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INDEPENDENT FACE AND VERTEX COVERS IN PLANE GRAPHS

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Two new notions of covering defined for plane graphs, a vertex-independent face cover (VIFC) of vertices, and a face-independent vertex cover (FIVC) of faces are studied in this paper. The main result provides necessary and sufficient conditions for a maximal embedded graph to have a FIVC. The proof uses special paths of triangles in such graphs. Moreover, VIFC and FIVC sets are investigated in outerplanar and Halin graphs.

1. Introduction

Let $G = (V, E)$ be a 2-connected planar graph embedded in the plane. A subset $W \subset V$ of vertices is called a *face-independent vertex cover* (simply, *FIVC*) of faces of G if every face of G has exactly one vertex in W. Note that if G is maximal planar then the face-independence of vertices is equivalent to the independence in the usual sense. A FIVC in G corresponds in the geometric dual G^* of G to a set of faces $\mathscr{C}_w$ which we call a *vertex-independent face cover* (simply, *VIFC*) of vertices of G^* since no two faces of $\mathscr{C}_w$ share a vertex and every vertex of G^* is contained in exactly one face of $\mathscr{C}_w$. A VIFC in G is nothing else than a 2-factor of G which consists of facial cycles. Figure 1(a) shows a graph G and its dual together with a FIVC in G and the corresponding VIFC in G^*. Note that G has also a VIFC and G^* has a FIVC, see Fig. 1(b).

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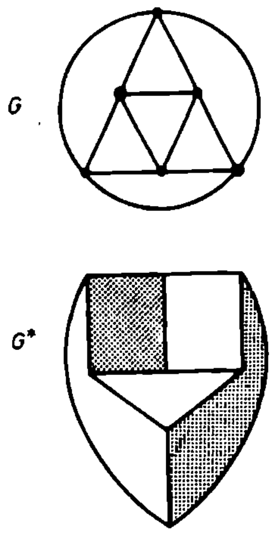

Fig. 1(a)

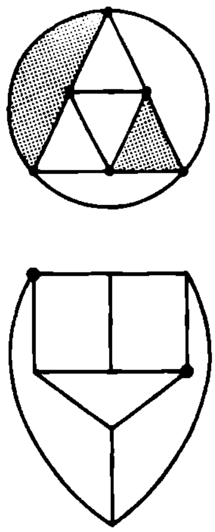
Fig. 1(b)

In contrast to almost all other notions of covering sets (of cycles and/or vertices), a graph may not have a FIVC and/or a VIFC. Figure 2 shows a member G of an infinite class of graphs which have neither FIVC nor VIFC, and whose dual graphs G^* have none either.

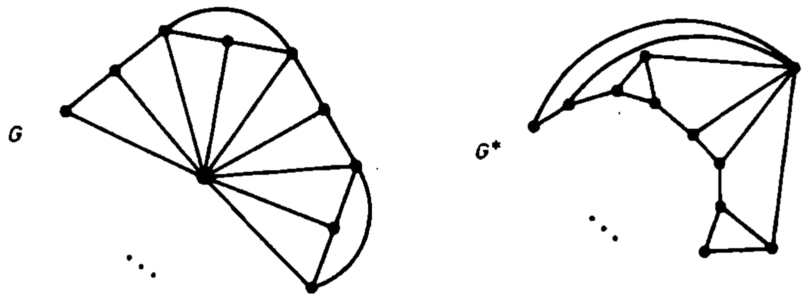

Fig. 2

Covering sets FIVC and VIFC have been originally introduced in [5] in a weaker form. A subset W of vertices in a plane graph is a *weak FIVC* if W covers all faces of G and at most one face of G contains more than one vertex of W. It is easy to see that the graph G in Fig. 2 has a weak FIVC. A weak FIVC in a plane graph G generates a weak VIFC in the dual graph G^* of G. Therefore, a weak FIVC consists of a set of faces of G^* which cover all vertices of G^* and at most one vertex belongs to more than one face chosen. In [5], weak FIVC's (called there just FIVC) were related to other weak VIFC's defined in extended dual graphs. If G is a 2-connected plane graph, then the *extended dual graph* G^+ of G can be obtained from G^* by splitting the vertex v_{ext} of G^* corresponding to the exterior face of G into $\deg(v_{ext})$ copies and drawing a cycle through them. It is clear that a weak FIVC of G corresponds to a set of faces which cover all interior vertices of G^* in G^+. The study in [5] was motivated by the traveling salesman problem on Halin graphs. Every Hamilton cycle in a Halin graph H generates a set of faces $\mathscr{D}$ which cover all interior vertices in H and corresponds to a weak FIVC in the *weak dual graph* $H^- = G^* - \{v_{ext}\}$ of H which is an outerplanar graph.

