1 + \dots + i_r = n \\ i_j > 0}} A^{i_1} \otimes \dots \otimes A^{i_r}$$

for $n \geq r$ and $A_r^n = 0$ for $n < r$. Thus we have

7.7. COROLLARY. *If G is an R -module, then $SP^n \otimes G, A^n \otimes G, \text{Hom}_R(SP^n, G), \text{Hom}_R(A^n, G)$ are functors of degree n .*

Hence, in view of Corollary 5.11 and Theorem 6.10, we get

7.8. COROLLARY. *If G is an R -module, then stable derived functors of the functors $SP^n \otimes G, A^n \otimes G, \widetilde{SP} \otimes G, \widetilde{A} \otimes G, \text{Hom}_R(SP^n, G), \text{Hom}_R(A^n, G), \text{Hom}_R(\widetilde{SP}, G), \text{Hom}_R(\widetilde{A}, G)$ form connected and exact sequences of functors.*

Now we consider the functors

$$(7.9) \quad F, \tilde{F}: \mathcal{M}_R \rightarrow \mathcal{M}_R$$

defined as follows. $F(M)$ is the free R -module generated by an R -module M and $\tilde{F}(M) = F(M)/F(0)$. The action of F and \tilde{F} on R -homomorphisms is defined in the natural way. Clearly, $\tilde{F}(0) = 0$ and it is easy to check that

$$\tilde{F}(M \oplus N) = \tilde{F}(M) \oplus \tilde{F}(N) \oplus \tilde{F}(M) \otimes \tilde{F}(N).$$

(2) $T \otimes G = T(\cdot) \otimes G$.

Hence, $\tilde{F}_2(M, N) = \tilde{F}(M) \otimes \tilde{F}(N)$ for R -modules M, N and therefore $\deg \tilde{F} = \infty$.

If $R = Z$ is the ring of integers, then it is proved in ([5]; 4.16) that the left stable derived functors of the functors \tilde{F} and \tilde{SP} are naturally equivalent. This fact and Corollary 7.8 yield

7.10. COROLLARY. *In the category of abelian groups $\mathcal{A}b$ the left stable derived functors of the functor F form a connected and exact sequence of covariant functors.*

Let T^n be one of the functors $SP^n, A^n, n \geq 2$. Then, using formulas (7.5), (7.6) and (4.5), we get

$$T^n(M \oplus N) = T^n(M) \oplus T_2^n(M, N) \oplus T^n(N) = \bigoplus_{i=0}^n T^i(M) \otimes T^{n-i}(N)$$

for any R -modules M, N . Thus, for $N = M$, there exist natural injections

$$u_{i,n-i}: T^i(M) \otimes T^{n-i}(M) \rightarrow T_2^n(M, M)$$

and natural projections

$$p_{i,n-i}: T^n(M \oplus M) \rightarrow T^i(M) \otimes T^{n-i}(M)$$

for $i = 1, 2, \dots, n-1$.

7.11. LEMMA. *Let $d: M \rightarrow M \oplus M, \bar{d}: M \oplus M \rightarrow M$ denote the diagonal and codiagonal homomorphisms, respectively, and let α be the composition*

$$(7.12) \quad T_2^n(M, M) \xrightarrow{\lambda_2(M, M)} T^n(M \oplus M) \xrightarrow{(T^n \bar{d})} T^n(M).$$

If we put

$$(7.13) \quad \begin{cases} \alpha_{i,n-i} = \alpha u_{i,n-i}: T^i(M) \otimes T^{n-i}(M) \rightarrow T^n(M), \\ \beta_{i,n-i} = p_{i,n-i} T^n(d): T^n(M) \rightarrow T^i(M) \otimes T^{n-i}(M) \end{cases}$$

for $i = 1, 2, \dots, n-1$, then

$$(7.14) \quad (\alpha_{i,n-i} \cdot \beta_{i,n-i})x = \binom{n}{i} x$$

for any $x \in T^n(M)$ and $i = 1, 2, \dots, n-1$.

Proof. Let $T^n = A^n$ and let $x = m_1 \wedge m_2 \wedge \dots \wedge m_n \in A^n(M)$. Setting $m_j^0 = (m_j, 0), m_j^1 = (0, m_j) \in M \oplus M$ for $j = 1, 2, \dots, n$, we get

$$\begin{aligned} & \alpha_{i,n-i} \beta_{i,n-i} (m_1 \wedge m_2 \wedge \dots \wedge m_n) \\ &= A^n(\bar{d}) \lambda_2(M, M) u_{i,n-i} p_{i,n-i} ((m_1^0 + m_1^1) \dots (m_n^0 + m_n^1)) \\ &= A^n(\bar{d}) \lambda_2(M, M) u_{i,n-i} p_{i,n-i} \left(\sum_{\varepsilon_1=0}^1 \dots \sum_{\varepsilon_n=0}^1 m_1^{\varepsilon_1} \wedge \dots \wedge m_n^{\varepsilon_n} \right). \end{aligned}$$

It is easy to see that the map

$$\lambda_2(M, M)u_{i,n-i}p_{i,n-i}: A^n(M \oplus M) \rightarrow A^n(M \oplus M)$$

carries the above sum into the sum of all elements $m_1^{\varepsilon_1} \wedge \dots \wedge m_n^{\varepsilon_n}$ with $\varepsilon_k = 0$ for exactly i different indexes k . Since in this sum there are $\binom{n}{i}$ such elements and clearly

$$A^n(\bar{d})(m_1^{\varepsilon_1} \wedge \dots \wedge m_n^{\varepsilon_n}) = m_1 \wedge \dots \wedge m_n,$$

then (7.14) follows. The proof for $T^n = SP^n$ is similar⁽³⁾.

7.15. THEOREM. Let T^n be one of the functors SP^n , A^n ; let M be an R -module and X a left complex of R -modules. Then, for any integer q , the following equalities hold

(i) $D_q^s T^n(X) = L_q^s T^n(M) = 0$, if n is not a prime power.

(ii) $p \cdot D_q^s T^n(X) = p \cdot L_q^s T^n(M) = 0$, if $n = p^r$, $r > 0$, where p is a prime.

Proof. Fix $q \geq 0$ and put $Q = KS^q X$, $SQ = KS^{q+1} X$, where K is a functor defined in [5], 3.2. Let α be the composed simplicial map

$$(7.16) \quad T_2^n(Q, Q) \xrightarrow{\lambda_2(Q, Q)} T^n(Q \oplus Q) \xrightarrow{T^n(\bar{d})} T^n(Q)$$

where $\bar{d}: Q \oplus Q \rightarrow Q$ is a codiagonal simplicial map and $\lambda_2(Q, Q)$ is the natural injection. It is clear that α is induced by the map (7.12) for $M = Q_j$, $j = 0, 1, 2, \dots$. Since Q is a q -trivial simplicial complex and $2q \leq 3q$, then by [5], 6.11 we have the exact sequence

$$(7.17) \quad H_{2q} T_2^n(Q, Q) \xrightarrow{\alpha_*} H_{2q} T^n(Q) \xrightarrow{\sigma_{2q}(X)} H_{2q+1} T^n(SQ) \rightarrow 0,$$

where $\alpha_* = H_{2q}(\alpha)$, since $H_{2q-1} T_2^n(Q, Q) = 0$ by Lemma 3.24. Therefore, using the formula (3.19) for $n = q+1$, and the exactness of the above sequence we have

$$(7.18) \quad D_q^s T^n(X) \simeq D_{2q+1} T^n(S^{q+1} X) \simeq H_{2q+1} T^n(SQ) \simeq H_{2q+1} T^n(Q) / \text{Im} \alpha_*.$$

Then the theorem is a consequence of the following relations:

$$(7.19) \quad \begin{aligned} H_{2q} T^n(Q) &= \text{Im} \alpha_*, \text{ if } n \text{ is not a prime power,} \\ p \cdot H_{2q} T^n(Q) &\subset \text{Im} \alpha_*, \text{ if } n = p^r, r > 0, \text{ where } p \text{ is a prime,} \end{aligned}$$

which may be proved as in [5], 10.1, using Lemma 7.11.

As an immediate consequence of Theorem 7.15 we obtain

7.20. COROLLARY. Let R be a commutative k -algebra, where k is a field of characteristic p . If $n \neq p^r$, $r > 0$, then $L_q^s SP^n$ and $L_q^s A^n$ are zero functors

⁽³⁾ Lemma 7.11 for $T^n = SP^n$ is proved in [5], § 10.

for all q . In particular, if $p = 0$, then all left stable derived functors of the functors $SP^n, \Lambda^n, \widetilde{SP}, \widetilde{\Lambda}$ are zero functors.

