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WELL-POSEDNESS OF THE DIRICHLET PROBLEM AND HOMOTOPY CLASSIFICATION OF ELLIPTIC SYSTEMS OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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The Dirichlet problem for any elliptic second-order equation with sufficiently smooth coefficients in a domain with sufficiently smooth boundary is always Fredholm [7]. For systems of second-order equations the situation is much more complicated. As shows the example of A. V. Bitsadze [2], the Dirichlet problem for systems of two second-order equations may not even be normally solvable in the sense of Hausdorff. Systems of second-order equations were divided by A. V. Bitsadze into two classes: strongly connected and weakly connected systems [2]. For a strongly connected system without lower order terms the well-posedness of the Dirichlet problem is always violated in some half-plane [10], and for a weakly connected system the Dirichlet problem is well-posed. So far, elliptic systems of second-order partial differential equations with constant coefficients have been studied fairly thoroughly [7]. Compared with the system of A. V. Bitsadze, nothing new arises in the general case. Multidimensional systems of equations with constant coefficients and systems with variable coefficients are not well investigated.

The example of A. V. Bitsadze has considerably stimulated the investigations in the theory of elliptic systems of partial differential equations and has shown that such systems should be classified more accurately. The first result in the classification of elliptic systems was the singling out of the class of strongly elliptic systems [13] for which the solvability character of the basic boundary problems remains the same as for one elliptic equation. For well-posed classical boundary problems the index is a homotopy invariant, therefore it is natural to state the problem of homotopy classification of systems of partial differential equations [5]. For systems of partial differential equations with two independent variables this problem has been solved [4], but for multidimensional elliptic systems there are only various estimates of the number of homotopy classes [3].

The definition of a strongly connected elliptic system of second-order partial differential equations with two independent variables and with constant coefficients is given via the structure of the general solution of the system [2]; moreover, it is not explicitly expressed by means of the coefficients of the system. This makes it more difficult to generalize the notion of strong connectedness to multidimensional systems. Since for strongly connected systems with two independent variables there is always violation of the noetherity of the Dirichlet problem, we make this property a basis of our generalization of the notion of strong connectedness to the multidimensional case.

DEFINITION. An elliptic system of second-order partial differential equations with constant coefficients is called *strongly connected* if there exists a half-space $H = \{\alpha_1 x_1 + \ldots + \alpha_n x_n > 0\}$ such that for the Dirichlet problem in this half-space the noetherity is violated. Violation of noetherity of the Dirichlet problem for the given system implies that the homogeneous problem has an infinite set of linearly independent solutions, or that for the solvability of the nonhomogeneous problem it is necessary to impose an infinite set of orthogonality conditions on the problems under consideration.

At present, all the facts known about the number of distinct homotopy classes of multidimensional second-order elliptic systems are based on the construction of examples of strongly connected systems not homotopic to each other [3, 9]. Therefore the investigation and construction of the classification of strongly connected systems is an important step in the homotopy classification of elliptic systems. Overdetermination or underdetermination measure of the Dirichlet problem is a fairly simple characteristic of strongly connected systems, therefore in the investigation of elliptic systems it is important to have a description of these characteristics and to know their dependence on the structure of the system. Here we consider the properties of a number of concrete strongly connected systems and mention some questions arising when investigating general elliptic systems with constant coefficients.

Among multidimensional strongly connected systems there is a well-known system which is obtained from the system

$$-\Delta u_j + \lambda \frac{\partial}{\partial x_j} \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0, \quad j = 1, \ldots, n, \lambda > 1, \tag{1}$$

for $\lambda = 2$, and for $n = 2$ reduces to the system of A. V. Bitsadze; therefore it is natural to consider it to be a multidimensional analogue of the system of A. V. Bitsadze [14]. It is easy to show that all solutions of system (1) which are regular in the half-space $H = \{\alpha_1 x_1 + \ldots + \alpha_n x_n > 0\}$ may be written in the form

$$u_j = \varphi_j + (\alpha_1 x_1 + \ldots + \alpha_n x_n) \frac{\partial \psi}{\partial x_j}, \quad j = 1, \ldots, n, \tag{2}$$

where φ_j, $j = 1, \ldots, n$, and ψ are arbitrary harmonic functions regular in the half-space H connected by the relation

$$\lambda \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_i} = (2-\lambda) \sum_{i=1}^{n} \alpha_i \frac{\partial \psi}{\partial x_i}. \tag{3}$$

