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GENERALIZED ANALYTIC FUNCTIONS AND A STRONG QUASI-ANALYTICITY PRINCIPLE

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The aim of my lecture is to present some results from the theory of generalized analytic functions (GAF for short). I will pay a special attention to deriving from the theory of GAF's a kind of quasi-analyticity principle. Let me say that GAF's behave in a simple way under basic algebraic and differential operations, and analytic change of variables. As pointed out by B. Ziemian, they form a natural subclass of distributions conormal to zero, whose symbols have a very explicit form (cf. [Zie]). Let me also say that GAF's appear naturally as solutions to singular differential equations, and practically all special functions, after suitable change of variables, are examples of GAF's.

The starting point for the theory of GAF's is the observation that a function u analytic in the open disc $B(r) = \{x \in \mathbb{C} : |x| < r\}$ can be written in the form

$$u(x) = S[x^{\cdot}], \tag{1}$$

where

$$S = \sum_{\alpha=0}^{\infty} a_\alpha \delta_{(\alpha)}$$

is a functional acting on the function $\alpha \to x^\alpha$ and the coefficients $a_\alpha, \alpha \in \mathbb{N}_0$, satisfy the condition: for every $\kappa > 0$ there exists $C < \infty$ such that $|a_\alpha| \le C r_{-\kappa}^{-\alpha}$, where $r_{-\kappa} = re^{-\kappa}$.

Looking at the formula (1), we see that for S we can choose an element of a certain class of functionals Q', acting on functions $\alpha \to x^\alpha$, getting a GAF from the class GAF(Q') determined by Q'. For example, if we take $S = Y$ (the

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Heaviside function) then

$$u(x) = \int\limits_0^\infty x^\alpha \, d\alpha = \frac{1}{-\ln x}, \qquad 0 < x < 1,$$

is a GAF but not an analytic function in $B(1)$.

Let us start our review of GAF's from practically the most general class, namely the class determined by the space of analytic functionals $Q'(\Gamma, r)$ with carrier in Γ. Here Γ is a cone set

$$\Gamma = \{\zeta \in \mathbb{C} : k_1 \operatorname{Re}(\zeta - w) \le \operatorname{Im}(\zeta - w) \le k_2 \operatorname{Re}(\zeta - w)\},$$

where $w \in \mathbb{C}$ is a vertex of Γ and $k_1 \le 0 \le k_2$. In the case when $\Gamma = [w, \infty)$ the theory of such functionals was given by M. Morimoto (cf. [Mo]); the general case was studied in [Ły 1].

The space $Q'(\Gamma, r)$ is defined as the dual space of

$$Q(\Gamma, r) = \varinjlim_{\delta > 0, \kappa > 0} Q_b(\Gamma_\delta, r_{-\kappa}),$$

where for $\delta > 0$ and $\kappa > 0$,

$$Q_b(\Gamma_\delta, r_{-\kappa}) = \{\varphi \in C^0(\overline{\Gamma}_\delta) \cap \mathcal{O}(\Gamma_\delta) : \sup_{\zeta \in \overline{\Gamma}_\delta} |\varphi(\zeta) r_{-\kappa}^\zeta| < \infty\}$$

and Γ_δ denotes the δ-neighbourhood of Γ.

Observe that the function $\zeta \to x^\zeta$ belongs to $Q(\Gamma, r)$ if and only if

$$x \in \Lambda_r^\Gamma = \{x \in \widetilde{\mathbb{C}} : |x| < r \, \exp(\min(k_1 \arg x, k_2 \arg x))\},$$

where $\widetilde{\mathbb{C}}$ denotes the universal covering space of $\mathbb{C} \setminus \{0\}$.

Thus, we can define the Taylor transform of $S \in Q'(\Gamma, r)$ by

$$\mathcal{T}S(x) = S[x^\cdot] \qquad \text{for } x \in \Lambda_r^\Gamma. \tag{2}$$

We have the following characterization of the image of $Q'(\Gamma, r)$ under the Taylor transformation:

THEOREM 1 ([Ły 1], Theorems 4 and 5). *The Taylor transformation gives an isomorphism between $Q'(\Gamma, r)$ and the space* GAF$(Q'(\Gamma, r))$ *of functions $u \in \mathcal{O}(\Lambda_r^\Gamma)$ such that for every $\varepsilon > 0$ and $\kappa > 0$ there exists $C < \infty$ such that*

$$|u(x)| \le C \exp H_{-\Gamma_\varepsilon}(-\ln|x|, \arg x) \qquad \text{for } x \in \Lambda_{r_{-\kappa}}^\Gamma,$$

where H_A denotes the supporting function of a set $A \subset \mathbb{C}$.