Yet another generalization of a VIFC known as a *face cover* can be obtained by allowing faces to overlap on vertices. This notion has recently been considered by several authors (see [2] and [3]). Every plane graph has a face cover but it was independently proved in [2] and [3] that the face cover problem is NP-complete in general. Several other versions of this problem are also NP-complete, as is verifying if a plane graph has a VIFC [4].

The main interest of this paper is focused on necessary and/or sufficient conditions for a graph to have a FIVC and/or a VIFC.

2. FIVC in maximal planar graphs

Let $G = (V, E)$ be a maximal plane graph. Hence, its dual graph $G^* = (V^*, E^*)$ is a 3-valent and 3-connected plane graph. If additionally, G is Eulerian (i.e., every vertex is of even degree) then it has a 3-coloring $\{V_1, V_2, V_3\}$ [6], i.e., $V = V_1 \cup V_2 \cup V_3$ and $V_i \cap V_j = \emptyset$, and each color class contains exactly one vertex from each face of G. Therefore, an Eulerian maximal plane graph G has exactly three FIVC of faces, each of which generates in G^* a VIFC of vertices.

THEOREM 1. *Every Eulerian maximal plane graph G has exactly three FIVC of faces which correspond to exactly three VIFC of vertices in its dual graph G^*.*

Note that from the theorem about 1-factors in bipartite regular graphs (see [1], p. 28) it only follows that G^* contains a cover of vertices by vertex disjoint cycles which, however, are not necessarily facial.

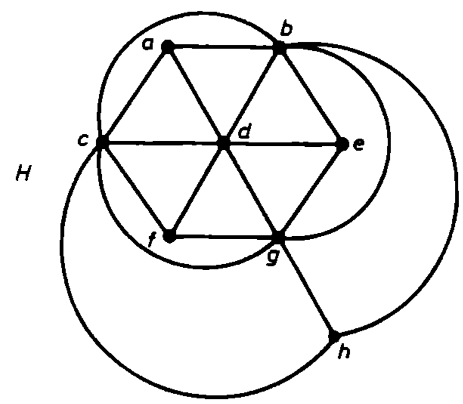

Fig. 3

We now turn our attention to maximal plane graphs with odd degree vertices. Figure 3 shows such a graph H. It has a FIVC of faces $A = \{a, e, f, h\}$ which, however, is not a color class of H in any of its 4-chromatic colorings and no color class of a chromatic coloring of H is its FIVC. Therefore, although a FIVC is a color class of a graph it may not be generated by any chromatic coloring. Note that H of Fig. 3 has no VIFC.

Every vertex together with its neighbors generate a wheel in a maximal plane graph. For an even degree vertex v, there are two ways to cover the faces containing v: either by taking v or by taking every second neighbor of v on the circle. If v is of odd degree, then the latter set is not independent and v must be chosen to every FIVC of faces in G. Hence, all odd degree vertices of G belong to every FIVC of G. Therefore no two such vertices can be adjacent, but this is not sufficient.

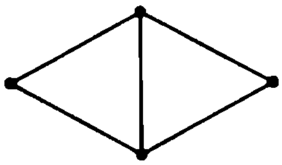

Fig. 4(a). A diamond

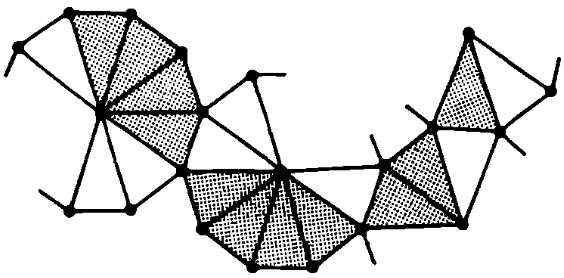

Fig. 4(b). A t-path

A pair of triangles sharing an edge is called a *diamond*, see Fig. 4(a). Here and in what follows a *triangle* in a plane graph is understood as a triangular face. The vertices in a diamond which belong to only one triangle are called *terminal*. It is easy to see that we have

