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PROJECTIVE REPRESENTATIONS OF REFLECTION GROUPS

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§ 0. Introduction

I. Schur in his classic paper [16] developed a full theory on the projective characters of symmetric groups. Although the construction of some of the basic projective representations in that paper can now be seen to be essentially those for Clifford algebras, the projective representation theory of symmetric groups from the point of view of Clifford algebras was considered for the first time in 1962 in [8]; it was for this reason that they were called spin representations. The main purpose of this paper is to exploit Clifford algebra techniques as far as possible to construct irreducible projective (spin) representations of reflection groups.

§ 1. Projective representations of finite groups

Let G be a finite group. A mapping $T: G \to \mathrm{GL}(n, \mathbf{C})$ is called a *projective representation of G of degree n with factor set* α if

$$T(g)\,T(h) = \alpha(g, h)\,T(gh)$$

for all $g, h \in G$, where $\alpha(g, h) \in \mathbf{C}^\times$. Then it is easily verified that

$$\alpha(g, h)\,\alpha(gh, k) = \alpha(g, hk)\,\alpha(h, k) \tag{1.1}$$

for all $g, h, k \in G$. Let

$$Z^2(G, \mathbf{C}^\times) = \{\alpha\colon G \times G \to \mathbf{C}^\times \mid \alpha \text{ satisfies (1.1)}\};$$

then $Z^2(G, \mathbf{C}^\times)$ is a multiplicative abelian group (with the obvious composition). If we put

$$B^2(G, \mathbf{C}^\times) = \{\delta \in Z^2(G, \mathbf{C}^\times) \mid \delta(x, y) = \mu(x)\mu(y)\mu(xy)^{-1} \text{ for some } \mu\colon G \to \mathbf{C}^\times\},$$

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then $H^2(G, \mathbf{C}^\times) = Z^2(G, \mathbf{C}^\times)/B^2(G, \mathbf{C}^\times)$ is a finite abelian group called the *Schur multiplier* of G. A group $\tilde{G}$ with normal subgroups N such that

$$(1)\ \tilde{G}/N \cong G \text{ and } (2)\ N \subseteq Z(\tilde{G}) \cap \tilde{G}'$$

is called a *stem extension* of G (where $Z(G)$ and G' denote the centre and derived group of G respectively). If, in addition, $N \cong H^2(G, \mathbf{C}^\times)$, then $\tilde{G}$ is called a *representation group* of G.

Projective representations T and T' of G of the same degree and with factor sets α and α' respectively, are *projectively equivalent* if

$$T'(g) = \mu(g) P^{-1} T(g) P$$

for all $g \in G$ where $\mu\colon G \to \mathbf{C}^\times$ and $P \in \mathrm{GL}(n, \mathbf{C})$, that is $\alpha = \alpha'$ in $H^2(G, \mathbf{C}^\times)$. Schur (see Józefiak [6] and Karpilovsky [7]) showed that, given a projective representation with fixed factor set α then it can be "linearised" to an ordinary representation of a stem extension $\tilde{G}$ of G and vice versa. Furthermore, Schur showed that there exists a *finite* representation group $\tilde{G}$ of G such that all projective representation of G for any factor set α can be linearised to an ordinary representation of $\tilde{G}$. Thus, the problem of determining all the projective representations of a finite group G is reduced to determining the ordinary representations of a representation group, or, if we wish to concentrate on a certain fixed factor set, to determining the ordinary representations of an appropriate stem extension.

§ 2. Reflection groups and Coxeter groups

Let U be an l-dimensional *real* euclidean space with positive definite bilinear form (,) and orthonormal basis $\{u_1, \dots, u_l\}$. Let W be the reflection group generated by the reflections $\tau_i = \tau_{u_i}$ $(i = 1, \dots, l)$, where

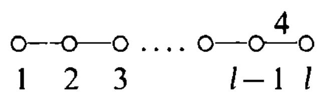
$$\tau_i u = u - \frac{2(u, u_i)}{|u_i|^2} u_i .$$

Then W is a *Coxeter group*, that is, W is the group generated by the $\tau_i (i = 1, \dots, l)$ subject to the relations

$$(\tau_i \tau_j)^{m_{ij}} = 1, \qquad m_{ij} \in \{n \mid n \in \mathbf{Z},\ n \geqslant 2\} \cup \{\infty\}.$$

The group W is completely determined by its *Coxeter diagram D*, which has vertex set $\{1, 2, \dots, l\}$ and edge set $\{\{i, j\} \mid m_{ij} \geqslant 3\}$ and where the edge $\{i, j\}$ is given the weight m_{ij} if $m_{ij} \geqslant 4$.

