POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE
(ROZPRAWY MATEMATYCZNE)

CIX

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On Banach ideals determined by Banach lattices and their applications

WARSZAWA 1973
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

INTRODUCTION*

In the present paper we investigate some operator ideals defined with the aid of a fixed Banach lattice. These operator ideals have their origin in the ideals of radonifying operators of Schwartz and can be considered as generalizations of the duals (or preduals) of these operators.

If X is a fixed Banach lattice and E and F are Banach spaces, then we call a bounded operator $T: E \to F$ *normable* by X, if for any bounded operator $A: F \to X$ AT maps the unit ball of E into an order bounded subset of X. The class of all operators normable by X is an operator ideal, which we call the *Schwartz ideal* determined by X. To understand why we define an operator ideal in this way, one should have the following picture in mind:

Imagine that X is a Banach lattice of some measurable functions on a probability space; call a cylindrical measure ν on E of type X, if there is stochastic function $A: E^* \to X$, determining it. If $T: E \to F$ is an operator so that T^* is normable by X, then given any cylindrical measure ν of type X, i. e. a bounded operator $A: E^* \to X$, we get that AT is the stochastic function determining $T(\nu)$. By assumption AT maps the ball of F^* into an order bounded set, meaning that $T(\nu)$ is a Radon measure.

This picture is a little simplified; in Section 5 we discuss it in detail.

We feel that the study of the present operator ideals is important of various reasons; the present setting enables us for example to make heavy use of the isomorphism theory of Banach spaces in problems concerning radonifying operators. Also we feel that the investigation of these operator ideals can be useful in the Banach space theory itself, f. ex. in problems concerning subspaces of L_p-spaces.

We now wish to indicate in greater detail the arrangement and the results of the present paper.

The Sections 1 and 2 are considered as preliminary sections. In Section 1 we give some definitions, notation and some elementary results

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* The present work in essentially the author's Ph. D- thesis, written at the Institute of Mathematics PAN, Warszaw, Poland, under the supervision of professor A. Pełczyński. A few additional notions and results are included in order to make this paper more self-contained.

concerning Banach space theory and certain operator ideals, defined by some summability properties of the operators in question with respect to unconditional bases. In Section 2 we state some facts on Banach lattices and prove a number of theorems on these, necessary for the study of the operator ideals, defined in the beginning. We end the section by giving an alternative proof of a lattice characterization of L_p-spaces due to Tzafriri; this result is very useful for us in Section 5.

In Section 3 we consider certain families of bounded and absolutely convex subsets of Banach spaces called ideal families, and we show how it is possible to define an operator ideal with the aid of such a family, and we investigate the basic properties of such an ideal. It turns out that if X is a Banach lattice, then the family of all symmetric order intervals in X is an ideal family, and the corresponding ideal is the Schwartz ideal determined by X. Though we are not going to investigate these families in the sequel, we have included this section to indicate, that the way we have constructed the Schwartz ideals above is only a special case of a more general construction of operator ideals, a construction which seems to be interesting to study in detail.

Section 4 is devoted to the detailed study of the Schwartz ideals defined in the beginning. We show for example that if the lattice structure an X is defined by an unconditional basis $\{x_n\}$ in X, then an operator is normable by X, if and only if its adjoint has certain summability properties with respect to $\{x_n\}$. Later we show that under rather mild restrictions on the Banach lattice X an operator is normable by X if and only if the adjoint operator satisfies certain summability conditions with respect to every sequence in X, consisting of mutually lattice disjoint elements of X. This theorem is very important since it reduces the general case to the much simpler case above. It implies f. ex. that an operator is normable by $L_p(\mu)$, $1 \leqslant p < \infty$, iff it has p-absolutely summing adjoint.

We also show by using the "principle of local reflexivity" that in theorems of the nature as the above cited it is possible to interchange the role of the operator and its adjoint. We end the section by discussing order bounded operators from a Banach space E into a Banach lattice X; we prove for example some theorems on compositions of order bounded operators with weakly compact operators; these theorems are all derived from a theorem proved here, which provides a kind of "universal" proof for all theorems of the type "weakly compact operators composed with an operator from a nice operator ideal". (f. ex. weakly compact operators composed with p-integral ones as considered by Persson).

In Section 5 we consider the case, where the Banach lattice in question is a Banach lattice of some measurable functions on a probability space. We show for example a representation theorem for order bounded operators in that case, a theorem which enables us to find the connection between radonifying operators and Schwartz ideals, as we have already mentioned. This connection enables us to formulate the Schwartz duality theorem for general Banach lattices in the language of Schwartz ideals. We end the section by proving that the validity of this theorem characterizes the L_p-spaces, $1 \leqslant p < \infty$, among Banach lattices.

In Section 6 we consider the space $L_p(0, 1)$, $1 < p < \infty$, where the lattice structure is defined by an unconditional basis in the space. Our main theorem says that if $\{x_n\}$ is an unconditional basis in $L_p(0, 1)$, $p \geqslant 2$, then an operator is normable by $\{x_n\}$ if and only if its adjoint is 2-absolutely summing.

For $1 < p < 2$ we have not been able to describe the Schwartz ideal determined by an unconditional basis in $L_p(0, 1)$, but we indicate that the answer is far from being as simple as above. We conclude the section by discussing some open problems, all of which are more or less connected with the one mentioned above.

Section 7 is devoted to some auxiliary results and open problems in the general theory.

Acknowledgement. I would like to thank professor A. Pełczyński for his kind advise, for many stimulating conversations during the preparation of this work and for his arranging of my stay in Poland. Also I will thank dr. S. Kwapień, who originally drew my attention to the present subject, and mgr. P. Wojtaszczyk for their willingness to discuss the subject with me.

Finally I would like to take this opportunity to express my gratitude to professor W. Zawadowski and mgr. H. Karwowski for their extreme helpfulness towards my family and me during our stay in Poland.

Section 1

NOTATION BASIC FACT AND PRELIMINARY RESULTS

In this paper the letter N' will stand for the set of natural numbers, R' the set of reals and C' the set of all complex numbers.

All vector spaces are assumed to be either over R' or C', and when it is of no importance to distinguish we shall simply use the term "*vector space over the scalar field*".

We shall call a function f from a measure space $(\Omega, \mathscr{S}, \mu)$ into a Banach space X *μ-measurable* if it is measurable in the sense of [9].

If Ω is a topological Hausdorff space and $\mu \geqslant 0$ is a finite Borel measure on Ω, then μ is called a *Radon measure*, if $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact}\}$ for all Borel sets $A \subseteq \Omega$. If Ω is a topological Hausdorff space, μ is a Radon on Ω and X is a Banach space, then it follows from the Lusin theorem (for a proof see [14], Theorem 7, page 203), that a function $f\colon \Omega \to X$ is μ-measurable (in our sense), if and only if there is a sequence $\{K_n\}$ of compact sets so that $\mu(K_n) \uparrow \mu(\Omega)$ and f restricted to each K_n is continuous.

The following notation for special Banach spaces is used:

If $1 \leqslant p < \infty$ then l_p denotes the space of all p-summable scalar sequences $\{a_n\}$, equipped with the norm $\|\{a_n\}\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$. l_∞ denotes the space of all bounded scalar sequences equipped with the supremum norm, and if S is a compact Hausdorff space then $C(S)$ denotes the space of all scalarvalued continuous functions on S equipped with the supremum norm. c_0 stands for the space of all sequences of scalars tending to zero, equipped with the supremum norm.

Further if $(\Omega, \mathscr{S}, \mu)$ is a measure space then $L_p(\mu)$, $1 \leqslant p < \infty$, stands for the space of all equivalence classes of scalarvalued μ-measurable functions f for which $\int_\Omega |f|^p d\mu < \infty$ and the space is equipped with the norm $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p}$. $L_\infty(\mu)$ is the space of all equivalence classes of essentially bounded μ-measurable functions, equipped with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{s\in\Omega} |f(s)|$.

Let X and Y be Banach spaces. When we use the term "operator from X to Y" we shall always suppose that the operator in question is linear; however we do not suppose continuity of the operator unless it is explicitly stated. The space of all bounded operators from X to Y is denoted by $B(X, Y)$ ($B(X)$, if $X = Y$).

By an *isomorphism* from X to Y we mean a bounded one-to-one operator from X to Y with closed range. We shall say that X and Y are *isomorphic*, if there is an isomorphism of X onto Y; in this case we define the distance $d(X, Y)$ between X and Y as:

$$d(X, Y) = \inf\{\|T\|\ \|T^{-1}\| \mid T \text{ isomorphism of } X \text{ onto } Y\}.$$

X and Y are called *isometric*, if there is an isomorphism T of X onto Y, so that $\|T\|\ \|T^{-1}\| = 1$.

A sequence $\{x_n\}$ in a Banach space X is called a *basis*, if to every element $x \in X$ there is a unique sequence of scalars $\{t_n\}$, so that

$$x = \sum_{n=1}^{\infty} t_n x_n.$$

A sequence $\{x_n\}$ in X is a *basic sequence*, if it is a basis in its closed linear span $[x_n]$. If $\{x_n\}$ is a basis in X, then the sequence $\{x_n^*\} \subseteq X^*$, defined by $x_n^*(x_k) = \delta_{kn}$ is called the *sequence biorthogonal* to $\{x_n\}$.

A sequence $\{x_n\}$ in X is an *unconditional basic sequence*, if it is a basic sequence and every expansion in it converges unconditionally.

If $\{x_n\}$ is a basis in X, then a sequence $\{y_n\} \subseteq X$ is called a *block basic sequence* with respect to $\{x_n\}$, if there exists a sequence $\{t_n\}$ of scalars and a sequence $\{p_n\}$ of natural numbers $p_n \leqslant p_{n+1}$, $n = 1, 2, \ldots, p_1 = 0$, so that $y_n = \sum_{k=p_n+1}^{p_{n+1}} t_k x_k$ for all $n \in N'$

The following proposition on basic sequences is due to Pełczyński and Bessaga [4].

1.1. PROPOSITION. *A sequence $\{x_n\}$ in a Banach space X is a basic sequence, if and only if there is a constant $K \geqslant 1$ so that for every pair (p, q) of natural numbers, $p \leqslant q$, and every q-tuple $(t_1, \ldots, t_q)$ of scalars the following inequality is satisfied*:

$$\Big\|\sum_{n=1}^{p} t_n x_n\Big\| \leqslant K \Big\|\sum_{n=1}^{q} t_n x_n\Big\|. \tag{1}$$

A sequence $\{x_n\} \subseteq X$ is an unconditional basic sequence if and only if there is a constant $K \geqslant 1$, so that for every pair (p, q) of natural numbers, $p \leqslant q$, and q-tuples $(t_1, \ldots, t_q)$, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q)$, $|\varepsilon_i| = 1$, $i = 1, 2, \ldots, q$

the inequality

$$\left\| \sum_{n=1}^{p} \varepsilon_n t_n x_n \right\| \leqslant K \left\| \sum_{n=1}^{q} t_n x_n \right\| \quad \textit{is valid.} \tag{2}$$

The infimum of all constants, which can be used in inequality (1) is called the *basis constant* of $\{x_n\}$, while the smallest possible constant, which can be used in (2) is called the *unconditional constant* of $\{x_n\}$.

A basis $\{x_n\}$ in X is called *boundedly complete*, if for every sequence $\{t_n\}$ of scalars $\sum_{n=1}^{\infty} t_n x_n$ is convergent whenever the set $\left\{\left\| \sum_{n=1}^{k} t_n x_n \right\| \,\middle|\, k \in N'\right\}$ is bounded. $\{x_n\}$ is said to be a *shrinking basis* if the sequence $\{x_n^*\} \subseteq X^*$, biorthogonal to $\{x_n\}$ is a basis of X^*.

The following proposition is due to James (for a proof see [6])

1.2. Proposition. *Let $\{x_n\}$ be an unconditional basis in X. $\{x_n\}$ is boundedly complete if and only if X contains no subspace isomorphic to c_0.*

$\{x_n\}$ is shrinking if and only if $\{x_n\}$ contains no subspace isomorphic to l_1.

If $\{x_n\}$ and $\{y_n\}$ are bases in X and Y respectively, then $\{x_n\}$ is equivalent to $\{y_n\}$ if the sequence $\sum_{n=1}^{\infty} t_n x_n$ converges in X if and only if $\sum_{n=1}^{\infty} t_n y_n$ converges in Y.

Finally a seminormalized basis $\{x_n\}$ is a basis so that $0 < \inf_n \|x_n\| \leqslant \sup \|x_n\| < \infty$. $\{x_n\}$ is a normalized basis if $\|x_n\| = 1$ for all n.

1.3. Definition. We shall say that a *Banach operator ideal* $\mathscr{A}$ is given, if for each pair X and Y of Banach spaces there is a linear subspace $\mathscr{A}(X, Y)$ of $B(X, Y)$ and a norm $a_{X,Y}$ on $\mathscr{A}(X, Y)$ turning it into a Banach space and so that the following conditions are satisfied:

(i) For every $T \in \mathscr{A}(X, Y)$, $\|T\| \leqslant a_{X,Y}(T)$.

(ii) If E, F, X and Y are Banach spaces and $S \in B(E, X)$, $T \in \mathscr{A}(X, Y)$ and $V \in B(Y, F)$, then $VTS \in \mathscr{A}(E, F)$ and

$$a_{E,F}(VTS) \leqslant \|S\| \; \|V\| a_{X,Y}(T).$$

(iii) If $T \in B(X, Y)$ is finite dimensional then $T \in \mathscr{A}(E, F)$.

Let now X be a Banach space with an unconditional basis $\{x_n\}$. We are now going to define a Banach operator ideal with the aid of this basis.

1.4. Definition. Let E and F be Banach spaces. An operator $T \in B(E, F)$ is called *$\{x_n\}$-absolutely summing*, if for all sequences $\{y_n\} \subseteq E$ so that $\sum_{n=1}^{\infty} y^*(y_n) x_n$ is convergent for all $y^* \in E^*$ we have that $\sum_{n=1}^{\infty} \|Ty_n\| x_n$ is convergent.

If E and F are dual spaces, say $E = Z_1^*$ and $F = Z_2^*$, then T is called *ω^*-$\{x_n\}$-absolutely summing* if for all sequences $\{y_n^*\} \subseteq Z_1^*$ so that $\sum_{n=1}^{\infty} y_n^*(y) x_n$ is convergent for all $y \in Z_1$ we have that $\sum_{n=1}^{\infty} \|Ty_n^*\| x_n$ is convergent.

The following propositions are very easy and we omit the proofs.

1.5. Proposition. *Let E and F be Banach spaces and $T \in B(E, F)$ an $\{x_n\}$-absolutely summing operator. Then there exists a constant $K \geqslant 0$, so that for all sequences $\{y_n\} \subseteq E$ with $\sum_{n=1}^{\infty} y^*(y_n) x_n$ convergent for all $y^* \in F^*$ the following inequality holds:*

$$\left\| \sum_{n=1}^{\infty} \|Ty_n\| x_n \right\| \leqslant K \sup_{\|y^*\| \leqslant 1} \left\| \sum_{n=1}^{\infty} y^*(y_n) x_n \right\|. \tag{1}$$

If $\{x_n\}$ is boundedly complete and $T \in B(E, F)$ then T is $\{x_n\}$-absolutely summing if and only if there is a constant $K \geqslant 0$, so that for any finite set $\{y_1, y_2, \ldots, y_k\} \subseteq E$ the following inequality holds

$$\left\| \sum_{n=1}^{k} \|Ty_n\| x_n \right\| \leqslant K \sup_{\|y^*\| \leqslant 1} \left\| \sum_{n=1}^{k} y^*(y_n) x_n \right\|. \tag{2}$$

Similar statements hold for ω^*-$\{x_n\}$-absolutely summing operators.

Let us denote the set of all $\{x_n\}$-absolutely summing operators from E to F by $\Pi_{\{x_n\}}(E, F)$. $\Pi_{\{x_n\}}(E, F)$ is readily seen to be a vector space and if we define the function $\pi_{\{x_n\}}\colon \Pi_{\{x_n\}}(E, F) \to R'_+ \cup \{0\}$ by

$$\pi_{\{x_n\}}(T) = \inf\{K \mid K \text{ satisfies (1)}\} \text{ for } T \in \Pi_{\{x_n\}}(E, F)$$

then $\pi_{\{x_n\}}$ is easily seen to be a norm on $\Pi_{\{x_n\}}(E, F)$.

In a similar manner we define the space $\Pi^{\omega^*}_{\{x_n\}}(E, F)$ and the norm $\pi^{\omega^*}_{\{x_n\}}$ for dual spaces E and F.

1.6. Proposition.

(i) *The class $\Pi_{\{x_n\}}$ of all $\{x_n\}$-absolutely summing operators is a Banach operator ideal.*

(ii) *For every pair E and F of dual Banach spaces $\Pi^{\omega^*}_{\{x_n\}}(E, F)$ is a Banach space under the norm $\pi^{\omega^*}_{\{x_n\}}$.*

(iii) *If $\{x_n\}$ is boundedly complete then $\Pi_{\{x_n\}}(E, F) = \Pi^{\omega^*}_{\{x_n\}}(E, F)$ for any pair E and F of dual Banach spaces, and $\pi_{\{x_n\}} \leqslant \pi^{\omega^*}_{\{x_n\}} \leqslant K\pi_{\{x_n\}}$ where K is the unconditional constant for $\{x_n\}$.*

In case $\{x_n\}$ is the unit vector basis of l_p $1 \leqslant p < \infty$, we shall use the term "p-absolutely summing operator" instead of "$\{x_n\}$-absolutely summing operator". These operators will play an important role in the present paper and for a detailed study of them we refer to Pietsch [25] and Persson and Pietsch [24].

Section 2

SOME RESULTS ON BANACH LATTICES

We assume that the reader is familiar with the notion of a vector lattice, as it appears in f. ex. ([29], Chapter V). (In this case the underlying vector space is of course assumed to be over the reals.)

If X is a vector lattice under the partial ordering $\leqslant$, and x and y are elements of X, then we put $x \vee y = \sup(x, y)$ and $x \wedge y = \inf(x, y)$. If $x \leqslant y$, then the order interval $[x, y]$ is set $\{z \in X \mid x \leqslant z \leqslant y\}$; an order interval of the form $[-x, x]$, $x \in X$ and $x \geqslant 0$ is called a symmetric order interval.

A set $A \subseteq X$ is said to be *order bounded*, if it is contained in some order interval. The vector lattice X is called *order complete*, if every order bounded set has a lowest upper bound (sup) and a greatest lower bound (inf).

If $x \in X$, then we define $x^+ = x \vee 0$, the positive part of x, and $x^- = -(x \wedge 0)$, the negative part of x. Clearly $x = x^+ - x^-$; the element $|x| = x^+ + x^-$ is called the absolute value of x.