By means of formulae (2) and (3) it is easy to find out that for $\lambda \neq 2$ the Dirichlet problem for system (1) in any half-space is solvable for any differentiable boundary data, and its solution is always unique. If $\lambda = 2$, then the homogeneous Dirichlet problem for system (1) in a half-space has an infinite set of linearly independent solutions of the form

$$u_j = (\alpha_1 x_1 + \ldots + \alpha_n x_n) \frac{\partial \psi}{\partial x_j}, \quad j = 1, \ldots, n.$$

The underdeterminedness of the homogeneous Dirichlet problem is characterized here by one, arbitrary harmonic function ψ regular in H. For $\lambda = 2$ equality (3) turns into a relation connecting the functions φ_j, $j = 1, \ldots, n$. To the system of this relation, for the solvability of the nonhomogeneous Dirichlet problem it is necessary to impose an infinite set of orthogonality conditions upon the boundary data.

Thus system (1) for $\lambda = 2$ is strongly connected, and for $\lambda \neq 2$ it is not so. This example shows that the property of the system to be strongly connected is not a homotopy invariant: it may disappear after a continuous deformation of the system.

For solutions regular in the ball $\Sigma = \{x_1^2 + \ldots + x_n^2 < R^2\}$ we have the representation [10]

$$u_j = \chi_j + \frac{1}{n-2}(x_1^2 + \ldots + x_n^2 - R^2)\frac{\partial \psi}{\partial x_j}, \quad j = 1, \ldots, n, \tag{4}$$

where $\chi_1, \ldots, \chi_n$, ψ are harmonic functions regular in the ball Σ and satisfying the relation

$$\frac{2(2-\lambda)}{n-2} \sum_{i=1}^{n} x_i \frac{\partial \psi}{\partial x_i} + 2\psi = \lambda \sum_{i=1}^{n} \frac{\partial \chi_i}{\partial x_i}. \tag{5}$$

Using formulae (4) and (5) it is easy to show that the Dirichlet problem in the ball Σ for system (1) is solvable for any differentiable boundary data and its solution is unique for all $\lambda > 1$ except

$$\lambda_k = 2 + \frac{n-2}{k}, \quad k = 1, 2, \ldots$$

For $\lambda = 2$ the Dirichlet problem is also solvable and the solution is unique, but it is necessary to have twice differentiable boundary data. For all $\lambda = \lambda_k$ the Dirichlet problem in Σ is Fredholm; moreover, with the growth of k the number of linearly independent solutions of the homogeneous Dirichlet problem grows unboundedly. For those λ it is sufficient to have differentiable boundary data.

The above example shows that for strongly connected systems various new phenomena in the character of the solvability of the Dirichlet problem may occur which have no analogies in the case of one second-order equation. Among such phenomena one should mention the loss of smoothness and the influence of lower order terms upon the solvability of boundary problems. The phenomenon of the influence of lower order terms is known for equations with two independent variables [7], and for multidimensional systems it can easily be shown by the example of the system

$$-\Delta u_j + 2\frac{\partial}{\partial x_j}\sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i} + cu_j = 0, \quad j = 1, \dots, n. \tag{6}$$

The system

$$-L(u_j) + \frac{\partial}{\partial x_j}\sum_{i=1}^{n}\sum_{k=1}^{n}\alpha_{ik}\frac{\partial u_i}{\partial x_k} = 0, \quad j = 1, \dots, n, \tag{7}$$

where L is a fixed elliptic operator with constant coefficients, is a direct generalization of system (1). Upon the jth equation of system (7) we act by the operator

$$l_j = \sum_{k=1}^{n}\alpha_{jk}\frac{\partial}{\partial x_k}$$

and add all the results. Then for the function

$$\Omega = \sum_{i,k=1}^{n}\alpha_{ik}\frac{\partial u_i}{\partial x_k}$$

we obtain the equation $(M-L)\Omega = 0$, where

$$M = \sum_{i,k=1}^{n}\alpha_{ik}\frac{\partial^2}{\partial x_i\,\partial x_k}.$$