7.21. THEOREM. Let R be a commutative principal ideal domain, let G, M be R -modules and X a left complex of free R -modules. If $T_G^n: \mathcal{M}_R \rightarrow \mathcal{M}_R$ ($n \geq 2$) is one of the functors $SP^n \otimes G, \Lambda^n \otimes G$, and $\hat{T}_G^n: \mathcal{M}_R \rightarrow \mathcal{M}_R$, ($n \geq 2$) is one of the functors $\text{Hom}_R(SP^n, G), \text{Hom}_R(\Lambda^n, G)$, then the following formulas hold:

- (a) $D_q^s T_G^n(X) \simeq (D_q^s T_R^n(X) \otimes G) \oplus \text{Tor}_1^R(D_{q-1}^s T_R^n(X), G);$
- (b) $D_q^g \hat{T}_G^n(X) \simeq \text{Hom}_R(D_q^s T_R^n(X), G) \oplus \text{Ext}_R^1(D_{q-1}^s T_R^n(X), G);$
- (a') $L_q^s T_G^n(M) \simeq (L_q^s T_R^n(M) \otimes G) \oplus \text{Tor}_1^R(L_{q-1}^s T_R^n(M), G);$
- (b') $R_s^g \hat{T}_G^n(M) \simeq \text{Hom}_R(L_q^s T_R^n(M), G) \oplus \text{Ext}_R^1(L_{q-1}^s T_R^n(M), G).$

If $R = Z$ is the ring of integers and $n = p^r$, $r > 0$, p is a prime, then

- (c) $L_q^s(SP^n \otimes_Z G) = R_s^g \text{Hom}_Z(SP^n, G) = 0 \quad \text{for } q < 2(n-1);$
- (d) $L_{2(n-1)}^s(SP^n \otimes_Z G)M = (M/pM) \otimes_Z G;$
- (e) $R_s^{2(n-1)} \text{Hom}_Z(SP^n, G)M = \text{Hom}_Z(M/pM, G).$

Proof. Let $Q = KS^{q+1}X$. Since the functor k (see 3.2) commutes with additive functors, then

$$D_q^s T_G^n(X) \simeq H_{2q+1}(T_R^n(Q) \otimes G).$$

It follows from the definition of the functors K and T_R^n , that $T_R^n(Q)$ is a simplicial complex of free R -modules. Hence, (a) follows from the Künneth Tensor Formula. Similarly one can prove (b). (a') and (b') follow immediately from (a) and (b). Finally, the equalities (c)–(e) follow from (a'), (b') and [5], 12.3, 12.10.

Certain information about the action of the functors $L_q^s SP^n$ and $L_q^s \Lambda^n$ on divisible abelian groups is given by the following

7.22. PROPOSITION. Let $T^n: \mathcal{M}_Z \rightarrow \mathcal{M}_Z$ be one of the functors SP^n, Λ^n . If A is an abelian group and W is the additive group of rationals, then for any integer q we have

- (i) $L_q^s T^n(A) = 0$, whenever A is divisible and torsion-free;
- (ii) $L_q^s T^n(A/kA) = 0$, if $(k, n) = 1$ and A is k -torsion-free;
- (iii) $L_q^s T^n(W/Z) \simeq L_q^s T^n(Z_{p^\infty})$, if $n = p^r$, $r > 0$, p is a prime.

Proof. In view of Theorem 7.15 (i), we may assume that $n = p^r$, $r > 0$ and p is a prime.

Ad (i). If A is a divisible and torsion-free group, then the homomorphism $p: A \rightarrow A$ given by $p(x) = px$ for $x \in A$ is an automorphism and hence $L_q^s T^n(p)$ is an automorphism. On the other hand, $L_q^s T^n(p) = p \cdot 1_{L_q^s T^n(A)} = 0$, by the additivity of the functors $L_q^s T^n$ and Theorem 7.15 (ii). Therefore (i) follows.

Ad (ii). It follows from Corollary 7.8 that the exact sequence

$$0 \rightarrow A \xrightarrow{k} A \rightarrow A/kA \rightarrow 0$$

induces a long exact sequence

$$\dots \rightarrow L_q^s T^n(A) \xrightarrow{k_*} L_q^s T^n(A) \rightarrow L_q^s T^n(A/kA) \rightarrow L_{q-1}^s T^n(A) \rightarrow \dots$$

with $k_*(x) = kx$, by the additivity of the functors $L_q^s T^n$. Consequently, if $(k, n) = 1$, then k_* is an automorphism by Theorem 7.15, and thus (ii) is proved.

Ad (iii). Since $W/Z = \bigoplus_t Z_{t^\infty}$, where t runs over all prime integers and $Z_{t^\infty} = \varinjlim_m Z_{t^m}$, then by Corollary 7.4 and the assertion (ii) of the proposition we have

$$L_q^s T^n(W/Z) \simeq \bigoplus_t \varinjlim_m L_q^s T^n(Z_{t^m}) \simeq \varinjlim_m L_q^s T^n(Z_{t^m}) \simeq L_q^s T^n(Z_{t^\infty}).$$

This completes the proof of the proposition.

7.23. DEFINITION. Let n be the natural integer. $T: \mathcal{M}_R \rightarrow \mathcal{M}_R$ is called an n -homogenous functor, if $T(rf) = r^n T(f)$ for any $r \in R$ and any R -homomorphism $f: M \rightarrow N$.

Examples of n -homogenous functors are SP^n and A^n .

7.24. PROPOSITION. *The left stable derived functors of the n -homogenous functor are n -homogenous, too.*

Proof. Follows from the definition of stable derived functors.

7.25. COROLLARY. *If $T: \mathcal{M}_R \rightarrow \mathcal{M}_R$ is an n -homogenous functor, then $(k^n - k)L_q^s T(M) = 0$ for any integers k, q and any R -module M .*

Proof. The additivity of the functors $L_q^s T$ and the last proposition give

$$k \cdot L_q^s T(1_M) = L_q^s T(k) = k^n \cdot L_q^s T(1_M),$$

which proves the corollary.

8. On J. H. C. Whitehead's functor Γ

Following J. H. C. Whitehead [15] we define a functor $\Gamma: \mathcal{M}_R \rightarrow \mathcal{M}_R$ and compare its stable derived functors with stable derived functors of the 2nd symmetric power functor SP^2 .

We say that the function $\varphi: M \rightarrow N$, between R -modules M, N , is *R-quadratic*, iff

$$(a) \varphi(rm) = r^2\varphi(m) \text{ for } r \in R \text{ and } m \in M,$$

(b) the associated symmetric function $\Delta\varphi: M \times M \rightarrow N$ defined by $\Delta\varphi(m, m') = \varphi(m + m') - \varphi(m) - \varphi(m')$, $m, m' \in M$, is bilinear, i.e. $\Delta\varphi(\cdot, m'): M \rightarrow N$ is an R -homomorphism for any $m' \in M$.

We denote by $\text{Quad}_R(M, N)$ the R -module of all quadratic functions from M to N . Clearly, $\text{Quad}_R(M, \cdot): \mathcal{M}_R \rightarrow \mathcal{M}_R$ is a covariant, additive and left exact functor, and $\text{Quad}_R(\cdot, N): \mathcal{M}_R \rightarrow \mathcal{M}_R$ is a contravariant non-additive functor.

If M is an R -module, then the R -module $\Gamma(M)$ is defined as the quotient module $F(M)/F_0(M)$, where $F(M)$ is the free R -module freely generated by the symbols $\omega(m)$, $m \in M$, and $F_0(M)$ is the submodule of $F(M)$ generated by the elements

$$\begin{aligned} & \omega(rm) - r^2\omega(m), \\ & \omega(m_1 + m_2 + m_3) - \omega(m_1 + m_2) - \omega(m_1 + m_3) - \omega(m_2 + m_3) + \omega(m_1) + \\ & \qquad \qquad \qquad + \omega(m_2) + \omega(m_3), \\ & \omega(rm + m_1) - r\omega(m + m_1) - \omega(rm) + r\omega(m) - \omega(m_1) + r\omega(m_1) \end{aligned}$$

with $m, m_1, m_2, m_3 \in M$ and $r \in R$. Evidently, the map $\gamma: M \rightarrow \Gamma(M)$, given by $\gamma(m) = \omega(m) + F_0(M)$, is an R -quadratic function. If $h: M \rightarrow N$ is an R -homomorphism, then we define the homomorphism $\Gamma(h): \Gamma(M) \rightarrow \Gamma(N)$ by $\Gamma(h)\gamma(m) = \gamma(h(m))$ and thus $\Gamma: \mathcal{M}_R \rightarrow \mathcal{M}_R$ is a covariant functor.

8.1. LEMMA. *If $h: M \rightarrow N$ is an epimorphism of R -modules and $\Gamma(M)_0$ is a submodule of $\Gamma(M)$, generated by elements $\gamma(k)$, $\Delta\gamma(m, k)$ for $k \in \text{Ker } h$ and $m \in M$, then $\text{Ker } \Gamma(h) = \Gamma(M)_0$.*

Proof. Since clearly $\Gamma(M)_0 \subset \text{Ker } \Gamma(h)$, then we have the homomorphism

$$f: \Gamma(M)/\Gamma(M)_0 \rightarrow \Gamma(N)$$

given by $f(\gamma(m) + \Gamma(M)_0) = \gamma(h(m))$. Define the map

$$g: N \rightarrow \Gamma(M)/\Gamma(M)_0$$

setting $g(n) = \gamma(m) + \Gamma(M)_0$, where $m \in h^{-1}(n)$. If $m' \in M$ is another element such that $h(m') = n$, then $m' - m \in \text{Ker } h$ and thus

$$\gamma(m') - \gamma(m) = \Delta\gamma(m, m' - m) + \gamma(m' - m) \in \Gamma(M)_0.$$

Hence $g(n)$ is independent of the choice of $m \in h^{-1}(n)$. Moreover, it is easy to verify that g is a quadratic function; so it induces the homomorphism $\bar{g}: \Gamma(N) \rightarrow \Gamma(M)/\Gamma(M)_0$ such that $\bar{g}f = 1$ and $f\bar{g} = 1$. It follows that f is an isomorphism and thus $\Gamma(M)_0 = \text{Ker } \Gamma(h)$.

