POLSKA AKADEMIA NAUK • INSTYTUT MATEMATYCZNY

ROZPRAWY MATEMATYCZNE

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A. MOSTOWSKI

On invariant, dual invariant and absolute formulas

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WROCŁAWSKA DRUKARNIA NAUKOWA

Introduction

The present paper [1] deals with properties of relations expressible in the simple theory of types. For the sake of simplicity we consider only one, say ternary, relation N. Let Φ be a formula of the simple theory of types with one free variable V whose type is that of ternary relations between individuals. Speaking intuitively the formula Φ "expresses" a property of ternary relations. This intuitive formulation, however, is inaccurate because the "meaning" of Φ depends on the meaning of logical and extra-logical constants which occur in Φ. Hence, before we can speak of the meaning of Φ, we must first choose a model $\mathcal{M}$ of the theory of types. Usually we have in mind a standard model, i.e. one whose individuals form a set X, whose sets of 1st type are arbitrary subsets of X and, generally, whose sets of the $k+1$st type are arbitrary sets of sets of the kth type. Let us denote this model by $St(X)$. It is easy to see that the property expressed by Φ in the standard model may depend on X. To avoid this complication we will speak not of properties of N alone but of pairs (M, N) where M is a set containing the field of N. When we speak about the property of (M, N) "expressed" by Φ we mean the property:

(0.1) *N satisfies Φ in the standard model $St(M)$.*

Now we know (cf. Henkin [3]) that there are non-standard (or general) models of type theory [2]. Let $\mathcal{M}$ be such a model and let the set of individuals of $\mathcal{M}$ be again M. It is easy to define when an element n of the model represents N in $\mathcal{M}$: this means simply that $N(x, y, z)$ is equivalent to the statement:

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(1) The results presented in sections 1-6 of this paper were reported on at the International Congress of Mathematicians, Edinburgh, 1958; the results presented in sections 7-8 were reported on at the Mathematical Symposium held in Berlin in November 1960.

(2) General models are understood as sequences consisting of collections C_i (which interpret universes of various types) and relations R_i holding between the elements of collections C_i and C_{i+1}; these relations interpret the notion of belonging of an element to another element of the next higher type. In Henkin's treatment of the subject C_i is always a collection of objects of type i and R_i is the ε relation, $i = 1, 2, \ldots$ Some of the subsequent definitions could be slightly simplified if we adhered to Henkin's notion of the general model. Our more abstract approach, however, will be needed in section 6 below.

There is an element u of $\mathcal{M}$ which satisfies in $\mathcal{M}$ the formulas "u is an ordered triple of x, y, z" and "u is an element of n" (3).

Now if $\mathcal{M}$ is a general model and n an element of $\mathcal{M}$ which represents N in $\mathcal{M}$, then we may take the statement

(0.2) *n satisfies the formula Φ in $\mathcal{M}$*

as an explication of the loose statement that N has the property expressed by Φ. With this explication the property expressed by Φ depends in general on the model $\mathcal{M}$.

We call Φ a formula *absolute* with respect to the class K of (general) models if for every pair (M, N) and every model $\mathcal{M}$ of K whose set of individuals is M and which contains an element n representing N conditions (0.1) and (0.2) are equivalent (cf. [7])(4). If $(0.2) \rightarrow (0.1)$, then we call Φ *invariant* with respect to K; if $(0.1) \rightarrow (0.2)$, then we call Φ *dual invariant* (cf. [5] and [8]).

We can now formulate the problem discussed in this paper. If Φ is an absolute formula, what can be said about the property (0.1)?

We shall show (theorem 6.3) that if K is the family of models which safisfy a recursively enumerable set of formulas true in each standard model, then the property is elementary (i.e. can be defined within the first order logic). We also show that (0.1) is elementary under the following assumptions: (1) K is the family of models which satisfy the set of all formulas true in each standard model, (2) M is infinite (theorem 7.4). We also obtain a characterization of (0.1) in the case of invariant and dual invariant formulas (theorems 6.1 and 6.2).

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(3) Had we adopted the notion of general model as defined by Henkin, we could simply say that n represents N in $\mathcal{M}$ if $n = N$.

(4) The definition given in this paper lacked precision and should be replaced by the definition given above (or by the still more formal definition given below on p. 12).

We take this opportunity to correct certain statements made in [7]:

(a) The definition of models (p. 33, line 5 from bottom) is incorrect and should be replaced by the following: a model is an ordered triple $\langle A_0, \mathfrak{A}, a\rangle$ where $\mathfrak{A}$ is a set of subsets of A_0 and a a set of subsets of A.

(b) The statement made at the bottom of p. 34 to the effect that Λ contains the axiom of extensionality is incorrect. It is true, however, that if Λ satisfies (2) then the set Λ' obtained from Λ by adjunction of axiom (3) also satisfies (2); thus we can replace Λ by the larger set Λ' in the whole subsequent proof.

(c) On p. 41 it is stated that the property of being inductive is an absolute property of a set. This is clearly false: the property is dually invariant but not absolute.

1. Lemmas concerning first order formulas

We denote functional variables by Roman capitals and individual variables by lower case Roman letters. First order formulas are denoted by German capitals. We assume that the identity predicate may occur in the formulas. To avoid confusion we use the colon as the symbol for the identity of two formulas.

If $\mathfrak{F}$: $\mathfrak{F}(\mathrm{P}_1, \ldots, \mathrm{P}_k, \mathrm{x}_1, \ldots, \mathrm{x}_l)$ has the free variables indicated, if I is a set, $R_1, \ldots, R_k$ are relation with fields contained in I such that R_j has as many arguments as P_j, $j = 1, \ldots, k$, and if $a_1, \ldots, a_l$ are elements of I, then the formula $\models_I \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_l]$ means that $\mathfrak{F}$ is satisfied in I by the assignment which correlates R_j with P_j, a_i with x_i ($j = 1, \ldots, k$, $i = 1, \ldots, l$) and the identity relation restricted to I with the identity predicate.

The formula $R \varepsilon \mathscr{B}(I)$ will mean that the field of R is contained in I.

1.1. *For every* $\mathfrak{F}$: $\mathfrak{F}(\mathrm{P}_1, \ldots, \mathrm{P}_k)$ *with the free variables* $\mathrm{P}_1, \ldots, \mathrm{P}_k$ *there is a formula* $\mathfrak{G}(\mathrm{P}_1, \ldots, \mathrm{P}_k)$ *such that for arbitrary sets* I, J *and relations* $R_1, \ldots, R_k$ *if* $J - I$ *is infinite and* $R_1, \ldots, R_k \varepsilon \mathscr{B}(I)$, *then*

$$\models_J \mathfrak{F}[R_1, \ldots, R_k] \equiv \models_I \mathfrak{G}[R_1, \ldots, R_k].$$

Proof. We shall formulate a more general theorem. Call a formula *numerical* if it contains no predicate variables. Let $\mathfrak{F}$ be a first order formula whose free variables are some of the variables $\mathrm{P}_1, \ldots, \mathrm{P}_k$, $\mathrm{x}_1, \ldots, \mathrm{x}_m$. We shall prove that for every partition

$$\{1, \ldots, m\} = \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \mathrm{I} \cup \mathrm{II}$$

there are: an integer h, first order formulas $\mathfrak{G}_s(\mathrm{P}_1, \ldots, \mathrm{P}_k, \mathrm{x}_{i_1}, \ldots, \mathrm{x}_{i_p})$ and numerical formulas $\mathfrak{N}_s(\mathrm{x}_{j_1}, \ldots, \mathrm{x}_{j_q})$, $s = 1, \ldots, h$ such that if $I \subseteq J$, $R_1, \ldots, R_k \varepsilon \mathscr{B}(I)$, $a_{i_1}, \ldots, a_{i_p} \varepsilon I$, and $a_{j_1}, \ldots, a_{j_q} \varepsilon J - I$, then

$$\models_J \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_m] \equiv \bigvee_{s \leqslant h} \models_I \mathfrak{G}_s[R_1, \ldots, R_k, a_{i_1}, \ldots, a_{i_p}] \,\&\, \models_{J-I} \mathfrak{N}_s[a_{j_1}, \ldots, a_{j_q}].$$

($\bigvee$ is of course an abbreviation for repeated alternations).

CASE I. $\mathfrak{F}$ is atomic, $\mathfrak{F}$: $P_j(x_{t_1}, \ldots, x_{t_r})$. We take $h = 1$, $\mathfrak{N}_1$: $(x)(x = x)$ and $\mathfrak{G}_1$: $\mathfrak{F}$ if $t_1, \ldots, t_r \varepsilon \mathrm{I}$, $\mathfrak{G}_1$: $\sim(x_{t_1} = x_{t_1}) \& \ldots \& \sim(x_{t_r} = x_{t_r})$ if at least one $t_i \varepsilon \mathrm{II}$.

CASE II. $\mathfrak{F}$ has the form $x_u = x_v$. We distinguish several subcases:

IIa. $u, v \varepsilon \mathrm{I}$; $h = 1$, $\mathfrak{G}_1 : \mathfrak{F}$, $\mathfrak{N}_1 : (x_1)(x_1 = x_1)$.

IIb. $u \varepsilon \mathrm{I}$, $v \varepsilon \mathrm{II}$; $h = 1$, $\mathfrak{G}_1 : \sim(x_u = x_u)$, $\mathfrak{N}_1 : \sim(x_v = x_v)$.

IIc. $u \varepsilon \mathrm{II}$, $v \varepsilon \mathrm{I}$; (as above).

IId. $u, v \varepsilon \mathrm{II}$; $h = 1$, $\mathfrak{G}_1 : (x)(x = x)$, $\mathfrak{N}_1 : (x_u = x_v)$.

CASE III. The theorem is valid for a formula $\mathfrak{F}$. We shall show that it is valid for the formula $\sim\mathfrak{F}$. Indeed,

$$\models_J \sim\mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_m] \equiv \bigwedge_{s \leqslant h} (\models_I \sim\mathfrak{G}_s[R_1, \ldots, R_k, a_{i_1}, \ldots, a_{i_p}] \vee \models_{J-I} \sim\mathfrak{N}_s[a_{j_1}, \ldots, a_{j_q}])$$

and the right-hand side of this equivalence can (by means of Boolean transformations) be reduced to the required form.

CASE IV. The theorem is valid for formulas $\mathfrak{F}^1, \mathfrak{F}^2$. Multiplying both sides of the equivalences

$$\models_J \mathfrak{F}^i[R_1, \ldots, R_k, a_1, \ldots, a_m] \equiv \bigvee_s \models_I \mathfrak{G}_s^i[R_1, \ldots, R_k, a_{i_1}, \ldots, a_{i_p}] \& \models_{J-I} \mathfrak{N}_s^i[a_{j_1}, \ldots, a_{j_q}], \quad i = 1, 2,$$

and performing suitable Boolean transformations on the right-hand side, we obtain the theorem for the formula $\mathfrak{F}^1 \& \mathfrak{F}^2$.

CASE V. The theorem is valid for a formula $\mathfrak{F}$. We shall prove it for the formula $\mathfrak{F}_1$: $(\mathsf{E}x_i)\mathfrak{F}$. We obviously have

$$\models_J \mathfrak{F}_1[R_1, \ldots, R_k, a_1, \ldots, a_{i-1}, a_{i+1}, , \ldots, a_m] \equiv$$
$$(\mathsf{E}a_i)_I \models_J \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_n] \vee (\mathsf{E}a_i)_{J-I} \models_J \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_m]$$

and thus it is sufficient to reduce both formulas on the right-hand side to the desired form.

Va. Reduction of the first formula. Change the given partition into a new one, $\mathrm{I}' \cup \mathrm{II}'$, which differs from $\mathrm{I} \cup \mathrm{II}$ only by assigning i to the class I'. Hence $i \varepsilon \mathrm{I}'$ and consequently $i \notin \mathrm{II}'$. Put $\mathrm{I}' = \{i, i_1', \ldots, i_{p'}'\}$, $\mathrm{II}' = \{j_1', \ldots, j_{q'}'\}$. According to the inductive assumption there are: an integer h', formulas $\mathfrak{G}_s'(P_1, \ldots, P_k, x_i, x_{i_1'}, \ldots, x_{i_{p'}'})$ and numerical formulas $\mathfrak{N}_s'(x_{j_1'}, \ldots, x_{j_{q'}'})$ $(s \leqslant h')$ such that if $I \subseteq J$, $R_1, \ldots, R_k \varepsilon \mathscr{B}(I)$, $a_i, a_{i_1'}, \ldots, a_{i_{p'}'} \varepsilon I$, $a_{j_1'}, \ldots, a_{j_{q'}'} \varepsilon J - I$, then

$$\models_J \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_m] \equiv \bigvee_{s \leqslant h'} \models_I \mathfrak{G}_s'[R_1, \ldots, R_k, a_i, a_{i_1'}, \ldots, a_{i_{p'}'}] \& \models_{J-I} \mathfrak{N}_s'[a_{j_1'}, \ldots, a_{j_{q'}'}],$$

whence

$$(\mathsf{E}\, a_i)_I \models_J \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_m] \equiv$$
$$\bigvee_{s \leqslant h'} \models_I \mathfrak{H}_s[R_1, \ldots, R_k, a_{i'_1}, \ldots, a_{i'_{p'}}] \,\&\, \models_{J-I} \mathfrak{N}'_s[a_{j'_1}, \ldots, a_{j'_{q'}}]$$

where $\mathfrak{H}_s$: $(\mathsf{E}x_i)\mathfrak{G}'_s$. Since x_i is free neither in $\mathfrak{H}_s$ nor in $\mathfrak{N}'_s$, the right-hand side can be written as

$$\bigvee_{s \leqslant h'} \models_I \mathfrak{H}_s[R_1, \ldots, R_k, a_{i_1}, \ldots, a_{i_p}] \,\&\, \models_{J-I} \mathfrak{N}'_s[a_{j_1}, \ldots, a_{j_q}],$$

where we assume that $a_{i_1}, \ldots, a_{i_p} \varepsilon I$, $a_{j_1}, \ldots, a_{j_q} \varepsilon J-I$.

Vb. Reduction of the second formula. Change the given partition into $\mathrm{I}'' \cup \mathrm{II}''$, in which $i \varepsilon \mathrm{II}''$. Put $\mathrm{I}'' = \{i''_1, \ldots, i''_p\}$, $\mathrm{II}'' = \{i, j''_1, \ldots, j''_q\}$. According to the inductive assumption there are: an integer h'', first order formulas $\mathfrak{G}''_s(\mathrm{P}_1, \ldots, \mathrm{P}_k, x_{i''_1}, \ldots, x_{i''_{p''}})$ and numerical formulas $\mathfrak{N}''_s(x_i, x_{j''_1}, \ldots, x_{j''_{q''}})$ $(s \leqslant h'')$ such that if $I \subseteq J$, $R_1, \ldots, R_k \varepsilon \mathscr{B}(I)$, $a_{i''_1}, \ldots, a_{i''_{p''}} \varepsilon I$, a_i, $a_{j''_1}, \ldots, a_{i''_{q''}} \varepsilon J-I$, then

$$\models_J \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_m] \equiv$$
$$\bigvee_{s \leqslant h''} \models_I \mathfrak{G}''_s[R_1, \ldots, R_k, a_{i''_1}, \ldots, a_{i''_{p''}}] \,\&\, \models_{J-I} \mathfrak{N}''_s[a_i, a_{j''_1}, \ldots, a_{j''_{q''}}].$$

It follows that

$$(\mathsf{E}\, a_i)_J \models_{J-I} \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_m] \equiv$$
$$\bigvee_{s \leqslant h''} \models_I \mathfrak{G}''_s[R_1, \ldots, R_k, a_{i''_1}, \ldots, a_{i''_{p''}}] \,\&\, \models_{J-I} \mathfrak{N}^*_s[a_{j''_1}, \ldots, a_{j''_{q''}}]$$

where $\mathfrak{N}^*_s$: $(\mathsf{E}x_i)\mathfrak{N}''_s$.

Since x_i is free neither in $\mathfrak{G}''_s$ nor in $\mathfrak{N}^*_s$, we can write the right-hand side as

$$\bigvee_{s \leqslant h''} \models_I \mathfrak{G}''_s[R_1, \ldots, R_k, a_{i_1}, \ldots, a_{i_p}] \,\&\, \models_{J-I} \mathfrak{N}^*_s[a_{j_1}, \ldots, a_{j_q}],$$

where we assume that $a_{i_1}, \ldots, a_{i_p} \varepsilon I$, $a_{j_1}, \ldots, a_{j_q} \varepsilon J-I$.

Assume now that $\mathfrak{F}$ has no free individual variables. We obtain an integer h, closed formulas $\mathfrak{G}_s(\mathrm{P}_1, \ldots, \mathrm{P}_k)$ and closed numerical formulas $\mathfrak{N}_s$ $(s \leqslant h)$ such that for arbitrary $I \subseteq J$, $R_1, \ldots, R_k \varepsilon \mathscr{B}(I)$

$$\models_J \mathfrak{F}[R_1, \ldots, R_k] \equiv \bigvee_{s \leqslant h} \models_I \mathfrak{G}_s[R_1, \ldots, R_k] \,\&\, \models_{J-I} \mathfrak{N}_s.$$

If $J-I$ is infinite, then the $\mathfrak{N}_s$ have definite truth values $\mathfrak{T}_s$ in $J-I$ independent of I, J. The formula $\mathfrak{G} : \mathfrak{G}_1 \,\&\, \mathfrak{T}_1 \vee \ldots \vee \mathfrak{G}_h \,\&\, \mathfrak{T}_h$ satisfies the equivalence stated in 1.1.

Let $\mathfrak{F}$ be a formula and H a one-argument predicate variable. We denote by $\mathfrak{F}_{\mathrm{rel}}(\mathrm{H}, \ldots)$ the formula resulting from $\mathfrak{F}$ by the relativization of all quantifiers to H.

1.2. *If $H \subseteq I$ and $R_1, \ldots, R_k \varepsilon \mathscr{B}(H)$, then*

$$\models_I \mathfrak{F}_{\mathrm{rel}}[H, R_1, \ldots, R_k] \equiv \models_H \mathfrak{F}[R_1, \ldots, R_k].$$

Proof by induction on the number of operators in $\mathfrak{F}$.

We introduce the following abbreviations:

$$\mathrm{H} \subseteq \mathrm{I} : (\mathrm{x})[\mathrm{H}(\mathrm{x}) \supset \mathrm{I}(\mathrm{x})],$$

$$\mathrm{G}_1, \ldots, \mathrm{G}_k \varepsilon \mathscr{B}(\mathrm{H}) : \bigwedge_{s \leqslant k} (\mathrm{x}_1, \ldots, \mathrm{x}_{p_s})[\mathrm{G}_s(\mathrm{x}_1, \ldots, \mathrm{x}_{p_s}) \supset \mathrm{H}(\mathrm{x}_1) \& \ldots \& \mathrm{H}(\mathrm{x}_{p_s})]$$

(where p_s is the number of arguments of G_s),

$$\infty(\mathrm{J}-\mathrm{I}, \mathrm{R}) : (\mathrm{x}, \mathrm{y})[\mathrm{R}(\mathrm{x}, \mathrm{y}) \supset \mathrm{J}(\mathrm{x}) \& \sim \mathrm{I}(\mathrm{x}) \& \mathrm{J}(\mathrm{y}) \& \sim \mathrm{I}(\mathrm{y})] \&$$
$$(\mathrm{x}, \mathrm{y}, \mathrm{z})[\mathrm{R}(\mathrm{x}, \mathrm{y}) \& \mathrm{R}(\mathrm{y}, \mathrm{z}) \supset \mathrm{R}(\mathrm{x}, \mathrm{z})] \& (\mathrm{x})[\mathrm{J}(\mathrm{x}) \& \sim \mathrm{I}(\mathrm{x}) \supset (\mathrm{E}\mathrm{y})\mathrm{R}(\mathrm{x}, \mathrm{y})] \&$$
$$(\mathrm{x})[\sim \mathrm{R}(\mathrm{x}, \mathrm{x})] \& (\mathrm{E}\mathrm{x})[\mathrm{J}(\mathrm{x}) \& \sim \mathrm{I}(\mathrm{x})].$$

1.3. *If $I, J, R \varepsilon \mathscr{B}(U)$ and $\models_U \infty[J-I, R]$, then $J-I$ is infinite*

Let $\mathfrak{F}, \mathfrak{G}$ be formulas with the functional variables $\mathrm{P}_1, \ldots, \mathrm{P}_k$ and the free individual variables $\mathrm{x}_1, \ldots, \mathrm{x}_l$. We say that $\mathfrak{F}$ *implies* $\mathfrak{G}$ and write $\mathfrak{F} \Rightarrow \mathfrak{G}$ if for every I, every $R_1, \ldots, R_k \varepsilon \mathscr{B}(I)$ and every $a_1, \ldots, a_l \varepsilon I$ the formula $\models_I \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_l]$ implies $\models_I \mathfrak{G}[R_1, \ldots, R_k, a_1, \ldots, a_l]$.