Note here that even such general GAF's appear as solutions to singular differential equations. An example ([Ły 1], Example 5) is given by the function

$$u(x) = \exp\left\{ -\frac{\lambda}{1+\delta}(-\ln x)^{1+\delta} \right\}, \qquad \delta > 0, \ \lambda > 0,$$

which belongs to $\mathrm{GAF}(Q'(\Gamma,1))$ with

$$\Gamma = \left\{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| \le \operatorname{ctg}\left(\frac{\pi}{2(1+\delta)}\right)\operatorname{Re}\zeta\right\}$$

and solves the equation

$$(-\ln x)^{\delta} x \frac{du}{dx} = \lambda u.$$

In the sequel we limit our considerations to the case when Γ is a half line. By Theorem 1 we get

Corollary 1. *Let $\Gamma = [w,\infty)$ with $w \in \mathbb{R}$. Then $\Lambda_r^{\Gamma} = \widetilde{B}(r)$ is the universal covering space of $B(r) \setminus \{0\}$ and*

$$\mathrm{GAF}\,(Q'(\Gamma,r)) = \{u \in \mathcal{O}(\widetilde{B}(r)) : \text{for every } \varepsilon > 0,\ \kappa > 0 \text{ there exists } C < \infty \text{ such that } |u(x) \le C|x|^{w-\varepsilon}\exp\varepsilon|\arg x| \text{ for } |x| \le r_{-\kappa}\}.$$

Unfortunately, the GAF's considered above do not in general satisfy the following quasi-analyticity property.

Property A. *If u given by* (1) *is flat of arbitrary order $m \in \mathbb{N}$ on $(0,r)$, then $u \equiv 0$.*

The lack of this property is due to the existence of analytic functionals supported by the point $\{\infty\}$, which introduce non-zero functions flat of infinite order.

To obtain GAF's with Property A we can take the space $L'_{(\ln r)}(\Gamma)$ of Laplace distributions in place of $Q'(\Gamma,r)$. Recall that $L'_{(\ln r)}(\Gamma)$ is the dual space of

$$L_{(\ln r)}(\Gamma) = \varinjlim_{a<\ln r}\ \varprojlim_{k\in\mathbb{N}_0} L_{a,k}(\Gamma),$$

where for any $a \in \mathbb{R}$ and $k \in \mathbb{N}_0$,

$$L_{a,k}(\Gamma) = \{\varphi \in C^{\infty}(\Gamma) : \sup_{y\in\Gamma}\sup_{\alpha\le k}|e^{-ay}D^{\alpha}\varphi(y)| < \infty\}.$$

Observe that the function

$$\Gamma \ni y \to \exp_z(y) = e^{yz}$$

belongs to $L_{(\ln r)}$ if ond only if $\operatorname{Re} z < \ln r$. Thus we can define the Laplace transform of $S \in L'_{(\ln r)}(\Gamma)$ by

$$\mathcal{L}S(z) = S[\exp_z] \quad \text{for } \operatorname{Re} z < \ln r.$$

As in the proof of Theorem 9.1 of [Sz-Zie], we find that $\mathcal{L}S$ is a holomorphic function on $\{\operatorname{Re} z < \ln r\}$, and for every $\kappa > 0$ there exist $C < \infty$ and $m_\kappa \in \mathbb{N}_0$ such that

$$\mathcal{L}S(z) \le C(|\operatorname{Re} z - \ln r| + |\operatorname{Im} z|)^{m_\kappa} e^{w\operatorname{Re} z} \quad \text{for } \operatorname{Re} z \le \ln r - \kappa. \tag{3}$$