It is easy to see that for any R -quadratic function $\varphi: M \rightarrow N$ there exists a unique R -homomorphism $h: \Gamma(M) \rightarrow N$ such that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & \Gamma(M) \\ & \searrow \varphi & \downarrow h \\ & & N \end{array}$$

Then it follows that

$$\bar{\gamma}(M, N): \text{Hom}_R(\Gamma(M), N) \rightarrow \text{Quad}_R(M, N)$$

defined by $\bar{\gamma}(M, N)h = h \cdot \gamma$ for $h \in \text{Hom}_R(\Gamma(M), N)$ is an R -isomorphism and determines a natural equivalence of functors of two variables

$$(8.2) \quad \bar{\gamma}(\cdot, \cdot): \text{Hom}_R(\Gamma(\cdot), \cdot) \rightarrow \text{Quad}_R(\cdot, \cdot).$$

Moreover, it is not difficult to check, that there exists a natural equivalence of functors

$$\text{Quad}_R(\varinjlim(\cdot), N) \simeq \varprojlim \text{Quad}_R(\cdot, N)$$

for any R -module N . Hence

$$\text{Hom}_R(\Gamma(\varinjlim), N) \simeq \text{Hom}_R(\varinjlim \Gamma, N)$$

and thus we have

8.3. COROLLARY. *The functor Γ commutes with direct limits.*

Similarly to the proof of Theorem 13.1 in [7] one can prove:

8.4. PROPOSITION. *The 2nd cross-effect functor of the functor Γ is naturally equivalent to the tensor product functor.*

Then, by (8.2), we have

8.5. COROLLARY. $\text{deg Quad}_R(\cdot, N) \leq 2$ for any R -module N .

Repeating the argument of Section 5. p. 62 in [15] one shows:

8.6. PROPOSITION. *Let I be a well-ordered set. If P is a free R -module, freely generated by elements e_i , $i \in I$, then $\Gamma(P)$ is free and the elements $\gamma(e_i)$, $\Delta\gamma(e_i, e_j)$, $i < j$, $i, j \in I$, are its free generators.*

8.7. THEOREM. (a) $R^q \text{Quad}_R(M, \cdot)(N) \simeq \text{Ext}_R^q(\Gamma(M), N)$.

(b) *If R is a principal ideal domain, then*

$$R_s^q \text{Quad}_R(\cdot, N)(M) \simeq \text{Hom}_R(L_q^s \Gamma(M), N) \oplus \text{Ext}_R^1(L_{q-1}^s \Gamma(M), N)$$

for any R -modules M, N .

Proof. (a) is a consequence of (8.2). The assertion (b) may be proved similarly as the isomorphism (a) in Theorem 7.21 using 8.2, 8.6 and the Universal Coefficients Theorem.

Let X be a left complex of R -modules and put $Q = KS^{q+1}X$, where $q \geq 0$. If

$$\alpha: Q \otimes Q \rightarrow \Gamma(Q), \quad \beta: \Gamma(Q) \rightarrow Q \otimes Q$$

are simplicial maps given by

$$\alpha(x \otimes y) = \Delta\gamma(x, y), \quad \beta(\gamma(x)) = x \otimes x,$$

respectively, then, by [5], 6.11, and Lemma 3.24, we have the exact sequence

$$(8.8) \quad H_{2q}(Q \otimes Q) \xrightarrow{H_{2q}(\alpha)} H_{2q}\Gamma(Q) \rightarrow H_{2q+1}\Gamma(SQ) \rightarrow 0$$

Since clearly $\alpha \cdot \beta = 2 \cdot 1_{\Gamma(Q)}$, then we may prove as in [5], 10.1 that

$$(8.9) \quad 2 \cdot H_{2q}\Gamma(Q) \subset \text{Im } H_{2q}(\alpha).$$

In particular, if X is a projective resolution of an R -module M , then using formula (3.19) for $n = q + 1$ and the exactness of the sequence (8.8), we have

$$L_q^s \Gamma(M) \simeq H_{2q}\Gamma(Q) / \text{Im } H_{2q}(\alpha).$$

Thus we have proved:

8.10. COROLLARY. $2 \cdot L_q^s \Gamma(M) = 0$ for any R -module M and $q \geq 0$.

Consider the following exact sequence

$$(8.11) \quad 0 \rightarrow W(M) \xrightarrow{r(M)} SP^2(M) \xrightarrow{t(M)} \Gamma(M) \xrightarrow{s(M)} U(M) \rightarrow 0$$

where $t(M)(m \vee m') = \Delta\gamma(m, m')$ for $(m \vee m') \in SP^2(M)$ and

$$W(M) = \text{Kert}(M), \quad U(M) = \text{Cokert}(M).$$

One can easily check that $t(M)$ determines the natural transformation of functors $t: SP^2 \rightarrow \Gamma$ and thus U and W are covariant functors.

8.12. COROLLARY. *The functors U and W commute with direct limits.*

Proof. Since SP^2 and Γ commute with direct limits, then the corollary follows from the definition of U and W , and the exactness of the direct limit functor.

8.13. PROPOSITION. *U is a right exact functor.*

Proof. Let $0 \rightarrow K \xrightarrow{j} M \xrightarrow{h} N \rightarrow 0$ be an exact sequence of R -modules. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \uparrow & & \uparrow & & \uparrow \\
 U(K) & \xrightarrow{U(j)} & U(M) & \xrightarrow{U(h)} & U(N) & \\
 \uparrow s & & \uparrow s & & \uparrow s & \\
 \Gamma(K) & \xrightarrow{\Gamma(j)} & \Gamma(M) & \xrightarrow{\Gamma(h)} & \Gamma(N) & \\
 \uparrow t & & \uparrow t & & \uparrow t & \\
 SP^2(K) & \xrightarrow{SP^2(j)} & SP^2(M) & \xrightarrow{SP^2(h)} & SP^2(N) &
 \end{array}$$

Clearly, $U(h)$ is an epimorphism and $\text{Im } U(j) \subset \text{Ker } U(h)$. Let $x \in \text{Ker } U(h)$. Since $x = s(y)$ for some $y \in \Gamma(M)$ and $\Gamma(h)y \in \text{Ker } s = \text{Im } t$, $\Gamma(h)y = tSP^2(h)(z)$ for some $z \in SP^2(M)$, because $SP^2(h)$ is an epimorphism. Therefore, $y - t(z) \in \text{Ker } \Gamma(h)$ and by Lemma 8.1, we have

$$y - t(z) = \sum_i r_i \gamma(k_i) + \sum_j r'_j \Delta\gamma(m_j, k'_j)$$

for some $k_i, k'_j \in \text{Ker } h$, $m_j \in M$ and $r_i, r'_j \in R$. But

$$\sum_j r'_j \Delta\gamma(m_j, k'_j) = t\left(\sum_j r'_j (m_j \vee k'_j)\right)$$

and $k_i = j(s_i)$, where $s_i \in K$. Then setting $u = \sum_i r_i \gamma(s_i) \in \Gamma(K)$ we get

$$U(j)s(u) = s\Gamma(j)(u) = s\left(\sum_i r_i \gamma(k_i)\right) = s(y) = x.$$

Hence $\text{Ker } U(h) \subset \text{Im } U(j)$ and the proof is completed.

8.14. COROLLARY. U and W are additive functors.

Proof. The corollary for the functor U follows immediately from the last proposition and for the functor W may be proved directly.

8.15. PROPOSITION. Let F and F' be a free modules with bases $\{e_i\}_{i \in I}$ and $\{e'_j\}_{j \in J}$, respectively, and let $f: F \rightarrow F'$ be an R -homomorphism. Then

(a) $U(F) \simeq \text{Coker}(F \xrightarrow{2} F)$, $W(F) \simeq \text{Ker}(F \xrightarrow{2} F)$,

(b) if $f(e_i) = \sum_j r_{ij} e'_j$, $r_{ij} \in R$, and $\bar{f}: F \rightarrow F'$ is the homomorphism given by $\bar{f}(e_i) = \sum_j r_{ij}^2 e'_j$, then we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(F) & \longrightarrow & F & \xrightarrow{2} & F & \longrightarrow & U(F) & \longrightarrow & 0 \\ & & \downarrow w(f) & & \downarrow \bar{f} & & \downarrow \bar{f} & & \downarrow U(f) & & \\ 0 & \longrightarrow & W(F') & \longrightarrow & F' & \xrightarrow{2} & F' & \longrightarrow & U(F') & \longrightarrow & 0. \end{array}$$

Proof. If I is a well-ordered set, then $SP^2(F)$ is a free module with basis $\{e_i \vee e_j\}_{i \leq j; i, j \in I}$ and

$$\begin{aligned} t(F)(e_i \vee e_j) &= \Delta\gamma(e_i, e_j) \quad \text{for } i < j, \\ t(F)(e_i \vee e_i) &= 2\gamma(e_i) \quad \text{for all } i \in I. \end{aligned}$$

Therefore the assertion (a) follows from Proposition 8.6. To prove (b) observe that

$$\begin{aligned} \Gamma(f)\gamma(e_i) &= \sum_j r_{ij}^2 \gamma(e'_j) + \sum_{j < k} r_{ij} r_{ik} \Delta\gamma(e'_j, e'_k) \\ &= \sum_j r_{ij}^2 \gamma(e'_j) + t(F')\left(\sum_{j < k} r_{ij} r_{ik} (e'_j \vee e'_k)\right). \end{aligned}$$

It then follows that (b) holds for U . The proof for the functor W is similar.

