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HYPERSURFACES WITH SINGULAR LOCUS A PLANE CURVE AND TRANSVERSAL TYPE A_1

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1. Introduction

In [Si-1] we studied functions $f\colon (C^{n+1}, 0) \to (C, 0)$ where the singular locus Σ was a smooth line and with transversal singularities on $\Sigma - \{0\}$ of type A_1. We called those singularities *isolated line singularities*. In this paper we generalize this to the case, where Σ is a plane curve in C^{n+1}.

We study the topology of the Milnor fibre with the help of a generic approximation, which has Σ as part of the critical locus, and where only special types of singularities are allowed:

(a) A_1-points; local formula $w_0^2 + \ldots + w_n^2$;

(b) A_∞-points; local formula $w_1^2 + \ldots + w_n^2$;

(c) D_∞-points; local formula $w_0 w_1^2 + w_2^2 + \ldots + w_n^2$;

(d) central type; local formula $u \cdot g^2 + w_2^2 + \ldots + w_n^2$ where $g(x, y) = 0$ is a reduced equation of the plane curve Σ and u is a unit.

The existence of the deformations follows from work of Pellikaan [Pe].

The homotopy type of the local Milnor fibres of the above elementary types are as follows:

(a) A_1-points: S^n;

(b) A_∞-points: S^{n-1};

(c) D_∞-points: S^n;

(d) central type: $S^{n-1} \vee S^n \vee \ldots \vee S^n$.

The A_∞-points occur in 1-dimensional bundles along the critical set Σ.

By methods similar to Lê (cf. [Br]) in the isolated singularity case, we construct the Milnor fibre of f by gluing together the local contributions. Our main result is:

THEOREM 3.11. *Let Σ be a plane curve and $f\colon (C^{n+1}, 0) \to (C, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ and transversal type A_1 on*

$\Sigma - \{0\}$ and let $^{\#}D_{\infty} > 0$, then the homotopy type of the Milnor fibre F of f is a bouquet of $\mu_n(f)$ n-spheres, where

$$\mu_n(f) = 2\mu(\Sigma) + {}^{\#}A_1 + 2\,{}^{\#}D_{\infty} - 1;$$

$\mu(\Sigma)$ = Milnor number of Σ, $^{\#}D_{\infty}$ = number of D_{∞}-points in the generic approximation with Σ fixed, $^{\#}A_1$ = number of A_1-points in the generic approximation with Σ fixed.

The proof is similar to [Si-1].

The method of the proof gives no result in the case $^{\#}D_{\infty} = 0$. For this case and the more general case that Σ is a 1-dimensional complete intersection singularity (icis) and transversal type A_1 on $\Sigma - \{0\}$ we refer to [Si-2]. In that paper we first compute the homology of the Milnor fibre in terms of $\mu(\Sigma)$, $^{\#}D_{\infty}$ and $^{\#}A_1$. From the homology and additional information about the fundamental group from [Lê–Sa] we can determine the homotopy type of the Milnor fibre, which is as follows:

$$S^n \vee \ldots \vee S^n \quad \text{if } {}^{\#}D_{\infty} > 0,$$

$$S^{n-1} \vee S^n \vee \ldots \vee S^n \quad \text{if } {}^{\#}D_{\infty} = 0.$$

As general references for singularities of functions $C^{n+1} \to C$ we mention the book of Arnol'd–Gusein Zade–Varchenko [Ar]. For the topology of singularities we refer to [Mi] and [Lo]. For non-isolated singularities, see also [Lê] and [Yo].

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2. Generic approximations of the function

2.1. Let Σ be a 1-dimensional complete intersection with isolated singularity at $0 \in C^{n+1}$. We consider f: $(C^{n+1}, 0) \to (C, 0)$ a holomorphic function germ with critical locus $\Sigma(f) = \Sigma$. This situation is treated (in more generality) in the thesis of Pellikaan [Pe]. On every branch of $\Sigma(f)$ there is if $z \neq 0$ a well-defined transversal singularity type.

Let $g_1, \ldots, g_n$ define the complete intersection Σ as a reduced algebraic set and let $I = (g_1, \ldots, g_n)$. Then we have:

$$f \text{ is singular on } \Sigma \Leftrightarrow f \in I^2 \quad ([\text{Pe}]\, \text{I } 1.6).$$

In this case we can write $f = \sum h_{ij} g_i g_j$ with $h_{ij} \in \mathcal{O}_{n+1}$ and $h_{ij} = h_{ji}$. On I and I^2 acts the subgroup $\mathcal{D}_{\Sigma}$ of $\mathcal{D}$ defined by $\mathcal{D}_{\Sigma} = \{\varphi \in \mathcal{D} \mid \varphi^*(I) \subset I\}$. Let $\tau_{\Sigma}(f)$

be the tangent space to the $\mathcal{G}_\Sigma$-orbit and $J_f = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right)$ the Jacobian ideal of f.