A number of relations between the algebraic structure and the order structure in a vector lattice are valid; for details we refer to [22] and [29].

We define a Banach lattice X to be a vector lattice with a complete norm $\|\cdot\|$, so that there is a constant $K \geqslant 1$ so that

$$(*) \qquad |x| \leqslant |y| \Rightarrow \|x\| \leqslant K\|y\|, \qquad x, y \in X.$$

The smallest constant, which can be used in $(*)$ is called the *lattice constant* with respect to $\|\cdot\|$. If the lattice constant is 1, then the norm is called a *lattice norm*. In any Banach lattice X it is possible to find a lattice norm, which is equivalent to the original norm. Indeed, define a new norm on X, by

$$|||x||| = \sup_{|y| \leqslant |x|} \|y\|, \qquad x \in X$$

clearly $|||\cdot|||$ is a lattice norm and it satisfies

$$\|x\| \leqslant |||x||| \leqslant K\|x\|, \qquad x \in X.$$

where K is the lattice constant relative to $\|\cdot\|$.

In the sequel, when we use the term "*Banach lattice* X", we shall assume that the given norm in X is a lattice norm. We do that with *one exception*, namely the one we explain now:

If X is a Banach space over the reals with an unconditional basis $\{x_n\}$ and biorthogonal sequence $\{x_n^*\}$, then X is a Banach lattice under the partial ordering $x \leqslant y$ if $x_n^*(x) \leqslant x_n^*(y)$ for all $n \in N'$.

It is easy to see that the lattice constant in this case is the same as the unconditional constant of the basis. When a Banach lattice has been defined in such a way, we find it more convenient to keep the original norm instead of going to the lattice norm.

A directed subset A of a Banach lattice X is a set with the property that if x and y are in A, then there is a $z \in A$ with $x \leqslant z$ and $y \leqslant z$. A directed set A can be considered as a net (generalized sequence) with the identity in A as the indexing map, therefore when we say that a directed set in a Banach lattice is convergent, we simply mean that it converges, considered as a net.

2.1. DEFINITION. A Banach lattice is called *boundedly complete*, if every norm bounded directed set is convergent.

We recall that a subspace M of a vector lattice X is called a *sublattice*, if $x, y \in M$ implies $x \vee y \in M$. It is easy to see that a Banach lattice always canonically can be imbedded into its double dual as a sublattice.

The following proposition gives a characterization of boundedly complete Banach lattices. The equivalence (a) $\Leftrightarrow$ (b) is trivial and that (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) follows from Tzafriri ([36], Theorem 14).

2.2. PROPOSITION. *If X is a Banach lattice then the following statements are equivalent*:

(a) *X is boundedly complete.*

(b) *Every norm bounded increasing sequence is convergent.*

(c) *No subspace of X is isomorphic to c_0.*

(d) *For every directed subset $A \subseteq X$, which is majorized in X^{**} we have* $\sup_{X^{**}} A \in X$ (*hence* $\sup_X A$ *exists and* $\sup_{X^{**}} A = \sup_X A$).

Remark. It follows from the above proposition that all reflexive lattice and all abstract L-spaces are boundedly complete Banach lattices. Further if X is a Banach space with a boundedly complete unconditional basis $\{x_n\}$, then X is boundedly complete considered as a lattice under the ordering induced by $\{x_n\}$.

In the rest of this section let X denote a fixed Banach lattice.

Another notion on Banach lattices we are going to use often in the sequel is the following:

2.3. DEFINITION. X is called *of minimal type*, if every directed subset of X, which is majorized, converges.

A number of theorems on Banach lattices of minimal type may be found in ([29], Chapter V). However the following proposition and its corollary, which we are going to use frequently in the sequel seem to be unknown.

2.4. PROPOSITION. *Let X be of minimal type and let K be a compact Hausdorff space. Then every positive operator from $C(K)$ into X is weakly compact.*

Proof. If $T\colon C(K) \to X$ is a positive operator and $\{f_n\} \subseteq C(K)$, $f_n \leqslant f_{n+1} \leqslant 1$ for all n, then it follows from the minimality of X, that Tf_n is convergent. From Grothendieck [12], Theorem 6 it now follows that T is weakly compact. ■

2.5. COROLLARY. *X is of minimal type, if and only if every order interval in X is weakly compact.*

Proof. Suppose first that X is of minimal type and let $x \in X$, $x \geqslant 0$. By the Kakutani representation theorem on abstract M-spaces [16], there is a compact Hausdorff space K and a positive operator $T\colon C(K) \to X$ mapping the unit ball of $C(K)$ onto $[-x, x]$. From Proposition 2.4 it now follows that $[-x, x]$ is weakly compact.

Next suppose that every order interval is weakly compact and let $A \subseteq X$ be a directed set, which is majorized. Let $x_0 \in A$ and consider the set

$$A_{x_0} = \{x \in A \mid x_0 \leqslant x\}.$$

If we can prove that A_{x_0} is convergent then also A must converge.

Now for every $x^* \in X^*$, $x^* \geqslant 0$ we get that $x^*(A_{x_0})$ is convergent and hence A_{x_0} is a weak Cauchy net in X; since it is contained in a weakly compact set, it is weakly convergent to x, say. From the theorem of Dini ([29], Chapter V, § 4.3) we now get that A_{x_0} is convergent to x in norm. ■

Remark. Clearly all boundedly complete Banach lattices are of minimal type, and so are all Banach lattices where the order is defined by an unconditional basis. An abstract M-space is not of minimal type unless it is finite-dimensional.

We recall that two elements x and y in X are called disjoint ($x \perp y$), if $|x| \wedge |y| = 0$. If $A \subseteq X$ then $A^\perp$ denotes the set of all those $y \in X$ for which $y \perp x$ for all $x \in A$.

2.6. PROPOSITION. *If $\{x_n\} \subseteq X$ is a sequence of mutually disjoint element then $\{x_n\}$ is an unconditional basic sequence in X with unconditional constant* 1.

Proof. Let p and q be natural numbers with $p \leqslant q$ and let $(t_1, t_2, \ldots, t_q) \in R^q$, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q) \in R^q$ with $|\varepsilon_i| = 1$, $i = 1, 2, \ldots, q$.

Then we have:

$$\Big|\sum_{n=1}^{p} \varepsilon_n t_n x_n\Big| = \sum_{n=1}^{p} |\varepsilon_n|\, |t_n|\, |x_n| \leqslant \sum_{n=1}^{q} |t_n|\, |x_n| = \Big|\sum_{n=1}^{q} t_n x_n\Big|$$

and hence

$$\Big\| \sum_{n=1}^{p} \varepsilon_n t_n x_n \Big\| \leqslant \Big\| \sum_{n=1}^{q} t_n x_n \Big\|. \quad ■$$

Remark. The proof of Proposition 2.6 also shows that if $\{x_n\} \subseteq X$ is a sequence of mutually disjoint elements then the sequence $\{|x_n|\}$ and $\{x_n\}$ are isometrically equivalent basic sequences.

The next proposition is very usefull for us in the sequel

2.7. PROPOSITION. *Let X be of minimal type and let $\{x_n\}$, $x_n \geqslant 0$ $n = 1, 2, \ldots$, be a sequence of mutually disjoint elements of X. Denote $\overline{\operatorname{span}}\{x_n\} = [x_n]$.*

Then the order in $[x_n]$, defined by the unconditional basis $\{x_n\}$ agrees with the order induced by X.

Furthermore, if $A \subseteq [x_n]$ and A is order bounded in X, then A is order bounded in $[x_n]$ as well and $\sup_X A = \sup_{[x_n]} A$.

Proof. Let $\{x_n^*\} \subseteq [x_n]^*$ be the sequence biorthogonal to $\{x_n\}$.

Since X is of minimal type X is also order complete, hence for each $n \in N$ there exists a bounded projection $P_n\colon X \to x_n^{\perp\perp}$ so that

$$0 \leqslant P_n x \leqslant x \text{ for all } x \in X, \qquad x \geqslant 0$$

(see [29], Chapter V, § 7.3).

If now $x \in [x_n]$, $x \geqslant 0$, then $P_n x \geqslant 0$ for all $n \in N$ but since the x_n's are mutually disjoint we get $P_n x = x_n^*(x) x_n$ and hence $x_n^*(x) \geqslant 0$.

Since clearly $x_n^*(x) \geqslant 0$ for all n implies $x \geqslant 0$ we have proved that the two orders agree on $[x_n]$.

To prove the second assertion let first x and y be is X; we wish to prove that $x \vee y \in [x_n]$. Clearly it is no restriction to assume that $x \geqslant 0$ and $y \geqslant 0$. Put $z = \sup_X(x, y)$. Then for each $n \in N'$:

$$x_n^*(x) x_n \leqslant x \leqslant z, \qquad x_x^*(y) x_n \leqslant y \leqslant z$$

and hence

$$\max\big(x_n^*(x), x_n(y)\big) x_n \leqslant z \qquad \text{for all } n \in N'$$

and therefore since the x_n's are mutually disjoint.

$$\sup_{[x_n]}(x, y) = \sum_{n=1}^{\infty} \max\big(x_n(x), x_n(y)\big) x_n \leqslant z$$

from which we conclude that $\sup_{[x_n]}(x, y) = \sup_X(x, y)$.

Using the above we easily get that the statement holds for a finite set of elements; therefore to prove it for an arbitrary order bounded (in X) subset of $[x_n]$ it is enough to show that if $A \subseteq [x_n]$ is a directed set, which is majorized in X, then $\sup_X A \in [x_n]$, but this follows trivially from the minimality of X and the fact that $[x_n]$ is closed in X. ■

2.8. DEFINITION. X is said *to have sufficiently many disjoint elements* if for any finite dimensional subspace $E \subseteq X$ and any $\varepsilon > 0$ there is an operator $T \in B(E, X)$ so that

(i) $\|x - Tx\| \leqslant \varepsilon \|x\|$ for all $x \in E$.

(ii) T has a representation of the form:

$$Tx = \sum_{n=1}^{k} x_x^*(x) x_n, \qquad x \in E$$

and where $\{x_n^*\}_{n=1}^k \subseteq E^*$ and $\{x_n\}_{n=1}^k \subseteq X$, $x_n \perp x_m$ for $n \neq m$.

2.9. PROPOSITION. *If X is order complete then X has sufficiently many disjoint elements.*

Proof. Let $E \subseteq X$ be a finite dimensional subspace of X and let $\varepsilon > 0$ be given. We can then choose $x_0 \in X$ with $x_0 \geqslant 0$, $\|x_0\| = 1$ so that

$$E \subseteq \operatorname{span}[-x_0, x_0]. \tag{1}$$

Let us denote $\operatorname{span}[-x_0, x_0]$ by $X_{[-x_0, x_0]}$ when it is equipped with the norm having $[-x_0, x_0]$ as unit ball and let $I\colon X_{[-x_0,x_0]} \to X$ be the formal identity map.

When equipped with the order induced by X, $X_{[-x_0,x_0]}$ is readily seen to be an abstract M-space, hence there is a compact Hausdorff space K and a lattice isometry U of $C(K)$ onto $X_{[-x_0,x_0]}$. It is easily seen that the order completness of X implies that of $X_{[-x_0,x_0]}$ and therefore K is stonian (see f. ex. [2]).

If $\mathscr{A}$ denotes the family of all indicator functions for clopen sets in K, then from the above we infer that

$$\overline{\operatorname{span}\mathscr{A}} = C(K). \tag{2}$$

It is readily checked that if $A, B \in \mathscr{A}$ with $A \cap B = \emptyset$ then

$$I \cup (1_A) \wedge IU(1_B) = \emptyset. \tag{3}$$

From (1) and (2) we get that there exists a finite dimensional operator $S\colon E \to C(K)$ with a representation of the form

$$Sx = \sum_{i=1}^{n} x_i^*(x) 1_{A_i}, \qquad x \in E \tag{4}$$

where $\{x_1^*, x_2^*, \ldots, x_n^*\} \subseteq E^*$ and $\{A_i \mid i = 1, 2, \ldots, n\} \subseteq \mathscr{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ and so that

$$|||Sx - U^{-1}I^{-1}(x)||| \leqslant \varepsilon \|x\|, \; x \in E \tag{5}$$

($|||\cdot|||$ denoting the sup-norm in $C(K)$).

If $T\colon E\to X$ is defined by

(6) $$Tx = \sum_{i=1}^{n} x_i^*(x) IU(1_{A_i}), \qquad x\in E$$

then T is of the required from and

(7) $$\|Tx-x\| = \|IU(U^{-1}I^{-1}x-Sx)\| \leqslant \|I\|\ \|U\|\varepsilon\|x\| \leqslant \varepsilon\|x\|$$

for $x\in E$. ■

The final theorem we are going to prove in the present section gives a lattice characterization of the spaces $L_p(\mu)$ $1\leqslant p<\infty$ and $c_0(\Gamma)$. The result is due to Tzafriri [35], who proved it with the aid of boolean algebras of projections. However, we would like here to give an alternative proof, which seems to be a little shorter and more direct.

2.10. THEOREM. *Let X be a Banach lattice of minimal type, so that there is a p, $1\leqslant p<\infty$ with the property that every normalized sequence consisting of mutually disjoint elements is equivalent to the unit vector basis of l_p. Then there is a measure space $(\Omega,\mathscr{S},\mu)$, so that X is lattice isomorphic to $L_p(\mu)$.*

If every normalized sequence consisting of mutually disjoint elements is equivalent to the unit vector basis of c_0, then there is a set Γ, so that X is lattice isomorphic to $c_0(\Gamma)$.

Before we can prove the theorem, we need the following lemma, which for the case $1\leqslant p<\infty$ is due to Tzafriri [35].

2.11. LEMMA. *Let X be a Banach lattice satisfying the conditions in the theorem. Then there are positive constants K_1 and K_2 so that for any finite or infinite sequence $\{x_n\}$ of mutually disjoint elements we have*

$$K_1\Big\|\sum_n x_n\Big\| \leqslant \Big(\sum_n \|x_n\|^p\Big)^{1/p} \leqslant K_2\Big\|\sum_n x_n\Big\|$$

$$\Big(K_1\Big\|\sum_n x_n\Big\| \leqslant \sup_n \|x_n\| \leqslant K_2\Big\|\sum_n x_n\Big\| \qquad \text{if } p=\infty\Big).$$

Proof. The right inequality can for $1\leqslant p<\infty$ be proved as by Tzafriri [35]; for $p=\infty$ it is trivial.

To prove the left inequality let $\mathscr{F}$ denote the class of all finite sets of normalized, mutually disjoint elements of X, each $F\in\mathscr{F}$ being indexed in some way. If $x\in X$ we let P_x denote the band projection onto the band generated by x (see [29], Chapter V). For $F\in\mathscr{F}$, $F=\{x_1,x_2,\ldots,x_n\}$ let $T_F\colon l_p\to \operatorname{span} F$ be defined by:

$$T_F a = \sum_{i=1}^{n} a(i)x_i, \qquad a\in l_p.$$

If we can prove that $\sup_{F\in\mathscr{F}} \|T_F\| < \infty$ we are done; hence suppose that this is not the case. By induction we can then construct an increasing sequence $\{n_k\}$, $n_0 = 0$, of natural numbers, a sequence $\{a_n\}$ of reals and a sequence $\{x_n\} \subseteq X$ of mutually disjoint normalized elements so that:

(i) $$\Big\| \sum_{n=n_k+1}^{n_{k+1}} a_n x_n \Big\| \geqslant 2^{k+1}, \qquad k = 0, 1, 2, \ldots$$

(ii) $$\|\{a_n\}_{n=n_k+1}^{n_{k+1}}\|_p \leqslant 1, \qquad k = 0, 1, 2, \ldots$$

(iii) $$\sup_{F\in\mathscr{F}} \Big\|\Big(I - \sum_{n=1}^{n_k} P_{x_n}\Big) T_F\Big\| = \infty, \qquad k = 1, 2, \ldots$$

Suppose $n_1, \ldots, n_k, x_1, \ldots, x_{n_k}$ and $a_1, \ldots, a_{n_k}$, have been defined to satisfy (i), (ii) and (iii) and put $Q_{n_k} = I - \sum_{n=1}^{n_k} P_{x_n}$. By (iii) we can then find an $F_1 \in \mathscr{F}$ so that $\|Q_{n_k} T_{F_1}\| \geqslant 2^{k+1} + 1$.

Since $\|Q_{n_k} T_F\| \leqslant \sum_{x\in F_1} \|P_x Q_{n_k} T_F\| + \Big\|\Big(I - \sum_{x\in F_1} P_x\Big) Q_{n_k} T_F\Big\|$ for $F \in \mathscr{F}$, we get by (iii) that there is an $x_0 \in F_1$, so that

(1) $$\sup_{F\in\mathscr{F}} \Big\|\Big(I - \sum_{x\in F_1\setminus\{x_0\}} P_x\Big) Q_{n_k} T_F\Big\| = \infty$$

(if $\sup_{F\in\mathscr{F}} \Big\|\Big(I - \sum_{x\in F_1} P_x\Big) Q_{n_k} T_F\Big\| = \infty$, then we just choose x_0 arbitrarily in F_1). Put $F_2 = F_1 \setminus \{x_0\}$; clearly

(2) $$\|Q_{n_k} T_{F_2}\| \geqslant 2^{k+1}.$$

Let $F_2 = \{y_{n_k+1}, y_{n_k+2}, \ldots, y_{n_{k+1}}\}$ and define

$$x_n = \frac{Q_{n_k} y_n}{\|Q_{n_k} y_n\|}, \qquad n = n_k + 1, n_k + 2, \ldots, n_{k+1}.$$

Clearly these elements are mutually disjoint and contained in $\{x_1, \ldots, x_{n_k}\}^{\perp}$.

By (2) there is a set $\{b_n\}_{n=n_k+1}^{n_{k+1}} \subseteq R'$, so that

(3) $$2^{k+1} \leqslant \Big\| \sum_{n=n_k+1}^{n_{k+1}} b_n Q_{n_k} y_n \Big\| = \sum_{n=n_k+1}^{n_{k+1}} b_n \|Q_{n_k} y_n\| x_n,$$

(4) $$\|\{b_n\}_{n=n_k+1}^{n_{k+1}}\|_p \leqslant 1.$$

Define $a_n = b_n \|Q_{n_k} y_n\|$ $n = n_k + 1, \ldots, n_{k+1}$. Since $\|Q_{n_k} y_n\| \leqslant 1$ we get that

(5) $$\|\{a_n\}_{n=n_k+1}^{n_{k+1}}\|_p \leqslant 1.$$

Further from (1) we get

$$\sup_{F\in\mathscr{F}}\Big\|\Big(I-\sum_{n=1}^{n_{k+1}}P_{x_n}\Big)T_F\Big\| = \sup_{F\in\mathscr{F}}\Big\|\Big(Q_{n_k}-\sum_{n=n_k+1}^{n_{k+1}}P_{x_n}\Big)T_F\Big\|$$

$$\sup_{F\in\mathscr{F}}\Big\|\Big(Q_{n_k}-Q_{n_k}\sum_{n=n_k+1}^{n_{k+1}}P_{y_n}\Big)T_F\Big\| = \infty.$$

This completes the induction.