8.16. THEOREM. *If 2 is a non-zero-divisor in R , then the following sequence is exact in the sense of Definition 3.43.*

$$\dots \rightarrow L_q^s SP^2 \xrightarrow{t_*} L_q^s \Gamma \xrightarrow{s_*} L_q U \xrightarrow{d_q} L_{q-1}^s SP^2 \rightarrow \dots$$

Proof. It follows from the last proposition and our assumptions that W vanishes on projective modules. Hence the theorem follows from the exactness of the sequence (8.11) and Corollary 3.47.

8.17. COROLLARY. *If 2 is a non-zero-divisor in R and $q \geq 2$, then*

- (a) $L_q^s \Gamma(M) \simeq L_q^s SP^2(M)$ for $q > n$, whenever $\text{w.dim}_R M \leq n$,
- (b) $L_q^s \Gamma \simeq L_q^s SP^2$ for $q \geq 2$, if $\text{w.gl.dim } R \leq 1$.

Proof. (b) follows from (a); (a) is a consequence of Theorem 8.16 and the following

8.18. LEMMA. *If $\text{w.dim}_R M \leq n$, then $L_q U(M) = 0$ for $q > n$.*

Proof. Clearly, $L_q U(P) = 0$ for P projective and $q \geq 1$. Hence, $L_q U(M) = 0$ for M flat and $q \geq 1$, because $L_q U$ commute with direct limits (see Corollary 8.12) and M is a direct limit of projective modules by [11]. Therefore the assertion is proved for $n = 0$. In the general case we have an exact sequence

$$0 \rightarrow K \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

with P_0, \dots, P_{n-1} projective and K flat. Then $L_{n+1} U(M) \simeq L_1 U(K) = 0$ and the proof is completed.

9. Computation of the modules $L_q^s SP^2(R)$, $L_q^s \Lambda^2(R)$ and $L_q^s \Gamma(R)$

Let P denote the complex given by

$$(9.1) \quad P_n = \begin{cases} 0 & \text{for } n \neq 0, \\ R & \text{for } n = 0, \end{cases}$$

and let $Q = KS^q P$, $SQ = KS^{q+1} P$, where $q > 0$ is a fixed integer.

If T is one of the functors $SP^2, \Lambda^2: \mathcal{M}_R \rightarrow \mathcal{M}_R$, then it follows from (7.5) and (7.6) that the second cross-effect functor T_2 of the functor T is naturally equivalent to the tensor product functor $\otimes = \otimes_R$. Hence, the exact sequence (7.17) gives for $n = 2$ and $X = P$ the exact sequence

$$(9.2) \quad H_{2q}(Q \otimes Q) \xrightarrow{H_{2q}(a)} H_{2q} T(Q) \xrightarrow{\sigma_{2q}(Q)} H_{2q+1} T(SQ) \rightarrow 0,$$

where $\alpha: T_2(Q, Q) \rightarrow T(Q)$ is the simplicial map (7.16). Since P is a projective resolution of the R -module R , then using formula (7.18) for $X = P$ we obtain

$$(9.3) \quad L_q^s T(R) \simeq H_{2q} T(Q) / \text{Im } H_{2q}(a).$$

Now we compute the modules $H_{2q}(Q \otimes Q)$, $H_{2q}T(Q)$, $\text{Im}H_{2q}(a)$ and $\text{Ker}H_{2q}(a)$.

By the definition of the functor K ([5]; 3.2), the R -modules Q_j for $j \leq q$ and Q_{2q} of the simplicial object Q have the following form

$$(9.4) \quad \begin{cases} Q_j = 0 & \text{for } j < q, \\ Q_q = R[1], & \text{where } [1]: [q] \rightarrow [q] \text{ is the identity map,} \end{cases}$$

$$(9.5) \quad Q_{2q} = \bigoplus_{\eta} R[\eta],$$

where η runs over all non-decreasing surjections from $[2q]$ to $[q]$ and $R[\eta]$ is a cyclic free R -module, with the free generator η . Furthermore, the simplicial map $Q(\eta): Q_q \rightarrow Q_{2q}$ induced by the non-decreasing surjection $\eta: [2q] \rightarrow [q]$ is defined by

$$(9.6) \quad Q(\eta)[1] = [\eta].$$

By Lemma 5.1, Chapter VIII in [12], any non-decreasing surjection $\eta: [2q] \rightarrow [q]$ has the unique factorization

$$\eta = \eta_q^{i_1} \eta_{2q-1}^{i_q}$$

with $0 \leq i_1 < i_2 < \dots < i_q < 2q$, where the maps $\eta_m^l: [m+1] \rightarrow [m]$ are defined by

$$\eta_m^l(i) = \begin{cases} i & \text{for } i \leq l, \\ i-1 & \text{for } i > l. \end{cases}$$

Therefore

$$(9.7) \quad Q_{2q} = \bigoplus_{0 \leq i_1 < \dots < i_q < 2q} R[\eta^{i_1} \dots \eta^{i_q}]$$

(we omit the lower indices).

Let P_{2q} denotes the set of all (q, q) -shuffles (μ, ν) ; i.e. such permutations $(\mu, \nu) = (\mu_1, \dots, \mu_q, \nu_1, \dots, \nu_q)$ of the set $\{0, 1, 2, \dots, 2q-1\}$ that $\mu_1 < \dots < \mu_q$ and $\nu_1 < \dots < \nu_q$ (see [12], Ch. VIII). For any shuffle (μ, ν) the signature $\varepsilon(\mu)$ of (μ, ν) is defined by

$$\varepsilon(\mu) = \sum_{i=1}^q \mu_i - (i-1).$$

9.8. LEMMA. Let $Q = KS^qP$, where $q \geq 0$ and P is the complex defined by (9.1). Then $H_{2q}(Q \otimes Q) = H_{2q}N(Q \otimes Q)$ is a cyclic free R -module with a free generator $e \in B_{2q}$, where

$$(9.9) \quad e = \begin{cases} [1_0] \otimes [1_0] \in Q_0 \otimes Q_0 & \text{for } q = 0; 1_0: [0] \rightarrow [0] \text{ is the identity map,} \\ \sum_{(\mu, \nu) \in P_{2q}} (-1)^{\varepsilon(\mu)} [\eta^{\mu_1} \dots \eta^{\mu_q}] \otimes [\eta^{\nu_1} \dots \eta^{\nu_q}] \in N(Q \otimes Q)_{2q} & \text{for } q > 0, \end{cases}$$

and B_{2q} denotes the R -module of $2q$ -boundaries of the complex $N(Q \otimes Q)$.

Proof. Consider the Eilenberg–MacLane’s complex map (see [6], 5.3)

$$\mathcal{V}: NQ \otimes NQ \rightarrow N(Q \otimes Q).$$

By Theorem 2.1a in [7], \mathcal{V} is a homotopy equivalence. Thus

$$H_{2q}(\mathcal{V}): H_{2q}(NQ \otimes NQ) \rightarrow H_{2q}N(Q \otimes Q)$$

is an isomorphism. But $NQ = NKS^qP = S^qP$, so

$$H_{2q}(NQ \otimes NQ) \simeq H_{2q}(S^qP \otimes S^qP) \simeq R$$

and it is easy to see that the class (modulo boundaries) of the element $[1] \otimes [1] \in (NQ \otimes NQ)_{2q} = (NQ)_q \otimes (NQ)_q = Q_q \otimes Q_q = R[1] \otimes R[1]$ is a free generator of $H_{2q}(NQ \otimes NQ)$, where $1: [q] \rightarrow [q]$ is the identity map. Hence $H_{2q}N(Q \otimes Q)$ is a cyclic free R -module, with the free generator $\mathcal{V}([1] \otimes [1]) + B_{2q}$. It follows from the definition of \mathcal{V} that

$$\mathcal{V}([1] \otimes [1]) = [1_0] \otimes [1_0] \quad \text{for } q = 0; \quad 1_0: [0] \rightarrow [0]$$

is the identity and

$$\mathcal{V}([1] \otimes [1]) = \sum_{(\mu, \nu) \in P_{2q}} (-1)^{\epsilon(\mu)} (s_{\mu_q} \dots s_{\mu_1} [1]) \otimes (s_{\nu_q} \dots s_{\nu_1} [1])$$

for $q > 0$, where $s_i = Q(\eta^i)$ is the i th degeneracy operator of the simplicial object Q . Moreover, formula (9.6) implies that

$$\begin{aligned} s_{\mu_q} \dots s_{\mu_1} [1] &= Q(\eta^{\mu_1} \dots \eta^{\mu_q})[1] = [\eta^{\mu_1} \dots \eta^{\mu_q}], \\ s_{\nu_q} \dots s_{\nu_1} [1] &= [\eta^{\nu_1} \dots \eta^{\nu_q}]. \end{aligned}$$

Consequently, $\mathcal{V}([1] \otimes [1]) = e$ and the Lemma follows.