Every reflection group corresponds to a root system Φ whose simple system we denote by π; the corresponding reflection group will be denoted by $W(\Phi)$. For example, if Φ is of type B_l, then the corresponding Coxeter diagram is



and

$$W(B_l) = \langle \tau_1, \ldots, \tau_l | \tau_i^2 = 1 \ (i = 1, \ldots, l), \ (\tau_i \tau_{i+1})^3 = 1 \ (i = 1, \ldots, l-2),$$
$$(\tau_{l-1} \tau_l)^4 = 1, \ (\tau_i \tau_j)^2 = 1 \ |j-i| \geqslant 2 \rangle.$$

§ 3. Schur multipliers of Coxeter groups

The Schur multipliers of finite reflection groups were determined by Ihara and Yokonuma [5] and certain infinite discrete reflection groups by Yokonuma [19]. The methods depended on ad-hoc arguments and essentially followed the pattern laid by Schur [16] who determined the Schur multiplier of the special case $W(A_l)$, namely the symmetric groups S_{l+1} of degree $l+1$. More recently, Howlett [4] has given a unified treatment, which shows how the Schur multiplier can be expressed in terms of the Coxeter diagram. In order to describe his result we need the following definition.

Let $A_2 = \{\{i, j\} | m_{ij} = 2\}$. Write $\{i, j\} \approx \{i, j'\}$ if $\{i, j\}, \{i, j'\} \in A_2$ and $m_{jj'}$ is odd. Let $\sim$ be the equivalence relation on A_2 induced by $\approx$, that is, $\{i, j\} \sim \{i', j'\}$ if and only if there is a sequence $\{i_0, j_0\}, \{i_1, j_1\}, \ldots, \{i_k, j_k\}$ in A_2 with $\{i_0, j_0\} = \{i, j\}$, $\{i_k, j_k\} = \{i', j'\}$ and $\{i_{r-1}, j_{r-1}\} \approx \{i_r, j_r\}$ for $r = 1, \ldots, k$. Then, Howlett proved the following theorem.

THEOREM 1. *If W is a Coxeter group with Coxeter diagram D, then $H^2(W, \mathbf{C}^\times)$ is an elementary abelian 2-group of rank $n_2(D)+n_3(D)+n_4(D)-n_1(D)$ where*

$n_1(D)$ = number of vertices of D,
$n_2(D)$ = number of edges of D of finite weight,
$n_3(D)$ = number of equivalence classes of $\sim$ on A_2,
$n_4(D)$ = number of connected components of D', the graph obtained from D by deleting all edges of even weight and all edges of infinite weight.

For example, for Weyl groups of type B_l, we have

$$n_1(B_l) = l, \quad n_2(B_l) = l-1, \quad n_4(B_l) = 2 \quad \text{and}$$

$$n_3(B_l) = \begin{cases} 0 & \text{if } l = 2, \\ 1 & \text{if } l = 3, \\ 2 & \text{if } l \geqslant 4. \end{cases}$$

Thus, it follows that

$$H^2(W(B_l), \mathbf{C}^\times) \cong \begin{cases} \mathbf{Z}_2 & \text{if } l = 2, \\ \mathbf{Z}_2^2 & \text{if } l = 3, \\ \mathbf{Z}_2^3 & \text{if } l \geqslant 4. \end{cases}$$

However, Ihara and Yokonuma are more explicit and give generators and relations for a representation group which are required, as will be seen later, when projective representations are constructed. For example, the group

$$R(B_l) = \langle r_1, \ldots, r_l, \alpha_1, \alpha_2, \alpha_3 | r_i^2 = 1\,(i = 1, \ldots, l), (r_i r_{i+1})^3 = 1\,(i = 1, \ldots, l-2),$$
$$(r_i r_j)^2 = \alpha_1\ (i, j = 1, \ldots, l-1), |j-i| \geqslant 2, (r_i r_l)^2 = \alpha_2\ (i = 1, \ldots, l-2),$$
$$(r_{l-1} r_l)^4 = \alpha_3,\ \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1,\ \alpha_i \alpha_j = \alpha_j \alpha_i\ (i, j = 1, 2, 3),$$
$$\alpha_i r_j = r_j \alpha_i,\ i = 1, 2, 3,\ j = 1, \ldots, l\rangle$$

is the representation group of $W(B_l)$ for $l \geqslant 4$. The group $\langle(\alpha_1, \alpha_2, \alpha_3)\rangle \cong H^2(W(B_l), \mathbf{C}^\times)$, and we shall denote the corresponding factor sets as $(\alpha_1, \alpha_2, \alpha_3) = (\pm 1, \pm 1, \pm 1)$.

