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L. AUSSENHOFER

Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups

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Abstract

For a topological group G, the group G^* of continuous homomorphisms (characters) into $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ endowed with the compact-open topology is called the *character group* of G and G is named (*Pontryagin*) *reflexive* if the canonical homomorphism $\alpha_G : G \to G^{**}, x \mapsto (\chi \mapsto \chi(x))$, is a topological isomorphism. A comprehensive exposition of duality theory is given here.

In particular, settings closely related to the theory of vector spaces (like local quasi-convexity and the corresponding hull) are studied and their relevance is pointed out. This is followed by an investigation of Pontryagin reflexivity of locally convex vector spaces, which generalizes the well known fact that every Banach space is a reflexive group. However, the spaces $\mathrm{L}^p([0,1])$ (for $p > 1$) contain proper closed subgroups which are not reflexive and have (topologically) the same character group as the whole space. On the other hand, every character group can be embedded into a group of the form $\mathcal{C}(X, \mathbb{T})$. It is proved that for every hemicompact k-space X (in particular, for every character group of an abelian metrizable group), this group is reflexive.

In the second part a self-contained introduction to the theory of nuclear groups (which has been introduced by W. Banaszczyk in [9]) is given. It is shown that the completion of a nuclear group is again nuclear and that the α corresponding to a complete nuclear group is surjective. In particular, every Čech-complete nuclear group is (strongly) reflexive. At the end, a simplified proof of the Bochner Theorem for nuclear groups is given.

Dedicated to Gerhard Turnwald

Introduction

For an abelian topological group G, the group G^* of continuous homomorphisms (characters) of G into the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, endowed with the compact-open topology, is named the character group of G. There is a canonical homomorphism

$$\alpha_G : G \to G^{**}, \quad x \mapsto (\chi \mapsto \chi(x)).$$

If α_G is a topological isomorphism then G is called reflexive. Obviously, every reflexive group is an abelian Hausdorff group. Examples of reflexive groups are:

- locally compact abelian groups (Pontryagin, van Kampen; [66] and [39]),
- the additive group of a Banach space (Smith; [77]), and
- products of reflexive groups (Kaplan; [40]).

The aim of this paper is to present new examples of reflexive groups and to study properties connected with reflexivity.

In Chapter 1, we introduce Nachbin–Shirota spaces, a class of topological spaces including all paracompact spaces (1.13), and give an example of a locally compact space which is not a Nachbin–Shirota space (1.21). We prove some results concerning almost metrizable and Čech-complete spaces, which will enable us to give (in Chapter 2) a characterization of abelian almost metrizable and Čech-complete groups (2.20 and 2.21).

In Chapter 3, we collect some elementary properties of homomorphism groups, among other things, a version of the Arzelà–Ascoli Theorem (3.5).

Afterwards, we specialize the situation and consider the group of continuous homomorphisms of an abelian metrizable group G into $A := \mathbb{R}^n \times \mathbb{T}^m \times D$ (where $n, m \in \mathbb{N}_0$ and D is a discrete abelian group). Then we obtain: if $n = 0$ and $D = \{e\}$ or if G is connected, then $\operatorname{Hom}(H, A) \cong \operatorname{Hom}(G, A)$ for every dense subgroup H of G (4.5), and $\operatorname{Hom}(G, A)$ is a hemicompact k-space (4.7). From that we conclude that every reflexive metrizable group must be complete.

In Chapter 5, we gather some elementary properties of reflexive groups. In addition to this, we prove decomposition results for higher character groups:

- If both α_G and α_{G^*} are continuous then $G^{***} = \alpha_{G^*}(G^*) \odot \alpha_G(G)^{\perp}$ (5.22).
- If H is a subgroup of a topological group G such that α_H is continuous, $\alpha_{G/H}$ is injective, and α_G is an open isomorphism then $H^{**} = \alpha_H(H) \odot \ker \iota^{**}$ where $\iota : H \to G$ denotes the canonical embedding (5.24).

Furthermore, we give an example of a compactly generated subgroup H of the sequence space ℓ^2 admitting a character which cannot be extended to a character of the whole group (5.27). An analogous situation cannot occur in the realm of nuclear vector spaces (cf. 20.13).

A necessary condition for a group G to be reflexive is that G is locally quasi-convex. This notion generalizes the description of closed, symmetric, convex subsets of a topological vector space obtained from the Hahn–Banach Theorem. We give proofs of the following permanence properties: subgroups, products, (arbitrary) sums and the completion of locally quasi-convex groups are again locally quasi-convex (6.8 and 6.17). On the other hand, Hausdorff quotients of locally quasi-convex groups need not be locally quasi-convex. Moreover, every abelian Hausdorff group is topologically isomorphic to the quotient of a locally quasi-convex group (12.9).

In Chapter 7, we study properties of the quasi-convex hull (this is the smallest quasi-convex set containing a given set). If G is a topological group such that α_G is injective then the quasi-convex hull of every finite set is finite (7.11), the quasi-convex hull of a singleton is smallest possible, namely $\{e, x, -x\}$ (7.8). The quasi-convex hull of a totally bounded subset in a locally quasi-convex Hausdorff group G is again totally bounded (analogously to the situation for locally convex vector spaces) and if, in addition, G is complete then the quasi-convex hull of every compact subset is compact (7.12).

In [47], Kye claimed to characterize those locally convex spaces V whose additive group is reflexive. Here, we show that the condition stated to be equivalent to the continuity of α_V is sufficient but not necessary (8.13 and 8.12). In addition, we show that the character group of a Banach space endowed with its weak topology is a reflexive group (8.26). At the end of Chapter 8 we observe that the complete metrizable sequence space ℓ^p for $0 < p < 1$ is not a locally quasi-convex group. Moreover, α_{ℓ^p} is a continuous monomorphism. In particular, ℓ^p cannot be a reflexive group (8.27). This gives a negative answer to a question asked by Pestov ([63]). (An even stronger counterexample is given in 11.15.)

In Chapter 9, we introduce locally convex vector groups, prove some of their properties, and give a new class of examples (9.12).

In Chapter 10, we show that every locally quasi-convex group is topologically isomorphic to a Hausdorff quotient of a subgroup of a locally convex vector group (10.1). Moreover, we show that every locally quasi-convex group can be embedded into a product of metrizable locally quasi-convex groups (10.6). A similar theorem for locally convex vector groups shows that every complete metrizable locally convex vector group is reflexive ((15.7) in [8] or 10.8).

This result essentially depends on the linear structure. The example given in Chapter 11 illustrates this fact: the character group of the almost everywhere integer-valued functions $\mathrm{L}^p_{\mathbb{Z}}([0,1])$ (for $p > 1$) coincides with the character group of the Banach space $\mathrm{L}^p([0,1])$ (11.14). In particular, the locally quasi-convex complete metrizable group $\mathrm{L}^p_{\mathbb{Z}}([0,1])$ is not reflexive.

In Chapter 12, we study the free abelian topological group $A(X)$ over a completely regular space X and give a neighbourhood basis for the unit element (12.6) consisting of quasi-convex sets (12.9).

If K is a compact space then the character group of $A(K)$ is topologically isomorphic to the group $\mathcal{C}(K, \mathbb{T})$ (15.1). The aim of Chapter 13 is to prove that this group is reflexive.

In Chapter 14, we generalize the above result and show that for every hemicompact k-space X the group $\mathcal{C}(X, \mathbb{T})$ endowed with the compact-open topology is reflexive (14.9). Furthermore, we give an example which illustrates that the k-space assumption cannot be omitted (14.14).

In Chapter 15, we study the reflexivity of free abelian topological groups. If X is a Nachbin–Shirota space then the character group of $A(X)$ is topologically isomorphic to $\mathcal{C}(X, \mathbb{T})$ (15.1) and $A(X)$ is reflexive if X is a punctiform (that is, every non-empty compact connected subspace is a singleton) Nachbin–Shirota space and a k-space. Conversely, if $A(X)$ is reflexive then the completely regular space X must be punctiform (15.4).

In the second part of this paper we treat nuclear groups which were introduced by W. Banaszczyk in [8]. This class includes all nuclear vector spaces (20.6(i)) and all locally compact abelian groups (20.8), it is closed under forming products, subgroups, Hausdorff quotients and countable sums (20.7).

In Chapter 16, we give a short survey of the properties of nuclear groups and stress the dichotomy between infinite-dimensional Banach spaces and nuclear vector spaces.

The definition of a nuclear group involves the Kolmogorov diameter. Therefore, we gather (in Chapters 17, 18, and 19) auxiliary results which make a treatment of nuclear groups possible.

Finally, in Chapter 20, we provide proofs for the following facts:

- Every nuclear group is locally quasi-convex ((8.5) in [8] or 20.15).
- Every character of a subgroup of a nuclear group can be extended to a character of the whole group ((8.3) in [8] or 20.13).
- A topological vector space is a nuclear group if and only if it is a nuclear vector space ((8.9) in [8] or 20.20).

The group $\mathcal{C}(X, \mathbb{T})$ is a nuclear group if and only if all compact subsets of X are finite (20.31). We show that every Čech-complete nuclear group is reflexive (20.40). Hence, only in the trivial case when K is finite, the main result of Chapter 13 is a consequence of the above theorem.

In Chapter 21, we show that every nuclear group can be embedded into a product of complete metrizable nuclear groups (21.3). As a consequence we deduce that the completion of every nuclear group is nuclear (21.4) and that for every complete nuclear group G, the canonical homomorphism α_G is an open isomorphism (21.5).

At the end, we give an analytic application. More precisely, a simpler proof of the Bochner Theorem for nuclear groups than the one given by W. Banaszczyk in [8] is presented.

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Notation

$\mathbb{N}$	$\{1, 2, \ldots\}$, the set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{T}$	the multiplicative group $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$\mathcal{U}_X(x)$	the neighbourhood filter of $x \in X$ in the topological space X
$\mathcal{C}(X, Y)$	the set of continuous functions between the topological spaces X and Y
$G = H_1 \odot H_2$	the inner direct product (2.1)
(S, U)	$\{f \in \mathcal{C}(X, Y) : f(S) \subseteq U\}$ for $S \subseteq X$ and $U \subseteq Y$
$\mathrm{Hom}(G, A)$	the group of continuous homomorphisms between abelian topological groups G and A
$\mathrm{Hom}_{\mathrm{p}}(G, A)$	the group $\mathrm{Hom}(G, A)$ endowed with the topology of uniform convergence on all finite subsets of G
$\mathrm{Hom}_{\mathrm{c}}(G, A)$	the group $\mathrm{Hom}(G, A)$ endowed with the compact-open topology
$\mathrm{Hom}_{\mathrm{tb}}(G, A)$	the group $\mathrm{Hom}(G, A)$ endowed with the topology of uniform convergence on all totally bounded subsets of G
G^*	the character group of the abelian topological group G
G^*_{p}	the character group endowed with the topology of uniform convergence on all finite subsets of G
G^*_{c}	the character group endowed with the compact-open topology
G^*_{tb}	the character group endowed with the topology of uniform convergence on all totally bounded subsets of G
α_G	$\alpha_G : G \to G^{**},\ x \mapsto (\chi \mapsto \chi(x))$ (for a topological group G)
R	$\{z \in \mathbb{T} : \mathrm{Re}\, z \geq 0\}$
$P(S, V)$	$\{\chi \in G^* : \chi(S) \subseteq V\}$ for a subset S of the abelian topological group G and $V \subseteq \mathbb{T}$
S^0	$P(S, R)$
V_n	$\{e^{2\pi i t} : \lvert t\rvert \leq 1/(4n)\}$ $(n \in \mathbb{N})$
$\mathrm{qc}(A)$	$\bigcap_{\chi \in A^0} \chi^{-1}(R)$
$\mathrm{co}(B)$	the convex hull of a subset of a vector space
$A(X)$	the free abelian topological group over X (12.1)
$L(X)$	the free locally convex vector space over X (12.4)
$\mathrm{d}_k(X, Y)$	the k-th Kolmogorov diameter of X w.r.t. Y, $k \in \mathbb{N}$ (cf. 18.1)
$\mathrm{Gl}(n, \mathbb{R})$	the group of invertible $n \times n$ matrices
B_p	$\{x \in V : p(x) \leq 1\}$ where p is a seminorm on a vector space V

1. Auxiliary results in topology

In this chapter we recall some facts on hemicompact spaces, k-spaces, Nachbin–Shirota, Dieudonné complete, almost metrizable, and Čech-complete spaces.

Definition 1.1. A Hausdorff space X is called *hemicompact* if there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets such that every compact subset of X is contained in one of them.

Let $(X,\mathcal{O})$ be a Hausdorff space. $\mathcal{O}_{\mathrm{k}}:=\{O\subseteq X : O\cap K$ is open with respect to K for every compact $K\subseteq X\}$ is called the *k-refinement* of $(X,\mathcal{O})$. The space $(X,\mathcal{O})$ is called a *k-space* if $\mathcal{O}=\mathcal{O}_{\mathrm{k}}$.

Remarks 1.2. (i) $\mathcal{O}_{\mathrm{k}}$ is a topology with $\mathcal{O}\subseteq\mathcal{O}_{\mathrm{k}}$.

(ii) The compact subsets of $(X,\mathcal{O})$ and $(X,\mathcal{O}_{\mathrm{k}})$ coincide. [For every compact subset $K\subseteq X$ (with respect to $\mathcal{O}$) the topologies induced by $\mathcal{O}$ and $\mathcal{O}_{\mathrm{k}}$ are the same.]

(iii) $(X,\mathcal{O}_{\mathrm{k}})$ is a k-space. [This follows directly from (ii).]

(iv) Every σ-compact locally compact space is hemicompact. [Considering the Aleksandrov compactification, Theorem (3.3.4) in [26] implies the assertion.]

Examples 1.3. Every metrizable and every locally compact space is a k-space. [See e.g. [26], Theorems (3.3.20) and (3.3.18) on p. 152.]

Theorem 1.4 (Whitehead). *The product of a k-space and a locally compact space is again a k-space.*

Proof. This is Theorem (3.3.27) in [26].

Proposition 1.5. *Let X be a k-space. A function f from X into a topological space Y is continuous if and only if $f|_K$ is continuous for every compact subset $K\subseteq X$.*

Proof. See e.g. [26], p. 152, Theorem (3.3.21).

Corollary 1.6. *Every continuous mapping $f:(X,\mathcal{O})\to(Y,\widetilde{\mathcal{O}})$ between Hausdorff spaces induces a continuous function $f_{\mathrm{k}}:(X,\mathcal{O}_{\mathrm{k}})\to(Y,\widetilde{\mathcal{O}}_{\mathrm{k}})$. In particular, the k-refinement of a homogeneous Hausdorff space is again a homogeneous space.*

Proof. The continuity of f_{k} is a consequence of 1.2(ii)–(iii) and 1.5. The rest of the assertion follows immediately.

Proposition 1.7. *Every hemicompact k-space is normal.*

Proof. Let A and B be disjoint closed subsets of X and $(K_n)_{n\in\mathbb{N}}$ an increasing sequence of compact subsets of X as in the definition. Put $A_n:=A\cap K_n$ and $B_n:=B\cap K_n$. Since K_1 is normal, there exists (by the Urysohn Lemma) a continuous function $f_1:K_1\to[0,1]$ such that $f_1(A_1)\subseteq\{0\}$ and $f_1(B_1)\subseteq\{1\}$. Suppose we have constructed continuous functions $f_j:K_j\to[0,1]$ $(j\in\{1,\dots,n\})$ such that $f_j|_{K_{j-1}}=f_{j-1}$ $(j\in\{2,\dots,n\})$ and $f_j(A_j)\subseteq\{0\}$ and $f_j(B_j)\subseteq\{1\}$. Since $(A_{n+1}\cup B_{n+1})\cap K_n=A_n\cup B_n$ and both $A_{n+1}\cup B_{n+1}$ and K_n are closed (in K_{n+1}) there exists, by construction, a continuous function $\widetilde{f}_{n+1}:(A_{n+1}\cup B_{n+1})\cup K_n\to[0,1]$ extending f_n such that $\widetilde{f}_{n+1}(A_{n+1})\subseteq\{0\}$ and $\widetilde{f}_{n+1}(B_{n+1})\subseteq\{1\}$. According to the Tietze extension theorem, there exists a continuous function $f_{n+1}:K_{n+1}\to[0,1]$ extending $\widetilde{f}_{n+1}$. The function $f:X\to[0,1]$ defined by

$f(x) := f_n(x)$ for $x \in K_n$ is therefore well defined. Since X is a hemicompact k-space, f is continuous. This shows that X is normal.

Corollary 1.8. *Every hemicompact k-space X is paracompact.*

Proof. According to 1.7, X is regular and it is obviously a Lindelöf space. The assertion follows from Theorem (5.1.2) in [26], p. 300.

Definition 1.9. A subset B of a topological space X is called *functionally bounded* in X if, for all $f \in \mathcal{C}(X, \mathbb{R})$, the image $f(B)$ is a bounded subset of $\mathbb{R}$.

A completely regular space X is called a *Nachbin–Shirota space* if every closed functionally bounded subset $B \subseteq X$ is compact.

In 1954, it was proved by Nachbin and Shirota (cf. [56] and [75]) that for a completely regular space X, the function space $\mathcal{C}(X, \mathbb{R})$ endowed with the compact-open topology is barrelled if and only if X is a Nachbin–Shirota space.

Remark 1.10. By Tietze's extension theorem, we have for a normal space X: a closed subset $B \subseteq X$ is functionally bounded in X if and only if it is functionally bounded in itself.

Lemma 1.11. *Let $(U_i)_{i \in I}$ be a locally finite family in the topological space X. For every compact subset $K \subseteq X$, the set $\{i \in I : U_i \cap K \neq \emptyset\}$ is finite.*

Proof. For every $x \in K$, there exists $U_x \in \mathcal{U}(x)$ such that $I_x := \{i \in I : U_x \cap U_i \neq \emptyset\}$ is finite. For a finite subset $F \subseteq K$ such that $K \subseteq \bigcup_{x \in F} U_x$ we get $\{i \in I : K \cap U_i \neq \emptyset\} \subseteq \bigcup_{x \in F} I_x$. Hence the assertion follows.

Lemma 1.12. *Let $(U_i)_{i \in I}$ be a locally finite family in the topological space X and let $f_i : X \to \mathbb{R}$ be continuous functions satisfying $f_i(X \setminus U_i) \subseteq \{0\}$. Then $f := \sum_{i \in I} f_i$ is also continuous.*

Proof. Let $x \in X$ and let $U \in \mathcal{U}(x)$ be open such that $U \cap U_i = \emptyset$ for all $i \in I \setminus F$ where F is a finite subset of I. For $y \in U$ we have $f_i(y) = 0$ for all $i \in I \setminus F$ and hence $f|_U = (\sum_{i \in F} f_i)|_U$. This shows that f is continuous.

Proposition 1.13. *Every paracompact space X is a Nachbin–Shirota space.*

Proof. Since X is normal (cf. Theorem (5.1.5) in [26], p. 300), we may assume that X is functionally bounded (1.10).

Let $(U_j)_{j \in J}$ be an open cover of X. By assumption, there exists an open, locally finite refinement $\mathcal{V} = (V_i)_{i \in I}$. It suffices to show that $\mathcal{V}$ has a finite subcover. Suppose the contrary. Inductively, we construct sequences $(V_n)_{n \in \mathbb{N}}$ in $\mathcal{V}$, $(x_n)_{n \in \mathbb{N}}$ in X, $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{C}(X, \mathbb{R})$ and a sequence of open sets $(W_n)_{n \in \mathbb{N}}$ satisfying the following conditions:

(i) $x_n \in W_n \subseteq V_n$,
(ii) $\overline{W}_n$ intersects only finitely many V_i non-trivially,
(iii) $W_n \cap (\bigcup_{k<n} W_k) = \emptyset$,
(iv) $f_n(x_n) = n$, and $X \setminus W_n \subseteq f_n^{-1}(\{0\})$.

Suppose the first $n \in \mathbb{N}_0$ members of the sequences have been chosen. Since $\bigcup_{k \leq n} \overline{W}_k$ intersects only finitely many V_i non-trivially and, by assumption, $\mathcal{V}$ has no finite subcover, there exists $x_{n+1} \in X \setminus \bigcup_{k \leq n} \overline{W}_k$. Choose $V_{n+1} \in \mathcal{V}$ containing x_{n+1}. By hypothesis, there exists an open W_{n+1} satisfying (i) to (iii). Since X is completely regular, there exists a continuous function f_{n+1} satisfying condition (iv).

Because of (i), $(W_n)_{n\in\mathbb{N}}$ is a locally finite family of open subsets, and it is a consequence of (iv) and 1.12 that $f := \sum_{n\in\mathbb{N}} f_n$ is (well defined and) continuous. But $f(X)$ is not bounded, which contradicts our hypothesis.

EXAMPLES 1.14. (i) Every metrizable space is paracompact. [See e.g. [26], p. 300, Theorem (5.1.3).]

(ii) Every σ-compact locally compact space is paracompact. [Every σ-compact locally compact space is a hemicompact k-space. (This follows from 1.2,(iv) and 1.3.) According to 1.8, it is paracompact.]

PROPOSITION 1.15. *The category of Nachbin–Shirota spaces is closed under forming closed subspaces, products, and topological sums.*

PROOF. Let X_0 be a closed subspace of the Nachbin–Shirota space X and let B be a closed and functionally bounded subset of X_0. Then B is functionally bounded in X since, otherwise, it could not be functionally bounded in X_0. By assumption, B is compact.

Let X_i ($i \in I$, where I is a non-empty index set) be Nachbin–Shirota spaces and let B be a closed and functionally bounded subset of $X := \prod_{i\in I} X_i$. By the Tikhonov theorem, it suffices to show that $\overline{\pi_i(B)}$ is a compact subset of X_i if $\pi_i : X \to X_i$ denotes the canonical projection. By assumption, it suffices to show that $\overline{\pi_i(B)}$ is functionally bounded in X_i (for all $i \in I$). For $f_i \in \mathcal{C}(X_i, \mathbb{R})$ the composition $f_i \circ \pi_i$ belongs to $\mathcal{C}(X, \mathbb{R})$ and, by the continuity of f_i, the image $f_i(\overline{\pi_i(B)}) \subseteq \overline{f_i(\pi_i(B))}$ is bounded.

Let X_i ($i \in I$) be Nachbin–Shirota spaces and let $B = \bigcup_{i\in I} B_i$ be a closed functionally bounded subset of $\bigcup_{i\in I} X_i$. If infinitely many B_i are non-empty, then B cannot be functionally bounded. Since all B_i are closed and functionally bounded in X_i, they are compact and so is the finite union of the non-empty B_i.

REMARKS 1.16. (i) For every subset $A \subseteq \mathbb{R}$ which is not (functionally) bounded, there exists an infinite discrete family $(U_i)_{i\in I}$ of open sets such that $U_i \cap A \neq \emptyset$ for all $i \in I$.

(ii) Inverse images of discrete families of open sets under continuous mappings are discrete families of open sets.

(iii) For every subset A of a topological space which is not functionally bounded, there exists an infinite discrete family $(U_i)_{i\in I}$ of open sets such that $U_i \cap A \neq \emptyset$ for all $i \in I$. [This is a direct consequence of (i) and (ii).]

DEFINITION 1.17. A completely regular space X is *Dieudonné complete* if its topology is induced by a complete uniformity.

REMARK 1.18. It was shown in [23] that every Dieudonné complete space is homeomorphic to a closed subspace of a product of metrizable spaces. Hence, according to 1.14(i), 1.13 and 1.15, every Dieudonné complete space is a Nachbin–Shirota space. Below we give an elementary proof.

PROPOSITION 1.19. *Every Dieudonné complete space is a Nachbin–Shirota space.*

Proof. Let $\mathcal{U}$ be a complete uniformity on X. Since a subset of X is compact if and only if it is closed and totally bounded (cf. Theorem (8.3.16) in [26]), we have to show that every functionally bounded subset of X is totally bounded. Assume that $A \subseteq X$ is not totally bounded. This means: There exists a (symmetric) member U of the uniformity such that for every finite subset $F \subseteq X$ we have $A \not\subseteq \{y \in X : \exists x \in F$ such that $(x,y) \in U\}$. Inductively, we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in A with $x_n \notin \bigcup_{k=1}^{n-1}\{x \in X : (x_k, x) \in U\}$. There exists a symmetric member W of $\mathcal{U}$ such that $W^4 \subseteq U$. Hence $W_n := \{x \in X : (x, x_n) \in W\}$ are disjoint neighbourhoods of the x_n. Moreover, $(W_n)_{n\in\mathbb{N}}$ forms a discrete family in X. [For $x \in X$ and $W_x := \{y \in X : (x,y) \in W\}$ we get $|\{n \in \mathbb{N} : W_x \cap W_n \neq \emptyset\}| \leq 1$. Indeed, suppose there exist $y_n \in W_x \cap W_n$ and $y_m \in W_x \cap W_m$. This means $(x_n, y_n) \in W$, $(y_n, x) \in W$, $(x, y_m) \in W$ and $(y_m, x_m) \in W$, which implies $(x_n, x_m) \in W^4 \subseteq U$ and hence $n = m$.]