1.4. *$\mathfrak{F} \Rightarrow \mathfrak{G}$ holds if and only if $\models_I \mathfrak{F}[R_1, \ldots, R_k, a_1, \ldots, a_l]$ implies $\models_I \mathfrak{G}[R_1, \ldots, R_k, a_1, \ldots, a_l]$ for an arbitrary at most denumerable I, arbitrary $R_1, \ldots, R_k \varepsilon \mathscr{B}(I)$ and arbitrary $a_1, \ldots, a_l \varepsilon I$.*

1.5. *Let $\mathfrak{F}(\mathrm{M}, \mathrm{N}, \mathrm{Q}_1, \ldots, \mathrm{Q}_k)$, $\mathfrak{G}(\mathrm{M}, \mathrm{N}, \mathrm{S}_1, \ldots, \mathrm{S}_l)$ be closed formulas with the variables indicated* (M *has* 1 *and* N 3 *arguments*). *Assume that*

(1.5.1) $$(\mathrm{M} \subseteq \mathrm{J}) \& \infty(\mathrm{J}-\mathrm{M}, \mathrm{R}) \& \mathrm{Q}_1, \ldots, \mathrm{Q}_k \varepsilon \mathscr{B}(\mathrm{J}) \& \mathrm{N} \varepsilon \mathscr{B}(\mathrm{M}) \& \mathfrak{F}_{\mathrm{rel}}(\mathrm{J}, \mathrm{M}, \mathrm{N}, \mathrm{Q}_1, \ldots, \mathrm{Q}_k)$$
$$\Rightarrow \mathrm{M} \subseteq \mathrm{K} \& \infty(\mathrm{K}-\mathrm{M}, \mathrm{S}) \& \mathrm{S}_1, \ldots, \mathrm{S}_l \varepsilon \mathscr{B}(\mathrm{K}) \& \mathrm{N} \varepsilon \mathscr{B}(\mathrm{M}) \supset \mathfrak{G}_{\mathrm{rel}}(\mathrm{K}, \mathrm{M}, \mathrm{N}, \mathrm{S}_1, \ldots, \mathrm{S}_l).$$

Under these assumptions there is a formula $\mathfrak{E}(\mathrm{M}, \mathrm{N})$ such that for an arbitrary set M and an arbitrary ternary relation $N \varepsilon \mathscr{B}(M)$

$$(1.5.2) \rightarrow (1.5.3) \rightarrow (1.5.4),$$

where

(1.5.2) *there is a $J \supseteq M$ and $Q_1, \ldots, Q_k \varepsilon \mathscr{B}(J)$ such that $J-M$ is infinite and*

$$\models_J \mathfrak{F}[M, N, Q_1, \ldots, Q_k],$$

(1.5.3) $$\models_M \mathfrak{E}[M, N],$$

(1.5.4) *for an arbitrary set* $K \supseteq M$ *and arbitrary* $S_1, \ldots, S_l \varepsilon \mathscr{B}(K)$, *if* $K-M$ *is infinite, then* $\models_K \mathfrak{G}[M, N, S_1, \ldots, S_l]$.

Proof. By the interpolation lemma ([1], [6]) there is a formula $\mathfrak{D}(\mathrm{M}, \mathrm{N})$ such that

(1.5.5) $\mathrm{M} \subseteq \mathrm{J} \,\&\, \infty(\mathrm{J}-\mathrm{M}, \mathrm{R}) \,\&\, \mathrm{Q}_1, \ldots, \mathrm{Q}_k \varepsilon \mathscr{B}(\mathrm{J}) \,\&\, \mathrm{N} \varepsilon \mathscr{B}(\mathrm{M}) \,\&$
$\mathfrak{F}_{\mathrm{rel}}(\mathrm{J}, \mathrm{M}, \mathrm{N}, \mathrm{Q}_1, \ldots, \mathrm{Q}_k)$

(1.5.6) $\Rightarrow \mathfrak{D}(\mathrm{M}, \mathrm{N})$,

(1.5.7) $\Rightarrow \mathrm{M} \subseteq \mathrm{K} \,\&\, \infty(\mathrm{K}-\mathrm{M}, \mathrm{S}) \,\&\, \mathrm{S}_1, \ldots, \mathrm{S}_l \varepsilon \mathscr{B}(\mathrm{K}) \,\&\, \mathrm{N} \varepsilon \mathscr{B}(\mathrm{M}) \supset$
$\mathfrak{G}_{\mathrm{rel}}(\mathrm{K}, \mathrm{M}, \mathrm{N}, \mathrm{S}_1, \ldots, \mathrm{S}_l)$.

Determine $\mathfrak{C}$ by 1.1 so that for arbitrary sets H, M and an arbitrary relation $N \varepsilon \mathscr{B}(M)$ if $H-M$ is infinite then

$$\models_H \mathfrak{D}[M, N] \equiv \models_M \mathfrak{C}[M, N].$$

If (1.5.2) holds, then $J, M, N, Q_1, \ldots, Q_k$ satisfy (1.5.5) in J (cf. 1.2) and hence $\models_J \mathfrak{D}[M, N]$, whence ($J-M$ being infinite) we obtain (1.5.3). If (1.5.3) holds and K is a set such that $K-M$ is infinite, then $\models_K \mathfrak{D}[M, N]$. If S is an ordering of $K-M$ and $S_1, \ldots, S_l \varepsilon \mathscr{B}(K)$, then the antecedent of (1.5.7) is satisfied (in K) and hence so is its consequent, whence by 1.2 we obtain $\models_K \mathfrak{G}[M, N, S_1, \ldots, S_l]$. This proves (1.5.4).

2. Representability of recursively enumerable sets

Let $\mathscr{Q}$ be the theory described in [10]. We adjoin to its axioms a sentence stating that every integer n is uniquely representable as $1/2(x+y)(x+y+1)+y$. Let A, B, C, D, E be functional variables with 1, 1, 2, 3, 3 arguments. Write axioms of $\mathscr{Q}$ (together with the additional axiom) using predicates $\mathrm{x} = 0, \mathrm{x} = \mathrm{y}+1, \mathrm{x} = \mathrm{y}+\mathrm{z}, \mathrm{x} = \mathrm{y}\cdot\mathrm{z}$ instead of functions. Replace these predicates by $\mathrm{B}(\mathrm{x})$, $\mathrm{C}(\mathrm{y}, \mathrm{x})$, $\mathrm{D}(\mathrm{y}, \mathrm{z}, \mathrm{x})$, $\mathrm{E}(\mathrm{y}, \mathrm{z}, \mathrm{x})$ and relativize all quantifiers to A. Call the resulting formula $\mathfrak{F}_0(\mathrm{A}, \ldots, \mathrm{E})$.

We shall denote by $A_0, \ldots, E_0$ the standard model of $\mathfrak{F}_0$ in which A_0 is the set of integers and relations $B_0, \ldots, E_0$ have their arithmetical meaning.

Instead of numerals we use formulas $\mathfrak{Z}_n(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{x})$ defined by induction as follows:

$$\mathfrak{Z}_0\colon \mathrm{B}(\mathrm{x}) \,\&\, \mathrm{A}(\mathrm{x}); \qquad \mathfrak{Z}_{n+1}\colon \mathrm{A}(\mathrm{x}) \,\&\, [(\mathrm{E}\mathrm{y})\, \mathfrak{Z}_n(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{y}) \,\&\, \mathrm{C}(\mathrm{y}, \mathrm{x})].$$

2.1. $\mathfrak{F}_0 \Rightarrow (\mathrm{E}!\mathrm{x})\mathfrak{Z}_n(\mathrm{x})$.

2.2. *For every recursive function* $f(n_1, \ldots, n_p)$ *there is a formula* $\mathfrak{G}_f\colon \mathfrak{G}_f(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{x}_1, \ldots, \mathrm{x}_p, \mathrm{y})$ *such that*

$$\mathfrak{F}_0 \,\&\, \mathfrak{Z}_{n_1}(\mathrm{x}_1) \,\&\ldots\&\, \mathfrak{Z}_{n_p}(\mathrm{x}_p) \Rightarrow \mathfrak{G}\,(\mathrm{x}_1, \ldots, \mathrm{x}_p, \mathrm{y}) \equiv \mathfrak{Z}_{f(n_1, \ldots, n_p)}(\mathrm{y}).$$

For $n = 1$ this theorem is proved in [10]. For $n > 1$ it results from the case $n = 1$ and from the provability (in our extension of the theory $\mathcal{Q}$) of theorems about the representability of p-tuples of integers by single integers.

2.3. *For every recursively enumerable set X of p-tuples of integers there is a formula $\mathfrak{X}_X(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{x}_1, \ldots, \mathrm{x}_p, \mathrm{y})$ such that*

(2.3.1) $$(k_1, \ldots, k_p) \varepsilon X \equiv (\mathsf{E}l)[\mathfrak{F}_0 \,\&\, \mathfrak{Z}_{k_1}(\mathrm{x}_1) \,\&\ldots\&\, \mathfrak{Z}_{k_p}(\mathrm{x}_p) \,\&\, \mathfrak{Z}_l(\mathrm{y}) \Rightarrow \mathfrak{X}_X(\mathrm{x}_1, \ldots, \mathrm{x}_p, \mathrm{y})],$$

(2.3.2) *for arbitrary $k_1, \ldots, k_p, l$ either*

$$\mathfrak{F}_0 \,\&\, \mathfrak{Z}_{k_1}(\mathrm{x}_1) \,\&\ldots\&\, \mathfrak{Z}_{k_p}(\mathrm{x}_p) \,\&\, \mathfrak{Z}_l(\mathrm{y}) \Rightarrow \mathfrak{X}_X(\mathrm{x}_1, \ldots, \mathrm{x}_p, \mathrm{y})$$

or

$$\mathfrak{F}_0 \,\&\, \mathfrak{Z}_{k_1}(\mathrm{x}_1) \,\&\ldots\&\, \mathfrak{Z}_{k_p}(\mathrm{x}_p) \,\&\, \mathfrak{Z}_l(\mathrm{y}) \Rightarrow \sim \mathfrak{X}_X(\mathrm{x}_1, \ldots, \mathrm{x}_p, \mathrm{y}).$$

Proof. Let f be a recursive function such that $(k_1, \ldots, k_p) \varepsilon X \equiv (\mathsf{E}l)[f(k_1, \ldots, k_p, l) = 0]$ and put $\mathfrak{X}_X$: $(\mathsf{E}\mathrm{z})[\mathfrak{G}_f(\mathrm{x}_1, \ldots, \mathrm{x}_p, \mathrm{y}, \mathrm{z}) \,\&\, \mathfrak{Z}_0(\mathrm{z})]$.

3. Simple theory of types

In the version of the type theory adopted here all formulas are built from the constants $|$, E and variables V_j^k, $k, j = 0, 1, 2, \ldots$ (cf. [9]). Formulas are expressions which belong to the least class containing "atomic" formulas $\mathrm{V}_m^{k+1}\,\mathrm{V}_n^k$ ($k, m, n = 0, 1, 2, \ldots$) and containing $|\Phi_1\Phi_2$ and $\mathsf{E}\mathrm{V}_m^k\Phi_1$ whenever it contains Φ_1, Φ_2 ($k, m = 0, 1, 2, \ldots$). The set of formulas is denoted by T_ω. We also denote by T_ω the whole system of the theory of types; the (semantical) notion of consequence in T_ω will be described below. We assume the usual abbreviations for connectives and quantifiers definable by means of the stroke and the existential quantifier. Formulas of T_ω will be denoted by Greek capitals. The Gödel number of Φ will be denoted by $\ulcorner\Phi\urcorner$ and the formula with the Gödel number n by $\breve{n}$. We assume that $\ulcorner\Phi\urcorner$ is larger than the (upper and lower) indices of any variable which occurs in Φ.

We introduce the following abbreviations[5]:

$$\Delta_n(\mathrm{V}_i^n, \mathrm{V}_j^n) : (\mathrm{V}_1^{n+1})[\mathrm{V}_1^{n+1}\mathrm{V}_i^n \equiv \mathrm{V}_1^{n+1}\mathrm{V}_j^n]$$

(V_i^n and V_j^n are identical).

$$\mathrm{A}_n(\mathrm{V}_i^{n+1}; \mathrm{V}_j^n, \mathrm{V}_k^n) : (\mathrm{V}_p^n)\{\mathrm{V}_i^{n+1}\mathrm{V}_p^n \equiv [\Delta_n(\mathrm{V}_p^n, \mathrm{V}_j^n) \vee \Delta_n(\mathrm{V}_p^n, \mathrm{V}_k^n)]\}$$

($p = j + k + 1$; V_i^{n+1} is the (unordered) pair of $\mathrm{V}_j^n, \mathrm{V}_k^n$).

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(5) The connectives of the propositional calculus and the general quantifier are to be thought of as defined by means of the stroke and the existential quantifier.

$T_0(V_i^0; V_j^0) : \Delta_0(V_i^0, V_j^0)$,

$T_{n+1}(V_i^{n+1}; V_j^0) : (EV_k^n)[T_n(V_k^n; V_j^0) \& A_{n+1}(V_i^{n+1}; V_k^n, V_k^n)]$

$(k = i+j+n+1;$ V_i^{n+1} is the unit set of the unit set ... of $V_j^0)$.

$B_1(V_i^0, V_j^0) : \Delta_0(V_i^0, V_j^0)$,

$B_{n+1}(V_i^{2n}; V_{j_1}^0, \ldots, V_{j_{n+1}}^0)$:

$$(EV_p^{2(n-1)}V_q^{2(n-1)}V_r^{2n-1}V_s^{2n-1})[T_{2(n-1)}(V_q^{2(n-1)}; V_{j_{n+1}}^0) \& B_n(V_q^{2(n-1)}; V_{j_1}^0, \ldots, V_{j_n}^0) \& A_{2n-1}(V_r^{2n-1}; V_p^{2(n-1)}, V_q^{2(n-1)}) \& A_{2n-1}(V_s^{2n-1}, V_q^{2(n-1)}, V_q^{2(n-1)}) \& A_{2n}(V_i^{2n}; V_r^{2n-1}, V_s^{2n-1})]$$

$(p = \max(j_1, \ldots, j_{n+1})+1,$ $q = p+1,$ $r = q+1,$ $s = r+1;$ V_i^{2n} is the ordered $n+1$-tuple with the elements $V_{j_1}^0, \ldots, V_{j_{n+1}}^0)$.

A *model* of T_ω is an ordered pair $\mathscr{M} = \langle \boldsymbol{R}, \boldsymbol{S} \rangle$ consisting of a sequence $\boldsymbol{R} = (R_0, R_1, \ldots)$ of sets and of a sequence $\boldsymbol{S} = (S_0, S_1, \ldots)$ of binary relations. R_j is the range of the variables of the j-th type and S_j interprets the ε-relation between elements of types j and $j+1$. In order to be a model $\mathscr{M}$ must satisfy the following conditions (in which A_0 denotes the set of integers, cf. p. 7):

1. $(j)\{j \varepsilon A_0 \supset [j = 0 \supset (Ex)(x \varepsilon R_j)]\}$,
2. $(j, k)[j, k \varepsilon A_0 \supset (j+1 = k \supset (x, y)\{(xS_j y) \supset [(x \varepsilon R_j) \& (y \varepsilon R_k)]\})]$,
3. $(j, k)(j, k \varepsilon A_0 \supset \{j \neq k \supset (x)[(x \varepsilon R_j) \supset \sim (x \varepsilon R_k)]\})$,
4. $(j, k)[j, k \varepsilon A_0 \supset (j+1 = k \supset (y, y')\{y \varepsilon R_k \& y' \varepsilon R_k \& (x)[x \varepsilon R_j \supset (xS_j y \equiv xS_j y')] \supset (y = y')\})]$,
5. $(j, k)\big[j, k \varepsilon A_0 \supset \big(j+1 = k \supset (x, x')\big\{(x \varepsilon R_j) \& (x' \varepsilon R_j) \supset (Ey)\big[(y \varepsilon R_k) \& (t)\big((t \varepsilon R_j) \supset \{(tS_j y) \equiv [(t = x) \vee (t = x')]\}\big)\big]\big\}\big)\big]$.

We assume as known the notion of safisfaction in a model. We write $\models_{\mathscr{M}} \Phi[a_1, \ldots, a_k]$ for "$a_1, \ldots, a_k$ satisfy Φ in $\mathscr{M}$". It is of course assumed that Φ has k free variables $V_{j_1}^{i_1}, \ldots, V_{j_k}^{i_k}$ and that $a_s \varepsilon R_{i_s}$ for $s = 1, \ldots, k$. The notion of consequence in T_ω is defined as usual: Φ follows from a set Z of closed formulas if Φ is valid in every $\mathscr{M}$ in which all the formulas of Z are valid.

If R_{j+1} is the family of all subsets of R_j and S_j is the ε relation between members of R_j and R_{j+1} then $\mathscr{M}$ is called a *standard model* and is denoted by $St(R_0)$.

Let M be a set and N a ternary relation such that $N \varepsilon \mathscr{B}(M)$. We say that the pair (M, N) is contained in $\mathscr{M}$ if $M = R_0$ and there is an m in R_5 such that

(3.1) $(x)[x \varepsilon R_4 \supset (xS_4 m \supset (Ey, z, t)\{y \varepsilon R_0 \& z \varepsilon R_0 \& t \varepsilon R_0 \& \models_{\mathscr{M}} B_3[x; y, z, t]\})]$,

$$(3.2)\quad (y, z, t)\{y \varepsilon R_0 \,\&\, z \varepsilon R_0 \,\&\, t \varepsilon R_0 \supset [N(y, z, t) \equiv (\mathsf{E}x)(x S_4 m \,\&\, \models_{\mathscr{M}} \mathrm{B}_3[x; y, z, t])]\}.$$

If an m satisfies the above conditions then we say that m *represents* N in $\mathscr{M}$ [6]. It is easy to prove that there is at most one such element in R_5.

We can now express precisely the definitions which were sketched in the introduction.

Lez $\mathscr{K}$ be a class of models, Φ a formula of T_ω with the unique free variable V_1^5. We say that Φ is (a) invariant, (b) dual invariant, (c) absolute with respect to $\mathscr{K}$ if for every model $\mathscr{M}$ in $\mathscr{K}$ and every pair (M, N) contained in $\mathscr{M}$ and represented therein by an element n

(a) $\models_{\mathscr{M}} \Phi[n]$ *implies* $\models_{St(M)} \Phi[N]$,

(b) $\models_{St(M)} \Phi[N]$ *implies* $\models_{\mathscr{M}} \Phi[n]$,

(c) $\models_{St(M)} \Phi[N]$ *is equivalent to* $\models_{\mathscr{M}} \Phi[n]$.

