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THE ŁOJASIEWICZ EXPONENT OF AN ANALYTIC MAPPING OF TWO COMPLEX VARIABLES AT AN ISOLATED ZERO

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1. Introduction

Let $H = (f, g)\colon U \to \mathbf{C}^2$, $0 \in U \subset \mathbf{C}^2$, be a holomorphic mapping having an isolated zero at the origin. By the *Łojasiewicz exponent of the mapping H at the point* 0 we mean the number

$$\lambda(H) = \inf\{v \in \mathbf{R}\colon \exists A > 0, \exists B > 0, \forall |z| < B, A|z|^v \leqslant |H(z)|\}.$$

This exponent plays an important part in the theory of singularities. In the case where H is the gradient of a holomorphic function h, the complex $E(\lambda(H)+1)$-th jet of h at 0 is C^0-sufficient (here $E(v)$ denotes the greatest integer $\leqslant v$). This means that any holomorphic function $\tilde{h}$ with the same complex $E(\lambda(H)+1)$-th jet is of the form $\tilde{h} = h\circ\Phi$ where Φ is a homeomorphism of a neighbourhood of $0 \in \mathbf{C}^2$ (see [1]).

The Łojasiewicz exponent has been studied by several authors. In [7] Lejeune-Jalabert and Teissier (in the multi-dimensional case) proved that $\lambda(H)$ is a rational number, that the infimum in the defining formula of $\lambda(H)$ is attained and that there exists an analytic path through 0 on which H has order of growth equal to $\lambda(H)$. They also gave other characterizations of $\lambda(H)$ which are, however, not easy to use for an effective calculation.

In [6] Kuo and Lu obtained an effective formula for $\lambda(H)$ in the case of the mapping H being the gradient of a holomorphic function h. They expressed $\lambda(H)$ in terms of the Puiseux expansion of h.

In the general case $\lambda(H)$ was estimated in [2], [8] in terms of the multiplicity of the mapping and the orders of its components, the estimates in [3] were given in terms of the intersection multiplicities of factors of f and g having the same tangents and their orders.

Recently, Płoski ([9]) obtained, under the assumption that f and g are irreducible, an exact formula for $\lambda(H)$, dependent on the multiplicity of H and the orders of f and g.

The aim of the present paper is to show that in the general case one can also obtain an exact formula for $\lambda(H)$ in terms of the intersection multiplicities of irreducible factors of f and g and their orders. From this result we immediately derive the above-mentioned results of Lejeune-Jalabert and Teissier (in the two-dimensional case), the result of Płoski and an effective method for calculating the Łojasiewicz exponent.

The proof of the fundamental formula is carried out according to the elementary "horn neighbourhoods" method used by Kuo and Lu in [6].

2. Notation and definitions

If f is a holomorphic function in a neighbourhood of the point $0 \in \boldsymbol{C}^k$, $k = 1, 2$, and $f(z) = \sum_{i=n}^{\infty} f_i(z)$, $f_n \neq 0$, where f_i is a homogeneous polynomial of degree i, then the number n is called the *order of the function* f *at the point* $0 \in \boldsymbol{C}^k$ and denoted by $\operatorname{ord} f$. When $f = 0$, we put $\operatorname{ord} f = \infty$.

By $\mathcal{O}^2$ we denote the ring of germs of holomorphic functions at the point $0 \in \boldsymbol{C}^2$. If f is a holomorphic function in a neighbourhood of the point $0 \in \boldsymbol{C}^2$, we denote by $\hat{f}$ the germ in $\mathcal{O}^2$ generated by f. Further notations concerning germs of holomorphic functions will be taken after Hervé (see [5]).

Let f and g be functions holomorphic in a neighbourhood of the origin in $\boldsymbol{C}^2$ and having a common isolated zero there. The multiplicity of this zero (cf. [4]) will be denoted by $\mu(f, g)$. If $H = (f, g)$, then, instead of $\mu(f, g)$, we write $\mu(H)$.

If $z = (x, y) \in \boldsymbol{C}^2$, then $|z| = \max(|x|, |y|)$.