9.10. PROPOSITION. *Let T be one of the functors SP^2 and Λ^2 . If $Q = KS^qP$, $q \geq 0$, and $\alpha: Q \otimes Q \rightarrow T(Q)$ is the composed simplicial map (7.16) for $n = 2$ and $X = P$, then the following statements hold:*

(i) *if $T = SP^2$, then*

$$\text{Im } H_{2q}(\alpha) = \begin{cases} H_0 SP^2(Q) \simeq R & \text{for } q = 0, \\ 2 \cdot H_{2q} SP^2(Q) & \text{for } q = 2l, \quad l > 0, \\ 0 & \text{for } q = 2l+1, \quad l \geq 0, \end{cases}$$

$$\text{Ker } H_{2q}(\alpha) = \begin{cases} 0 & \text{for } q = 0, \\ R & \text{for } q = 2l+1, \quad l \geq 0, \\ {}_2R & \text{for } q = 2l, \quad l > 0; \end{cases}$$

(ii) if $T = A^2$, then

$$\operatorname{Im} H_{2q}(a) = \begin{cases} H_0 A^2(Q) = 0 & \text{for } q = 0 \\ 0 & \text{for } q = 2l, l > 0 \\ 2 \cdot H_{2q} A^2(Q) & \text{for } q = 2l+1, l \geq 0 \end{cases}$$

$$\operatorname{Ker} H_{2q}(a) = \begin{cases} R & \text{for } q = 2l, l \geq 0 \\ {}_2R & \text{for } q = 2l+1, l \geq 0 \end{cases}$$

where ${}_2R = \{r \in R, 2r = 0\}$.

Proof. According to Lemma 9.8, $\operatorname{Im} H_{2q}(a)$ is generated by the element $H_{2q} N(a)(e + B_{2q}) \in H_{2q} T(Q)$. But $\deg T = 2$ and $(NQ)_i = 0$ for $i > q$, thus it follows from [5], 4.23 that $NT(Q)_{2q+1} = 0$. Hence $H_{2q} NT(Q) \simeq \operatorname{Ker} d_{2q}$ and $H_{2q} N(a)(e + B_{2q}) = \alpha(e) \in \operatorname{Ker} d_{2q}$, where e is the element (9.9).

In particular, we have

$$H_0 NT(Q) = \operatorname{Ker} d_0 = T(Q_0) = T(R[1_0]),$$

where $1_0: [0] \rightarrow [0]$ is the identity map; and thus the assertion (ii) follows for $q = 0$, because $A^2(R) = 0$. If $T = SP^2$, then $H_0 NT(Q)$ is a cyclic free R -module generated by the element $[1_0] \otimes [1_0]$ and hence $H_0 N(a)$ is an isomorphism by Lemma 8.8 and the equality $H_0 N(a)e = [1_0] \otimes [1_0]$.

Now let $q \geq 1$. We compute the element $\alpha(e)$. First observe that α and the natural epimorphism $Q \otimes Q \rightarrow T(Q)$ coincide. Therefore we have

$$\alpha(e) = \sum_{(\mu, \nu) \in P_{2q}} (-1)^{\varepsilon(\mu)} [\eta^{\mu_1} \dots \eta^{\mu_q}] \vee_{\wedge} [\eta^{\nu_1} \dots \eta^{\nu_q}],$$

where we write \vee if $T = SP^2$ and \wedge if $T = A^2$. Let $\varphi: P_{2q} \rightarrow P_{2q}$ be the natural involution defined by $\varphi(\mu, \nu) = (\nu, \mu)$. Making use of φ we can define an equivalence relation \sim . Namely

$$(a \sim b) \Leftrightarrow (a = b \text{ or } a = \varphi(b)) \quad \text{for } a, b \in P_{2q}.$$

If B is any set of representatives of all classes of equivalence (with respect to this equivalence relation, of course), then it is clear that $B \cup \varphi(B) = P_{2q}$ and $B \cap \varphi(B) = \emptyset$. Hence

$$\begin{aligned} \alpha(e) &= \sum_{(\mu, \nu) \in B} (-1)^{\varepsilon(\mu)} [\eta^{\mu_1} \dots \eta^{\mu_q}] \vee_{\wedge} [\eta^{\nu_1} \dots \eta^{\nu_q}] + \\ &\quad + \sum_{(\mu, \nu) \in \varphi(B)} (-1)^{\varepsilon(\mu)} [\eta^{\mu_1} \dots \eta^{\mu_q}] \vee_{\wedge} [\eta^{\nu_1} \dots \eta^{\nu_q}] \\ &= \sum_{(\nu, \mu) \in \varphi(B)} (-1)^{\varepsilon(\mu)} [\eta^{\mu_1} \dots \eta^{\mu_q}] \vee_{\wedge} [\eta^{\nu_1} \dots \eta^{\nu_q}] + \\ &\quad + \sum_{(\nu, \mu) \in \varphi(B)} (-1)^{\varepsilon(\nu)} [\eta^{\nu_1} \dots \eta^{\nu_q}] \vee_{\wedge} [\eta^{\mu_1} \dots \eta^{\mu_q}] \\ &= \begin{cases} \sum_{(\nu, \mu) \in \varphi(B)} [(-1)^{\varepsilon(\mu)} + (-1)^{\varepsilon(\nu)}] ([\eta^{\mu_1} \dots \eta^{\mu_q}] \vee [\eta^{\nu_1} \dots \eta^{\nu_q}]), \\ \sum_{(\nu, \mu) \in \varphi(B)} [(-1)^{\varepsilon(\mu)} - (-1)^{\varepsilon(\nu)}] ([\eta^{\mu_1} \dots \eta^{\mu_q}] \wedge [\eta^{\nu_1} \dots \eta^{\nu_q}]), \end{cases} \end{aligned}$$

because $x \vee y = y \vee x$ and $x \wedge y = -(y \wedge x)$. Since $\{\mu_1, \dots, \mu_q, \nu_1, \dots, \nu_q\} = \{0, 1, 2, \dots, 2q-1\}$, then $\varepsilon(\mu) + \varepsilon(\nu) = q^2$. Consequently

$$(9.11) \quad (-1)^{\varepsilon(\mu)} + (-1)^{\varepsilon(\nu)} = \begin{cases} 0 & \text{for } q = 2l+1, l \geq 0, \\ \pm 2 & \text{for } q = 2l, l > 0, \end{cases}$$

$$(9.12) \quad (-1)^{\varepsilon(\mu)} - (-1)^{\varepsilon(\nu)} = \begin{cases} \pm 2 & \text{for } q = 2l+1, l \geq 0, \\ 0 & \text{for } q = 2l, l > 0, \end{cases}$$

and thus for $T = SP^2$ we have

$$(9.13) \quad H_{2q}(\alpha) = \begin{cases} 0 & \text{for } q = 2l+1, l \geq 0, \\ 2e_1 & \text{for } q = 2l, l > 0, \end{cases}$$

where

$$(9.14) \quad e_1 = \sum_{(\nu, \mu) \in \varphi(B)} \pm [\eta^{\mu_1} \dots \eta^{\mu_q}] \vee [\eta^{\nu_1} \dots \eta^{\nu_q}].$$

Since $H_{2q}(Q \otimes Q)$ is a cyclic free module generated by the element e (see 9.9), and the elements $[\eta^{\mu_1} \dots \eta^{\mu_q}] \vee [\eta^{\nu_1} \dots \eta^{\nu_q}]$, $(\nu, \mu) \in \varphi(\beta)$, are distinct free generators of the free R -module $SP^2(Q)_{2q}$, then the second part of assertion (i) follows from formula (9.13).

Now we prove the first part of (i). It follows from (7.19) for $n = 2$ and $X = P$ that $2 \cdot H_{2q} SP^2(Q) \subset \text{Im } H_{2q}(\alpha)$. Therefore, in view of formula (9.13), it is sufficient to show that $e_1 \in H_{2q} NSP^2(Q)$, i.e. $d_{2q}(e_1) = 0$ and

$$e_1 \in NSP^2(Q) = \bigcap_{i=1}^{2q} \text{Ker } \tilde{\varepsilon}_i,$$

where $\varepsilon_i: [2q-1] \rightarrow [2q]$, $i = 0, 1, \dots, 2q$, are non-decreasing maps (3.3). Since $d_{2q}(e_1) = \tilde{\varepsilon}_0(e_1)$, we must show that $\tilde{\varepsilon}_i(e_1) = 0$ for $i = 0, 1, \dots, 2q$.

First assume that $R = Z$ is the ring of integers. Then

$$2e_1 \in (NSP^2(Q))_{2q} \cap \text{Ker } d_{2q} = \bigcap_{i=0}^{2q} \text{Ker } \tilde{\varepsilon}_i$$

implies $\tilde{\varepsilon}_i(2e_1) = 2\tilde{\varepsilon}_i(e_1) = 0$ for $i = 0, 1, \dots, 2q$. Hence $\tilde{\varepsilon}_i(e_1) = 0$, because $2\tilde{\varepsilon}_i(e_1) \in NSP^2(Q)_{2q-1} \subset SP^2(Q)_{2q-1}$ and $SP^2(Q)_{2q-1}$ is a free group.

In the general case, let P^Z denote the left complex P for $R = Z$ (see 9.1). Then the Z -homomorphism $j: Z \rightarrow R$ defined by $j(1) = 1$ induces the complex map $j: P^Z \rightarrow P$ and hence we get a simplicial map of simplicial abelian groups

$$h = SP^2(KS^q(j)): SP^2(Q^Z) \rightarrow SP^2(Q)$$

such that $e_1 = h_{2q}(e_1^Z)$, where $e_1^Z \in NSP^2(Q^Z)$ is the element (9.14) for $R = Z$. Then the equalities $\tilde{\varepsilon}_i(e_1^Z) = 0$, $i = 0, 1, \dots, 2q$, imply

$$\tilde{\varepsilon}_i(e_1) = \tilde{\varepsilon}_i h_{2q}(e_1^Z) = h_{2q-1} \tilde{\varepsilon}_i(e_1^Z) = 0$$

for $i = 0, 1, \dots, 2q$ and thus $e_1 \in H_{2q} NSP^2(Q)$. This completes the proof of assertion (i) of the proposition. Similarly one can prove (ii) using formula (9.12).