Furthermore, there are continuous functions $f_n \in \mathcal{C}(X,\mathbb{R})$ such that $f_n(x_n) = n$ and $f_n(X \setminus W_n) \subseteq \{0\}$. According to 1.12, $f := \sum_{n\in\mathbb{N}} f_n$ is a continuous function such that $f(A)$ is not bounded. This completes the proof.

Lemma 1.20. *Let* ω_1 *denote the first uncountable ordinal number, put* $\Omega_1 := \{\omega \leq \omega_1\}$ *and set* $X := (\Omega_1 \times \Omega_1) \setminus \{(\omega_1, \omega_1)\}$. *Every equicontinuous pointwise bounded family* S *in* $\mathcal{C}(X,\mathbb{R})$ *is the restriction to* X *of an equicontinuous pointwise bounded family* $\widetilde{S}$ *in* $\mathcal{C}(\Omega_1 \times \Omega_1, \mathbb{R})$.

Proof. Consider

$$(*)\qquad \forall n \in \mathbb{N}\ \exists \gamma_n < \omega_1 \text{ such that } |s(\gamma_n,\gamma_n) - s(\alpha,\alpha')| \leq 1/n\ \forall \gamma_n \leq \alpha, \alpha' < \omega_1 \text{ and } \forall s \in S.$$

Suppose $(*)$ is correct. If γ_0 $(< \omega_1)$ denotes the least upper bound of the γ_n, we get $s(\alpha,\alpha') = s(\gamma_0,\gamma_0)$ for all $\alpha,\alpha' \geq \gamma_0$ and all $s \in S$. This implies the assertion.

If $(*)$ does not hold, then there exists $n_0 \in \mathbb{N}$ such that for all $\gamma < \omega_1$ there are $\alpha, \alpha' \geq \gamma$ and $s_\gamma \in S$ such that $|s_\gamma(\alpha,\alpha') - s_\gamma(\gamma,\gamma)| \geq 1/n_0$. Inductively, we can find an increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ and sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\alpha'_n)_{n\in\mathbb{N}}$ such that $\gamma_n \leq \alpha_n, \alpha'_n$ and $\gamma_{n+1} \geq \alpha_n, \alpha'_n$ and satisfying $|s_n(\alpha_n,\alpha'_n) - s_n(\gamma_n,\gamma_n)| \geq 1/n_0$ (for all $n \in \mathbb{N}$) for suitable $s_n \in S$. Let γ_0 be the least upper bound for $\gamma_n, \alpha_n, \alpha'_n$. Every neighbourhood of (γ_0,γ_0) in X contains almost all (γ_n,γ_n) and (α_n,α'_n). Since $\{(\alpha,\alpha') : |s(\alpha,\alpha') - s(\gamma_0,\gamma_0)| < 1/(2n_0)\ \forall s \in S\}$ is a neighbourhood of (γ_0,γ_0) which contains at most one of the points (α_n,α'_n) or (γ_n,γ_n), this leads to the desired contradiction.

Example 1.21. There exists a locally compact space X which is not a Nachbin–Shirota space. [Let X be as in 1.20. It is well known that $\Omega_1 \setminus \{\omega_1\}$ is not compact and hence X is not compact. According to 1.20, X is functionally bounded (in itself). Hence X is a locally compact space which is not a Nachbin–Shirota space.]

Definition 1.22. A Hausdorff space X is called *almost metrizable* if every $x \in X$ is contained in a compact subset having a countable neighbourhood basis in X.

Examples 1.23. (i) Every metrizable and every compact space is almost metrizable.

(ii) It is a consequence of the Aleksandrov compactification that every locally compact space, being an open subset of a compact space, is Čech-complete (which means it can be represented as a countable intersection of open subsets of a compact space).

Proposition 1.24. *Every almost metrizable space is a k-space.*

Proof. Let A be a subset of an almost metrizable space X such that the intersection of A with every compact subset of X is compact. Suppose there exists $x \in \overline{A} \setminus A$. By assumption, there is a compact set K having a countable neighbourhood basis $(U_n)_{n\in\mathbb{N}}$ such that $x \in K$. Since $A \cap K$ is compact and $x \notin A \cap K$, there exists $U \in \mathcal{U}_X(x)$ such that $\emptyset = \overline{U} \cap A \cap K$. For every $n \in \mathbb{N}$, there exists $x_n \in A \cap U \cap \bigcap_{j=1}^{n} U_j \subseteq A \cap \overline{U}$. This implies $x_n \notin K$. For $B := \{x_n : n \in \mathbb{N}\}$, the set $K \cup B$ is compact. [Every open cover of $K \cup B$ contains a finite family that covers K and which therefore contains $U_n \supseteq \{x_m : m \geq n\}$ for suitable $n \in \mathbb{N}$. This implies that $K \cup B$ is compact.] Furthermore, $B = A \cap \overline{U} \cap B = (A \cap \overline{U} \cap K) \cup (A \cap \overline{U} \cap B) = A \cap \overline{U} \cap (K \cup B)$ is, by assumption, compact. Hence $X \setminus B$ is an open superset of K. This leads to the contradiction $U_n \subseteq X \setminus B$ for suitable $n \in \mathbb{N}$ and $x_n \in B \cap U_n$.

Proposition 1.25. (i) *Every Čech-complete space is a Baire k-space.*

(ii) *A metrizable space is completely metrizable if and only if it is Čech-complete.*

(iii) *Every countable product of Čech-complete spaces is Čech-complete.*

Proof. (i) follows from (3.9.3) and (3.9.5) in [26] and (ii) is a consequence of (4.3.26) in [26]. (iii) is Theorem (3.9.8) in [26].

Proposition 1.26. *Let X and Y be completely regular spaces and let f be a perfect mapping of X onto Y. Then X is Čech-complete if and only if Y is Čech-complete.*

Proof. This is Theorem (3.9.10) in [26].

Lemma 1.27. *Let $(K_i)_{i\in I}$ be a family of compact subsets of the topological space X and let O be an open superset of $\bigcap_{i\in I} K_i$. Then there exists a finite subset $F \subseteq I$ such that $\bigcap_{i\in F} K_i \subseteq O$.*

Proof. $\bigcap_{i\in I} K_i \cap (X \setminus O) = \emptyset$ implies the existence of a finite set $F \subseteq I$ such that $\bigcap_{i\in F} K_i \cap (X \setminus O) = \emptyset$. The assertion follows.

Lemma 1.28. *Let K be a compact subset of the Hausdorff space X having a countable neighbourhood basis $(U_n)_{n\in\mathbb{N}}$. For an arbitrary $x \in K$ and every sequence $(W_n)_{n\in\mathbb{N}}$ in $\mathcal{U}_X(x)$, there exists a sequence $(\widetilde{W}_n)_{n\in\mathbb{N}}$ in $\mathcal{U}_X(x)$ such that*

(i) $\widetilde{W}_{n+1} \subseteq \widetilde{W}_n$,

(ii) $K \cap \overline{\widetilde{W}_{n+1}} \subseteq K \cap \widetilde{W}_n$, *and*

(iii) $\widetilde{W}_n \subseteq U_n \cap W_n$.

For each such sequence $(\widetilde{W}_n)_{n\in\mathbb{N}}$, the set $K_0 := \bigcap_{n\in\mathbb{N}} \widetilde{W}_n$ is a compact subset of $\bigcap_{n\in\mathbb{N}} W_n$, $(\widetilde{W}_n)_{n\in\mathbb{N}}$ is a neighbourhood basis of K_0, and $x \in K_0 = \bigcap_{n\in\mathbb{N}}(K \cap \overline{\widetilde{W}_n}) = \bigcap_{n\in\mathbb{N}} \overline{\widetilde{W}_n}$.

Proof. Suppose first that we already have a sequence $(\widetilde{W}_n)_{n\in\mathbb{N}}$ in $\mathcal{U}_X(x)$ with the properties (i) to (iii). Then $K_0 = \bigcap_{n\in\mathbb{N}} \widetilde{W}_n \overset{\text{(iii)}}{\subseteq} \bigcap_{n\in\mathbb{N}}(U_n \cap W_n) \subseteq K$ implies $K_0 = \bigcap_{n\in\mathbb{N}}(K \cap \widetilde{W}_n) \overset{\text{(ii)}}{=} \bigcap_{n\in\mathbb{N}}(K \cap \overline{\widetilde{W}_n})$. So K_0 is compact. (iii) implies $K_0 \subseteq \bigcap_{n\in\mathbb{N}} W_n$. Since $K \cap \overline{\widetilde{W}_n} \downarrow K_0$, it is a consequence of 1.27 that for every $U \in \mathcal{U}_X(K_0)$ there exists an $n_0 \in \mathbb{N}$ such that $K \cap \overline{\widetilde{W}_n} \subseteq U$ for all $n \geq n_0$. Hence $K \subseteq U \cup (X \setminus \overline{\widetilde{W}_{n_0}})$. By assumption, there exists $k \geq n_0$ such that $U_k \subseteq U \cup (X \setminus \overline{\widetilde{W}_{n_0}})$ and hence $\widetilde{W}_k \subseteq U_k \subseteq U \cup (X \setminus \widetilde{W}_{n_0})$, which implies $\widetilde{W}_k \subseteq U$. This shows that $(\widetilde{W}_n)_{n\in\mathbb{N}}$ forms a neighbourhood basis of K_0. Since X is a Hausdorff space and K_0 is compact, every $y \in X \setminus K_0$ can be separated from K_0 by open sets. This implies that the intersection of all closed neighbourhoods of K_0 equals K_0. So we get $K_0 = \bigcap_{n\in\mathbb{N}} \overline{\widetilde{W}_n}$.

It remains to construct a sequence $(\widetilde{W}_n)_{n\in\mathbb{N}}$ satisfying (i) to (iii). Choose an arbitrary $\widetilde{W}_1 \in \mathcal{U}_X(x)$ contained in $U_1 \cap W_1$. Suppose now that $\widetilde{W}_1, \ldots, \widetilde{W}_n$ have already been constructed. Since $x \in K$ and X is a Hausdorff space, we have $\widetilde{W}_n \supseteq \{x\} = \bigcap_{V\in\mathcal{U}_X(x)} \overline{V} = \bigcap_{V\in\mathcal{U}_X(x)} (\overline{V} \cap K)$. As a consequence of 1.27, we get $\overline{V} \cap K \subseteq \widetilde{W}_n$ for a suitable $V \in \mathcal{U}_X(x)$. This implies that $\widetilde{W}_{n+1} := V \cap U_{n+1} \cap W_{n+1} \cap \widetilde{W}_n$ has the desired properties.

Proposition 1.29. *Every closed subspace and every union of G_δ subsets of an almost metrizable space X is again almost metrizable.*

Proof. Let A be a closed subspace of X and $x \in A$. Let $K \subseteq X$ be a compact set containing x and having a countable neighbourhood basis $(U_n)_{n\in\mathbb{N}}$ in X. Then x belongs to the compact set $A \cap K$ and $(A \cap U_n)_{n\in\mathbb{N}}$ is a neighbourhood basis of $A \cap K$. [Let $U \subseteq X$ be open such that $A \cap K \subseteq U$. Then $K \subseteq U \cup (X \setminus A)$ and hence (by assumption) $U_n \subseteq U \cup (X \setminus A)$ (for suitable $n \in \mathbb{N}$). This implies $A \cap U_n \subseteq U \cap A$.] The first assertion follows.

To prove the second, consider open subsets W_n of X ($n \in \mathbb{N}$) and $x \in \bigcap_{n\in\mathbb{N}} W_n$. Let $K \subseteq X$ be a compact set containing x and having a countable neighbourhood basis $(U_n)_{n\in\mathbb{N}}$ in X. It is a consequence of 1.28 that x belongs to a compact set $K_0 \subseteq \bigcap_{n\in\mathbb{N}} W_n$ which has a countable neighbourhood basis in X and hence in every subspace containing K_0. This implies that any union of G_δ subsets of X is almost metrizable.

Corollary 1.30. *Every Čech-complete and*, *in particular*, *every locally compact space is almost metrizable.*

Proof. This follows from the definition of Čech-complete spaces, 1.23(ii), and 1.29.

Theorem 1.31. *Every continuous open mapping $f : X \to Y$ of a Čech-complete space X onto a Hausdorff space Y is compact-covering.* (*This means that every compact subset of Y is the image of a compact subset of X under f.*)

Proof. This is Theorem 1.2 in [3], p. 207.

Theorem 1.32 (Arzelà–Ascoli). *Let C be a subset of the function space $\mathcal{C}(X,Y)$ endowed with the compact-open topology where X is a k-space and $(Y,\mathcal{U})$ is a uniform space. C is compact if and only if it is closed*, *equicontinuous*, *and for every $x \in X$ the set $\{f(x) : f \in C\}$ is contained in a compact subset of Y.*

Proof. See [26], p. 443, Theorem (8.2.10).

Notes 1.33. 1.7 was proved by Warner (cf. [86], p. 267, Lemma 1.2). Nachbin and Shirota mentioned that paracompact spaces are Nachbin–Shirota spaces (cf. [56] and [75]). The proof given here was motivated by Proposition 2 in [4]. The formulations of 1.19 and Example 1.21 were pointed out to me by K. Yamada. The proof of 1.20 is similar to Example (3.1.27) in [26], p. 130. The statements on Čech-complete and almost metrizable spaces are a modification of a manuscript by G. Turnwald. Literature concerning this topic can be found in [19] and [69], Chapter 13.

2. Auxiliary results for topological groups

In this chapter we collect some basic facts on topological groups. Besides that, we use the results on almost metrizable and Čech-complete spaces proved in the last section to give a characterization of almost metrizable and of Čech-complete groups (2.20 and 2.21).

Definition 2.1. Let H_1, H_2 be subgroups of a topological group G. If $H_1 \times H_2 \to G$, $(x_1, x_2) \mapsto x_1 + x_2$, is a topological isomorphism then G is named the *inner direct sum* of H_1 and H_2. We write $G = H_1 \oplus H_2$. (If the operation of G is written multiplicatively then G is said to be the *inner direct product* of H_1 and H_2 and we use the notation $G = H_1 \odot H_2$.)

Theorem 2.2. *Let K be a closed subgroup of $\mathbb{R}^n$. There exists a maximal subspace V contained in K. Moreover, for every linear subspace $W \leq \mathbb{R}^n$ such that $\mathbb{R}^n = V \oplus W$ we have $K = V \oplus (K \cap W)$ and $K \cap W$ is a discrete subgroup of $\mathbb{R}^n$.*

Proof. This is Theorem 2 in Chapter VII, §1.2 of [18].

Proposition 2.3. *Let H be a subgroup of an abelian group G and let D be an abelian divisible group. For every homomorphism $\varphi : H \to D$ there exists a homomorphism $\varphi' : G \to D$ extending φ.*

Proof. This is (A.7) in [34].

Lemma 2.4. *Let H_1 and H_2 be subgroups (linear subspaces) of an abelian group (of a vector space). For all homomorphisms (linear functions) $\varphi_j : H_j \to G$ into an abelian group (a vector space) G $(j \in \{1, 2\})$ satisfying $\varphi_1|_{H_1 \cap H_2} = \varphi_2|_{H_1 \cap H_2}$ the function $\varphi : H_1 + H_2 \to G$, $h_1 + h_2 \mapsto \varphi_1(h_1) + \varphi_2(h_2)$ $(h_j \in H_j)$, is a well defined homomorphism (linear function) extending φ_1 and φ_2.*

Proof. This follows by a straightforward calculation.

Lemma 2.5. *Let G be an abelian group and let $\mathcal{U}$ be a non-empty subset of the power set of G satisfying*

(i) *$e \in U$ for all $U \in \mathcal{U}$,*
(ii) *$\forall U \in \mathcal{U}$ $\exists V \in \mathcal{U}$ such that $-V \subseteq U$,*
(iii) *$\forall U \in \mathcal{U}$ $\exists V \in \mathcal{U}$ such that $V + V \subseteq U$, and*
(iv) *$\forall U, V \in \mathcal{U}$ $\exists W \in \mathcal{U}$ such that $W \subseteq U \cap V$.*

Then there exists a unique group topology $\mathcal{O}$ on G such that $\mathcal{U}$ is a neighbourhood basis of the unit element. $(G, \mathcal{O})$ is a Hausdorff space if and only if $\bigcap_{U \in \mathcal{U}} U = \{e\}$.

Proof. See e.g. [17], Chapter III, §1.2, Proposition 1 on p. 222.

Proposition 2.6. *For every Hausdorff space X and every abelian Hausdorff group A, the group $\mathcal{C}(X, A)$ of continuous functions endowed with the compact-open topology $\mathcal{O}_{\mathrm{co}}$*

is an abelian Hausdorff group. The sets of the form

$$(K,U) := \{f \in \mathcal{C}(X,A) : f(K) \subseteq U\},$$

where K *is a compact subset of* X *and* $U \in \mathcal{U}_A(e)$, *form a neighbourhood basis of the unit element of* $\mathcal{C}(X,A)$.

Proof. According to 2.5, the sets (K,U), where $K \subseteq X$ is a compact subset and $U \in \mathcal{U}_A(e)$, form a neighbourhood basis for a group topology $\mathcal{O}_{\text{gr}}$ on $\mathcal{C}(X,A)$. It suffices to show that $\mathcal{O}_{\text{co}} = \mathcal{O}_{\text{gr}}$.

Let $f \in \bigcap_{j=1}^{n}(K_j,O_j)$ where $n \in \mathbb{N}$, $K_j \subseteq X$ is compact and $O_j \subseteq A$ is open. There exists a symmetric neighbourhood $V \in \mathcal{U}_A(e)$ such that $f(K_j) + V \subseteq O_j$ for all $j \in \{1,\dots,n\}$. Then

$$f \in f + \Big(\bigcup_{j=1}^{n} K_j, V\Big) \subseteq \bigcap_{j=1}^{n}(K_j,O_j).$$

[For $f' \in (\bigcup_{j=1}^{n} K_j, V)$ and $x \in K_j$ we have $(f+f')(x) \in f(K_j) + V \subseteq O_j$.]

Conversely, let $f \in \mathcal{C}(X,A)$, let $K \subseteq X$ be compact, and let $V \in \mathcal{U}_A(e)$ be open and symmetric. There exists an open symmetric $\widetilde{V} \in \mathcal{U}_A(e)$ such that $\widetilde{V} + \widetilde{V} \subseteq V$. Furthermore, for $x \in K$, there exists $U_x \in \mathcal{U}_K(x)$ such that $f(x) - f(x') \in \widetilde{V}$ for all $x' \in \overline{U}_x$. For finite $F \subseteq K$ such that $K \subseteq \bigcup_{x \in F} U_x$, we get

$$f \in \bigcap_{x \in F}(K \cap \overline{U}_x, f(x) + \widetilde{V}) \subseteq f + (K,V).$$

[For $f' \in \bigcap_{x \in F}(K \cap \overline{U}_x, f(x) + \widetilde{V})$ and $x' \in K$, there exists $x \in F$ such that $x' \in U_x$. Hence $(f'-f)(x') \in f(x) + \widetilde{V} - f(x') \in \widetilde{V} + \widetilde{V} \subseteq V$.]

Proposition 2.7. *A topological group is metrizable if and only if it is a Hausdorff space and the unit element has a countable neighbourhood basis.*

Proof. See (8.3) in [34].

Proposition 2.8. *For every hemicompact space* X *and every abelian metrizable group* A, *the group* $G := \mathcal{C}(X,A)$ (*endowed with the compact-open topology*) *is metrizable.*

Proof. Because of 2.7, it is sufficient to show that G is a Hausdorff space having a countable neighbourhood basis of e.

Let $f_1, f_2 : X \to A$ be continuous functions and let $x_0 \in X$ be such that $f_1(x_0) \neq f_2(x_0)$. Since A is a Hausdorff space, there exist disjoint open neighbourhoods $U_j \in \mathcal{U}_A(f_j(x_0))$ ($j \in \{1,2\}$). Hence $(\{x_0\}, U_j)$ (for $j \in \{1,2\}$) are disjoint open neighbourhoods of f_1 and f_2.

Let $(K_n)_{n \in \mathbb{N}}$ be an increasing sequence of compact subsets of X (as in the definition) and let $(U_m)_{m \in \mathbb{N}}$ be a countable neighbourhood basis of e in A. It is easy to see that the sets (K_n, U_m) (where $n, m \in \mathbb{N}$) form a neighbourhood basis of the unit element of G. Hence the assertion follows.

Notation 2.9. Let $(G_i)_{i \in I}$ (where I is a non-empty index set) be a family of abelian Hausdorff groups and put

$$G := \sum_{i\in I} G_i = \Big\{(x_i)_{i\in I} \in \prod_{i\in I} G_i : x_i = e \text{ for almost all } i\Big\}.$$

Fix $U_i \in \mathcal{U}_{G_i}(e)$ and let

$$U(U_i)_{i\in I} := \Big\{(x_i)_{i\in I} \in \sum_{i\in I} G_i\text{: } \exists F \subseteq I \text{ finite: } x_i = e\ \forall i \notin F$$
$$\forall i \in F\ \exists k_i \in \mathbb{N}\text{: } kx_i \in U_i\ \forall k \in \{1,\dots,k_i\} \text{ and } \sum_{i\in F} \frac{1}{k_i} < 1\Big\}.$$

It was proved in [40] (pp. 652ff.) that the above sets form a neighbourhood basis of e of a Hausdorff group topology on G (cf. 2.5). It is called the *asterisk topology* and is, in general, finer than the rectangular topology on G. If I is countable, it coincides with the rectangular topology ([40], p. 654).

Remark 2.10. The asterisk topology on sums of Hausdorff groups is the analog of the locally convex direct sum of locally convex vector spaces. [Cf. [74], p. 55.]

Notation 2.11. A net $(x_j)_{j\in J}$ in an abelian topological group G is named a *Cauchy net* if for every $U \in \mathcal{U}_G(e)$ there exists $j_0 \in J$ such that $x_j - x_{j'} \in U$ for all $j, j' \geq j_0$. An abelian Hausdorff group G is called *complete* if every Cauchy net converges. (This is equivalent to: the uniformity generated by the neighbourhoods of e is complete.)

Remark 2.12. The topology of a complete abelian metrizable group is generated by a complete metric. [Cf. Theorem (8.21) in [34], p. 68.]

Theorem 2.13. *For every abelian Hausdorff group G, there exists a complete abelian Hausdorff group $\widetilde{G}$ and a homomorphic embedding $\varphi : G \to \widetilde{G}$ such that $\varphi(G)$ is dense in $\widetilde{G}$. Up to topological isomorphism, $\widetilde{G}$ is unique. If G is metrizable, so is $\widetilde{G}$.*

$\widetilde{G}$ is called the *completion* of G.

Proof. See e.g. Theorem (10.15) in [69], p. 181.

Lemma 2.14. *Every completely metrizable subset M of a metrizable space X is a G_δ subset of X.*

Proof. See e.g. Theorem (4.3.24) in [26].

Corollary 2.15. *The intersection of two dense completely metrizable subsets of a nonempty metrizable space X is not empty.*

Proof. Since every completely metrizable space is a Baire space and 2.14 holds, it is sufficient to show that the intersection of a dense Baire space $B \subseteq X$ with a dense G_δ subset $A \subseteq X$ is not empty.

Let $U_n \subseteq X$ be open subsets such that $A = \bigcap_{n\in\mathbb{N}} U_n$. Since B and U_n are dense in X, the intersections $B \cap U_n$ are dense open subsets of B. Hence $A \cap B = \bigcap_{n\in\mathbb{N}}(U_n \cap B)$ is not empty.

Theorem 2.16. *If G is an abelian metrizable group such that its topology is induced by a complete (not necessarily invariant) metric ϱ, then G is a complete group.*

Proof. Let $\widetilde{G}$ denote the completion of G, which is metrizable according to 2.13. We are to show that $G = \widetilde{G}$.

We observe that for every $x \in \widetilde{G}$ the completely metrizable space xG is dense in $\widetilde{G}$. Corollary 2.15 implies that $xG \cap G \neq \emptyset$ for every $x \in \widetilde{G}$, which means that $x \in G$. Hence the assertion follows.