3. The main results

Let U be a neighbourhood of $0 \in \boldsymbol{C}^2$ and $H = (f, g)\colon U \to \boldsymbol{C}^2$ a holomorphic mapping. Let $f = f_1 \ldots f_r$, $g = g_1 \ldots g_s$ in U, where $\hat{f} = \hat{f}_1 \ldots \hat{f}_r$, $\hat{g} = \hat{g}_1 \ldots \hat{g}_s$ are factorizations of $\hat{f}$ and $\hat{g}$ into non-invertible and irreducible factors in $\mathcal{O}^2$. Let $\Gamma_i = \{z \in U\colon f_i(z) = 0\}$ for $i = 1, \ldots, r$ and $\Gamma_{r+j} = \{z \in U\colon g_j(z) = 0\}$ for $j = 1, \ldots, s$.

MAIN THEOREM. *If H has an isolated zero at the origin, then*:

(i) $\lambda(H) = \max_{i,j=1}^{r,s} \big(\mu(f_i, g)/\operatorname{ord} f_i,\ \mu(f, g_j)/\operatorname{ord} g_j\big)$;

(ii) $\lambda(H) \in \{v \in \boldsymbol{R}\colon \exists A > 0,\ \exists B > 0,\ \forall |z| < B,\ A|z|^v \leqslant |H(z)|\}$;

(iii) *if* $\lambda(H) = \mu(f_p, g)/\operatorname{ord} f_p$ *or* $\lambda(H) = \mu(f, g_q)/\operatorname{ord} g_q$, *then* $|H(z)| \sim |z|^{\lambda(H)}$ *on* Γ_p *or* Γ_{r+q}.

From the above theorem we immediately obtain the following corollaries.

COROLLARY 3.1 (cf. [2]). *If $H = (f, g)$ satisfies the assumptions of the Main Theorem, then*:

(a) $\lambda(H) \geqslant \mu(f, g)/\min(\operatorname{ord} f, \operatorname{ord} g)$;

(b) $\lambda(H) \leqslant \max(\operatorname{ord} f, \operatorname{ord} g) + \mu(f, g) - \operatorname{ord} f \operatorname{ord} g$.

COROLLARY 3.2 (cf. [7]). *If $H = (f, g)$ satisfies the assumptions of the Main Theorem, then $\lambda(H)$ is a rational number, the infimum in the defining formula of $\lambda(H)$ is attained and there exists an analytic path through 0 on which the order of growth of H equals $\lambda(H)$.*

COROLLARY 3.3 (cf. [9]). *If $H = (f, g)$ satisfies the assumptions of the Main Theorem and $\hat{f}$, $\hat{g}$ are irreducible germs in C^2, then $\lambda(H) = \mu(H)/\min(\operatorname{ord} f, \operatorname{ord} g)$.*

4. Auxiliary results

Let $H = (f, g)$ satisfy the same assumptions as in the preceding section, $m = \operatorname{ord} f$, $n = \operatorname{ord} g$ and let f, g be distinguished pseudopolynomials of the form

$$(4.1)\qquad \begin{aligned} f(x, y) &= x^m + a_1(y)x^{m-1} + \ldots + a_m(y),\\ g(x, y) &= x^n + b_1(y)x^{n-1} + \ldots + b_n(y). \end{aligned}$$

Let D be the least common multiple of $\operatorname{ord} f_1, \ldots, \operatorname{ord} f_r, \operatorname{ord} g_1, \ldots, \operatorname{ord} g_s$.

LEMMA 4.1. *There exist holomorphic functions $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_n$ in a neighbourhood of $0 \in C$, such that*:

(a) $\operatorname{ord} \alpha_i \geqslant D$, $\operatorname{ord} \beta_j \geqslant D$ *and*

$$(4.2)\qquad \begin{aligned} f(x, t^D) &= \prod_{k=1}^{m} (x - \alpha_k(t)),\\ g(x, t^D) &= \prod_{l=1}^{n} (x - \beta_l(t)); \end{aligned}$$

(b) *for every k, there exists an i such that*

$$(4.3)\qquad \sum_{j=1}^{n} \operatorname{ord}(\alpha_k - \beta_j) = D\mu(f_i, g)/\operatorname{ord} f_i$$

and, for every i, there exists some k such that (4.3) *holds*;