4. Formalization of the satisfaction relation

It will be convenient to present formally the basic semantical definitions. Let $\mathscr{M}$ be a model. We call an *S-system for* $\mathscr{M}$ a system consisting *of a relation* U, of a doubly infinite sequence of binary relations $W_{k,l}$ and of a sequence of sets Y_n and satisfying the following conditions 6-13:

6. $(\mathsf{E}f) f \varepsilon U$,
7. $(j, k)\{j, k \varepsilon A_0 \supset (f)[f \varepsilon U \supset (\mathsf{E}!\, a) W_{k,j}(f, a)]\}$,
8. $(j, k)\{j, k \varepsilon A_0 \supset (f, a)[f \varepsilon U \& W_{k,j}(f, a) \supset a \varepsilon R_k]\}$,
9. $(j)\{j \varepsilon A_0 \supset (f)[f \varepsilon Y_j \supset f \varepsilon U]\}$,
10. $(j, k, l, m, n)\{j, k, l, m, n \varepsilon A_0 \supset [k = j+1 \,\&\, n = \ulcorner \mathrm{V}_l^k \mathrm{V}_m^j \urcorner \supset (f)(f \varepsilon U \supset \{f \varepsilon Y_n \equiv (a, b)[W_{j,m}(f, a) \& W_{k,l}(f, b) \supset a S_j b]\})]\}$,
11. $(j, k, l)[j, k, l \varepsilon A_0 \supset (l = \ulcorner | \breve{j} \breve{k} \urcorner \supset (f)\{f \varepsilon U \supset [f \varepsilon Y_l \equiv (f \notin Y_j \vee f \notin Y_k)]\})]$.

Definition:

$$\Omega_{p,q,m}(f', f, a) \equiv W_{p,q}(f', a) \& (p_1, q_1)(p_1, q_1 \varepsilon A_0 \supset \{p_1 \leqslant m \,\&\, q_1 \leqslant m \,\&\, (p_1 \neq p \vee q_1 \neq q) \supset (b)[W_{p_1,q_1}(f, b) \equiv W_{p_1,q_1}(f', b)]\}).$$

12. $(p, q, m)(p, q, m \varepsilon A_0 \supset (f, a)\{f \varepsilon U \& a \varepsilon R_p \supset (\mathsf{E}f')[f' \varepsilon U \& \Omega_{p,q,m}(f', f, a)]\})$,
13. $(p, q, m, n)\{p, q, m, n \varepsilon A_0 \supset [n = \ulcorner \mathsf{E} \mathrm{V}_q^p \breve{m} \urcorner \supset (f)(f \varepsilon U \supset \{f \varepsilon Y_n \equiv (\dot{\mathsf{E}} a, f')[f' \varepsilon U \& \Omega_{p,q,m}(f', f, a) \& f' \varepsilon Y_m]\})]\}$.

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[6] Sometimes we shall also say that m represents the whole pair (M, N).

A particular S-system for $\mathscr{M}$ is obtained as follows:

U is the set U^0 of mappings f such that $f V_l^k \varepsilon R_k$; $W_{k,l}^0(f, a)$ means that $f(V_l^k) = a$; Y_n^0 is the set of $f \varepsilon U^0$ such that $\breve{n}$ is satisfied by the assignment $V_l^k \to f(V_l^k)$[6a]. In this case the formula $\Omega_{p,q,m}(f', f, a)$ means that $f'(V_q^p) = a$ and $f'(V_{q_1}^{p_1}) = f(V_{q_1}^{p_1})$ for arbitrary p_1, q_1 such that $p_1 \leqslant m$, $q_1 \leqslant m$, $(p_1, q_1) \neq (p, q)$. The S-system U^0, $W_{k,l}^0$, Y_n^0 is called the *standard S-system for* $\mathscr{M}$. In the sequel we assume that $\mathscr{M} = \langle \boldsymbol{R}, \boldsymbol{S} \rangle$ is a model and (U, W, Y) any S-system for $\mathscr{M}$.

4.1. *Let $i_1, \ldots, i_k, j_1, \ldots, j_k$ be integers and let $a_s \varepsilon R_{i_s}$ for $s = 1, 2, \ldots, k$. Then there is an f in U such that $f(V_{j_s}^{i_s}) = a_s$ for $s = 1, \ldots, k$.*

Proof: By 6 and 12.

4.2. *If Φ is a formula of T_ω with the free variables $V_{j_1}^{i_1}, \ldots, V_{j_k}^{i_k}$, if $a_s \varepsilon R_{i_s}$ for $s = 1, 2, \ldots, k$ and if $f \varepsilon U$ and $f(V_{j_s}^{i_s}) = a_s$ for $s = 1, 2, \ldots, k$, then*

$$f \varepsilon Y_{\ulcorner \Phi \urcorner} \equiv \models_{\mathscr{M}} \Phi[a_1, \ldots, a_k].$$

Proof: By induction on the length of Φ using 10, 11, 13 and the remark that $i_s, j_s \leqslant \ulcorner \Phi \urcorner$ for $s = 1, 2, \ldots, k$.

Now let $\mathrm{A}, \mathrm{U}, \mathrm{R}, \mathrm{Y}, \mathrm{S}, \mathrm{W}$ be predicate variables with 1, 1, 2, 2, 3, 4 arguments. Note that conditions 1-11 have the form

$$(j, k, l, m, n)\{j, k, l, m, n \varepsilon A_0 \supset [(j, k, l, m, n) \varepsilon R \supset \Lambda]\}$$

where R is a recursive set of quintuples of integers and Λ is a formula built from the simple formulas of the form $a \varepsilon R_k$, $a S_j b$, $f \varepsilon U$, $f \varepsilon Y_n$, $W_{k,l}(f, a)$ by means of connectives and quantifiers (actually all the variables j, k, l, m, n occur only in 10, in other formulas some of them are lacking). We replace $(j, k, \ldots, n) \varepsilon R$ by a suitable first order formula which defines this relation (in the sense in which we can say of the formula $\mathfrak{G}_f$ of 2.1 that it defines f). Further we replace in Λ $a \varepsilon R_k$ by $\mathrm{R(k, a)}$, $a S_j b$ by $\mathrm{S(j, a, b)}$, $f \varepsilon U$ by $\mathrm{U(f)}$ and $f \varepsilon Y_n$ by $\mathrm{Y(n, f)}$. Finally replace $j, k, l, m, n \varepsilon A_0$ by $\mathrm{A(j) \& \ldots \& A(n)}$. In this way we obtain 11 formulas $\mathfrak{B}_1$-$\mathfrak{B}_{11}$:

$\mathfrak{B}_1$: $\mathrm{(j)\{A(j) \supset [B(j) \supset (Ex)R(j, x)]\}}$,

$\mathfrak{B}_2$: $\mathrm{(j, k)\big(A(j) \& A(k) \supset \{C(j, k) \supset (x, y)[S(j, x, y) \supset R(j, x) \& R(k, y)]\}\big)}$,

$\mathfrak{B}_3$: $\mathrm{(j, k)\big(A(j) \& A(k) \supset \{j \neq k \supset (x)[R(j, x) \supset \sim R(k, x)]\}\big)}$,

$\mathfrak{B}_4$: $\mathrm{(j, k)\big\{A(j) \& A(k) \supset \big[C(j, k) \supset (y, y')\big(R(k, y) \& R(k, y') \& \& (x)\{R(j, x) \supset [S(j, x, y) \equiv S(j, x, y')]\} \supset (y = y')\big)\big]\big\}}$,

$\mathfrak{B}_5$: $\mathrm{(j, k)\Big[A(j) \& A(k) \supset \Big(C(j, k) \supset (x, x')\big\{R(j, x) \& R(j, x') \supset (Ey)\big[R(k, y) \& (t)\big(R(j, t) \supset \{S(j, t, y) \equiv [(t = x) \vee (t = x')]\}\big)\big]\big\}\Big)\Big]}$,

$\mathfrak{B}_6$: $\mathrm{(Ef)\,U(f)}$,

[6a] If n is not a Gödel number of a formula, then $Y_n^0 = U^0$.

$\mathfrak{B}_7$: $(j, k)\{A(j) \& A(k) \supset (f)[U(f) \supset (E!\, a)W(k, j, f, a)]\}$,
$\mathfrak{B}_8$: $(j, k)\{A(j) \& A(k) \supset (f, a)[U(f) \& W(k, j, f, a) \supset R(k, a)\}]$,
$\mathfrak{B}_9$: $(j)\{A(j) \supset (f)[Y(j, f) \supset U(f)]\}$,
$\mathfrak{B}_{10}$: $(j, k, l, m, n)\{A(j) \& A(k) \& A(l) \& A(m) \& A(n) \supset [\mathfrak{G}^{(1)}(A, \ldots, E, j, \ldots, n) \& C(j, k) \supset (f)(U(f) \supset \{Y(n, f) \equiv (a, b)[W(j, m, f, a) \& W(k, l, f, b) \supset S(j, a, b)]\})]\}$,
$\mathfrak{B}_{11}$: $(j, k, l)\{A(j) \& A(k) \& A(l) \supset [\mathfrak{G}^{(2)}(A, \ldots, E, j, k, l) \supset (f)(U(f) \supset \{Y(l, f) \equiv [\sim Y(j, f) \vee \sim Y(k, f)]\})]\}$.

We denote by $\mathfrak{H}(p, q, m, f', f, a)$ the formula

$$W(p, q, f', a) \& (p_1, q_1, r, s)\{A(p_1) \& A(q_1) \& A(r) \& A(s) \& D(p_1, r, m) \& D(q_1, s, m) \& (p_1 \neq p \vee q_1 \neq q) \supset (b)[W(p_1, q_1, f, b) \equiv W(p_1, q_1, f', b)]\},$$

and by $\mathfrak{B}_{12}$, $\mathfrak{B}_{13}$ the formulas

$\mathfrak{B}_{12}$: $(p, q, m)(A(p) \& A(q) \& A(m) \supset (f, a)\{U(f) \& R(p, a) \supset (Ef')[U(f') \& \mathfrak{H}(p, q, m, f', f, a)]\})$,

$\mathfrak{B}_{13}$: $(p, q, m, n)\{A(p) \& A(q) \& A(m) \& A(n) \supset [\mathfrak{G}^{(3)}(A, \ldots, E, m, p, q, n) \supset (f)(U(f) \supset \{Y(n, f) \equiv (Ea, f')[U(f') \& \mathfrak{H}(p, q, m, f', f, a) \& Y(m, f')]\})]\}$.

The formulas $\mathfrak{G}^{(1)}$, $\mathfrak{G}^{(2)}$, $\mathfrak{G}^{(3)}$ are formulas $\mathfrak{G}_f$ of 2.2 corresponding to the recursive functions $\ulcorner V_l^k V_m^j \urcorner$, $\ulcorner |\breve{k}\breve{l}| \urcorner$, $\ulcorner E V_q^p \breve{m} \urcorner$.

We shall now prove some theorems in which we discuss relations between S-systems and arbitrary models of $\mathfrak{B}$: $\mathfrak{B}_1 \& \ldots \& \mathfrak{B}_{13}$ (7).

Let $A_0, \ldots, E_0$ be the standard model of $\mathfrak{F}_0$, $\mathcal{M} = \langle \boldsymbol{R}, \boldsymbol{S} \rangle$ a model of T_ω and U^0, $W_{k,l}^0$, Y_n^0 the standard S-system for $\mathcal{M}$. Define relations R^*, S^*, U^*, W^*, Y^* as follows:

$$R^*(x, y) \equiv x \varepsilon A_0 \& y \varepsilon R_x,$$
$$S^*(x, y, z) \equiv x \varepsilon A_0 \& y S_x z,$$
$$U^*(f) \equiv f \varepsilon U^0,$$
$$W^*(x, y, z, t) \equiv x \varepsilon A_0 \& y \varepsilon A_0 \& W_{x,y}^0(z, t),$$
$$Y^*(x, y) \equiv x \varepsilon A_0 \& y \varepsilon Y_x^0.$$

We say that the system $A_0, \ldots, E_0, R^*, \ldots, Y^*$ is *determined* by $\mathcal{M}$. The field of these relations we denote by I^*.

4.3. *$I^* \supset R_0$ and the set $I^* - R_0$ is infinite.*

Proof. Sets R_j are disjoint (by 3) and non void by 1 and 5. The field of R^* contains all these sets.

4.4. $\models_{I^*} \mathfrak{B}[A_0, \ldots, E_0, R^*, S^*, U^*, W^*, Y^*]$.

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(7) The method consisting in expressing the semantics of a formal system in the first order calculus and considering models of the axiomatic system thus obtained was invented and first used by Trahtenbrot [11].

Proof. $\models_I \mathfrak{B}_i[A_0, \ldots, E_0, R^*, \ldots, Y^*]$ follows from the fact that the standard S-system for $\mathcal{M}$ satisfies the i-th of the conditions 1-13. We will verify this, say, for $\mathfrak{B}_{13}$. Assume that m, n, p, q are integers and that $\models_I \mathfrak{G}^{(3)}[A_0, \ldots, E_0, m, p, q, n]$. Since $\models_I \mathfrak{F}_0[A_0, \ldots, E_0]$ we obtain by 2.2 $n = \ulcorner \mathsf{E}V_q^p \breve{m} \urcorner$. Assume further that $f \varepsilon U^0$, i.e. that f is a mapping of variables into $R_0 \cup R_1 \cup \ldots$ such that $f(V_l^k) \varepsilon R_k$ for arbitrary k, l. If $Y^*(n, f)$, then f satisfies the formula $\mathsf{E}V_q^p \breve{m}$ (by 4.2) and hence there are an a in R_p and a mapping f' such that $\Omega_{p,q,m}(f', f, a)$ and f' satisfies $\breve{m}$. Hence $f' \varepsilon U^0$, $Y^*(m, f')$. Finally we easily see that

$$(4.4.1) \qquad \models_I \mathfrak{H}[A_0, \ldots, E_0, R^*, \ldots, Y^*, p, q, m, f', f, a].$$

Conversely, if there are f' and a such that $f' \varepsilon U^0$, $Y^*(m, f')$ and (4.4.1), then $\Omega_{p,q,m}(f', f, a)$ and $f' \varepsilon Y_m$, whence $f \varepsilon Y_n$ and $Y^*(n, f)$, which proves that $\models_I \mathfrak{B}_{13}[A_0, \ldots, E_0, R^*, \ldots, Y^*]$.

The proof for the formulas $\mathfrak{B}_1$-$\mathfrak{B}_{12}$ is similar.

Theorem 4.4 shows that every model $\mathcal{M}$ determines a model for the formula $\mathfrak{B}$. The next theorem shows that, conversely, certain models of $\mathfrak{B}$ determine a model $\mathcal{M}$.

Let $A, \ldots, E, R, S, U, W, Y$ be relations with the same numbers of arguments as $A_0, \ldots, E_0, R^*, \ldots, Y^*$. Let I be the union of fields of these relations. Define sets and relations $R_k, S_k, W_{k,l}, Y_n$ as follows

$$R_k = \{x \colon (\mathsf{E}y)[y \varepsilon A \,\&\, \models_I \mathfrak{Z}_k[A, B, C, y] \,\&\, R(y, x)]\},$$

$$S_k = \{(x, y) \colon (\mathsf{E}z)[z \varepsilon A \,\&\, \models_I \mathfrak{Z}_k[A, B, C, z] \,\&\, S(z, x, y)]\},$$

$$(4.4.2) \qquad W_{k,l} = \{(x, y) \colon (\mathsf{E}z, t)[z \varepsilon A \,\&\, t \varepsilon A \,\&\, \models_I \mathfrak{Z}_k[A, B, C, z] \,\&\, \\ \&\, \models_I \mathfrak{Z}_l[A, B, C, t] \,\&\, W(z, t, x, y)]\},$$

$$Y_n = \{x \colon (\mathsf{E}y)[y \varepsilon A \,\&\, \models_I \mathfrak{Z}_n[A, B, C, y] \,\&\, Y(y, x)]\}.$$

4.5. *If* $\models_I \mathfrak{F}_0[A, \ldots, E]$ *and* $\models_I \mathfrak{B}[A, \ldots, E, R, S, U, W, Y]$, *then* $\mathcal{M} = (R_0, R_1, \ldots, S_0, S_1, \ldots)$ *is a model of* T_ω *and* U, $W_{k,l}$, Y_n *form an S-system for* $\mathcal{M}$.

Proof. Condition 1: $\models_I \mathfrak{B}_1$ can be written as $(j)\{\mathfrak{Z}_0[j] \supset (\mathsf{E}x) R(j, x)\}$. Since there is a j such that $\models_I \mathfrak{Z}_0[j]$ and $j \varepsilon A$, we infer from $\models_I \mathfrak{B}_1$ that I contains an x such that $R(j, x)$. This proves $x \varepsilon R_0$, i.e. condition 1.

Condition 2. Assume that j is an integer, $k = j+1$, $x S_j y$. Hence there is a z in A such that $\models_I \mathfrak{Z}_j[A, B, C, z]$ and $S(z, x, y)$. Since $\mathfrak{Z}_j(\mathrm{z}) \Rightarrow \mathrm{A}(\mathrm{z})$, we obtain $z \varepsilon A$. Let t be such that $C(z, t)$; the existence of t is ensured by the assumption $\models_I \mathfrak{F}_0[A, E]$. The same assumption yields $t \varepsilon A$; from the definition of $\mathfrak{Z}_{j+1}$ we obtain $\models_I \mathfrak{Z}_{j+1}[A, B, C, t]$. From $\models_I \mathfrak{B}_2[A, \ldots, E, R, S, U, W, Y]$ we further obtain $R(z, x)$ and $R(t, y)$, i.e. $x \varepsilon R_j$ and $y \varepsilon R_{j+1}$.

Conditions 3-9 are proved similarly.

We shall also discuss proofs of conditions 10 and 13 because of their slightly more complicated form (conditions 11 and 12 can be treated similarly).

Condition 10. Let j, k, l, m, n be integers such that $k = j+1$ and $n = \ulcorner V_l^k V_m^j \urcorner$. Since $\models_A \mathfrak{F}_0[A, \ldots, E]$, there are elements s, t, u, v, w of A such that

$$\models_A \mathfrak{Z}_j[s], \quad \models_A \mathfrak{Z}_k[t], \quad \models_A \mathfrak{Z}_l[u], \quad \models_A \mathfrak{Z}_m[v], \quad \models_A \mathfrak{Z}_n[w].$$

Since all the quantifiers in $\mathfrak{F}_0$ and in $\mathfrak{Z}_i$ $(i = 0, 1, \ldots)$ are relativized to A, we can replace here $\models_A$ by $\models_I$. By 2.2

$$\models_I \mathfrak{G}^{(1)}[A, \ldots, E, s, t, u, v, w] \equiv n = \ulcorner V_l^k V_m^j \urcorner$$

and obviously $C(t, s)$. Hence we obtain (by 1.2)

$$\models_I \mathfrak{G}^{(1)}[A, \ldots, E, s, t, u, v, w],$$

and since $\models_I \mathfrak{B}_{10}$, we infer that the formula

$$(\mathrm{f})[\mathrm{U}(\mathrm{f}) \supset \{\mathrm{Y}(\mathrm{w}, \mathrm{f}) \equiv (\mathrm{a}, \mathrm{b})[\mathrm{W}(\mathrm{s}, \mathrm{v}, \mathrm{f}, \mathrm{a}) \,\&\, \mathrm{W}(\mathrm{t}, \mathrm{u}, \mathrm{f}, \mathrm{b}) \supset \mathrm{S}(\mathrm{s}, \mathrm{a}, \mathrm{b})]\}]$$

is satisfied in I if u, v, w, s, t are interpreted as u, v, w, s, t. From this we infer that for an arbitrary f in U

$$f \varepsilon Y_n \equiv (a, b)_I [W_{j,m}(f, a) \,\&\, W_{k,l}(f, b) \supset a S_j b].$$

This proves that condition 10 is satisfied.