Proposition 2.17. *Every continuous homomorphism* $\varphi : H_1 \to G_2$, *where* H_1 *is a subgroup of the abelian Hausdorff group* G_1 *and* G_2 *is a complete abelian Hausdorff group, can be extended to a continuous homomorphism* $\overline{H}_1 \to G_2$.

Proof. The idea of the proof is pointed out in (23.30) in [34], p. 369.

Proposition 2.18. *Let* K *be a compact subgroup of the abelian Hausdorff group* G. *The canonical projection* $\pi : G \to G/K$ *is perfect and closed, and the inverse image under* π *of each compact subset of* G/K *is a compact subset of* G.

Proof. This is a consequence of (5.18) in [34] and (3.7.2) in [26].

Remark 2.19. Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing set of symmetric neighbourhoods of a topological group such that $U_{n+1} + U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} U_n$ is a subgroup.

Proposition 2.20. *An abelian Hausdorff group* G *is an almost metrizable space if and only if there exists a compact subgroup* $H \leq G$ *such that* G/H *is metrizable.*

Proof. Assume first that G is almost metrizable. There exists a compact set K containing e which has a countable neighbourhood basis $(U_n)_{n\in\mathbb{N}}$. It is possible to find symmetric neighbourhoods $W_n \in \mathcal{U}_G(e)$ such that $KW_n \subseteq U_n$ (cf. (4.10) in [34]) and $W_{n+1}^2 \subseteq W_n = W_n^{-1}$. Then we get $\overline{W}_{n+1} \subseteq W_n$ and $H := \bigcap_{n\in\mathbb{N}} W_n = \bigcap_{n\in\mathbb{N}} \overline{W}_n$ is a closed subgroup of G (2.19). As a consequence of 1.28 (setting $W_n = G = X$), H is compact and $(W_n)_{n\in\mathbb{N}}$ is a neighbourhood basis of H. Let $\pi : G \to G/H$ denote the canonical projection. For each open neighbourhood $W \in \mathcal{U}_{G/H}(e)$ the inverse image satisfies $H \subseteq \pi^{-1}(W)$ and hence $W_n \subseteq \pi^{-1}(W)$ for suitable $n \in \mathbb{N}$. Since π is open, the sets $(\pi(W_n))_{n\in\mathbb{N}}$ form a neighbourhood basis of $e_{G/H}$. Now, 2.7 implies that G/H is metrizable.

Conversely, assume there exists a compact subgroup H such that G/H is metrizable. Let $\pi : G \to G/H$ denote the canonical projection and let $(W_n)_{n\in\mathbb{N}}$ be a countable neighbourhood basis of $\pi(e)$. For an open set W including H we get $\pi(W) \in \mathcal{U}_{G/H}(e)$, since π is open. Hence $W_n \subseteq \pi(W)$ for suitable $n \in \mathbb{N}$. This shows that $(\pi^{-1}(W_n))_{n\in\mathbb{N}}$ forms a neighbourhood basis of H. The assertion follows, since G is homogeneous.

Corollary 2.21. *An abelian Hausdorff group* G *is Čech-complete if and only if there exists a compact subgroup* H *such that* G/H *is complete and metrizable.*

Proof. By assumption, or according to 1.30 and 2.20, there exists a compact subgroup H of G such that G/H is metrizable. The canonical homomorphism $\pi : G \to G/H$ is perfect (2.18). Hence G is Čech-complete if and only if G/H is Čech-complete (1.26). The assertion follows from 1.25(ii) and 2.12 and 2.16.

Notes 2.22. 2.6 generalizes Lemma 1.7 in [59]. The characterization of Čech-complete and of almost metrizable groups as well as 2.16 and 2.15 (due to Pasynkov [60] and [61]; cf. also [19]) are taken from a manuscript of G. Turnwald.

3. Elementary properties of homomorphism groups

The aim of this chapter is to establish some elementary facts on homomorphism groups. Assume that $(G,+)$ and $(A,+)$ are abelian topological groups and further that A is a Hausdorff space.

Let $\mathrm{Hom}(G,A)$ denote the set of all continuous homomorphisms $G \to A$. Let $\mathcal{S}$ be a family of subsets of G which is closed under forming finite unions. According to 2.5, the sets of the form

$$P(S,V) := \{\chi \in \mathrm{Hom}(G,A) : \chi(S) \subseteq V\}$$

where $S \in \mathcal{S}$ and $V \in \mathcal{U}_A(e)$ form a neighbourhood basis of the unit element for a group topology on $\mathrm{Hom}(G,A)$; it is named the *topology of uniform convergence on all elements of* $\mathcal{S}$.

This topology is Hausdorff if $\bigcup_{S \in \mathcal{S}} S = G$.

By $\mathrm{Hom}_\mathrm{p}(G,A)$ $(\mathrm{Hom}_\mathrm{tb}(G,A))$ we denote the homomorphism group endowed with the topology of uniform convergence on all finite (totally bounded) subsets of G. By $\mathrm{Hom}_\mathrm{c}(G,A)$ we denote the group endowed with the compact-open topology. (Even if G is not a Hausdorff space, the compact-open topology is a Hausdorff topology on $\mathrm{Hom}(G,A)$ and the group operations are continuous with respect to this topology.) If no confusion can arise, we write $\mathrm{Hom}(G,A)$ instead of $\mathrm{Hom}_\mathrm{c}(G,A)$.

REMARK 3.1. (i) If G is a Hausdorff group then the compact-open topology on $\mathrm{Hom}(G,A)$ coincides with the topology of uniform convergence on all compact subsets of G. [Since $\mathrm{Hom}(G,A)$ is a subgroup of $\mathcal{C}(G,A)$, this is a consequence of 2.6.]

(ii) $\mathrm{Hom}_\mathrm{p}(G,A)$ can be considered as a subgroup of A^G (endowed with the product topology). [Let G_d denote the group G endowed with the discrete topology. Since $\mathrm{Hom}_\mathrm{p}(G,A)$ can be identified with a subgroup of $\mathrm{Hom}_\mathrm{c}(G_\mathrm{d},A)$, the assertion follows from (i).]

NOTATION 3.2. Suppose now that A is a locally compact abelian group which has a compact symmetric neighbourhood U_0 that contains only the trivial subgroup. Put $S^0 := P(S,U_0)$.

In the sequel, the following two cases will be important:

- $A = \mathbb{T}$ and $U_0 = \{z \in \mathbb{T} : \mathrm{Re}\, z \geq 0\}$, and
- $A = \mathbb{R}$ and $U_0 = [-1,1]$

REMARK 3.3. Let A and $U_0 \in \mathcal{U}_A(e)$ be as above.

(i) The sets $U_n := \{x \in U_0 : kx \in U_0 \ \forall k \in \{1,\dots,2^n\}\}$ form a neighbourhood basis of $e \in A$. [Fix an open neighbourhood $U \in \mathcal{U}_A(e)$. The sets U_n are compact, symmetric and decreasing. For $x \in \bigcap_{n\in\mathbb{N}} U_n$ we get $\langle x\rangle_\mathbb{Z} \subseteq \bigcap_{n\in\mathbb{N}} U_n$ and hence $\bigcap_{n\in\mathbb{N}} U_n = \{e\}$. 1.27 implies $U_n \subseteq U$ for a suitable $n \in \mathbb{N}$.]

(ii) For every subgroup $H \leq G$, we have $H^0 = \{\chi \in \mathrm{Hom}(G,A) : \chi(H) = \{e\}\}$. In particular, if G is compact then $\mathrm{Hom}_\mathrm{c}(G,A)$ is discrete. [For $\chi \in H^0$, we see that $\chi(H)$ is a subgroup of A contained in U_0. By assumption, it must be trivial.]

(iii) It is a consequence of some structure theorems for locally compact abelian groups that $A \cong \mathbb{R}^n \times \mathbb{T}^m \times D$ where $n, m \in \mathbb{N}_0$ and D is a discrete abelian group. [Cf. [42], p. 518 and Theorem (9.8) in [34].]

Proposition 3.4. (i) *For every abelian Hausdorff group G, the sets of the form S^0 where S is a finite* (*compact*, *totally bounded*) *subset of G form a neighbourhood basis of e in* $\mathrm{Hom}_{\mathrm{p}}(G,A)$ ($\mathrm{Hom}_{\mathrm{c}}(G,A)$, $\mathrm{Hom}_{\mathrm{tb}}(G,A)$).

(ii) *A homomorphism $\chi : G \to A$ of the topological group G into A is continuous if and only if there exists $W \in \mathcal{U}_G(e)$ such that $\chi \in W^0$.*

Proof. (i) For $V \in \mathcal{U}_{\mathrm{Hom}(G,A)}(e)$, where $\mathrm{Hom}(G,A)$ is endowed with the topology of uniform convergence on all finite (compact, totally bounded) subsets of G, there exist a finite (compact, totally bounded) subset $S \subseteq G$ and $U \in \mathcal{U}_A(e)$ such that $P(S,U) \subseteq V$. Furthermore, there exists $n \in \mathbb{N}$ such that $U_n \subseteq U$ (3.3(i)) and hence $P(S,U_n) \subseteq P(S,U) \subseteq V$. The set $\widetilde{S} := (S \cup \{e\})^{2^n} := \{\sum_{j=1}^{2^n} x_j : x_j \in S \cup \{e\}\}$ is finite (compact, totally bounded) and for $\chi \in \widetilde{S}^0$ and $x \in S$, we get $\chi(\sum_{j=1}^{k} x) \in U_0$ for $k \in \{1,\dots,2^n\}$, which means $\chi(x) \in U_n$. Hence $\widetilde{S}^0 \subseteq P(S,U_n) \subseteq V$.

(ii) The condition is obviously necessary. To prove the sufficiency, we only have to show that $\chi^{-1}(U_n) \in \mathcal{U}_G(e)$ for all $n \in \mathbb{N}$ (3.3(i)). By assumption, there exists $W \in \mathcal{U}_G(e)$ such that $W^{2^n} \subseteq \chi^{-1}(U_0)$. For $x \in W$ we get $\chi(\sum_{j=1}^{k} x) \in U_0$ for $k \in \{1,\dots,2^n\}$, which implies $\chi(W) \subseteq U_n$ (3.3(i) again).

In the sequel, we will often need the assumption that

($*$) G is a Hausdorff group and A is compact, or G contains no proper open subgroup.

Proposition 3.5. *Suppose* ($*$) *holds. Let $W \in \mathcal{U}_G(e)$. Then W^0 is a compact subset of* $\mathrm{Hom}_{\mathrm{tb}}(G,A)$ (*and hence of* $\mathrm{Hom}_{\mathrm{c}}(G,A)$ *and* $\mathrm{Hom}_{\mathrm{p}}(G,A)$).

Proof. The following canonical homomorphisms are continuous:

$$\mathrm{Hom}_{\mathrm{tb}}(G,A) \to \mathrm{Hom}_{\mathrm{c}}(G,A) \to \mathrm{Hom}_{\mathrm{p}}(G,A) \to A^G.$$

The last homomorphism is an embedding (3.1(ii)). The image of W^0 in A^G is $\{\chi : G \to A : \chi$ is a homomorphism and $\chi(W) \subseteq U_0\}$ (3.4(ii)). It is closed in A^G, since U_0 is closed in A and the set of homomorphisms of G into A is a closed subgroup of A^G. If A is compact then $\{\chi(x) : \chi \in W^0\}$ has compact closure in A for all $x \in G$. On the other hand, if G contains no proper open subgroup, then each $x \in G$ can be represented in the form $x = \sum_{j=1}^{n} x_j$ (for suitable $n \in \mathbb{N}$ and $x_j \in W \cup (-W)$) and hence $\{\chi(x) : \chi \in W^0\}$ is contained in the compact set U_0^n. Now it is a consequence of the Tikhonov Theorem that W^0, considered as a subset of A^G or $\mathrm{Hom}_{\mathrm{p}}(G,A)$, is compact.

Hence it suffices to show that the topologies on W^0 induced by $\mathrm{Hom}_{\mathrm{tb}}(G,A)$ and $\mathrm{Hom}_{\mathrm{p}}(G,A)$ coincide. Let therefore $\chi_0 \in W^0$ and $V \in \mathcal{U}_{\mathrm{Hom}_{\mathrm{tb}}(G,A)}(\chi_0)$. According to 3.4(i), there exists a totally bounded subset $S \subseteq G$ such that $V' := (\chi_0 + S^0) \cap W^0 \subseteq V$. According to 3.3(i), there exists $n \in \mathbb{N}$ such that $U_n + U_n + U_n \subseteq U_0$. For $W' \in \mathcal{U}_G(e)$ such that $(W')^{2^n} \subseteq W$, there exist $s_1,\dots,s_m \in S$ such that $S \subseteq \bigcup_{j=1}^{m}(s_j + W')$. Then we have $(\chi_0 + P(\{s_1,\dots,s_m\},U_n)) \cap W^0 \subseteq V'$. [Choose any χ in the set on the left. Since $\chi, \chi_0 \in W^0$, it is a consequence of 3.3(i) that $\chi(W') \subseteq U_n$ and $\chi_0(W') \subseteq U_n$.

Since each $s \in S$ is of the form $s = s_j + w$ for suitable $j \in \{1, \dots, m\}$ and $w \in W'$, we get $(\chi - \chi_0)(s) = (\chi - \chi_0)(s_j) + \chi(w) - \chi_0(w) \in U_n + U_n + U_n \subseteq U_0$.] This proves the assertion.

Example 3.6. Set $A := \mathbb{R}$ and $G := \mathbb{R}_{\mathrm{d}}$ (the reals with the discrete topology). Then $\{0\}^0 = \mathrm{Hom}(G, A)$ (3.3(iii)), which cannot be compact, since $\{\chi(1) : \chi \in \{0\}^0\} = \mathbb{R}$ is not totally bounded. This shows that the assumption $(*)$ in 3.5 cannot be omitted.

Notes 3.7. The special cases of 3.4 and 3.5 where $A = \mathbb{T}$ are standard.

4. Homomorphism groups of abelian metrizable groups

In this chapter we examine the case of homomorphism groups $\mathrm{Hom}(G, A)$ where G is an abelian metrizable group.

As in the last chapter, let A be a locally compact abelian group which has a compact neighbourhood U_0 that contains only the trivial subgroup. We retain condition $(*)$.

Lemma 4.1. *Let H be a dense subgroup of the abelian Hausdorff group G and let $\iota : H \to G$ be the canonical embedding. Then $\iota^* : \mathrm{Hom}_{\mathrm{c}}(G, A) \to \mathrm{Hom}_{\mathrm{c}}(H, A), \chi \mapsto \chi \circ \iota$, is a continuous isomorphism.*

Proof. Since ι is continuous and has dense image, ι^* is a continuous monomorphism. The surjectivity of ι^* follows from 2.17. (A, being a locally compact abelian group, is complete!)

Lemma 4.2. *Suppose $(*)$ holds. Let $W_2 \subseteq W_1$ be open neighbourhoods of e in an abelian Hausdorff group G, let $V \subseteq \mathrm{Hom}_{\mathrm{c}}(G, A)$ be an open neighbourhood of e in the k-refinement of $\mathrm{Hom}_{\mathrm{c}}(G, A)$, and let D be a dense subspace of G. For every $M \subseteq G$ satisfying $(M \cup W_1)^0 \subseteq V$, there exists a finite subset $F \subseteq W_1 \cap D$ such that $(M \cup F \cup W_2)^0 \subseteq V$.*

Proof. Let $\mathcal{F}$ be the set of all finite subsets of $W_1 \cap D$. Assume $V_F := (M \cup F \cup W_2)^0 \cap (G^* \setminus V) = M^0 \cap F^0 \cap (G^* \setminus V) \cap W_2^0 \neq \emptyset$ for all $F \in \mathcal{F}$. Since V_F is a closed subset of the compact set W_2^0 (3.5) and, for $F_1, \dots, F_n \in \mathcal{F}$, we have $\bigcap_{j=1}^n V_{F_j} \supseteq V_{\bigcup_{j=1}^n F_j}$, we get $\bigcap_{F \in \mathcal{F}} V_F \neq \emptyset$. From $\bigcap_{F \in \mathcal{F}} F^0 = (\bigcup_{F \in \mathcal{F}} F)^0 = (W_1 \cap D)^0 = W_1^0$ (since $W_1 \cap D$ is dense in W_1) we obtain $\emptyset \neq M^0 \cap (\bigcap_{F \in \mathcal{F}} F^0) \cap W_2^0 \cap (G^* \setminus V) = M^0 \cap W_1^0 \cap (G^* \setminus V)$, contradicting the assumption.

Theorem 4.3. *Assume $(*)$ holds. Let G be an abelian metrizable group. For every neighbourhood V of e in the k-refinement of $\mathrm{Hom}_{\mathrm{c}}(G, A)$ and every dense subspace $D \subseteq G$, there is a sequence $(a_n)_{n \in \mathbb{N}}$ in D converging to e such that $\{a_n : n \in \mathbb{N}\}^0 \subseteq V$.*

Proof. We may assume that V is an open neighbourhood of e in the k-refinement. Let $(W_n)_{n \in \mathbb{N}}$ be an open neighbourhood basis of e in G such that $W_1 = G$ and $W_{n+1} \subseteq W_n$ (for all $n \in \mathbb{N}$). Since $\{e\} = W_1^0 \subseteq V$ (3.3(ii)), we can inductively find finite subsets $F_n \subseteq D \cap W_n$ such that $(\bigcup_{k<n} F_k)^0 \cap W_n^0 \subseteq V$ (4.2). Since $D \cap W_n \neq \emptyset$ for all $n \in \mathbb{N}$ we may assume $F_n \neq \emptyset$ as well. Hence there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in D converging to e and satisfying $\{a_n : n \in \mathbb{N}\} = \bigcup_{k \in \mathbb{N}} F_k$. From $V \supseteq \bigcup_{n \in \mathbb{N}}((\bigcup_{k<n} F_k)^0 \cap W_n^0) \supseteq (\bigcup_{k \in \mathbb{N}} F_k)^0 \cap \bigcup_{n \in \mathbb{N}} W_n^0 = \{a_n : n \in \mathbb{N}\}^0$, the assertion follows.

Corollary 4.4. *Suppose* $(*)$ *holds. Let* D *be a dense subspace of the abelian metrizable group* G*. The sets of the form* $\{a_n : n \in \mathbb{N}\}^0$*, where* $(a_n)_{n\in\mathbb{N}}$ *in* D *tends to* e_G*, form a neighbourhood basis of* e *in* $\mathrm{Hom_c}(G, A)$*.*

Proof. This is a direct consequence of 4.3.

Corollary 4.5. *Suppose* $(*)$ *holds. Let* $\iota : H \to G$ *denote the canonical embedding where* H *is a dense subgroup of an abelian metrizable group* G*. Then* $\iota^* : \mathrm{Hom_c}(G, A) \to \mathrm{Hom_c}(H, A)$*,* $\chi \mapsto \chi \circ \iota$*, is a topological isomorphism. In particular,* $\mathrm{Hom_c}(G, \mathbb{T}) \cong \mathrm{Hom_c}(H, \mathbb{T})$ *and, if* G *is connected,* $\mathrm{Hom_c}(G, \mathbb{R}) \cong \mathrm{Hom_c}(H, \mathbb{R})$*.*

Proof. Because of 4.1, it is sufficient to show that ι^* is open. Therefore, let $V \in \mathcal{U}_{\mathrm{Hom_c}(G,A)}(e)$. According to 4.3 (or 4.4), there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in H converging to e such that $\{h_n : n \in \mathbb{N}\}^0 = (\{h_n : n \in \mathbb{N}\} \cup \{e\})^0 \subseteq V$. Since $\{h_n : n \in \mathbb{N}\} \cup \{e\}$ is compact, the assertion follows.

Example 4.6 (Turnwald). Let H denote the set of integers endowed with the weak topology. (This is the group topology induced by the homomorphism $\mathbb{Z} \to \mathbb{T}^{\mathbb{T}}$, $k \mapsto (z^k)_{z\in\mathbb{T}}$.) G, the completion of H, is a compact group and hence $\mathrm{Hom_c}(G, \mathbb{T})$ is discrete (3.3(ii)). According to the Glicksberg theorem on weakly compact sets (cf. [30] and [14]), $\mathrm{Hom_c}(H, \mathbb{T}) \cong \mathrm{Hom_c}(\mathbb{Z}, \mathbb{T}) \cong \mathbb{T}$ is compact (and not discrete).

This example shows that in the above corollary "metrizable" cannot be replaced by "compact".

Corollary 4.7. *Under the assumptions of the theorem we have: For every abelian metrizable group* G*, the homomorphism group* $\mathrm{Hom_c}(G, A)$ *is a hemicompact k-space. In particular,* $\mathrm{Hom_c}(G, \mathbb{T})$ *and, if* G *is connected,* $\mathrm{Hom_c}(G, \mathbb{R})$ *are hemicompact k-spaces.*

Proof. 4.3 implies that for every neighbourhood W of e in the k-refinement of $\mathrm{Hom_c}(G, A)$ there is a compact subset $K \subseteq G$ such that $K^0 \subseteq W$. Since the k-refinement of $\mathrm{Hom_c}(G, A)$ is homogeneous (1.6) it coincides with the given topology.

According to 1.3 and 1.32, every compact subset of $\mathrm{Hom_c}(G, A)$ is equicontinuous and hence contained in W^0 for a suitable $W \in \mathcal{U}_G(e)$. Since G has a countable neighbourhood basis $(W_n)_{n\in\mathbb{N}}$ of e, it follows from 3.5 that the polars $(W_n^0)_{n\in\mathbb{N}}$ form a cobasis for the compact subsets of $\mathrm{Hom_c}(G, A)$.

Remark 4.8. If G is an abelian Hausdorff group and a hemicompact space then $\mathrm{Hom_c}(G, A)$ is metrizable. [This follows immediately from 3.4(i) and 2.7.]

Lemma 4.9. *Assume* $(*)$ *holds. If* G *is a k-space then the compact subsets of* $\mathrm{Hom_c}(G, A)$ *and* $\mathrm{Hom_{tb}}(G, A)$ *coincide.*

Proof. It suffices to show that every compact subset K of $\mathrm{Hom_c}(G, A)$ is contained in a compact subset S of $\mathrm{Hom_{tb}}(G, A)$. By 1.32, K is equicontinuous, which means that there exists a neighbourhood $W \in \mathcal{U}_G(e)$ such that $K \subseteq W^0$. Now, 3.5 implies that W^0 is a compact subset of $\mathrm{Hom_{tb}}(G, A)$.

Theorem 4.10. *Let* H *be a dense subgroup of the abelian metrizable group* G*. Suppose that* A *is compact or that* H *and* G *are connected. Then*

$$\mathrm{Hom_{tb}}(H, A) = \mathrm{Hom_c}(H, A) \cong \mathrm{Hom_c}(G, A) = \mathrm{Hom_{tb}}(G, A).$$

Proof. Because of 4.5, it suffices to show that $\mathrm{Hom_c}(G,A) = \mathrm{Hom_{tb}}(G,A)$ is valid. Therefore, let S be a totally bounded subset of G. It is a consequence of 4.9 that S^0 is a neighbourhood of e in the k-refinement of $\mathrm{Hom_c}(G,A)$ and hence, because of 4.7, a neighbourhood of e with respect to the original topology.

Proposition 4.11. *If G is an abelian Hausdorff group and a k-space then* $\mathrm{Hom_c}(G,A)$ *is complete*; *in particular*, $\mathrm{Hom_c}(G,\mathbb{T})$ *is complete.*

Proof. Let $(\chi_j)_{j\in J}$ be a Cauchy net in $\mathrm{Hom_c}(G,A)$. It converges uniformly on every compact subset of G to a homomorphism $\chi : G \to A$. Since the restriction of χ to every compact subset is continuous, 1.5 implies that $\chi \in \mathrm{Hom}(G,A)$ and hence it is the desired limit.

Notes 4.12. 4.1 is a standard result, 4.2 and 4.3 generalize the Lemma on p. 150 and the Theorem on p. 151 of [74]. (Cf. also [23].) 4.6 is taken from [82]. For the case $A = \mathbb{T}$, the assertion in 4.7 concerning hemicompactness and 4.8 have been proved by Noble ([58]). 4.10 and 4.11 generalize (17.4) and (1.11) in [8].

The results of this chapter for $A = \mathbb{T}$ have been found independently by M. Chasco and they have recently been published ([21]).