(c) *for every l, there exists some j such that*

$$(4.4)\qquad \sum_{i=1}^{m} \operatorname{ord}(\beta_l - \alpha_i) = D\mu(f, g_j)/\operatorname{ord} g_j$$

and, for every j, there exists an l such that (4.4) *holds.*

Proof. We may assume that f_i, $i = 1, \ldots, r$, and g_j, $j = 1, \ldots, s$, are also distinguished pseudopolynomials of x of degrees m_i and n_j, respectively. Let (U_i^*, Φ_i^*, W_i^*) and $(U_j^{**}, \Phi_j^{**}, W_j^{**})$, where $\Phi_i^*(t) = (\varphi_i^*(t), t^{m_i})$, $\Phi_j^{**}(t) = (\varphi_j^{**}(t), t^{n_j})$, be canonical parametrizations of the set of zeros of f_i, g_j in a neighbourhood of $0 \in C^2$ (cf. [4]). It is easy to see that $\operatorname{ord} \varphi_i^* \geqslant m_i$, $\operatorname{ord} \varphi_j^{**} \geqslant n_j$ and

$$\text{(4.5)} \qquad \begin{aligned} f_i(x, t^{m_i}) &= \prod_{p=1}^{m_i} (x - \varphi_i^*(\varepsilon_i^p t)), \\ g_j(x, t^{n_j}) &= \prod_{q=1}^{n_j} (x - \varphi_j^{**}(\eta_j^q t)), \end{aligned}$$

where ε_i, η_j are the m_i-th and n_j-th primitive roots of unity, respectively. Hence

$$\text{(4.6)} \qquad \begin{aligned} f(x, t^D) &= \prod_{i=1}^{r} \prod_{p=1}^{m_i} (x - \varphi_i^*(\varepsilon_i^p t^{D/m_i})), \\ g(x, t^D) &= \prod_{j=1}^{s} \prod_{q=1}^{n_j} (x - \varphi_j^{**}(\eta_j^q t^{D/n_j})), \end{aligned}$$

which gives (a). Moreover, from (4.6) and (4.2) it follows that, for every k, there exist i, p such that $\alpha_k(t) = \varphi_i^*(\varepsilon_i^p t^{D/m_i})$ and vice versa. From the definition of the parametric multiplicity (cf. [4]) we have

$$\mu(f_i, g) = \operatorname{ord} g \circ \Phi_i^*(\varepsilon_i^p t).$$

Hence

$$(D/m_i)\, \mu(f_i, g) = \operatorname{ord} g \circ \Phi_i^*(\varepsilon_i^p t^{D/m_i}) = \operatorname{ord} g\big(\alpha_k(t), t^D\big).$$

On the other hand, (4.2) implies

$$\operatorname{ord} g\big(\alpha_k(t), t^D\big) = \sum_{j=1}^{n} \operatorname{ord}(\alpha_k - \beta_j),$$

which gives (b). We show (c) in an analogous way.

Now we formulate and prove a lemma playing a key part in the proof of the Main Theorem. We shall carry it out according to the "horn neighbourhoods" method.

Let us assume that (4.2) holds in a polydisc $\tilde{P} = \{(x, t): |x| < \tilde{\varrho},\ |t| < \tilde{\varrho}^{1/D}\}$. It is easy to see that there exist positive numbers c and d such that

$$\text{(4.7)} \qquad c\,|t|^{\operatorname{ord}(\alpha_i - \beta_j)} \leqslant |\alpha_i(t) - \beta_j(t)| \leqslant d\,|t|^{\operatorname{ord}(\alpha_i - \beta_j)}$$

and

$$\text{(4.8)} \qquad |\beta_l(t) - \beta_j(t)| \leqslant d\,|t|^{\operatorname{ord}(\beta_l - \beta_j)}$$

for $|t| < \tilde{\varrho}^{1/D}$, $i = 1, \ldots, m$, $l, j = 1, \ldots, n$, provided $\beta_l \neq \beta_j$. Take a positive number w such that $w < c$ and put $P = \{(x, t): |x| < \varrho, |t| < \varrho^{1/D}, \varrho = \min(\tilde{\varrho}, w/2d)\}$. Let us denote

$$(4.9) \qquad v = \max_{i,j=1}^{r,s} \left(\mu(f_i, g)/\operatorname{ord} f_i, \mu(f, g_j)/\operatorname{ord} g_j\right).$$