Define

$$j(f) = \dim_C \frac{I}{J_f} \quad \text{and} \quad c(f) = \dim_C \frac{I^2}{\tau_\Sigma(f)},$$

the Jacobi number and the codimension.

2.2. PROPOSITION ([Pe]). *Equivalent are:*

(a) $c(f) < \infty$;

(b) $j(f) < \infty$;

(c) *the transversal type of f along $\Sigma - \{0\}$ is A_1.*

Moreover if $c(f) < \infty$ then f is finitely determined inside I^2.

2.3. We specialize now to the case that Σ is a plane curve. We can choose coordinates

$$(z_0, z_1, \ldots, z_n) = (x, y, z_2, \ldots, z_n)$$

such that Σ is given by

$$g_1 = g(x, y) = 0,$$

$$g_2 = z_2 = 0, \ldots, g_n = z_n = 0.$$

So $I = (g, z_2, \ldots, z_n)$.

If f is $\mathcal{G}_\Sigma$-equivalent with $u \cdot g^2 + z_2^2 + \ldots + z_n^2$ then f is called of *central singularity type* (u unit in $\mathcal{O}_{(x,y)}$). According to the splitting lemma [Gr-Me] we can suppose in general:

$$f(x, y, z) = f'(x, y, z_2, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2$$

with $f' \in (x, y, z_1, \ldots, z_r)^3$.

2.4. Consider the following deformation of f with fixed critical locus (cf. [Pe] (7.18))

$$f_s(z) = f(x, y, z) + \sum_{k,l} a_{k,l} g_k g_l + \sum_{i,k} b_{ik} z_i g_k$$

where we choose the matrix $(a_{k,l})$ in diagonal form with diagonal elements $(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$ and $s \in S = \{(\lambda_1, \ldots, \lambda_r), (b_{ik})\}$.

PROPOSITION. *There exists a dense open subset V of S and an open neighbourhood U of 0 in C^{n+1} such that for all $s \in V$ sufficiently small*

(i) *f_s has only A_1-singularities in $U \setminus \Sigma$;*

(ii) *f_s has only A_∞- and D_∞-singularities on $U \cap \Sigma - \{0\}$;*

(iii) *f_s has the central singularity type in* 0;

(iv) $j_f = {}^{\#}\{A_1\text{-points of } f_s \text{ on } U\setminus\Sigma\} + {}^{\#}\{D_\infty\text{-points of } f_s \text{ on } U\cap\Sigma\} + j_{f_s,0}$ *where $j_{f_s,0}$ is the Jacobi number of the central singularity.*

Proof. Almost all the assertions are shown by Pellikaan ([Pe], I (7.18)). The special choice of the matrix (a_{ij}) doesn't influence his proof. A computation shows that the 2-jet-extension

$$j^2 F:\ C^{n+1}\times S \to J^2_{(0)}(C^{n+1}, C)$$

is transversal to the A_1-stratum outside Σ and to the D_∞-stratum on $\Sigma-\{0\}$. (For details cf. [Pe]). The assertions (i) and (ii) follow as an application of Sard's theorem.

For (iii) a more carefull analysis of Pellikaans proof is necessary. Write $f = h_{ij}g_i g_j$ where

$$(h_{ij}) = \begin{pmatrix} * & 0 \\ 0 & I_{n-r}\end{pmatrix}.$$

Since the 2-jet of f is equal to $z_{r+1}^2+\ldots+z_n^2$ this implies that $h_{ij}\in\mathfrak{m}$ if $2\leqslant i\leqslant r$ and $2\leqslant j\leqslant r$ ($\mathfrak{m}$ is the maximal ideal).