5. Some results in duality theory

In this chapter we recall the basic definitions and facts of duality theory. Using the notation of the last chapter, we put for an abelian topological group G,

$$G^*_{\mathrm{p}} := \mathrm{Hom_p}(G,\mathbb{T}), \quad G^*_{\mathrm{c}} := \mathrm{Hom_c}(G,\mathbb{T}), \quad G^*_{\mathrm{tb}} := \mathrm{Hom_{tb}}(G,\mathbb{T}).$$

G^* is called the *character group* of G and the continuous homomorphisms $\chi : G \to \mathbb{T}$ are named *characters.* If no confusion can arise we write G^* instead of G^*_{c}.

Taking into consideration 3.4(i), a neighbourhood basis of e_{G^*} for a Hausdorff group G is given by the sets

$$K^0 := P(K, \{z \in \mathbb{T} : \operatorname{Re} z \geq 0\})$$

where $K \subseteq G$ is compact.

We use the abbreviation

$$R := \{z \in \mathbb{T} : \operatorname{Re} z \geq 0\}.$$

Since G^* is an abelian Hausdorff group (this follows from the remarks preceding 3.1), we are able to form the character group of G^*. This gives rise to the canonical homomorphism

$$\alpha_G : G \to G^{**}, \quad x \mapsto (\chi \mapsto \chi(x)).$$

Observe that $G^* \to \mathbb{T}$, $\chi \mapsto \chi(x)$, is a character of G^* for all $x \in G$ since $\{x\}$ is a compact subset of G. Hence α_G is well defined.

Definition 5.1. A topological group G is called (*Pontryagin*) *reflexive* if α_G is a topological isomorphism.

As a consequence of 4.11 and 4.7 we have:

Proposition 5.2. *Every reflexive metrizable group is complete.*

We will see in 8.11 that there exist reflexive groups which are not complete.

Theorem 5.3. *Every locally compact abelian group is reflexive.*

Proof. This is the famous Pontryagin–van Kampen theorem. See e.g. Theorem 52 in [67], p. 273 or Theorem (24.8) in [34], p. 378 or Theorem 23 in [55], p. 84 or Theorem (1.7.2.) in [72], p. 28.

Theorem 5.4. *Products of reflexive groups are reflexive. Moreover, for abelian Hausdorff groups G_i $(i \in I)$, the following mappings are topological isomorphisms*:

$$\Big(\sum_{i\in I} G_i\Big)^* \to \prod_{i\in I} G_i^*, \quad \chi \mapsto (\chi \circ \varphi_i)_{i\in I},$$

where $\varphi_i : G_i \to \sum_{j\in I} G_j$ denotes the canonical embedding and $\sum_{i\in I} G_i$ is endowed with the asterisk topology; *and*

$$\Big(\prod_{i\in I} G_i\Big)^* \to \sum_{i\in I} G_i^*, \quad \chi \mapsto (\chi \circ \psi_i)_{i\in I},$$

where $\psi_i : G_i \to \prod_{j\in I} G_j$ denotes the canonical embedding and $\sum_{i\in I} G_i^*$ is endowed with the asterisk topology.

Proof. This is the main theorem of [40].

In the case of vector spaces there exists an important connection between the character group and the topological dual:

Proposition 5.5. *Let V be a topological vector space. The mapping $\varrho : V' \to V^*$, $f \mapsto e^{2\pi i f}$, is an isomorphism. Moreover, if V is a Hausdorff space and both the topological dual V' and the character group V^* are endowed with the compact-open topology then ϱ is a topological isomorphism.*

Proof. A proof can be found in [8], p. 18, Proposition (2.3). See also [77], Lemma 1.

Theorem 5.6. *Every Banach space is a* (*Pontryagin*) *reflexive group.*

Proof. This was first proved by Smith (cf. [77]). Another proof can be found in [8], p. 140, Proposition (15.2). (Cf. also 8.18.)

Notation 5.7. Let G be a topological group and let H be a Hausdorff group. Every continuous homomorphism $\varphi : G \to H$ gives rise to a continuous homomorphism $\varphi^* : H^* \to G^*$, $\chi \mapsto \chi \circ \varphi$. It is called the *dual homomorphism* of φ.

For a subgroup G_0 of a topological group G we put $G_0^\perp := \{\chi \in G^* : \chi(G_0) = \{1\}\}$ (cf. 3.3(ii)); $G_0^\perp$ is called the *annihilator* of G_0 (in G^*).

Remark 5.8. For a topological group G we have $\alpha_G^* \circ \alpha_{G^*} = \mathrm{id}_{G^*}$. [For $\chi \in G^*$ and $x \in G$ we have $\alpha_G^* \circ \alpha_{G^*}(\chi)(x) = \alpha_{G^*}(\chi)(\alpha_G(x)) = \alpha_G(x)(\chi) = \chi(x)$.]

Proposition 5.9. *The character group of a reflexive group G is reflexive.*

Proof. The assertion follows from 5.8.

Proposition 5.10. *For a Hausdorff group G, the canonical homomorphism α_G is continuous if and only if every compact subset of G^* is equicontinuous.*

Proof. α_G is continuous if and only if for every compact subset of $K \subseteq G^*$ and every $V \in \mathcal{U}_{\mathbb{T}}(1)$ the inverse image $\alpha_G^{-1}(P(K,V))$ belongs to $\mathcal{U}_G(e)$. This is equivalent to: For every compact K and every V there exists $U \in \mathcal{U}_G(e)$ such that $\alpha_G(U) \subseteq P(K,V)$ ($\Leftrightarrow \chi(x) = \alpha_G(x)(\chi) \in V$ for all $\chi \in K$ and $x \in U$). The assertion follows.

Proposition 5.11. *For every Hausdorff group G, the restriction of α_G to compact subsets of G is continuous.*

Proof. See e.g. Lemma 1 in [46], p. 68 or Theorem 2.3 in [58], p. 556.

Corollary 5.12. *If an abelian topological group G is a k-space then α_G is continuous. In particular, if G is metrizable then α_G is continuous.*

Proof. This follows from 5.11 and 1.5. Alternatively: This is a consequence of 1.32 and 5.10. Observe that every metrizable space is a k-space (1.3).

Proposition 5.13. *Let G be a Hausdorff group. If α_G is an open isomorphism, then α_{G^*} is continuous.*

Proof. We have to show that every compact subset K of G^{**} is equicontinuous (5.10). Fix $V \in \mathcal{U}_{\mathbb{T}}(1)$. By assumption, $\alpha_G^{-1}(K)$ is a compact subset of G. Since $P(\alpha_G^{-1}(K),V) \in \mathcal{U}_{G^*}(e)$, it remains to observe that $K \subseteq P(P(\alpha_G^{-1}(K),V),V)$: let $\alpha_G(x) \in K$ and $\chi \in P(\alpha_G^{-1}(K),V)$. Then $\alpha_G(x)(\chi) = \chi(x) \in V$.

Definition 5.14. A subgroup H of a topological group G is called *dually closed* in G if for every $x \in G \setminus H$ there exists a character $\chi \in H^{\perp}$ such that $\chi(x) \neq 1$.

H is named *dually embedded* in G if every character of H can be extended to a character of G.

Remark 5.15. Every dually closed subgroup is closed.

Let H be a subgroup of a Hausdorff group G. Then H is dually closed in G if and only if $\alpha_{G/H}$ is injective.

If $\iota : H \to G$ denotes the embedding, then H is dually embedded in G if and only if ι^* is surjective.

Every open subgroup H of an abelian topological group G is dually closed and dually embedded. [This is an easy consequence of the fact that α_G is injective for every discrete abelian group G and 2.3.]

Proposition 5.16. *For a continuous homomorphism φ from the abelian topological group G into the Hausdorff group H, we have $\varphi(G)^{\perp} = \ker \varphi^*$.*

Proof. This follows immediately from the definitions.

Lemma 5.17. *Let $\varphi : G \to H$ be a continuous homomorphism between abelian Hausdorff groups. If φ is compact-covering then φ^* is an embedding.*

Proof. Since φ is a continuous surjection, φ^* is a continuous monomorphism (5.16). So it remains to show that φ^* is open with respect to its image. Let therefore K_H be a compact subset of H and let $V \in \mathcal{U}_{\mathbb{T}}(1)$. For a compact subset $K_G \subseteq G$ satisfying $\varphi(K_G) = K_H$ we get $\varphi^*(P(K_H,V)) \supseteq \varphi^*(H^*) \cap P(K_G,V)$, which implies the assertion.

Theorem 5.18. *Let $\varphi : G \to H$ be a continuous open homomorphism with compact kernel between the abelian Hausdorff groups G and H. Then φ^* is a continuous open homomorphism with compact kernel.*

Proof. This is Lemma (2.5) in [13].

Definition 5.19. A topological group G is called a *c-group* if α_G is continuous. It is named a *c^∞-group* if all successively formed character groups are c-groups.

Proposition 5.20. *The character group of an abelian almost metrizable group G is a k-space and G^{**} is Čech-complete.*

Proof. According to 2.20, there exists a compact subgroup $H \leq G$ such that G/H is metrizable. It follows from 2.18 that the canonical projection $\pi : G \to G/H$ is compact-covering and it is a consequence of 5.17 that π^* is an embedding. One easily verifies that the image of π^* is the open subgroup $H^\perp$ (3.3(ii)). Hence $H^\perp$ is topologically isomorphic to the k-space $(G/H)^*$ (4.7). Topologically, G^* is the sum of translates of $H^\perp$ and hence a k-space. (Cf. Theorem (3.3.26) in [26].)

According to 5.18, the dual homomorphism of the embedding $H^\perp \to G^*$ is a continuous open homomorphism with compact kernel $H^{\perp\perp}$ (5.18 and 5.16); furthermore, it is surjective (5.16 and 5.15). Hence $G^{**}/H^{\perp\perp} \cong (H^\perp)^* \cong (G/H)^{**}$. It follows from 4.7, 4.8 and 4.11 that $(G/H)^{**}$ is a complete metrizable group. Hence the assertion is a consequence of 2.21.

Corollary 5.21. *Every abelian almost metrizable group is a c^∞-group.*

Proof. According to 1.24, every abelian almost metrizable group G is a k-space. It follows from 5.20 that G^* is a k-space. According to 5.12, α_G and α_{G^*} are continuous. The assertion is a consequence of 5.20 and 1.30.

Proposition 5.22 (Vilenkin). *Let G be a topological group such that α_G and α_{G^*} are continuous. Then $G^{***} = \alpha_{G^*}(G^*) \odot \alpha_G(G)^\perp$.*

Proof. Consider the mapping $\pi : G^{***} \to \alpha_{G^*}(G^*)$, $\eta \mapsto \alpha_{G^*}(\alpha_G^*(\eta))$. By assumption, π is a continuous homomorphism. 5.8 implies $\alpha_G^*(\alpha_{G^*}(\chi)) = \chi$ (for all $\chi \in G^*$). Hence we get $\pi(\alpha_{G^*}(\chi)) = \alpha_{G^*}(\chi)$. This means $\pi|_{\alpha_{G^*}(G^*)} = \mathrm{id}_{\alpha_{G^*}(G^*)}$. Since $\eta \in \ker \pi$ if and only if $\alpha_G^*(\eta)(x) = \eta(\alpha_G(x)) = 1$ for all $x \in G$ we get $\ker \pi = \alpha_G(G)^\perp$. This implies the assertion.

Remark 5.23. I am not aware of any topological group G where $\alpha_G(G)^\perp \neq \{e_{G^{***}}\}$.

Proposition 5.24. *Let H be a dually closed subgroup of a Hausdorff group G such that α_G is an open isomorphism. Let $\iota : H \to G$ denote the embedding. Then α_H is an open isomorphism if and only if ι^{**} is injective. Moreover, if α_H is continuous, then $H^{**} = \alpha_H(H) \odot \ker \iota^{**}$.*

Proof. Consider the following commutative (!) diagram:

$$\begin{tikzcd}
H \arrow[r, "\iota"] \arrow[d, "\alpha_H"'] & G \arrow[d, "\alpha_G"] \\
H^{**} \arrow[r, "\iota^{**}"'] & G^{**}
\end{tikzcd}$$

Since α_G is injective, so is α_H. For $U \in \mathcal{U}_G(e)$ we get $\alpha_H(U \cap H) = \alpha_H(H) \cap {\iota^{**}}^{-1}(\alpha_G(U))$. ["$\subseteq$" is trivial. For $x \in H$ such that $\iota^{**}(\alpha_H(x)) \in \alpha_G(U)$ we get $\alpha_G(x) \in \alpha_G(U)$ and hence $x \in U$, which implies "$\supseteq$".] This shows that α_H is open with respect to its image.

Assume that α_H is onto. For $x \in H$ such that $\alpha_H(x) \in \ker \iota^{**}$ we have $\alpha_G(x) = \iota^{**}(\alpha_H(x)) = e_{G^{**}}$. Hence $x = e$ and ι^{**} is injective.

Conversely, assume that $\ker \iota^{**}$ is trivial. Fix $\kappa \in H^{**}$. By assumption, there exists $x \in G$ such that $\alpha_G(x) = \iota^{**}(\kappa)$. It suffices to show that $x \in H$. Otherwise, there exists $\chi \in H^{\perp}$ such that $1 \neq \chi(x) = \alpha_G(x)(\chi) = \iota^{**}(\kappa)(\chi) = \kappa(\iota^{*}(\chi)) = \kappa(\chi|_H) = 1$, which gives the desired contradiction.

Assume now that α_H is continuous. Since H is dually closed in G, we have $\alpha_G^{-1}(H^{\perp\perp}) = H$ and (in general) $\iota^{**}(H^{**}) \subseteq H^{\perp\perp}$. By assumption, the mapping $\pi : H^{**} \to \alpha_H(H)$, $\tau \mapsto \alpha_H(\alpha_G^{-1}(\iota^{**}(\tau)))$, is a (well defined) continuous homomorphism. For $x \in H$, we get $\iota^{**}(\alpha_H(x)) = \alpha_G(x)$; this implies that $\pi|_{\alpha_H(H)} = \mathrm{id}_{\alpha_H(H)}$. Since α_H is injective, we get $\ker \pi = \ker \iota^{**}$, which implies the assertion.

Corollary 5.25 (Noble). *Let H be a dually closed and dually embedded subgroup of a topological group G. If α_G is an open isomorphism, then so is α_H.*

Proof. Let $\iota : H \to G$ denote the embedding. By assumption, ι^{*} is surjective (5.15) and hence ι^{**} is injective (5.16). Now, the assertion follows from the above proposition.

Proposition 5.26. *Let $(a_n)_{n\in\mathbb{N}}$ be an orthogonal system in ℓ^2. Then $H := \overline{\langle a_n : n \in \mathbb{N}\rangle_{\mathbb{Z}}}$ is a dually closed subgroup which is reflexive.*

Proof. Let $Y \subset \ell^2$ be an orthonormal system such that $\{a_n/\|a_n\| : n \in \mathbb{N}\} \cup Y$ is a complete orthonormal system. Then we get

$$H = \{x \in \ell^2 : x \cdot y = 0 \ \forall y \in Y \text{ and } x \cdot a_n \in \|a_n\|^2 \cdot \mathbb{Z} \ \forall n \in \mathbb{N}\}$$

(where $x \cdot y$ denotes the standard inner product). "$\subseteq$" is trivial. To prove the converse inclusion, observe that every $x \in \ell^2$ has the form $x = \sum_{n\in\mathbb{N}} (x \cdot a_n) a_n/\|a_n\|^2 + \sum_{y\in Y} (x \cdot y) y$. Now it follows easily that H is dually closed.

Let $\iota : H \to \ell^2$ denote the embedding. Because of 5.24 and 5.16, H is reflexive if we can show that the image of ι^{*} is dense. For $\chi \in H^{*}$, there exists $\varepsilon > 0$ such that $\chi(H \cap \varepsilon B) \subseteq R$ (where B denotes the closed unit ball of ℓ^2). According to 4.4, it suffices to show that for every sequence $(x_m)_{m\in\mathbb{N}}$ in $\langle a_n : n \in \mathbb{N}\rangle_{\mathbb{Z}}$ tending to 0, there exists $y := \sum_{n=1}^{N} \eta_n a_n \in \ell^2$ such that $\exp(2\pi i y \cdot x_m)\overline{\chi(x_m)} \in R$. Let $x_m = \sum_{n\in\mathbb{N}} k_{m,n} a_n$ (and $k_{m,n} \in \mathbb{Z}$). Then $x_m \to 0$ is equivalent to $\sum_{n\in\mathbb{N}} k_{m,n}^2 \|a_n\|^2 \to 0$ as $m \to \infty$; hence, there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{n\in\mathbb{N}} k_{m,n}^2 \|a_n\|^2 \le \varepsilon^2 \quad \forall m \ge m_0$$

and there exists $N \in \mathbb{N}$ such that

$$\sum_{n>N} k_{m,n}^2 \|a_n\|^2 < \varepsilon^2 \quad \forall m \in \{1, \ldots, m_0\}.$$

For $\eta_n \in \mathbb{R}$ such that $e^{2\pi i \eta_n} = \chi(a_n)$ and $y = \sum_{n=1}^{N} (\eta_n/\|a_n\|^2) a_n$, we get

$$\begin{aligned}\exp(2\pi i y\cdot x_m)\overline{\chi(x_m)} &= \exp\Big(2\pi i\sum_{n=1}^{N}\eta_n k_{m,n}\Big)\overline{\prod_{n\in\mathbb{N}}\chi(a_n)^{k_{m,n}}}\\ &= \overline{\chi\Big(\sum_{n>N}k_{m,n}a_n\Big)}\in R,\end{aligned}$$

since $\sum_{n>N}k_{m,n}a_n\in H\cap\varepsilon B$ for all $m\in\mathbb{N}$.

Remark 5.27. The above proposition shows that the closure of $H:=\langle(1/\sqrt{n})e_n\rangle_{\mathbb{Z}}$ is a dually closed subgroup of ℓ^2 which is reflexive. But H is not dually embedded in ℓ^2:

$$\chi:H\to\mathbb{T},\quad \sum_{n\in\mathbb{N}}\frac{k_n}{\sqrt{n}}e_n\mapsto\exp\Big(2\pi i\sum_{n\in\mathbb{N}}\frac{k_n}{n}\Big),$$

is a character, since $|\sum_{n\in\mathbb{N}}k_n/n|\le\sum_{n\in\mathbb{N}}k_n^2/n=\|\sum_{n\in\mathbb{N}}(k_n/\sqrt{n})e_n\|^2$. Since $\sum_{n\ge2}1/n=\infty$, there exists no sequence $(\eta_n)\in\ell^2$ such that $\exp(2\pi i\eta_n/\sqrt{n})=\exp(2\pi i\cdot 1/n)=\chi((1/\sqrt{n})e_n)$ for all $n\in\mathbb{N}$. According to 5.5, H is not dually embedded in ℓ^2.

It follows from 4.5 that $H^*\cong\overline{H}^*$. This shows that there exist reflexive, dually closed subgroups of reflexive groups which are not dually embedded.

Observe further that H is a compactly generated subgroup.

Lemma 5.28. *For every projective system* $\{G_i,\varphi_{i_1i_2},I\}$ *of topological groups* G_i *having the property that* α_{G_i} *is injective, the projective limit* G_0 *is dually closed in* $\prod_{i\in I}G_i$. *More precisely,* G_0 *is the intersection of the kernels of all characters of the form* $(x_i)\mapsto\chi_{i_1}(\varphi_{i_1i_2}(x_{i_2})-x_{i_1})$ *where* $\chi_{i_1}\in G_{i_1}^*$ *and* $i_1\le i_2\in I$.

Proof. Let $(y_i)_{i\in I}\in\prod_{i\in I}G_i\setminus G_0$. There are $i_1,i_2\in I$ such that $\varphi_{i_1i_2}(y_{i_2})\ne y_{i_1}$ and, by assumption, a character $\chi_{i_1}\in G_{i_1}^*$ such that $\chi_{i_1}(\varphi_{i_1i_2}(y_{i_2})-y_{i_1})\ne1$. Hence $\chi:(x_i)_{i\in I}\mapsto\chi_{i_1}(\varphi_{i_1i_2}(x_{i_2})-x_{i_1})$ is a character belonging to $G_0^{\perp}$ and $\chi((y_i)_{i\in I})\ne1$. Conversely, G_0 is contained in the kernel of each of these characters.

Notes 5.29. I am indebted to W. Banaszczyk for providing me with a translation of [85] which was originally published in Russian. This led to 5.22; the proof given here is much simpler. 5.8, 5.9, 5.10, 5.15 and 5.16 are standard results. 5.25 is a consequence of the proof of Theorem (3.1) in [58], p. 558. 5.24, which generalizes this result, is new. 5.26 and 5.27 show that 5.24 is really stronger than 5.25. The first parts of 5.20 and 5.21 were obtained jointly with G. Turnwald.

6. Locally quasi-convex groups

In the realm of topological vector spaces it is important to consider locally convex spaces. Since it is not clear how to define linear combinations in an arbitrary topological group, we use the description of convexity given by the Hahn–Banach Theorem to define something similar for topological groups. Vilenkin was the first who defined locally quasi-convex groups ([85]) for abelian groups with a boundedness (cf. §18 in [8]). Afterwards, W. Banaszczyk generalized this setting to arbitrary topological groups and pointed out its importance in connection with duality theory.

Definition 6.1. A subset A of a topological group G is called *quasi-convex* if for every $x \in G \setminus A$, there is a $\chi \in A^0$ such that $\chi(x) \notin R$.

A topological group G is called *locally quasi-convex* if it has a neighbourhood basis of e consisting of quasi-convex sets.

If we had taken another (symmetric and sufficiently small) neighbourhood of $1 \in \mathbb{T}$, the quasi-convex sets would change. The setting of local quasi-convexity does not depend on the neighbourhood selected.

Obviously, local quasi-convexity is invariant under topological isomorphisms.

Proposition 6.2. (i) *$A \subseteq G$ is quasi-convex if and only if $A = \bigcap_{\chi \in A^0} \chi^{-1}(R)$. In particular, every quasi-convex set is closed, symmetric and contains* $\ker \alpha_G$.

(ii) *The intersection of any family of quasi-convex subsets of a group is quasi-convex.*

(iii) *Let $\varphi : G \to H$ be a continuous homomorphism where G and H are topological groups. For every quasi-convex subset $A \subseteq H$ the inverse image $\varphi^{-1}(A)$ is a quasi-convex subset of G.*

(iv) *For arbitrary $A \subseteq G$, the set* $\mathrm{qc}(A) := \bigcap_{\chi \in A^0} \chi^{-1}(R)$ *is the smallest quasi-convex set containing A; it is called the* quasi-convex hull *of A.*

(v) *For every $A \subseteq G$, we have* $A^0 = (\mathrm{qc}(A))^0$.

Proof. (i) to (iii) are trivial.

(iv) Let $C \supseteq A$ be quasi-convex. For $x \notin C$ there exists $\chi \in C^0$ ($\subseteq A^0$) such that $\chi(x) \notin R$. Hence $x \notin \bigcap_{\chi \in A^0} \chi^{-1}(R)$. According to (ii) and (iii), the latter set is quasi-convex. (Observe that $R \subset \mathbb{T}$ is quasi-convex.) Hence (iv) follows.

(v) Since $A \subseteq \mathrm{qc}(A)$, "$\supseteq$" is clear. The other inclusion follows from (iv).

Lemma 6.3. *For $V_n := \{e^{2\pi i t} : |t| \le 1/(4n)\}$ ($n \in \mathbb{N}$) we have: $z \in V_n$ if and only if $z^k \in V_1 = R$ for all $k \in \{1, \ldots, n\}$.*

Proof. The first condition is obviously sufficient for the second. So, choose any $z = e^{2\pi i t}$ ($t \in [-1/2, 1/2]$) such that $z^k \in R$ for $k \in \{1, \ldots, n\}$. We have $|t| \le 1/4$ since $z \in R$. We may assume $t \ne 0$. Hence there exists $m \in \mathbb{N}$ such that $1/(4(m+1)) < |t| \le 1/(4m)$. This implies $z^{m+1} \notin R$ and hence $m \ge n$. This in turn shows that $|t| \le 1/(4n)$ and $z \in V_n$.