LEMMA 4.2. *There exists a constant $A_1 > 0$ such that*

$$(4.10) \qquad |H(x, t^D)| \geqslant A_1 |t|^{vD} \quad \textit{for } (x, t) \in P.$$

Proof. We first show that (4.10) holds in any horn neighbourhood F_k of the form

$$F_k = \{(x, t) \in P: |x - \alpha_k(t)| \leqslant w|t|^{\varkappa_k}\}$$

where $\varkappa_k = \max_j \operatorname{ord}(\alpha_k - \beta_j)$. From the definition of F_k and from (4.7) we have

$$|x - \beta_j(t)| \geqslant c|t|^{\operatorname{ord}(\alpha_k - \beta_j)} - w|t|^{\varkappa_k}$$
$$\geqslant (c - w)|t|^{\operatorname{ord}(\alpha_k - \beta_j)}.$$

Hence and from (4.2) we get

$$|g(x, t^D)| \geqslant (c - w)^n |t|^{\sum_{j=1}^{n} \operatorname{ord}(\alpha_k - \beta_j)}.$$

By Lemma 4.1 (b), there exists an i such that

$$|g(x, t^D)| \geqslant (c - w)^n |t|^{D\mu(f_i, g)/\operatorname{ord} f_i}.$$

Hence and from the definition of v we get (4.10) in F_k, where $A_1 = (c - w)^n$.

Let

$$F_{l,q} = \{(x, t) \in P: |x - \beta_l(t)| \leqslant w|t|^{\operatorname{ord}(\beta_l - \beta_q)}\}$$

for $\beta_l \neq \beta_q$ and

$$\tilde{F}_{l,q} = F_{l,q} - \bigcup_{k=1}^{m} F_k - \bigcup_{p,k} F_{p,k},$$

where p, k run over all indices such that $F_{p,k}$ is a proper subset of $F_{l,q}$. We now show that (4.10) holds in $\tilde{F}_{l,q}$ for any l, q such that $\beta_l \neq \beta_q$. Fix $i \in \{1, \ldots, m\}$. Three cases can occur. In the first one, if $\operatorname{ord}(\beta_l - \alpha_i) = \varkappa_i$, we have the inequality

$$(4.11) \qquad |x - \alpha_i(t)| \geqslant w|t|^{\operatorname{ord}(\beta_l - \alpha_i)}.$$

In the second case, if

$$\operatorname{ord}(\beta_l - \alpha_i) \leqslant \operatorname{ord}(\beta_l - \beta_q),$$

then from the definition of $\tilde{F}_{l,q}$ and from (4.7) we get

$$|x-\alpha_i(t)| \geqslant c|t|^{\operatorname{ord}(\alpha_i-\beta_l)} - w|t|^{\operatorname{ord}(\beta_l-\beta_q)} \geqslant (c-w)|t|^{\operatorname{ord}(\beta_l-\alpha_i)}. \tag{4.12}$$

In the third case, if

$$\operatorname{ord}(\beta_l-\beta_q) < \operatorname{ord}(\beta_l-\alpha_i) < \operatorname{ord}(\beta_p-\alpha_i) = \varkappa_i,$$

we easily check that $F_{p,l} \subset F_{l,q}$. In fact, if $(x, t) \in F_{p,l}$, then

$$|x-\beta_p(t)| \leqslant w|t|^{\operatorname{ord}(\beta_p-\beta_l)}.$$

Hence and from (4.8) we have in this case

$$|x-\beta_l(t)| \leqslant (w+d)|t|^{\operatorname{ord}(\beta_p-\beta_l)} \leqslant w|t|^{\operatorname{ord}(\beta_l-\beta_q)},$$