Let $f_s = \sum H_{ij}g_i g_j$ where

$$H_{ij} = h_{ij}+a_{ij}+\sum z_t b_{it}\delta_{ij} = h_{ij}+(\lambda_i+\sum z_t b_{it})\delta_{ij}.$$

Remark that:

$$H_{ij} = h_{ij} \quad \text{for } i\neq j$$

and

$$H_{ii} = h_{ii}+\lambda_i+\sum z_t b_{it}$$

so

$$\begin{aligned} H_{ii} &\equiv \lambda_i \pmod{\mathfrak{m}} && \text{if } 2\leqslant i\leqslant r,\\ H_{ii} &= 0 && \text{if } i\geqslant r+1,\\ H_{ij} &\equiv 0 \pmod{\mathfrak{m}} && \text{if } i\geqslant 2 \text{ and } j\geqslant 2.\end{aligned}$$

For λ_i sufficiently general we can suppose that H_{ii} are units. Moreover we can suppose: $\det H_{ij}$ is invertible (this corresponds to Pellikaan's statement $\delta_{f_s,0}=0$).

We next transform the matrix (H_{ij}) into a normal form with standard technics from quadratic forms. We first treat z_2. Set

$$\bar z_2 := z_2+\frac{H_{23}}{H_{22}}z_3+\ldots+\frac{H_{2n}}{H_{22}}z_n+\frac{H_{1n}}{H_{22}}g$$

which defines a coordinate transformation on $(C^{n+1}, 0)$. Remark $\bar{z}_2 \equiv z_2 \pmod{\mathfrak{m}}$. Also mod $\mathfrak{m}$ we have:

$$(H_{ij}) \equiv \left(\begin{array}{c|ccc} * & * & & * \\ \hline * & \lambda_2 & & 0 \\ & & \ddots & \\ * & 0 & & \lambda_n \end{array}\right)$$

The form of this matrix and the $\lambda_2, \ldots, \lambda_n$ are not changed by this coordinate transformation (nb. $\lambda_{r+1} = \ldots = \lambda_n = 1$).

We treat $z_3, \ldots, z_n$ in the same way and get

$$f_s = H_{11}^* g^2 + \lambda_2 \bar{z}_2^2 + \ldots + \lambda_n \bar{z}_n^2.$$

Since $\bar{z}_k \equiv z_k \pmod{\mathfrak{m}}$ we have $(\bar{z}_2, \ldots, \bar{z}_n, g) = (z_2, \ldots, z_n, g)$. Moreover $\det(H_{ij})$ remains invertible. So H_{11}^* is a unit in $\mathcal{O}_{n+1}$. Since f_s is finitely determined we can change coordinates again (by completing squares) such that H_{11}^* is a function of x and y only. So f_s is right-equivalent to

$$u \cdot g^2 + z_2^2 + \ldots + z_n^2 \quad (u \text{ unit in } \mathcal{O}_{(x,y)}). \qquad \square$$

2.5. *Remark*. It can happen that $^{\#}D_{\infty} = 0$. In the case of isolated line singularities, this only happens for type A_{∞}.

C. Cox showed me the examples

$$f = xyz + z^p \quad (p \geqslant 2).$$

The critical locus is the union of the x-axis and the y-axis, the transversal type is A_1. The deformation

$$f_s = xyz + z^p + sz^2$$

has the properties:

$$^{\#}D_{\infty} = 0, \quad ^{\#}A_1 = p-2,$$

and has central type for $s \neq 0$.

Also $f = z \cdot g + z^p$ $(p \geqslant 2)$, where $g = 0$ is a plane curve, has the property $^{\#}D_{\infty} = 0$.

Pellikaan [Pe] showed in Lemma 1.7.17, that if Σ is a reduced 1-dimensional complete intersection, defined by the ideal I and $^{\#}D_{\infty} = 0$, then there exist generators $g_1, \ldots, g_n$ of I such that f is equivalent to $g_1^2 + \ldots + g_n^2$. This shows that in general there are plenty possible f with $^{\#}D_{\infty} = 0$. At the other hand it is not difficult to degenerate such f to $\xi \cdot g_n^2 + \ldots + g_n^2$ $(\xi \in \mathfrak{m})$ with $^{\#}D_{\infty} > 0$.

2.6. *Remark.* It is in principle possible to classify the non-isolated singularities of this paper in the same way as isolated singularities. Proposition 2.2 makes this possible.

For the case that Σ is a smooth line, we refer to [Si-1].

In the case that Σ is of type A_1:

$$g(x, y) = xy = 0, \quad z = 0$$

the beginning of a list of singularities is as follows:

Type	f	corank	j_f	c_f	$^\# A_1$	$^\# D_\infty$
$T_{\infty,\infty,2}$	x^2y^2	2	1	0	0	0
	$x^2y^2\ (y+x^n)\ (n \geqslant 1)$	2	$2n+2$	n	n	$n+1$
$T_{\infty,\infty,r}$	$xyz+z^r\ (r \geqslant 3)$	3	$r-1$	$r-2$	$r-2$	0
	$xz^2+yz^2+x^2y^2$	3	5	2	2	2

The list contains all simple singularities and all singularities with $^\# D_\infty = 0$.