It is easy to verify that for the ellipticity of system (7) the ellipticity of the operator M is necessary and sufficient. If the characteristic form of the operator M,

$$\chi(\Xi) = \sum_{i,j=1}^{n}\alpha_{ij}\xi_i\xi_j,$$

is not divisible by the characteristic form of the operator L,

$$\Lambda(\Xi) = \sum_{i,j=1}^{n}a_{ij}\xi_i\xi_j,$$

then it is easy to show that all solutions of system (7) which are regular in some domain D may be written as

$$u_j = \varphi_j + \frac{\partial \psi}{\partial x_j}, \quad j = 1, \ldots, n, \tag{8}$$

where φ_j, $j = 1, \ldots, n$, are solutions of the equation $L(\varphi_j) = 0$ regular in the domain D and connected by the relation

$$\sum_{i,j=1}^{n} \alpha_{ij} \frac{\partial \varphi_i}{\partial x_j} = 0, \tag{9}$$

and ψ is a solution of the equation $N(\psi) = 0$ regular in the domain D. Moreover, $N = M - L$. If the domain D is a half-space, the Dirichlet problem for system (7) may be investigated by means of the Fourier transformation, using formulae (8) and (9). This allows one to determine which systems of the form (7) are strongly connected.

Consider a specific case of system (7) in a three-dimensional space:

$$\begin{aligned} -\Delta u + \frac{\partial}{\partial x} \Omega(u, v, w) &= 0, \\ -\Delta v + \frac{\partial}{\partial y} \Omega(u, v, w) &= 0, \\ -\Delta w + \frac{\partial}{\partial z} \Omega(u, v, w) &= 0, \end{aligned} \tag{10}$$

$$\Omega = \alpha_1 u_x + \beta u_y + \gamma u_z - \beta v_x + \alpha_2 v_y + \delta v_z - \gamma w_x - \delta w_y + \alpha_3 w_z .$$

Here L is the Laplace operator, and the operator N has the form

$$N = (\alpha_3 - 1) \frac{\partial^2}{\partial z^2} + (\alpha_2 - 1) \frac{\partial^2}{\partial y^2} + (\alpha_1 - 1) \frac{\partial^2}{\partial x^2} .$$

Therefore system (10) is elliptic when the α_i, $i = 1, 2, 3$, are all smaller than 1 or all larger than 1. Formulae (8), equality (9) being taken into account, may now be written in the form

$$\begin{aligned} u &= \gamma \varphi_x + \delta \varphi_y - \alpha_3 \varphi_z + \omega_x, \\ v &= \gamma \psi_x + \delta \psi_y - \alpha_3 \psi_z + \omega_y, \\ w &= \alpha_1 \varphi_x + \beta \varphi_y + \gamma \varphi_z - \beta \psi_x + \alpha_2 \psi_y + \delta \psi_z + \omega_z, \end{aligned} \tag{11}$$

where φ and ψ are arbitrary harmonic functions, and ω is a solution of the equation $N(\omega) = 0$.

We will study the Dirichlet problem for system (10) in the half-space $H = \{z > 0\}$ with the conditions

$$u = f_1, \quad v = f_2, \quad w = f_3 \tag{12}$$

on the boundary of H, by means of the Fourier transformations in the variables x and y. The Fourier transforms of the functions φ, ψ and ω, bounded on infinity, have the form

$$A(\xi, \eta)\exp(-\varrho z), \quad B(\xi, \eta)\exp(-\varrho z), \quad C(\xi, \eta)\exp(-\varrho_1 z),$$
$$\varrho = (\xi^2+\eta^2)^{1/2}, \quad \varrho_1 = \{(\alpha_3-1)^{-1}[(\alpha_1-1)\xi^2+(\alpha_2-1)\eta^2]\}^{1/2}.$$