Define now

$$c_n = a_n\cdot 2^{-k-1} \quad \text{for } n_k < n \leqslant n_{k+1},\ k = 1, 2, \ldots$$

From (ii) it follows that $\{c_n\}\in l_p(\{c_n\}\in c_0)$ on the other hand from (i) it follows that $\sum_{n=1}^{\infty} c_n x_n$ is divergent, thus contradicting the assumption of the lemma.

Proof of Theorem 2.10.

Case 1. Assume that X has a Freudenthal unit x_0, $\|x_0\| = 1$ $x_0 \geqslant 0$ (i. e. $X = \overline{\mathrm{span}}[0, x_0]$, or equivalently, since X is of minimal type, x_0 has the property $|x| \wedge x_0 = 0 \Rightarrow x = 0$, see f. ex. ([29], § 7.7)).

Define

$$\mathscr{A}_X = \{e\in X \mid 0\leqslant e\leqslant x_0,\ e\wedge(x_0-e) = 0\}.$$

It is easy to see that $\mathscr{A}_X$ is a boolean algebra, and hence using the Stone representation theorem (see f. ex. [32]) we can identify it with the algebra $\mathscr{F}(\Omega)$ of all clopen subsets of some 0-dimensional compact set Ω. In the following we shall not distinguish between $\mathscr{A}_X$ and $\mathscr{F}(\Omega)$.

It follows from the above lemma that to each partition $Q = \{e_1, e_2, \ldots$ $\ldots, e_n\}$ of Ω into finitely many disjoint clopen subsets there is an isomorphism T: $\mathrm{span}Q \to l_p^n$ so that

$$K_1\|x\| \leqslant \|T_Q x\| \leqslant K_2\|x\|, \qquad x\in \mathrm{span}Q. \tag{1}$$

$$T_Q\left(\frac{e_i}{\|e_i\|}\right) = \delta_i, \qquad (\delta_i \text{ the } i'\text{ th unit vector of } l_p^n). \tag{2}$$

If $x\in \mathrm{span}\mathscr{A}_X$, then we define for each partition Q of the above kind

$$\varrho_Q(x) = \begin{cases} \|T_Q x\| & \text{if } x\in \mathrm{span}Q,\\ 0 & \text{else.}\end{cases}$$

If Q_1 and Q_2 are partitions of Ω as above then we say that Q_2 *is finer than* Q_1 and write $Q_1 \precsim Q_2$, if every element of Q_1 is the union of some elements of Q_2. Clearly $\precsim$ defines a partial ordering in the family of all partitions of Ω into finitely many disjoint clopen sets, which is readily seen to be filtered upwards.

Define

$$\Pi = [0, K]^{\{x \in \operatorname{span} \mathscr{A}_X \mid \|x\| \leqslant 1\}}$$

Π is compact when equipped with the product topology. We define $\pi_Q \in \Pi$ by

$$\pi_Q(x) = \varrho_Q(x), \qquad \|x\| \leqslant 1, \; x \in \operatorname{span} \mathscr{A}_X$$

$\{\pi_Q\}$ is a net on Π and hence we can find a convergent subset; let us for simplicity assume that $\{\pi_Q\}$ itself is convergent. Since for every $x \in \operatorname{span} \mathscr{A}_X$ there is a partition Q_x, so that for all partitions Q, $Q_X \precsim Q$ we have $x \in \operatorname{span} Q$ and hence $\varrho_Q(x) = \|T_Q x\|$ we infer that $\{\varrho_Q(x)\}$ is convergent for all $x \in \operatorname{span} \mathscr{A}_X$. Put

$$\varrho(x) = \lim \varrho_Q(x), \qquad x \in \operatorname{span} \mathscr{A}_X.$$

It follows from the above that

$$K_1 \|x\| \leqslant \varrho(x) \leqslant K_2 \|x\|, \qquad x \in \operatorname{span} \mathscr{A}_X.$$

It is easy to check that ϱ is in fact a norm on $\operatorname{span} \mathscr{A}_X$.

If $x, y \in \operatorname{span} \mathscr{A}_X$ with $|x| \wedge |y| = 0$, then there is a partition Q_0 so that $x, y \in \operatorname{span} Q$ for all Q, $Q_0 \precsim Q$, and disjoint there, but this means that

$$\varrho_Q(x+y)^p = \|T_Q(x+y\|^p = \|T_Q x\|^p + \|T_Q y\|^p = \varrho_Q(x)^p + \varrho_Q(y)^p$$

and hence

$$(*) \qquad \varrho(x+y)^p = \varrho(x)^p + \varrho(y)^p$$

there the pth powers mean max, in case $p = \infty$). Since $\overline{\operatorname{span}} \mathscr{A}_X = X$ if follows that ϱ can be extended by continuity to a norm on X, equivalent to $\|\cdot\|$. From $(*)$ together with a result of Bohnenblust [5] we infer that if $1 \leqslant p < \infty$, then (X, ϱ) is lattice isometric to $L_p(\mu)$, where μ is the measure on the σ-algebra generated by $\mathscr{F}(\Omega)$ given by

$$\mu(e) = \varrho(e)^p, \qquad e \in \mathscr{A}_X.$$

If $p = \infty$, we get from [5] that there is a set Γ, so that (X, ϱ) is lattice isometric to $c_0(\Gamma)$.

The general case.

Let $\{x_\alpha \mid \alpha \in I\}$ be a maximal set of mutually disjoint elements (such a set exists by Zorn's lemma). Define

$$X_\alpha = x_\alpha^{\perp\perp}, \qquad \alpha \in I.$$

Since for each $\alpha \in I$, x_α is a Freudenthal unit in X_α, we get by the first part that each X_α is lattice isomorphic to some $L_p(\mu_\alpha)$ (some $c_0(\Gamma_\alpha)$, if $p = \infty$) and using the above lemma again, we observe that there is a constant M and lattice isomorphisms T_α of X_α onto $L_p(\mu_\alpha)$ (onto $c_0(\Gamma_\alpha)$) with $\|T_\alpha\|\,\|T_\alpha^{-1}\|\leqslant M$. Using our assumption on X and the fact that X is the ordered direct sum of the X_α's we get

$$X \overset{\text{lattice}}{\longrightarrow} \Big(\sum_\alpha X_\alpha\Big)_{l_p} \overset{\text{lattice}}{\longrightarrow} \big(L_p(\mu_\alpha)\big)_{l_p} \quad \text{if } 1\leqslant p<\infty \text{ and}$$

$$X \overset{\text{lattice}}{\longrightarrow} \Big(\sum_\alpha X_\alpha\Big)_{c_0} \overset{\text{lattice}}{\longrightarrow} \Big(\sum_\alpha c_0(\Gamma_\alpha)\Big)_{c_0} \quad \text{for } p=\infty.$$

This finishes the proof, since the latter spaces are lattice isomorphic to $L_p(\mu)$ for some μ, respective $c_0(\Gamma)$ for some Γ.

2.12. COROLLARY. *Let X be a Banach lattice of minimal type, in which all normalized sequence consisting of mutually disjoint elements are equivalent. Then either there is a p, $1\leqslant p<\infty$ and a measure space $(\Omega, \mathscr{S}, \mu)$ so that X is lattice isomorphic to $L_p(\mu)$ or there is a set Γ with X lattice isomorphic to $c_0(\Gamma)$.*

Proof. The assumptions imply that if $\{x_n\}$ is a normalized sequence in X, consisting of mutually disjoint elements, then every normalized block basis with respect to $\{x_n\}$ is equivalent to $\{x_n\}$; hence $\{x_n\}$ is perfectly homogeneous and by a theorem of Zippin [37] either there is a p, $1\leqslant p<\infty$ so that $\{x_n\}$ is equivalent to the unit vector basis of l_p, or $\{x_n\}$ is equivalent to the unit vector basis of c_0. Obviously we get the same p for all sequences $\{x_n\}$, and we can apply the above theorem. ■

Remark. It can be shown that if $1\leqslant p<\infty$, then the other assumptions in Theorem 2.10 already imply that X is of minimal type, hence this assumption is dispensable for finite p. However we only need to apply the theorem to cases, where we know that X is of minimal type, and therefore we do not go into that. Note that minimality is not dispensable, if $p=\infty$.

Section 3

IDEAL FAMILIES OF SETS IN A BANACH SPACE AND SCHWARTZ OPERATOR IDEALS

In this section let X denote a Banach space over either the reals of the complex numbers.

If B is a non-empty, bounded, closed and absolutely convex subset of X we denote by X_B the space $\bigcup_{n=1}^{\infty} nB$ equipped with the norm having B as unit ball. It is easily seen that X_B is a Banach space. If $I_B\colon X_B \to X$ is the formal identity map, then the number $\|I_B\|$ will be called the radius of B.

3.1. Definition. A family $\mathscr{F}$ of non-empty, closed, bounded and absolutely convex subsets of X is called a *pre-ideal family*, if the following conditions are satisfied:

(i) $\forall A \in \mathscr{F}, \forall \lambda > 0\ \lambda A \in \mathscr{F}$,

(ii) $\forall A, B \in \mathscr{F},\ A + B \in \mathscr{F}$,

(iii) If $\{A_n\} \subseteq \mathscr{F}$ and $\sum_{n=1}^{\infty} \|I_{A_n}\| < \infty$, then there exists a $B \in \mathscr{F}$, so that

$$\|I_B\| \leqslant \sum_{n=1}^{\infty} \|I_{A_n}\| \quad \text{and} \quad \sum_{n=1}^{k} A_n \subseteq B \quad \text{for all } k \in N.$$

3.2. Definition. Let $\mathscr{F}$ be an pre-ideal family in X, and let E be a Banach space. An operator $T\colon E \to X$ is called *$\mathscr{F}$-bounded*, if there is an $A \in \mathscr{F}$, so that $Tx \in A$ for all $x \in E$, $\|x\| \leqslant 1$.

The set of all $\mathscr{F}$-bounded operators from E to X will be denoted by $\mathscr{B}_{\mathscr{F}}(E, X)$, and if $T \in \mathscr{B}_{\mathscr{F}}(E, X)$ then we define the number $b_{\mathscr{F}}(T)$ by

$$b_{\mathscr{F}}(T) = \inf\{\|I_A\| \mid Tx \in A \ \text{for}\ \|x\| \leqslant 1 \ \text{and}\ A \in \mathscr{F}\}.$$

Remark. It is easy to see that if the pre-ideal family $\mathscr{F}$ contains a neighbourhood of 0 in X, then $\mathscr{B}_{\mathscr{F}}(E, X) = B(E, X)$ and $b_{\mathscr{F}}(T) = \|T\|$ for all $T \in B(E, X)$.

3.3. Theorem. *Let $\mathscr{F}$ be a pre-ideal family in X and let E be a Banach space, then:*

(i) *Every operator* $T \in \mathcal{B}_{\mathcal{F}}(E, X)$ *is bounded and* $\|T\| \leqslant b_{\mathcal{F}}(T)$,

(ii) $\mathcal{B}_{\mathcal{F}}(E, X)$ *is a vector space,*

(iii) $b_{\mathcal{F}}$ *is a norm on* $\mathcal{B}_{\mathcal{F}}(E, X)$ *turning it into a Banach space.*

Proof. (i). If $T \in \mathcal{B}_{\mathcal{F}}(E, X)$, then there is an $A \in \mathcal{F}$, so that T admits the factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "T_1"'] & & X \\ & X_A \arrow[ur, "I_A"'] & \end{tikzcd}$$

where $\|T_1\| \leqslant 1$. From this (i) follows immediately.

It is trivial that $\mathcal{B}_{\mathcal{F}}(E, X)$ is a vector space and that $b_{\mathcal{F}}$ is a norm on $\mathcal{B}_{\mathcal{F}}(E, X)$, so let us prove the completeness of $b_{\mathcal{F}}$. In view of (i) it is enough to prove that if $\{T_n\} \subseteq \mathcal{B}_{\mathcal{F}}(E, X)$ so that $\sum_{n=1}^{\infty} b_{\mathcal{F}}(T_n) < \infty$ then the operator $T = \sum_{n=1}^{\infty} T_n$ is $\mathcal{F}$-bounded and

$$b_{\mathcal{F}}(T) \leqslant \sum_{n=1}^{\infty} b_{\mathcal{F}}(T_n).$$

Hence let $\{T_n\}$ be a sequence with the above properties and let $\varepsilon > 0$ be arbitrary. Choose a sequence $\{A_n\} \subseteq \mathcal{F}$ so that

$$\|I_{A_n}\| \leqslant b_{\mathcal{F}}(T_n) + \varepsilon \cdot 2^{-n} \quad \text{for } n \in N',$$

$$T_n x \in A_n \quad \text{for all } n \in N' \text{ and all } x \in E,\ \|x\| \leqslant 1.$$

Then $\sum_{n=1}^{\infty} \|I_{A_n}\| \leqslant \sum_{n=1}^{\infty} b_{\mathcal{F}}(T_n) + \varepsilon$ and hence from property (iii) of $\mathcal{F}$ we can find a $B \in \mathcal{F}$, so that

$$\|I_B\| \leqslant \sum_{n=1}^{\infty} \|I_{A_n}\|.$$

$$\sum_{n=1}^{k} T_n x \in B \quad \text{for all } k \in N' \text{ and all } x \in E,\ \|x\| \leqslant 1$$

and hence since B is closed $Tx \in B$ for all $x \in E$, $\|x\| \leqslant 1$; Furthemore, by the above inequalities we get

$$b_{\mathcal{F}}(T) \leqslant \|I_B\| \leqslant \sum_{n=1}^{\infty} b_{\mathcal{F}}(T_n) + \varepsilon$$

which proves the statement, since ε was arbitrary.

3.4. Definition. A pre-ideal family $\mathcal{F}$ in X is called an *ideal family* if for every $x \in X$ there is an $A \in \mathcal{F}$, so that $x \in A$.

Let now E and F be Banach spaces and let $\mathcal{F}$ be an ideal family in X; we introduce the following concept.

3.5. DEFINITION. A linear operator $T: E \to F$ is called *normable by $\mathscr{F}$*, if for any bounded linear operator $S: F \to X$, $S \circ T \in \mathscr{B}_{\mathscr{F}}(E, X)$.

The set $\mathscr{S}_{\mathscr{F}}(E, F)$ of all linear operators from E to F, which are normable by $\mathscr{F}$ is called the *Schwartz ideal* determined by $\mathscr{F}$.

If $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ then the number

$$s_{\mathscr{F}}(T) = \sup\{b_{\mathscr{F}}(S \circ T) \mid S \in B(F, X),\ \|S\| \leqslant 1\}$$

is called the *Schwartz norm* of T.

Remark. We can of course also define $\mathscr{S}_{\mathscr{F}}(E, F)$ for a pre-ideal family $\mathscr{F}$, but it is easy to see that $\mathscr{S}_{\mathscr{F}}(E, F) \neq \{0\}$ if and only if $\mathscr{F}$ is an ideal family.

3.6. THEOREM.

(i) *If $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ then $s_{\mathscr{F}}(T) < \infty$.*

(ii) *If $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ then T is bounded and $\|T\| \leqslant s_{\mathscr{F}}(T)$.*

(iii) *$\mathscr{S}_{\mathscr{F}}(E, F)$ is a vector space and $s_{\mathscr{F}}$ is a norm on $\mathscr{S}_{\mathscr{F}}(E, F)$, turning it into a Banach space.*

(iv) *If G is another Banach space then $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ and $S \in B(F, G)$ imply $ST \in \mathscr{S}_{\mathscr{F}}(E, G)$ and $s_{\mathscr{F}}(ST) \leqslant \|S\| s_{\mathscr{F}}(T)$. $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ and $S: B(G, E)$ imply $TS \in \mathscr{S}_{\mathscr{F}}(G, F)$ and $s_{\mathscr{F}}(TS) \leqslant \|S\| s_{\mathscr{F}}(T)$.*

Proof. Let $T \in \mathscr{S}_{\mathscr{F}}(E, F)$. To prove (i), we must show that the operator $A_T: B(F, X) \to \mathscr{B}_{\mathscr{F}}(E, X)$ defined by

$$A_T(S) = S \circ T, \qquad S \in B(F, X)$$

is bounded.

A_T is clearly continuous, when $\mathscr{B}_{\mathscr{F}}(E, X)$ is equipped with the operator norm, but since this norm is coarser than $b_{\mathscr{F}}$ it follows that A_T is closed. The closed graph theorem and Theorem 3.3 now show that A_T is bounded.

To show (ii) let $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ and let $y^* \in F^*$, $\|y^*\| = 1$ and $z \in X$, $\|z\| = 1$ be chosen arbitrarily. Define $S: F \to X$ by

$$Sy = y^*(y) z.$$

Clearly $\|S\| \leqslant 1$ and if $x \in E$, $\|x\| \leqslant 1$ we get:

$$|y^*(Tx)| \leqslant \|STx\| \leqslant b_{\mathscr{F}}(ST) \leqslant s_{\mathscr{F}}(T)$$

which proves (ii), since y^* was arbitrary.

(iii) is trivial, except perhaps the completeness of $s_{\mathscr{F}}$, but this is an easy consequence of the similar statement on $b_{\mathscr{F}}$ in Theorem 3.3 and the definition of $s_{\mathscr{F}}$.

(iv). Let $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ and $S \in B(F, G)$; for an arbitrary operator $S_1 \in B(G, X)$ we get

$$(S_1 \circ S) \circ T \in \mathscr{B}_{\mathscr{F}}(E, X)$$

hence $S \circ T \in \mathscr{S}_{\mathscr{F}}(E, F)$ and

$$s_{\mathscr{F}}(ST) = \sup_{\|S_1\| \leqslant 1} b_{\mathscr{F}}(S_1 \circ S \circ T) = \|S\| \sup_{\|S_1\| \leqslant 1} b_{\mathscr{F}}\left(S_1 \circ \frac{S}{\|S\|} \circ T\right) \leqslant \|S\| s_{\mathscr{F}}(T).$$

If $T \in \mathscr{S}_{\mathscr{F}}(E, F)$ and $S \in B(G, E)$ then we have for $S_1 \in B(F, X)$, $\|S_1\| \leqslant 1$, that $S_1 T \in \mathscr{B}_{\mathscr{F}}(E, X)$ and hence $S_1 TS \in \mathscr{B}_{\mathscr{F}}(G, X)$. This shows that $TS \in \mathscr{S}_{\mathscr{F}}(G, X)$ with

$$s_{\mathscr{F}}(TS) = \sup_{\|S_1\| \leqslant 1} b_{\mathscr{F}}(S_1 TS) \leqslant \|S\| s_{\mathscr{F}}(T)$$

which proves the assertion. ■

3.7. DEFINITION. Let X and Y be Banach spaces and $\mathscr{F}_1$ and $\mathscr{F}_2$ ideal families in X and Y respectively. $\mathscr{F}_1$ is said to be *coarser norming* that $\mathscr{F}_2$($\mathscr{F}_2$ finer norming than $\mathscr{F}_1$) if for any pair E, F of Banach spaces we have

(i) $S_{\mathscr{F}_2}(E, F) \subseteq \mathscr{S}_{\mathscr{F}_1}(E, F)$.