2.7. We next are interested in the Milnor fibres. Let f: $(C^{n+1}, 0) \to (C, 0)$ and let ε_0 be an admissible radius for the Milnor fibration, that is $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_0$ holds

$$f^{-1}(0) \pitchfork \partial B_\varepsilon \quad \text{(as a stratified set)}.$$

For each admissible $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$f^{-1}(t) \pitchfork \partial B_\varepsilon \quad \text{for all } 0 < t \leqslant \delta(\varepsilon).$$

We fix now $\varepsilon \leqslant \varepsilon_0$ and consider $0 < \delta \leqslant \delta(\varepsilon)$ and take the representative

$$f\colon X_\Delta = f^{-1}(\Delta) \cap B_\varepsilon \to \Delta$$

where Δ is a disc of radius δ.

LEMMA. *Let f_s be as above. Consider the restriction*

$$f_s\colon X_{\Delta,s} := f_s^{-1}(\Delta) \cap B_\varepsilon \to \Delta.$$

For $s \in S$ and $\delta > 0$ sufficiently small we have:

(1) *$f_s^{-1}(t) \pitchfork \partial B_\varepsilon$ for all $t \in \Delta$;*

(2) *above the boundary circles $\partial\Delta$ the fibrations induced by f and f_s are equivalent;*

(3) *X_Δ and $X_{\Delta,s}$ are homeomorphic.*

Proof, cf. [Si-2]. □

3. The homotopy type of the Milnor fibre

3.1. From now on we choose s such that f_s: $X_{\Delta,s} \to \Delta$ satisfies the conditions of Proposition 2.4 and Lemma 2.7.

We ommit the suffix s and write again

$$f\colon X_\Delta \to \Delta.$$

The critical set of f consits of

(a) The 1-dimensional icis Σ, where local singularities are A_∞, D_∞ or the central type.

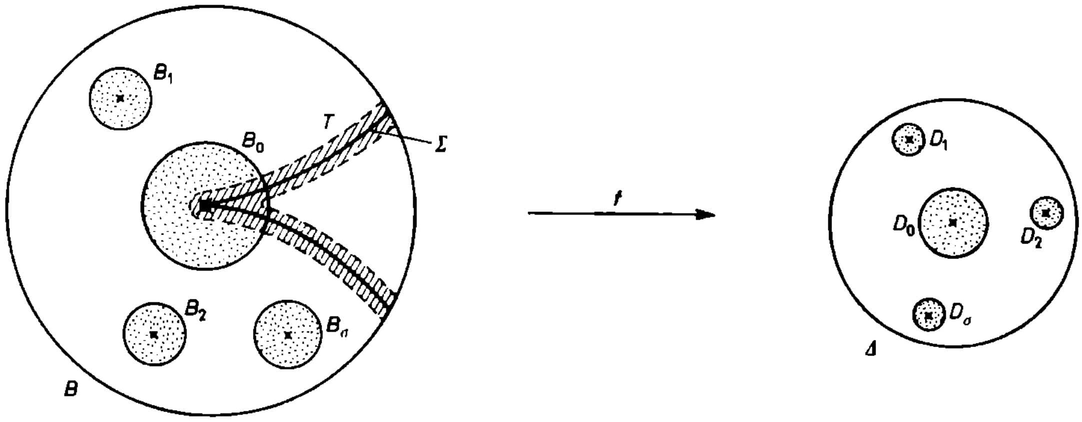
(b) isolated points $\{c_1, \ldots, c_\sigma\}$ where the local singularity is of type A_1.

We can suppose that all critical values of f are different (this is mostly for notational convenience). The critical value 0 corresponds to the non-isolated singularities on Σ. We follow now the construction in [Si-1].

3.2. Define $B_0, B_1, \ldots, B_\sigma$ disjoint $(2n+2)$-balls around $c_0 = 0$, $c_0 = 0, c_1, \ldots, c_\sigma$ and inside $B = B_\varepsilon$. Let $D_0, \ldots, D_\sigma$ be disjoint 2-discs around $f(c_0), \ldots, f(c_\sigma)$ and inside $D = D_\eta$ chosen in such a way that we get locally

$$f\colon B_i \cap f^{-1}(D_i) \to D_i$$

which are Milnor fibrations above $D_i - \{f(c_i)\}$.