§ 4. Clifford algebras, Pin groups and their representations

The basic references for the material in this section are Atiyah, Bott and Shapiro [1] and Morris [11].

Let f be a symmetric, nondegenerate, bilinear form on $V = \mathbf{R}^n$ and let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard $\mathbf{R}$-basis for V. Let $C(V, f)$ be the Clifford algebra of V and f, that is, $C(V, f)$ may be regarded as the real polynomial algebra generated by 1, ε_i $(i = 1, \ldots, n)$ subject to the relations

$$\varepsilon_i^2 = f(\varepsilon_i, \varepsilon_i)\,1, \quad \varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0\ (i \neq j).$$

Then $(C(V, f)\colon \mathbf{R}) = 2^n$ and $\{\varepsilon_{i_1} \ldots \varepsilon_{i_s} | 1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant n\}$ is an $\mathbf{R}$-basis for $C(V, f)$. Furthermore

$$C(V, f) = C_0(V, f) \oplus C_1(V, f)$$

is a $\mathbf{Z}_2$-graded algebra, where $C_0(V, f)$ and $C_1(V, f)$ contain the elements of odd and even degrees respectively. Now, let $\alpha\colon C(V, f) \to C(V, f)$ be the automorphism of $C(V, f)$ defined by

$$\alpha(c_i) = (-1)^i c_i \quad (i = 1, 2), \quad c_i \in C_i(V, f)$$

and $t\colon C(V, f) \to C(V, f)$ be the antiautomorphism of $C(V, f)$ defined by

$$(\varepsilon_{i_1} \ldots \varepsilon_{i_s})^t = \varepsilon_{i_s} \ldots \varepsilon_{i_1}.$$

Then, the so-called *Pinorial group* is the group

$$\mathrm{Pin}(V, f) = \{x \in C(V, f) | \alpha(x) v x^{-1} \in V \text{ for all } v \in V,\ (\alpha(x^t) x)^2 = 1\}.$$

Furthermore, define $\varrho\colon \mathrm{Pin}(V, f) \to \mathrm{Aut}\, V$ by

$$\varrho(x) v = \alpha(x) v x^{-1}$$

for all $x \in \mathrm{Pin}(V, f)$, $v \in V$.

We now consider two special cases where f corresponds to the positive definite quadratic form $x_1^2 + \ldots + x_n^2$ and the negative definite quadratic form $-x_1^2 - \ldots - x_n^2$. The corresponding Clifford algebras, Pinorial groups, etc. will be denoted by $C^{\pm}(n)$, $\mathrm{Pin}^{\pm}(n)$, etc. The following basic theorem can be proved.

THEOREM 2. *The sequences*

$$1 \to \mathbf{Z}_2 \to \mathrm{Pin}^{\pm}(n) \overset{\varrho^{\pm}}{\to} O(n) \to 1$$

are exact and $\mathrm{Pin}^{+}(n)$ *and* $\mathrm{Pin}^{-}(n)$ *are nonisomorphic stem extensions of* $O(n)$.

From now on, unless there is a good reason for doing otherwise, we shall state our results in terms of $\mathrm{Pin}^{+}(n)$ only (which will be denoted by $\mathrm{Pin}(n)$); a corresponding result can always be stated for $\mathrm{Pin}^{-}(n)$. For later purposes, it will be useful to give the elements of $\mathrm{Pin}(n)$ which are mapped by ϱ onto certain elements of $O(n)$. If $v \in V$ with $f(v, v) \neq 0$ and τ_v is the corresponding reflection in $O(n)$, then if $x_v = (f(v, v))^{-1} v$