(ii) There is a constant $K > 0$ so that

$$s_{\mathscr{F}_1}(T) \leqslant K s_{\mathscr{F}_2}(T) \quad \text{for all } T \in \mathscr{S}_{\mathscr{F}_1}(E, F).$$

$\mathscr{F}_1$ and $\mathscr{F}_2$ are said to be *equinorming*, if $\mathscr{F}_1$ is both coarser and finer norming than $\mathscr{F}_2$.

We shall in the sequal often see examples of ideal families which can be compared in the above sense.

Let us end the present section with the following proposition

3.8. PROPOSITION. *Let $\mathscr{F}$ be an ideal family in a Banach space X. The following statements are equivalent.*

(i) *For every Banach space E $B(E, X) = \mathscr{S}_{\mathscr{F}}(E, X)$.*

(ii) *For every Banach space E $B(E, X) = \mathscr{B}_{\mathscr{F}}(E, X)$.*

(iii) *$\mathscr{F}$ contains a 0-neighbourhood of X.*

Proof. (i) $\Rightarrow$ (ii) is clear, since $\mathscr{S}_{\mathscr{F}}(E, X) \subseteq \mathscr{B}_{\mathscr{F}}(E, X)$.

(ii) $\Rightarrow$ (iii) follows easily from the definition of $\mathscr{B}_{\mathscr{F}}(E, X)$ since (ii) implies that the identity in B is $\mathscr{F}$-bounded.

(iii) $\Rightarrow$ (i) follows directly from the definitions.

Section 4

SCHWARTZ IDEALS DETERMINED BY A BANACH LATTICE

In this section we shall assume that all Banach spaces in question are real Banach spaces.

If X is a Banach lattice and $\mathscr{F}$ is the family of all symmetric order intervals in X, then it is easy to see that $\mathscr{F}$ is an ideal family in X. We shall denote the Schwartz ideal determined by $\mathscr{F}$ by $\mathscr{S}_X$ instead of $\mathscr{S}_{\mathscr{F}}$, when no confusion about the order structure in X can be made. Also if E and F are Banach spaces and $T \in \mathscr{S}_X(E, F)$ we shall say that T is normable by X, and we denote the Schwartz norm of T by $s_X(T)$ (1).

If X is a Banach space with an unconditional basis $\{x_n\}$, then X is a Banach lattice under the ordering discussed in Section 1, and we shall in that case use the notation $\mathscr{S}_{\{x_n\}}$ for the Schwartz ideal, $s_{\{x_n\}}$ for the Schwartz norm, and use the expression "T normable by $\{x_n\}$" e. t. c.

Our first proposition is a corollary of Proposition 3.8 and the Kakutani representation theorem for abstract M-spaces.

4.1. PROPOSITION. *If X is a Banach lattice, then the following statements are equivalent.*

(i) *For every Banach space E, $B(E, X) = \mathscr{S}_X(E, X)$.*

(ii) *For every Banach space E $B(E, X) = \mathscr{B}_X(E, X)$.*

(iii) *There is a compact Hausdorff space S, so that X is order isomorphic to $C(S)$.*

Proof. It is enough to prove that (iii) of 3.8 is equivalent to (iii) here. If (iii) of 3.8 is assumed, then there is an $x \in X$, $0 \leqslant x$, so that X is order isomorphic to $X_{[-x,x]}$. The latter space is easily seen to be an abstract M-space with unit and hence by the Kakutani theorem on representation of such spaces [16], we get that $X_{[-x,x]}$ is order isometric to $C(S)$ for some compact Hausdorff space S. Clearly (iii) implies (iii) of 3.8. ∎

Before we treat the case of general Banach lattices let us look a little on the unconditional basis case.

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(1) We shall use the term order bounded operator instead of $\mathscr{F}$-bounded operator and write $\mathscr{B}_X$ instead of $\mathscr{B}_{\mathscr{F}}$.

About this case we have the following theorem.

4.2. THEOREM. *Let X be a Banach space with an unconditional basis $\{x_n\}$ with biorthogonal sequence $\{x_n^*\}$ and let E and F be Banach spaces and $T \in B(E, F)$. Then T is normable by $\{x_n\}$ if and only if T^* is ω^*-absolutely $\{x_n\}$-summing, and in this case we have $s_X(T) = \pi^{\omega^*}_{\{x_n\}}(T^*)$.*

Proof. Suppose first that T is normable by $\{x_n\}$ and let $\{y_n^*\} \subseteq F^*$ be a sequence so that $\sum_{n=1}^{\infty} y_n^*(y) x_n$ is convergent $y \in F$.

Let us define

$$Ay = \sum_{n=1}^{\infty} y_n^*(y) x_n \quad \text{for all } y \in F.$$

Using the closed graph theorem it is readily verified that A is a bounded linear operator from F to X.

By the assumption on T there exists an element $x \in X$, so that

$$(1) \qquad |x_n^*(ATy)| \leqslant x_n^*(x) \quad \text{for all } n \in N \text{ and for all } y \in E \text{ with } \|y\| \leqslant 1.$$

From (1) we get immediately: $\|T^* y_n^*\| \leqslant x_n^*(x)$ for all $n \in N$, and therefore $\sum_{n=1}^{\infty} \|T^* y_n^*\| x_n$ is convergent.

It is also easily verified that $b_{\{x_n\}}(AT) = \left\|\sum_{n=1}^{\infty}\right\| T^* y_n^* \| x_n\|$ and therefore by the definition of $s_{\{x_n\}}$

$$\left\|\sum_{n=1}^{\infty}\right\| T^* y_n^* \| x_n\| \leqslant s_{\{x_n\}}(T) \|A\| = s_{\{x_n\}}(T) \sup_{\|y\| \leqslant 1} \left\|\sum_{n=1}^{\infty} y_n^*(y) x_n\right\|$$

which proves that $\pi^{\omega^*}_{\{x_n\}}(T^*) \leqslant s_{\{x_n\}}(T)$.

To prove the assertion in the other direction let $A \in B(F, X)$. Since

$$Ay = \sum_{n=1}^{\infty} (A^* x_n^*)(y) x_n \quad \text{for all } y \in F$$

we get by assumption that: $\sum_{n=1}^{\infty} \|T^* A^* x_n^*\| x_n$ is convergent and

$$(2) \qquad \left\|\sum_{n=1}^{\infty}\right\| T^* A^* x_n^* \| x_n\| \leqslant \pi^{\omega^*}_{\{x_n\}}(T^*) \|A\|.$$

Since for all $n \in N$ and all $y \in E$, $\|y\| \leqslant 1$ we have

$$|x_n^*(ATy)| \leqslant \|T^* A^* x_n^*\|$$

it follows that T is normable by $\{x_n\}$; furthermore, from (2) we get that $s_{\{x_n\}}(T) \leqslant \pi^{\omega^*}_{\{x_n\}}(T^*)$. ∎

4.3. COROLLARY. *Let E and F be Banach spaces and $1 \leqslant p < \infty$. An operator $T \in B(E, F)$ is normable by the unit vector basis of l_p, if and only if T^* is p-absolutely summing.*

We now turn our attention again to the case of a general Banach lattice. Using the results stated in Section 2 we shall prove that under rather weak restrictions on the Banach lattice X it is possible to describe the operators normable by X by certain summability conditions of their adjoints, conditions similar to those of Theorem 4.2.

In the rest of this section let X denote an arbitrary Banach lattice and E and F Banach spaces.

Our first result in the direction described above is:

THEOREM. 4.4 *Let X be of minimal type. If $T \in \mathscr{S}_X(E, F)$ then $T \in \mathscr{S}_{\{x_n\}}(E, F)$ for any sequence $\{x_n\} \subseteq X$, consisting of mutually disjoint elements, and*

$$s_{\{x_n\}}(T) \leqslant s_X(T).$$

In other words X is is finer norming than any sequence $\{x_n\}$ of mutually disjoint elements of X.

Proof. Let $\{x_n\} \subseteq X$ so that $x_n \perp x_m$ for $n \neq m$. Since equivalent bases are equinorming and since clearly there is an isometry of $[x_n]$ onto $[|x_n|]$, carrying x_n to $|x_n|$ for all $n \in N$, it is no restriction to assume that $x_n \geqslant 0$ for all n.

Let $T \in \mathscr{S}_X(E, F)$ and $A \in B(F, [x_n])$. Since by assumption on T the set:

$$M = \{|ATx| \;\big|\; x \in E, \; \|x\| \leqslant 1\}$$

is order bounded in X, it follows from Proposition 2.7 that it is also order bounded in $[x_n]$, hence AT is order bounded. Furthermore from 2.7

$$b_{\{x_n\}}(AT) = \|\sup_{[x_n]} M\| = \|\sup_X M\| = b_X(AT) \leqslant \|A\| s_X(T)$$

and from this we get that

$$s_{\{x_n\}}(T) \leqslant s_X(T). \quad \blacksquare$$

Remark. It is easy to see that the requirement that X is of minimal type is not dispensable in the above theorem; indeed, let $I\colon c_0 \to c_0$ be the identity map, it is clearly normable by l_∞, but not by c_0.

Let us list a few corollares of Theorem 4.4.

4.5. COROLLARY. *Let X be a Banach space with an unconditional basis $\{x_n\}$. Then $\{x_n\}$ is finer norming that any block basic sequence with respect to $\{x_n\}$. Further if $T \in \mathscr{S}_{\{x_n\}}(E, F)$ and $\{z_n\}$ is a block basis with respect to $\{x_n\}$ then*

$$s_{\{z_n\}}(T) \leqslant K^2 s_{\{x_n\}}(T)$$

when K denotes the unconditional constant of $\{x_n\}$.

The corollary follows immediately from 4.4; let us just mention that the entrance of the unconditional constant for $\{x_n\}$ in the above inequality is caused by the fact that we do not use the lattice norm in X.

Remark. We would like to mention here, that if an operator is $\{x_n\}$-summing for an unconditional basis $\{x_n\}$ then it is also summing with respect to any block basis sequence of $\{x_n\}$, hence 4.5 follows from 4.2. However we feel that the picture of the whole situation becomes more clear if we take it out as a corollary of the more general 4.4 of the same type.

4.6. COROLLARY. *Let X be a Banach space with an unconditional basis $\{x_n\}$. If X contains a subspace of l_p, for some $p, l = p < \infty$ then every operator normable by $\{x_n\}$ has absolutely p-summing adjoint.*

Proof. The properties of the unit vector basis of l_p and the results of [4] give that it is equivalent to a block basis of $\{x_n\}$ hence the conclusion follows from Corollary 4.5. ■

Corollary 4.6 shall be very usefull for us in the sequal.

The next theorem gives conditions for the converse of Theorem 4.4 to hold.

4.7. THEOREM. *Let X be a boundedly complete Banach lattice having the metric approximation property and let $T \in B(E, F)$.*

The following statements are equivalent.

(i) $T \in \mathscr{S}_X(E, F)$.

(ii) $T \in \mathscr{S}_{\{x_n\}}(E, F)$ *for every sequence $\{x_n\} \subseteq X$ of mutually disjoin elements and* $\sup\limits_{\{x_n\}} s_{\{x_n\}}(T) < \infty$.

(iii) *There exists a constant $K \geqslant 0$ so that for all finite sets $\{x_1, x_2, \ldots, x_n\}$ of mutually disjoint elements form X and all finite sets $\{y_1^*, \ldots, y_n^*\} \subseteq F^*$ we have:*

$$\Big\|\sum_{i=1}^{n}\|T^* y_i^*\| \, \|x_i\|\Big\| \leqslant K \sup_{\|y\|\leqslant 1} \Big\|\sum_{i=1}^{n} y_i^*(y) x_i\Big\|.$$

Further, if $T \in \mathscr{S}_X(E, F)$, then

$$s_X(T) = \inf\{K \mid K \textit{ satisfies } \text{(iii)}\} = \sup_{\{x_n\}} s_{\{x_n\}}(T)$$

where the sup *is extended over all finite or infinite sequences of mutually disjoint elements.*

Proof. (i) $\Rightarrow$ (ii) follows immediately from Theorem 4.4. since every boundedly complete Banach lattice is of minimal type.

(ii) $\Rightarrow$ (iii) follows from Theorem 4.2, since every basic sequence $\{x_n\}$ consisting of mutually disjoint elements is boundedly complete in the case considered.

Furthermore from 4.2 it follows that

$$\sup_{\{x_n\}} s_{\{x_n\}}(T) = \inf\{K \mid K \text{ satisfies (iii)}\}.$$

(iii) $\Rightarrow$ (i). Suppose first that $A\colon F \to X$ is a finite-dimensional operator with a representation:

$$Ax = \sum_{k=1}^{n} x_k^*(x)\, x_k \quad \text{for } x \in F \tag{1}$$

where $\{x_1^*, \ldots, x_n^*\} \subseteq F^*$ and $x_1, x_2, \ldots, x_n$ are mutually disjoint elements from X.

For all $y \in E$, $\|y\| \leqslant 1$ we have:

$$\begin{aligned} |ATy| &= \Big|\sum_{k=1}^{n} (T^* x_k^*)(y)\, x_k\Big| = \sum_{k=1}^{n} |(T^* x_k^*)(y)|\, |x_k| \\ &\leqslant \sum_{k=1}^{n} \|T^* x_k^*\|\, |x_k| = \Big|\sum_{k=1}^{n} \|T^* x_k^*\| \cdot x_k\Big|. \end{aligned} \tag{2}$$

Since

$$\Big\|\sum_{k=1}^{n}\Big\| T^* x_k^* \|x_k\| \leqslant K \sup_{\|y\| \leqslant 1} \Big\|\sum_{k=1}^{n} x_k^*(y)\, x_k\Big\| = K\|A\|$$

we get for every finite subset $\{y_1, \ldots, y_n\}$ of the unit ball of E:

$$\|\, |ATy_1| \vee |ATy_2| \vee \quad \vee |ATy_n|\, \| \leqslant K\|A\|. \tag{3}$$

If $A\colon F \to X$ is an arbitrary bounded finite-dimensional operator, then it follows from Proposition 2.9 that A is a uniform limit of a sequence of finite-dimensional operators of the form (1); since the lattice operations in X are continuous it follows that inequality (3) is valid for A.

Let now $A \in B(E, X)$; since X has the metric approximation property; these is a net $\{A_t \mid t \in I\}$ of finite-dimensional operators with $\|A_t\| \leqslant \|A\|$ for all $t \in I$ so that

$$\lim_t A_t x = Ax \quad \text{for all } x \in F.$$

From this it follows that (3) is valid for A.

Since X is boundedly complete, it follows from (3) that the set

$$C_A = \{ATx \mid \|x\| \leqslant 1\}$$

is order bounded in X for every $A \in B(F, X)$, hence $T \in \mathscr{S}_X(E, F)$.

Further it follows from (3) that

$$b_X(AT) = \|\sup C_A\| \leqslant K\|A\| \tag{4}$$

since sup C_A is the limit of the net consisting of all supremas of finitely many clements from C_A; hence

$$s_X(T) \leqslant K. \tag{5}$$

To prove the last statement let $T \in \mathcal{S}_X(E, F)$. From the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) it follows that

$$\inf\{ K \mid K \text{ satisfies (iii)}\} = \sup_{\{x_n\}} s_{\{x_n\}}(T) \leqslant s_X(T). \tag{6}$$

Combining that with (5) from the implication (iii) $\Rightarrow$ (i) we get the desired result. ■

Remark. It is easy to see that Theorem 4.7 is still valid if we assume that F^* has the metric approximation property instead of X.

Our first corollary of 4.7 was first proved by Kwapień [17]; it is the "functional-analysis version" of a theorem of Schwartz on cylindrical measures [31].

4.8. Corollary. *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let $1 \leqslant p < \infty$. We consider $L_p(\mu)$ as a Banach lattice under the ordering "$\leqslant$ a. e.". If E and F are Banach spaces and $T \in B(E, F)$, then $T \in \mathcal{S}_{L_p(\mu)}(E, F)$ if and only if T^* is p-absolutely summing.*

Further in that case

$$s_{L_p(\mu)}(T) = \Pi_p(T^*).$$

Proof. $L_p(\mu)$ is boundedly complete, if $1 \leqslant p < \infty$, and has the metric approximation property, so we can apply 4.7. It follows immediately from Theorem 4.2, that it is enough to consider sequences $\{x_n\}$ of mutually disjoint elements from $L_p(\mu)$, for which $\|x_n\| = 1$ for all n in Theorem 4.7.

It is easy to see that every such sequence is 1-equivalent to the unit vector basis of l_p hence the corollary follows immediately from 4.7 and Corollary 4.3. ■

From the above corollary we see that Theorem 4.7 is in fact a generalization of a theorem of Schwartz on cylindrical measures stating that the ideal of p-radonifying operators and that of p-absolutely summing operators almost coincide.

We shall in the next section study the relation between the theory developed here and the theory of cylindrical measures a little more detailed.

The next proposition we are going to prove can also be taken out as a Corollary of 4.7 in case X has the metric approximation property, however we prefer here to give a direct proof of a formally stronger statement.