LEMMA 2. *If a maximal plane graph G has a FIVC W then for every diamond H in G, either both terminal vertices of H or none of them belongs to W.*

As a generalization of diamonds, we now introduce t-paths which will then be used to characterize graphs with FIVC's. A *t-path* in a maximal plane graph G is a sequence $p = (T_1, \ldots, T_k)$ of distinct triangles such that:

(i) T_i and T_{i+1} form a diamond in G for $i = 1, 3, \ldots, 2\lfloor k/2 \rfloor - 1$, i.e., T_i shares an edge with T_{i+1}. Let H_i denote the diamond consisting of T_{2i-1} and T_{2i} for $i = 1, 2, \ldots, \lfloor k/2 \rfloor$.

(ii) The terminal vertex of H_1 which belongs to T_1 does not belong to any other triangle of p and the other terminal vertex of H_1 is a terminal vertex of H_2. For every $i = 2, \ldots, \lfloor k/2 \rfloor - 1$, the diamond H_i shares its terminal vertices with those of H_{i-1} and H_{i+1}. For $l = \lfloor k/2 \rfloor$, one terminal vertex of H_l is also a terminal vertex of H_{l-1}. Moreover, if k is odd then the other terminal vertex of H_l belongs to T_k.

(iii) H_i and H_{i+1} may share at most one edge, $i = 1, 2, \ldots, \lfloor k/2 \rfloor - 1$.

A diamond is a t-path and another t-path is shown in Fig. 4(b). Let $p = (T_1, \ldots, T_k)$ be a t-path and let $(H_1, \ldots, H_l)$ be the sequence of the corresponding diamonds. The *length* of p is equal to k. The terminal vertices in H_i's are called *even* with respect to p and the other ones are called *odd*. The terminal vertex of H_1 which belongs to T_1 is called an *initial* vertex of p. If k is even then the terminal vertex of H_l which belongs to T_k is called an *end-vertex* of p, and when k is odd then each of the two vertices of T_k which is not a terminal vertex of H_l is an *end-vertex* of p. For the sake of simplicity, if u is the initial vertex of p and v is an end-vertex of p then we shall say that p *connects* u and v.

LEMMA 3. *Every wheel W in a maximal plane graph G has zero or an even number of triangles in common with a t-path $p = (T_1, \ldots, T_k)$ of G except possibly when W contains T_1 or T_k.*

Proof. Let $p = (T_1, \ldots, T_k)$ be a t-path in a maximal plane graph G. It is evident that any two triangles which share a vertex (or/and an edge) in G are contained in one wheel. Therefore, any two consecutive triangles of p belong to one wheel, in particular, each diamond of p is contained in a wheel and if two consecutive diamonds share an edge they are also contained in one wheel. The wheel which is centered at the initial or at an end-vertex of p may contain an odd number of triangles of p, in particular it may contain only T_1 or T_k. Note, however, that T_1 and T_k may belong to one wheel of G.

A t-path in G can be constructed from a sequence of adjacent wheels (two wheels are *adjacent* if they share exactly one edge). We simply take in each wheel a section consisting of an even number of triangles and such that these sections in two consecutive wheels share exactly one vertex. It is easy to see that the resulting sequence of triangles is a t-path.

It is clear that if the initial vertex v of a t-path p belongs to a FIVC U of a maximal plane graph G then every even vertex of p has to belong to U and no odd vertex of p can be in U. Hence, if v is of odd degree then no odd vertex of p is of odd degree. As a main result of this section we shall prove that the converse also holds.

THEOREM 4. *A maximal plane graph G has a FIVC if and only if no two odd degree vertices in G are connected by an odd-length t-path.*

Proof. If G has no odd-degree vertices, i.e., if G is Eulerian then the result follows by Theorem 1.

Let $G = (V, E)$ be a maximal plane graph with a set $W \subseteq V$ of odd-degree vertices, where $|W| \geqslant 2$. If G has a FIVC U then evidently all vertices of W belong to U, so no two vertices in W can be connected by an odd-length t-path.

Assume now that G has at least two odd-degree vertices such that no two of them are connected by an odd-length t-path. To construct a FIVC U in G we first add to U all elements of W and call them even. Then we show that any other vertex of G, $v \in V-W$, gets either label even or label odd on every t-path from a vertex in W to v. The vertices in W together with those labeled even constitute a FIVC.