Example 6.4. The connected quasi-convex neighbourhoods of $1 \in \mathbb{T}$ are exactly V_n ($n \in \mathbb{N}$) and $\mathbb{T}$. [Obviously, $\mathbb{T}$ is quasi-convex. Since $V_n = \bigcap_{k=1}^{n} \{z \in \mathbb{T} : z^k \in R\}$ (6.3) these sets are quasi-convex (and connected). Conversely, let $V \ne \mathbb{T}$ be a quasi-convex connected neighbourhood. Being homeomorphic to a real interval (and symmetric and closed) it must have the form $\{e^{2\pi i t} : |t| \le x\}$ for a suitable $x \in\,]0, 1/2[$. One easily verifies that $\{e^{2\pi i t} : |t| \le y\}$ cannot be quasi-convex for $y \in\,]1/4, 1/2[$. Hence we have $x \le 1/4$. Let $k \in \mathbb{N}$ be maximal with the property that $x \le 1/(4k)$. Since V is connected, we have $V^0 = \{\chi_j : |j| \le k\}$ (where $\chi_j : \mathbb{T} \to \mathbb{T}$, $z \mapsto z^j$). This implies $V = \bigcap_{|j| \le k} \chi_k^{-1}(R) = V_k$ (see 6.3).]

Proposition 6.5. *A Hausdorff topological vector space is locally convex if and only if it is a locally quasi-convex group.*

Proof. This is Proposition (2.4) in [8].

Proposition 6.6. *Every character group of a Hausdorff group G endowed with the topology of uniform convergence on all compact* (*or totally bounded or finite*) *subsets is locally quasi-convex. In particular*, *every reflexive group is locally quasi-convex.*

Proof. According to 6.2(ii) and (iii), the polar $S^0=\bigcap_{x\in S}\alpha_G(x)^{-1}(R)$ of an arbitrary subset S of G is quasi-convex. The assertion follows from 3.4(i), 6.2(ii) and (iii).

Lemma 6.7. *For every finite family* $(N_j)_{j\in F}\in\mathbb{N}^F$ *such that* $\sum_{j\in F}1/(N_j+1)\geq 1$, *there exists a subset* $F_0\subseteq F$ *such that*

$$1\leq\sum_{j\in F_0}\frac{1}{N_j+1}<\sum_{j\in F_0}\frac{1}{N_j}\leq 2.$$

Proof. Let $F_0\subseteq F$ be minimal with the property that $\sum_{j\in F_0}1/(N_j+1)\geq 1$ (this means that $\sum_{j\in F'}1/(N_j+1)<1$ for every proper subset $F'\subset F_0$). We claim that F_0 has the desired properties. Put $F_1:=\{j\in F_0:N_j=1\}$.

If $F_1=\emptyset$, then $N_j\geq 2$ (for all $j\in F_0$) and hence

$$\sum_{j\in F_0}\frac{1}{N_j}\leq\frac{3}{2}\sum_{j\in F_0}\frac{1}{N_j+1}<\frac{3}{2}\left(1+\frac{1}{3}\right)=2$$

(by the minimality of F_0).

Let $|F_1|=1$. One easily checks that $\sum_{j\in F_0\setminus F_1}1/(N_j+1)\leq 2/3$ and hence

$$\sum_{j\in F_0}\frac{1}{N_j}\leq 1+\frac{3}{2}\sum_{j\in F_0\setminus F_1}\frac{1}{N_j+1}\leq 2.$$

If $|F_1|=2$, the assertion is trivial since this implies $F_0=F_1$.

By the minimality of F_0, the case $|F_1|>2$ cannot occur. This completes the proof.

Proposition 6.8. (i) *Every subgroup H of a locally quasi-convex group G is locally quasi-convex.*

(ii) *Every product of locally quasi-convex groups is locally quasi-convex.*

(iii) *Every sum of abelian locally quasi-convex Hausdorff groups* (*endowed with the asterisk topology*) *is locally quasi-convex.*

Proof. (i) For every quasi-convex subset $U\subseteq G$ the set $U\cap H$ is quasi-convex in H. Hence (i) is obvious.

(ii) It suffices to show that the product of quasi-convex sets $C_j\subseteq G_j$ is quasi-convex in the product $\prod_{j\in J}G_j$. For $(x_j)_{j\in J}\in\prod_{j\in J}G_j\setminus\prod_{j\in J}C_j$ there exists $j_0\in J$ such that $x_{j_0}\notin C_{j_0}$. By assumption, there is a character $\chi\in G_{j_0}^*$ such that $\chi(C_{j_0})\subseteq R$ and $\chi(x_{j_0})\notin R$. Hence $\chi\circ\pi_{j_0}$, where π_{j_0} is the canonical projection, has the desired properties.

(iii) Let G_j be locally quasi-convex and put $G:=\sum_{j\in J}G_j$. The assertion follows if we can prove that for arbitrary quasi-convex neighbourhoods $W_j\in\mathcal{U}_{G_j}(e)$ and $\widetilde{W}_j:=W_j+W_j$,

$$\mathrm{qc}(U_W)\subseteq U_{\widetilde{W}}$$

where U_W and $U_{\widetilde{W}}$ are the neighbourhoods in the asterisk topology associated with $(W_j)_{j\in J}$ and $(V_j)_{j\in J}$ (cf. 2.9).

Identifying G^* with $\prod_{j\in J} G_j^*$ (5.4), we get

$$(*)\qquad \prod_{j\in J} W_j^0 \subseteq U_W^0 \overset{6.2(\mathrm{v})}{=} (\mathrm{qc}(U_W))^0.$$

[For $(y_j)_{j\in J} \in U_W$ there exist a finite set $F \subseteq J$ and $(n_j)_{j\in F} \in \mathbb{N}^F$ such that $\sum_{j\in F} 1/n_j < 1$, $y_j = e$ for $j \in J \setminus F$ and $ky_j \in W_j$ for $k \in \{1, \ldots, n_j\}$ and $j \in F$. Hence, for $\chi_j \in W_j^0$, $\chi_j^k(y_j) = \chi_j(ky_j) \in R$ for $k \in \{1, \ldots, n_j\}$, which implies $\chi_j(y_j) \in V_{n_j}$ (6.3). For arbitrary $\chi_j \in W_j^0$, we get $\prod_{j\in J} \chi_j(y_j) = \prod_{j\in F} \chi_j(y_j) \in \prod_{j\in F} V_{n_j} \subseteq R$. (The last inclusion is an easy consequence of the functional equation of the exponential mapping.)]

Fix $(x_j)_{j\in J} \in \mathrm{qc}(U_W)$. For arbitrary $j_0 \in J$ and $\chi_{j_0} \in W_{j_0}^0$ and $\chi_j = e_{G_j^*}$ for $j \in J \setminus \{j_0\}$, we see as a consequence of $(*)$ that $\prod_{j\in J} \chi_j(x_j) = \chi_{j_0}(x_{j_0}) \in R$. Hence, $N_j := \sup\{n \in \mathbb{N} : \chi_j(x_j) \in V_n \ \forall \chi_j \in W_j^0\} \in \mathbb{N} \cup \{\infty\}$ is well defined. Obviously, $F' := \{j \in J : N_j < \infty\}$ is contained in the finite set $F := \{j \in J : x_j \neq e\}$. We get $\sum_{j\in F'} 1/(N_j + 1) < 1$.

[Suppose the contrary. According to 6.7, there exists a subset $F_0 \subseteq F'$ such that

$$1 \leq \sum_{j\in F_0} \frac{1}{N_j + 1} \leq \sum_{j\in F_0} \frac{1}{N_j} \leq 2.$$

By the definition of the N_j, we can find $\chi_j \in W_j^0$ for $j \in F_0$ such that $\chi_j(x_j) = e^{2\pi i t_j}$ where

$$\frac{1}{4(N_j + 1)} < t_j \leq \frac{1}{4N_j}.$$

Letting χ_j be the trivial character for $j \notin F_0$, we get

$$\prod_{j\in J} \chi_j(x_j) = \prod_{j\in F_0} \chi_j(x_j) = \exp\Big(2\pi i \sum_{j\in F_0} t_j\Big) \notin R$$

since

$$\frac{1}{4} \leq \sum_{j\in F_0} \frac{1}{4(N_j + 1)} < \sum_{j\in F_0} t_j \leq \sum_{j\in F_0} \frac{1}{4N_j} \leq \frac{1}{2}.$$

By $(*)$, $(\chi_j)_{j\in J}$ belongs to $(\mathrm{qc}(U_W))^0$. This leads to a contradiction.]

Trivially, $\sum_{j\in F'} 1/N_j < 2$. For $j \in F \setminus F'$, it is possible to choose $N_j \in \mathbb{N}$ such that $\sum_{j\in F} 1/N_j < 2$. For $j \in F$ and $k \in \{1, \ldots, N_j\}$ and $\chi_j \in W_j^0$ we get $\chi_j(kx_j) = \chi_j^k(x_j) \in R$ (according to 6.3), which implies $kx_j \in W_j$, since the W_j are quasi-convex. Hence $kx_j \in \widetilde{W_j}$ for $1 \leq k \leq 2N_j$. This proves the assertion.

Remark 6.9. For every subset $A \subseteq G$ we have $\alpha_G(\mathrm{qc}\, A) = A^{00} \cap \alpha_G(G)$. [This is a direct consequence of 6.2(iv).]

Proposition 6.10. *Let G be an abelian locally quasi-convex Hausdorff group. Then α_G is injective and $\alpha : G \to \alpha_G(G)$, $x \mapsto \alpha_G(x)$, is open. Moreover, the mapping $\gamma : G \to (G_{\mathrm{tb}}^*)_{\mathrm{c}}^*$, $x \mapsto (\chi \mapsto \chi(x))$, is open with respect to $\gamma(G)$.*

Conversely: If G is an abelian Hausdorff group such that α_G is an embedding, then G is locally quasi-convex.

Proof. Obviously, α_G is injective. Let $U \in \mathcal{U}_G(e)$ be quasi-convex. Since $\alpha_G(U) = U^{00} \cap \alpha_G(G)$ and U^0 is a compact subset of G^* (3.5) and hence $U^{00} \in \mathcal{U}_{G^{**}}(e)$, the first assertion follows.

Analogously, $\gamma(U) = \gamma(G) \cap U^{00}$. According to 3.5, U^0 is a compact subset of G^*_{tb} and hence $U^{00} \in \mathcal{U}_{(G^*_{\mathrm{tb}})^*_{\mathrm{c}}}$.

Conversely, assume that α_G is an embedding. For each $U \in \mathcal{U}_G(e)$ there exists a compact subset $K \subseteq G^*$ such that $\alpha_G(U) \supseteq K^0 \cap \alpha_G(G)$ (3.4). Since α_G is continuous, K is equicontinuous (5.10) and hence $K \subseteq V^0$ for a suitable $V \in \mathcal{U}_G(e)$. We get $\alpha_G(U) \supseteq V^{00} \cap \alpha_G(G) = \alpha_G(\mathrm{qc}(V))$ (6.9). Hence $\mathrm{qc}(V) \subseteq U$ and the assertion follows.

Remark 6.11. In 8.27 we will give an example of a complete metrizable abelian topological group G which has sufficiently many characters (this means α_G is injective) which is not locally quasi-convex.

Proposition 6.12. *Let G be a complete locally quasi-convex Hausdorff group such that α_G is continuous. Then $\alpha_G : G \to G^{**}$ is an embedding with closed image.*

Proof. It follows immediately from 6.10 that α_G is an embedding. This implies that $\alpha_G(G)$ is a complete and hence closed subgroup of G^{**}.

Corollary 6.13. *Let G be a complete locally quasi-convex Hausdorff group with continuous α_G. Then G is locally compact if and only if G^* is locally compact.*

Proof. Obviously, the condition is sufficient. Let therefore G^* be locally compact. Then G^{**} is locally compact. Hence G, being topologically isomorphic to a closed subgroup of G^{**} (6.12), is locally compact as well.

Lemma 6.14. *Let A and B be subsets of an abelian Hausdorff group G. Then $(A \cup \{e\}) + (A \cup \{e\}) \subseteq B$ implies $\mathrm{qc}(A) + \mathrm{qc}(A) \subseteq \mathrm{qc}(B)$.*

Proof. Let $x_1, x_2 \in \mathrm{qc}(A)$ and $\chi \in (\mathrm{qc}(B))^0$. According to 6.3, $\chi(A) \subseteq V_2$ and, as a consequence of 6.2(iii) and 6.4, we get $\chi(\mathrm{qc}(A)) \subseteq V_2$. Hence $\chi(x_1 + x_2) \subseteq V_2 \cdot V_2 = R$, which implies the assertion.

Lemma 6.15. *Let H be a dense subgroup of the abelian Hausdorff group G. The sets $(\overline{W}^G)_{W \in \mathcal{U}_H(e)}$ form a neighbourhood basis of e in G.*

Proof. This is standard. (Cf. Theorem (1.3.6) in [26].)

Lemma 6.16. *Let H be a locally quasi-convex group which is dense in the abelian Hausdorff group G. Then G is also locally quasi-convex.*

Proof. (1) For every quasi-convex neighbourhood $W \in \mathcal{U}_H(e)$, we have $\overline{W}^G \subseteq \mathrm{qc}_G(W) \in \mathcal{U}_G(e)$ and $\mathrm{qc}_G(W) \cap H = W$. (Here, qc_G denotes the quasi-convex hull formed with respect to G.) [The first inclusion is a consequence of the fact that $\mathrm{qc}_G(W)$ is closed in G. Now, 6.15 implies $\mathrm{qc}_G(W) \in \mathcal{U}_G(e)$ and it is a consequence of 2.17 that $(\mathrm{qc}_G(W)) \cap H = \bigcap_{\widetilde{\chi} \in G^* : \widetilde{\chi}(W) \subseteq R} (\widetilde{\chi}^{-1}(R) \cap H) = \bigcap_{\chi \in H^* : \chi(W) \subseteq R} \chi^{-1}(R) = W$ (since W is quasi-convex).]

(2) For arbitrary $U \in \mathcal{U}_G(e)$, we have $U \subseteq \overline{(U+U) \cap H}$. [Let $x \in U$ and let $\widetilde{U} \in \mathcal{U}_G(e)$ be arbitrary such that $\widetilde{U} \subseteq U$. Then $(x+\widetilde{U}) \cap (U+U) \cap H = (x+\widetilde{U}) \cap H \neq \emptyset$, which implies (2).]

Now, let $U \in \mathcal{U}_G(e)$ be arbitrary. By assumption and according to 6.15, there exist quasi-convex neighbourhoods $W, W_1 \in \mathcal{U}_H(e)$ such that $\overline{W}^G \subseteq U$ and $W_1 + W_1 \subseteq W$. Hence, $\mathrm{qc}_G(W_1) \overset{(1),(2)}{\subseteq} \overline{((\mathrm{qc}_G)W_1 + (\mathrm{qc}_G)W_1) \cap H} \overset{6.14}{\subseteq} \overline{(\mathrm{qc}_G)W \cap H} \overset{(1)}{=} \overline{W} \subseteq U$. The assertion follows from (1).

Corollary 6.17. *The completion of each abelian, locally quasi-convex Hausdorff group G is again locally quasi-convex.*

Proof. This is an immediate consequence of (2.13 and) 6.16.

Proposition 6.18. *Let $(G, \mathcal{O})$ be an abelian Hausdorff group. The sets of the form $(\mathrm{qc}(U))_{U \in \mathcal{U}_G(e)}$ form a neighbourhood basis of a locally quasi-convex group topology $\mathcal{O}_{\mathrm{qc}}$ of G which is coarser than the given one and which coincides with $\mathcal{O}$ if and only if $(G, \mathcal{O})$ is locally quasi-convex.*

$(G, \mathcal{O}_{\mathrm{qc}})$ is a Hausdorff space if and only if α_G is injective.

Algebraically, the character groups of $(G, \mathcal{O})$ and $(G, \mathcal{O}_{\mathrm{qc}})$ are the same. If G is metrizable and α_G is injective, then $(G, \mathcal{O})^* = (G, \mathcal{O}_{\mathrm{qc}})^*$.

Proof. Since every quasi-convex set is symmetric and contains $\{e\}$, it follows from 6.14 and 2.5 that the sets $(\mathrm{qc}(U))_{U \in \mathcal{U}_G(e)}$ form a neighbourhood basis of a group topology $\mathcal{O}_{\mathrm{qc}}$ on G. (Observe that $\mathrm{qc}(U_1 \cap \ldots \cap U_n) \subseteq \mathrm{qc}(U_1) \cap \ldots \cap \mathrm{qc}(U_n)$.) This topology is obviously coarser than $\mathcal{O}$. They coincide if and only if G is locally quasi-convex.

The equality

$$\bigcap_{U \in \mathcal{U}_G(e)} \mathrm{qc}(U) = \bigcap_{U \in \mathcal{U}_G(e)} \Big(\bigcap_{\chi \in U^0} \chi^{-1}(R) \Big) = \bigcap_{\chi \in G^*} \chi^{-1}(R) = \ker \alpha_G$$

and 2.5 imply that $(G, \mathcal{O}_{\mathrm{qc}})$ is a Hausdorff space if and only if α_G is injective.

6.2(iii) implies that, algebraically, the character groups of $(G, \mathcal{O})$ and $(G, \mathcal{O}_{\mathrm{qc}})$ are the same. This shows that $(G, \mathcal{O}_{\mathrm{qc}})$ is locally quasi-convex.

Assume now that G is metrizable and that α_G is injective. Then, according to 2.7, $\mathcal{O}_{\mathrm{qc}}$ is also metrizable. Obviously, $(G, \mathcal{O}_{\mathrm{qc}})^* \to (G, \mathcal{O})^*$ is a continuous isomorphism. It is a consequence of 4.7 that both character groups are k-spaces. Hence it suffices to show that every compact subset K of $(G, \mathcal{O})^*$ is compact with respect to $(G, \mathcal{O}_{\mathrm{qc}})^*$. According to 5.12 and 5.10, K is contained in the polar U^0 for a suitable $U \in \mathcal{U}_G(e)$. Since $(\mathrm{qc}(U))^0 = U^0$ by 6.2(v), the assertion follows from 3.5.

Notes 6.19. Similar results as treated in this section can be found in [85]. 6.2 is a standard result. Our exposition of 6.3 is taken from a lecture on "Duality theory of abelian topological groups" (in German) from G. Turnwald. The assertions of 6.6 and 6.8(iii) can be found in [8] on p. 2 and p. 10. The first assertion of 6.10 is Lemma (14.1) in [8]. Corollary 6.17 is taken from [15], the proof of 6.16 is very similar to the proof given there. 6.18 generalizes Section 3 of [20].

7. Properties of the quasi-convex hull

In this chapter we examine the quasi-convex hull of a singleton (7.8), a finite set (7.11) and of totally bounded and compact subsets (7.12) of locally quasi-convex groups.

Notation 7.1. Let $Z_l < \mathbb{T}$ be the group of l-th roots of unity. Then $Z_l \cong \mathbb{Z}/l\mathbb{Z}$.

Theorem 7.2. *Let G be an abelian topological group such that α_G is injective. For torsion free elements $x_1, \dots, x_n$ and torsion elements $y_1, \dots, y_m$ $(n, m \in \mathbb{N}_0)$ satisfying*

$$\sum_{k=1}^{n} k_j x_j + \sum_{j=1}^{m} \alpha_j y_j = 0 \ (k_j \in \mathbb{Z},\ 0 \le \alpha_j < \operatorname{ord} y_j) \Rightarrow k_j = \alpha_j = 0 \ \forall j$$

the mapping

$$\varphi : G^* \to \mathbb{T}^n \times Z_{l_1} \times \ldots \times Z_{l_m}, \quad \chi \mapsto (\chi(x_1), \dots, \chi(x_n), \chi(y_1), \dots, \chi(y_m)),$$

has dense image.

Proof. Let $\varphi^* : \mathbb{Z}^n \times \mathbb{Z}_{l_1} \times \ldots \times \mathbb{Z}_{l_m} \to G^{**}$, $(k_1, \dots, k_m, \alpha_1, \dots, \alpha_m) \mapsto (\chi \mapsto \prod_{j=1}^{n} \chi(x_j)^{k_j} \cdot \prod_{j=1}^{m} \chi(y_j)^{\alpha_j})$ (where $l_j := \operatorname{ord} y_j$), be the dual homomorphism of φ (composed with the canonical identifications). Since every closed subgroup of $\mathbb{T}^n \times \mathbb{Z}_{l_1} \times \ldots \times \mathbb{Z}_{l_m}$ is dually closed and $\ker \varphi^* = (\operatorname{im} \varphi)^\perp$ (5.16), it suffices to show that φ^* is injective. $\varphi^*((k_1, \dots, k_n, \alpha_1, \dots, \alpha_m)) = e_{G^{**}}$ means

$$\chi\Big(\sum_{j=1}^{n} k_j x_j + \sum_{j=1}^{m} \alpha_j y_j\Big) = \prod_{j=1}^{n} \chi(x_j)^{k_j} \prod_{j=1}^{m} \chi(y_j)^{\alpha_j} = 1$$

for all $\chi \in G^*$. Since α_G is injective the last equality implies $\sum_{j=1}^{n} k_j x_j + \sum_{j=1}^{m} \alpha_j y_j = 0$ and the assumption yields $k_j = \alpha_j = 0$. Hence φ^* is injective and the assertion follows.

It will be tacitly used that every subgroup of a compact group is dually embedded and that every closed subgroup is dually closed.

Lemma 7.3. *For $z \in \mathbb{T}$ and $w \in \mathbb{T} \setminus \{1, z, \overline{z}\}$ there exists $k \in \mathbb{Z}$ such that $\operatorname{Re} z^k \ge 0$ and $\operatorname{Re} w^k < 0$. (This is equivalent to: There exists $\chi \in \mathbb{T}^*$ such that $\chi(z) \in R$ and $\chi(w) \notin R$.)*

Proof. Let $x, y \in [0, 1[$ be such that $z = \exp(2\pi i x)$ and $w = \exp(2\pi i y)$.

1. If x is rational then $\langle z \rangle_{\mathbb{Z}}$ is finite and hence a closed subgroup of $\mathbb{T}$.

(a) Let $w \notin \langle z \rangle$. There is a $\chi \in \mathbb{T}^*$ such that $\chi|_{\langle z \rangle} \equiv 1$ and $\chi(w) \ne 1$. Hence χ^k, for suitable $k \in \mathbb{N}$, has the desired properties.

(b) Let $w \in \langle z \rangle$ and put $l := |\langle z \rangle|$. The cases $l \in \{1, 2, 3\}$ cannot occur. Let therefore $l \ge 4$. There exists $\chi \in \mathbb{T}^*$ such that $\chi(z) = \exp(2\pi i/l)$ and $m \in \{2, \dots, l-2\}$ such that $\chi(w) = \exp(2\pi i m/l)$. Replacing w by $\overline{w}$ (if necessary) we may assume $m \le l/2$. If $m > l/4$ the assertion follows. Since $m \ge 2$, we may assume $m \le l/4$ and $l \ge 8$. Let $k \in \mathbb{N}$ be minimal with the property that $k \cdot m > l/4$. Suppose that $k > l/4$. This would imply

$$m \cdot (k-1) > m \cdot (l/4 - 1) \ge l/2 - 2 \ge l/4$$

contradicting our assumption. Hence χ^k has the desired properties.

2. Let x be irrational. Assume there exist $(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that

$$kx + ly = m \in \mathbb{Z} \quad \text{and} \quad \gcd(k,l,m) = 1.$$

We must have $l \neq 0$.

(a) Let y be rational. We have $y = m/l$ and $k = 0$ since x is irrational. There exists $n_0 \in \mathbb{N}$ such that $n_0 y \in]1/4, 3/4[+ \mathbb{Z}$. Since lx is irrational (observe that $l \neq 0$) there is, according to 7.2, an $n \in \mathbb{N}$ such that $n \cdot (lx) + n_0 \cdot x \in]-1/4, 1/4[+ \mathbb{Z}$. Hence $n \cdot l + n_0$ has the desired properties.

(b) Let y (and x) be irrational. This implies $k \neq 0$.

Assume that $\gcd(k,l) = 1$. Replacing w by $\overline{w}$, we get $z^k = w^l$. Hence $|k| = |l| = 1$ is not possible. We first show that there exists $u \in \mathbb{T}$ such that $u^l = z$ and $u^k = w$.

There is a $\widetilde{u} \in \mathbb{T}$ such that $\widetilde{u}^l = z$ and hence $\widetilde{u}^{kl} = z^k = w^l$. Hence $\widetilde{u}^k \cdot \zeta_l = w$ for a suitable l-th root of unity ζ_l. Since $\gcd(k,l) = 1$, there exists an l-th root of unity $\widetilde{\zeta_l}$ such that $\zeta_l = (\widetilde{\zeta_l})^k$. This implies $(\widetilde{u} \cdot \widetilde{\zeta_l})^k = w$ and, of course, $(\widetilde{u} \cdot \widetilde{\zeta_l})^l = z$. Hence $u := \widetilde{u} \cdot \widetilde{\zeta_l}$ has the above properties.