$$\varrho(x_v) = \tau_v.$$

Put

$$E_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix} \quad (i = 1, \dots, m) \quad \text{and} \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$(4.1)\quad \begin{cases} \varrho^{-1}(E_1 \dotplus \dots \dotplus E_m) \\ \qquad = \varrho^{-1}(1 \dotplus E_1 \dotplus \dots \dotplus E_m) = \pm \prod_{j=1}^{m} (\cos\tfrac{1}{2}\theta_j - \varepsilon_{2j-1}\varepsilon_{2j}\sin\tfrac{1}{2}\theta_j), \\ \varrho^{-1}(E_0 \dotplus E_2 \dotplus \dots \dotplus E_m) \;= \pm\varepsilon_2 \prod_{j=2}^{m} (\cos\tfrac{1}{2}\theta_j - \varepsilon_{2j-1}\varepsilon_{2j}\sin\tfrac{1}{2}\theta_j), \\ \varrho^{-1}(-1 \dotplus E_1 \dotplus \dots \dotplus E_m) = \pm\varepsilon_{2m+1} \prod_{j=1}^{m} (\cos\tfrac{1}{2}\theta_j - \varepsilon_{2j-1}\varepsilon_{2j}\sin\tfrac{1}{2}\theta_j), \end{cases}$$

that is, we have the preimages of the toral elements of $O(n)$, where $n = 2m$ or $2m+1$.

If $n_1, \dots, n_k$ are positive integers such that $n = n_1 + n_2 + \dots + n_k$ and if $O(n_1, \dots, n_k) = O(n_1) \times \dots \times O(n_k)$, then put

$$\mathrm{Pin}(n_1, \dots, n_k) = \varrho^{-1}(O(n_1, \dots, n_k));$$

then $\mathrm{Pin}(n_1, \dots, n_k)$ is a stem extension of the subgroup $O(n_1, \dots, n_k)$ of $O(n)$. (Incidentally, $\mathrm{Pin}(n_1, \dots, n_r)$ is isomorphic to the graded tensor product $\mathrm{Pin}(n_1) \hat{\otimes} \dots \hat{\otimes} \mathrm{Pin}(n_k)$ (see [11]), but this fact need not be used in the subsequent representation theory).

We now consider the representations of $C(n)$ and $\mathrm{Pin}(n)$.

If $n = 2m$ is even, then $C(n)$ has one irreducible representation P_0 of degree 2^m and if $n = 2m+1$ is odd, then $C(n)$ has two inequivalent representations $P_{\pm 1}$ of degree 2^m. These representations can be given explicitly as follows:

Let

$$\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \varrho = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

be the Pauli matrices and let

$$M_{2r-1} = \tau^{\otimes(r-1)} \otimes \varrho \otimes \varepsilon^{\otimes(m-r)}, \quad M_{2r} = \tau^{\otimes(r-1)} \otimes \sigma \otimes \varepsilon^{\otimes(m-r)} \quad (r = 1, 2, \ldots, m)$$

and

$$M_{2m+1} = \tau^{\otimes m}.$$

If

$$\delta = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \pm 1 & \text{if } n \text{ is odd,} \end{cases}$$

put

$$P_\delta(\varepsilon_i) = M_i \quad (i = 1, \ldots, n),$$

then P_δ is an irreducible representation of $C(n)$. (If n is odd, the second irreducible representation is given by $P_{-1}(\varepsilon_i) = -P_1(\varepsilon_i)$ $(i = 1, \ldots, n)$.) In fact, $P_{\pm\delta}|_{\mathrm{Pin}(n)}$ and $P_{\pm\delta}|_{\mathrm{Pin}(n_1,\ldots,n_k)}$ are also irreducible representations, and are often referred to as *spin* representations of $O(n)$ or $O(n_1, \ldots, n_k)$, which will be denoted by $\bar{P}_{\pm\delta}$. In particular,

$$\text{(4.2)} \quad \begin{cases} \bar{P}_\delta(E_1 \dotplus \ldots \dotplus E_m) \\ \qquad = \bar{P}_\delta(1 \dotplus E_1 \dotplus \ldots \dotplus E_m) = \prod_{r=1}^{m} \otimes \begin{bmatrix} e^{i\theta_r/2} & 0 \\ 0 & e^{-i\theta_r/2} \end{bmatrix}, \\ \bar{P}_\delta(E_0 \dotplus E_2 \dotplus \ldots \dotplus E_m) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes \prod_{r=2}^{m} \otimes \begin{bmatrix} e^{i\theta_r/2} & 0 \\ 0 & e^{-i\theta_r/2} \end{bmatrix}, \\ \bar{P}_\delta(-1 \dotplus E_1 \dotplus \ldots \dotplus E_m) = \prod_{r=1}^{m} \otimes \begin{bmatrix} e^{i\theta_r/2} & 0 \\ 0 & -e^{-i\theta_r/2} \end{bmatrix}. \end{cases}$$