Passing to Fourier transforms in (11) and (12) and substituting (11) into (12), we obtain a system of linear equations for the functions A, B and C with determinant given by the formula

$$D(\xi, \eta) = (\varrho_1-\varrho)[-\alpha_3\varrho+i(\gamma\xi+\delta\eta)][i(\gamma\xi+\delta\eta)+\alpha_3\varrho_1-\varrho_1-\varrho]. \tag{13}$$

The determinant $D(\xi, \eta)$ may be zero at points different from the point $\xi = \eta = 0$ just due to the first and the third factor, and it may be identically zero only due to the third factor. For $D(\xi, \eta)$ to be identically zero, it is necessary and sufficient that

$$\gamma\xi+\delta\eta \equiv 0, \quad (\alpha_3-1)\varrho_1-\varrho = 0,$$

hence it follows that

$$\gamma = \delta = 0, \quad \alpha_1 = \alpha_2, \quad \alpha_3 = \alpha_2(\alpha_2-1)^{-1}. \tag{14}$$

From the ellipticity of the operator N it follows moreover that $\alpha_1 > 1$, $\alpha_2 > 1$, $\alpha_3 > 1$. Conditions (14) guarantee the strong connectedness of system (10). However, system (10) may also be strongly connected when conditions (14) are violated, since the noetherity of the Dirichlet problem for (10) may be violated in another half-space, not necessarily in the half-space H. For instance, for $\beta = \gamma = \delta = 0$ the noetherity of the Dirichlet problem for (10) is violated in the half-space $G = \{x > 0\}$ if

$$\alpha_2 = \alpha_3, \quad \alpha_1 = \alpha_2(\alpha_2-1)^{-1}.$$

Thus generally speaking, from the strong connectedness of a system there does not follow the violation of the noetherity of the Dirichlet problem for the system in any half-space.

For $\alpha_1 > 1$, $\alpha_2 > 1$ and $\alpha_3 > 1$ system (10) is homotopic to a three-dimensional system (1) with $\lambda > 1$. Concerning the above example it is natural to ask the following two questions. Is it possible to deform two strongly connected systems contained in the same homotopy class into each other without leaving the set of strongly connected systems? May strongly connected systems fill in an open set in a homotopy class?

For $\alpha_3 < 1$, the determinant $D(\xi, \eta)$ may be zero only due to the first factor, but this does not affect the well-posedness of the Dirichlet problem since system (10) for $\alpha_1 < 1$, $\alpha_2 < 1$, $\alpha_3 < 1$ is strongly elliptic. For $\alpha_1 > 1$, $\alpha_2 > 1$, $\alpha_3 > 1$, this determinant may be zero if

$$\gamma\xi+\delta\eta = 0, \quad (\alpha_3-1)\varrho_1 = \varrho. \tag{15}$$

Equalities (15) may hold simultaneously at points different from the point $\xi = \eta = 0$ iff the quadratic form

$$\sigma(\xi, \eta) = \left(\alpha_1 - \frac{\alpha_3}{\alpha_3 - 1}\right)\xi^2 + \left(\alpha_2 - \frac{\alpha_3}{\alpha_3 - 1}\right)\eta^2$$

is not of fixed sign. The function D is zero on the straight line $l = \{\gamma\xi + \delta\eta = 0,\ \gamma^2 + \delta^2 \neq 0\}$ if

$$\delta^2\left(\alpha_1 - \frac{\alpha_3}{\alpha_3 - 1}\right) + \gamma^2\left(\alpha_2 - \frac{\alpha_3}{\alpha_3 - 1}\right) = 0,$$

but if $\gamma = \delta = 0$ and the form $\sigma(\xi, \eta)$ is not of fixed sign, then D is zero on two straight lines. These zeros of the function D undoubtedly influence the solvability character of the Dirichlet problem, but their influence requires a further study. Note that for strongly connected systems $\sigma(\xi, \eta) \equiv 0$. The cases when D is zero at some points different from $\xi = \eta = 0$, but is not identically zero, have no analogies in the theory of systems with two independent variables, their study is therefore of great interest since new phenomena may be discovered here.