4.9. Proposition. *Let E and F be Banach spaces and $T \in B(E, F)$. If T^* is 1-absolutely summing, then T is normable by any boundedly complete Banach lattice X.*

Further, in that case we have:

$$s_X(T) \leqslant \Pi_1(T^*).$$

Proof. Let K denote the unit ball in F^{**} equipped with the ω^*-

topology. Since T^* is 1-absolutely summing, there is a positive Radon measure μ on K so that

$$\|T^* y^*\| \leqslant \int_K |y^{**}(y^*)|\, d\mu(y^{**}) \tag{1}$$

and

$$\mu(K) = \Pi_1(T^*). \tag{2}$$

Let now X be a boundedly complete Banach lattice, and $A \in B(F, X)$. Let us define $f\colon X^* \to R'$ by

$$f(x^*) = \int_K |A^{**} y^{**}|(x^*)\, d\mu(y^{**}). \tag{3}$$

It is easy to see that $f \in X^{**}$, $f \geqslant 0$ and that

$$\|f\| \leqslant \|A^{**}\| \mu(K) = \|A\| \Pi_1(T^*).$$

For every $x^* \in X^*$, $x^* \geqslant 0$ and every $x \in E$ with $\|x\| \leqslant 1$ we get

$$\begin{aligned} |x^*(ATx)| &\leqslant \|T^* A^* x^*\| \leqslant \int_K |(A^{**} y^{**})(x^*)|\, d\mu(y^{**}) \\ &\leqslant \int_K |A^{**} y^{**}|(x^*)\, d\mu(y^{**}) = f(x^*). \end{aligned} \tag{4}$$

Hence identifying X with a sublattice of X^{**} in the canonical manner, we get

$$|ATx| \leqslant f \quad \text{for all } x \in E,\ \|x\| \leqslant 1. \tag{5}$$

From Theorem 2.2 it now follows that $T \in \mathscr{S}_X(E, F)$.
Further from the same proposition we get

$$b_X(AT) \leqslant \|f\| \leqslant \|A\| \Pi_1(T^*) \tag{6}$$

hence

$$s_X(T) \leqslant \Pi_1(T^*). \quad \blacksquare$$

Remark. The reason for that we can avoid assuming that X has the metric approximation in the above proposition is of course that when T^* is absolutely summing, then we have an "integral inequality" for T^*, and hence we need not go to finite-dimensional subspaces of X.

We now state a few other corollaries.

4.10. Corollary. *Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and let X be a boundedly complete Banach lattice. Under these circumstances every bounded linear operator from $L_1(\mu)$ into X maps order bounded sets into order bounded sets.*

Proof. Let $A \in B(L_1(\mu), X)$ and let $f \in L_1(\mu)$, $f \geqslant 0$.

Let us define $T\colon L_\infty(\mu)\to L_1(\mu)$ by

$$Tg = f\cdot g \quad \text{for all } g\in L_\infty(\mu).$$

It is easy to see that

$$T^*g = fg \quad \text{for all } g\in L_\infty(\mu)$$

and hence it follows that T^* is absolutely summing. By 4.9 T is normable by X and therefore we have that the set

$$A([-f, f]) = AT\{g\in L_\infty(\mu) \mid \|g\|_\infty \leqslant 1\}$$

is order bounded in X.

Remark. Corollary 4.10 was proved by Grothendieck [11] in case $X = L_1(\nu)$ for some measure ν.

In a manner similar to the above we can prove:

4.11. Corollary. *Let* $(\Omega, \mathscr{S}, \mu)$ *be a finite measure space and let* $1\leqslant p<\infty$. *If* p' *is defined by* $\dfrac{1}{p}+\dfrac{1}{p'}=1$, *then the formal identity map of* $L_{p'}(\mu)$ *into* $L_1(\mu)$ *is normable by* $L_p(\mu)$.

4.12. Corollary. *Let* H *be a Hilbert space,* X *a Banach lattice and* $T\in B(H, X)$. *Then*

(i) *If* T *is order bounded then* T^* *is absolutely summing, and hence* $T\in \mathscr{S}_Y(H, X)$ *for every boundedly order complete Banach lattice* Y.

(ii) *If* X *is boundedly order complete, then* T *is order bounded, if and only if* $T\in\mathscr{S}_X(H, X)$.

(iii) *If* $T\in\mathscr{S}_X(H, X)$, *then* $S\circ T$ *is a Hilbert Schmidt operator for every* $S\in B(X, H)$.

Proof. The only thing we have to prove is the first part of (i), since the rest then will follow from Proposition 4.9.

If T is order bounded, then there is a compact Hausdorff space K so that T admits a factorization

$$\begin{tikzcd} H \arrow[rr, "T"] \arrow[dr, "T_1"'] & & X \\ & C(K) \arrow[ur, "T_2"'] & \end{tikzcd}$$

where T_2 is positive.

Hence T^* has a factorization

$$\begin{tikzcd} X^* \arrow[rr, "T^*"] \arrow[dr, "T_2^*"'] & & H^* \\ & C(K)^* \arrow[ur, "T_1^*"'] & \end{tikzcd}$$

but $C(K)^* = L_1(\mu)$ for some measure μ and therefore T_1^* is 1-absolutely summing by a result of Grothendieck (which states that a bounded linear operator from an L_1-space into a Hilbert space is absolutely summing; for proofs of this see [11] and [18]). ■

We now wish to show that it is possible to interchange the role of T and T^* in Theorem 4.7. To do this we need the following lemma, the proof of which is dependent on the principle of local reflexivity of Lindenstrauss and Rosenthal [19] and it's strengthened form by Rosenthal, Zippin and Johnson [13]:

4.13. LEMMA. *Let E and X be Banach spaces so that X has the K-metric approximation property. Let $F \subseteq E^*$ be a finite dimensional subspace and let $\varepsilon > 0$ and $T \in B(E^*, X)$ be given. Then there exists a finite dimensional operator $S \in B(X^*, E)$ so that*

(i) *$S^*(E^*) \subseteq X$ (X canonically imbedded into X^{**}),*

(ii) *$\|T^* x^* - S^* x^*\| \leqslant \varepsilon \|x^*\|$ for all $x^* \in F$,*

(iii) *$\|S\| \leqslant K(1+\varepsilon)\|T\|$.*

Remark. We shall say that the Banach space X has the K-metric approximation property, if the identity operator on X can be approximated uniformly on compact sets by finite dimensional operators with norm $\leqslant K$.

Proof of Lemma 4.13. Since X has the K-metric approximation property there is a finite dimensional operator $R \in B(E^*, X)$ so that

$$\|Tx^* - Rx^*\| \leqslant \varepsilon \|x^*\| \quad \text{for } x^* \in F, \tag{1}$$

$$\|R\| \leqslant K\|T\|. \tag{2}$$

By the principle of local reflexivity [13], Section 3 there is an invertible operator $U\colon R^*(X^*) \to E$ so that

$$\|U\| \leqslant 1+\varepsilon, \qquad \|U^{-1}\| \leqslant 1, \tag{3}$$

$$U_{|R^*X^* \cap E} = \text{identity}, \tag{4}$$

$$x^* U R^* z^* = (R^* z^*)(x^*) \quad \text{for all } x^* \in F \text{ and } z^* \in X^*. \tag{5}$$

If we define $S = U R^*$, then S^* maps E^* into X and from (5) we infer that $S^* x^* = Rx^*$ for all $x^* \in F$. Furthermore from (3): $\|S\| \leqslant \|R\|(1+ +\varepsilon) \leqslant K(1+\varepsilon)\ \|T\|$. ■

We are now able to prove:

4.14. THEOREM. *Let X be a Banach space with a boundedly complete unconditional basis $\{x_n\}$.*

If E and F are Banach spaces and $T \in B(E, F)$, then T^* is normable by $\{x_n\}$ if and only if T is $\{x_n\}$-absolutely summing and in this case.

$$\pi_{\{x_n\}}(T) \leqslant s_{\{x_n\}}(T^*) \leqslant K\pi_{\{x_n\}}(T)$$

where K is the unconditional constant for $\{x_n\}$.

Proof. It follows immediately from the earlier results in this section that if T^* is normable by $\{x_n\}$ then T^{**} and hence T is $\{x_n\}$-absolutely summing, and

$$s_{\{x_n\}}(T^*) = \pi^{\omega^*}_{\{x_n\}}(T^{**}) \geqslant \pi_{\{x_n\}}(T).$$

Assume now that T is $\{x_n\}$-absolutely summing.

Let first $A \in B(X^*, E)$ with $A^*(E^*) \subseteq X$.

If $\{y_1^*, y_2^*, \ldots, y_k^*\} \subseteq F^*$ $\|y_i^*\| \leqslant 1$ $i = 1, 2, \ldots$ then

$$(1) \quad \| \, |A^*T^*y_1^*| \vee |A^*T^*y_2^*| \vee \quad \vee |A^*T^*y_k^*| \, \| \leqslant \Big\| \sum_{n=1}^{\infty} \|TAx_n^*\| x_n \Big\|$$

$$\leqslant \pi_{\{x_n\}}(T) \sup_{\|y^*\| \leqslant 1} \Big\| \sum_{n=1}^{\infty} x_n^*(A^*y^*) x_n \Big\| = \pi_{\{x_n\}}(T) \|A\|$$

where $\{x_n^*\}$ is the sequence biorthogonal to $\{x_n\}$.

If now $A \in B(E^*, X)$ is arbitrary and $\varepsilon > 0$ then it follows from Lemma 4.13 that there is a net $\{A_t \mid t \in I\}$ of finite dimensional operators from X^* to E so that

$$(2) \qquad A_t^*(E^*) \subseteq X \quad \text{for all } t \in I,$$

$$(3) \qquad A_t^* y^* \to Ay^* \quad \text{for all } y^* \in E^*,$$

$$(4) \qquad \|A_t\| \leqslant K(1+\varepsilon) \|A\|.$$

From this together with (1) it follows that for any finite set $\{y_1^*, y_2^*, \ldots$ $\ldots, y_k^*\} \subseteq F^*$ with $\|y_i\| \leqslant 1$ we have

$$\| \, |AT^*y_1^*| \vee |AT^*y_2^*| \vee \quad \vee |AT^*y_k^*| \, \| \leqslant K\pi_{\{x_n\}}(T) \|A\|.$$

Hence AT^* is order bounded and

$$b_{\{x_n\}}(AT^*) \leqslant K\pi_{\{x_n\}}(T) \|A\|$$

and therefore T^* is normable by $\{x_n\}$ with:

$$s_{\{x_n\}}(T^*) \leqslant K\pi_{\{x_n\}}(T). \quad \blacksquare$$

4.15. Corollary. *Let X be a Banach space with an unconditional boundedly complete basis $\{x_n\}$.*

If E and F are Banach spaces and $T \in B(E, F)$, then T is $\{x_n\}$-absolutely summing if and only if T^{**} is $\{x_n\}$-absolutely summing and in this case

$$\pi_{\{x_n\}}(T) \leqslant \pi_{\{x_n\}}(T^{**}) \leqslant K\pi_{\{x_n\}}(T).$$

Combining Theorem 4.14 with Theorem 4.7 we get

4.16. Theorem. *Let X be a boundedly complete Banach lattice which has the metric approximation property, and let E and F be Banach spaces with $T \in B(E, F)$. Then*

(i) *T is $\{x_n\}$-absolutely summing with respect to every sequence $\{x_n\}$ of mutually disjoint elements from X and $\sup\limits_{\{x_n\}} \pi_{\{x_n\}}(T) < \infty$ if and only if T^* is normable by X and in that case $s_X(T^*) = \sup\limits_{\{x_n\}} \pi_{\{x_n\}}(T)$.*

(ii) *T is normable by X if and only if T^{**} is normable by X and in that case $s_X(T) = s_X(T^{**})$.*

In the rest of this section we shall study order bounded operators a little more closely. In the rest of the section we shall suppose that X is a Banch lattice of minimal type and E is a Banach space.

Our first proposition is the following:

4.17. PROPOSITION. *If $T \in \mathscr{B}_X(E, X)$ then T^* has the following property:*

(*) *For every boundedly complete Banach lattice Y and every operator $A \in B(E^*, Y)$ AT^* maps order bounded sets in X^* into order bounded set in Y.*

Proof. Since T is order bounded it admits a factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "T_1"'] & & X \\ & C(K) \arrow[ur, "T_2"'] & \end{tikzcd}$$

for some compact Hausdorff space K, and where T_2 is positive. By dualization:

$$\begin{tikzcd} X^* \arrow[rr, "T^*"] \arrow[dr, "T_2^*"'] & & E^* \\ & C(K)^* \arrow[ur, "T_1^*"'] & \end{tikzcd}$$

Since also T_2^* is positive T_2^* maps order bounded sets into order bounded sets, and since $C(K)^*$ is an L_1-space the result follows from Corollary 4.10.

4.18. PROPOSITION. *If $T \in \mathscr{B}_X(E, X)$ then T maps weak Cauchy sequences into norm convergent ones.*

Proof. Let K, T_1 and T_2 have the same meaning as in the previous proposition. Since X is of minimal type T_2 is weakly compact by 2.5 and then by the Dunford–Pettis theorem [9] T_2 has the property stated. This finishes the proof.

Using Proposition 4.18 and the theorem of Eberlein we easily get that if $S\colon F \to E$ is weakly compact and $T \in \mathscr{B}_X(E, X)$ then TS is compact. We are however going to prove a result considerably stronger than this.

A few words about the next theorem. We are actually only going to use the proof of it in the sequal but we have decided to state it separately to stress the fact that it is not dependent on lattice theoretical considerations. As an sole exception in this part of the section the letter X shall stand for a Banach space in the next theorem (and only in this). We have done that of notational reasons.

4.19. THEOREM. *Let E and X be Banach spaces and K a compact Hausdorff space. Let $S\colon E \to C(K)$ and $T\colon C(K) \to X$ be weakly compact operators. Then there is a sequence $\{A_n\}$ of finite-dimensional operators from E to X so that $\|A_n - TS\| \to 0$ for $n \to \infty$.*

Furthermore there exists a positive $\mu \in C(K)^*$ so that the A_n's can be chosen to have a factorization of the form:

$$\begin{tikzcd} E \arrow[rr, "A_n"] \arrow[dr, "S_n"'] & & X \\ & L_\infty(\mu) \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

where S_n is a finite dimensional operator and $\tilde{T}$ is a weakly compact operator so that $\overline{\tilde{T}(U)} = \overline{T(V)}$. U and V denoting the unit balls in $L_\infty(\mu)$, respectively $C(K)$.

Proof. Since T is weakly compact, T^* is also weakly compact. Hence by the characterization of weakly compact sets in $C(K)^*$ (see [2] and also Rosenthal [27], Lemma 1.3) there is a positive measure $\mu \in C(K)^*$ so that

$$T^* X^* \subseteq L_1(\mu) \tag{1}$$

where $L_1(\mu)$ via the Radon–Nikodym theorem is indentified with the subspace of $C(K)^*$, consisting of all measures absolutely continuous with respect to μ.

Let $I\colon L_1(\mu) \to C(K)^*$ denote the canonical injection and define $T_1\colon X^* \to L_1(\mu)$ by $T^* = I \circ T_1$ and put $\tilde{T} = T_1^*$. By the weak compactness of T we get that $T^{**}\big(C(K)^{**}\big) \subseteq X$ and hence since I^* is onto we get that $\tilde{T}\big(L_\infty(\mu)\big) \subseteq X$ and putting $J = I^*_{|C(K)}$ we easily get $T = \tilde{T}J$. Clearly $\tilde{T}$ is weakly compact and has the properties stated.

Let for each $t \in K$ δ_t denote the Dirac measure at the point t and define $\varphi\colon K \to E^*$ by

$$\varphi(t) = S^* \delta_t. \tag{1}$$

φ is continuous from K into $\big(E^*, \sigma(E^*, E)\big)$ but since $\varphi(K)$ is contained in a weakly compact subset of E^*, it is continuous when E^* is equipped with its weak topology as well. It is wellknown that under these circumstances φ is μ-measurable and hence there is a sequence $\{\varphi_n\}$ of E^*-valued simple functions on K so that

$$\|\varphi_n(t)\| \leqslant 2\|\varphi(t)\| \leqslant 2\|S^*\| \quad \text{for almost all } t \in K(\mu) \tag{2}$$

$$\varphi_n \to \varphi \text{ in } \mu\text{-measure}. \tag{3}$$

Let us define $B_n\colon L_1(\mu) \to E^*$ by

$$B_n f = \int f\varphi_n \, d\mu, \qquad f \in L_1(\mu),\ n = 1, 2, \ldots \tag{4}$$

From the weak compactness of T^* we infer that

$$(\tilde{T}B_n^*|_E)^* = B_n \circ T_1. \tag{5}$$

Put $S_n = B_n^*|_E$ and $A_n = \tilde{T}S_n$.

Let now $\varepsilon > 0$ be given arbitrarily. Since T_1 is weakly compact there is an $\delta > 0$ so that

$$\mu(A) < \delta \Rightarrow 3\|S^*\| \int_A |T_1 x^*| \, d\mu \leqslant \varepsilon,\ x^* \in X^*,\ \|x^*\| \leqslant 1. \tag{6}$$

Choosing now the number n_0 so large that $\mu(C_n) < \delta$ for all $n \geqslant n_0$ where

$$C_n = \{t \in K \mid \ \|\varphi(t) - \varphi_n(t)\| > \varepsilon\}$$

we get for all $x \in E$ and all $x^* \in X^*$ with $\|x\| \leqslant 1$, $\|x^*\| \leqslant 1$.

$$\begin{aligned} |\langle x^*, TSx - A_n x\rangle| &= \left|\int \cdot (T_1 x^*)(t) \langle \varphi(t) - \varphi_n(t), x\rangle \, d\mu(t)\right| \\ &\leqslant \int |(T_1 x^*)(t)| \, \|\varphi(t) - \varphi_n(t)\| \, d\mu(t) \\ &\leqslant \varepsilon \int_{K \setminus C_n} |T_1 x^*| \, d\mu + \varepsilon \leqslant \varepsilon(\|T\| + 1). \end{aligned} \tag{7}$$

Hence $\|A_n - TS\| \leqslant \varepsilon(\|T\| + 1)$ for $n \geqslant n_0$ and we have proved the assertion.