Let $w \in W$ and $v \in V-W$. We first take a shortest path q between v and w and show how to transform q into a t-path. Let $q = (w = u_0, u_1, \ldots, u_l = v)$ and we may assume that every vertex u_i $(1 \leqslant i \leqslant l)$ is of even degree. Any two consecutive edges $\{u_i, u_{i+1}\}$ and $\{u_{i+1}, u_{i+2}\}$ $(i = 0, 1, \ldots, l-2)$ of q belong to a wheel: denote it by F_{i+1}. Two consecutive wheels overlap on two triangles.

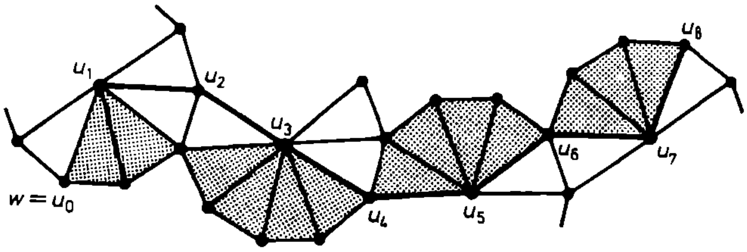

Fig. 5

Figure 5 illustrates a construction of a corresponding t-path q^t for q, depending on the distance between the vertices u_i and u_{i+2} on the circle of the wheel F_{i+1}. We always take an even (or zero) number of triangles from each wheel F_i except possibly the last wheel which contains the last vertex u_l of the path q (see Lemma 3 for the relevant property of t-paths). Using this t-path we determine the parity of the vertex v and label it either even or odd.

To complete the proof we have to show that every $v \in V-W$ gets the same label regardless of the t-path chosen. To this end, we show that there is no triangle in G which contains two vertices that could have been labeled even. Assume that G has a triangle $T = (v_1, v_2, v_3)$ whose two vertices v_1 and v_2 got labeled even. We may assume that all vertices of T are of even degree. Hence, there exist two vertices w_1, w_2 in W and two t-paths q_1 and q_2 from w_1 to v_1

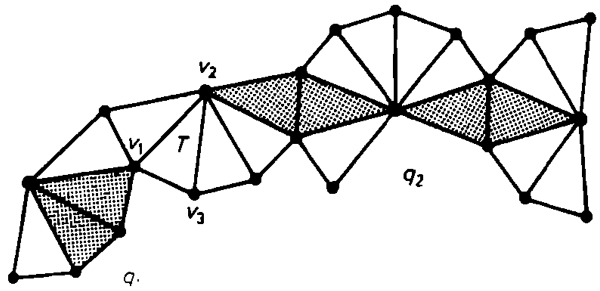

Fig. 6

and from w_2 to v_2, resp., in which v_1 and v_2 are both even. Now we use one of the t-paths, say q_1, to extend the other t-path q_2 so that the resulting t-path r connects w_2 to w_1 and w_1 is odd in r — a contradiction since w_1 and w_2 are of odd degree. To find the t-path r, note first that extending q_2 with the triangle T results in a t-path in which w_1 is odd. Then using consecutive wheels associated with q_1 we can further extend q_2 so that a vertex with even (odd) label in q_1 will get odd (even) label in r (see Fig. 6). Finally, w_1 is reached as an odd-labeled vertex.

COROLLARY 5. *A maximal plane graph G has exactly three or at most one FIVC of faces. The latter occurs when G contains odd degree vertices.*

Transforming the result of Theorem 4 to the dual graphs of maximal plane graphs we obtain

COROLLARY 6. *A 3-connected cubic plane graph has a 2-factor consisting of facial cycles if and only if no two odd length faces are connected by an even sequence of faces.*

3. VIFC and FIVC in special planar graphs

In this section we investigate the existence of VIFC and FIVC sets in special families of planar graphs: outerplanar and Halin graphs.

3.1. Outerplanar graphs

We assume that an outerplanar graph is given in its most natural outerplane embedding.

Weak VIFC. By the definition, the exterior face covers all vertices of an outerplane graph.

Weak FIVC. It was demonstrated in [5] that there exists a one-to-one correspondence between Hamilton cycles in a Halin graph G and weak FIVC's of faces of the corresponding weak dual G^- of G which is an outerplane graph.

VIFC. As in the case of weak VIFC's, the exterior face covers all vertices of an outerplane graph.