and if ζ_δ is the character of $\bar{P}_\delta$, then

$$\text{(4.3)} \quad \begin{cases} \zeta_\delta(E_1 \dotplus \ldots \dotplus E_m) \\ \qquad = \zeta_\delta(1 \dotplus E_1 \dotplus \ldots \dotplus E_m) = \prod_{r=1}^{m} (2\cos\tfrac{1}{2}\theta_r), \\ \zeta_\delta(E_0 \dotplus E_2 \dotplus \ldots \dotplus E_m) = 0, \\ \zeta_\delta(-1 \dotplus E_1 \dotplus \ldots \dotplus E_m) = i^m \prod_{r=1}^{m} (2\sin\tfrac{1}{2}\theta_r). \end{cases}$$

In addition, if $u \in V = \mathbf{R}^n$, then $u = \sum_{i=1}^{n} \alpha_i \varepsilon_i$, $\alpha_i \in \mathbf{R}$ and thus

$$\text{(4.4)} \quad \bar{P}_\delta(\tau_u) = P_\delta(u) = \frac{1}{|u|} \sum_{i=1}^{n} \alpha_i M_i.$$

§ 5. Projective representations of reflection groups

We now show that projective representations of reflection groups may be obtained by considering suitable embeddings of reflection groups in the appropriate orthogonal groups.

If τ: $W(\Phi) \to O(n)$ is an embedding of $W(\Phi)$ in $O(n)$, let $X^{\tau}(\Phi) = \varrho^{-1}(\tau(W(\Phi)))$. Then we need to determine whether $X^{\tau}(\Phi)$ is indeed a stem extension of $W(\Phi)$.

Let η be the natural embedding of $W(\Phi)$ in $O(l)$ and put $x_i = \varrho_+^{-1}(\tau_j)$, $y_j = \varrho_-^{-1}(\tau_j)$ $(j = 1, \ldots, l)$. Then we have the following theorem [11]:

THEOREM 3.

$$X^{+\eta}(\Phi) = \langle x_j \; (j = 1, \ldots, l),\; c \,|\, (x_i x_j)^{m_{ij}} = c^{m_{ij}-1},\; x_i c = c x_i,\; c^2 = 1\rangle,$$

$$X^{-\eta}(\Phi) = \langle y_j \; (j = 1, \ldots, l),\; d \,|\, (y_i y_j)^{m_{ij}} = d,\; y_i d = d y_i,\; d^2 = 1\rangle.$$

The groups $X^{\pm\eta}(\Phi)$ are stem extensions if and only if m_{ij} is even for some $i, j \in \Phi$.

This means, in particular, that we can determine easily from the Coxeter diagram when these groups are stem extensions. We obtain the following results.

COROLLARY. *If Φ corresponds to the finite reflection group, the groups $X^{\pm\eta}(\Phi)$ are stem extensions if and only if Φ is of type A_l $(l \geqslant 3)$, B_l $(l \geqslant 2)$, D_l $(l \geqslant 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4 or $I_2(p)$ $(p = 8, 10, 12, \ldots)$.*

We note that if $\Phi = A_l$ $(l \geqslant 3)$, then $X^{\pm\eta}(A_l)$ are the two representation groups of the symmetric groups given by Schur [16] (see also Józefiak [6], who gives a lucid and readable modern exposition of Schur's classic but difficult paper and Stembridge [17] who gives a new treatment and extends the results considerably).

We now consider the restriction of the irreducible representation P_δ of $C(n)$ to $X^{\eta}(\Phi)$. Unfortunately, the root systems Φ are not always presented in the most suitable form so as to give irreducible spin representations of $X^{\eta}(\Phi)$ (or $W(\Phi)$). For example, if $\Phi = A_l$, then the root system is embedded in $\mathbf{R}^{l+1}$ and if η' is the natural permutation representation of $S_{l+1} \cong W(A_l)$, then the root system

$$A_l = \{u_r = \varepsilon_r - \varepsilon_{r+1} \;\; (r = 1, \ldots, l)\},$$

that is, η' may be regarded as an embedding of $W(A_l)$ in $O(l+1)$. Thus, by (4.4)

$$\bar{P}_\delta^{\eta'}(\tau_r) = \bar{P}_\delta^{\eta'}(r, r+1) = \frac{1}{\sqrt{2}}(M_r - M_{r+1}) \qquad (r = 1, \ldots, l) \tag{5.1}$$

which gives a spin representation of $W(A_l)$ of degree $2^{[(l+1)/2]}$. This representation is not however irreducible in general. However, what can be proved is the following:

THEOREM 4. *If* $P_\mathfrak{z}^\eta = P_\delta \downarrow X^\eta(\Phi)$, *then* $\bar{P}_\mathfrak{z}^\eta$ *is an irreducible spin representation of* $W(\Phi)$ *if and only if* Φ *is embedded in* $\mathbf{R}^l$.