If X is a Banach lattice of minimal type and E is a Banach space we denote by $N_X(E, X)$ the closure in $\mathscr{B}_X(E, X)$ of the finite dimensional operators. As a corollary of the above theorem we can prove:

4.20. Theorem. *Let E and F be Banach spaces and X a Banach lattice of minimal type. If $A \in \mathscr{B}_X(F, X)$ and $B\colon E \to F$ is a weakly compact operator then $AB \in N_X(E, X)$.*

Proof. Since A is order bounded and X is of minimal type, A admits a factorization through a $C(K)$-space say

$$\begin{tikzcd} F \arrow[rr, "A"] \arrow[dr, "A_1"'] & & X \\ & C(K) \arrow[ur, "T"'] & \end{tikzcd}$$

where T is a positive weakly compact operator so that

$$|Ax| \leqslant T(1) \quad \text{for all } x \in F,\ \|x\| \leqslant 1. \tag{1}$$

Setting $S = A_1 B$ it is readily seen that we are in the same situation as in the previous theorem; we shall use the same terminology and make the same constructions as there. However, instead of inequality (7) we proceed as follows:

Let $x \in E$, $\|x\| \leqslant 1$ and $x^* \in X^*$, $x^* \geqslant 0$ $\|x^*\| \leqslant 1$, we get:

$$(2) \qquad \begin{aligned} |\langle x^*, ABx - A_n x\rangle| &= |\langle x^*, TSx - A_n x\rangle| \\ &\leqslant \int (T_1 x^*)(t) \|\varphi_n(t) - \varphi(t)\| d\mu(t) \\ &\leqslant \langle x^*, \tilde{T}(\|\varphi(\cdot) - \varphi_n(\cdot)\|)\rangle \end{aligned}$$

and hence

$$|ABx - A_n x| \leqslant \tilde{T}(\|\varphi(\cdot) - \varphi_n(\cdot)\|).$$

Calculating now the norm of this element we get for $n \geqslant n_0$

$$\begin{aligned} \|\tilde{T}(\|\varphi(\cdot) - \varphi_n(\cdot)\|)\| &= \sup_{\|x^*\| \leqslant 1} \left|\int (T_1 x^*)(t) \|\varphi(t) - \varphi_n(t)\| d\mu(t)\right. \\ &\leqslant \varepsilon + \sup_{\|x^*\| \leqslant 1} \int_{K \setminus C_n} |(T_1 x^*)(t)| \|\varphi(t) - \varphi_n(t)\| d\mu(t) \leqslant \varepsilon \|T\| + \varepsilon. \end{aligned}$$

Hence for $n \geqslant n_0$

$$b_X(AB - A_n) \leqslant \varepsilon(\|T\| + 1). \qquad \blacksquare$$

As a corollary of this result we infer:

4.21. Corollary. *Let E be a reflexive Banach space then $N_X(E, X) = \mathscr{B}_X(E, X)$.*

In particular all order bounded operators are compact.

4.22. Theorem. *Let E and F be Banach spaces with E^* separable and let K be a compact Hausdorff space. Let S: $E \to C(K)$ be a bounded operator and T: $C(K) \to F$ a weakly compact operator. Then there is a sequence $\{A_n\}$ of finite dimensional operators from E to F so that $\|A_n - TS\| \to 0$ for $n \to \infty$.*

Further the A_n's can be chosen to satisfy conditions similar to those of Theorem 4.19.

In particular TS is compact.

Proof. In the terminology of Theorem 4.19 we have only to show that the map φ defined there is μ-measurable (note, that this was the only place in the proof, where the weak compactness of S was used). Using the separability of E^* we infer that $x^{**}\varphi(\cdot)$ is μ-measurable for all $x^{**} \in E^{**}$, and since φ is essentially separable-valued it follows that φ is μ-measurable. We can now continue as in the proof of Theorem 4.19. $\blacksquare$

From 4.22 we now obtain:

4.23. Theorem. *Let E be a Banach space with separable dual and let X be a Banach lattice of minimal type. Then $N_X(E, X) = \mathscr{B}_X(E, X)$.*

From the last theorems it is seen that in case X is of minimal type then the order bounded operators behave like the p-integral operators and the space $N_X(E, X)$ plays the role of the p-nuclear operators.

In the next section we shall see that any L_p-order bounded operator is in fact p-integral and every operator in $N_{L_p}(E, L_p)$ is p-nuclear (follows from our representation theorem in the next section together with a result of Persson [23]). However due to the factorization scheme of general p-integral operators between Banach spaces E and F it is readily seen that the statement "$N_p(E, F) = I_p(E, F)$ if E^* is separable or E is reflexive" easily can be derived from our Theorems 4.20, 4.23 and Corollary 4.21. Hence these can be considered as generalizations of the Theorems 4 and 5 and Corollary 1 in [23].

All these theorems in turn stem from the two "universal" Theorems 4.19 and 4.22.

Let us finally discuss the following example.

EXAMPLE. The Banach lattices $c_0(\Gamma)$.

It is wellknown and also easy to see that if Γ is an arbitrary set, then a closed subset of $c_0(\Gamma)$ is compact, if and only if is order bounded. This yields that if E is a Banach space and $T \in B(E, c_0(\Gamma))$, then T is order bounded, if and only if it is compact.

We can then show

THEOREM A. *Let E and F be Banach space and Γ some set. Then*

(i) $\mathscr{S}_{c_0(\Gamma)}(E, F) = \mathscr{S}_{c_0}(E, F)$.

(ii) *$T \in \mathscr{S}_{c_0}(E, F)$ if and only if T^* maps ω^*-sequentially compact subsets of F^* into relatively compact subsets of E^*.*

(iii) *If $T \in B(E, F)$ is compact, then $T \in \mathscr{S}_{c_0}(E, F)$.*

(iv) *If F is either separable or reflexive, then $T \in \mathscr{S}_{c_0}(E, F)$ if and only if T is compact.*

Proof. (i). Let $T \in \mathscr{S}_{c_0}(E, F)$ and let $A \in B(F, c_0(\Gamma))$. Put

$$f(t) = \sup\{|(AT)(x)(t)| \mid \|x\| \leqslant 1\}.$$

If $\{t_n\} \subseteq \Gamma$ is an arbitrary infinite sequence, then it follows from the assumption on T, that $f(t_n) \to 0$ $n \to \infty$. Therefore $f \in c_0(\Gamma)$. The other direction of (i) follows from Theorem 4.2.

(ii) follows immediately from Theorem 4.2.

(iii) is trivial in view of the remarks above.

(iv) If F is separable, then the unit ball of F^* is ω^*-sequentially compact, hence from (ii) we get that if $T \in \mathscr{S}_{c_0}(E, F)$, then T^* is compact and therefore T is compact.

If F is reflexive then the unit ball of F^* is ω^*-sequentially compact by Eberlein's theorem and we get the conclusion as before. ■

It follows from Amir and Lindenstrauss [1] that the conditions on F in (iv) can be replaced by the more general condition that F is weakly compactly generated.

Let us also mention that it follows from the Dunford Pettis theorem that every weakly compact operator $T \in B(l_\infty, l_\infty)$ is normable by c_0, since every operator from l_∞ to c_0 is weakly compact. Furthermore from 4.19 and 4.22 it follows that if E is either reflexive or has separable dual then $B(E, l_\infty) = \mathscr{S}_{c_0}(E, l_\infty)$.

Note finally that a theorem for c_0 analogous to 4.14 is untrue; indeed in l_1 weak convergence and norm convergence of sequences coincide, but since l_∞ has a quotient isomorphic to l_2 [18], there is a non-compact operator from l_∞ to c_0, and therefore the identity in l_∞ is not normable by c_0.

Section 5

BANACH FUNCTION LATTICES AND THE DUALITY THEOREM OF SCHWARTZ

In this section we shall study the operator ideals defined before in case X is a Banach lattice of measurable functions on some probability space $(\Omega, \mathcal{S}, \mu)$ and make the connection between the ideals defined here and the ideals of radonifying operators defined by Schwartz (see for example [31]). Further we shall prove that the so-called Schwartz duality theorem actually characterizes the L_p-spaces among lattices.

Let $(\Omega, \mathcal{S}, \mu)$ be a probability space and let $L_0(\mu)$ be the space of all equivalence classes (mod μ) of μ-measurable real valued functions defined on Ω. We equip $L_0(\mu)$ with the natural ordering "$\leqslant$ a. e." and we let $L_0^+(\mu)$ denote the positive cone of this ordering.

We recall [20] that a function $\varrho\colon L_0^+(\mu) \to [0, \infty]$ is called a *function norm* on $L_0(\mu)$, if the following conditions are satisfied:

(I) $\varrho(f) = 0$ if and only if $f = 0$ a. e., $\varrho(f_1+f_2) \leqslant \varrho(f_1) + \varrho(f_2)$, $\varrho(af) = a\varrho(f)$ when $a \geqslant 0$.

(II) If $\{f_n\} \subseteq L_0^+(\mu)$ $f_n \uparrow f$ a. e. then $\varrho(f_n) \uparrow \varrho(f)$.

(III) If $A \in \mathcal{S}$, then $\varrho(1_A) < \infty$.

(IV) There is a constant $K \geqslant 0$ so that

$$\int f d\mu \leqslant K\varrho(f) \quad \text{for } f \in L_0^+(\mu).$$

It is wellknown that the set

$$L_\varrho(\mu) = \{f \in L_0(\mu) \mid \varrho(|f|) < \infty\}$$

is a subspace of $L_0(\mu)$ and that the function $f \to \varrho(|f|)$ is a norm on $L_\varrho(\mu)$ turning it into a Banach space. Further under the ordering "$\leqslant$ a. e." $L_\varrho(\mu)$ is a Banach lattice (general reference [20]).

Our first theorem is the following:

5.1. THEOREM. *Let $(\Omega, \mathcal{S}, \mu)$ be a probability space and let ϱ be a function norm on $L_0(\mu)$. If E is a Banach space and $T \in B(E, L_\varrho(\mu))$ then T is $L_\varrho(\mu)$-order bounded if and only if there is a function $\varphi\colon \Omega \to E^*$ with the properties:*

(i) $(Tx)(t) = \langle \varphi(t), x \rangle$ *for all* $x \in E$ *and almost all* $t \in \Omega$.

(ii) *There is a function* $f \in L_\varrho(\mu)$ *so that* $\|\varphi(t)\| \leqslant f(t)$ *for almost all* $t \in \Omega$.

Proof. The "if" part is trivial so let us suppose that T is $L_\varrho(\mu)$-order bounded.

We can then find an $f \in L_\varrho(\mu)$ so that

$$|Tx| \leqslant f\|x\| \quad \text{for all } x \in E. \tag{1}$$

Let us define the operator $S\colon E \to L_\infty(\mu)$ by

$$(Sx)(t) = \begin{cases} \dfrac{(Tx)(t)}{f(t)} & \text{if } f(t) \neq 0, \\ 0 & \text{else}. \end{cases} \tag{2}$$

We have clearly

$$\|Sx\|_\infty = \operatorname*{ess\,sup}_{t \in \Omega} |(Sx)(t)| \leqslant \|x\| \quad \text{for } x \in E \tag{3}$$

so $\|S\| \leqslant 1$.

By the lifting theorem of Tulcea [34] we can find a $\psi\colon \Omega \to L_\infty(\mu)^*$ so that

$$\|\psi(t)\| \leqslant 1 \quad \text{for almost all } t \in \Omega, \tag{4}$$

$$g(t) = \langle g, \psi(t) \rangle, \quad g \in L_\infty(\mu) \text{ and almost all } t \in \Omega. \tag{5}$$

Let us now define:

$$\varphi(t) = f(t) S^*\big(\psi(t)\big). \tag{6}$$

Then for $x \in E$

$$\langle \varphi(t), x \rangle = f(t) \langle \psi(t), Sx \rangle = f(t)(Sx)(t) = (Tx)(t) \tag{7}$$

for almost all $t \in \Omega$.

$$\|\varphi(t)\| \leqslant f(t)\, \|S^*\|\, \|\psi(t)\| \leqslant f(t) \quad \text{a. a. } t \in \Omega. \tag{8}$$

This finishes the proof.

Remark. In general the function φ will not be μ-measurable, and it need not be uniquely determined (mod μ). However it is easily seen that if $T \in \mathscr{B}_{L_\varrho(\mu)}\big(E, L_\varrho(\mu)\big)$ and φ and ψ are two measurable functions satisfying (i) and (ii) of 5.1, then $\varphi = \psi$ a. e.

We have the following proposition concerning the possibility of representing order bounded operators by measurable functions.

5.2. Proposition. *Let* E, μ *and* ϱ *be as in* 5.1, *let* $T \in \mathscr{B}_{L_\varrho(\mu)}\big(E, L_\varrho(\mu)\big)$ *and let* φ *be associated with* T *according to* 5.1.

(i) *If* E^* *is separable, then* φ *is measurable, and hence* φ *is uniquely determined by* $T(\operatorname{mod} \mu)$.

(ii) *If E is reflexive, then there is a measurable function $\bar{\varphi}$ satisfying* (i) *and* (ii) *of* 5.1.

Proof. Case (i) is trivial, since φ has separable range.

(ii). Let $f \in L_\varrho(\mu)$ so that $\|\varphi(t)\| \leqslant f(t)$ for almost all $t \in \Omega$, if $I\colon L_\varrho(\mu) \to L_1(\mu)$ is the formal identity map, then $S = IT \in \mathscr{B}_{L_1(\mu)}(E, L_1(\mu))$. By corollary 4.21 there is a sequence $\{S_n\}$ of finite-dimensional operators. so that $S_n \to S$ in $\mathscr{B}_{L_1(\mu)}(E, L_1(\mu))$ i.e., there is a sequence $\{h_n\} \subseteq L_1(\mu)$ with $\|\{h_n\}\|_{L_1(\mu)} \to 0$ and so that for all $x \in E$:

$$(1) \qquad |(Sx)(t) - (S_n x)(t)| \leqslant h_n(t) \quad \text{a. a. } t \in \Omega.$$

Without loss of generality we can assume that also $h_n \to 0$ a. e. Clearly there exist measurable functions $\varphi_n\colon \Omega \to E^*$ so that for $x \in E$.

$$(2) \qquad (S_n x)(t) = \langle \varphi_n(t), x\rangle \quad \text{a. a. } t \in \Omega$$

hence for all $x \in E$ and all natural numbers n and m

$$(3) \qquad |\langle \varphi_n(t) - \varphi_m(t), x\rangle| \leqslant h_n(t) + h_m(t) \quad \text{a. a. } t \in \Omega.$$

Since $\varphi_n - \varphi_m$ as a mesurable function has essentially separable range we can conclude using the technique from [9], Theorem III.6.11 that

$$\|\varphi_n(t) - \varphi_m(t)\| \leqslant h_n(t) + h_m(t) \quad \text{a. a. } t \in \Omega$$

hence there is a measurable function $\bar{\varphi}\colon \Omega \to E^*$ with $\varphi_n(t) \to \bar{\varphi}(t)$ a. e. and therefore for $x \in E$.

$$(4) \qquad (Sx)(t) = \lim (S_n x)(t) = \lim \langle \varphi_n(t), x\rangle = \langle \bar{\varphi}(t), x\rangle \quad \text{a. a. } t \in \Omega$$

which implies that

$$(Tx)(t) = \langle \bar{\varphi}(t), x\rangle \quad \text{a. a. } t \in \Omega.$$

But since $|Tx| \leqslant f$ a. e. and $\bar{\varphi}$ has essentially separable range we can conclude that $\|\bar{\varphi}(\cdot)\| \leqslant f$ a. e. ■

Throughout this section when we speak about a function norm ϱ on $L_0(\mu)$ we shall also assume that ϱ is absolutely continuous, i. e.

(V) If $(E_n) \subseteq \mathscr{S}$ and $\mu(E_n) \to 0$ then $\varrho(|f| \cdot 1_{E_n}) \to 0$ for every $f \in L_\varrho(\mu)$.

As examples of $L_\varrho(\mu)$-spaces satisfying (I)–(V) we can mention all L_p-spaces $1 \leqslant p < \infty$ and all Orlicz function spaces $L_M(\mu)$, where M is an Orlicz function satisfying the Δ_2-condition for large values of the argument. For further information on the $L_\varrho(\mu)$-spaces we refer to [20], which is our standard reference on that subject.

It is easy to see that if ϱ is a function norm on $L_0(\mu)$ then the Banach lattice $L_\varrho(\mu)$ is boundedly complete. Indeed if $\mathscr{A} \subseteq L_\varrho(\mu)$ is a directed set with $f \geqslant 0$ for all $f \in \mathscr{A}$, and $\sup\{\varrho(f) \mid f \in \mathscr{A}\} < \infty$ then also $\sup\{\int f d\mu \mid f \in \mathscr{A}\} = a < \infty$; we may then find a sequence $\{f_n\} \subseteq \mathscr{A}$, so that $\int f_n d\mu \uparrow a$ and hence there is a function $f \in L_1(\mu)$ so that $f_n \uparrow f$. It is easy to see if $g \in \mathscr{A}$, then $g \leqslant f$ a. e., also $f \in L_\varrho(\mu)$; by (II) and by (V) we get that ϱ satisfies "the monotone convergence theorem of Lebesgue" [20], and $\mathscr{A}$ converges to f.

Our Theorem 5.1 shows that the ideal of operators normable by $L_p(\mu)$, $1 \leqslant p < \infty$ for some probability measure μ coincides with the operator ideal considered by Badrikian [3], Kwapień [17], Schwartz [31] and others under the name of p-decomposable operators, and they have discussed in detail the connection between these operators and the p-radonifying operators (cf. [31] for the definition).

Our next theorem shows that a similar connection holds in case we are working with function norms. Before we can prove it we need a few notational remarks and definitions from the theory of radonifying operators. (The basic reference is [31].)

Let E be a Banach space over the reals.

Suppose that for all $n \in N'$ and all $T \in B(E, R'^n)$ we have given a Radon measure μ_T in R'^n. If for all n, m N' $m \leqslant n$ and all $S \in B(R'^n, R'^m)$ we have $\mu_{ST} = S(\mu_T)$ for $T \in B(E, R'^n)$, then we say that the system $\{\mu_T \mid T \in B(E, R'^n),\ n \in N'\}$ defines a cylindrical measure μ on E.

Let E and F be Banach spaces and $T \in B(E, F)$. If μ is a cylindrical measure on E determined by the system $\{\mu_S \mid S \in B(E, R'^n),\ n \in N'\}$ then we denote by $T(\mu)$ the cylindrical measure determined by the system $\{\mu_{ST} \mid S \in B(F, R'^n),\ n \in N'\}$.

Clearly every Radon measure on a Banach space is a cylindrical measure, and hence if μ is a cylindrical measure on E, we can write that μ is determined by the system $\{T(\mu) \mid T \in B(E, R'^n),\ n \in N'\}$.

If Ω is a topological Hausdorff space with a Radon probability μ and $A\colon E^* \to L_0(\mu)$ is a linear operator, then A defines a cylindrical measure ν_A on E; ν_A is given by

$$(x_1^*, x_2^*, \ldots, x_n^*)(\nu_A) = (Ax_1^*, \ldots, Ax_n^*)(\mu) \qquad \{x_1^*, \ldots, x_n^*\} \subseteq E^*\ n \in N'\}.$$

It is wellknown that for every cylindrical measure ν on E, there is a pair (Ω, μ) and an A as above so that $\nu = \nu_A$. If $A\colon E^* \to L_0(\Omega_0, \mu_0)$ and $B\colon E^* \to L_0(\Omega_\varkappa \mu_\varkappa)$ are linear maps then $\nu_A = \nu_B$ if and only if A and B have the same marginal distributions (i. e. $(Ax_1^*, \ldots, Ax_n^*)(\mu_1) = (Bx_1^*, \ldots, Bx_n^*)(\mu_2)$ for all finite sets $\{x_1^*, \ldots, x_n^*\} \subseteq E^*$).