Condition 13. Let p, q, m, n be integers such that $n = \ulcorner \mathsf{E} V_q^p \breve{m} \urcorner$. As above, we determine elements s, t, u, v of A such that

$$\models_A \mathfrak{Z}_p[s], \quad \models_A \mathfrak{Z}_q[t]. \quad \models_A \mathfrak{Z}_m[u], \quad \models_A \mathfrak{Z}_n[v]$$

and show that the formula

(4.5.1) $\quad (\mathrm{f})\{\mathrm{U}(\mathrm{f}) \supset [\mathrm{Y}(\mathrm{v}, \mathrm{f}) \equiv (\mathsf{E}\mathrm{a}, \mathrm{f}')[\mathrm{U}(\mathrm{f}') \,\&\, \mathfrak{H}(\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{f}', \mathrm{f}, \mathrm{a}) \,\&\, \mathrm{Y}(\mathrm{u}, \mathrm{f}')]]\}$

is satisfied in I if u, v, s, t are interpreted as u, v, s, t.

Now let f be an element of U. We have to show that

(4.5.2) $\quad f \varepsilon Y_n \equiv (\mathsf{E}a, f')[f' \varepsilon U \,\&\, \Omega_{p,q,m}(f', f, a) \,\&\, f' \varepsilon Y_m].$

Assume that $f \varepsilon Y_n$. Since (4.5.1) is true in I, there are a, f' in I such that $f' \varepsilon U$, $f' \varepsilon Y_m$ and $\models_I \mathfrak{H}[s, t, u, f', f, a]$. The last condition implies that $W_{p,q}(f', a)$ and that

$$(b)[W(s_1, t_1, f, b) \equiv W(s_1, t_1, f', b)]$$

for arbitrary elements s_1, t_1 of A satisfying (for certain r, w in A) $D(s_1, r, u)$, $D(t_1, w, u)$ and $s_1 \neq s$ or $t_1 \neq t$. Now we notice that

$$\mathfrak{F}_0 \,\&\, \mathfrak{Z}_m[u] \,\&\, (\mathfrak{Z}_0[s_1] \vee \mathfrak{Z}_1[s_1] \vee \ldots \vee \mathfrak{Z}_m[s_1]) \Rightarrow (\mathrm{E}r)\, D(s_1, r, u)$$

([10], p. 54). It follows that for arbitrary integers $p_1, q_1 \leqslant m$ such that $p_1 \neq p$ or $q_1 \neq q$ we have

$$(b)[W_{p_1,q_1}(f, b) \equiv W_{p_1,q_1}(f', b)],$$

i.e. that $\Omega_{p,q,m}(f', f, a)$.

Conversely if f', a are such that $f' \varepsilon U$, $f' \varepsilon Y_m$ and $\Omega_{p,q,m}(f', f, a)$, then we show, as above, that $\models_I \mathfrak{H}[s, t, u, f', f, a]$ and hence, by $\models_I \mathfrak{B}_{13}$, that $f \varepsilon Y_n$. Equivalence (4.5.2) is thus proved.

Remark. Theorem 4.5 represents the crucial step towards our final result. The circumstance which makes possible the proof of this theorem is the fact that conditions 1-13 have a form of general statements concerning integers and that in the formulas following the initial general quantifiers either there are no bound number variables (the case of conditions 1-11) or at most such number variables as are bound by a quantifier with limited scope (the cases of conditions 12 and 13, in which there are bound number variables in the formulas $\Omega_{p,q,m}(f', f, a)$).

Definition. The model $\mathcal{M}$ and the S-system U, $W_{k,l}$, Y_n defined in (4.4.2) are said to be determined by relations $A, \ldots, E, R, S, U, W, Y$.

5. Formulas $\mathfrak{M}_Z$ and $\mathfrak{N}_\Phi$

Let Z be a recursively enumerable set of (Gödel numbers of) closed formulas of T_ω and let $\mathfrak{M}_Z(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{Y})$ be the formula

$$(\mathrm{x}, \mathrm{y}, \mathrm{f})[\mathrm{A}(\mathrm{x}) \,\&\, \mathrm{A}(\mathrm{y}) \,\&\, \mathrm{U}(\mathrm{f}) \,\&\, \mathfrak{X}_Z(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{y}, \mathrm{x}) \supset \mathrm{Y}(\mathrm{x}, \mathrm{f})].$$

5.1. *If $\mathcal{M}$ is a model of T_ω and $A_0, \ldots, E_0, R^*, S^*, U^*, W^*, Y^*$ is the system determined by $\mathcal{M}$, then $\models_{I^*} \mathfrak{M}_Z[A_0, \ldots, E_0, R^*, \ldots, Y^*]$ if and only if $\models_{\mathcal{M}} \Theta$ for every Θ in Z.*

Proof. Assume $\models_{I^*} \mathfrak{M}_Z$ and let $\Theta \varepsilon Z$. It follows by 2.3 that there is an integer l such that

$$\mathfrak{F}_0 \,\&\, \mathfrak{Z}_{\ulcorner\Theta\urcorner}(\mathrm{x}) \,\&\, \mathfrak{Z}_l(\mathrm{y}) \Rightarrow \mathfrak{X}_Z(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{y}, \mathrm{x}).$$

Since $\models_{I^*} \mathfrak{Z}_{\ulcorner\Theta\urcorner}[\ulcorner\Theta\urcorner]$ and $\models_{I^*} \mathfrak{Z}_l[l]$, we obtain from $\models_{I^*} \mathfrak{M}_Z$

$$f \varepsilon U^* \supset Y^*(\ulcorner\Theta\urcorner, f),$$

i.e. $f \varepsilon Y^*_{\ulcorner\Theta\urcorner}$. Since U^*, $W^*_{k,l}$, Y^*_n form an S-system, we find by 4.2 that $\models_{\mathcal{M}} \Theta$.

Assume now that $\models_{\mathcal{M}} \Theta$ for every Θ in Z, that x, y are elements of I^* such that $x, y \,\varepsilon\, A_0$ and $\models_{I^*} \mathfrak{X}_Z[A_0, \ldots, E_0, y, x]$ and that $f \,\varepsilon\, U^*$. Since $\models_{I^*} \mathfrak{F}_0[A_0, \ldots, E_0]$ we infer by 2.3 that $x \,\varepsilon\, Z$ and hence $f \,\varepsilon\, Y_x$, i.e. $Y^*(x, f)$. This proves that $\models_{I^*} \mathfrak{M}_Z[A_0, \ldots, E_0, R^*, S^*, U^*, W^*, Y^*]$.

5.2. *If $A, \ldots, E, R, S, U, W, Y$ are relations whose fields are contained in I such that $\models_I \mathfrak{F}_0[A, \ldots, E]$, $\models_I \mathfrak{B}[A, \ldots, E, R, \ldots, Y]$ and $\models_I \mathfrak{M}_Z[A, \ldots, E, R, \ldots, Y]$ and if $\mathcal{M}$ is a model determined by these relations, then $\models_{\mathcal{M}} \Theta$ for every Θ in Z.*

Proof. Let $\Theta \,\varepsilon\, Z$; determine l such that

$$\mathfrak{F}_0 \,\&\, \mathfrak{Z}_{\ulcorner\Theta\urcorner}(\mathrm{x}) \,\&\, \mathfrak{Z}_l(\mathrm{y}) \Rightarrow \mathfrak{X}_Z(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{y}, \mathrm{x}).$$

Since A contains elements x, y such that $\models_I \mathfrak{Z}_{\ulcorner\Theta\urcorner}[x]$ and $\models_I \mathfrak{Z}_l[y]$ and since $\models_I \mathfrak{F}_0[A, \ldots, E]$, we infer that $\models_I \mathfrak{X}_Z[A, \ldots, E, y, x]$. Since $\models_I \mathfrak{M}_Z[A, \ldots, Y]$ we infer that if $f \varepsilon U$, then $Y(x, f)$. We thus obtain $x \,\varepsilon\, A \,\&\, \models_I \mathfrak{Z}_{\ulcorner\Theta\urcorner}[x] \,\&\, Y(x, f)$, which proves (cf. definitions (4.4.2)) that $f \,\varepsilon\, Y_{\ulcorner\Theta\urcorner}$ and hence by 4.2 $\models_{\mathcal{M}} \Theta$.

Remark. If Z consists of but one formula Θ, then instead of $\mathfrak{M}_Z$ we can use the formula $\mathfrak{M}_\Theta$: $(\mathrm{x}, \mathrm{f})[\mathfrak{Z}_{\ulcorner\Theta\urcorner}(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{x}) \supset \mathrm{Y}(\mathrm{x}, \mathrm{f})]$.

Let Φ be a formula of T_ω with the single free variable V_1^5. We shall define two formulas $\mathfrak{N}_\Phi$ and $\overline{\mathfrak{N}}_\Phi$ with the same predicate variables $\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \mathrm{S}, \mathrm{U}, \mathrm{W}, \mathrm{Y}, \mathrm{M}, \mathrm{N}$. The intuitive contents of the formula $\mathfrak{N}_\Phi$ is that the model $\mathcal{M}$ determined by $A, \ldots, E, R, \ldots, Y$ contains the pair (M, N) and that the element representing N in M satisfies Φ in $\mathcal{M}$. The intuitive contents of $\overline{\mathfrak{N}}_\Phi$ is similar: it says that if $\mathcal{M}$ contains the pair (M, N), then the element representing N in $\mathcal{M}$ satisfies Φ in $\mathcal{M}$.

Put

$$\mathfrak{N}'\colon (\mathrm{Ex})\{\mathfrak{Z}_0(\mathrm{x}) \,\&\, (\mathrm{y})[\mathrm{M}(\mathrm{y}) \equiv \mathrm{R}(\mathrm{x}, \mathrm{y})] \,\&\, (\mathrm{y}, \mathrm{z}, \mathrm{t})[\mathrm{N}(\mathrm{y}, \mathrm{z}, \mathrm{t}) \supset \\ \supset \mathrm{M}(\mathrm{y}) \,\&\, \mathrm{M}(\mathrm{z}) \,\&\, \mathrm{M}(\mathrm{t})]\}.$$

$$\mathfrak{K}\colon \mathfrak{Z}_0(\mathrm{x}_0) \,\&\, \mathfrak{Z}_1(\mathrm{x}_1) \,\&\ldots\&\, \mathfrak{Z}_5(\mathrm{x}_5) \,\&\, \mathfrak{Z}_{\ulcorner\Phi\urcorner}(\mathrm{y}) \,\& \\ \&\, \mathfrak{Z}_{\ulcorner \mathrm{EV}_1^0 \mathrm{EV}_2^0 \mathrm{EV}_3^0 \mathrm{B}_3(\mathrm{V}_1^4;\, \mathrm{V}_1^0, \mathrm{V}_2^0, \mathrm{V}_3^0)\urcorner}(\mathrm{z}) \,\&\, \mathfrak{Z}_{\ulcorner \mathrm{B}_3(\mathrm{V}_1^4;\, \mathrm{V}_1^0, \mathrm{V}_2^0, \mathrm{V}_3^0)\urcorner}(\mathrm{t}),$$

$$\mathfrak{L}\colon (\mathrm{s})\{\mathrm{S}(\mathrm{x}_4, \mathrm{s}, \mathrm{n}) \supset (\mathrm{Ef})[\mathrm{U}(\mathrm{f}) \,\&\, \mathrm{W}(\mathrm{x}_4, \mathrm{x}_1, \mathrm{f}, \mathrm{s}) \,\&\, \mathrm{Y}(\mathrm{z}, \mathrm{f})]\},$$

$$\mathfrak{P}\colon (\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3)\big(\mathrm{R}(\mathrm{x}_0, \mathrm{t}_1) \,\&\, \mathrm{R}(\mathrm{x}_0, \mathrm{t}_2) \,\&\, \mathrm{R}(\mathrm{x}_0, \mathrm{t}_3) \supset \{\mathrm{N}(\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3) \equiv \\ (\mathrm{Eu}, \mathrm{f})[\mathrm{S}(\mathrm{x}_4, \mathrm{u}, \mathrm{n}) \,\&\, \mathrm{U}(\mathrm{f}) \,\&\, \mathrm{W}(\mathrm{x}_0, \mathrm{x}_1, \mathrm{f}, \mathrm{t}_1) \,\&\, \mathrm{W}(\mathrm{x}_0, \mathrm{x}_2, \mathrm{f}, \mathrm{t}_2) \,\& \\ \mathrm{W}(\mathrm{x}_0, \mathrm{x}_3, \mathrm{f}, \mathrm{t}_3) \,\&\, \mathrm{W}(\mathrm{x}_4, \mathrm{x}_1, \mathrm{f}, \mathrm{u}) \,\&\, \mathrm{Y}(\mathrm{t}, \mathrm{f})]\}\big),$$

$$\mathfrak{Q}\colon (\mathrm{Ef})[\mathrm{U}(\mathrm{f}) \,\&\, \mathrm{W}(\mathrm{x}_5, \mathrm{x}_1, \mathrm{f}, \mathrm{n}) \,\&\, \mathrm{Y}(\mathrm{y}, \mathrm{f})],$$

$$\mathfrak{N}_\Phi\colon \mathfrak{N}' \,\&\, (\mathrm{Ex}_0, \mathrm{x}_1, \ldots, \mathrm{x}_5, \mathrm{y}, \mathrm{z}, \mathrm{t}, \mathrm{n})[\mathfrak{K} \,\&\, \mathrm{R}(\mathrm{x}_5, \mathrm{n}) \,\&\, \mathfrak{L} \,\&\, \mathfrak{P} \,\&\, \mathfrak{Q}],$$

$$\overline{\mathfrak{N}}_\Phi\colon \mathfrak{N}' \,\&\, (\mathrm{x}_0, \ldots, \mathrm{x}_5, \mathrm{y}, \mathrm{z}, \mathrm{t}, \mathrm{n})[\mathfrak{K} \,\&\, \mathrm{R}(\mathrm{x}_5, \mathrm{n}) \,\&\, \mathfrak{L} \,\&\, \mathfrak{P} \supset \mathfrak{Q}].$$

5.3. *Let $\mathscr{M}$ be a model, $A_0, \ldots, E_0, R^*, \ldots, Y^*$ the system determined by $\mathscr{M}$ and (M, N) a pair (consisting of a set and of a ternary relation in $\mathscr{B}(M)$) such that $M \subseteq I^*$. The following conditions are then equivalent:*

(5.3.1) *(M, N) is contained in $\mathscr{M}$ and the element n which represents N in $\mathscr{M}$ satisfies $\models_{\mathscr{M}} \Phi[n]$;*

(5.3.2) $$\models_{I^*} \mathfrak{N}_\Phi[A_0, \ldots, E_0, R^*, \ldots, Y^*, M, N].$$

Proof. Since $M \subseteq I^*$, it easily follows from the definition that $\models_{I^*} \mathfrak{N}'$ is equivalent to $M = R_0$ and $N \subseteq R_0^3$. Obviously A_0 contains elements $x_0, \ldots, x_5, y, z, t$ satisfying $\mathfrak{K}$. These elements are simply the integers $0, \ldots, 5$, $\ulcorner\Phi\urcorner$, $\ulcorner \mathsf{E}V_1^0\mathsf{E}V_2^0\mathsf{E}V_3^0 B_3(V_1^4; V_1^0, V_2^0, V_3^0)\urcorner$, $\ulcorner B_3(V_1^4; V_1^0, V_2^0, V_3^0)\urcorner$. Taking these integers as interpretations of $\mathrm{x}_0, \ldots, \mathrm{x}_5, \mathrm{y}, \ldots, \mathrm{t}$ we see that $\models_{I^*} \mathfrak{L}$ is equivalent to the statement that if sS_4n (in the model $\mathscr{M}$), then s satisfies the formula $\mathsf{E}V_1^0\mathsf{E}V_2^0\mathsf{E}V_3^0 B_3(V_1^4; V_1^0, V_2^0, V_3^0)$ in $\mathscr{M}$. Similarly, $\models_{I^*} \mathfrak{P}$ is equivalent to the statement that for arbitrary t_1, t_2, t_3 in R_0 the relation $N(t_1, t_2, t_3)$ holds if and only if there is an element u of R_4 such that $\models_{\mathscr{M}} B_3[u; t_1, t_2, t_3]$ and uS_4n. Thus $\models_{I^*} \mathfrak{N}' \,\&\, \mathfrak{K} \,\&\, \mathfrak{L} \,\&\, \mathfrak{P} \,\&\, \mathrm{R}(x_5, n)$ states that the pair (M, N) is contained in $\mathscr{M}$ and that n represents N in $\mathscr{M}$. Finally, $\models_{I^*} \mathfrak{Q}[n]$ is equivalent to $\models_{I^*} \Phi[n]$ by 4.2. It follows that (5.3.2) is equivalent to (5.3.1).

5.4. *Let $\mathscr{M}, A_0, \ldots, E_0, R^*, \ldots, Y^*, M, N$ be as in 5.3 and assume that (M, N) is contained in $\mathscr{M}$ and that n represents N in $\mathscr{M}$. Then conditions*

$$\models_{\mathscr{M}} \Phi[n], \qquad \models_{I^*} \overline{\mathfrak{N}}_\Phi[A_0, \ldots, E_0, R^*, \ldots, Y^*, M, N]$$

are equivalent.

Proof is similar to that of 5.3.

In the remaining lemmas of this section we assume that I is a set, $A, \ldots, E, R, \ldots, Y, M, N$ relations in $\mathscr{B}(I)$ such that $\models_I \mathfrak{F}_0[A, \ldots, E]$ and $\models_I \mathfrak{B}[A, \ldots, E, R, \ldots, Y]$. $\mathscr{M}$ denotes the model determined by $(A, \ldots, E, R, \ldots, Y)$; as usual, $\mathscr{M} = (R_0, R_1, \ldots; S_0, S_1, \ldots)$. $U, W_{k,l}, Y_n$ denotes the S-system for $\mathscr{M}$ which is determined by $(A, \ldots, E, R, \ldots, Y)$ according to (4.4.2).

5.5. *$\models_I \mathfrak{N}'[A, B, R, M, N]$ is equivalent to $(M = R_0) \& (N \subseteq M^3)$.*

Proof: obvious.

5.6. *If $\models_I \mathfrak{K}[A, B, C, x_0, \ldots, x_5, y, z, t]$ and $R(x_5, n)$, then $n \in R_5$ and $\models_I \mathfrak{N}' \,\&\, \mathfrak{L} \,\&\, \mathfrak{P}[A, B, R, S, U, W, Y, M, N, x_0, \ldots, x_4, z, t, n]$ is equivalent to the following condition: the pair (M, N) is contained in $\mathscr{M}$ and n represents N in M.*

Proof. From $=_I \mathfrak{L}$ it follows that if $sS_4 n$, then there is an f in U such that $W_{4,1}(f, s)$ and $f \varepsilon Y_{\ulcorner \mathrm{EV}_1^0 \mathrm{EV}_2^0 \mathrm{EV}_3^0 \mathrm{B}_3(\mathrm{V}_1^4; \mathrm{V}_1^0, \mathrm{V}_2^0, \mathrm{V}_3^0) \urcorner}$. Hence s satisfies in $\mathscr{M}$ the formula $\mathsf{E}\mathrm{V}_1^0 \mathsf{E}\mathrm{V}_2^0 \mathsf{E}\mathrm{V}_3^0 \mathrm{B}_3(\mathrm{V}_1^4; \mathrm{V}_1^0, \mathrm{V}_2^0, \mathrm{V}_3^0)$, i.e. there are $t_1, t_2, t_3 \varepsilon R_0$ such that $\models_{\mathscr{M}} \mathrm{B}_3[s; t_1, t_2, t_3]$. From $\models_I \mathfrak{P}$ it follows that if $t_1, t_2, t_3 \varepsilon R_0$ then $N(t_1, t_2, t_3)$ is equivalent to the existence of an element u such that $uS_4 n$ and $\models_{\mathscr{M}} \mathrm{B}_3[u; t_1, t_2, t_3]$. Together with 5.6 this proves that the pair (M, N) is contained in $\mathscr{M}$ and that n represents N in $\mathscr{M}$.