To every elliptic system of first-order partial differential equations with constant coefficients one may, in a standard way, associate a strongly connected system of second-order equations [11]. We take a system of m first-order equations in a space of n independent variables in the form

$$\frac{\partial u}{\partial x_1} + \sum_{i=2}^{n} A_i \frac{\partial u}{\partial x_i} = 0, \tag{16}$$

where $u = (u_1, \ldots, u_m)$ is a vector and A_i are quadratic $(m \times m)$-matrices. For the ellipticity of the system it is necessary that m be an even number. Denote the operator on the left-hand side of the system by

$$D = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} A_i \frac{\partial}{\partial x_i}$$

and consider the system of second-order equations

$$D^2 v = 0, \qquad v = (v_1, \ldots, v_m). \tag{17}$$

It is easy to show that all solutions of system (17) which are regular in the half-space $H = \{x_1 > 0\}$ may be represented as $v = u + x_1 w$ with $Du = 0$ and $Dw = 0$. Hence it follows that the homogeneous Dirichlet problem for (17) in the half-space H has an infinite set of linearly independent solutions of the form $v = x_1 w$, where w is a solution of the first-order system $Dw = 0$, regular in H; and for the solvability of the nonhomogeneous problem it is necessary that the vector $f = (f_1, \ldots, f_m)$ of boundary data be the boundary value of a solution of the first-order system $Du = 0$, regular in H. It is in this way that the first examples of multidimensional strongly connected systems [1, 6] have been constructed; moreover, the initial first-order systems taken were multi-dimensional generalizations of the Cauchy–Riemann system.

In [15] the following generalization of the Moisil–Theodoresco system is given:

$$
\begin{aligned}
u_x + v_y + w_z - as_x - bs_y &= 0,\\
s_x - v_z + w_x + au_x + bu_y &= 0,\\
s_y + u_z - w_x + av_x + bv_y &= 0,\\
s_z + v_x - u_y + aw_x + bw_y &= 0.
\end{aligned}
$$

System (17) constructed by means of this first-order system falls into an equation with respect to the function s,

$$L(s) \equiv \Delta s + a^2 s_{xx} + 2ab s_{xy} + b^2 s_{yy} = 0,$$

and the system of three second-order equations

$$
\begin{aligned}
&-L(u) + 2\frac{\partial}{\partial x}(u_x + v_y + w_z) + 2a\frac{\partial}{\partial x}(au_x + bu_y - v_z + w_y)\\
&\qquad\qquad + 2b\frac{\partial}{\partial y}(au_x + bu_y - v_z + w_y) = 0,\\
&-L(v) + 2\frac{\partial}{\partial y}(u_x + v_y + w_z) + 2a\frac{\partial}{\partial x}(av_x + bv_y + u_z - w_x)\\
&\qquad\qquad + 2b\frac{\partial}{\partial y}(av_x + bv_y + u_z - w_x) = 0,\\
&-L(w) + 2\frac{\partial}{\partial z}(u_x + v_y + w_z) + 2a\frac{\partial}{\partial x}(aw_x + bw_y + v_x - u_y)\\
&\qquad\qquad + 2b\frac{\partial}{\partial y}(aw_x + bw_y + v_x - u_y) = 0.
\end{aligned}
\tag{18}
$$

This system is strongly connected.

The fact that system (17) may decompose as above makes the study of strongly connected second-order systems more difficult because distinct first-order systems may generate equivalent strongly connected systems. Using this property, V. I. Shevchenko [8] constructed an example of a strongly connected system of three equations

$$
\begin{aligned}
-\Delta u + 2\frac{\partial}{\partial x}(u_x + v_y + w_z) + 2\frac{\partial}{\partial t}(u_t + v_x - w_y) &= 0,\\
-\Delta v + 2\frac{\partial}{\partial y}(u_x + v_y + w_z) + 2\frac{\partial}{\partial t}(v_t + w_x - u_z) &= 0,\\
-\Delta w + 2\frac{\partial}{\partial z}(u_x + v_y + w_z) + 2\frac{\partial}{\partial t}(w_t - v_x + u_y) &= 0
\end{aligned}
\tag{19}
$$

with four independent variables.