We now introduce the following definition

5.3. Definition. Let E be a Banach space and Ω a topological space with a Radon probability μ. If ϱ is a function norm on $L_0(\mu)$, then a Radon measure ν on E is said *to be of order* ϱ, if there is a measurable function $\varphi\colon \Omega \to E$ with $\|\varphi(\cdot)\| \in L_\varrho(\mu)$ so that $\varphi(\mu) = \nu$. A cylindrical measure ν is of called of *type* ϱ, if there is an $A \in B(E^*, L_\varrho(\mu))$ with $\nu = \nu_A$.

If F is another Banach space and $T \in B(E, F)$ then T is called *ϱ-radonifying*, if T maps all cylindrical measures of type ϱ into Radon measures of order ϱ.

Remark. If $\varrho = \|\cdot\|_\varrho$ $1 \leqslant p < \infty$, then we get exactly the p-radonifying operators of [31].

5.4. THEOREM. *If one of the two conditions:*

(i) *F is reflexive.*

(ii) *$F = G^*$, where G^* is separable, is satisfied, then an operator $T \in B(E, F)$ is ϱ-radonifying if and only if T^* is normable by $L_\varrho(\mu)$.*

Proof. Assume that T^* is normable by $L_\varrho(\mu)$.

Case 1. F reflexive. Let ν be a cylindrical measure on E, and let $A \in B(E^*, L_\varrho(\mu))$ be an operator determining ν. By Theorem 5.1 and Proposition 5.2 there is a measurable function $\varphi\colon \Omega \to F$ so that

$$(AT^*x^*)(t) = \langle x^*, \varphi(t)\rangle \quad \text{for almost all } t,\ x^* \in F^*, \tag{1}$$

$$\|\varphi(\cdot)\| \in L_\varrho(\mu). \tag{2}$$

It follows that the operator AT^* determines the cylindrical measure $T(\nu)$. Further from (1) we infer that $\varphi(\mu) = T(\nu)$. ■

Case 2. $F = G^*$, G^* separable. Let ν be a cylindrical measure on E of type ϱ and let $A \in B(E^*, L_\varrho(\mu))$ be an operator determining ν and let $\varphi\colon \Omega \to G^{***}$ be associated to AT^* as in 5.1.

Considering G^* as a subspace of G^{***} there is a natural projection P of G^{***} onto G^*. Define

$$\psi(t) = P(\varphi(t)), \quad t \in \Omega.$$

Since G^* is separable ψ is μ-measurable. We define $S \in B(G, L_\varrho(\mu))$ formal by

$$(Sg)(t) = \langle \psi(t), g\rangle, \quad t \in \Omega, g \in G$$

hence $S = AT^*_{|G}$.

It is easily seen from this that if $I\colon G^* \to (G^*, \sigma(G^*, G))$ is the identity map, then

$$(IT)(\nu) = I(\psi(\mu)).$$

Continuing now as in ([31], Proposition XI, 2.1) we infer from this that $T(\nu) = \psi(\mu)$. ■

The following proposition is special case of 4.16.

5.5. PROPOSITION. *Let E and F be Banach spaces, where F satisfies one of the conditions in 5.4. An operator $T \in B(E, F)$ is ϱ-radonifying if and only if T is $\{f_n\}$-absolutely summing with respect to every sequence $\{f_n\} \subseteq L_\varrho(\mu)$, where the functions f_n have disjoint supports.*

Remark 1. It follows f. ex. from [26] that if ϱ is a function norm on $L_0(\mu)$, then $L_\varrho(\mu)$ has the metric approximation property.

Remark 2. In general there is probably a connection between operators $T \in B(E, F)$ with T^* normable by $L_\varrho(\mu)$ and operators $T \in B(E, F)$ mapping cylindrical measures on E of type ϱ into Radon measure on $(F^{**}, \sigma(F^{**}, F^*))$. However we did not check it.

In the theory of p-radonifying operators the so called Schwartz duality theorem [31] plays an essential role, and it is therefore of interest to investigate, if this result can be extended to other classes of radonifying operators; more generally the above investigations of the connection between the Schwartz ideals and ideals of radonifying operators give us an idea of how to formulate the Schwartz duality theorem for general Banach lattices; it can be stated in general as follows: "Let E be a Banach space and X be a Banach lattice. If $T\colon E \to X$ is order bounded, then T^* is normable by X". It is then natural to ask for which Banach lattices X such a statement is true for all Banach spaces E. We are going to show that this property actually characterizes the L_p-spaces $1 \leqslant p < \infty$ among boundedly complete Banach lattices.

Of notational reasons we introduce the following definition:

5.6. Definition. A Banach space X is called an *$(\mathscr{S})$-lattice* if the following implication holds for every Banach space E

$$T \in \mathscr{B}_X(E, X) \Rightarrow T^* \in \mathscr{S}_X(X^*, E^*).$$

The fact that the lattices $L_p(\mu)$, $1 \leqslant p < \infty$, μ finite are $(\mathscr{S})$-lattices is exactly the Schwartz duality theorem. Kwapień's proof [17] of this is perhaps the easiest in the present context, since his formulation of the Schwartz duality theorem is the same as ours.

The following proposition is slightly stronger:

5.7. Proposition. *Let $(\Omega, \mathscr{S}, \mu)$, be a measure space, and let E be a Banach space. If $(1 \leqslant p < \infty)$ and $T \in \mathscr{B}_{L_p(\mu)}(E, L_p(\mu))$ then T is p-integral.*

Proof. Since every order bounded subset of $L_p(\mu)$ is contained in a band in $L_p(\mu)$, which is lattice isomorphic to $L_p(\nu)$ for some finite ν, it is no loss of generality to assume μ finite. If $T\colon E \to L_p(\mu)$ is order bounded then T has a factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "S_1"'] & & L_p(\mu) \\ & L_\infty(\mu) \arrow[ur, "S_2"'] & \end{tikzcd}$$

where S_2 is order bounded (cf. Theorem 5.1). Hence by Kwapień's version of the duality theorem, it follows that S_2 is p-absolutely summing, and from Persson and Pietsch [24], Theorem 46, we get that in fact S_2 is p-integral, and therefore T is also p-integral. ■

Remark. Combining 5.7 with 4.4 we get that $L_p(\mu), 1 \leqslant p < \infty$, μ arbitrary, is an $(\mathscr{S})$-lattice. Before we continue towards our aim let us also mention.

5.8. PROPOSITION. *Let X be a boundedly complete Banach lattice and let E be a Banach space. If $T \in B(E, X)$ is 1-integral then T is order bounded.*

Proof. A direct application of the factorization of 1-integral maps [24] and Corollary 4.10.

We can now prove

5.9. THEOREM. *If X is a Banach lattice of minimal type then the following statements are equivalent.*

(i) *X is an $(\mathscr{S})$-lattice.*

(ii) $T \in \mathscr{B}_X(c_0, X) \Rightarrow T^* \in \mathscr{S}_X(X^*, l_1)$.

(iii) *Either there is a measure space $(\Omega, \mathscr{S}, \mu)$ and a p, $1 \leqslant p < \infty$ so that X is lattice isomorphic to $L_p(\mu)$ or there is a set Γ with X lattice isomorphic to $c_0(\Gamma)$.*

(iv) *If $T \in \mathscr{B}(X^*, X)$ and $T^*X^* \subseteq X$, then $T^* \in \mathscr{S}_X(X^*, X)$.*

Proof. (i) $\Rightarrow$ (ii) is trivial; for $L_p(\mu)$, $1 \leqslant p < \infty$, (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) follow from 5.7 and for $c_0(\Gamma)$ these implications are trivial.

(ii) $\Rightarrow$ (iii): Let $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq X$ be two normalized sequences, each consisting of mutually disjoint positive elements. We want to show that $\{x_n\}$ is equivalent to $\{y_n\}$

To this end let $x \in [x_n]$ with $x = \sum_{n=1}^{\infty} t_n x_n$. and define $T\colon c_0 \to X$ by

$$Tf = \sum_{n=1}^{\infty} t_n f(n) x_n, \quad f \in c_0.$$

T is clearly order bounded, and using our assumption together with Theorem 4.4, we get that $T^{**} \in \Pi^{\omega^*}_{\{y_n\}}(l_\infty, X)$ and hence $T \in \Pi_{\{y_n\}}(c_0, X)$.

Let $\{e_n\}$ denote the unit vector basis of c_0; since $\sum_{n=1}^{\infty} g(n) y_n$ is convergent for all $g \in l_1$ we get that the series

$$\sum_{n=1}^{\infty} \|Te_n\| y_n = \sum_{n=1}^{\infty} t_n y_n$$

is convergent.

By interchanging the role of $\{x_n\}$ and $\{y_n\}$ in the above argument, we obtain that $\{x_n\}$ is equivalent to $\{y_n\}$; hence the conclusion follows from Theorem 2.12.

If we prove that (iv) $\Rightarrow$ (ii) then we are done. Hence suppose (iv) and let $T \in \mathscr{B}_X(c_0, X)$ and $A \in B(l_1, X)$. Since X is of minimal type we get that $T^{**}(l_\infty) \subseteq X$ and that T^{**} is order bounded, hence $T^{**}A^* \in \mathscr{B}_X(X^*, X)$; it is easily checked that $(T^{**}A^*)^*(X^*) \subseteq X$ and that $(T^{**}A^*)^* = AT^*$. By assumption AT^* is then order bounded and thus, since A was arbitrary $T^* \in \mathscr{S}_X(X^*, l_1)$. ■

Let us conclude the present section with an application of the above theorem:

Let $(\Omega, \mathscr{S}, \mu)$ be a probability space and let ϱ be a function norm on $L_0(\mu)$. If $f \in L_0(\mu \times \mu)$. and $x \in \Omega$ then we define $f_x \in L_0(\mu)$ by $f_x(y) = f(x, y)$, $y \in \Omega$ and we put $\varrho_y(f(x, y)) = \varrho(f_x)$. It is easy to see that the function $x \to \varrho_y(f(x, y))$ is μ-measurable. We shall say that ϱ satisfies the Fubini inequality, if there exist constants $K_1 > 0$ and $K_2 > 0$ so that

$$K_1 \varrho_x \varrho_y(f(x, y)) \leqslant \varrho_y \varrho_x(f(x, y)) \leqslant K_2 \varrho_x \varrho_y(f(x, y))$$

for all f $L_0^+(\mu \times \mu)$.

The following theorem is a consequence of Theorem 5.9.

5.10. THEOREM. *If $(\Omega, \mathscr{S}, \mu)$ is a probability space, and ϱ is a function norm on $L_0(\mu)$ satisfying the Fubini inequality, then there exists a positivt measure ν on $\mathscr{S}$, having the same zero-sets as μ and a p, $1 \leqslant p < \infty$, so thae $L_\varrho(\mu)$ is lattice isomorphic to $L_p(\nu)$.*

More specyfically: as sets $L_\varrho(\mu)$ and $L_p(\nu)$ are equal and the formal identity map is a topological isomorphism.

Proof. We want to show that under the assumptions $L_\varrho(\mu)$ is an $(\mathscr{S})$-lattice. Hence let E be a Banach space and $T \in \mathscr{B}_{L_\varrho(\mu)}(E, L_\varrho(\mu))$. By Theorem 5.1 there is a φ: $\Omega \to E^*$ so that for $x \in E$

(i) $(Tx)(t) = \langle \varphi(t), x \rangle$ for almost all $t \in \Omega$.

(ii) $\|\varphi(t)\| \leqslant f(t)$ for almost all $t \in \Omega$ where $f \in L_\varrho(\mu)$.

Let now $\{f_n\}$ be a sequence of positive mutually disjoint elements of $L_\varrho(\mu)$ and let $\{x_n\} \subseteq E$ so that $\sum_{n=1}^{\infty} x^*(x_n) f_n$ is convergent for all $x^* \in E^*$. If k is a natural number, then:

$$\begin{aligned}
\varrho\Big(\sum_{n=1}^{k} \varrho(|Tx_n|) f_n\Big) &= \varrho_s\Big(\sum_{n=1}^{k} \varrho_t(|\langle \varphi(t), x_n \rangle|) f_n(s)\Big) \\
&= \varrho_s \varrho_t\Big(\sum_{n=1}^{k} |\langle \varphi(t), x_n \rangle| f_n(s)\Big) \\
&\leqslant K_2 \varrho_t \varrho_s\Big(\sum_{n=1}^{k} |\langle \varphi(t), x_n \rangle| f_n(s)\Big) \\
&\leqslant K_2 \varrho(f) \sup_{\|x^*\| \leqslant 1} \varrho\Big(\sum_{h=1}^{k} |x^*(x_n)| f_n\Big)
\end{aligned}$$

hence T is $\{f_n\}$-absolutely summing.

Hence we have proved that $L_\varrho(\mu)$ is an $(\mathscr{S})$-lattice and from Theorem 5.9 it follows that $L_\varrho(\mu)$ is lattice isomorphic to $L_p(\nu)$ for some p, $1 \leqslant p < \infty$ and some measure ν. However from the proof of 5.9, the proof of Theorem 2.10 and the results of Bohnenblust it follows that we can find an equivalent norm ϱ_1 on $L_\varrho(\mu)$ in the present case, so that the measure ν can be chosen to be defined on $\mathscr{S}$ by the formula

$$\nu(A) = \varrho_1(1_A), \qquad A \in \mathscr{S}.$$

Further it follows from the quoted places that then $L_\varrho(\mu)$ and $L_p(\nu)$ will be equal as sets and the formal identity map will be a lattice isomorphism. Hence obviously μ and ν also have the same zero-sets. ■

Section 6

SCHWARTZ IDEALS DETERMINED BY UNCONDITIONAL BASIC SEQUENCES IN $L_p(0, 1)$

In this section we determine completely the Schwartz ideals defined by an unconditional basic sequence $\{x_n\}$ in $L_p(0,1)$, $p \geqslant 2$ so that $[x_n]$ contains an isomorph of a Hilbert space, and we give some partial results for $1 \leqslant p < 2$ and for basic sequences, whose closed linear span does not contain a Hilbert space. We prove for example that if $\{x_n\}$ is an unconditional basis in $L_p(0,1)$, $p \geqslant 2$ then an operator is normable by $\{x_n\}$, if and only if its adjoint in 2-absolutely summing.

To prove our main theorem we need the following lemma on unconditional basic sequences in L_p-spaces, which can be found in [10]. However, for the sake of completeness we prefer to give the following proof.

6.1. Lemma. *If $\{x_n\}$ is an unconditional basic sequence in $L_p(0,1)$ $1 \leqslant p < \infty$, then there exist constants m_p and M_p, so that:*

$$m_p \Big\| \Big(\sum_{k=1}^{n} |a_k|^2 |x_k|^2 \Big)^{1/2} \Big\|_p \leqslant \Big\| \sum_{k=1}^{n} a_k x_k \Big\|_p \leqslant M_p \Big\| \Big(\sum_{k=1}^{n} |a_k|^2 |x_k|^2 \Big)^{1/2} \Big\|_p \tag{1}$$

for all $n = 1, 2, \ldots$ and all n-tuples $(a_1, a_2, \ldots, a_n) \in R^n$.

Proof. Let $\{r_k\}$ be the sequence of Rademacher functions on $[0,1]$, i. e.

$$r_k(t) = \operatorname{sign} \sin 2^k \pi t, \qquad t \in [0,1],\ k = 1, 2, \ldots$$

We recall that by the Khinchin inequality (see for example [18]), we can find constants A_p and B_p so that

$$A_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2} \leqslant \Big(\int_0^1 \Big| \sum_{k=1}^{n} a_k r_k(t) \Big|^p dt \Big)^{1/p} \leqslant B_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2} \tag{2}$$

for every natural number n and all n-tuples $(a_1, a_2, a_n) \in R^n$.

Let K be the unconditional constant for $\{x_n\}$. If $a_1, a_2, \ldots, a_n$ are scalars, then by (2)

$$(3)\qquad A_p\Big(\sum_{k=1}^{n}|a_k|^2|x_k(s)|^2\Big)^{1/2} \leqslant \Big(\int_0^1\Big|\sum_{k=1}^{n}a_k r_k(t)x_k(s)\Big|^p\,dt\Big)^{1/p} \leqslant B_p\Big(\sum_{k=1}^{n}|a_k|^2|x_k(s)|^2\Big)^{1/p}$$

Taking the p-norm on both sides of (3) and using the Fubini theorem we get

$$(4)\qquad A_p\Big\|\Big(\sum_{k=1}^{n}|a_k|^2|x_k|^2\Big)^{1/2}\Big\|_p \leqslant \Big(\int_0^1\Big\|\sum_{k=1}^{n}a_k r_k(t)x_k\Big\|_p^p\,dt\Big)^{1/p} \leqslant B_p\Big\|\Big(\sum_{k=1}^{n}|a_k|^2|x_k|^2\Big)^{1/2}\Big\|_p.$$

For almost all $t \in [0,1]$ we have

$$(5)\qquad \frac{1}{K}\Big\|\sum_{k=1}^{n}r_k(t)a_k x_k\Big\|_p \leqslant \Big\|\sum_{k=1}^{n}a_k x_k\Big\|_p \leqslant K\Big\|\sum_{k=1}^{n}a_k r_k(t)x_k\Big\|_p.$$

Combining (4) and (5) we get the existence of the constants m_p and M_p and inequality (1). ■

Our main theorem of this section is:

6.2. THEOREM. *Let $p \geqslant 2$ and let $\{x_n\}$ be an unconditional basic sequence in $L_p(0,1)$. We have*

(i) *Every linear operator with 2-absolutely summing adjoint is normable by $\{x_n\}$.*

(ii) *If $[x_n]$ contains a subspace isomorphic to l_2, then a linear operator is normable by $\{x_n\}$ if and only if it has 2-absolutely summing adjoint.*

(iii) *Every linear operator with p-absolutely summing adjoint is normable by $\{x_n\}$ ($\|x_n\| = 1$, $n = 1, 2, \ldots$) if and only if $\{x_n\}$ is equivalent to the unit vector basis of l_p.*

Proof. Throughout the proof let E an F be Banach spaces and let $T \in B(E, F)$.