Let $n > |l| + |k|$ be an odd number and $V \in \mathcal{U}_{\mathbb{T}}(1)$ be such that for each $\zeta \in Z_n \cap R$ we have $\zeta \cdot V^n \subseteq R$ and for each $\zeta' \in Z_n \setminus R$ we have $\zeta' \cdot V^n \cap R = \emptyset$. Let ζ_n be a primitive n-th root of unity. Since ζ_n^l and ζ_n^k satisfy the assumptions of part 1, there is an $r \in \{1, \ldots, n-1\}$ such that $\zeta_n^{lr} \in R$ and $\zeta_n^{kr} \notin R$. According to 7.2, there is an $s \in \mathbb{Z}$ such that $u^s \in \zeta_n^r \cdot V$. This implies $z^s = u^{l \cdot s} \in \zeta_n^{l \cdot r} \cdot V^{|l|} \subseteq R$ and $w^s = u^{k \cdot s} \in \zeta_n^{k \cdot r} \cdot V^{|k|}$, which means $w^s \notin R$.

Let $k = dk'$ and $l = dl'$ where $\gcd(d,m) = \gcd(k',l') = 1$. If $w^d \notin \{1, z^d, \overline{z}^d\}$ we can apply the above argument. Otherwise, we may assume $w^d = z^d$. Hence there exists a primitive d'-th root of unity ζ with $d' \mid d$ such that $w = \zeta z$. Choose $n_0 \in \{1, \ldots, d'\}$ such that $\zeta^{n_0} \notin R$. According to 7.2, there exists $n_1 \in \mathbb{N}$ such that $z^{n_1 d' + n_0}$ is very near to 1, more precisely, such that $w^n = z^n \zeta^{n_0} \notin R$ and $z^n \in R$ for $n := n_1 d' + n_0$.

3. Let x and y be irrational numbers such that $kx + ly \in \mathbb{Z}$ implies $k = l = 0$. According to 7.2, the sequence $\{(z^n, w^n) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$.

Now the assertion follows.

Lemma 7.4. (i) *For all* $z \in \mathbb{T} \setminus \{1\}$ *there exists* $k \in \mathbb{Z}$ *such that* $z^k \in \{e^{2\pi i t} : t \in [1/3, 1/2]\}$.

(ii) *For each* $z \in \mathbb{T} \setminus (\{-1\} \cup Z_5)$ *there exists* $k \in \mathbb{Z}$ *such that* $z^k \in \{e^{2\pi i t} : t \in [1/4, 1/3]\}$.

Proof. Let $x \in]0,1[$ be such that $e^{2\pi i x} = z$. If x is irrational, the assertion follows from 7.2. Otherwise, there are $k, n \in \mathbb{N}$ such that $x = k/n$ and $\gcd(k,n) = 1$. This means that z is a primitive n-th root of unity. Hence, for $n \geq 6$ (or $n \geq 12$), (i) (or (ii)) is trivial. For $2 \leq n \leq 5$ (or $3 \leq n \leq 11$, $n \neq 5$) the assertion follows from an easy computation since there are m and $l \in \mathbb{Z}$ such that $lk + mn = 1$ ($\Leftrightarrow lk/n \in 1/n + \mathbb{Z}$).

Corollary 7.5. (i) *For all* $z_1, z_2 \in \mathbb{T} \setminus \{1\}$ *there exist* $k_1, k_2 \in \mathbb{Z}$ *such that* $z_1^{k_1} z_2^{k_2} \in R$ *and* $z_1^{k_1} \notin R$.

(ii) *For all* $z_1, z_2 \in \mathbb{T} \setminus \{-1, 1\}$ *there exist* $k_1, k_2 \in \mathbb{Z}$ *such that* $z_1^{k_1} z_2^{k_2} \in R$ *and* $z_1^{k_1} z_2^{-k_2} \notin R$.

Proof. (i) According to 7.4(i), there are k, l in $\mathbb{Z}$ such that $z_1^k, z_2^l \in \{e^{2\pi it} : t \in [1/3, 1/2]\}$. Setting $k_2 := l$ if $z_1^k z_2^l \in R$ and $k_2 := 2l$ otherwise and $k_1 := k$, it is easy to see that k_1, k_2 have the desired properties.

(ii) It is a consequence of 7.4(ii) that there exist $k_1, k_2 \in \mathbb{Z}$ such that $z_1^{k_1} \in \{e^{2\pi it} : t \in [2/3, 3/4]\}$ if $z_1 \notin Z_5$ and $z_1^{k_1} = e^{2\pi i \cdot 4/5}$ otherwise, and $z_2^{k_2} \in \{e^{2\pi it} : t \in [1/4, 1/3]\}$ if $z_2 \notin Z_5$ and $z_2^{k_2} = e^{2\pi i \cdot 2/5}$ otherwise. An easy computation shows that k_1, k_2 have the desired properties.

Lemma 7.6. *For all* $z = (z_1, z_2, z_3) \in \mathbb{T}^3$ *and* $w = (w_1, w_2, w_3) \in \mathbb{T}^3 \setminus \{(1,1,1), z, z^{-1}\}$ *there exist* $(k_1, k_2, k_3) \in \mathbb{Z}^3$ *such that* $\prod_{j=1}^3 z_j^{k_j} \in R$ *and* $\prod_{j=1}^3 w_j^{k_j} \notin R$.

Proof. If

$$(*) \qquad \exists j \in \{1,2,3\} \text{ such that } w_j \neq 1, z_j, \overline{z}_j,$$

is valid, the assertion follows from 7.3. Renumbering, and considering w^{-1} instead of w, we may assume that

$$1 \neq w_1 = z_1 \quad \text{and} \quad w_2 \neq z_2.$$

Assume first that $w_2 = 1$. According to 7.5(i), there exist $(k_1, k_2, 0) \in \mathbb{Z}^3$ with the desired properties.

According to $(*)$, we may assume now that $1 \neq w_2 = \overline{z}_2 \neq z_2$. Furthermore, $w_1 = -1$ implies $w_3 \neq \overline{z}_3$. If $w_3 = 1$, the assertion follows similarly to the case $w_2 = 1$.

It remains to consider the case

$$1 \neq w_2 = \overline{z}_2 \neq z_2 \quad \text{and} \quad 1 \neq w_j = z_j \neq \overline{z}_j \quad \text{for } j = 1 \text{ or } j = 3.$$

7.5(ii) implies the existence of $k_2, k_j \in \mathbb{Z}$ such that $z_j^{k_j} z_2^{k_2} \in R$ and $w_j^{k_j} w_2^{k_2} \notin R$. Hence the assertion follows.

Theorem 7.7. *For an abelian topological group* G, *the following assertions are equivalent*:

(i) α_G *is injective.*

(ii) *For all* $x \in G$ *and* $y \in G \setminus \{e, x, x^{-1}\}$, *there exists* $\chi \in G^*$ *such that* $\chi(x) \in R$ *and* $\chi(y) \notin R$.

Proof. (ii)⇒(i) is trivial. (Set $x = e$.)

(i)⇒(ii). By hypothesis, there are $\chi_1, \chi_2, \chi_3 \in G^*$ such that $\chi_1(y) \neq 1$, $\chi_2(x) \neq \chi_2(y)$ and $\chi_3(x) \neq \overline{\chi_3(y)} = \chi_3(-y)$. The assertion follows from Lemma 7.6 since, for $k_j \in \mathbb{Z}$, the mapping $g \mapsto \prod_{j=1}^3 \chi_j^{k_j}(g)$ is a character of G.

In terms of quasi-convex hulls, the above theorem reads as follows:

Theorem 7.8. *Let* G *be an abelian Hausdorff group. The following assertions are equivalent*:

(i) α_G *is injective.*

(ii) *For every* $x \in G$ *we have* $\mathrm{qc}(\{x\}) = \{e, x, x^{-1}\}$.

(iii) $\{e\}$ *is quasi-convex.*

Example 7.9. $\mathrm{qc}(\{-1, e^{2\pi i/3}\}) = \bigcap_{k \in 6 \cdot \mathbb{Z}} \{z \in \mathbb{T} : z^k \in R\} = \{e^{2\pi ik/6} : k = 0, 1, \ldots, 5\}$.

This example shows that the quasi-convex hull of a finite subset $F \subseteq \mathbb{T}$ is in general strictly larger than $F \cup F^{-1} \cup \{1\}$.

Lemma 7.10. *Let A be a non-empty index set. Consider a finite subset $F \subseteq \mathbb{T}^A$. Then* $\mathrm{qc}(F)$ *is a finite subset contained in* $\langle F\rangle_{\mathbb{Z}}$.

Proof. For $(k_\alpha)_\alpha = K \in \mathbb{Z}^{(A)}$ and $(z_\alpha)_\alpha = z \in \mathbb{T}^A$ we set $z^K := \prod_{\alpha\in A} z_\alpha^{k_\alpha}$. Since $\langle F\rangle_{\mathbb{Z}}$ is a finitely generated subgroup, there exist $z_1, \ldots, z_l \in \mathbb{T}^A$ ($l \in \mathbb{N}_0$) and a finite torsion group T such that $\langle F\rangle_{\mathbb{Z}} = \langle z_1, \ldots, z_l\rangle_{\mathbb{Z}} \cdot T$ (algebraically) and $\prod_{j=1}^{l} z_j^{k_j} = (1)$ ($k_j \in \mathbb{Z}$) implies $k_j = 0$ for $j = 1, \ldots, l$. Set

$$m_1 := |T| \quad\text{and}\quad m_2 := \max\Big\{\sum_{j=1}^{l} |k_{z,j}| : z \in F \text{ and } z \cdot \prod_{j=1}^{l} z_j^{k_{z,j}} \in T\Big\}.$$

Then $F \subseteq T \cdot \{\prod_{j=1}^{l} z_j^{k_j} : \sum_{j=1}^{l} |k_j| \le m_2\} =: \widetilde{F}$. If suffices to show that $\widetilde{F}$ is quasi-convex. Let $w \notin \widetilde{F}$.

1. Assume first that $|\langle w\rangle_{\mathbb{Z}}| < \infty$. Since $w \notin T$, there exists $K_0 \in \mathbb{Z}^{(A)}$ such that $t^{K_0} = 1$ for all $t \in T$ and $w^{K_0} \notin R$. It is a consequence of 7.2 that there exists $\widetilde{K} \in \mathbb{Z}^{(A)}$ such that $z_j^K \in \{e^{2\pi i t} : t \in \left]-1/(4m_2), 1/(4m_2)\right[\}$ for $K := m_1 \cdot \operatorname{ord} w \cdot \widetilde{K} + K_0$ and $j \in \{1, \ldots, l\}$. For $z \in \widetilde{F}$, we have $z^K \in R$ and $w^K \notin R$.

2. If $|\langle w\rangle_{\mathbb{Z}}| = \infty$ and $\langle w\rangle_{\mathbb{Z}} \cap \langle F\rangle_{\mathbb{Z}} = \{(1)\}$ then $(w, z_1, \ldots, z_l)$ satisfies the hypotheses of 7.2. Hence there exists $K \in \mathbb{Z}^{(A)}$ such that $w^{m_1 K} \notin R$ and $z_j^{m_1 K} \in \{e^{2\pi i t} : t \in \left]-1/(4m_2), 1/(4m_2)\right[\}$. So $m_1 \cdot K$ has the desired properties.

3. Now, let $|\langle w\rangle_{\mathbb{Z}}| = \infty$ and assume that $\langle w\rangle_{\mathbb{Z}} \cap \langle F\rangle_{\mathbb{Z}} \ne \{(1)\}$. Let μ be minimal with the property that there are $\mu_1, \ldots, \mu_l \in \mathbb{Z}$ such that $o := \operatorname{ord}(s) < \infty$ for $s := w^\mu \prod_{j=1}^{l} z_j^{\mu_j}$. If $\mu \mid \mu_j$ for all j then $\mu = 1$. (Otherwise, $\operatorname{ord}(w \prod_{j=1}^{l} z_j^{\mu_j/\mu}) < \infty$, which contradicts the choice of μ.)

(a) Assume first $\mu = 1$.

If $s \in T$ then $\sum_{j=1}^{l} |\mu_j| > m_2$. There exists $K' \in \mathbb{Z}^{(A)}$ such that for $K := m_1 \cdot K'$ and

$$I := \left]\frac{1}{4\sum_{j=1}^{l}|\mu_j|}, \min\left(\frac{1}{4m_2}, \frac{1}{2\sum_{j=1}^{l}|\mu_j|}\right)\right[$$

we have $z_j^K \in \{e^{2\pi i t} : t \in I\}$ if $\mu_j < 0$ and $z_j^K \in \{e^{-2\pi i t} : t \in I\}$ if $\mu_j \ge 0$. Hence $w^K = \prod_{j=1}^{l} z_j^{-\mu_j K}$, which means $w^K \notin R$. For $z = \prod_{j=1}^{l} z_j^{k_j} \cdot t \in \widetilde{F}$ ($t \in T$), we have $z^K = \prod_{j=1}^{l} z_j^{k_j \cdot K} \subseteq \{e^{2\pi i t} : |t| < 1/(4m_2)\}^{m_2}$ and hence $z^K \in R$.

If $s \notin T$, then we can find $K_0 \in \mathbb{Z}^{(A)}$ such that $s^{K_0} \notin R$ and $t^{K_0} = 1$ for all $t \in T$. There exists a symmetric open neighbourhood $V \in \mathcal{U}_{\mathbb{T}}(1)$ such that $(s^{K_0} V^{\sum_{j=1}^{l}|\mu_j|}) \cap R = \emptyset$ and $V^{m_2} \subseteq R$, and further, $K' \in \mathbb{Z}^{(A)}$ such that $z_j^K \in V$ for $K := K_0 + o m_1 K'$ and $j = 1, \ldots, l$. We get $w^K = s^{K_0} \prod_{j=1}^{l} z_j^{-K\mu_j} \in s^{K_0} V^{\sum_{j=1}^{l}|\mu_j|}$ hence $w^K \notin R$. For $z = \prod_{j=1}^{l} z_j^{k_j} \cdot t \in \widetilde{F}$ ($t \in T$), we have $z^K = \prod_{j=1}^{l} z_j^{k_j K} \in V^{\sum_{j=1}^{l}|k_j|} \subseteq V^{m_2} \subseteq R$.

(b) Now we may assume that μ does not divide μ_1. There exists a μ-th root of unity $\zeta \in \mathbb{T}$ such that $\zeta^{-\mu_1} \notin R$. Let $V \in \mathcal{U}_{\mathbb{T}}(1)$ be open and symmetric such that $(\zeta^{-\mu_1} V^{\sum_{j=1}^{l}|\mu_j|}) \cap R = \emptyset$ and $V^{\mu m_2} \subseteq R$. According to 7.2, there exists $K' \in \mathbb{Z}^{(A)}$

such that $z_1^{om_1K'} \in \zeta V$ and $z_j^{om_1K'} \in V$ for $j = 2, \dots, l$; put $K := om_1\mu K'$.Then $w^K = \prod_{j=1}^l z_j^{-\mu_j om_1 K'} \in \zeta^{-\mu_1} V^{|\mu_1|} \cdot V^{\sum_{j=2}^l |\mu_j|} \subseteq \zeta^{-\mu_1} V^{\sum_{j=1}^l |\mu_j|}$ implies $w^K \notin R$. For $z = \prod_{j=1}^l z_j^{k_j} \cdot t \in \widetilde{F}$ $(t \in T)$, we get $z^K = \prod_{j=1}^l z_j^{k_j om_1 \mu K'} \subseteq V^{\mu \sum_{j=1}^l |k_j|} \subseteq R$.

Now the assertion follows.

Theorem 7.11. *Let G be a topological group such that α_G is injective. Then the quasi-convex hull of every finite set is finite.*

Proof. Let $F \subseteq G$ be a finite subset. By assumption, $\beta : G \to \mathbb{T}^{G^*}$, $x \mapsto (\chi(x))_{\chi \in G^*}$, is a continuous monomorphism. Since $\mathrm{qc}(\beta(F))$ is a finite subset of $\beta(G)$ (7.10), the assertion follows from 6.2(iii).

Theorem 7.12. *Let G be an abelian locally quasi-convex Hausdorff group. The quasi-convex hull of every totally bounded set is totally bounded. If, in addition, G is complete, the quasi-convex hull of every compact set is compact.*

Proof. Let $S \subseteq G$ be totally bounded. $S^0 \in \mathcal{U}_{G^*_{\mathrm{tb}}}$ implies that S^{00} is a compact subset of $(G^*_{\mathrm{tb}})^*_{\mathrm{c}}$. For γ as in 6.10, we get $\mathrm{qc}(S) = \gamma^{-1}(S^{00}) = \gamma^{-1}(\gamma(G) \cap S^{00})$. According to 6.10, the mapping $\gamma(G) \to G$, $\gamma(x) \mapsto x$, is well defined and continuous. Hence $\mathrm{qc}(S)$ is totally bounded.

If G is complete, a subset of G is compact if and only if it is totally bounded and closed. Hence the assertion follows.

Notes 7.13. 7.2 is a strengthening of Kronecker's Approximation Theorem (cf. [34], Theorem (26.14)). I wish to thank G. Turnwald for helpful discussions which led to 7.3. Observe that an analogue to 7.12 holds for locally convex vector spaces.

8. Reflexivity of locally convex vector spaces

The aim of this chapter is to examine Pontryagin reflexivity of real locally convex spaces. Since every (Pontryagin) reflexive group is locally quasi-convex (6.6) and every locally quasi-convex Hausdorff topological vector space is locally convex (6.5), we consider locally convex spaces only.

Since there exists a non-equivalent definition of reflexivity in the realm of locally convex spaces, we call, in order to avoid ambiguities, a locally convex space *Pontryagin reflexive* if α_V is a topological isomorphism. Besides that, we use the term *reflexive vector space* in the meaning of [44], p. 304, or [74], p. 144.

Moreover, the following inclusion holds:

Theorem 8.1. *The additive group of every reflexive linear space is Pontryagin reflexive.*

Proof. See e.g. [8], (15.3) or [77].

In the sequel, we tacitly use the equivalence of norms on every finite-dimensional vector space.

Definition 8.2. We call a subset A of a vector space *radial* if it absorbs every finite subset F; this means there exists $\lambda_0 > 0$ such that $F \subseteq \lambda \cdot A$ for all $\lambda \geq \lambda_0$.

A subset C is named *circled* if $\lambda \cdot C \subseteq C$ for every $|\lambda| \leq 1$.

A closed, convex, circled, radial subset B of a locally convex space is called a *barrel.*

A locally convex Hausdorff vector space in which every barrel is a neighbourhood of 0 is called *barrelled.*

PROPOSITION 8.3. *Let* $(V, \mathcal{O})$ *be a real locally convex Hausdorff vector space and let* $V_{\mathrm{k}} := (V, \mathcal{O}_{\mathrm{k}})$ *denote the k-refinement. Then we have*:

(i) *The addition* $V_{\mathrm{k}} \times V_{\mathrm{k}} \to V_{\mathrm{k}}$ *is separately continuous.* (*This is equivalent to*: *The translations are homeomorphisms.*)

(ii) *The scalar multiplication* $\mathbb{R} \times V_{\mathrm{k}} \to V_{\mathrm{k}}$ *is continuous.*

(iii) *Every neighbourhood of* 0 *in* V_{k} *is radial.*

(iv) *There exists a neighbourhood basis of* 0 *in* V_{k} *consisting of circled sets.*

PROOF. (i) follows from 1.6.

(ii) According to 1.4, the space $\mathbb{R} \times V_{\mathrm{k}}$ is a k-space. Since the restriction of the scalar multiplication to compact subsets is continuous, the assertion follows from 1.5 (and 1.2(ii)).

(iii) Let F be a finite subset of V and let K be a compact neighbourhood of 0 in the finite-dimensional space $\langle F \rangle_{\mathbb{R}}$ (endowed with the topology induced by $\mathcal{O}$). For every $U \in \mathcal{U}_{V_{\mathrm{k}}}(0)$, there exists $M > 0$ such that $K \subseteq M \cdot U$. Since K absorbs F, so does U.

(iv) Fix $U \in \mathcal{U}_{V_{\mathrm{k}}}(0)$. According to (ii), there exist $\delta > 0$ and $U_0 \in \mathcal{U}_{V_{\mathrm{k}}}(0)$ such that $\delta' \cdot U_0 \subseteq U$ for all $\delta' \in]-\delta, \delta[$. The union $U_1 := \bigcup_{\delta' \in]-\delta,\delta[} \delta' \cdot U_0$ is a circled neighbourhood of 0 in V_{k} contained in U.

LEMMA 8.4. *Let* V *be a vector space and let* $\mathcal{U}$ *be a filter basis in* V *satisfying*:

(i) *for all* $U \in \mathcal{U}$, *there exists* $V \in \mathcal{U}$ *such that* $V + V \subseteq U$,

(ii) *every* $U \in \mathcal{U}$ *is radial and circled.*

Then $\mathcal{U}$ *is a neighbourhood basis of* 0 *of a vector space topology on* V. *This topology is a Hausdorff space if and only if* $\bigcap_{U \in \mathcal{U}} U = \{0\}$.

PROOF. This follows from 2.5 and Lemma (1.2) in [74], p. 14.

PROPOSITION 8.5. *Let* $(V, \mathcal{O})$ *be a real locally convex Hausdorff vector space. The convex hulls* $\mathrm{co}(U)$ *of all circled neighbourhoods of* 0 *in* V_{k} *form a neighbourhood basis of a locally convex Hausdorff vector space topology* $\mathcal{O}_{\mathrm{ck}}$ *on* V. *The topology* $\mathcal{O}_{\mathrm{ck}}$ *is the finest locally convex vector space topology having the same compact subsets as* $(V, \mathcal{O})$.

PROOF. Let $\mathcal{U}_0$ denote the set of all circled neighbourhoods of 0 in V_{k}. In order to prove that $\mathcal{O}_{\mathrm{ck}}$ is a vector space topology, it suffices to verify that $(\mathrm{co}(U))_{U \in \mathcal{U}_0}$ satisfies the conditions of 8.4. Since $U \in \mathcal{U}_0$ implies $\frac{1}{2}U \in \mathcal{U}_0$ (8.3(ii)) and $\frac{1}{2}\mathrm{co}(U) = \mathrm{co}\big(\frac{1}{2}U\big)$, we get $\mathrm{co}\big(\frac{1}{2}U\big) + \mathrm{co}\big(\frac{1}{2}U\big) = \mathrm{co}(U)$, which shows (i). Condition (ii) follows immediately from 8.3(iii).

Because of 8.3(iv), the identity mappings $(V, \mathcal{O}_{\mathrm{k}}) \to (V, \mathcal{O}_{\mathrm{ck}}) \to (V, \mathcal{O})$ are continuous. This shows that $\mathcal{O}_{\mathrm{ck}}$ is a Hausdorff topology on V. Because of 1.2(ii), the compact subsets of $(V, \mathcal{O}_{\mathrm{k}})$ and $(V, \mathcal{O})$, and hence of $(V, \mathcal{O}_{\mathrm{ck}})$, coincide. By construction, $\mathcal{O}_{\mathrm{ck}}$ is the finest locally convex vector space topology with this property.

Notation 8.6. A real locally convex Hausdorff space $(V,\mathcal{O})$ is called a *ck-space* if $\mathcal{O}$ and $\mathcal{O}_{\text{ck}}$ coincide.

Example 8.7. Examples of non-trivial ck-spaces can be found in Section 5 of [29].

Definition 8.8. A barrelled space V in which every closed bounded set is compact is called a *Montel space.*

Theorem 8.9. *Every Montel space is a reflexive vector space.*

Proof. See [44], §27, 2(1), p. 372.

Theorem 8.10. *There exists a real Montel space which is not complete and in which every compact subset is contained in a finite-dimensional subspace.*

Proof. See [1] or [45].

Corollary 8.11. *There exists a* (*Pontryagin*) *reflexive group which is not complete.*

Proof. This follows from 8.1, 8.9 and 8.10.

Corollary 8.12. *There exists a locally convex vector space V* (*more precisely, a Montel space*) *such that α_V is continuous but $V \neq V_{\text{ck}}$.*

Proof. Let $(V,\mathcal{O})$ be a Montel space having the properties stated in 8.10. Since V is a reflexive vector space (8.9), its additive group is Pontryagin reflexive (8.1). Hence α_V is continuous.