that is to say, $(x, t) \in F_{l,q}$. In consequence, from (4.7) we have

$$|x-\alpha_i(t)| \geqslant w|t|^{\operatorname{ord}(\beta_p-\beta_l)} - d|t|^{\operatorname{ord}(\alpha_i-\beta_p)} \geqslant (w/2)|t|^{\operatorname{ord}(\beta_l-\alpha_i)}. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we find that, for $(x, t) \in \tilde{F}_{l,q}$,

$$|x-\alpha_i(t)| \geqslant A_2|t|^{\operatorname{ord}(\beta_l-\alpha_i)}, \qquad A_2 = \min(w/2,\, c-w),$$

and so,

$$|f(x, t^D)| \geqslant A_2^m |t|^{\sum_{i=1}^{m} \operatorname{ord}(\beta_l-\alpha_i)}.$$

By Lemma 4.1 (c), there exists a j such that

$$|f(x, t^D)| \geqslant A_2^m |t|^{D\mu(f,g_j)/\operatorname{ord} g_j}.$$

Hence and from the definition of v we get (4.10) in $\tilde{F}_{l,q}$, where $A_1 = A_2^m$.

Since (4.10) holds in $\tilde{F}_{l,q}$ for all l, q, therefore it also holds in $\bigcup_{l,q} F_{l,q}$.

To complete the proof, it suffices to establish (4.10) in the complement of $\bigcup_k F_k \cup \bigcup_{l,q} F_{l,q}$. Take $i \in \{1, \ldots, m\}$. Two cases are possible. In the first one, if

$$\min_l \operatorname{ord}(\beta_l-\alpha_i) = \max_l \operatorname{ord}(\beta_l-\alpha_i) = \varkappa_i,$$

we have

$$|x-\alpha_i(t)| \geqslant w|t|^{\varkappa_i} = w|t|^{\min_l \operatorname{ord}(\beta_l-\alpha_i)}. \tag{4.14}$$

In the second case, if

$$\operatorname{ord}(\beta_q-\alpha_i) = \min_l \operatorname{ord}(\beta_l-\alpha_i) < \operatorname{ord}(\beta_p-\alpha_i) = \varkappa_i,$$

we have

$$(4.15)\qquad |x-\alpha_i(t)| \geqslant w|t|^{\operatorname{ord}(\beta_p-\beta_q)} - d|t|^{\operatorname{ord}(\beta_p-\alpha_i)}$$
$$\geqslant (w/2)|t|^{\min_l \operatorname{ord}(\beta_l-\alpha_i)}.$$

From (4.14) and (4.15) we get in the general case

$$|x-\alpha_i(t)| \geqslant (w/2)|t|^{\min_l \operatorname{ord}(\beta_l-\alpha_i)},$$

and so,

$$|f(x, t^D)| \geqslant (w/2)^m |t|^{\sum_{i=1}^{m} \min_l \operatorname{ord}(\beta_l-\alpha_i)}$$
$$\geqslant (w/2)^m |t|^{\min_l \sum_{i=1}^{m} \operatorname{ord}(\beta_l-\alpha_i)}.$$

Further, from Lemma 4.1(c) we have

$$|f(x, t^D)| \geqslant (w/2)^m |t|^{\min_j D\mu(f, g_j)/\operatorname{ord} g_j}.$$

Hence and from the definition of v we get also (4.10) in this case, where $A_1 = (w/2)^m$.

This concludes the proof of the lemma. □

5. Proof of the Main Theorem

Since the Łojasiewicz exponent, as well as the multiplicities $\mu(f_i, g)$, $\mu(f, g_j)$, and $\operatorname{ord} f$, $\operatorname{ord} g$, $\operatorname{ord} f_i$, $\operatorname{ord} g_j$, $i = 1, \ldots, r$, $j = 1, \ldots, s$, are invariants of linear automorphisms of C^2, we may assume that $\operatorname{ord} f(x, 0) = \operatorname{ord} f(x, y)$ and $\operatorname{ord} g(x, 0) = \operatorname{ord} g(x, y)$. Then, by the Weierstrass Preparation Theorem, there exist distinguished pseudopolynomials $\tilde{f}$, $\tilde{g}$ associated with f, g. By putting $\tilde{H} = (\tilde{f}, \tilde{g})$, it is easy to check that $\lambda(H) = \lambda(\tilde{H})$. Then, without loss of generality, we may assume that f and g are distinguished pseudopolynomials of form (4.1).