Let $\Sigma^* = \overline{\Sigma - B_0}$. The number of topological components of Σ^* is equal to the number of irreducible branches of Σ. Each branch Σ_k^* $(k = 1, \ldots, r)$ is a disc with one hole.

3.3. We want to construct a nice tube neighbourhood of Σ^*. To do this we consider the map $w\colon (C^{n+1}, 0) \to (C^n, 0)$ defined by

$$w_1 = g,$$

$$w_2 = z_2,$$

$$\cdots\cdots$$

$$w_n = z_n.$$

Let

$$r(z) = |w_1(z)|^2 + \ldots + |w_n(z)|^2.$$

Remark that

$$r^{-1}(0) = w^{-1}(0) = \Sigma.$$

For $\varepsilon^* > 0$ sufficiently small, define:

$$T = \{z \in \overline{B \setminus B_0} \mid r(z) \leqslant \varepsilon^*\},$$
$$\partial T = \{z \in B \setminus B_0 \mid r(z) = \varepsilon^*\}$$

T and ∂T have topological components, which we denote by $T_1, \ldots, T_r$ and $\partial T_1, \ldots, \partial T_r$, where the numbering corresponds to the branches $\Sigma_1^*, \ldots, \Sigma_r^*$.

3.4. LEMMA. *$T = \Sigma^* \times Q^n$ where Q^n is a closed n-ball in C^n.*

Proof. The lemma follows from the Ehresmann fibration theorem since f is submersive on T and so are its restrictions to $T \cap \partial B_\varepsilon$ and $T \cap \partial B_0$. □

3.5. LEMMA. (a) *There exist ε^* such that for all $0 < \varepsilon \leqslant \varepsilon^*$*

$$f^{-1}(0) \pitchfork \partial B_0 \quad \text{and} \quad f^{-1}(0) \pitchfork \partial T.$$

(b) *For every $0 < \varepsilon \leqslant \varepsilon^*$ there exist a $\tau = \tau(\varepsilon)$ such that for all $0 < |t| \leqslant \tau$*

$$f^{-1}(t) \pitchfork \partial B_0 \quad \text{and} \quad f^{-1}(t) \pitchfork \partial T.$$

Proof. Application of the curve selection lemma and the openness of the transversality conditions. □

3.6. Along Σ we have 3 types of singularities: A_∞, D_∞ and central type. In each case we consider the pair, consisting of the Milnor fibre of f and the Milnor fibre of the restriction of f to a nearby slice transversal to Σ. The topology of these pairs can be described as follows:

A_∞: Milnor pair $\overset{h}{\simeq} (S^{n-1}, S^{n-1})$,

D_∞: Milnor pair $\overset{h}{\simeq} (S^n, S^{n-1})$,

central: Milnor pair $\overset{h}{\simeq} (S^{n-1} \vee S^n \vee \ldots \vee S^n, S^{n-1})$.

The first two cases are treated in [Si-1], for the central type we refer to the next proposition.

3.7. PROPOSITION. *Let f be of central type*

$$f = u \cdot g^2 + z_2^2 + \ldots + z_n^2,$$

and let f' be the restriction of f to a transversal slice $w_0 =$ const at a point of $\partial B_0 \cap \Sigma_k^*$. The pair consisting of the Milnor fibres of f and f' is homotopy *equivalent to the pair* $(S^{n-1} \vee S^n \vee \ldots \vee S^n, S^{n-1})$ *where the wedge contains* $2\mu(\Sigma)$ *copies of* S^n *and one* S^{n-1}.

Proof. We first consider $n = 1$. It is sufficient to consider

$$f(x, y) = g(x, y)^2 = 1.$$

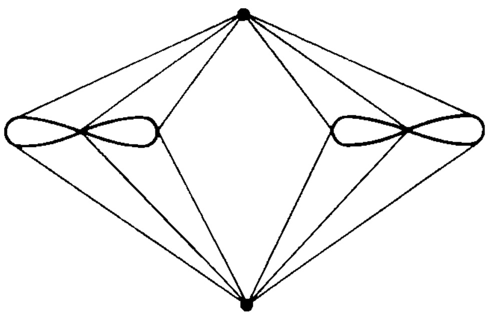
We get two components: $g(x, y) = 1$ and $g(x, y) = -1$, each corresponding to a Milnor fibre of the curve $g = 0$. This Milnor fibre is a bouquet of $\mu(\Sigma)$ n-spheres.