Applying exact sequence (8.8), formula (8.9) and Proposition 8.6, similarly as above for the Whitehead's functor Γ one can prove:

9.15. PROPOSITION. *If $H_{2q}(\alpha)$ is the homomorphism from (8.8) for the complex $X = P$ defined by (9.1), then*

$$\begin{aligned} \text{Im } H_{2q}(\alpha) &= \begin{cases} 2 \cdot H_{2q} \Gamma(Q) & \text{for } q = 2l, l \geq 0, \\ 0 & \text{for } q = 2l+1, l \geq 0, \end{cases} \\ \text{Ker } H_{2q}(\alpha) &= \begin{cases} {}_2R & \text{for } q = 2l, l \geq 0, \\ R & \text{for } q = 2l+1, l \geq 0. \end{cases} \end{aligned}$$

The main result of this section is the following

9.16. THEOREM. *If R is a commutative ring with identity, then*

$$L_{q-2}^s \Gamma(R) \simeq L_{q-1}^s \Lambda^2(R) \simeq L_q^s SP^2(R) \simeq \begin{cases} 0 & \text{for } q \leq 1, \\ R/2R & \text{for } q = 2l, l \geq 1, \\ {}_2R & \text{for } q = 2l+1, l \geq 1. \end{cases}$$

Proof. We prove the theorem for SP^2 . The proof for Λ^2 and Γ is similar.

In view of Corollary 3.36 and Proposition 9.10, we may assume $q \geq 1$ (see 9.3). Consider the simplicial object $Q' = KS^{q-1}P$, where P is the complex from (9.1) and let $\alpha': Q' \otimes Q' \rightarrow SP^2(Q')$ be the composed simplicial map (7.16) for $Q = Q'$ and $T^n = SP^2$. Since $SQ' = Q$, then by ([5], 6.11) we get the exact sequence

$$\dots \rightarrow H_{2q-1} SP^2(Q') \xrightarrow{\sigma} H_{2q} SP^2(Q) \xrightarrow{d_{2q}} H_{2(q-1)}(Q' \otimes Q') \xrightarrow{H_{2q}(\alpha')} H_{2q-2} SP^2(Q').$$

But $\deg SP^2 = 2$ and $(NQ')_i = 0$ for $i > (q-1)$, then $H_{2q-1} SP^2(Q') = 0$ by [5], 4.23 and therefore

$$H_{2q} SP^2(Q) \simeq \text{Ker } H_{2(q-1)}(\alpha').$$

Hence, by Proposition 9.10 and formula (9.3), we get

$$L_q^s SP^2(R) \simeq \text{Ker } H_{2(q-1)}(\alpha') / 2 \cdot \text{Ker } H_{2(q-1)}(\alpha') \simeq R/2R$$

for $q = 2l, l \geq 1$, and

$$L_q^s SP^2(R) \simeq \text{Ker } H_{2(q-1)}(\alpha') \simeq \begin{cases} 0 & \text{for } q = 1, \\ {}_2R & \text{for } q = 2l+1, l \geq 1. \end{cases}$$

A generalization of the first part of assertion (c) in Theorem 7.21 is the following

9.17. COROLLARY. *If M is an R -module, then*

$$L_1^s SP^2(M) = L_0^s SP^2(M) = L_0^s \Lambda^2(M) = 0.$$

Proof. Choose an exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

with F free. Then from Corollary 7.8 we derive the exact sequence

$$L_1^s SP^2(F) \rightarrow L_1^s SP^2(M) \rightarrow L_0^s SP^2(K) \rightarrow L_0^s SP^2(F) \rightarrow L_0^s SP^2(M) \rightarrow 0.$$

Since $L_q^s SP^2$ are additive functors, then $L_0^s SP^2(F) = L_1^s SP^2(F) = 0$ by the last Theorem. Thus $L_0^s SP^2(N) = 0$ for any R -module N and therefore $L_0^s SP^2(K) = L_1^s SP^2(M) = 0$. Similarly, $L_0^s \Lambda^2(M) = 0$ and the corollary follows.

10. Computation of the functors

$L_q^s SP^2$, $L_q^s \Lambda^2$ and $L_q^s \Gamma$

Let $\mathcal{F}r(R)$ and $\mathcal{F}r_1(R)$ denote the categories of all free and all cyclic free R -modules, respectively, together with their R -homomorphisms. If U and W are the functors from (8.11), then it follows from Theorem 9.16 and Propositions 7.24 and 8.15 that

$$(10.1) \quad \begin{aligned} L_{q-2}^s \Gamma|_{\mathcal{F}r_1(R)} &\simeq L_{q-1}^s \Lambda^2|_{\mathcal{F}r_1(R)} \simeq L_q^s SP^2|_{\mathcal{F}r_1(R)} \simeq \\ &\simeq \begin{cases} 0 & \text{for } q \leq 1, \\ U|_{\mathcal{F}r_1(R)} & \text{for } q = 2l, l > 0, \\ W|_{\mathcal{F}r_1(R)} & \text{for } q = 2l+1, l > 0, \end{cases} \end{aligned}$$

where $|_{\mathcal{F}r_1(R)}$ denotes the restriction to $\mathcal{F}r_1(R)$.

In this section we will see that in the above formula the category $\mathcal{F}r_1(R)$ may be replaced by $\mathcal{F}r(R)$. Moreover, we compute the functors $L_q^s SP^2$, $L_q^s \Lambda^2$ and $L_q^s \Gamma$, whenever R has the following two properties:

- (a) the element 2 is a non-zero-divisor in R ,
- (b) $r^2 - r \in 2R$ for any $r \in R$.

In what follows we use the following notations. If $f_i: M_i \rightarrow M$ are any homomorphisms, then $\bigoplus_i f_i: \bigoplus_i M_i \rightarrow M$ is the unique homomorphism such that its composition with the natural injections $i_k: M_k \rightarrow \bigoplus_i M_i$ are f_k . Dually, if $g_j: M \rightarrow M_j$ are any homomorphisms, then $\prod_j g_j: M \rightarrow \prod_j M_j$ is the unique homomorphism such that its compositions with the natural projections $p_k: \prod_j M_j \rightarrow M_k$ are g_k .

10.2. THEOREM. If U and W are the functors from (8.11), then

$$L_{q-2}^s \Gamma|_{\mathcal{F}r(R)} \simeq L_{q-1}^s A^2|_{\mathcal{F}r(R)} \simeq L_q^s SP^2|_{\mathcal{F}r(R)} \simeq \begin{cases} 0 & \text{for } q \leq 1, \\ U|_{\mathcal{F}r(R)} & \text{for } q = 2l, l \geq 1, \\ W|_{\mathcal{F}r(R)} & \text{for } q = 2l+1, l \geq 1. \end{cases}$$

We conclude from formula (10.1) that the theorem follows from the following

10.3. PROPOSITION. Let $L, H: \mathcal{M}_R \rightarrow \mathcal{M}_R$ be a covariant functors and let L_2, H_2, L_1, H_1 be the restrictions of the functors L, H to the category $\mathcal{F}r(R)$ and $\mathcal{F}r_1(R)$, respectively. Then

(a) if L_2 and H_2 commute with arbitrary direct sums, then any natural equivalence of functors $t_1: L_1 \rightarrow H_1$ may be uniquely extended to a natural equivalence of functors $t_2: L_2 \rightarrow H_2$;

(b) if L and H are right exact functors, then any natural equivalence of functors $t_2: L_2 \rightarrow H_2$ may be uniquely extended to a natural equivalence of functors $t: L \rightarrow H$.

Proof. First we prove (a). If $F \simeq \bigoplus_{n \in I} Rx_n$ is a free module with free basis $\{x_n\}_{n \in I}$ and $i_n: Rx_n \rightarrow F, p_n: F \rightarrow Rx_n$ denote the natural injections and projections respectively, then we define $t_2(F): L_2(F) \rightarrow H_2(F)$ as the composed homomorphism

$$L_2(F) \xrightarrow{\left(\bigoplus_n L_2(i_n)\right)^{-1}} \bigoplus_n L_2(Rx_n) \xrightarrow{\bigoplus_n t_1(Rx_n)} \bigoplus_n H_2(Rx_n) \xrightarrow{\bigoplus_n H_2(p_n)} H_2(F).$$

If $f: (F, \{x_n\}) \rightarrow (F', \{y_m\})$ is a homomorphism of free modules and n is fixed, then

$$f_{m,n} = p'_m f i_n: Rx_n \rightarrow Ry_m$$

are zero homomorphisms for all but a finite number of m . Hence $\prod_n L_2(f_{m,n}): L_2(Rx_n) \rightarrow \prod_n L_2(Ry_m)$ induces the homomorphism (denoted by the same symbol)

$$\prod_n L_2(f_{m,n}): L_2(Rx_n) \rightarrow \bigoplus_n L_2(Ry_m) \subset \prod_n L_2(Ry_m).$$

Let

$$\varphi = \bigoplus_n \left(\prod_m L_2(f_{m,n}) \right): \bigoplus_n L_2(Rx_n) \rightarrow \bigoplus_n L_2(Ry_m),$$

$$\psi = \bigoplus_n \left(\prod_m H_2(f_{m,n}) \right): \bigoplus_n H_2(Rx_n) \rightarrow \bigoplus_n H_2(Ry_m).$$