In one specific case, in the above-mentioned way, one may construct a family of strongly connected systems depending on a parameter. Consider the first-order system

$$Du \equiv \frac{\partial u}{\partial x_1} + \sum_{i=2}^{n} A_i \frac{\partial u}{\partial x_i} = 0, \qquad u = (u_1, \ldots, u_m),\ m = 2k,$$

such that the matrix $E\xi_1 + \sum_{i=2}^{n} A_i \xi_i$ is skew-symmetric, and introduce the operator

$$\bar{D}u = \frac{\partial u}{\partial x_1} + \sum_{i=2}^{n} A_i^* \frac{\partial u}{\partial x_i},$$

where A_i^* is the transposed matrix. It is obvious that $D+\bar{D} = 2E(\partial/\partial t)$ where E is the unit matrix, and the operator $D\bar{D} = \bar{D}D$ is strongly elliptic since the product of a matrix by its transpose is positive-definite. Put

$$Du - \bar{D}u = 2Bu = 2\sum_{i=2}^{n} A_i \frac{\partial u}{\partial x_i},$$

since for skew-symmetric matrices we have $A_i^* = -A_i$. Hence

$$Du = \frac{\partial u}{\partial x_1} + Bu, \qquad \bar{D}u = \frac{\partial u}{\partial x_1} - Bu,$$

$$Du + \lambda\bar{D}u = (1+\lambda)\frac{\partial u}{\partial x_1} + (1-\lambda)Bu,$$

where λ is a real parameter. Thus for $\lambda < 1$ the operator $D+\lambda\bar{D}$, upon substituting the independent variable $x_1 = (1+\lambda)(1-\lambda)^{-1}\tau$, may be reduced to the operator D, and for $\lambda > 1$, upon substituting $x_1 = (\lambda+1)(\lambda-1)^{-1}\tau$, this operator is reduced to the operator $\bar{D}$. By these properties of the operators D and $\bar{D}$ it is easy to write out the representation of solutions of the systems

(20) $$(D+\lambda\bar{D})(Du+\mu\bar{D}u) = 0, \qquad \lambda \neq \mu.$$

A solution of system (20) is expressed as follows:

$$u = \Phi_1\left(\frac{1-\lambda}{1+\lambda}x_1, \ldots, x_n\right) + \Phi_2\left(\frac{1-\mu}{1+\mu}x_1, \ldots, x_n\right), \qquad \lambda < 1,\ \mu < 1,$$

(21) $$u = \Psi\left(\frac{\lambda-1}{\lambda+1}x_1, \ldots, x_n\right) + \Phi\left(\frac{1-\mu}{1+\mu}x_1, \ldots, x_n\right), \qquad \lambda > 1,\ \mu < 1,$$

$$u = \Psi_1\left(\frac{\lambda-1}{\lambda+1}x_1, \ldots, x_n\right) + \Psi_2\left(\frac{\mu-1}{\mu+1}x_1, \ldots, x_n\right), \qquad \lambda > 1,\ \mu > 1,$$

where Φ and Ψ are arbitrary solutions of the systems

$$D\Phi = 0 \quad \text{and} \quad D\Psi = 0$$

respectively. From formulae (21) one may easily conclude that for $\lambda < 1$, $\mu < 1$ and $\lambda > 1$, $\mu > 1$, the Dirichlet problem in the half-space $x_1 > 0$ for system (20) is not Noetherian, i.e. for these values of the parameters system (20) is strongly connected.

To system (16) there corresponds the matrix $A(\xi) = E\xi_1 + \sum_{i=2}^{n} A_i \xi_i$, and to system (17) the matrix $B(\xi) = [A(\xi)]^2$. It is obvious that system (17) falls into systems containing less than m unknown functions iff the matrix B has a block-diagonal structure. This reduces the problem of decomposition of system (17) to a purely algebraic problem.

To be able to say anything about strongly connected systems, it is important to investigate the solvability character of boundary problems for non-strongly connected systems which are nevertheless homotopic to strongly connected ones, and to expose the new phenomena which appear for multidimensional systems. It is particularly important to study non-strongly elliptic systems with variable coefficients which are not well investigated even in the two-dimensional case.

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