At points of $\partial B_0 \cap \Sigma_k^*$ we can use w_0 and g as local coordinates from 3.4. So the transversal Milnor fibre is given by

$$f = g^2 = 1, \quad w_0 \text{ const}$$

and consists of two points, one in each of the two components of the Milnor fibre of f.

So the pair is homotopy equivalent to $((S^1 \vee \ldots \vee S^1) \cup (S^1 \vee \ldots \vee S^1), S^0)$. If $n = 2$ we have to take the double suspension of the spaces and we get $((S^2 \vee \ldots \vee S^2) \cup (S^2 \vee \ldots \vee S^2), S^1)$ where the copies of $S^2 \vee \ldots \vee S^2$ are connected in 2 points of S^1.



This pair is homotopy equivalent to $(S^1 \vee S^2 \vee \ldots \vee S^2, S^1)$. For $n \geqslant 3$ further double suspension gives the result. □

3.8. For B_0, T and D_0 small enough we define for $t \in \partial D_0$

$$F^* = f^{-1}(t) \cap T,$$

$$F_k^* = f^{-1}(t) \cap T_k,$$

$$F^c = f^{-1}(t) \cap B_0,$$

$$F^0 = F^* \cup F^c.$$

We use coordinates $(w_0, w_1, \ldots, w_n)$ in T with $w_0 \in \Sigma^*$ and $(w_1, \ldots, w_n) \in Q^n$. Consider the projection, which we can suppose to be holomorphic

$$w_0 \colon T_k \to \Sigma_k^*$$

and its restriction to F_k^*. This projection is singular at point s of $\Gamma \cap F_k^*$ where Γ is the polar curve of f with respect to w_0 and is given by

$$\frac{\partial f}{\partial w_1} = \ldots = \frac{\partial f}{\partial w_n} = 0.$$

Since Γ cuts Σ^* only in the D_∞-points of f ([Si-1]) it follows that

$$w_0 \colon F_k^* \to \Sigma_k^*$$

can only be singular in the neighbourhoods of D_∞-points of f.

Let $S_{k,1}, \ldots, S_{k,\tau_k}$ be small disjoint discs around the D_∞-points in Σ_k^*. Set

$$S_k = \bigcup_i S_{k,i}, \qquad M_k = \overline{\Sigma_k^* \setminus S_k}.$$

We also suppose that

$$f\colon w_0^{-1}(S_{k,i}) \cap f^{-1}(D_0) \to D_0$$

satisfies the Milnor conditions with respect to the polyball $S_{k,i} \times Q^n$.

3.9. LEMMA. *For the diameter of T sufficiently small the projection*

$$w_0 \colon F_k^* \to \Sigma_k^*$$

is locally trivial above M_k with fibre equivalent to the Milnor fibre of the quadratic singularity: $w_1^2 + \ldots + w_n^2$.

Proof. For families of quasi-homogeneous singularities there is a stable radius for the Milnor construction (cf. [Ok] or [Os]). This implies that the various transversality conditions are satisfied and the lemma follows from Ehresmann's fibration theorem. □

3.10. PROPOSITION. *Let $^\# D_\infty > 0$, then F^0 is homotopy equivalent to the union of the Milnor fibre of the central singularity and the Milnor fibres of the D_∞-singularities, glued together along a common S^{n-1}. So*

$$F^0 \overset{h}{\simeq} S^n \vee \ldots \vee S^n; \qquad b_n(F^0) = 2\mu(\Sigma) + 2\,{}^\# D_\infty - 1.$$

If $^\# D_\infty = 0$ then

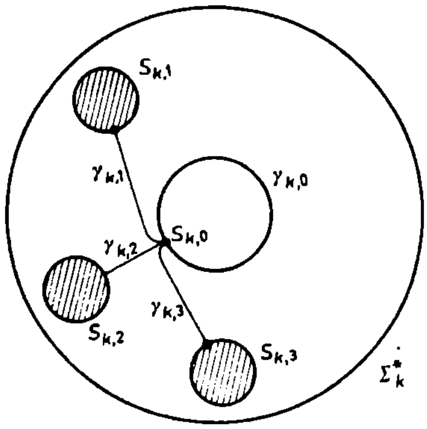
$$F^0 \overset{h}{\simeq} F^c \overset{h}{\simeq} S^n \vee \ldots \vee S^n \vee S^{n-1}; \qquad b_n(F^0) = 2\mu(\Sigma).$$