Let $U = \{z: |z| < \varrho\}$. We first show that there exists a positive constant A such that

$$(5.1)\qquad |H(z)| \geqslant A|z|^v \quad \text{for } z \in U,$$

where $v = \max_{i,j=1}^{r,s} (\mu(f_i, g)/\operatorname{ord} f_i,\ \mu(f, g_j)/\operatorname{ord} g_j)$. From Lemma 4.1(a) it follows that there exists a constant $e > 0$ such that, for each $i \in \{1, \ldots, m\}$,

$$(5.2)\qquad |\alpha_i(t)| \leqslant e|t|^D.$$

Let us consider two cases. In the first one, if $|x| \geq (e+1)|y|$, we put $y = t^D$ and then, by (4.2) and (5.2), we obtain

$$|f(x, t^D)| \geq \big(|x|/(e+1)\big)^m \geq \big(|x|/(e+1)\big)^v.$$

Hence we get (5.1), where $A = \big(1/(e+1)\big)^v$. In the second case, if $|x| < (e+1)|y|$, then putting $t^D = y$ in Lemma 4.2, we get $|H(z)| \geq A_1|y|^v$. Hence we also obtain (5.1), where $A = A_1/(e+1)^v$.

Let us now assume that, for some $p \in \{1, \ldots, r\}$ or $q \in \{1, \ldots, s\}$, we have $v = \mu(f_p, g)/\mathrm{ord} f_p$ or $v = \mu(f, g_q)/\mathrm{ord}\, g_q$. Let (U_p^*, Φ_p^*, W_p^*) be a canonical parametrization of the set of zeros of f_p in a neighbourhood of $0 \in C^2$, such that $W_p^* \subset U$. It is easy to see that $|H \circ \Phi_p^*(t)| \sim |t|^{\mu(f_p, g)}$ and $|\Phi_p^*(t)| \sim |t|^{\mathrm{ord} f_p}$. Hence, on the curve Γ_p we have $|H(z)| \sim |z|^v$. Analogously, in the second case $|H(z)| \sim |z|^v$ on the curve Γ_{r+q}. Hence and from the definition of $\lambda(H)$ we conclude that $\lambda(H) \geq v$. On the other hand, by (5.1), we have $\lambda(H) \leq v$. In consequence, $\lambda(H) = v$ and the proof of (i) is completed.

Condition (ii) follows from (5.1), and (iii) is obvious. This ends the proof of the theorem. □

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References

[1] S. H. Chang and Y. C. Lu, *On C^0-sufficiency of complex jets*, Canad. J. Math. 25 (1973), 874–880.

[2] J. Chądzyński, *On the order of an isolated zero of a holomorphic mapping*, Bull. Pol. Acad. Sci. Math. 31 (1983), 121–128.

[3] J. Chądzyński and T. Krasiński, *The Noether exponent and the Łojasiewicz exponent I, II*. Bull. Soc. Sci. Lett. Łódź. 36,11 (1986), 16 pp.; 36. 14 (1986), 10 pp.

[4] J. Chądzyński, T. Krasiński and W. Kryszewski, *On the parametric and algebraic multiplicities of an isolated zero of a holomorphic mapping*, Lecture Notes in Math. 1039, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo 1983, 88–101.

[5] M. Hervé, *Several Complex Variables*, Oxford University Press, Oxford 1963.

[6] T. C. Kuo and Y. C. Lu, *On analytic function germs of two complex variables*, Topology 16 (1977), 299–310.

[7] M. Lejeune-Jalabert et B. Teissier, *Cloture integrale des idéaux et equisingularité*, Centre de Mathématiques, École Polytechnique, 1974.

[8] A. Płoski, *Sur l'exposant d'une application analytique I, II*, Bull. Pol. Acad. Sci. Math. 32 (1984), 669–673; 33 (1985), 123–127.

[9] —, *Remarque sur la multiplicité d'intersection des branches planes*, Bull. Pol. Acad. Sci. Math. 33 (1985), 601–605.

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