In fact, suitable root systems can be given for all the irreducible root systems; for example,

$$A_l = \{\sqrt{(r-1)}\,\varepsilon_{r-1} - \sqrt{(r+1)}\,\varepsilon_r \ (r = 1, \ldots, l),\ \varepsilon_0 = 0\}$$

and

$$E_6 = \{\varepsilon_r - \varepsilon_{r+1}\ (r = 1, 2, 3, 4),\ \varepsilon_4 + \varepsilon_5,\ \tfrac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_5 - \sqrt{3}\,\varepsilon_6)\}.$$

For most of the remaining cases, the root systems given in Bourbaki [2] are already in a suitable form. (For the other exceptions see [9, 11].)

Again, in the particular case $\Phi = A_l$, applying (4.4) to the above root system gives

$$\bar{P}_\mathfrak{z}^\eta(\tau_r) = \bar{P}_\mathfrak{z}^\eta(r, r+1) = \sqrt{\frac{r-1}{2r}}\, M_{r-1} - \sqrt{\frac{r+1}{2r}}\, M_r \qquad (r = 1, \ldots, l) \tag{5.2}$$

which is precisely the irreducible basic spin representation given by Schur [16] (see also Józefiak [6]).

We also note that formula (4.3) can be used to give the value of the basic spin character on all the classes of conjugate elements of $W(\Phi)$, all that is required are the eigenvalues of the elements in each class – information which is readily available from the usual way of parameterising the class in terms of partitions. Detailed results are to be found in [10, 11]. These again generalize the results obtained by Schur for symmetric groups.

In addition we note Nazarov's [14] recent results which have now generalized in a striking way Schur's construction of the basic spin representation of the symmetric group to give a complete set of irreducible spin representations. However, his construction also features the "nasty" coefficients which appear in (5.2), in contrast to the simplicity of (5.1). In fact, the representation $\bar{P}_\mathfrak{z}^{\eta\prime}$ defined in (5.1) is the irreducible *graded* basic spin representation of S_n. It would be interesting to see whether Nazarov's construction can be modified to give the remaining irreducible graded spin representations of S_n.

We now consider the root system $\Phi = B_l$, where

$$B_l = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \ldots,\ \varepsilon_{l-1} - \varepsilon_l,\ 2\varepsilon_l\}.$$

In this case the basic spin representation, again denoted by $\bar{P}_\mathfrak{z}^\eta$, is given by

$$\bar{P}_\mathfrak{z}^\eta(\tau_r) = \frac{1}{\sqrt{2}}(M_r - M_{r+1}) \qquad (r = 1, \ldots, l-1), \qquad \bar{P}_\mathfrak{z}^\eta(\tau_l) = M_l.$$

This representation, in the notation of § 3, is the projective representation of $W(B_l)$ corresponding to the factor set $(-1, -1, -1)$. In fact, a complete set of irreducible projective representations for this factor set can now be easily given (see Read [15]).

THEOREM 5. *If T^λ denotes an irreducible ordinary representation of S_n corresponding to the partition λ of n (written $\lambda \vdash n$) then*

$$\{\bar{P}^\eta_{\pm\delta} \otimes T^\lambda \mid \lambda \vdash n\}$$

is a complete set of irreducible projective representations of $W(B_l)$ with factor set $(-1, -1, -1)$.

We now show that further basic projective representations of $W(B_l)$ for other factor sets can be obtained by taking other embeddings of $W(B_l)$ in suitable orthogonal groups.