(i) Assume that T^* is 2-absolutely summing, and let K denote the unit ball of F^{**} equipped with the ω^*-topology. There is a positive Radon measure μ on K, so that

$$(1)\qquad \|T^*y^*\| \leqslant \Big(\int_K |y^{**}, (y^*)|^2\,d\mu(y^{**})\Big)^{\frac{1}{2}}.$$

Let now $\{y_n^*\} \subseteq F^*$ so that

$$\sum_{n=1}^{\infty} y_n^*(y) x_n \text{ is convergent for all } y \in F. \tag{2}$$

In view of Theorem 4.2 it is enough to prove that $\sum_{n=1}^{\infty} \|T^* y_n^*\| x_n$ is convergent, and hence, since $\{x_n\}$ is boundedly complete, it follows from Lemma 6.1, that it is enough to prove that there is a constant M, so that for all $k = 1, 2, \ldots$ we have

$$\Big\| \Big(\sum_{n=1}^{k} \big\| T^* y_n^* \big\|^2 . |x_n(t)|^2 \Big)^{1/2} \Big\| \leqslant M. \tag{3}$$

Now by (1) we get for every $t \in [0, 1]$.

$$\Big(\sum_{k=1}^{n} \|T^* y_n^*\|^2 |x_n(t)|^2 \Big)^{1/2} \leqslant \Big(\int_K \sum_{n=1}^{k} |y^{**}(y_n^*)|^2 \, |x_n(t)|^2 d\mu(y^{**}) \Big)^{1/2}$$

and hence:

$$\begin{aligned}
&\int_0^1 \Big(\sum_{n=1}^{k} \|T^* y_n^*\|^2 |x_n(t)|^2 \Big)^{p/2} dt \\
&\leqslant \int_0^1 \Big(\int_K \sum_{n=1}^{k} |y^{**}(y_n^*)|^2 |x_n(t)|^2 d\mu(y^{**}) \Big)^{p/2} dt \\
&\leqslant \int_0^1 \int_K \Big(\sum_{n=1}^{k} |y^{**}(y_n^*)|^2 |x_n(t)|^2 \Big)^{p/2} d\mu(y^{**}) \, dt \\
&\leqslant \int_K \int_0^1 \Big(\sum_{n=1}^{k} |y^{**}(y_n^*)|^2 |x_n(t)|^2 \Big)^{p/2} dt \, d\mu(y^{**}) \\
&\leqslant \frac{1}{m_p^p} \int_K \Big\| \sum_{n=1}^{k} |y^{**}(y_n^*)| x_n \Big\|_p^p d\mu(y^{**}) \\
&\leqslant m_p^{-p} \mu(K) \sup_{\|y\| \leqslant 1} \Big\| \sum_{n=1}^{\infty} |y_n^*(y)| x_n \Big\|_p^p < \infty.
\end{aligned} \tag{4}$$

This proves (i).

(ii) The "if" part follows from (i), and the "only if" part follows from 4.4.

(iii) The "if" part is trivial, so let us assume that every operator with p-absolutely summing adjoint is normable by $\{x_n\}$.

Let $a \in l_p$, and let p^* denote the dual number to p, i. e. $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$.

Define $S\colon l_{p*} \to l_1$ by

$$Sg = ga \quad \text{for all } g \in l_{p*}.$$

Since S^* is p-absolutely summing (it is in fact even p-nuclear) we get by assumption that S is normable by $\{x_n\}$. Let $\{e_n\}$ denote the unit vector basis of c_0. From Theorem 4.2 we get that there is a constant K, so that for all $k = 1, 2, \ldots$

$$\Big\| \sum_{n=1}^{k} |a(n)| x_n \Big\| = \Big\| \sum_{n=1}^{k} \|S^*_{e_n}\| x_n \Big\| \leqslant K \sup_{\substack{\|f\|\leqslant 1 \\ f \in l_1}} \Big\| \sum_{n=1}^{k} f(e_n) x_n \Big\|.$$

Hence $\sum_{n=1}^{\infty} a_n x_n$ is convergent.

On the other hand from Lemma 6.1 we get that if $\sum_{n=1}^{\infty} a_n x_n$ is convergent, then for all $k = 1, 2, \ldots$

$$\sum_{n=1}^{k} |a_n|^p = \sum_{n=1}^{k} |a_n|^p \|x_n\|_p^p = \Big\| \Big(\sum_{n=1}^{k} |a_n|^p |x_n|^p \Big)^{1/p} \Big\|_p^p$$

$$\leqslant \Big\| \Big(\sum_{n=1}^{k} |a_n|^2 |x_n|^2 \Big)^{1/2} \Big\|_p^p \leqslant \frac{1}{m_p^p} \Big\| \sum_{n=1}^{k} a_n x_n \Big\|_p^p$$

and hence $\sum_{n=1}^{\infty} |a_n|^p < \infty$.

The above shows that $\{x_n\}$ is equivalent to the unit vector basis of l_p.

As a special case of Theorem 6.2 we get

6.3. Corollary. *Let $p \geqslant 2$ and let $\{x_n\}$ be an unconditional basis in $L_p(0, 1)$. A linear operator is normable by $\{x_n\}$ if and only if it has 2-absolutely summing adjoint.*

Remark. Note that the assumption $p \geqslant 2$ was heavily used in the proof of Theorem 6.2, both in (i) and (iii).

6.4. Corollary. *Let $p > 2$ and let X be an $\mathscr{L}_p$-space with an unconditional basis $\{x_n\}$. If E is an $\mathscr{L}_q$-space with $q \geqslant 2$ and F is an $\mathscr{L}_1$-space, then every bounded operator from E to F is normable by $\{x_n\}$.*

Proof. A direct application of 6.2 (i) and Theorem 4.3 of [18]

Using 4.4 we get immediately:

6.5. Proposition. *If $\{x_n\}$ is an unconditional basis in $L_p(0, 1)$ $1 < p < \infty$, then every linear operator which is normable by $\{x_n\}$ has p-absolutely summing adjoint.*

Let $p, q > 1$ and $\{x_n\}$ and $\{y_n\}$ unconditional bases in $L_p(0, 1)$ and $L_q(0, 1)$ respectively; it follows from Theorem 6.2 and its corollary that if $p, q \geqslant 2$ then $\{x_n\}$ and $\{y_n\}$ are equinorming. On the other hand we see from 6.5, that if $p < 2$ and $p < q$, then $\{x_n\}$ is finer norming than $\{y_n\}$, but $\{x_n\}$ and $\{y_n\}$ are not equinorming.

We would also like to point but that in Proposition 6.5 the term "basis" can be interchanged with the term "basic sequence" if $p \geqslant 2$, since it follows from [15] that any basic sequence in $L_p(0, 1)$, $p \geqslant 2$ contains an isomorph of either l_p or l_2. However this interchangement is not possible in case $1 < p < 2$, since here $L_p(0, 1)$ contains a subspace isometric to l_r for each r, $p \leqslant r \leqslant 2$, [15].

Our main problem in this section is of course

6.6. PROBLEM. *What is the Schwartz ideal determined by an unconditional basis in* $L_p(0, 1)$, $1 < p < 2$.

Or a more concrete one:

6.7. PROBLEM. *Let* $\{x_n\}$ *be an unconditional basis in* $L_p(0, 1)$, $1 < p < 2$. *Is an operator normable by* $\{x_n\}$ *if and only if its adjoint is* p*-absolutely summing?*

On unconditional basic sequences in $L_1(0, 1)$ we have the following proposition, which we state without proof (the proof is almost trivial).

6.8. PROPOSITION. *Let* $\{x_n\}$ *be an unconditional basic sequence in* $L_1(0, 1)$ *so that* $[x_n]$ *is non-reflexive.*

Then an operator is normable by $\{x_n\}$ *if and only if it has* 1*-absolutely summing adjoint.*

The problem of finding the Schwartz ideal determined by an unconditional basic sequence $\{x_n\} \in L_1(0, 1)$ for which $[x_n]$ is reflexive is exactly the same as the problem of finding the Schwartz ideals determined by unconditional basic sequences in the spaces $L_p(0, 1)$, $1 < p < 2$ Indeed $L_1(0, 1)$ contains subspaces isometric to $L_p(0, 1)$ for all p, $1 \leqslant p < 2$, and on the other hand it was proved recently by Rosenthal [28] that every reflexive subspace of $L_1(0, 1)$ is isomorphic to a subspace of $L_p(0, 1)$ for some p, $1 < p \leqslant 2$.

Let us finally state the following problem on basic sequences in $L_p(0, 1)$, $p > 2$.

6.9. PROBLEM. *Let* $\{x_n\}$ *be an unconditional basic sequence in* $L_p(0, 1)$, $p > 2$. *Does one of the following two possibilities occur:*

(i) *The class of operators normable by* $\{x_n\}$ *is exactly the class of operators with* 2*-absolutely summing adjoints.*

(ii) *The class of operators normable by* $\{x_n\}$ *is exactly the class of operators with* p*-absolutely summing adjoints.*

In view of Theorem 6.2 it is natural to ask, the question: "Let $\{x_n\}$ be an unconditional basic sequence in $L_p(0, 1)$, $p > 2$, so that $[x_n]$ does not contain a Hilbert space. Is every p-abolutely summing operator normable by $\{x_n\}$?" The answer to this question is negative as it is seen from the next proposition.

6.10. PROPOSITION. *In the space l_p, $p > 2$, there is an unconditional basis $\{x_n\}$ so that the Schwartz ideal determined by $\{x_n\}$ is exactly the ideal of operators with 2-absolutely summing adjoints.*

Proof. It was proved by Pełczyński [21] that the space $X_p = \left(\sum_{n=1}^{\infty} l_2^n\right)_{l_p}$ is isomorphic to l_p, hence the natural basis in X_p is equivalent to an unconditional basis $\{x_n\}$ in l_p, and it is clearly not equivalent to the unit vector basis of l_p. Since $\{x_n\}$ contains the spaces l_2^n in blocks, it is readily seen that every operator normable by $\{x_n\}$ has 2-absolutely summing adjoint. The rest now follows from the proof of Theorem 6.2. ■

Since also the space X_p defined above is isomorphic to l_p, in case $1 < p < 2$, the following question is of vital interest for problem 6.6.

6.11. PROBLEM. *What is the Schwartz ideal determined by the natural basis in X_p in case $1 < p < 2$.*

Inspired from Proposition 6.10 we can pose

6.12. PROBLEM. *Let $\{x_n\}$ be an unconditional basis in $L_p(0, 1)$ $1 < p < 2$. Does there exist an unconditional basis $\{y_n\}$ in l_p, so that $\{x_n\}$ and $\{y_n\}$ are equinorming?*

Remark. For $p \geq 2$ it follows from 6.3 and 6.10 that the answer to 6.12 is positive.

If $\{x_n\}$ is a basis in $L_p(0, 1)$ $1 \leq p < \infty$ and $E_n = \text{span}\,\{x_1, \ldots, x_n\}$, then $\left(\sum_{n=1}^{\infty} E_n\right)_{l_p}$ is isomorphic to l_p. Indeed there are finite-dimensional subspaces $F_n \subseteq L_p(0,1)$ so that $d(l_p^{\dim F_n}, F_n) \leq 2$ and $E_n \subseteq F_n$, and hence

$$\left(\sum_{n=1}^{\infty} E_n\right)_{l_p} \subseteq \left(\sum_{n=1}^{\infty} F_n\right)_{l_p} \sim l_p$$

but since there are uniformly bounded projections P_n of F_n onto E_n, we also get that $\left(\sum_n E_n\right)_{l_p}$ is isomorphic to a complemented subspace of l_p and hence isomorphic to l_p.

The next problem is somewhat more concrete than the preceding one.

6.13. PROBLEM. *Let $\{x_n\}$ be an unconditional basis in $L_p(0, 1)$ $1 < p < \infty$, and let E_n be defined as above. Are $\{x_n\}$ and the natural basis in $\left(\sum_n E_n\right)_{l_p}$ equinorming?*

The Problems 6.12 and 6.13 have corresponding problems for basic sequences. Let us end this section by stating these.

6.14. PROBLEM. *Let $\{x_n\}$ be an unconditional basic sequence in $L_p(0, 1)$;*

$1 < p < \infty$. *Does there exist an unconditional basic sequence* $\{y_n\}$ *in* l_p *so that* $\{x_n\}$ *and* $\{y_n\}$ *are equinorming?*

If $\{x_n\}$ is a basic sequence in $L_p(0, 1)$ we define the space E_n as before and using the first part of the argument preceding 6.12 we get that $(\sum_n E_n)_{l_p}$ is isomorphic to a subspace of l_p.

Hence the following problem is more concrete that 6.12.

6.15. PROBLEM. *Let* $\{x_n\}$ *be an unconditional basic sequence in* $L_p(0, 1)$ $1 < p < \infty$. *Are* $\{x_n\}$ *and the natural basis in* $(\sum_n E_n)_{l_p}$ *equinorming?*

Section 7

SOME CONCLUDING REMARKS AND SOME OPEN PROBLEMS

It seems, as the assumptions in Theorem 4.7 can be weakened; here we do not think so much on the approximation assumption as on the boundedly completeness of X. We can pose

7.1. Problem. *Can the condition "X boundedly complete" be weakened in Theorem* 4.7 ?

Is 4.7 *true, if X is just of minimal type* ?

The following problem is very interesting

7.2. Problem. *Let X be a Banach lattice of minimal type and let E be a Banach space, so that $B(E, X) = \mathscr{B}_X(E, X)$. Is E finite dimensional*? *What is the situation, if $X = c_0$*?

It follows from the results in Section 4 that to solve Problem 7.2 it is enough to consider the case, where the lattice structure in X is defined by an unconditional basis $\{x_n\}$ in X. If $\{x_n\}$ is boundedly complete and X contains a subspace isomorphic to l_p for some p, $1 \leqslant p < \infty$, then it follows from Corollary 4.6 that E is finite-dimensional; hence in particular, if $\{x_n\}$ is boundedly complete, but X is not reflexive, then E is finite dimensional. However it is a wellknown problem, whether every reflexive Banach space with an unconditional boundedly complete basis contains a subspace isomorphic to l_p for some $p, 1 < p < \infty$. We feel that 7.2 might be easier than this problem and has a positive solution.

In case $X = c_0$ almost nothing is known. If E has the property stated in 7.2, then ω^*-convergence and norm convergence of sequences in E^* coincide, and therefore if the unit ball in E^* is ω^*-sequentially compact (f. ex. if E is weakly compactly generated, see [1] and the example in Section 4), then E is finite dimensional.

It follows from the example in Section 4, that if $B(E, c_0) = \mathscr{B}_{c_0}(E, c_0)$, then every operator from E to a separable Banach space is compact, and hence all separable quotients of E are finite dimensional. This gives the link to the problem, whether every Banach space has a separable, infinite dimensional quotient space.

The next problem is of a similar nature as 7.2.

7.3. PROBLEM. *Let E and F be Banach spaces and X a Banach lattice of minimal type. Suppose that $\mathscr{S}_X(E, F) = B(E, F)$. What can be said about E and F?*

Here very little is known, even in case $X = l_p$ $1 \leqslant p < \infty$. If $X = l_1$ there are some partial solutions [18]. If $X = c_0$, then it is easy to see that if E is either reflexive or has separable dual and F is a $C(K)$-space, which is a Grothendieck space (i. e. ω^*-convergence and weak convergence of sequences in F^* coincide; l_∞ is such a space), then $B(E, F) = \mathscr{S}_{c_0}(E, F)$. We make the following conjecture.

CONJECTURE. If $\mathscr{S}_{c_0}(E, F) = B(E, F)$, then E is either reflexive or has separable dual, and F is isomorphic to a $C(K)$-space, which is a Grothendieck space.

Clearly a positive verification of this conjecture would imply a positive solution to 7.2.

7.4. PROBLEM. *Let $\mathscr{A}$ be an operator ideal. How can one decide, whether or not $\mathscr{A}$ is determined by a Banach lattice?*

Another series of problems is concerned with determining the Schwartz ideals for infinite product of Banach lattices.

Let $\{X_n\}$ be a sequence of Banach lattices and let $1 \leqslant p \leqslant \infty$, and put $X = (\sum_{n=1}^{\infty} X_n) l_p$ $(X = (\sum_{n=1}^{\infty} X_n)_{c_0}$ if $p = \infty)$. X is a Banach lattice when equipped with the product ordering. We may ask:

7.5. PROBLEM. *Under which assumptions can the Schwartz ideal be determined by an expression, involving the ideals $\mathscr{S}_{X_n}$ and the ideal of p-absolutely summing operators?*

Problem 7.5 is probably easier, if we assume that the order structure in every X_n is induced by an unconditional basis in X_n; the product ordering will namely then be induced by the natural unconditional basis in X, defined by the given bases in the X_n's.

Problem 7.5 can also be posed in a much more general manner, involving the sum of a sequence of lattices in the sense of some unconditional basis; such sums are considered in [8].

It is clear that the solution of Problem 7.5 will have a great impact on the problems we considered in Section 6 for $1 < p < 2$. The following special case of 7.5 is of particular interest for these problems:

7.6. PROBLEM. *Let $1 < p < 2$. What is the Schwartz ideal determined by the lattice $(\sum_{n=1}^{\infty} l_2)_{l_p}$?*

We recall that if $1 \leqslant p \leqslant q < \infty$ and E and F are Banach spaces, then an operator $T \in B(E, F)$ is called *(p, q)-summing* if there is a constant $K \geqslant 0$ so that

$$\Big(\sum_{i=1}^{n}\|Tx_i\|^q\Big)^{1/q} \leqslant K \sup_{\|x^*\|\leqslant 1}\Big(\sum_{i=1}^{n}|x^*(x_i)|^p\Big)^{1/p}$$

for every finite set $\{x_1, x_2, \ldots, x_n\} \subseteq E$.

It is easy to see that if $T \in B(E, F)$ then T^* is (p, q)-summing, if and only if T has the property: For every operator $A\colon F \to l_p$ IAT maps the unit ball of E into an order bounded set in l_q, I denoting the formal identity map of l_p into l_q.

Using the same technique as in Section 4 we easily get that the roles of T and T^* in the above statement can be interchanged. From this we can for example prove that if T is (p, q)-summing, then so is T^{**}. It seems that such a proof is simpler than the proof of Simmons [33], though we also (as he does) use the principle of local reflexivity.

In general it would be of interest to see, if the above (rather simple) characterization of (p, q)-summing operators can be useful in the theory of these operators. For example, does it have any impact on the problems concerning composition rules for (p, q)-summing operators?

Concerning Section 5 we can make the following remarks:

In [7] E. Dubinsky and M. S. Ramanujan define Λ-nuclear and Λ-absolutely summing operators for certain sequence spaces Λ. If Λ is the sequence space associated to some unconditional basis $\{x_n\}$ in a Banach space, then "Λ-absolutely summing" is the same as our term "$\{x_n\}$-absolutely summing". Furthermore an $\{x_n\}$-order bounded operator is Λ-nuclear in their terminology. They ask for which Banach spaces E and F it is true that every Λ-nuclear operator is Λ-absolutely summing. Using the theorem of Zippin and the technique of Theorem 5.10 we get the following partial result concerning the above question.

7.7. Proposition. *If Λ is the sequence space associated to an unconditional basis in a Banach space then every Λ-nuclear operator is Λ-absolutely summing, if and only if $\Lambda = l_p$ for some p, $1 \leqslant p < \infty$ or c_0.*

We conclude the present section here. For problems and remarks o Section 6, we refer to that section itself.

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