Let $\mathcal{A}$ denote the set of all symmetric convex radial subsets in V. According to 8.4, $\mathcal{A}$ forms a neighbourhood basis of 0 of a locally convex vector space topology on V which induces the same topology as $\mathcal{O}$ on every finite-dimensional subspace and hence on every compact subset of V. This topology is the finest locally convex vector space topology on V, hence, it must coincide with V_{ck}. Since $(V,\mathcal{O})$ was assumed not to be complete, it remains to prove that V_{ck} is complete.

Therefore, let $(x_j)_{j\in J}$ be a Cauchy net in V_{ck} and let $(b_i : i \in I)$ be a Hamel basis of V. For $(n_i)_{i\in I} \in \mathbb{N}^I$, there exists $j_{(n_i)}$ such that $x_j - x_{j'} \in A_{(n_i)} := \operatorname{conv}\{\pm(1/n_i)b_i : i \in I\}$ for all $j, j' \geq j_{(n_i)}$. For $x_j = \sum_{i\in I} \beta_{ij} b_i$, this is equivalent to $\sum_{i\in I} n_i|\beta_{ij} - \beta_{ij'}| \leq 1$ for all $j, j' \geq j_{(n_i)}$. In particular, the limit $\beta_i := \lim_{j\in J} \beta_{ij}$ exists. Let $I_0 := \{i \in I : \beta_i \neq 0\}$. Choose $(n_i)_{i\in I} \in \mathbb{N}^I$ such that $1/n_i < |\beta_i|$ for all $i \in I_0$. For $j_0 := j_{(n_i)}$ (as above) and $i \in I_0$ we get $1/n_i < |\beta_i| \leq |\beta_{ij_0} - \beta_i| + |\beta_{ij_0}| \leq 1/n_i + |\beta_{ij_0}|$. This implies that $\beta_{ij_0} \neq 0$ for all $i \in I_0$. Hence I_0 must be finite. For arbitrary $A \in \mathcal{A}$, there exists $(n_i)_{i\in I} \in \mathbb{N}^I$ such that $A_{(n_i)} \subseteq A$. For $j, j' \geq j_{(n_i)}$ we have $x_j - x_{j'} \in A_{(n_i)}$, which is equivalent to $\sum_{i\in I} n_i|\beta_{ij} - \beta_{ij'}| \leq 1$. This yields $\sum_{i\in I} n_i|\beta_{ij} - \beta_i| \leq 1$. Hence $\sum_{i\in I} \beta_i b_i$ is the limit of the given Cauchy net.

Proposition 8.13. *For every ck-space V, the evaluation mapping α_V is continuous.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
V \arrow[r, "\text{id}"] \arrow[d, "\beta_V"'] & V \arrow[d, "\alpha_V"] \\
(V'_{\text{c}})'_{\text{c}} \arrow[r, "\varphi"'] & V^{**}
\end{tikzcd}$$

where $\beta_V : V \to (V'_c)'_c$, $x \mapsto (f \mapsto f(x))$. Applying 5.5 to V and V', we see that the diagram commutes and that φ is a topological isomorphism. Hence, α_V is continuous if and only if β_V is continuous.

We have to show that for an arbitrary compact subset $K \subseteq V'_c$ and every $\eta > 0$ the set $K_0 := \{x \in V : |f(x)| \le \eta \ \forall f \in K\}$ is a neighbourhood of 0. (Replacing, if necessary, K by $(1/\eta) \cdot K$, we may assume that $\eta = 1$.) Obviously, K_0 is closed, symmetric and convex. It remains to prove that K_0 is a neighbourhood of 0 in the k-refinement. We show that $W := \{x \in V : |f(x)| < 1 \ \forall f \in K\}$ is an open neighbourhood of 0 in the k-refinement. Suppose the contrary. This means that there exists a compact subset $S \subseteq V$ and $x \in S \cap W$ such that for every neighbourhood $U \in \mathcal{U}_S(x)$ we have $U \not\subseteq W$. This is equivalent to

$$\forall U \in \mathcal{U}_S(x) \ \exists x_U \in U \text{ and } f_U \in K \text{ such that } |f_U(x_U)| \ge 1.$$

Since $(x_U)_{U \in \mathcal{U}_S(x)}$ converges to x and the net $(f_U)_{U \in \mathcal{U}_S(x)}$ belongs to the compact set K, the net $(x_U, f_U)_{U \in \mathcal{U}_S(x)}$ has an accumulation point (x, f) where $f \in K$. It follows from $x \in W$ that $\varepsilon := 1 - |f(x)| > 0$. Hence, $\widetilde{U} := \{\widetilde{x} \in S : 1 - |f(\widetilde{x})| > \varepsilon/2\}$ is an open neighbourhood of x in S. Since S is compact, the set $P(S, \varepsilon/2) := \{\widetilde{f} \in V' : \widetilde{f}(S) \subseteq {]{-\varepsilon/2}, \varepsilon/2[}\}$ is an open neighbourhood of 0 in V'_c. Hence, there exists an open neighbourhood $U \in \mathcal{U}_S(x)$ such that $U \subseteq \widetilde{U}$ and $f_U \in f + P(S, \varepsilon/2)$. By definition, $|f_U(x_U)| \ge 1$. On the other hand, $|f_U(x_U)| = |(f_U - f)(x_U) + f(x_U)| < \varepsilon/2 + |f(x_U)| < \varepsilon/2 + 1 - \varepsilon/2 = 1$, contradicting the choice of x_U and f_U. This completes the proof.

Remark 8.14. It was claimed by Kye ([47]) that α_V is continuous if and only if V is a ck-space. This contradicts 8.12.

Proposition 8.15. *Let U be a symmetric radial subset of a vector space V. For every homomorphism $\chi : V \to \mathbb{T}$ satisfying $\chi(U) \subseteq R$, there exists a unique linear functional $f : V \to \mathbb{R}$ such that $e^{2\pi i f} = \chi$ and* $\sup\{|f(x)| : x \in U\} \le 1/4$.

Proof. This follows from Proposition (2.2) in [8].

Proposition 8.16. *Let V be a real locally convex space. α_V is surjective if and only if the closed convex hull of every compact subset is weakly compact.*

Proof. This is Proposition (15.1) in [8]. Cf. also Theorem 3.2 of [47].

Corollary 8.17. *For every complete locally convex space V, the canonical homomorphism α_V is surjective.*

Proof. Since both the convex hull (cf. Theorem (3.20)(b) in [73]) and the closure of every totally bounded set are again totally bounded, it remains to observe that a subset is compact if and only if it is totally bounded and closed (cf. Theorem (8.3.16) in [26]). So the closed convex hull of every compact set is compact and hence weakly compact.

Theorem 8.18. *Every complete real locally convex Hausdorff vector space V such that α_V is continuous is a Pontryagin reflexive group. In particular, every Banach space is a Pontryagin reflexive group.*

Proof. According to 6.5, V is locally quasi-convex. Hence 6.10 and 8.17 imply that α_V is an open isomorphism. By assumption or according to 5.12, α_V is continuous.

Theorem 8.19 (Hahn–Banach). *Let p be a seminorm on a vector space E and let E_0 be a linear subspace of E. For every linear functional $f_0 : E_0 \to \mathbb{R}$ satisfying $s := \sup\{|f(x)| : x \in E_0 \cap B_p\} < \infty$, there exists a linear functional $f : E \to \mathbb{R}$ extending f_0 and satisfying $s = \sup\{|f(x)| : x \in B_p\}$ (where $B_p := \{x \in E : p(x) \leq 1\}$).*

Proof. This is Theorem 1 in §7 of [16], p. 101.

Proposition 8.20. *Every finite-dimensional subspace U and every closed finite-codimensional subspace W of a locally convex vector space V is an inner direct summand.*

Proof. This is Lemma 4.21 in [73].

Corollary 8.21. *For linearly independent vectors $(x_1, \ldots, x_n)$ (and $n \in \mathbb{N}_0$) of a Banach space V, the group $V/\langle x_1, \ldots, x_n\rangle_{\mathbb{Z}}$ is reflexive.*

Proof. According to 8.20, there exists a closed subspace $V_0 \cong V/\langle x_1, \ldots, x_n\rangle_{\mathbb{R}}$ of V such that $V = V_0 \oplus \langle x_1, \ldots, x_n\rangle_{\mathbb{R}}$. Obviously, V_0 is a Banach space. It is easily checked that $V/\langle x_1, \ldots, x_n\rangle_{\mathbb{Z}}$ is topologically isomorphic to $V_0 \oplus \mathbb{T}^n$ and hence reflexive (8.18 and 5.4).

As an application of the above theory, we prove that the dual of a Banach space endowed with the weak (vector space) topology is Pontryagin reflexive.

Theorem 8.22 (Krein–Shmul'yan). *The closed convex hull of every weakly compact subset of a Banach space is weakly compact.*

Proof. See e.g. [24], p. 434, or [87].

Proposition 8.23. *For every Banach space E, the identity mapping $\varphi : (E_{\mathrm{n}})'_{\mathrm{b}} \to (E_{\mathrm{w}})'_{\mathrm{c}}$ (where E_{n} denotes the Banach space with its norm topology, E_{w} stands for the weak vector space topology on E, and the index* b *denotes the strong dual) is a continuous isomorphism. It is open if and only if E is a reflexive vector space.*

Proof. It is clear that φ is an algebraic isomorphism. Since every weakly compact set is weakly bounded and, due to the Mackey Theorem (cf. 3.18 in [73]), also bounded in the original topology, it follows that φ is continuous.

Recall that E_{n} is a reflexive vector space if and only if the closed unit ball (and hence every closed bounded subset) is compact in E_{w} (cf. Satz 70.4 in [33]).

Assume first that E is a reflexive vector space. The above characterization implies immediately that φ is open. Conversely, assume that for every bounded set B in E_{n}, there exists a weakly compact set K such that $\varphi(B^0) \supseteq K^0$. This implies $\overline{\mathrm{co}(B)} \subseteq \overline{\mathrm{co}(K)}$. According to 8.22, $\overline{\mathrm{co}(K)}$ is weakly compact and so is $\overline{\mathrm{co}(B)}$. The assertion follows again from the above characterization.

Proposition 8.24. *Let E_{w} be an infinite-dimensional Banach space endowed with the weak vector space topology. Then $\alpha_{E_{\mathrm{w}}}$ is an open isomorphism which is not continuous.*

Proof. According to 8.23, the identity mapping $\varphi : (E_{\mathrm{n}})'_{\mathrm{b}} \to (E_{\mathrm{w}})'_{\mathrm{c}}$ is a continuous isomorphism. Since $(E_{\mathrm{n}})'_{\mathrm{b}}$ is an infinite-dimensional Banach space, it contains a compact subset which is not contained in a finite-dimensional subspace (e.g. a suitable convergent sequence). On the other hand, since every neighbourhood of 0 in E_{w} contains a linear subspace of finite codimension, every equicontinuous subset of $(E_{\mathrm{w}})'_{\mathrm{c}}$ is contained in a finite-dimensional subspace. 5.10 and 5.5 imply that $\alpha_{E_{\mathrm{w}}}$ cannot be continuous.

The surjectivity of $\alpha_{E_{\mathrm{w}}}$ is a consequence of 8.22 and 8.16. (Observe that the weak topology on E_{w} is again E_{w}.) Since E_{w} is locally convex, the assertion follows from 6.5 and 6.10.

Lemma 8.25. *Let $\varphi : G_1 \to G_2$ be a continuous homomorphism between abelian Hausdorff groups. Suppose that φ and φ^* are surjective, α_{G_1} is continuous, and α_{G_2} is an open isomorphism. Then G_2^* is reflexive.*

Proof. Because of 5.13, 6.6 and 6.10, it suffices to show that $\alpha_{G_2^*}$ is surjective. Consider the following commutative diagram:

$$\begin{tikzcd}
G_1 \arrow[r, "\varphi"] \arrow[d, "\alpha_{G_1}"'] & G_2 \arrow[d, "\alpha_{G_2}"] \\
G_1^{**} \arrow[r, "\varphi^{**}"'] & G_2^{**}
\end{tikzcd}$$

For $\eta \in G_2^{***}$, the composition $\eta \circ \varphi^{**} \circ \alpha_{G_1}$ belongs to G_1^*. By assumption, there exists $\chi_2 \in G_2^*$ such that $\eta \circ \alpha_{G_2} \circ \varphi = \eta \circ \varphi^{**} \circ \alpha_{G_1} = \varphi^*(\chi_2) = \chi_2 \circ \varphi$. Since φ is surjective, we get $\eta \circ \alpha_{G_2} = \chi_2 \overset{5.8}{=} \alpha_{G_2^*}(\chi_2) \circ \alpha_{G_2}$. The surjectivity of α_{G_2} implies that $\eta = \alpha_{G_2^*}(\chi_2)$. This completes the proof.

Theorem 8.26. *Let E_{w} denote a real Banach space with the weak vector space topology. The dual space $(E_{\mathrm{w}})'_{\mathrm{c}}$ is a Pontryagin reflexive group.*

Proof. Let E_{n} be the Banach space endowed with the norm topology. For $G_1 = E_{\mathrm{n}}$, $G_2 = E_{\mathrm{w}}$, and $\varphi = \mathrm{id}$, the hypotheses of 8.25 are satisfied (8.24, 8.18, and 5.5). Hence $(E_{\mathrm{w}})^*$ is Pontryagin reflexive and the assertion follows from 5.5.

Example 8.27. There exists a complete metrizable group which has sufficiently many continuous characters and which is not locally quasi-convex: The sequence space ℓ^p for $p \in \,]0,1[$ has these properties. [It is well known that ℓ^p is a complete metrizable vector space which has sufficiently many continuous characters and which is not locally convex. According to 6.5, ℓ^p is not locally quasi-convex.]

Remark 8.28. Since every reflexive group is locally quasi-convex, the above example gives a negative answer to two questions:

Noble asked (in 1970) whether every complete k-group G such that α_G is injective is reflexive. In [57], Nickolas proved that the group $A([0,1])$ (the free abelian group generated by $[0,1]$) has these properties and is not reflexive.

In 1995, Pestov strengthened the above question: Is every Čech-complete group G with sufficiently many characters a reflexive group?

In 11.15, we give an example of a complete metrizable group G which is locally quasi-convex but α_G is not surjective.

Notes 8.29. ck-spaces were examined by Frölicher ([29]). The example given in 8.12 was also motivated by [29]. The formulation of 8.13 is weaker than Lemma 2.2 in [47]. The proof is considerably different. The part concerning the continuity in 8.24 is taken from [8], (15.9); it is due to Mackey and a proof can be found in [2]. I am indebted to G. Turnwald for an inspiring discussion which led to 8.27.

9. Locally convex vector groups

In this chapter we introduce locally convex vector groups. These objects are important to characterize locally quasi-convex groups (10.1). Besides other things, we prove that every locally convex vector group is locally quasi-convex and we give an example of a locally convex vector group which illustrates the connection with locally convex vector spaces.

Definition 9.1 (Raĭkov). Let V be a real vector space and let $\mathcal{O}$ be a Hausdorff group topology on V. $(V,\mathcal{O})$ is called a *locally convex vector group* if there exists a neighbourhood basis of 0 consisting of symmetric convex sets.

Example 9.2. $(\mathbb{R}^n,\mathcal{O})$ $(n\in\mathbb{N})$ is a locally convex vector group if and only if there exist a basis $(x_1,\dots,x_n)$ of $\mathbb{R}^n$ and $m\in\{0,\dots,n\}$ such that $\mathcal{O}$ induces the usual topology on $\langle x_1,\dots,x_m\rangle_{\mathbb{R}}$ and the discrete topology on $\langle x_{m+1},\dots,x_n\rangle_{\mathbb{R}}$. [Cf. Proposition 3 in [41], p. 533 or Satz 2 in [50], p. 15.]

Remarks 9.3. (i) Let V be a locally convex vector group. For every $\lambda\in\mathbb{R}$, the mapping $V\to V$, $x\mapsto\lambda x$, is continuous and for $\lambda\neq 0$ it is a homeomorphism. [It suffices to show that the function is continuous for $\lambda>0$. Let $n\in\mathbb{N}_0$ be maximal with $n\le\lambda$. For $U\in\mathcal{U}_V(0)$, there exists a symmetric convex neighbourhood $\widetilde{U}\in\mathcal{U}_V(0)$ such that $(n+1)\widetilde{U}\subseteq U$ (this means that $u_1,\dots,u_{n+1}\in\widetilde{U}$ implies $\sum_{j=1}^{n+1}u_j\in U$). Hence, $\lambda\widetilde{U}=(n+(\lambda-n))\widetilde{U}\subseteq n\widetilde{U}+(n-\lambda)\widetilde{U}\subseteq n\widetilde{U}+\widetilde{U}\subseteq U$.]

(ii) The class of locally convex vector groups is closed under forming subspaces, quotients by closed subspaces, and arbitrary products.

(iii) Every real locally convex Hausdorff vector space is a locally convex vector group.

Theorem 9.4 (Kenderov). *For every subspace V_0 of the locally convex vector group $(V,\mathcal{O})$ and every continuous linear functional $f_0: V_0\to\mathbb{R}$, there exists a continuous linear functional $f: V\to\mathbb{R}$ extending f_0.*

Proof. Since f_0 is continuous, there exists a symmetric convex neighbourhood $U\in\mathcal{U}_V(0)$ such that $f_0(U\cap V_0)\subseteq[-1,1]$. The family $\left(\frac{1}{n}U\right)_{n\in\mathbb{N}}$ forms a neighbourhood basis of 0 in $\langle U\rangle_{\mathbb{R}}$ of a (not necessarily Hausdorff) vector space topology $\mathcal{O}_U$ (8.4) which is coarser than the topology induced by $\mathcal{O}$ (9.3(i)). With respect to this topology, the mapping $f_U:=f_0|_{V_0\cap\langle U\rangle_{\mathbb{R}}}$ is continuous (3.4(ii)). According to the Hahn–Banach Theorem, there exists a continuous linear functional $\widetilde{f_U}:\langle U\rangle_{\mathbb{R}}\to\mathbb{R}$ extending f_U. Furthermore, $\widetilde{f}:\langle U\rangle_{\mathbb{R}}+V_0\to\mathbb{R}$, $x+y\mapsto\widetilde{f_U}(x)+f_0(y)$ (where $x\in\langle U\rangle_{\mathbb{R}}$ and $y\in V_0$), is a well defined linear mapping (2.4) which is continuous since $\widetilde{f}|_{\langle U\rangle_{\mathbb{R}}}$ is continuous (with respect to $\mathcal{O}_U$ and hence with respect to $\mathcal{O}$) and $\langle U\rangle_{\mathbb{R}}$ is an open subspace of V. An arbitrary linear extension f of $\widetilde{f}$ has the desired properties.

Corollary 9.5. *Every closed subspace V_0 of a locally convex vector group V is dually closed.*

Proof. Fix $x_0 \in V \setminus V_0$. Since V/V_0 is a locally convex vector group (9.3(ii)), there exists, according to 9.4, a linear form $f : V \to \mathbb{R}$ such that $f(V_0) = \{0\}$ and $f(x_0) = 1/2$. Hence $\chi := \exp(2\pi i f)$ has the desired properties.

Proposition 9.6. *Every subspace* V_0 *of a locally convex vector group* $(V, \mathcal{O})$ *is dually embedded.*

Proof. Let $\chi_0 : V_0 \to \mathbb{T}$ be a character. There exists a symmetric convex neighbourhood $U \in \mathcal{U}_V(0)$ such that $\chi(V_0 \cap U) \subseteq R$. Proposition 3.4(ii) implies that $\chi_0|_{V_0 \cap \langle U \rangle_{\mathbb{R}}}$ is continuous with respect to the topology induced by the vector space topology $\mathcal{O}_U$ on $\langle U \rangle_{\mathbb{R}}$ which is determined by the neighbourhood basis $\left(\frac{1}{n}U\right)_{n \in \mathbb{N}}$ (8.4). According to 5.5 (or 8.15), there exists a linear functional $f_U : \langle U \rangle_{\mathbb{R}} \cap V_0 \to \mathbb{R}$ which is continuous with respect to $\mathcal{O}_U$ and such that $\exp(2\pi i f_U) = \chi_0|_{V_0 \cap \langle U \rangle_{\mathbb{R}}}$. By the Hahn–Banach Theorem, there exists a linear functional $f : \langle U \rangle_{\mathbb{R}} \to \mathbb{R}$ extending f_U which is continuous with respect to $\mathcal{O}_U$ and hence with respect to $\mathcal{O}$. According to 2.4, the mapping $\widetilde{\chi} : V_0 + \langle U \rangle_{\mathbb{R}} \to \mathbb{T}$, $x + y \mapsto \chi_0(x) \cdot \exp(2\pi i f(y))$ (for $x \in V_0$ and $y \in \langle U \rangle_{\mathbb{R}}$), is a well defined homomorphism. It is continuous since its restriction to the open subspace $\langle U \rangle_{\mathbb{R}}$ is continuous. Hence an arbitrary homomorphism χ extending $\widetilde{\chi}$ (2.3) is the desired extension of χ_0.

Proposition 9.7. *Let* U *be a symmetric convex neighbourhood of* 0 *in a locally convex vector group* V. *If* $\sigma(V, V')$ *denotes the weak topology on* V (*this is the topology induced by all continuous linear functionals*) *then* $\overline{\frac{1}{2}U}^{\sigma(V,V')} \subseteq U$. *In particular*, *the weakly closed symmetric convex neighbourhoods form a neighbourhood basis of* 0.

Proof. This is Proposition 4 in [41], p. 535.

Proposition 9.8. *Every weakly closed symmetric convex neighbourhood* U *of* 0 *in a locally convex vector group* V *is quasi-convex.*

Proof. For $x_0 \notin U$, there exist $f_1, \dots, f_n \in V'$ and $\varepsilon > 0$ such that $\{y \in V : |f_j(x_0) - f_j(y)| < \varepsilon \text{ for all } j \in \{1, \dots, n\}\} \cap U = \emptyset$. Put $F : V \to \mathbb{R}^n$, $x \mapsto (f_1(x), \dots, f_n(x))$. Obviously, $F(U)$ is a symmetric convex subset in $\mathbb{R}^n$ and $F(x_0) \notin \overline{F(U)}$. Hence, there exists a linear form $h : \mathbb{R}^n \to \mathbb{R}$ satisfying $h(F(U)) \subseteq [-1/4, 1/4]$ and $h(F(x_0)) \in]1/4, 3/4[$. The character $\chi := \exp(2\pi i (h \circ F))$ satisfies $\chi(U) \subseteq R$ and $\chi(x_0) \notin R$.

Corollary 9.9. *Every locally convex vector group* V *is locally quasi-convex.*

Proof. The assertion follows from 9.7 and 9.8.

Remark 9.10. The reals endowed with the weak (group) topology are a locally quasi-convex group (6.2) which is not locally compact. Hence it is not a locally convex vector group (9.2). This shows that a vector space V endowed with a locally quasi-convex Hausdorff topology need not be a locally convex vector group! Cf. 6.5 and 20.20.

Proposition 9.11. *Let* $(V, \mathcal{O})$ *be a locally convex vector group. The set of all symmetric*, *convex and radial neighbourhoods of* 0 *in* $\mathcal{O}$ *forms a neighbourhood basis of a locally convex vector space topology* $\mathcal{O}_V$ *on* V. *This is the finest locally convex vector space topology which is coarser than* $\mathcal{O}$.

For every locally convex vector space F *and every linear functional* $\varphi : V \to F$ *we have*: φ *is continuous with respect to* $\mathcal{O}$ *if and only if it is continuous with respect to* $\mathcal{O}_V$.

Proof. This is almost clear. See also [50], p. 23 or Behauptung (2.1) in [51].

Example 9.12. Let V_0 be a closed subspace of a real locally convex vector space $(V, \mathcal{O})$. If $\mathcal{W}$ denotes the set of all closed subspaces of finite codimension including V_0 and if $\mathcal{U}$ is a neighbourhood basis of 0 in V consisting of symmetric convex sets then $\mathcal{U}_0 := (U \cap W : U \in \mathcal{U}$ and $W \in \mathcal{W})$ forms a neighbourhood basis for a locally convex vector group topology $\mathcal{O}_{V_0}$ on V. [Applying 2.5, it remains to observe that the intersection of two closed finite-codimensional subspaces of V containing V_0 is of the same type.]

Since the topologies on V_0 induced by $\mathcal{O}$ and $\mathcal{O}_{V_0}$ coincide, it is clear that V_0 is connected. On the other hand, for every $x \in V \setminus V_0$, there exists $W \in \mathcal{W}$ such that $x \notin W$. Hence V_0 is the component of 0 in $(V, \mathcal{O}_{V_0})$.