Conversely, if (M, N) is contained in $\mathscr{M}$ and n represents N in $\mathscr{M}$, then by 5.5 we obtain $\models_I \mathfrak{N}'$. If $sS_4 n$, then s satisfies in $\mathscr{M}$ the formula $\mathsf{E}\mathrm{V}_1^0 \mathsf{E}\mathrm{V}_2^0 \mathsf{E}\mathrm{V}_3^0 \mathrm{B}_3(\mathrm{V}_1^4; \mathrm{V}_1^0, \mathrm{V}_2^0, \mathrm{V}_3^0)$, whence we obtain $\models_I \mathfrak{L}$. Finally, $N(t_1, t_2, t_3)$ is equivalent to the existence of an element u in R_4 such that $uS_4 n$ and $=_{\mathscr{M}} \mathrm{B}_3[u; t_1, t_2, t_3]$, which yields $\models_I \mathfrak{P}$.

5.7. $\models_I \mathfrak{N}_\Phi[A, \ldots, E, R, \ldots, M, N]$ *is equivalent to the following condition*: (M, N) *is contained in* $\mathscr{M}$ *and the element* n *which represents* N *in* $\mathscr{M}$ *satisfies* $\models_{\mathscr{M}} \Phi[n]$.

Proof. Obviously there are $x_0, \ldots, x_5, y, z, t$ which satisfy $\models_I \mathfrak{R}[A, B, C, x_0, \ldots, x_5, y, z, t]$. Hence, by 5.6, if $\models_I \mathfrak{N}_\Phi$, then (M, N) is contained in $\mathscr{M}$; if n represents N in $\mathscr{M}$, then $\models_I \mathfrak{Q}[U, W, Y, x_1, x_5, y, n]$, whence by 4.2 $\models_{\mathscr{M}} \Phi[n]$. Conversely, from $\models_{\mathscr{M}} \Phi[n]$ it follows that $=_I \mathfrak{Q}[U, W, Y, x_1, x_5, y, n]$. Using 5.6 we obtain $\models_I \mathfrak{N}_\Phi$.

5.8. *If* $\mathscr{M}$ *contains the pair* (M, N) *and* n *represents* N *in* $\mathscr{M}$, *then* $\models_I \overline{\mathfrak{N}}_\Phi[A, \ldots, E, R, \ldots, Y, M, N]$ *is equivalent to* $\models_{\mathscr{M}} \Phi[n]$.

Proof. If $\models_I \overline{\mathfrak{N}}_\Phi$, then the n which represents N in $\mathscr{M}$ satisfies $\models_I \mathfrak{Q}[U, W, Y, x_1, x_5, y, n]$ (where the meaning of x_1, x_5, y is as in 5.7) and hence $\models_{\mathscr{M}} \Phi[n]$. Conversely, assume that $n \varepsilon R_5$ and $\models_{\mathscr{M}} \Phi[n]$; obviously we have $\models_I \mathfrak{N}'$. If $x_0, \ldots, x_5, y, z, t$ satisfy $\mathfrak{R}$, and if $n' \varepsilon R_5$ and $\models_I \mathfrak{L} \& \mathfrak{P}[A, \ldots, E, R, \ldots, Y, M, N, x_1, \ldots, x_5, z, t, n']$, then n' represents N in $\mathscr{M}$, whence $n' = n$ and consequently $\models_{\mathscr{M}} \Phi[n']$, which yields $\models_I \mathfrak{Q}[U, W, Y, x_1, x_5, y, n']$. This proves that $\models_I \overline{\mathfrak{N}}_\Phi$.

6. A characterization of conditions expressed by invariant, dual invariant and absolute formulas

Let Z be a recursively enumerable set of closed formulas of T_ω such that every Θ in Z is valid in every model of the form $St(X)$ (i.e. in every standard model). Let $\mathscr{K}$ be the family of models $\mathscr{M}$ such that $\models_{\mathscr{M}} \Theta$ for every Θ in Z.

6.1. *If a formula* Φ *with one free variable* V_1^5 *is invariant with respect to* $\mathscr{K}$, *then there is a first order formula* $\mathfrak{F}(\mathrm{M}, \mathrm{N}, \mathrm{Q}_1, \ldots, \mathrm{Q}_k)$ *such that* *for an arbitrary pair*[8] (M, N) *the following conditions are equivalent*:

(6.1.1) $\models_{St(M)} \Phi[N]$,

(6.1.2) *there is a set I and relations $Q_1, \ldots, Q_k \varepsilon \mathscr{B}(I)$ such that $I \supseteq M$, $I-M$ is infinite and $\models_I \mathfrak{F}[M, N, Q_1, \ldots, Q_k]$.*

Proof. Take as $\mathfrak{F}$ the formula

$$\mathfrak{F}_0(\mathrm{A}, \ldots, \mathrm{E}) \,\&\, \mathfrak{B}(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \mathrm{S}, \mathrm{U}, \mathrm{W}, \mathrm{Y}) \,\&\, \mathfrak{M}_Z(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}) \,\&\, \mathfrak{N}_\Phi(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}, \mathrm{M}, \mathrm{N}).$$

Thus $k = 10$ and the $\mathrm{Q}_1, \ldots, \mathrm{Q}_k$ are $\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}$.

Assume (6.1.1). Consider the model $\mathscr{M} = St(M)$ and the system $A_0, \ldots, E_0, R^*, \ldots, Y^*$ determined by $\mathscr{M}$. Let I^* be, as usual, the union of fields of these relations. By 4.4 $\models_{I^*} \mathfrak{B}[A_0, \ldots, E_0, R^*, \ldots, Y^*]$ and by 4.3 $I^* \supseteq R_0 - M$ and the difference $I^* - M$ is infinite. Obviously $\models_{I^*} \mathfrak{F}_0[A_0, \ldots, E_0]$. By 5.1 and the assumption that every Θ in Z is valid in every standard model we obtain $\models_{I^*} \mathfrak{M}_Z[A_0, \ldots, E_0, R^*, \ldots, Y^*]$. Since N is the element of $\mathscr{M}$ which represents N in $\mathscr{M}$, we infer by 5.3 that $\models_{I^*} \mathfrak{N}_\Phi[A_0, \ldots, E_0, R^*, \ldots, Y^*, M, N]$. This proves the implication (6.1.1) $\rightarrow$ (6.1.2). Note that invariance of Φ has not been used in this proof.

Assume (6.1.2), i.e. assume that there are a set $I \supseteq M$ such that $I-M$ is infinite and relations $A, \ldots, E, R, \ldots, Y$ in $\mathscr{B}(I)$ such that

$\models_I \mathfrak{F}_0[A, \ldots, E]$, $\models_I \mathfrak{B}[A, \ldots, E, R, \ldots, Y]$, $\models_I \mathfrak{M}_Z[A, \ldots, E, R, \ldots, Y]$
and

$$\models_I \mathfrak{N}_\Phi[A, \ldots, E, R, \ldots, Y, M, N].$$

Let $\mathscr{M}$ be a model determined by $A, \ldots, E, R, \ldots, Y$. By 5.2 $\mathscr{M}$ belongs to $\mathscr{K}$ and according to 5.7 the pair (M, N) is contained in $\mathscr{M}$ and we have $\models_{\mathscr{M}} \Phi[n]$ for the element n which represents N in $\mathscr{M}$. Since Φ is invariant, we obtain (6.1.1).

6.2. *If a formula Φ with exactly one free variable V_1^5 is dual invariant with respect to $\mathscr{K}$, then there is a first order formula $\mathfrak{G}(\mathrm{M}, \mathrm{N}, \mathrm{S}_1, \ldots, \mathrm{S}_l)$ such that for every pair*[8] (M, N) *the following conditions are equivalent*:

(6.2.1) $\models_{St(M)} \Phi[N]$,

(6.2.2) *For arbitrary $K \supseteq M$ and arbitrary $S_1, \ldots, S_l$ in $\mathscr{B}(K)$ if $K-M$ is infinite, then $\models_K \mathfrak{G}[M, N, S_1, \ldots, S_l]$.*

Proof. Take as $\mathfrak{G}$ the formula

$$\mathfrak{F}_0(\mathrm{A}, \ldots, \mathrm{E}) \,\&\, \mathfrak{B}(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}) \,\&\, \mathfrak{M}_Z(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}) \,\&\, \mathfrak{N}'(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}, \mathrm{M}) \supset \mathfrak{N}_\Phi(\mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}, \mathrm{M}, \mathrm{N}).$$

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[8] A "pair" (M, N) always consists of a set and a ternary relation N whose field is contained in M.

Thus $l = 10$ and the $S_1, \ldots, S_l$ are $A, \ldots, E, R, \ldots, Y$. Assume (6.2.1) and let K be a set such that $K \supseteq M$ and $K - M$ is infinite. Choose arbitrary $A, \ldots, E, R, \ldots, Y$ in $\mathscr{B}(K)$. In order to prove that $\models_K \mathfrak{G}[M, N, A, \ldots, E, R, \ldots, Y]$ we assume that

(6.2.3) $\models_K \mathfrak{F}_0[A, \ldots, E], \quad \models_K \mathfrak{B}[A, \ldots, E, R, \ldots, Y],$

$$\models_K \mathfrak{M}_Z[A, \ldots, E, R, \ldots, Y], \quad \models_K \mathfrak{N}'[A, \ldots, E, R, \ldots, Y, M]$$

and deduce that

(6.2.4) $$\models_K \overline{\mathfrak{N}}_\Phi[A, \ldots, E, R, \ldots, Y, M, N].$$

Let us therefore assume that $x_0, \ldots, x_5, y, z, t, n$ are elements of K such that

(6.2.5) $\models_K \mathfrak{K}[A, B, C, x_0, \ldots, x_5, y, z, t], \quad R(x_5, n),$

$$\models_K \mathfrak{L} \& \mathfrak{P}[A, B, R, S, U, W, Y, M, N, x_0, \ldots, x_4, z, t, n].$$

We have to prove that $\models_K \mathfrak{Q}[U, W, Y, x_1, x_5, n]$. The assumptions (6.2.5) yield by 5.6 that (M, N) is contained in $\mathscr{M}$ and that n represents N in $\mathscr{M}$. By the dual invariance of Φ we infer that $\models_{\mathscr{M}} \Phi[n]$. This, however, is equivalent to $\models_K \mathfrak{Q}[U, W, Y, x_1, x_5, n]$. (6.2.4) is thus proved.

Now assume (6.2.2) and consider the model $\mathscr{M} = St(M)$. Take the system $A_0, \ldots, E_0, R^*, \ldots, Y^*$ determined by $\mathscr{M}$ and let I^* be the union of fields of these relations. By 4.3 $I^* \supseteq M$ and $I^* - M$ is infinite. From (6.2.2) we infer that $\models_{I^*} \mathfrak{G}[M, N, A_0, \ldots, E_0, R^*, \ldots, Y^*]$. Now we obviously have $\models_{I^*} \mathfrak{F}_0[A_0, \ldots, E_0]$; lemmas 4.4, 5.1 and 5.5 show that $\models_{I^*} \mathfrak{B}[A_0, \ldots, E_0, R^*, \ldots, Y^*]$, $\models_{I^*} \mathfrak{M}_Z[A_0, \ldots, E_0, R^*, \ldots, Y^*]$ and $\models_{I^*} \mathfrak{N}'[A_0, \ldots, E_0, R^*, \ldots, Y^*]$. According to the definition of $\overline{\mathfrak{N}}_\Phi$ we obtain $\models_{I^*} \overline{\mathfrak{N}}_\Phi[A_0, \ldots, E_0, R^*, \ldots, Y^*, M, N]$. Since the pair (M, N) is contained in $St(M) = \mathscr{M}$ and N is represented in $\mathscr{M}$, we obtain (6.2.1) by 5.8.

6.3. *If a formula Φ with exactly one free variable V_1^5 is absolute with respect to $\mathscr{K}$, then there is a first order formula $\mathfrak{E}(M, N)$ such that for an arbitrary pair (M, N) the conditions*

$$\models_{St(M)} \Phi[N] \quad \models_M \mathfrak{E}(M, N)$$

are equivalent.

Proof. Let $\mathfrak{F}$, $\mathfrak{G}$ be formulas whose existence is stated in 6.1, 6.2. We shall show that the formula (1.5.1) holds. Indeed, let U be a set, let J, K be subsets of U, and let $M \subseteq J$, $M \subseteq K$, $R, Q_1, \ldots, Q_k \varepsilon \mathscr{B}(J)$, $S, S_1, \ldots, S_l \varepsilon \mathscr{B}(K)$, $N \varepsilon \mathscr{B}(M)$. Let us assume that $\models_U \infty[J - M, R]$, $\models_U \infty[K - M, S]$ and $\models_U \mathfrak{F}_{\rm rel}[J, M, N, Q_1, \ldots, Q_k]$. It follows that $\models_J \mathfrak{F}[M, N, Q_1, \ldots, Q_k]$ and that $J - M$ is infinite, whence by 6.1

$\models_{S(M)} \Phi[N]$. By 6.2 we obtain $\models_K \mathfrak{G}[M, N, S_1, \ldots, S_l]$, since $K-M$ is infinite, whence $\models_U \mathfrak{G}_{\mathrm{rel}}[K, M, N, S_1, \ldots, S_l]$. Thus the assumptions of 1.5 are satisfied and we infer that there is a first order formula $\mathfrak{C}(M, N)$ such that (6.1.2) $\rightarrow \models_M \mathfrak{C}[M, N] \rightarrow$(6.2.2). Since (6.1.2) and (6.2.2) are both equivalent to (6.1.1), we infer that (6.1.1) is equivalent to $\models_M \mathfrak{C}[M, N]$.

7. The space of models

Put $\widehat{R}_j = \{2^j, 2^j\cdot 3, 2^j\cdot 5, \ldots\}$ for $j = 0, 1, 2, \ldots$ Our aim in this section is the proof of the following theorem:(9)

7.1. *There exists a set $\mathfrak{S}$ of models of T_ω such that*

(7.1.1) *Every model $\mathcal{M}$ in $\mathfrak{S}$ has the form*

$$(\widehat{R}_0, \widehat{R}_1, \ldots, S_0, S_1, \ldots);$$

(7.1.2) *If $\mathcal{M}$ is in $\mathfrak{S}$, then $2k+1\ S_0\, 2$ for every k;*

(7.1.3) *If $\mathcal{M}$ is in $\mathfrak{S}$, then 2^5 represents in $\mathcal{M}$ a ternary relation;*

(7.1.4) *A topology can be introduced in $\mathfrak{S}$ in such a way that $\mathfrak{S}$ becomes a compact Hausdorff space;*

(7.1.5) *For every formula Φ with the free variables $\mathrm{V}_{k_1}^{i_1}, \ldots, \mathrm{V}_{k_n}^{i_n}$ the set $\{\mathcal{M}\colon \models_{\mathcal{M}} \Phi[2^{i_1}(2k_1+1), \ldots, 2^{i_n}(2k_n+1)]\}$ is open and closed in $\mathfrak{S}$.*

(7.1.6) *If $\mathcal{M}' = (R_0', R_1', \ldots, S_0', S_1', \ldots)$ is a model of T_ω with denumerable R_j' $(j = 0, 1, 2, \ldots)$ and containing an element r_1 in R_1' such that $r_0 S_0' r_1$ for every r_0 in R_0' as well as an element r_5 in R_5' which represents in $\mathcal{M}'$ a ternary relation, then there is in $\mathfrak{S}$ a model $\mathcal{M}$ isomorphic with $\mathcal{M}'$ and such that r_1 corresponds to 2 and r_5 to 2^5.*

The proof of this theorem will be divided into several parts. First of all we shall construct an auxiliary system T_ω^∞ which can be said to result from T_ω by adjunction of symbols for arbitrary Skolem functions for formulas of T_ω.

The class of well-formed formulas of T_ω^∞ is the union $\bigcup_n K_n$, where K_n is defined by induction as follows. Let K_0 be the set of well-formed formulas of T_ω. Let the void set be the set of functors of K_0 and the set of all variables its set of terms. Now assume that $n \geqslant 0$ and that both K_n and the sets of its functors and terms are already defined.

We let a symbol $\mathfrak{f}_{\Phi,k,j}$ correspond to every formula Φ in $K_n - \bigcup_{j<n} K_j$ and to every V_j^k. The type of this symbol is $(p_1, \ldots, p_l; k)$, where $p_1, \ldots, p_l$ are the upper indices of the variables which are free in Φ and different from V_j^k. The set consisting of all functors of K_n

(9) This theorem is a reformulation of a joint result of A. Ehrenfeucht and the author; see [2]. A closely related result is contained in Beth [12], pp. 523-525. Beth's result is insofar weaker as he leaves undetermined the interpretation of $=$.

as well as of all symbols $\mathfrak{f}_{\Phi,k,j}$ is the set of functors of K_{n+1}. The terms of K_{n+1} are defined by induction: all terms of K_n are terms of K_{n+1} and their rank in K_{n+1} is 0. If $\mathfrak{f}$ is a functor of K_{n+1} of the type $(p_1, \ldots, p_l; k)$ and if $\tau_1, \ldots, \tau_l$ are terms of K_{n+1} whose ranks in K_{n+1} are $\leqslant r$ and whose types are $p_1, \ldots, p_l$, then $\mathfrak{f}\tau_1, \ldots \tau_l$ is a term of type k and of rank $\leqslant r$. Finally we define the class K_{n+1} itself: formulas in K_n are formulas of K_{n+1} and their rank in K_{n+1} is 0. If τ_1, τ_2 are terms of K_{n+1} whose types are $j+1$ and j, then $\tau_1\tau_2$ is a formula of K_{n+1} and its rank in K_{n+1} is 0. If Φ_1, Φ_2 are formulas of K_{n+1} whose ranks in K_{n+1} are $\leqslant r$, then $|\Phi_1\Phi_2$ and $\mathsf{E}\mathrm{V}_j^k\Phi_1$ are formulas of K_{n+1} whose ranks are $\leqslant r+1$. No other expression is a formula of K_{n+1}.

Skolem resolvents. Let Φ be a formula of T_ω^∞. We define a Skolem resolvent of Φ as follows. If Φ is an atomic formula, then Φ^{Sk} is Φ. If Φ is $|\Phi_1\Phi_2$, then Φ^{Sk} is $|\Phi_1^{\mathrm{Sk}}\Phi_2^{\mathrm{Sk}}$. If Φ is $\mathsf{E}\mathrm{V}_q^p\Phi_1$, then Φ^{Sk} is the formula $Sb\big(\mathrm{V}_q^p/\mathfrak{f}_{\Phi_1^{\mathrm{Sk}},p,q}(\mathrm{V}_{q_1}^{p_1}, \ldots, \mathrm{V}_{q_k}^{p_k})\Phi_1^{\mathrm{Sk}}\big)$[10] where $\mathrm{V}_{q_1}^{p_1}, \ldots, \mathrm{V}_{q_k}^{p_k}$ are all the free variables of Φ_1 which are different from V_q^p.