I. The first exploits the fact that $W(B_l) \cong \mathbf{Z}_2^l \rtimes W(A_{l-1})$. Via this isomorphism, we have an embedding

$$W(B_l) \xrightarrow{\nu} O(l-1)$$

and the corresponding basic projective (spin) representation, denoted by $\bar{P}^\nu_\delta$, corresponds to the factor set $(-1, 1, 1)$, is irreducible and is given by

$$\bar{P}^\nu_\delta(\tau_n) = \sqrt{\frac{r-1}{2r}}\, M_{r-1} - \sqrt{\frac{r+1}{2r}}\, M_r \qquad (r = 1, \ldots, l-1),$$

$$\bar{P}^\nu_\delta(\tau_l) = I \qquad \text{(the identity matrix)}.$$

In fact, the stem extension in this case is $X^\nu(\Phi) \cong \mathbf{Z}_2 \rtimes X^\eta(A_{l-1})$.

II. Let H be the normal subgroup of $W(B_l)$ which contains the elements in which the number of sign changes and the number of transpositions are both even. Then $W(B_l)/H \cong V_4$, the four group. This isomorphism results in an embedding $W(B_l) \xrightarrow{\mu} O(2)$ given by

$$\mu(\tau_i) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (i = 1, \ldots, l-1), \qquad \mu(\tau_l) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding basic projective (spin) representation, denoted by $\bar{P}^\mu_\delta$, is of degree 2, corresponds to the factor set $(1, 1, -1)$ and is given by

$$\bar{P}^\mu_\delta(\tau_r) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad (r = 1, \ldots, l-1), \qquad \bar{P}^\mu_\delta(\tau_l) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

El-Sharabasy [3] and Munir [13] have determined the remaining irreducible projective representations for this factor set.

III. Another orthogonal embedding is obtained by using the 1-dimensional character θ of $W(B_l)$ defined by

$$\theta(\sigma) = \begin{cases} 1 & \text{if } \sigma \in W(D_l), \\ -1 & \text{if } \sigma \in W(B_l) \setminus W(D_l). \end{cases}$$

Then define η': $W(B_l) \to O(l)$ by $\eta'(\sigma) = \theta(\sigma)\eta(\sigma)$ for all $\sigma \in W(B_l)$ (η is the natural embedding). Thus, in particular we have

$$\eta'(\tau_i) = \eta(\tau_i) = \tau_i \quad (i = 1, \ldots, l-1), \qquad \eta'(\tau_l) = -\eta(\tau_l) = -\tau_l.$$

We therefore need to know $\varrho^{-1}(-I_l)$; this is given by taking the special case $\theta_i = \pi$ $(i = 1, 2, \ldots, m)$ in (4.1); that is

$$\varrho(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_l) = -I_l,$$

which implies that

$$\varrho^{-1}(-\tau_l) = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_l \varepsilon_l = \varepsilon_1 \ldots \varepsilon_{l-1}.$$

Thus, if we put

$$x_r = \frac{1}{\sqrt{2}}(\varepsilon_r - \varepsilon_{r+1}) \quad (r = 1, \ldots, l-1), \qquad x_l = \varepsilon_1 \ldots \varepsilon_{l-1}$$

then

$$X^{\eta'}(B_l) = \langle x_1, \ldots, x_{l-1}, x_l, c \mid x_r^2 = 1 \ (r = 1, \ldots, l),\ c^2 = 1,$$
$$(x_r x_{r+1})^3 = 1 \ (r = 1, \ldots, l-2),\ (x_r x_s)^2 = c \ (r, s = 1, \ldots, l-1)\ |s-r| \geqslant 2,$$
$$(x_r x_l)^2 = c \ (r = 1, \ldots, l-2),\ (x_{l-1} x_l)^4 = 1,\ x_r c = c x_r \rangle$$

is the stem extension corresponding to the factor set $(-1, 1, -1)$ if l is even and $(-1, -1, -1)$ if l is odd.

Furthermore, the corresponding irreduoible projective representation for this factor set is given by

$$\bar{P}_{\delta}^{\eta'}(x_r) = \frac{1}{\sqrt{2}}(M_r - M_{r+1}) \quad (r = 1, \ldots, l-1),$$
$$\bar{P}_{\delta}^{\eta'}(x_l) = M_1 \ldots M_{l-1}.$$

§ 6. Real projective representations of reflection groups

In [12], the author and M. Makhool have explicitly calculated the irreducible *real* representations of Clifford algebras. (The position is more complicated in that the periodicity is now 8 rather than 2 in the complex case.) These have been exploited to obtain the irreducible basic real spin representations of all reflection groups. In addition, they will be used to modify Nazarov's work [14] to give a complete set of irreducible real spin representations of symmetric groups.

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