FIVC. Since all vertices belong to one face (in an outerplane embedding), a FIVC may consist of only one vertex. Therefore, an outerplane graph has a FIVC if and only if it is an extended fan (see Fig. 7(a)). In general, an outerplanar graph may not have a FIVC for any of its embeddings — Fig. 7(b) shows one such graph. The problem of when an outerplanar graph has an embedding (possibly different from an outerplane one) which admits a FIVC will be treated in a forthcoming paper [6] together with an efficient method for handling all embeddings of an outerplanar graph.

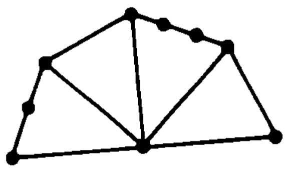

Fig. 7(a). An extended fan

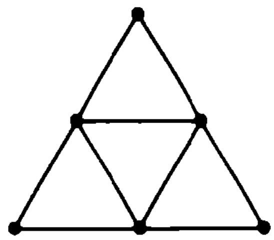

Fig. 7(b)

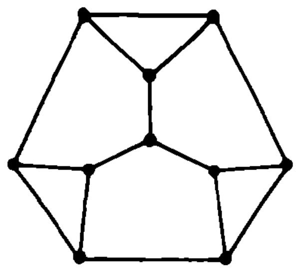

Fig. 7(c)

3.2. Halin graphs

Since every Halin graph is 3-connected, it has a unique set of faces. Therefore, without loss of generality we may assume that a Halin graph G is given with its longest face $C(G)$ as an exterior one.

Weak VIFC. It is easy to see that every Hamilton cycle in a Halin graph G generates the set of interior faces which cover all interior vertices of G (see also [5]). There exists a one-to-one correspondence between weak VIFC's of vertices in a Halin graph and weak FIVC's of faces in the corresponding outerplanar graph.

Weak FIVC. For every Halin graph G, except an odd wheel, we can construct a weak FIVC of all faces of G with more than one vertex on the exterior face. If the exterior cycle $C(G)$ is of even length then we choose every second vertex of $C(G)$. If the cycle $C(G)$ is of odd length then:

a) If G has a fan F with an even number of spokes then we take the center v of F and every second vertex from those which lie on $C(G)$ and do not belong to the same face with v.

b) If every fan of G has an odd number of spokes then we reduce G by contracting one fan F of G to a vertex. Let G' denote the resulting Halin graph and v' the contracted vertex. Note that $C(G')$ is also of odd length but G' may contain a fan with an even number of spokes. If so, we find a weak FIVC U' in G' as described above. Then a weak FIVC U of G is obtained from U' either by taking every second exterior vertex of F starting with the second one if $v' \notin U'$, or by replacing v' in U' by the center of F. If G' has no fan with an even number of spokes, we keep contracting fans until the current graph contains an even fan or is a wheel. In both cases, except an odd wheel, the graph has a weak FIVC which, as above, can be transformed to a weak FIVC in G.

VIFC. Let G be a Halin graph. First note that the exterior face of G does not belong to any VIFC of G since G has interior vertices and any other face shares an edge with the exterior face. Hence, it follows that every triangular face of G belongs to every VIFC and therefore G cannot have two such faces adjacent (i.e., sharing a vertex). Consider two triangular faces consecutive with regard to their order on the exterior face. To cover all exterior vertices laying between them we have to choose to a VIFC every second face located between the triangles, so there must be an odd number of such faces. When this is done for every pair of consecutive triangular faces, we finally check if any two nontriangular faces chosen are vertex disjoint. If so, G has a VIFC just constructed, otherwise it has no such face cover of vertices.

FIVC. Not every Halin graph has a FIVC, e.g. the graph in Fig. 7(c) has no such cover. However, this can be quite easily tested. First, note that if a fan in a Halin graph G has more than three spokes then its interior vertex must belong to every FIVC. Moreover, exactly one face can be covered by its exterior vertex, so in particular at most one triangle is covered by its noninterior vertex. To find if G has a FIVC, we root the interior tree $T(G)$ of G at one of the vertices of a fan F with the maximum number of spokes and attempt to generate a FIVC of G starting with a vertex which covers all faces of F. If F has more than three spokes we have only one choice (the center of F), otherwise each of the three vertices in F must be considered separately. Details of a search along $T(G)$ are left to the reader.

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