7.2. *Φ^{Sk} has the same free variables as Φ and does not have bound variables.*

Semi-frames. A semi-frame for T_ω is a sequence

$$(R_0, R_1, \ldots, S_0, S_1, \ldots)$$

where the R_j are sets and $S_j \subseteq R_j \times R_{j+1}$. A semi-frame for T_ω^∞ is a sequence

$$(R_0, R_1, \ldots, S_0, S_1, \ldots, T_{\mathfrak{f}}, \ldots)$$

where $\mathfrak{f}$ runs over functors of T_ω^∞ and R_j is a set, $S_j \subseteq R_j \times R_{j+1}$ for $j = 0, 1, 2, \ldots$ and where $T_{\mathfrak{f}}$ is a function from $R_{p_1} \times \ldots \times R_{p_k}$ to R_p for every functor $\mathfrak{f}$ of type $(p_1, \ldots, p_k; p)$.

The notions of satisfaction of a formula and of the value of a term in a semi-frame are assumed as known.

Let R_j^* be the set of terms of T_ω^∞ whose types are equal to j $(j = 0, 1, 2, \ldots)$. For a functor $\mathfrak{f}$ of type $(p_1, \ldots, p_k; p)$ we denote by $T_{\mathfrak{f}}^*$ the function from $R_{p_1}^* \times \ldots \times R_{p_k}^*$ to R_p defined thus: $T_{\mathfrak{f}}^*(\tau_1, \ldots, \tau_k) = \mathfrak{f}(\tau_1, \ldots, \tau_k)$.

The space $\mathfrak{S}^*$ of semi-frames. Let $\mathfrak{S}^*$ be the set of all semi-frames $(R_0^*, R_1^*, \ldots, S_0, S_1, \ldots, T_{\mathfrak{f}}^*, \ldots)$. We introduce a topology in $\mathfrak{S}^*$ by taking as an open basis sets $[\Phi]$ defined as follows: $\mathcal{M} \varepsilon [\Phi]$ if and only if Φ is satisfied in $\mathcal{M}$ by the assignment f_0 which correlates with each variable V_q^p the element V_q^p of R_p^*: we assume here that Φ is an open formula.

(10) $\mathrm{Sb}(\mathrm{V}_q^p/a)$ symbolizes the substitution of a for V_q^p.

7.3. *If $\mathcal{M} \varepsilon \mathfrak{S}^*$ and τ is a term of T_ω^∞, then the value of τ in $\mathcal{M}$ under the assignment f_0 is τ.*

Proof is by easy induction on the rank of τ.

7.4. *$\mathfrak{S}^*$ is a separable Hausdorff space and the neighbourhoods $[\Phi]$ are open and closed.*

The only fact which needs verification is the existence of disjoint neighbourhoods of two different semi-frames $\mathcal{M}'$ and $\mathcal{M}''$. If $\mathcal{M}' \neq \mathcal{M}''$, then there are j, τ_1, τ_2 such that either $\tau_1 S_j' \tau_2$ and τ_1 non-$S_j''\tau_2$ or τ_1 non-$S_j'\tau_2$ and $\tau_1 S_j''\tau_2$. Taking as Φ the formula $\tau_2\tau_1$, we obtain $\mathcal{M}'\varepsilon[\Phi]$ and $\mathcal{M}''\varepsilon \mathfrak{S}^* - [\Phi] = [\sim \Phi]$ or conversely.

7.5. *$\mathfrak{S}^*$ is compact.*

Proof. Let Φ_n be a sequence of open formulas of T_ω^∞ such that $\bigcap_{n\leqslant a} [\Phi_n] \neq 0$ for $a = 0, 1, 2, \ldots$ Let $\mathcal{F}$ be the filter of those closed and open subsets of $\mathfrak{S}^*$ which contain at least one of the sets $\bigcap_{n\leqslant a} [\Phi_n]$ and let $\mathcal{F}^*$ be an extension of $\mathcal{F}$ to a prime filter. Define relations S_j' by the equivalence

(7.5.1) $$\tau_1 S_j' \tau_2 \equiv [\tau_2\tau_1]\varepsilon \mathcal{F}^*$$

and let $\mathcal{M}'$ be the semi-frame $(R_0^*, R_1^*, \ldots, S_0', S_1', \ldots, T_\mathfrak{f}^*, \ldots)$. We prove by induction that for every open formula Ψ of T_ω^∞

(7.5.2) $$\mathcal{M}'\varepsilon[\Psi] \equiv [\Psi]\varepsilon \mathcal{F}^*.$$

By (7.5.1) this is true for atomic formulas. Indeed, if Ψ is the formula $\tau_2\tau_1$, then the left-hand side of (7.5.2) is equivalent to the following statement: the value of τ_1 in $\mathcal{M}'$ for the assignment f_0 bears the relation S_j' to the value of τ_2 for the same assignment. By 7.3 the values in question are τ_1 and τ_2 and so the left-hand side of (7.5.2) is equivalent to the left-hand side of (7.5.1), i.e. to the right-hand side of (7.5.2).

If (7.5.2) is true for the formulas Ψ_1 and Ψ_2, it is true for the formula $|\Psi_1\Psi_2$ because

$$\begin{aligned}\mathcal{M}'\varepsilon[|\Psi_1\Psi_2] &\equiv \mathcal{M}'\varepsilon(\mathfrak{S}^* - [\Psi_1]) \cup (\mathfrak{S}^* - [\Psi_2]) \\ &\equiv [\Psi_1] \text{ non-}\varepsilon \mathcal{F}^* \text{ or } [\Psi_2] \text{ non-}\varepsilon \mathcal{F}^* \\ &\equiv (\mathfrak{S}^* - [\Psi_1]) \cup (\mathfrak{S}^* - [\Psi_2])\varepsilon \mathcal{F}^* \equiv [|\Psi_1\Psi_2]\varepsilon \mathcal{F}^*.\end{aligned}$$

Auxiliary sets $\mathcal{E}_1$, $\mathcal{E}_2$ of formulas. Let p, q be integers and let Φ be an open formula of T_ω^∞ whose free variables different from V_q^p are $V_{q_1}^{p_1}$, $\ldots$, $V_{q_k}^{p_k}$. Consider the formula

$$\Phi \supset Sb\big(V_q^p/\mathfrak{f}_{\Phi,p,q}(V_{q_1}^{p_1}, \ldots, V_{q_k}^{p_k})\big)\Phi$$

and let $\mathcal{E}_{\Phi,p,q}$ be the set of all formulas obtained from it by substituting terms of types $p, p_1, \ldots, p_k$ for the variables V_q^p, $V_{q_1}^{p_1}, \ldots, V_{q_k}^{p_k}$. Let $\mathcal{E}_1$ be the union of all sets $\mathcal{E}_{\Phi,p,q}$.

Let $\mathscr{E}_2$ be the set of formulas Φ^{Sk} where Φ is one of the following formulas

$$(\mathrm{V}_0^n, \mathrm{V}_1^n)(\mathsf{E}\mathrm{V}_0^{n+1})\, \mathrm{A}_n(\mathrm{V}_0^{n+1}; \mathrm{V}_0^n, \mathrm{V}_1^n),$$
$$(\mathrm{V}_0^{n+1}, \mathrm{V}_1^{n+1})\{(\mathrm{V}_0^n)[\mathrm{V}_0^{n+1}\mathrm{V}_0^n \equiv \mathrm{V}_1^{n+1}\mathrm{V}_0^n] \equiv \Delta_{n+1}(\mathrm{V}_0^{n+1}, \mathrm{V}_1^{n+1})\},$$
$$(\mathsf{E}\mathrm{V}_0^p \ldots \mathrm{V}_k^p) \sim [\Delta_p(\mathrm{V}_0^p, \mathrm{V}_1^p) \vee \ldots \vee \Delta_p(\mathrm{V}_i^p, \mathrm{V}_j^p) \vee \ldots \vee \Delta_p(\mathrm{V}_{k-1}^p, \mathrm{V}_k^p)] \qquad (p, k = 0, 1, 2, \ldots),$$
$$(\mathrm{V}_0^0)\mathrm{V}_0^1\mathrm{V}_0^0,$$
$$(\mathrm{V}_0^4)[\mathrm{V}_0^5\mathrm{V}_0^4 \supset (\mathsf{E}\mathrm{V}_0^0, \mathrm{V}_1^0, \mathrm{V}_2^0)\mathrm{B}_3(\mathrm{V}_0^4; \mathrm{V}_0^0, \mathrm{V}_1^0, \mathrm{V}_2^0)].$$

Finally let $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2$.

A closed space $\mathfrak{P}$. Let $\mathfrak{P}$ be the space of all the semi-frames $\mathscr{M} = (R_0^*, R_1^*, \ldots, S_0, S_1, \ldots)$ such that the semi-frame of T_ω^∞

$$ext(\mathscr{M}) = (R_0^*, R_1^*, \ldots, S_0, S_1, \ldots, T_{\mathfrak{f}}^*, \ldots)$$

belongs to $\bigcap_{\Phi\varepsilon\mathscr{E}}[\Phi]$. The topology in $\mathfrak{P}$ is introduced via the one-one correspondence $\mathscr{M} \rightleftarrows ext(\mathscr{M})$. $\mathfrak{P}$ is obviously homeomorphic to a closed subset of $\mathfrak{S}^*$.

7.6. *If* $\mathscr{M}$ *belongs to* $\mathfrak{P}$ *then conditions* $\models_{\mathscr{M}} \Phi[\tau_1, \ldots, \tau_k]$ *and* $\models_{ext(\mathscr{M})} \Phi^{\mathrm{Sk}}[\tau_1, \ldots, \tau_k]$ *are equivalent for arbitrary formulas* Φ *with* k *free variables and for arbitrary* k *terms* $\tau_1, \ldots, \tau_k$ *of appropriate types.*

Proof. It is sufficient to show that if 7.6 holds for a formula Φ with the free variables $\mathrm{V}_{q_1}^{p_1}, \ldots, \mathrm{V}_{q_k}^{p_k}, \mathrm{V}_q^p$, it does so for the formula Ψ: $\mathsf{E}\mathrm{V}_q^p\Phi$.

Let us first assume that Ψ^{Sk} is satisfied in $ext(\mathscr{M})$ by the assignment $\mathrm{V}_{q_i}^{p_i} \to \tau_i$, $i = 1, 2, \ldots, k$. From the definition of Ψ^{Sk} it follows that Φ^{Sk} is satisfied in $ext(\mathscr{M})$ by the following assignment: $\mathrm{V}_{q_i}^{p_i} \to \tau_i$, $\mathrm{V}_q^p \to \mathfrak{f}_{\Phi^{\mathrm{Sk}},p,q}(\tau_1, \ldots, \tau_k)$. Hence we obtain $\models_{\mathscr{M}} \Psi[\tau_1, \ldots, \tau_k]$ using the inductive assumption.

Now let us assume that $\models_{\mathscr{M}} \Psi[\tau_1, \ldots, \tau_k]$, i.e., that there is a term τ in R_p^* such that $\models_{\mathscr{M}} \Phi[\tau, \tau_1, \ldots, \tau_k]$. Using the inductive assumption we obtain $\models_{ext(\mathscr{M})} \Phi^{\mathrm{Sk}}[\tau, \tau_1, \ldots, \tau_k]$, i.e. $ext(\mathscr{M})\varepsilon[\Phi^{\mathrm{Sk}}(\tau, \tau_1, \ldots, \tau_k)]$. Now we notice that

$$ext(\mathscr{M})\varepsilon\big[\Phi^{\mathrm{Sk}}(\tau, \tau_1, \ldots, \tau_k) \supset \Phi^{\mathrm{Sk}}\big(\mathfrak{f}_{\Phi^{\mathrm{Sk}}p,q}(\tau_1, \ldots, \tau_k), \tau_1, \ldots, \tau_k\big)\big]$$

since the formula in the square brackets belongs to $\mathscr{E}$. Hence we obtain

$$ext(\mathscr{M})\varepsilon\big[\Phi^{\mathrm{Sk}}\big(\mathfrak{f}_{\Phi^{\mathrm{Sk}},p,q}(\tau_1, \ldots, \tau_k), \tau_1, \ldots, \tau_k\big)\big], \text{ i.e., } \models_{ext(\mathscr{M})} \Psi^{\mathrm{Sk}}[\tau_1, \ldots, \tau_k].$$

7.7. If $\mathscr{M}\varepsilon\mathfrak{P}$ *and* $\Phi\varepsilon\mathscr{E}$, *then* $\models_{\mathscr{M}} \Phi[\mathrm{V}_{q_1}^{p_1}, \ldots, \mathrm{V}_{q_k}^{p_k}]$ *where* $\mathrm{V}_{q_i}^{p_i}$, $i = 1, 2, \ldots, k$ *are all the free variables of* Φ.

This is a direct corollary to 7.6.

7.8. *If* $\mathscr{M}\varepsilon\mathfrak{P}$, *if* Φ *is a formula and* $\tau_1, \ldots, \tau_k$ *are terms such that* $\models_{\mathscr{M}} \Phi[\tau_1, \ldots, \tau_k]$, *then there is a neighbourhood* U *of* $\mathscr{M}$ *in* $\mathfrak{P}$ *such that* $\models_{\mathscr{N}} \Phi[\tau_1, \ldots, \tau_k]$ *for every* $\mathscr{N}$ *in* $U \cap \mathfrak{P}$.

Proof. It is sufficient to take as U the neighbourhood in $\mathfrak{P}$ which corresponds (via the mapping $\mathscr{M} \rightleftarrows ext(\mathscr{M})$) to the neighbourhood $[\Phi^{\mathrm{Sk}}(\tau_1, \ldots, \tau_k)]$.

The equivalence relation $aeq_{\mathscr{M}}$. Let $\mathscr{M}$ be a semi-frame in $\mathfrak{P}$ and let $aeq_{\mathscr{M}}$ be a relation defined as follows:

$$\tau_1\, aeq_{\mathscr{M}}\, \tau_2 \equiv (\mathsf{E}p)\, \tau_1, \tau_2 \,\varepsilon\, R_p^* \,\&\, ext(\mathscr{M})\,\varepsilon\,[\Delta_p^{\mathrm{Sk}}(\tau_1, \tau_2)].$$

7.9. *If* $\mathscr{M} \varepsilon \mathfrak{P}$ *then* $aeq_{\mathscr{M}}$ *is an equivalence relation.*

Proof. The formula $(\Delta_p(\tau_1, \tau_2))^{\mathrm{Sk}}$ has the form $\mathfrak{e}\tau_1 \equiv \mathfrak{e}\tau_2$ where $\mathfrak{e} = \mathfrak{e}(\tau_1, \tau_2)$ is the functor $\mathfrak{f}_{\Phi, p+1, 0}$ and Φ: $\mathrm{V}_0^{p+1}\tau_1 \not\equiv \mathrm{V}_0^{p+1}\tau_2$. Since $[\mathfrak{f}\tau_1 \equiv \mathfrak{f}\tau_1] = \mathfrak{S}^*$ for every functor $\mathfrak{f}$, we infer that the relation $aeq_{\mathscr{M}}$ is reflexive.

If $\tau_1\, aeq_{\mathscr{M}}\, \tau_2$, then $ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_2)\tau_1 \equiv \mathfrak{e}(\tau_1, \tau_2)\tau_2]$. Since the formula $\mathfrak{e}(\tau_2, \tau_1)\tau_1 \not\equiv \mathfrak{e}(\tau_2, \tau_1)\tau_2 \supset \mathfrak{e}(\tau_1, \tau_2)\tau_1 \not\equiv \mathfrak{e}(\tau_1, \tau_2)\tau_2$ belongs to $\mathscr{E}_1$, we infer that if the relation $ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_2, \tau_1)\tau_1 \not\equiv \mathfrak{e}(\tau_2, \tau_1)\tau_2]$ were true, we would have $ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_2)\tau_1 \not\equiv \mathfrak{e}(\tau_1, \tau_2)\tau_2]$ and hence we would also have τ_1 *non* $aeq_{\mathscr{M}}\, \tau_2$. This proves that the above relation is false and hence that $\tau_2\, aeq_{\mathscr{M}}\, \tau_1$. Thus $aeq_{\mathscr{M}}$ is symmetric.

If $\tau_1\, aeq_{\mathscr{M}}\, \tau_2$ and $\tau_2\, aeq_{\mathscr{M}}\, \tau_3$ then

$$ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_2)\tau_1 \equiv \mathfrak{e}(\tau_1, \tau_2)\tau_2],$$
(7.9.1)
$$ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_2, \tau_3)\tau_2 \equiv \mathfrak{e}(\tau_2, \tau_3)\tau_3].$$

Assume that $ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_3)\tau_1]$. If we had $ext(M)$ *non*-$\varepsilon\,[\mathfrak{e}(\tau_1, \tau_3)\tau_2]$, then we would also have $ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_3)\tau_1 \not\equiv \mathfrak{e}(\tau_1, \tau_3)\tau_2]$, and since the formula $\mathfrak{e}(\tau_1, \tau_3)\tau_1 \not\equiv \mathfrak{e}(\tau_1, \tau_3)\tau_2 \supset \mathfrak{e}(\tau_1, \tau_2)\tau_1 \not\equiv \mathfrak{e}(\tau_1, \tau_2)\tau_1$ is in $\mathscr{E}_1$, we would obtain a contradiction with (7.9.1). Hence $ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_3)\tau_2]$. By a similar reasoning we obtain $ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_3)\tau_3]$ and thus

$$ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_3)\tau_1 \supset \mathfrak{e}(\tau_1, \tau_3)\tau_3].$$

We prove similarly that

$$ext(\mathscr{M})\,\varepsilon\,[\mathfrak{e}(\tau_1, \tau_3)\tau_3 \supset \mathfrak{e}(\tau_1, \tau_3)\tau_1]$$

and the transitivity of $aeq_{\mathscr{M}}$ is thus proved.

7.10. *If* $\mathscr{M} \varepsilon \mathfrak{P}$ *and* $\tau_1\, aeq_{\mathscr{M}}\, \tau_2$ *then*

(7.10.1) $$\models_{\mathscr{M}} \Phi[\ldots\tau_1\ldots] \equiv \models_{\mathscr{M}} \Phi[\ldots\tau_2\ldots]$$

for an arbitrary formula Φ *of* T_ω^∞.

Proof. In view of 7.6 it will be sufficient to prove (7.10.1) for open formulas replacing $\mathscr{M}$ by $ext(\mathscr{M})$. It is obvious that if the formula thus modified is true for Φ_1 and Φ_2, it remains true for $|\Phi_1\Phi_2$. Hence it remains to prove that for any σ

(7.10.2) $$ext(\mathcal{M})\varepsilon[\sigma\tau_1 \equiv \sigma\tau_2],$$

(7.10.3) $$ext(\mathcal{M})\varepsilon[\tau_1\sigma \equiv \tau_2\sigma].$$

(7.10.2) results at once from the definition of $aeq_{\mathcal{M}}$. If (7.10.3) were false, we would have $ext(\mathcal{M})\varepsilon[\sim\Phi^{\mathrm{Sk}}(\tau_1, \tau_2)]$ where Φ is the formula $(\mathrm{V}_0^n)[\mathrm{V}_1^{n+1}\mathrm{V}_0^n \equiv \mathrm{V}_2^{n+1}\mathrm{V}_0^n]$ and $n+1$ is the type of τ_1 and of τ_2. Because of $\tau_1\, aeq_{\mathcal{M}}\tau_2$ we have $ext(\mathcal{M})\varepsilon[\Delta_{n+1}(\tau_1, \tau_2)]$. It follows that $ext(\mathcal{M})$ *non*-ε $[(\mathrm{V}_0^{n+1}, \mathrm{V}_1^{n+1})\{\Phi^{\mathrm{Sk}} \equiv (\Delta_{n+1}(\mathrm{V}_0^{n+1}, \mathrm{V}_1^{n+1}))^{\mathrm{Sk}}\}]$, which contradicts the assumption $ext(\mathcal{M})\varepsilon[\Psi]$ for all Ψ in $\mathcal{E}_2$.