Proof. Let $\gamma_{k,0} = \Sigma_k^* \cap \partial B_0$ and $s_{k,0} \in \gamma_{k,0}$. Choose a system of paths $\gamma_{k,1}, \ldots, \gamma_{k,\tau_k}$ from $s_{k,0}$ to $S_{k,1}, \ldots, S_{k,\tau_k}$ (in the usual way; see the diagram). Set

$$\gamma_k = \bigcup_i \gamma_{k,i}.$$

$S_k \cup \gamma_k$ is a deformation retract of Σ_k^* and $\gamma_{k,0}$ is a deformation retract of γ_k. Since we can suppose that w_0 is locally trivial above M_k it follows from the homotopy lifting property that

$$(F_k^*, w_0^{-1}(\gamma_{k,0})) \overset{h}{\simeq} (w_0^{-1}(S_k \cup \gamma_k), w_0^{-1}(\gamma_k)).$$



If $^\# D_\infty > 0$ on Σ_k^* this is homotopy equivalent to $(w_0^{-1}(\gamma_k) \cup E_k, w_0^{-1}(\gamma_k))$ where E_k is the disjoint union of $2\,^\# D_\infty$ n-cells, which are attached to the vanishing cycle S^{n-1} in the standard way. The attachment takes place in $w_0^{-1}(s_{k,0})$.

If $^\# D_\infty = 0$ on Σ_k^* then $F_k^* \cup F^c \overset{h}{\simeq} F^c$.

In both cases

$$F^0 = F^c \cup F_1^* \cup \ldots \cup F_r^*.$$

From 3.7 we know that

$$F^c \overset{h}{\simeq} S^{n-1} \vee S^n \vee \ldots \vee S^n; \quad b_n(F^c) = 2\mu(\Sigma),$$

and that each $w_0^{-1}(s_{k,0})$ can up to homotopy be identified with the S^{n-1} of the wedge.

If $^\# D_\infty > 0$ on a Σ_k^* then this S^{n-1} is killed and we have

$$F^0 \overset{h}{\simeq} S^n \vee \ldots \vee S^n; \quad b_n(F^0) = 2\mu(\Sigma) + 2\,^\# D_\infty - 1.$$

If $^\# D_\infty = 0$ then $F^0 \overset{h}{\simeq} F^c$. □

3.11. THEOREM. *Let Σ be a plane curve and f: $(C^{n+1}, 0) \to (C, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ and transversal singularity type A_1 on $\Sigma - \{0\}$ and let $^\# D_\infty > 0$, then the homotopy type of the Milnor* *fibre F of f is a bouquet of $\mu_n(f)$ spheres S^n where*

$$\mu_n(f) = 2\mu(\Sigma) + {}^{\#}A_1 + 2\,{}^{\#}D_\infty - 1;$$

$\mu(\Sigma)$ = Milnor number of Σ, ${}^{\#}D_\infty$ = number of D_∞ points in the generic approximation with Σ fixed, ${}^{\#}A_1$ = number of A_1 points in the generic approximation with Σ fixed,

Proof. Take Δ, D_0, $D_1, \ldots, D_\sigma$ and B, B_0, $B_1, \ldots, B_\sigma$, T as before. Let $t \in \partial D_0$. Choose a system of paths $\psi_1, \ldots, \psi_\sigma$ from t to $D_1, \ldots, D_\sigma$. For $T \subset D$ we set $X_T = f^{-1}(T) \cap B$. As in the preceeding proposition there is a homotopy equivalence

$$(X_\Delta, X_t) \overset{h}{\simeq} (X_{D_0} \cup_{\psi_1} e_1^{n+1} \cup \ldots \cup_{\psi_\sigma} e_\sigma^{n+1}, X_t).$$

Moreover

$$(X_{D_0}, X_t) \overset{h}{\simeq} \left(X_{D_0} \cap (B_0 \cup T) \cup X_t, X_t\right).$$

Let ϕ_i: $S^n \to F^0 = X_t \cap (B_0 \cup T)$ represent the $2\mu(\Sigma) + 2\,{}^{\#}D_\infty - 1$ generators of $\pi_n(F^0)$. Use $\{\phi_i\}$ to attach $(n+1)$-cells f_i^{n+1} to F^0. The inclusion mapping

$$F^0 = X_t \cap (B_0 \cup T) \hookrightarrow X_{D_0} \cap (B_0 \cup T)$$

extends to a homotopy equivalence

$$F^0 \cup f_1^{n+1} \cup \ldots \cup f_q^{n+1} \to X_{D_0} \cap (B_0 \cup T)$$

since both spaces are contractable. So we get a homotopy equivalence:

$$(X_{D_0}, X_t) \overset{h}{\simeq} (X_t \cup f_1^{n+1} \cup \ldots \cup f_q^{n+1}, X_t);$$