Then we have the following diagram

$$\begin{array}{ccccccc}
 L_2(F) & \xleftarrow{\oplus_n L_2(i_n)} & \bigoplus_n L_2(Rx_n) & \xrightarrow{\oplus_n t_1(Rx_n)} & \bigoplus_n H_2(Rx_n) & \xrightarrow{\oplus_n H_2(i_n)} & H_2(F) \\
 \downarrow L_2(f) & & \downarrow \varphi & & \downarrow \psi & & \downarrow H_2(f) \\
 L_2(F') & \xleftarrow{\oplus_m L_2(i'_m)} & \bigoplus_m L_2(Ry_m) & \xrightarrow{\oplus_m t_1(Ry_m)} & \bigoplus_m H_2(Ry_m) & \xrightarrow{\oplus_m H_2(i'_m)} & H_2(F')
 \end{array}$$

Since t_1 is a natural transformation of functors, then the middle square is commutative. Moreover

$$\begin{aligned}
 L_2(f)L_2(i_n) &= L_2\left(\sum_n i'_m f_{m,n}\right) = \sum_m L_2(i'_m)L_2(f_{m,n}) \\
 &= \left(\bigoplus_m L_2(i'_m)\right)\left(\prod_m L_2(f_{m,n})\right) = \left(\bigoplus_m L_2(i'_m)\right)\varphi|_{L_2(Rx_n)}
 \end{aligned}$$

and thus the left square is commutative. Similarly the right square is commutative. Consequently, we get the commutative diagram

$$\begin{array}{ccc}
 L_2(F) & \xrightarrow{L_2(f)} & L_2(F') \\
 \downarrow t_2(F) & & \downarrow t_2(F') \\
 H_2(F) & \xrightarrow{H_2(f)} & H_2(F')
 \end{array}$$

which shows (for $f = 1_F$) that $t_2(F)$ is independent of the choice of a free basis in F , and that t_2 is a natural transformation of functors. Therefore (a) is proved.

Now we prove (b). Let M be an R -module and let

$$(10.4) \quad F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

be an exact sequence, with F_0, F_1 free. By our assumptions we have a commutative diagram

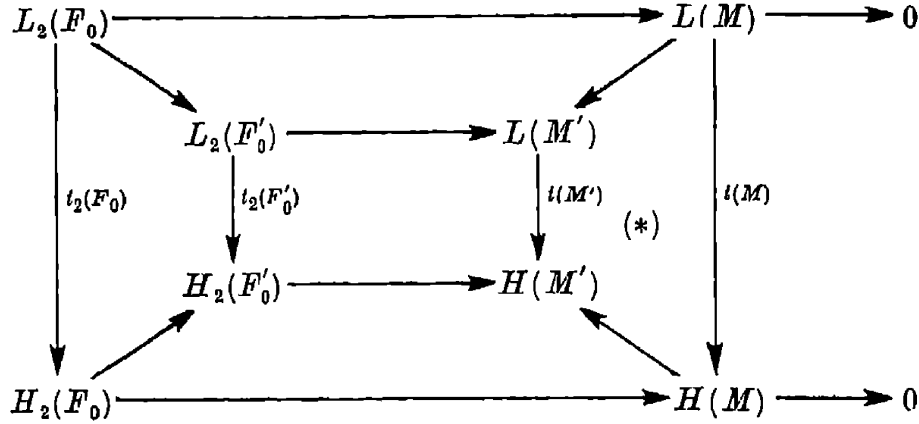
$$\begin{array}{ccccccc}
 L_2(F_1) & \longrightarrow & L_2(F_0) & \longrightarrow & L_2(M) & \longrightarrow & 0 \\
 \downarrow t_2(F_1) & & \downarrow t_2(F_0) & & & & \\
 H_2(F_1) & \longrightarrow & H_2(F_0) & \longrightarrow & H_2(M) & \longrightarrow & 0
 \end{array}$$

with exact rows and isomorphisms $t_2(F_1), t_2(F_0)$. Then there exists a unique isomorphism $t(M): L(M) \rightarrow H(M)$ making the above diagram commutative.

If $f: M \rightarrow M'$ is a homomorphism, then there exists a commutative diagram

$$\begin{array}{ccccccc}
 F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
 F'_0 & \longrightarrow & F'_0 & \longrightarrow & M' & \longrightarrow & 0
 \end{array}$$

with exact rows and F'_0, F'_1 free. Therefore, we have the following figure



We know that all squares except possibly (*) are commutative. Since $L_2(F_0) \rightarrow L(M)$ is an epimorphism, the diagram (*) is also commutative. This shows (for $f = 1_M$) that $t(M)$ does not depend on the choice of the sequence (10.4) and defines a natural equivalence of functors $t: L \rightarrow H$. This completes the proof.

10.5. COROLLARY. (a) $L_2^s SP^2 \simeq L_1^s A^2 \simeq L_0^s \Gamma \simeq U$.

(b) If 2 is a non-zero-divisor in R , then

$$L_3^s SP^2 \simeq L_2^s A^2 \simeq L_1^s \Gamma \simeq L_1 U,$$

where U is the functor from (8.11).

Proof. Since the left stable derived functors of the functors SP^2, A^2, Γ form a connected and exact sequence of functors, then $L_2^s SP^2, L_1^s A^2$ and $L_0^s \Gamma$ are right exact functors by Corollary 9.17 and the fact that $L_i^s \Gamma = 0$ for $i < 0$. Therefore (a) follows from Theorem 10.1 and Proposition 10.3 (b).

Now we prove (b). Let M be a module and let

$$0 \rightarrow K \rightarrow F \xrightarrow{h} M \rightarrow 0$$

be an exact sequence, with F free and $K = \text{Ker } h$. Then by (a) we have a commutative diagram

$$\begin{array}{ccccccc}
 L_3^s SP^2(F) & \longrightarrow & L_3^s SP^2(M) & \longrightarrow & L_2^s SP^2(K) & \longrightarrow & L_2^s SP^2(F) \\
 & & & & \wr & & \wr \\
 L_1 U(F) & \longrightarrow & L_1 U(M) & \longrightarrow & U(K) & \longrightarrow & U(F)
 \end{array}$$

with exact rows and $L_1 U(F) = 0$. Moreover, it follows from Theorem 10.2, Proposition 8.15 and our assumptions that $L_3^s SP^2(F) \simeq \text{Ker}(F \xrightarrow{2} F) = 0$. Therefore there exists a unique isomorphism $L_3^s SP^2(M) \simeq L_1 U(M)$, which completes the above diagram to a commutative diagram, and it is easy to see (see the proof of Proposition 10.3) that this isomorphism defines a natural equivalence of functors $L_3^s SP^2 \simeq L_1 U$. Similarly (b) follows for the functors $L_2^s A^2$ and $L_1^s \Gamma$.

10.6. COROLLARY. *Let R be a field. Then*

$$L_q^s SP^2 = L_q^s A^2 = L_q^s \Gamma = 0 \quad \text{for } q \in \mathbb{Z}$$

if the characteristic of R is different from 2, and

$$L_{q-2}^s \Gamma(M) \simeq L_{q-1}^s A^2(M) \simeq L_q^s SP^2(M) \simeq \begin{cases} 0 & \text{for } q \leq 1, \\ M & \text{for } q > 1, \end{cases}$$

in the opposite case.

Proof. The corollary follows immediately from Theorem 10.2 and the definition of the functors U and W (see Section 8).

Recall that a ring R is a Boolean ring, if $r^2 = r$ for any $r \in R$.

10.7. DEFINITION. Let r be a fixed element of a ring R . Then R is called an r -Boolean ring, iff R/rR is a Boolean ring.

Clearly Boolean rings are r -Boolean rings for any $r \in R$. Examples of 2-Boolean rings are the ring of integers, the ring of p -adic integers and rings such that 2 is invertible. It is clear that the class of n -Boolean rings, for fixed $n \in \mathbb{Z}$, is closed under finite direct sums, arbitrary direct products, homomorphic images and localizations.

Let $\cdot_2, \cdot_2': \mathcal{M}_R \rightarrow \mathcal{M}_R$ be covariant functors defined by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \cdot_2 M & \longrightarrow & M & \xrightarrow{2} & M & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \downarrow \cdot_2 f & & \downarrow f & & \downarrow f & & \downarrow f_2 & & \\ 0 & \longrightarrow & \cdot_2 N & \longrightarrow & N & \xrightarrow{2} & N & \longrightarrow & N_2 & \longrightarrow & 0 \end{array}$$

for any R -homomorphism $f: M \rightarrow N$. If we assume that R is a 2-Boolean ring, then the restricted functors $\cdot_2, \cdot_2': \mathcal{F}_r(R) \rightarrow \mathcal{M}_R$ and the functors $U, W: \mathcal{F}_r(R) \rightarrow \mathcal{M}_R$ from (8.11) coincide respectively. Then Theorem 10.2 yields the following

10.8. COROLLARY. *If R is a commutative 2-Boolean ring, then*

$$\begin{aligned} L_{q-2}^s \Gamma|_{\mathcal{F}_r(R)} &\simeq L_{q-1}^s A^2|_{\mathcal{F}_r(R)} \simeq L_q^s SP^2|_{\mathcal{F}_r(R)} \\ &\simeq \begin{cases} 0 & \text{for } q \leq 1, \\ \cdot_2 & \text{for } q = 2l, l \geq 1, \\ \cdot_2' & \text{for } q = 2l+1, l \geq 1. \end{cases} \end{aligned}$$

10.9. LEMMA. *If R is a 2-Boolean ring, then $U \simeq \cdot \otimes_R R_2$.*

Proof. It is clear that $\cdot \otimes_R R_2 \simeq \cdot_2$ and thus $U|_{\mathcal{F}_r(R)} \simeq \cdot \otimes_R R_2|_{\mathcal{F}_r(R)}$, whenever R is a 2-Boolean ring. Since U and $\cdot \otimes_R R_2$ are right exact functors, the lemma follows from Proposition 10.3(b).