The space $\mathfrak{S}$ (11). We now define a mapping of $\mathfrak{P}$ onto a class of semi-frames $(\widehat{R}_0, \widehat{R}_1, \ldots, S_0, S_1, \ldots)$.

Let
$$\mathrm{V}_0^j = \tau_{j,0}, \tau_{j,1}, \ldots$$
be a fixed sequence (without repetitions) containing all elements of R_j^* (i.e., all terms of T_ω^∞ of type j), $j = 0, 1, \ldots$ Let $\mathcal{M} = (R_0^*, R_1^*, \ldots, S_0^*, S_1^*, \ldots)$ be an element of $\mathfrak{S}^*$. We define a mapping $\varphi_{\mathcal{M}}\colon \tau_{j,k} \to c_{j,k}$ of R_j^* onto $\widehat{R}_j$ $(j = 0, 1, \ldots)$ by induction as follows: Put $\varphi_{\mathcal{M}}(\mathrm{V}_0^j) = 2^j$ and
$$\varphi_{\mathcal{M}}(\tau_{j,k+1}) = \min_p[p\,\varepsilon\widehat{R}_j\,\&\,p \neq \varphi_{\mathcal{M}}(\tau_{j,s}) \text{ for } s = 0, 1, \ldots, k]$$
if $\tau_{j,k+1}$ *non*-$aeq_{\mathcal{M}}\tau_{j,s}$ for $s = 0, 1, \ldots, k$ and
$$\varphi_{\mathcal{M}}(\tau_{j,k+1}) = \varphi_{\mathcal{M}}(\tau_{j,s})$$
if s is the least integer $\leqslant k$ such that $\tau_{j,k+1}\, aeq_{\mathcal{M}}\tau_{j,s}$.

$\varphi_{\mathcal{M}}$ maps R_j^* onto $\widehat{R}_j$. Otherwise there would be an s such that every term $\tau_{j,k}$ would bear the relation $aeq_{\mathcal{M}}$ to one of the terms $\tau_{j,i}$ with $i \leqslant s$. This would entail that the formula
$$\Phi : (\mathsf{E}\mathrm{V}_1^j, \ldots, \mathsf{E}\mathrm{V}_{s+1}^j)(\mathrm{V}_0^j)[\Delta_j(\mathrm{V}_0^j, \mathrm{V}_1^j) \vee \ldots \vee \Delta_j(\mathrm{V}_0^j, \mathrm{V}_{s+1}^j)]$$
is true in $\mathcal{M}$. Hence we would obtain $\mathcal{M}\varepsilon[\Phi^{\mathrm{Sk}}]$, which contradicts the assumption that $\mathcal{M}\varepsilon[\Psi]$ for every Ψ in $\mathcal{E}_2$.

The mapping $\varphi_{\mathcal{M}}$ is in general not one-to-one.

Let $\mu(\mathcal{M})$ be the semi-frame $(\widehat{R}_0, \widehat{R}_1, \ldots, S_0, S_1, \ldots)$, where $2^m(2p+1)S_m2^{m+1}(2q+1)$ is true if and only if there are terms $\tau_{m,k}, \tau_{m+1,l}$ such that $\varphi_{\mathcal{M}}(\tau_{m,k}) = 2^m(2p+1)$, $\varphi_{\mathcal{M}}(\tau_{m+1,l}) = 2^{m+1}(2q+1)$ and $\tau_{m,k}S_m^*\tau_{m+1,l}$.

From 7.10 it follows that S_m can also be defined in the following way: $2^m(2p+1)S_m2^{m+1}(2q+1)$ if and only if $\tau_{m,k}S_m^*\tau_{m+1,l}$ for arbitrary terms such that $\varphi_{\mathcal{M}}(\tau_{m,k}) = 2^m(2p+1)$ and $\varphi_{\mathcal{M}}(\tau_{m+1,l}) = 2^{m+1}(2q+1)$.

---

(11) The construction of $\mathfrak{S}$ outlined below is due to A. Ehrenfeucht.

Let $\mathfrak{S}$ be the set of all semi-frames $\mu(\mathcal{M})$ where $\mathcal{M}$ runs over the space $\mathfrak{P}$. Define neighbourhoods in $\mathfrak{S}$ as sets described in (7.1.5).

We are going to prove that all conditions of theorem 7.1 are satisfied with this choice of $\mathfrak{S}$.

First we are going to prove that $\mathfrak{S}$ consists of models, i.e., that every semi-frame $\mu(\mathcal{M})$ in $\mathfrak{S}$ satisfies conditions 1-5 of p. 11.

Conditions 1-3 are obvious.

Condition 4. Assume that $y, y' \varepsilon \widehat{R}_{j+1}$ and that for every x in $\widehat{R}_j$ the equivalence $xS_jy \equiv xS_jy'$ is true. Determine m, n such that $\varphi_{\mathcal{M}}(\tau_{j+1,m}) = y$, $\varphi_{\mathcal{M}}(\tau_{j+1,n}) = y'$. It follows that $\tau_{j,k}S_j^*\tau_{j+1,m} \equiv \tau_{j,k}S_j^*\tau_{j+1,n}$, for each k, whence $\tau_{j+1,m}\, aeq_{\mathcal{M}}\, \tau_{j+1,n}$ and consequently $y = y'$.

Condition 5. Let x, x' be arbitrary elements of $\widehat{R}_j$. Since the formula $(\mathrm{V}_0^j, \mathrm{V}_1^j)(\mathsf{E}\mathrm{V}_0^{j+1})\mathrm{A}_j(\mathrm{V}_0^{j+1}; \mathrm{V}_0^j, \mathrm{V}_1^j)$ belongs to $\mathcal{E}$, it is true in $\mathcal{M}$, and hence there is a term $\tau_{j+1,m}$ such that for every σ in R_j^*

$$\sigma S_j^* \tau_{j+1,m} \equiv \tau_{j,p}\, aeq_{\mathcal{M}}\, \sigma \vee \tau_{j,q}\, aeq_{\mathcal{M}}\, \sigma .$$

Assuming that $\varphi_{\mathcal{M}}(\tau_{j,p}) = x$ and $\varphi_{\mathcal{M}}(\tau_{j,q}) = x'$, we infer that the element $y = \varphi_{\mathcal{M}}(\tau_{j+1,m})$ satisfies the equivalence $tS_jy \equiv (x = t)$ or $(x' = t)$ for every t in $\widehat{R}_j$.

Before checking conditions (7.1.1)-(7.1.5) we prove one more lemma:

7.11. *If Φ is a formula of T_ω^∞ with the free variables $\mathrm{V}_{p_1}^{i_1}, \dots, \mathrm{V}_{p_k}^{i_k}$ and $\mathcal{M}$ is in $\mathfrak{P}$, then*

(7.11.1) $\models_{\mathcal{M}} \Phi[\tau_{i_1,n_1}, \dots, \tau_{i_k,n_k}] \equiv \models_{\mu(\mathcal{M})} \Phi[\varphi_{\mathcal{M}}(\tau_{i_1,n_1}), \dots, \varphi_{\mathcal{M}}(\tau_{i_k,n_k})]$.

The lemma is obvious for the atomic formulas because of the definition of the relations S_j. It is also obvious that if the lemma is true for two formulas Φ_1 and Φ_2, then it is true for the formula $|\Phi_1\Phi_2$. Finally let us assume that the lemma holds for the formula Ψ with the free variables $\mathrm{V}_j^i, \mathrm{V}_{p_1}^{i_1}, \dots, \mathrm{V}_{p_k}^{i_k}$ and let $\Phi: \mathsf{E}\mathrm{V}_j^i\Psi$.

The condition $\models_{\mathcal{M}} \Psi[\tau_{i,n}, \tau_{i_1,n_1}, \dots, \tau_{i_k,n_k}]$ implies the condition $\models_{\mu(\mathcal{M})} \Psi[\varphi_{\mathcal{M}}(\tau_{i,n}), \varphi_{\mathcal{M}}(\tau_{i_1,n_1}), \dots, \varphi_{\mathcal{M}}(\tau_{i_k,n_k})]$, whence we infer that the left-hand side of (7.11.1) implies the right-hand side. If the right-hand side of (7.11.1) is true, then there is an element x of $\widehat{R}_i$ such that $\models_{\mu(\mathcal{M})} \Psi[x, \varphi_{\mathcal{M}}(\tau_{i_1,n_1}), \dots, \varphi_{\mathcal{M}}(\tau_{i_k,n_k})]$. Since every element of $\widehat{R}_i$ is representable as $\varphi_{\mathcal{M}}(\tau_{i,n})$, we obtain, by the inductive assumption, $\models_{\mathcal{M}} \Psi[\tau_{i,n}, \tau_{i_1,n_1}, \dots, \tau_{i_k,n_k}]$, which proves that the left-hand side of (7.11.1) is true.

$\mathfrak{S}$ satisfies conditions (7.1.1)-(7.1.5).

(7.1.1) is obvious.

(7.1.2). If $\mu(\mathcal{M}) \varepsilon \mathfrak{S}$, then $\mathcal{M} \varepsilon \mathfrak{P}$ and hence $\mathcal{M} \varepsilon [(\mathrm{V}_0^0)\mathrm{V}_0^1\mathrm{V}_0^0]$; consequently $\tau_{0,k}S_0^*\mathrm{V}_0^1$ for $k = 0, 1, 2, \dots$, whence xS_02 for every x in $\widehat{R}_0$.

(7.1.3). If $\mu(\mathcal{M})\varepsilon\mathfrak{S}$ and $\mathcal{M}\varepsilon\mathfrak{P}$, then $\mathcal{M}\varepsilon[\Phi]$, where Φ is the formula $(\mathrm{V}_0^4)[\mathrm{V}_0^5\mathrm{V}_0^4\supset(\mathsf{E}\mathrm{V}_0^0\mathrm{V}_1^0\mathrm{V}_2^0)\mathrm{B}_3(\mathrm{V}_0^4;\mathrm{V}_0^0,\mathrm{V}_1^0,\mathrm{V}_2^0)]$. If $xS_4 2^5$, then there is a k such that $x=\varphi_{\mathcal{M}}(\tau_{4,k})$ and therefore there are p_1,p_2,p_3 such that $\models_{\mathcal{M}}\mathrm{B}_3[\tau_{4,k},\tau_{0,p_1},\tau_{0,p_2},\tau_{0,p_3}]$. Using lemma 7.11 and denoting $\varphi_{\mathcal{M}}(\tau_{0,p_i})$ by x_i, $i=1,2,3$, we obtain $\models_{\mu(\mathcal{M})}\mathrm{B}_3[x;x_1,x_2,x_3]$, whence we infer that 2^5 represents a ternary relation in $\mu(\mathcal{M})$.

(7.1.4). It can immediately be verified that Hausdorff's axioms hold in $\mathfrak{S}$. Since a continuous mapping of a compact space is itself compact, it is sufficient to show that the mapping μ is continuous.

In order to show this we first observe that a necessary and sufficient condition for the equation $\varphi_{\mathcal{M}}(\tau_{j,k})=2^j(2s+1)$ is the existence of $s+1$ integers $a_0,a_1,\ldots,a_s$ satisfying the following statements: $a_0<a_1<\ldots<a_s\leqslant k$; $\tau_{j,k}\,aeq_{\mathcal{M}}\,\tau_{j,a_s}$; τ_{j,a_p} *non* $aeq_{\mathcal{M}}\,\tau_{j,a_q}$ for $0\leqslant p<q\leqslant s$; for every $b\leqslant a_s$ there is a $p\leqslant s$ such that $\tau_{j,b}\,aeq_{\mathcal{M}}\,\tau_{j,a_p}$. It easily follows that there is a formula $\Omega_{j,k,s}$ of T_ω^∞ such that for every $\mathcal{M}$ in $\mathfrak{P}$ the equation $\varphi_{\mathcal{M}}(\tau_{j,k})=2^j(2s+1)$ is equivalent to $\mathcal{M}\varepsilon[\Omega_{j,k,s}]$. To obtain this formula it is clearly sufficient to build the disjunction of all formulas

$$\Delta_j(\tau_{j,k},\tau_{j,a_s})\,\&\sim[\bigvee_{0\leqslant m<n\leqslant s}\Delta_j(\tau_{j,a_m},\tau_{j,a_n})]\,\&\bigwedge_{0\leqslant b\leqslant a_s}\bigvee_{0\leqslant p\leqslant s}\Delta_j(\tau_{j,b},\tau_{j,a_p}),$$

where $(a_0,\ldots,a_s)$ runs over all sequences of integers $0,1,\ldots,k$ such that $a_0<a_1<\ldots<a_s$.

Now let Φ be a formula of T_ω with the free variables $\mathrm{V}_{q_1}^{p_1},\ldots,\mathrm{V}_{q_k}^{p_k}$ and assume that the model $\mathcal{M}_0=\mu(\mathcal{M})$ satisfies the following condition $\models_{\mathcal{M}_0}\Phi[2^{p_1}(2n_1+1),\ldots,2^{p_k}(2n_k+1)]$. Let τ_{p_s,m_s} be a term such that $\varphi_{\mathcal{M}}(\tau_{p_s,m_s})=2^{p_s}(2n_s+1)$, $s=1,2,\ldots,k$. It follows easily that $\mathcal{M}\varepsilon[\Omega_{p_s,m_s,n_s}]$ for $s=1,2,\ldots,k$ and hence

$$\mathcal{M}\varepsilon\bigcap_{s\leqslant k}[\Omega_{p_s,m_s,n_s}]\cap[\Phi(\tau_{p_1,m_1},\ldots,\tau_{p_k,m_k})].$$

If $\mathcal{M}'$ is any semi-frame which belongs to $\mathfrak{P}$ and is such that $ext(\mathcal{M}')$ belongs to the intersection at the right-hand side of this formula, then $\varphi_{\mathcal{M}'}(\tau_{p_s,m_s})=2^{p_s}(2n_s+1)$ for $s=1,2,\ldots,k$ and (in view of 7.3 and 7.11) $\models_{\mathcal{M}'}\Phi[2^{p_1}(2n_1+1),\ldots,2^{p_k}(2n_k+1)]$. This proves that if U is the neighbourhood (in $\mathfrak{S}$)

$$\{\mathcal{M}:\ \models_{\mathcal{M}}\Phi[2^{p_1}(2n_1+1),\ldots,2^{p_k}(2n_k+1)]\},$$

then $\mu^{-1}(U)$ contains an open set, i.e., that the mapping μ is continuous.

(7.1.5) is obvious.

(7.1.6). Let $\mathcal{M}'=(R_0',R_1',\ldots,S_0',S_1',\ldots)$ be a model of T_ω with denumerable sets R_j' and let r_1 be an element of R_1' such that $r_0S_0'r_1$ for every r_0 in R_0' and r_5 an element of R_5' which represents in $\mathcal{M}'$ a ternary relation.

We first construct a semi-frame $\mathcal{M}''$ of T_ω^∞ which differs from $\mathcal{M}'$ only in containing functions T_f'' which interpret the functors of T_ω^∞.

Let ε be a choice function for subsets of $\bigcup_j R'_j$. We define $T''_{\mathfrak{f}}$ for functors of K_r by induction on r. For $r = 0$ there are no such functors and our construction is void. Let us assume that $T''_{\mathfrak{f}}$ is already defined for functors of K_s, $s < r$ and let $\mathfrak{f} = \mathfrak{f}_{\Phi,p,q}$ be a functor of K_r. Thus Φ is a formula of K_{r-1} which is not a formula of K_{r-2}. By the inductive assumption the notion of satisfaction for Φ is already defined because functors which occur in Φ are all functors of K_{r-1}. Let $V^{p_1}_{q_1}, \ldots, V^{p_k}_{q_k}$ be all the free variables of Φ which are different from V^p_q, let $a_j \varepsilon R'_{p_j}$ for $j = 1, 2, \ldots, k$ and denote by $F_{a_1,\ldots,a_k}$ the set of a in R'_p which together with $a_1, \ldots, a_k$ satisfy Φ in $\mathscr{M}'$ (extended by the interpretations of functors of K_r). Define $T''_{\mathfrak{f}}(a_1, \ldots, a_k) = \varepsilon F_{a_1,\ldots,a_k}$ if $F_{a_1,\ldots,a_k} \neq 0$ and $T''_{\mathfrak{f}}(a_1, \ldots, a_k) = \varepsilon R'_p$ otherwise. The semi-frame $\mathscr{M}''$ is thus defined.

If Ψ is a formula in $\mathscr{E}_1$, then the closure of Ψ is true in $\mathscr{M}''$. Indeed let Ψ have the form

$$\Phi \supset Sb\big(V^p_q/\mathfrak{f}_{\Phi,p,q}(V^{p_1}_{q_1}, \ldots, V^{p_k}_{q_k})\big)\Phi$$

where Φ is an open formula of K_r. If $\models_{\mathscr{M}''} \Phi[a, a_1, \ldots, a_k]$, then a is in $F_{a_1,\ldots,a_k}$ and hence so is $T''_{\mathfrak{f}}(a_1, \ldots, a_k)$, where $\mathfrak{f}$ is an abbreviation of $\mathfrak{f}_{\Phi,p,q}$. It follows that $\models_{\mathscr{M}''} \Phi[T_{\mathfrak{f}}(a_1, \ldots, a_k), a_1, \ldots, a_k]$, which proves that the consequent of the formula Ψ is satisfied in $\mathscr{M}''$ if $V^{p_s}_{q_s}$ is interpreted as a_s, $s = 1, 2, \ldots, k$.

If Ψ is a formula of $\mathscr{E}_2$, then $\models_{\mathscr{M}''} \Psi[\ldots]$ where $\ldots$ denotes the assignment of r_1 to V^1_0 and of r_5 to V^5_0. For the first three formulas of $\mathscr{E}_2$ (see p. 26) this is obvious since these formulas are true in $\mathscr{M}'$ and do not contain functors. For the last two formulas our assertion follows from the assumptions concerning r_1 and r_5.

We now define a semi-frame

$$\mathscr{M}^* = (R^*_0, R^*_1, \ldots, S^*_0, S^*_1, \ldots, T^*_{\mathfrak{f}}, \ldots).$$

Let

$$t_{j,0}, t_{j,1}, \ldots$$

be a sequence without repetitions consisting of all elements of R'_j, $j = 0, 1, 2, \ldots$ We can assume that $t_{1,0} = r_1$ and $t_{5,0} = r_5$. Let τ be a term and $v(\tau)$ its value in $\mathscr{M}''$ under the interpretation $V^j_k \to t_{j,k}$, $j, k = 0, 1, 2, \ldots$ We define

$$\sigma S^*_j \tau \equiv \sigma \varepsilon R^*_j \,\&\, \tau \varepsilon R^*_{j+1} \,\&\, v(\sigma) S'_j v(\tau).$$

The semi-frame $\mathscr{M}^*$ is hereby defined. We shall show that it enjoys the following property: If Φ is a formula of T^∞_ω with the free variables $V^{p_1}_{q_1}, \ldots, V^{p_k}_{q_k}$ and $\tau_j \varepsilon R^*_{p_j}$ for $j = 1, 2, \ldots, k$, then

$$\models_{\mathscr{M}^*} \Phi[\tau_1, \ldots, \tau_k] \equiv \models_{\mathscr{M}''} \Phi[v(\tau_1), \ldots, v(\tau_k)].$$

According to the definition of the relations S_j^* this equivalence is true for atomic formulas. It is obvious that if the equivalence holds for the formulas Φ_1, Φ_2, it does so for the formula $|\Phi_1\Phi_2$. Now assume that the equivalence holds for the formula Ψ with the free variables $V_q^p, V_{q_1}^{p_1}, \ldots, V_{q_k}^{p_k}$ and let Φ be the formula $\mathsf{E}V_q^p\Psi$. If $\models_{\mathscr{M}^*}\Phi[\tau_1, \ldots, \tau_k]$, then there is a term τ in R_p^* such that $\models_{\mathscr{M}^*}\Psi[\tau, \tau_1, \ldots, \tau_k]$, whence by the inductive assumption $\models_{\mathscr{M}''}\Psi[\mathrm{v}(\tau), \mathrm{v}(\tau_1), \ldots, \mathrm{v}(\tau_k)]$ and we obtain $\models_{\mathscr{M}''}\Phi[\mathrm{v}(\tau_1), \ldots, \mathrm{v}(\tau_k)]$.