Fix $\kappa > 0$ and put

$$P_\kappa(z) = (z - \ln r - 1)^{m_\kappa + 2}.$$

Then by Lemma 9.1 of [Sz-Zie] the function

$$S_\kappa(y) = \frac{1}{2\pi i} \int\limits_{c+i\mathbb{R}} \frac{\mathcal{L}S(z)}{P_\kappa(z)} e^{-yz}\, dy$$

does not depend on the choice of $c \le \ln r - \kappa$; it is continuous on $\mathbb{R}$ with support in Γ and satisfies

$$|S_\kappa(y)| \le C_\kappa r_{-\kappa}^{-y} \quad \text{for } y \in \Gamma. \tag{4}$$

Furthermore, $S_\kappa \in L'_{(\ln r - \kappa)}(\Gamma)$ and

$$\mathcal{L}S_\kappa(z) = \frac{\mathcal{L}S(z)}{P_\kappa(z)} \quad \text{for } \operatorname{Re} z < \ln r - \kappa.$$

Since

$$\mathcal{L}(DS_\kappa)(z) = (-z)\mathcal{L}S_\kappa(z) \quad \text{for } \operatorname{Re} z < \ln r - \kappa,$$

we get the structure theorem for Laplace distributions.

THEOREM 2. *A distribution $S \in D'(\Gamma)$ is in $L'_{(\ln r)}(\Gamma)$ if and only if for every $\kappa > 0$ there exist a polynomial P_κ and a function S_κ continuous on $\mathbb{R}$ with support in Γ satisfying* (4), *and*

$$S = P_\kappa(D) S_\kappa \quad \text{in } L'_{(\ln r - \kappa)}(\Gamma).$$

Since

$$\mathcal{T}S(x) = \mathcal{L}S(\ln x) \quad \text{for } S \in L'_{(\ln r)}(\Gamma) \quad \text{and } x \in \widetilde{B}(r),$$

by (3) we get

THEOREM 3. *Let $S \in L'_{(\ln r)}(\Gamma)$. Put $u(x) = \mathcal{T}S(x)$ for $x \in \widetilde{B}(r)$. Then $u \in \mathcal{O}(\widetilde{B}(r))$, and for every $\kappa > 0$ there exist $C_\kappa < \infty$ and $m_\kappa \in \mathbb{N}$ such that*

$$|u(x)| \le C_\kappa \left(\left| \ln \frac{|x|}{r} \right| + |\arg x| \right)^{m_\kappa} |x|^w \quad \text{for } |x| \le r_{-\kappa}. \tag{5}$$

Our next aim is to give a converse of Theorem 3. To this end we characterize the Mellin transforms of functions holomorphic on $\widetilde{B}(r)$ and satisfying (5).

THEOREM 4. *Let $u \in \mathcal{O}(\widetilde{B}(r))$ satisfy* (5). *Then for every $t < r$ the local Mellin transform of u,*

$$\mathcal{M}_t u(z) = \int\limits_0^t u(x) x^{-z-1}\, dx,$$

defined for $\operatorname{Re} z < w$, extends holomorphically to a function $G_t \in \mathcal{O}(\mathbb{C} \setminus \Gamma)$ satisfying

$$|G_t(z)| \le \begin{cases} C_t\, t^{-\operatorname{Re} z} / |\operatorname{Im} z|^{m_t + 1} & \text{for } z \in \mathbb{C} \setminus \Gamma,\ 0 < |\operatorname{Im} z| \le 1, \\ C_t\, t^{-\operatorname{Re} z} / |\operatorname{Im} z| & \text{for } z \in \mathbb{C} \setminus \Gamma,\ |\operatorname{Im} z| \ge 1. \end{cases}$$

Furthermore, if $t_1 < t_2 < r$ *then the difference* $G_{t_1} - G_{t_2}$ *is an entire function satisfying*

$$|(G_{t_1} - G_{t_2})(z)| \le C \frac{\min(t_1^{-\operatorname{Re} z}, t_2^{-\operatorname{Re} z})}{1 + |\operatorname{Im} z|} \quad \text{for } z \in \mathbb{C}.$$

Proof. The holomorphic extension of $\mathcal{M}_t u$ onto the set $\{\pm \operatorname{Im} z > 0\}$ is given by

$$G_t^{\pm}(z) = \int_{\gamma^{\pm}(t)} u(x) x^{-z-1}\, dx, \tag{7}$$

where $\gamma^{\pm}(t) = \{x \in \widetilde{B}(r) : x = te^{\mp i\varphi}, 0 \le \varphi < \infty\}$.