10.10. THEOREM. *If R is a 2-Boolean ring such that 2 is a non-zero-divisor in R , then*

$$L_{q-2}^s \Gamma \simeq L_{q-1}^s \mathcal{A}^2 \simeq L_q^s SP^2 \simeq \begin{cases} 0 & \text{for } q \leq 1, \\ \cdot \otimes_R R_2 & \text{for } q = 2l, l \geq 1, \\ \text{Tor}_1^R(\cdot, R_2) & \text{for } q = 2l+1, l \geq 1. \end{cases}$$

Proof. We prove the theorem by induction on l . We fix our attention on the functor SP^2 . The proof for \mathcal{A}^2 and Γ is similar.

In view of Corollary 9.17, we may assume $q > 1$. Moreover, it follows from Corollary 10.5 that the theorem is true for $l = 1$. Now assume that $L_{2(l-1)}^s SP^2 \simeq \cdot \otimes_R R_2$ and $L_{2l-1}^s SP^2 \simeq \text{Tor}_1^R(\cdot, R_2)$ for some $l \geq 1$. Let M be an R -module and let

$$F_1 \xrightarrow{f} F_0 \rightarrow M \rightarrow 0$$

be an exact sequence with F_0, F_1 free. Then from the exact sequences

$$0 \rightarrow \text{Ker } f \rightarrow F_1 \rightarrow \text{Im } f \rightarrow 0,$$

$$0 \rightarrow \text{Im } f \rightarrow F_0 \rightarrow M \rightarrow 0,$$

we derive the exact sequences

$$L_{2l}^s SP^2(F_1) \rightarrow L_{2l}^s SP^2(\text{Im } f) \rightarrow L_{2l-1}^s SP^2(\text{Ker } f),$$

$$L_{2l}^s SP^2(\text{Im } f) \rightarrow L_{2l}^s SP^2(F_0) \rightarrow L_{2l}^s SP^2(M) \rightarrow L_{2l-1}^s SP^2(\text{Im } f),$$

where the last terms are zero, because $\text{Tor}_1^R(K, R_2) = 0$ whenever K is a submodule of a free module and 2 is a non-zero-divisor in R . Hence we have the commutative diagram

$$\begin{array}{ccccccc} L_{2l}^s SP^2(F_1) & \rightarrow & L_{2l}^s SP^2(F_0) & \rightarrow & L_{2l}^s SP^2(M) & \rightarrow & 0 \\ \wr & & \wr & & & & \\ F_1 \otimes_R R_2 & \rightarrow & F_0 \otimes_R R_2 & \rightarrow & M \otimes_R R_2 & \rightarrow & 0 \end{array}$$

with exact rows and thus the last terms are isomorphic. Moreover, this isomorphism defines a natural equivalence of functors $L_{2l}^s SP^2 \simeq \cdot \otimes_R R_2$. Finally, using the same type of arguments as in the proof of Corollary 10.5, we show that $L_{2l+1}^s SP^2 \simeq \text{Tor}_1^R(\cdot, R_2)$. This completes the proof.

11. Eilenberg-MacLane's stable homology and cohomology functors

Let $\mathcal{A}b$ denotes the category of abelian groups. Eilenberg and MacLane defined in [6] the homology and cohomology functors

$$H_q(\cdot, n, G), H^q(\cdot, n, G): \mathcal{A}b \rightarrow \mathcal{A}b \quad (q \in \mathbb{Z})$$

and natural transformations of functors

$$\begin{aligned}\sigma_{q,n}: H_q(\cdot, n, G) &\rightarrow H_{q+1}(\cdot, n+1, G), \\ \sigma^{q,n}: H^{q+1}(\cdot, n+1, G) &\rightarrow H^q(\cdot, n, G),\end{aligned}$$

for any abelian group G and positive integers q and n .

It permits to define the Eilenberg–MacLane’s stable homology and cohomology functors

$$H_q^s(\cdot, G), H_s^q(\cdot, G): \mathcal{A}b \rightarrow \mathcal{A}b \quad (q \in \mathbb{Z})$$

as follows:

$$\begin{aligned}H_q^s(\cdot, G) &= \varinjlim_n \{H_{q+n}(\cdot, n, G), \sigma_{q+n,n}\} \quad \text{for } q \geq 0, \\ H_s^q(\cdot, G) &= \varprojlim_n \{H^{q+n}(\cdot, n, G), \sigma^{q+n,n}\} \quad \text{for } q \geq 0, \\ H_q^s(\cdot, G) &= H_s^q(\cdot, G) = 0 \quad \text{for } q < 0.\end{aligned}$$

By Theorem 20.4 in [6], $\sigma_{q,n}$ and $\sigma^{q,n}$ are natural equivalences of functors for $2n > q \geq 0$. Then we may identify the functors $H_q^s(\cdot, G)$, $H_s^q(\cdot, G)$ with the functors $H_{q+n}(\cdot, n, G)$, $H^{q+n}(\cdot, n, G)$, respectively, for $n > q \geq 0$.

11.1. PROPOSITION. *For any abelian group G we have*

$$H_q^s(\cdot, G) \simeq L_q^s(\widetilde{SP} \otimes G), \quad H_s^q(\cdot, G) \simeq R_s^q \text{Hom}_{\mathbb{Z}}(\widetilde{SP}, G),$$

where \widetilde{SP} is the functor from (7.3).

Proof. Clearly, we may assume $q \geq 0$. Let π be an abelian group. It follows from [5]; 4.6 and the above remark that for $n > q \geq 0$ we have

$$H_q^s(\pi, G) \simeq H_{q+n}(\pi, n, G) \simeq H_{q+n}(\widetilde{SP}(X) \otimes G),$$

where $X = KS^n P$ and P is a projective resolution of π . But $H_i(SP^0(X) \otimes G) = 0$ for $i > 0$ implies

$$H_q^s(\pi, G) \simeq \bigoplus_{i \geq 0} H_{q+n}(SP^i(X) \otimes G) \simeq L_q^s(\widetilde{SP} \otimes G)(\pi).$$

Similarly, the second equality in the proposition holds.

A consequence of Corollary 7.8 and the above proposition is the following

11.2. THEOREM. *The functors $\{H_q^s(\cdot, G)\}_{q \in \mathbb{Z}}$, $\{H_s^q(\cdot, G)\}_{q \in \mathbb{Z}}$ form connected and exact sequences of covariant and contravariant functors, respectively.*

11.3. COROLLARY. *Let G be an abelian group. Then*

$$\begin{aligned}H_q^s(\cdot, G) &= \text{Tor}_q^{\mathbb{Z}}(\cdot, G) \quad \text{for } q = 0, 1, \\ H_s^q(\cdot, G) &= \text{Ext}_{\mathbb{Z}}^q(\cdot, G) \quad \text{for } q = 0, 1.\end{aligned}$$

Proof. By Theorem 7.21 (c) $L_q^s(SP^n \otimes G) = 0$ for $n > 1$ and $q = 0, 1$. Together with Proposition 11.1 this implies for $q = 0, 1$ the following equalities

$$\begin{aligned} H_q^s(\cdot, G) &\simeq L_q^s(\widetilde{SP} \otimes G) = L_q^s(\bigoplus_{n \geq 1} SP^n \otimes G) \simeq \bigoplus_{n \geq 1} L_q^s(SP^n \otimes G) \\ &\simeq L_q^s(SP^1 \otimes G) \simeq L_1(\cdot \otimes G) = \text{Tor}_1^Z(\cdot, G). \end{aligned}$$

A similar computation proves the second equality.

11.4. COROLLARY. *Let W denotes the additive group of rationals and π be a fixed torsion-free and divisible group. Then*

- (i) $H_q^s(\pi, G) = \text{Tor}_q^Z(\pi, G)$, $H_s^q(\pi, G) = \text{Ext}_Z^q(\pi, G)$ for $q \geq 0$;
- (ii) $H_{q+1}^s(W/Z, G) \simeq H_q^s(Z, G)$ for $q \geq 1$;
- (iii) $H_s^{q+1}(W/Z, G) \simeq H_s^q(Z, G)$ for $q \geq 1$ and G divisible.

Proof. Assertion (i) follows from Propositions 11.1, 7.22 (i) and Corollary 11.3. Concerning assertions (ii) and (iii), the exact sequence

$$0 \rightarrow Z \rightarrow W \rightarrow W/Z \rightarrow 0$$

induces the exact sequences (see Theorem 11.2)

$$\begin{aligned} \dots \rightarrow H_{q+1}^s(W, G) \rightarrow H_{q+1}^s(W/Z, G) \xrightarrow{d_q} H_q^s(Z, G) \rightarrow H_q^s(W, G) \rightarrow \dots, \\ \dots \rightarrow H_s^q(W, G) \rightarrow H_s^q(Z, G) \xrightarrow{d_q} H_s^{q+1}(W/Z, G) \rightarrow H_s^{q+1}(W, G) \rightarrow \dots \end{aligned}$$

The conclusion then follows from (i).

Remark. Applying the results of the last two sections one can easily compute the groups $H_q^s(\pi, G)$ and $H_s^q(\pi, G)$ for $q \leq 4$ (see [1]).

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