Conversely, if the last condition is satisfied, then there is an m such that $\models_{\mathscr{M}''}\Psi[t_{p,m}, \mathrm{v}(\tau_1), \ldots, \mathrm{v}(\tau_k)]$ and hence we obtain the relation $\models_{\mathscr{M}''}\Psi[\mathrm{v}(V_m^p), \mathrm{v}(\tau_1), \ldots, \mathrm{v}(\tau_k)]$. Using the inductive assumption we obtain $\models_{\mathscr{M}^*}\Psi[V_m^p, \tau_1, \ldots, \tau_k]$ and hence $\models_{\mathscr{M}^*}\Phi[\tau_1, \ldots, \tau_k]$.

From the equivalence just proved we infer that $\mathscr{M}^* \varepsilon \bigcap_{\Phi\varepsilon\mathscr{S}}[\Phi]$ and hence that the semi-frame

$$\mathscr{M}^{**} = (R_0^*, R_1^*, \ldots, S_0^*, S_1^*, \ldots)$$

(obtained from $\mathscr{M}^*$ by deleting the functions $T_\dagger^*$) belongs to $\mathfrak{P}$. Hence $\mathscr{M} = \mu(\mathscr{M}^{**})$ belongs to $\mathfrak{S}$. We shall show that the models $\mathscr{M}'$ and $\mathscr{M}$ are isomorphic.

Let $x = 2^j(2n+1)$ be an element of $\widehat{R}_j$ and let m be an integer such that $\varphi_{\mathscr{M}^{**}}(\tau_{j,m}) = x$. We let the element $\mathrm{v}(\tau_{j,m})$ of R_j' correspond to x. This element is independent of the particular value of m since from $\varphi_{\mathscr{M}^{**}}(\tau_{j,m}) = \varphi_{\mathscr{M}^{**}}(\tau_{j,s})$ it follows that $\tau_{j,m}\, aeq_{\mathscr{M}^{**}}\, \tau_{j,s}$, whence $\models_{\mathscr{M}^{**}}\Delta_j[\tau_{j,m}, \tau_{j,s}]$ and consequently $\models_{\mathscr{M}^*}\Delta_j[\tau_{j,m}, \tau_{j,s}]$; therefore $\models_{\mathscr{M}''}\Delta_j[\mathrm{v}(\tau_{j,m}), \mathrm{v}(\tau_{j,s})]$, which proves that $\mathrm{v}(\tau_{j,m}) = \mathrm{v}(\tau_{j,s})$.

The same argument read backwards shows that the mapping of $\mathscr{M}$ into $\mathscr{M}'$ defined above is one-to-one. It is a mapping onto since $\tau_{j,m}$ can be made to run over the whole of R_j^* as x runs over $\widehat{R}_j$. Finally

$$xS_jy \equiv 2^j(2n+1)S_j2^{j+1}(2p+1)$$
$$\equiv \varphi_{\mathscr{M}^{**}}(\tau_{j,m})S_j^*\varphi_{\mathscr{M}^{**}}(\tau_{j+1,q}) \equiv \mathrm{v}(\tau_{j,m})S_j'\mathrm{v}(\tau_{j+1,q}),$$

which proves the isomorphism of the models $\mathscr{M}'$ and $\mathscr{M}$.

8. A generalization of the results of section 6

In this section we generalize the theorems obtained in section 6 by dropping the assumption that $\mathscr{K}$ is a class of models characterized by a recursively enumerable set of formulas.

Let Z_0 be a set of formulas which are true in every standard model $St(M)$ with infinite M. We denote by $\Theta_1', \Theta_2', \ldots$ a sequence containing all formulas of Z_0 and put $\Theta_k : \Theta_1' \& \ldots \& \Theta_k'$. $\mathscr{K}_0$ denotes the family of models $\mathscr{M}$ in which all formulas Θ_k are true.

8.1. *If $\mathcal{M}_0$ is a limit in $\mathfrak{S}$ of $\mathcal{M}_k$ and 2^5 represents in $\mathcal{M}_k$ the relation N_k, then N_k is convergent to a relation N_0 which is represented in $\mathcal{M}_0$ by 2^5.*

Proof. $N_k \to N_0$ means that for arbitrary p, q, r in $\{1, 3, 5, \ldots\}$ $N_0(p, q, r)$ if and only if $N_s(p, q, r)$ for sufficiently large s. Let $p = 2i+1$, $q = 2j+1$, $r = 2k+1$ and consider the formula

$$\Xi\colon (\mathsf{E}V_p^4)[B_3(V_0^4; V_i^0, V_j^0, V_k^0) \& V_0^5 V_0^4].$$

2^5 represents in $\mathcal{M}_0$ a relation N_0; hence (cf. definition of representability on p. 12)

$$N_0(2i+1, 2j+1, 2k+1) \equiv \models_{\mathcal{M}_0} \Xi[2i+1, 2j+1, 2k+1, 2^5] \equiv \mathcal{M}_0 \varepsilon [\Xi].$$

Since $\mathcal{M}_0$ is the limit of $\mathcal{M}_s$ and $[\Xi]$ is open in $\mathfrak{S}$, the last condition is equivalent to

$$\mathcal{M}_s \varepsilon [\Xi] \textit{ for sufficiently large } s,$$

i.e.

$$N_s(2i+1, 2j+1, 2k+1) \textit{ for sufficiently large } s.$$

8.1 is thus proved.

Let Φ be a formula of T_ω with the unique free variable V_1^5. We call Φ an extensionally elementary formula (abbreviated: e.e. formula) if there is a first order closed formula $\mathfrak{E}(M, N)$ with the unary predicate variable M and ternary N such that for an arbitrary pair (M, N) with infinite M

$$\models_{St(M)} \Phi[N] \equiv \models_M \mathfrak{E}[M, N].$$

Let k be an integer and $\mathfrak{S}_k = \{\mathcal{M} : \mathcal{M} \varepsilon \mathfrak{S} \& \models_{\mathcal{M}} \Theta_k\}$.

8.2. *If Φ is not e.e., then there are models $\mathcal{M}'$, $\mathcal{M}''$ in $\mathfrak{S}_k$ such that $\models_{\mathcal{M}'} \Phi[2^5]$, $\models_{\mathcal{M}''} \sim\Phi[2^5]$, 2^5 represents in $\mathcal{M}'$ the same relation as in $\mathcal{M}''$.*

Proof. Let us assume that the theorem is false. It follows that:

(8.2.1) *for every pair (M, N) with denumerable and infinite M and every denumerable model $\mathcal{M}$ containing this pair and satisfying the conditions $\models_{\mathcal{M}} \Theta_k$, $\models_{\mathcal{M}} (\mathsf{E}V_0^1)(V_0^0)(V_0^1 V_0^0)$, the truth value of $\models_{\mathcal{M}} \Phi[n]$ (where n represents N in $\mathcal{M}$) is independent of $\mathcal{M}$.*

Indeed a denumerable model satisfying these conditions is isomorphic to a model of class $\mathfrak{S}_k$ in which 2^5 represents a relation isomorphic with N.

We shall now show that if Φ satisfies (8.2.1), then it is an e.e. formula. Let Π be a formula which is satisfied in a model $\mathcal{M}$ if and only if R_0 is infinite. We can take as Π for instance the axiom of infinity as formulated in [9]. Put $\Theta: \Pi \& \Theta_k \& (\mathsf{E}V_0^1)(V_0^0)V_0^1V_0^0$ and consider the formulas

$$\mathfrak{F}(\mathrm{M}, \mathrm{N}, \mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}): \mathfrak{F}_0 \& \mathfrak{M}_\Theta \& \mathfrak{B} \& \mathfrak{N}_\Phi,$$

$$\mathfrak{G}(\mathrm{M}, \mathrm{N}, \mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y}): \mathfrak{F}_0 \& \mathfrak{M}_\Theta \& \mathfrak{B} \& \mathfrak{N}' \supset \ddot{\mathfrak{N}}_\Phi.$$

We shall show that

(8.2.2) $$(\mathrm{M} \subseteq \mathrm{J}) \& \infty(\mathrm{J}-\mathrm{M}, \mathrm{Q}) \& \mathrm{A}, \ldots, \mathrm{E}, \mathrm{R}, \ldots, \mathrm{Y} \varepsilon \mathscr{B}(\mathrm{J}) \& \mathrm{N} \varepsilon \mathscr{B}(\mathrm{M}) \& \mathfrak{F}_{\mathrm{rel}}(\mathrm{J}, \mathrm{M}, \mathrm{N}, \mathrm{A}, \ldots, \mathrm{Y}) \Rightarrow$$
$$\Rightarrow (\mathrm{M} \subseteq \mathrm{K}) \& \infty(\mathrm{K}-\mathrm{M}, \mathrm{S}) \& \mathrm{A}', \ldots, \mathrm{E}', \mathrm{R}', \ldots, \mathrm{Y}' \varepsilon \mathscr{B}(\mathrm{K}) \& \mathrm{N} \varepsilon \mathscr{B}(\mathrm{M}) \supset \mathfrak{G}_{\mathrm{rel}}(\mathrm{K}, \mathrm{M}, \mathrm{N}, \mathrm{A}', \ldots, \mathrm{Y}').$$

Indeed, let U be a denumerable set, $J, K, M, N, A, \ldots, Y, A', \ldots Y', R, S \varepsilon \mathscr{B}(U)$ and assume that $M \subseteq J$, $M \subseteq K$, $\models_U \infty[J-M, Q]$, $\models_U \infty[K-M, S]$, $N \varepsilon \mathscr{B}(M)$ and $\models_U \mathfrak{F}_{\mathrm{rel}}[J, M, N, A, \ldots, Y]$. We have to show that $\models_U \mathfrak{G}_{\mathrm{rel}}[K, M, N, A', \ldots, Y']$, i.e. that the following formula holds: $\models_K \mathfrak{G}[M, N, A', \ldots, Y']$. We can therefore assume that

(8.2.3) $$\models_K \mathfrak{F}_0[A', \ldots, E'],\ \models_K \mathfrak{M}_\Theta[A', \ldots, Y'],\ \models_K \mathfrak{B}[A', \ldots, R'],\ \models_K \mathfrak{N}'[A', \ldots, E', M, N]$$

and have to derive $\models_K \ddot{\mathfrak{N}}_\Phi[A', \ldots, Y', M, N]$. To obtain this we assume that $x_0, \ldots, x_s, y, z, t, n'$ are elements of K such that

(8.2.4) $$\models_K \mathfrak{K}[A', B', C', x_0, \ldots, x_5, y, z, t],\ \models_K \mathfrak{L} \& \mathfrak{P}[A', B', R', S', U', W', Y', M, N, x_0, \ldots, x_5, y, z, t, n']$$

and have only to derive $\models_K \mathfrak{Q}[U', W', Y', x_1, x_5, y, n']$.

Since $\models_U \mathfrak{F}_{\mathrm{rel}}[J, M, N, A, \ldots, Y]$, we infer that $\models_J \mathfrak{F}[M, N, A, \ldots, Y]$, and hence by 5.7 that the model $\mathscr{M}$ determined by $A, \ldots, Y$ contains the pair (M, N) and that $\models_{\mathscr{M}} \Phi[n]$, where n is any element representing N in $\mathscr{M}$. Since $\models_{\mathscr{M}} \Theta$ (in view of $\models_J \mathfrak{M}_\Theta[A, \ldots, Y]$), we see that $\models_{\mathscr{M}} \Pi$ and hence that $\mathscr{M}$ is denumerable and infinite. Furthermore, $\models_{\mathscr{M}} \Theta_k$ and $\models_{\mathscr{M}} (\mathsf{E}V_0^1)(V_0^0)V_0^1V_0^0$. Thus by (8.2.1) we infer that if $\mathscr{M}'$ is any denumerable model which contains the pair (M, N) and satisfies $\models_{\mathscr{M}'} \Theta_k$, then $\models_{\mathscr{M}'} \Phi[n']$ where n' represents N in $\mathscr{M}'$. According to (8.2.3) and (8.2.4) the system $(A', \ldots, Y')$ determines a model $\mathscr{M}'$ which contains (M, N) and satisfies Θ_k, and in which n' represents N. Hence $\models_{\mathscr{M}'} \Phi[n']$. This, however, is equivalent to $\models_K \mathfrak{Q}[U', W', Y', x_1, x_5, y, n']$. Implication (8.2.2) is thus proved.

We can now apply 1.5 and obtain a first order formula $\mathfrak{E}(\mathrm{M},\mathrm{N})$ such that for any pair (M, N) each of the following conditions implies the next one:

there is a $J \supseteq M$ *and* $A, \ldots, Y \varepsilon \mathscr{B}(J)$ *such that* $J-M$ *is infinite and* $\models_J \mathfrak{F}[M, N, A, \ldots, Y]$;

$\models_M \mathfrak{E}(M, N)$;

for every $K \supseteq M$ *and arbitrary* $A', \ldots, Y' \varepsilon \mathscr{B}(K)$ *if* $K-M$ *is infinite, then* $\models_K \mathfrak{G}[M, N, A', \ldots, Y']$.

We shall now show that for an arbitrary pair (M, N) with infinite M

$$\models_{St(M)} \Phi[N] \equiv \models_M \mathfrak{E}[M, N].$$

Assume first that $\models_{St(M)}\Phi[N]$ and M is infinite. Let $\mathscr{M} = St(M)$ and let $(A_0, \ldots, E_0, R^*, \ldots, Y^*)$ be the system determined by $\mathscr{M}$. Denoting by I^* the union of fields of these relations, we infer that $I^* \supseteq M$, I^*-M is infinite and $\models_{I^*} \mathfrak{F}[M, N, A_0, \ldots, E_0, R^*, \ldots, Y^*]$, whence we obtain $\models_M \mathfrak{E}[M, N]$. Conversely, assume $\models_M \mathfrak{E}[M, N]$. Choose $K, A', \ldots, Y'$ so that $A' = A_0, \ldots, E' = E_0, R' = R^*, \ldots, Y' = Y^*$. Hence $K-M$ is infinite and $\models_K \mathfrak{G}[M, N, A_0, \ldots, E_0, R^*, \ldots, Y^*]$. This implies that $\models_K \overline{\mathfrak{N}_\Phi}[M, N, A_0, \ldots, E_0, R^*, \ldots, Y^*]$ since the antecedent of $\mathfrak{G}$, i.e. the formula $\mathfrak{F}_0 \& \mathfrak{M}_\Theta \& \mathfrak{B} \& \mathfrak{N}'$, is satisfied in K by $A_0, \ldots, E_0, R^*, \ldots, Y^*, M, N$. In view of 5.4 we obtain $\models_{St(M)}\Phi[N]$ because N is the element of $\mathscr{M}$ which represents N in $\mathscr{M}$.

Lemma 8.2 is thus proved.

8.3. *If* Φ *is absolute with respect to* $\mathscr{K}_0$, *then* Φ *is extensionally elementary.*

Proof. Assume that Φ is not an e.e. formula. Let $\mathscr{M}'_k$, $\mathscr{M}''_k$ be two sequences of models in $\mathfrak{S}$ such that for each k

$$\models_{\mathscr{M}'_k}\Theta_k, \ \models_{\mathscr{M}''_k}\Theta_k, \ \models_{\mathscr{M}'_k}\Phi[2^5], \ \models_{\mathscr{M}''_k} \sim \Phi[2^5],$$

2^5 *represents in* $\mathscr{M}'_k$ *the same relation* N_k *as in* $\mathscr{M}''_k$.

Let $\mathscr{M}'_0$, $\mathscr{M}''_0$ be models which are limits of subsequences $\mathscr{M}'_{i_n}$, $\mathscr{M}''_{i_n}$. By 8.1 2^5 represents in $\mathscr{M}'_0, \mathscr{M}''_0$ the same relation N_0. Since $\mathscr{M}'_{i_n} \varepsilon [\Theta_k]$ for sufficiently large n, we infer that $\mathscr{M}'_0 \varepsilon [\Theta_k]$; otherwise we would have $\mathscr{M}'_0 \varepsilon [\sim \Theta_k]$ and hence almost all $\mathscr{M}'_{i_n}$ would belong to $[\sim \Theta_k]$. Hence $\models_{\mathscr{M}'_0}\Theta_k$ for each k and consequently $\mathscr{M}'_0 \varepsilon \mathscr{K}_0$. Similarly we show that $\mathscr{M}''_0 \varepsilon \mathscr{K}_0$. From $\models_{\mathscr{M}'_k}\Phi[2^5]$ we obtain $\models_{\mathscr{M}'_0}\Phi[2^5]$ and similarly $\models_{\mathscr{M}''_0} \sim \Phi[2^5]$.

If Φ were absolute with respect to $\mathscr{K}_0$, then denoting by M_0 the set $\{1, 3, 5, \ldots\}$ we would have

$$\models_{St(M_0)}\Phi[N_0] \equiv \models_{\mathscr{M}}\Phi[n]$$

for any $\mathcal{M}$ in $\mathcal{K}_0$ which contains the pair (M_0, N_0) and in which n represents N_0. In particular we would obtain

$$\models_{\mathcal{M}_0'}\Phi[2^5] \equiv \models_{\mathcal{M}_0''}\Phi[2^5] \equiv \models_{St(M_0)}\Phi[N_0].$$

We obtain thus a contradiction which shows that a formula which is not an e.e. formula cannot be absolute with respect to $\mathcal{K}_0$ ([12]).

([12]) Professor D. Scott has pointed out that theorem 8.3 can be deduced from a theorem due to J. Keisler [4]. It can be shown quite easily that if Φ is a formula absolute for an elementary class K of models of T_ω, then the class of pairs (M, N) such that $\models_{St(M)}\Phi[N]$ is closed with respect to isomorphisms and ultraproducts. The same is true for the complementary class and hence, by theorem 2.11 of [4], both classes are elementary. Theorem 8.3 is thus proved. This proof uses the generalized continuum hypothesis since so does Keisler's proof of his theorem 2.11. Professor Scott also points out that it is possible to obtain a similar proof of theorem 8.3, which does not use the continuum hypothesis if one applies another characterization of **EC** classes.

Our elementary proof of theorem 8.3 was based on results which allowed us to obtain information concerning invariant and dually invariant formulas. We do not know whether methods borrowed from the theory of models (as understood by Tarski and his school) can yield the same information. In order to clarify this question it would be desirable to investigate whether theorems 6.1 and 6.2 can be extended to the case where $\mathcal{K}$ is an $\mathbf{EC}_\delta$ class, i.e., a class of models which satisfy an arbitrary (not necessarily a recursively enumerable) set of axioms.

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CONTENTS