X_Δ is obtained from X_t by attaching $2\mu(\Sigma) + {}^{\#}A_1 + 2\,{}^{\#}D_\infty - 1$ $(n+1)$-cells. So X_t is $(n-1)$-connected, since X_Δ is contractable. Since X_t has the homotopy type of a n-dimensional finite CW-complex, it follows that X_t has the homotopy type of a bouquet of $\mu_n(f) = 2\mu(\Sigma) + {}^{\#}A_1 + {}^{\#}2D_\infty - 1$ n-spheres. □

3.12. *Remark.* As we already mentioned in the introduction we showed in [Si-2] in a slightly different way that in case of Σ a 1 dimensional isolated complete intersection singularity (icis) the homotopy type of the Milnor fibre F is as follows:

$${}^{\#}D_\infty > 0\colon \quad S^n \vee \ldots \vee S^n, \quad b_n(F) = \mu(\Sigma) + {}^{\#}A_1 + 2\,{}^{\#}D_\infty - 1;$$

$${}^{\#}D_\infty = 0\colon \quad S^{n-1} \vee S^n \vee \ldots \vee S^n, \; b_n(F) = \mu(\Sigma) + {}^{\#}A_1,$$

where ${}^{\#}A_1$ and ${}^{\#}D_\infty$ denote the number of A_1-points, respectively D_∞-points in an approximation f_s, which deforms Σ into a smooth singular locus. So the notation $^{\#}A_1$ is used in [Si-2] in an other way and differs $\mu(\Sigma)$ with the notation in this paper.

3.13. EXAMPLE. $f = x^2y^2 + y^2z^2 + z^2x^2$.

Now the critical locus is not a plane curve, even not a complete intersection. It consist of the three coordinate axis in C^3. The transversal type is A_1.

Consider the deformation

$$f_s = x^2y^2 + y^2z^2 + z^2x^2 + sxyz.$$

The singular locus of f_s consists of

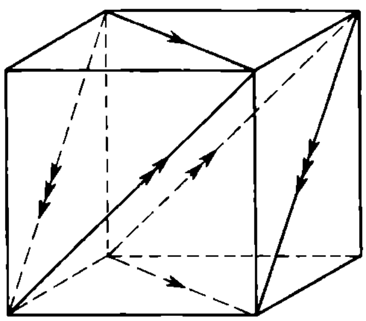
(a) four isolated A_1-points;

(b) the three coordinate axis with each two D_∞-points and a central singularity at the origin, which is equivalent to xyz.

The Milnor fibre of the central singularity is homotopy equivalent to a 2-torus. In fact in polar coordinates this fibre is given by:

$$|x| \cdot |y| \cdot |z| = 1,$$

$$\arg x + \arg y + \arg z \equiv 0 \bmod 2\pi \text{ (on 3-torus)}.$$



The transversal Milnor fibres (corresponding to the 3 axis) are three independent circles (up to homotopy) and indicated by —▸—, —▸▸—, and —▸▸▸—. Every two of them form a basis of $\pi_1 = H_1$.

The constructions of this paper, apply also to this example. For the definitions of the tubes along Σ^* one can use here the ordinary distance function.

The part F^0 of the Milnor fibre is homotopy equivalent to the union of the torus and the Milnor fibres of the D_∞-points, which are glued together along the torus above three 1-spheres on the torus. Since on every branch of Σ we have $^{\#}D_{\infty} > 0$, the generators of the fundamental group of the central singularity are killed and so

$$F^0 \overset{h}{\simeq} S^2 \vee \ldots \vee S^2, \quad b_2(F^0) = 11.$$

For the full Milnor fibre F we must also consider the contributions from the A_1-points and we get

$$F \overset{h}{\simeq} S^2 \vee \ldots \vee S^2, \quad b_2(F) = 15.$$

In this example

$$2\mu(\Sigma) + {}^{\#}A_1 + 2\,{}^{\#}D_\infty - 1 = 2\cdot 2 + 4 + 2\cdot 6 - 1 = 19$$

which is different from 15.

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