Indeed, by (5), we get $G_t^{\pm}(z) = \mathcal{M}_t u(z)$ for $\{\operatorname{Re} z < w\} \cap \{\pm \operatorname{Im} z > 0\}$. To prove (6), observe that, by (5) and (7),

$$|G_t^{\pm}(z)| \le C_t t^{-\operatorname{Re} z} \int_0^{\infty} (\ln r - \ln t + \varphi)^{m_t} e^{\pm\varphi \operatorname{Im} z}\, d\varphi \quad \text{for } \pm \operatorname{Im} z > 0.$$

Thus (6) follows easily from the formula

$$\int_0^{\infty} \varphi^k e^{-\varphi y}\, d\varphi = \frac{k!}{y^{1+k}} \quad \text{for } y > 0.$$

The second assertion is clear since

$$(G_{t_1} - G_{t_2})(z) = \int_{t_1}^{t_2} u(x) x^{-z-1} dx \quad \text{for } z \in \mathbb{C}.$$

By Theorem 4, Remark 11.2 of [Zie] and Theorem 11.1 of [Zie], we get

THEOREM 5. *Let* $u \in \mathcal{O}(\widetilde{B}(r))$ *satisfy* (5) *and* $t < r$. *Put*

$$S = \frac{1}{2\pi i}(\mathcal{M}_t u(\cdot + i0) - \mathcal{M}_t u(\cdot - i0)). \tag{8}$$

Then $S \in L'_{(\ln r)}(\Gamma)$ *and* $u(x) = \mathcal{T}S(x)$ *for* $x \in \widetilde{B}(r)$. *Thus, the Taylor transformation is an isomorphism of* $L'_{(\ln r)}$ *onto the space of functions* $u \in \mathcal{O}(\widetilde{B}(r))$ *satisfying* (5).

COROLLARY 2. *Functions from the class* GAF $(L'(\Gamma))$ *have Property A.*

Proof. Let $u(x) = \mathcal{T}S(x)$ for some $S \in L'_{(\ln r)}(\Gamma)$ and $x \in \widetilde{B}(r)$. Then by Theorem 3, $u \in \mathcal{O}(\widetilde{B}(r))$ and (5) holds. The assumption that u is flat of arbitrary order $m \in \mathbb{N}$ on $(0, r)$ implies that $\mathcal{M}_t u$ is an entire function. Thus, the difference of boundary values in (8) vanishes and consequently $u \equiv 0$.

THEOREM 6 (Strong quasi-analyticity principle). *Let* $F \in \mathcal{O}(\operatorname{Re} z \ge 0)$ *and suppose that, for some* $w \in \mathbb{R}$, $m \in \mathbb{N}$ *and* $C < \infty$,

$$|F(z)| \le C(1 + |z|)^m e^{w \operatorname{Re} z} \quad \textit{for } \operatorname{Re} z \ge 0. \tag{9}$$

If for every $l > 0$ there exists $C_l < \infty$ such that

$$|F(z)| \le C_l e^{-lz} \quad \text{for } z \in \mathbb{R}_+, \tag{10}$$

then $F \equiv 0$.

Proof. Put $u(x) = F(-\ln x)$ for $x \in \widetilde{\widetilde{B}}(1)$. Then $u \in \mathcal{O}(\widetilde{\widetilde{B}}(1))$ and $|u(x)| \le C(1 + |\ln|x|| + |\arg x|)^m |x|^w$ for $|x| \le 1$. This means that u is a GAF and by (10) it is flat of arbitrary order $l \in \mathbb{N}$ on $(0, r)$. Hence, by Property A, $u \equiv 0$ and consequently $F \equiv 0$.

Remark. One can define GAF's determined by the space of Laplace ultradistributions $L^{(M_p)'}_{(\ln r)}(\Gamma)$ with the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfying the conditions (M.1), (M.2), (M.3) of [Ko], thus getting a stronger version of Theorem 6 (cf. [Ly 2]). For example, if (M_p) is the Gevrey sequence $M_p = (p!)^s$ with $s > 1$, (9) can be replaced by the weaker condition:

$$|F(z)| \le C \exp(m|z|^{1/s} + w \operatorname{Re} z).$$

Another extension of the strong quasi-analyticity principle can be derived from the study of the Laplace distributions or ultradistributions supported by a convex proper cone in $\mathbb{R}^n$.

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