Moreover, it follows easily from 8.20 that the finest vector space topology on V which is coarser than $\mathcal{O}_{V_0}$ is again $\mathcal{O}$.

The above construction enables us to reduce questions concerning locally convex vector groups to locally convex spaces; for instance:

Applications 9.13. (i) There exists a locally convex vector group $(V, \mathcal{O})$ such that the component V_0 is not an inner direct summand. [It is well known that there exists a Banach space V which has a closed subspace V_0 which is not an inner direct summand. (See e.g. [73], p. 136.) For $\mathcal{O}_{V_0}$ as in 9.12, the assertion follows from 9.11.]

(ii) If $(V, \mathcal{O})$ is a complete locally convex vector space and if V_0 is a closed subspace of V then the locally convex vector group $(V, \mathcal{O}_{V_0})$ is also complete. [Let $(x_j)_{j \in J}$ be a Cauchy net in $(V, \mathcal{O}_{V_0})$. Obviously, $(x_j)_{j \in J}$ is also a Cauchy net in $(V, \mathcal{O})$. By assumption, there exists $x \in V$ such that $x_j \to x$ in $\mathcal{O}$. Since x has a neighbourhood basis in $\mathcal{O}_{V_0}$ consisting of sets which are closed with respect to $\mathcal{O}$ we find that $(x_j)_{j \in J}$ converges to x in $\mathcal{O}_{V_0}$.]

Note 9.14. 9.4 is taken from [41].

10. Two representations of locally quasi-convex groups

In this chapter we give two representations of locally quasi-convex groups. The first asserts that every locally quasi-convex group is topologically isomorphic to a Hausdorff quotient of a subgroup of a locally convex vector group (10.1). Secondly, we prove that every locally quasi-convex group can be embedded into a product of complete metrizable locally quasi-convex groups (10.6). Applying this embedding theorem to locally convex vector groups, we prove that α_V is an open isomorphism for every complete locally convex vector group V (10.8).

Theorem 10.1. *For every locally quasi-convex Hausdorff group G with a neighbourhood basis $\mathcal{U}$, there exist: a locally convex vector group V, a subgroup H of V, and a closed subgroup $K \leq H$ such that $G \cong H/K$. Moreover, V can be chosen so that $0 \in V$ has a neighbourhood basis of the cardinality of $\mathcal{U}$.*

The construction in the following proof will be taken up again in the proof of 20.32.

Proof. We may assume that $\mathcal{U}$ consists of quasi-convex sets. For $U \in \mathcal{U}$, we put

$$X_U := \{\kappa \in \mathbb{R}^{G^*} : (\exists x \in U : \varrho(\kappa(\chi)) = \chi(x) \text{ and } |\kappa(\chi)| \leq 1/4)\ \forall \chi \in U^0\}$$

and $Y_U := \operatorname{co} X_U$ where $\varrho : \mathbb{R} \to \mathbb{T},\ t \mapsto e^{2\pi i t}$, is the canonical projection.

Observe that

$$(*) \qquad |\kappa(\chi)| \leq 1/(4n) \quad \text{for all } \kappa \in Y_U \text{ and all } \chi \in P(U, V_n).$$

[Since $\{\chi, \chi^2, \ldots, \chi^n\} \subseteq U^0$, we get $|\widetilde{\kappa}(\chi)| \leq 1/(4n)$ for arbitrary $\widetilde{\kappa} \in X_U$ (see 6.3).]

For $\widetilde{U}, U \in \mathcal{U}$ such that $\widetilde{U} + \widetilde{U} \subseteq U$ we have $Y_{\widetilde{U}} + Y_{\widetilde{U}} \subseteq Y_U$. [It suffices to show that $X_{\widetilde{U}} + X_{\widetilde{U}} \subseteq X_U$. For $\kappa_1, \kappa_2 \in X_{\widetilde{U}}$, there exist $x_1, x_2 \in \widetilde{U}$ such that

$$\varrho(\kappa_j(\widetilde{\chi})) = \widetilde{\chi}(x_j) \quad \text{and} \quad |\kappa_j(\widetilde{\chi})| \leq 1/4 \quad \text{for } j \in \{1, 2\} \text{ and all } \widetilde{\chi} \in \widetilde{U}^0.$$

Since $U^0 \subseteq P(\widetilde{U}, V_2)$ (by 6.3), $(*)$ implies $|\kappa_j(\chi)| \leq 1/8$ and hence $|(\kappa_1 + \kappa_2)(\chi)| \leq 1/4$ for all $\chi \in U^0$. Furthermore, $\varrho((\kappa_1 + \kappa_2)(\chi)) = \chi(x_1) \cdot \chi(x_2) = \chi(x_1 + x_2)$, which implies the assertion, since $x_1 + x_2 \in U$.]

Obviously, for $\widetilde{U}, U \in \mathcal{U}$ such that $\widetilde{U} \subseteq U$ we have $Y_{\widetilde{U}} \subseteq Y_U$. Since all $U \in \mathcal{U}$ are symmetric, so are the Y_U. It is a consequence of 2.5 that $(Y_U)_{U \in \mathcal{U}}$ forms a neighbourhood basis of $(0)_{\chi \in G^*}$ of a group topology $\mathcal{O}$ on $V := \mathbb{R}^{G^*}$. For $\kappa \neq (0)$, there exists $\chi \in G^*$ such that $\kappa(\chi) \neq 0$. If $|\kappa(\chi)| > 1/4$ then $\kappa \notin Y_U$ for all U. Otherwise, for $n \in \mathbb{N}$ and $U \in \mathcal{U}$ such that $\kappa(\chi) \in]1/(4n), 1/4]$ and $\chi(U) \subseteq V_n$, we conclude from $(*)$ that $\kappa \notin Y_U$. This shows that $(V, \mathcal{O})$ is a Hausdorff space and the above construction yields that it is a locally convex vector group.

The induced mapping $\alpha : G \to \alpha_G(G),\ x \mapsto \alpha_G(x)$, is an isomorphism (6.10). Let $\widetilde{\varrho} : \mathbb{R}^{G^*} \to \mathbb{T}^{G^*},\ (x_\chi) \mapsto (\varrho(x_\chi))$, be the canonical epimorphism and put $H := \widetilde{\varrho}^{-1}(\alpha_G(G))$; then $K := \widetilde{\varrho}^{-1}(\{(1)_{\chi \in G^*}\})$ is a closed subgroup of H. Let $\pi : H \to H/K$ denote the canonical projection. The homomorphism $\varrho' : H/K \to \alpha_G(G)$ induced by $\widetilde{\varrho}$ is an isomorphism.

$$\begin{tikzcd}
 & H \arrow[d, "\widetilde{\varrho}|_H"'] \arrow[dr, "\pi"] & \\
G \arrow[r, "\alpha"'] & \alpha_G(G) & H/K \arrow[l, "\varrho'"]
\end{tikzcd}$$

In order to prove that $\alpha^{-1} \circ \varrho'$ is a topological isomorphism, it is enough to show

$$(**) \qquad (\varrho'(\pi(H \cap Y_U))) = \widetilde{\varrho}(H \cap Y_U) = \alpha(U)$$

for all $U \in \mathcal{U}$, since π is a continuous open epimorphism.

Fix $U \in \mathcal{U}$ and $x \in U$. For $\kappa \in H$ such that $\widetilde{\varrho}(\kappa) = \alpha(x)$ and $\chi \in U^0$, we get $\varrho(\kappa(\chi)) = \widetilde{\varrho}(\kappa)(\chi) = \alpha(x)(\chi) = \chi(x) \in R$. In addition, we may assume that $|\kappa(\chi)| \leq 1/4$ for all $\chi \in U^0$. This shows that $\kappa \in X_U$ and "$\supseteq$" follows.

Let $\kappa \in Y_U \cap H$. There exists $x \in G$ such that $\widetilde{\varrho}(\kappa) = \alpha(x)$. For $\chi \in U^0$ we get, according to $(*)$, $\chi(x) = \alpha(x)(\chi) = \widetilde{\varrho}(\kappa)(\chi) = \exp(2\pi i \kappa(\chi)) \in R$, which implies $x \in U$ since U was assumed to be quasi-convex. This completes the proof.

Lemma 10.2. *Let G be an abelian Hausdorff group and let $\mathcal{U}$ be a neighbourhood basis of the unit element. Assume that for each $U \in \mathcal{U}$, there exist*

(i) *a closed subgroup* $H_U \subseteq U$ *such that* $U = U + H_U$, *and*

(ii) *a group topology* $\mathcal{O}_{\mathcal{U}}$ *on* G/H_U *which is coarser than the quotient topology* (*this means that the canonical projection* $\pi_U : G \to (G/H_U, \mathcal{O}_U)$ *is continuous*) *and such that* $\pi_U(U)$ *is a neighbourhood of* $\pi_U(e)$ *with respect to* $\mathcal{O}_U$.

Then the mapping

$$\Phi : G \to \prod_{U\in\mathcal{U}} (G/H_U, \mathcal{O}_{\mathcal{U}}), \quad x \mapsto (\pi_U(x))_{U\in\mathcal{U}},$$

is an embedding and $\Phi(G)$ *is dually embedded in the product.*

Proof. Since all π_U are continuous, so is Φ.

For $x \in G \setminus \{e\}$, there is a $U \in \mathcal{U}$ such that $x \notin U$ and hence $x \notin H_U$, which implies $\pi_U(x) \neq \pi_U(e)$. So Φ is injective.

Fix $V \in \mathcal{U}$. Since $\pi_V(x) \in \pi_V(V)$ if and only if $x \in V + H_V \overset{\text{(i)}}{=} V$ we get $\Phi(V) = \{(\pi_U(x))_{U\in\mathcal{U}} : x \in G,\, \pi_V(x) \in \pi_V(V)\} = \Phi(G) \cap (\pi_V(V) \times \prod_{U\in\mathcal{U}\setminus\{V\}} G/H_U)$. Hence (ii) implies that Φ is an embedding.

For each $\chi \in \Phi(G)^*$ the composition $\chi \circ \Phi$ belongs to G^* and hence there is a $V \in \mathcal{U}$ such that $\chi(\Phi(V)) \subseteq R$. This implies $\chi \circ \Phi \in H_V^{\perp}$ (3.3(ii)). Hence $\widetilde{\chi} : G/H_V \to \mathbb{T}$, $x + H_V \mapsto \chi(\Phi(x))$, is well defined and it follows from (i), (ii) and 3.4(ii) that $\widetilde{\chi}$ is continuous with respect to $\mathcal{O}_V$. Let $\Pi_V : \prod_{U\in\mathcal{U}} G/H_U \to G/H_V$ denote the canonical projection. Then $\widetilde{\chi} \circ \Pi_V \in (\prod_{U\in\mathcal{U}} G/H_U)^*$ and, for $(\pi_U(x))_{U\in\mathcal{U}} \in \Phi(G)$, we have $\widetilde{\chi}(\Pi_V((\pi_U(x))_{U\in\mathcal{U}})) = \widetilde{\chi}(\pi_V(x)) = \chi(\Phi(x))$. This means $\widetilde{\chi} \circ \Pi_V|_{\Phi(G)} = \chi$ and hence $\Phi(G)$ is dually embedded in $\prod_{U\in\mathcal{U}} G/H_U$. This completes the proof.

Lemma 10.3. *Let* G *be an abelian topological group and let* $U = \bigcap_{\chi\in S} \chi^{-1}(V_n)$ (*for some* $n \in \mathbb{N}$ *and* $S \subseteq G^*$) *be a quasi-convex subset of* G. *Then* $H_U := \bigcap_{\chi\in S} \ker\chi$ *is the largest subgroup contained in* U; *it is closed and* $U = U + H_U$. *If* $\pi_U : G \to G/H_U$ *denotes the canonical projection then* $\pi_U(U)$ *contains no non-trivial subgroup of* G/H_U.

The set $W := \{x \in U : x + x \in U\}$ *is quasi-convex and equals* $\bigcap_{\chi\in S} \chi^{-1}(V_{2n})$; *furthermore*, $W + W \subseteq U$. *If* U *is a neighbourhood of the unit element*, *so is* W.

Proof. Obviously, H_U is a closed subgroup of G and a subset of U. Let $\widetilde{H} \leq G$ be contained in U. For all $\chi \in S$ we get $\chi(\widetilde{H}) = \{1\}$ (3.3(ii)), which implies $\widetilde{H} \leq H_U$.

$U \subseteq U + H_U$ is trivial. For each $\chi \in S$ we have $\chi(H_U) = \{1\}$. This implies $\chi(H_U + U) = \chi(U) \subseteq V_n$ and hence $U = \bigcap_{\chi\in S} \chi^{-1}(V_n) \supseteq U + H_U$.

For every subgroup $A \leq G/H_U$ contained in $\pi_U(U)$, the subgroup $\pi_U^{-1}(A)$ of G is contained in U and hence in H_U, which implies that $A = \{e_{G/H_U}\}$.

By the definition of U, we have $x \in W$ if and only if $\chi(x) \in V_n$ and $\chi(x + x) \in V_n$ for all $\chi \in S$; this is equivalent to $\chi(x) \in V_{2n}$ for all $\chi \in S$. Hence, according to 6.2(iii), (ii) and 6.4, $W = \bigcap_{\chi\in S} \chi^{-1}(V_{2n})$ is quasi-convex. For $x, y \in W$ and $\chi \in S$ we have $\chi(x + y) \in V_{2n} \cdot V_{2n} = V_n$, which implies $x + y \in U$. The assertion follows easily.

Proposition 10.4. *Let* $U = \bigcap_{\chi\in U^0} \chi^{-1}(V_1)$ *be a quasi-convex neighbourhood of the unit element of the abelian Hausdorff group* G. *For* H_U *as in* 10.3, *there exists a coarsest group topology* $\mathcal{T}_U$ *such that* $(G/H_U, \mathcal{T}_U)$ *is a locally quasi-convex Hausdorff group and such that the image of* U *under the canonical projection is a neighbourhood of* e_{G/H_U}.

A neighbourhood basis of e_{G/H_U} *in* $\mathcal{T}_U$ *is given by the sets* $(\bigcap_{\chi\in U^0}\chi^{-1}(V_{2^n}))_{n\in\mathbb{N}_0}$. *In particular*, $(G/H_U,\mathcal{T}_U)$ *is a metrizable group.*

Proof. Put $U_0 := U$ and, inductively, $U_{n+1} := \{x \in U_n : x + x \in U_n\}$. According to 10.3, the U_n are quasi-convex neighbourhoods containing H_U and satisfying $U_n = \bigcap_{\chi\in U^0}\chi^{-1}(V_{2^n})$. Hence $\bigcap_{n\in\mathbb{N}_0} U_n = \{x \in U : \chi(x) = 1\ \forall\chi \in U^0\} = H_U$. Let $\pi_U : G \to G/H_U$ denote the canonical projection. Applying 2.5, 10.3 and 2.19, it is easy to check that $(\pi_U(U_n))_{n\in\mathbb{N}_0}$ is a neighbourhood basis of $\pi_U(e)$ of a Hausdorff group topology $\mathcal{T}_U$ on G/H_U. Furthermore, $(G/H_U,\mathcal{T}_U)$ is locally quasi-convex. [Let $\pi_U(x) \notin \pi_U(U_n)$ for some $n \in \mathbb{N}_0$. Hence $x \notin U_n + H_U \overset{10.3}{=} U_n$, and therefore there is a $\chi \in G^*$ such that $\chi(U_n) \subseteq R$ and $\chi(x) \notin R$. Since $H_U \le \ker\chi$ (3.3(ii)), there is a continuous (with respect to the quotient topology) homomorphism $\widetilde{\chi} : G/H_U \to \mathbb{T}$ satisfying $\widetilde{\chi}\circ\pi_U = \chi$. Since $\widetilde{\chi}(\pi_U(U_n)) \subseteq R$, it follows that $\widetilde{\chi}$ is also continuous with respect to $\mathcal{T}_U$ (3.4(ii)).] Having a countable neighbourhood basis $(\pi_U(U_n))_{n\in\mathbb{N}_0}$ and being a Hausdorff space, $(G/H_U,\mathcal{T}_U)$ is a metrizable group (2.7).

Now, let $\mathcal{O}$ be any group topology on G/H_U such that $\pi_U(U)$ is a neighbourhood of $\pi_U(e)$. For a neighbourhood $\widetilde{V}$ of $\pi_U(e)$ (in $\mathcal{O}$) satisfying $\widetilde{V}^{2^n} \subseteq \pi_U(U)$, we get $\pi_U^{-1}(\widetilde{V}) \subseteq U_n$ (by 6.3) and hence $\pi_U(U_n) \supseteq \widetilde{V}$. The assertion follows.

Lemma 10.5. *Let* U *be a weakly closed* (*cf.* 9.7) *symmetric convex neighbourhood of a locally convex vector group* G. *Then* $H_U := \{x \in U : \lambda\cdot x \in U\ \forall\lambda \in \mathbb{R}\}$ *is a subspace and the largest subgroup contained in* U. *If* $\pi_U : G \to G/H_U$ *denotes the canonical projection then* $\pi_U(U)$ *contains only the trivial subgroup and* $(\pi_U((1/2^n)U))_{n\in\mathbb{N}}$ *forms a neighbourhood basis of a metrizable locally convex vector group topology* $\mathcal{T}_U$ *on* G/H_U. *Moreover*, $(G/H_U,\pi_U,\mathcal{T}_U)$ *satisfies the assumptions of* 10.2 *and* G/H_U *is topologically isomorphic to the product of a metrizable locally convex space and a discrete vector space.*

Proof. Let $\psi_U : \langle U\rangle_{\mathbb{R}} \to \mathbb{R}$, $x \mapsto \inf\{\lambda > 0 : x \in \lambda U\}$, denote the Minkowski functional corresponding to U. Since every subgroup contained in U is mapped onto $\{0\}$ under ψ_U, we see that for the union H_U of all subgroups in U, $H_U \subseteq \psi_U^{-1}(\{0\})$. Conversely, $\psi_U^{-1}(\{0\})$ is a vector space contained in U. This shows that equality holds. It is a consequence of 9.8 and 10.3 that H_U is closed and that $U = U + H_U$. For every subgroup $A \le V/H_U$ contained in $\pi_U(U)$ the subgroup $\pi_U^{-1}(A)$ is contained in $U + H_U = U$ and hence in H_U, which implies that A is trivial.

H_U being a subspace, the canonical projection $\pi_U : G \to G/H_U$ is a linear operator. Hence $\pi_U(U)$ is symmetric and convex and the sets $(\pi_U((1/2^n)U))_{n\in\mathbb{N}_0}$ form a neighbourhood basis of a group topology $\mathcal{T}_U$ on (the vector space) G/H_U (by 2.5). It follows from 2.19 and 10.3 that $(G/H_U,\mathcal{T}_U)$ is a Hausdorff space. By construction, it is a locally convex vector group. The open subspace $\pi_U(\langle U\rangle_{\mathbb{R}})$ is a metrizable locally convex vector space (2.7 and 8.4). Hence, any subspace $F_U \le G/H_U$ complementing $\pi_U(\langle U\rangle_{\mathbb{R}})$ algebraically and endowed with the discrete topology is a topological inner direct summand.

Theorem 10.6. *Every locally quasi-convex Hausdorff group* (*locally convex vector group*) G *can be embedded into a product of complete metrizable locally quasi-convex Hausdorff groups* (*of complete metrizable locally convex spaces and discrete spaces*) *such that the image is dually embedded. If* G *is complete the image is dually closed.*

Proof. Let $\mathcal{U}$ be a neighbourhood basis of e in G consisting of quasi-convex (weakly closed symmetric convex) sets. According to 10.3 and 10.4 (10.5), the sets U, H_U and $\mathcal{T}_U$ satisfy the conditions of 10.2. Hence $\Phi : G \to \prod_{U\in\mathcal{U}}(G/H_U, \mathcal{T}_U)$, $x \mapsto (\pi_U(x))_{U\in\mathcal{U}}$, is an embedding into a product of metrizable locally quasi-convex groups (metrizable locally convex vector groups which are topologically isomorphic to the product of a metrizable locally convex vector space and a discrete vector space) and $\Phi(G)$ is dually embedded.

Assume first that G is a locally quasi-convex group. For $V \subseteq U$, $V, U \in \mathcal{U}$, we put $\pi_{UV} : G/H_V \to G/H_U$, $x + H_V \mapsto x + H_U$. By the definition of H_U (cf. 10.3), this mapping is well defined. For $n \in \mathbb{N}_0$, we have

$$\begin{aligned}\pi_{UV}^{-1}\Big(\pi_U\Big(\bigcap_{\chi\in U^0}\chi^{-1}(V_{2^n})\Big)\Big) &= \bigcap_{\chi\in U^0}\{\pi_V(x) : \chi(x)\in V_{2^n}\} \supseteq \bigcap_{\chi\in V^0}\{\pi_V(x) : \chi(x)\in V_{2^n}\}\\ &= \pi_V\Big(\bigcap_{\chi\in V^0}\chi^{-1}(V_{2^n})\Big) \in \mathcal{U}_{G/H_V}(e).\end{aligned}$$

Hence π_{UV} is continuous. It follows from 2.13 and 6.17 that the completion $\widetilde{G/H_U}$ exists and that it is locally quasi-convex.

Let $\widetilde{\pi_{UV}} : \widetilde{G/H_V} \to \widetilde{G/H_U}$ denote the continuous homomorphism extending π_{UV} (see 2.17) and let $\iota : \prod_{U\in\mathcal{U}} G/H_U \to \prod_{U\in\mathcal{U}} \widetilde{G/H_U}$ be the canonical embedding. Obviously, $\iota \circ \Phi$ is an embedding. Since ι has dense image, it is a consequence of 2.17 (or 4.1) that $\iota(\Phi(G))$ is dually embedded in $\prod_{U\in\mathcal{U}} \widetilde{G/H_U}$.

Assume now that G is complete. Put $P := \varprojlim(G/H_U, \pi_{UV}, \mathcal{U})$ and $\widetilde{P} := \varprojlim(\widetilde{G/H_U}, \widetilde{\pi_{UV}}, \mathcal{U})$. It is clear that $\Phi(G) \subseteq P$ and that $\iota(P) \subseteq \widetilde{P}$. In order to prove that G is dually closed in $\prod_{U\in\mathcal{U}} \widetilde{G/H_U}$, it suffices to show that the other inclusions also hold (6.10 and 5.28). Therefore, let $(x_U + H_U)_{U\in\mathcal{U}} \in P$. The net $(x_U)_{U\in\mathcal{U}}$ is a Cauchy net in G. [For $U \in \mathcal{U}$ and $V, V' \in \mathcal{U}$ contained in U, we get $x_V - x_{V'} = (x_V - x_U) - (x_{V'} - x_U) \in H_U - H_U = H_U \subseteq U$.] One easily checks that $x := \lim_{U\in\mathcal{U}} x_U$ satisfies $\Phi(x) = (x_U + H_U)_{U\in\mathcal{U}}$. This shows that $\Phi(G) = P$. In order to prove $\iota(P) = \widetilde{P}$, it suffices to check that $\iota(P)$ is dense in $\widetilde{P}$ since $\iota(P) = \iota(\Phi(G))$ is complete and hence closed.

The sets $(\prod_{U\in\mathcal{U}\setminus\{U_0\}} \widetilde{G/H_U} \times \overline{\pi_{U_0}(U_0)}) \cap \widetilde{P}$ (where $U_0 \in \mathcal{U}$ and the closure is taken in $\widetilde{G/H_{U_0}}$) form a neighbourhood basis of e in $\widetilde{P}$. [According to 6.15, the sets $(\prod_{j=1}^n \overline{W}_j \times \prod_{U\in\mathcal{U}\setminus\{U_1,\dots,U_n\}} G_U/H_U) \cap \widetilde{P}$, where W_j is a neighbourhood of e in $(G/H_{U_j}, \mathcal{T}_{U_j})$ and $n\in\mathbb{N}$, form a neighbourhood basis of e in $\widetilde{P}$. For $U_0 \in \mathcal{U}$ contained in $\bigcap_{j=1}^n(\pi_{U_j}^{-1}(W_j)\cap U_j)$, we have $\widetilde{\pi}_{U_jU_0}(\overline{\pi_{U_0}(U_0)}) \subseteq \overline{\pi_{U_jU_0}(\pi_{U_0}(U_0))} \subseteq \overline{W}_j$. This shows that the above neighbourhood contains $(\prod_{U\in\mathcal{U}\setminus U_0} \widetilde{G/H_U} \times \overline{\pi_{U_0}(U_0)}) \cap \widetilde{P}$.]

For $(\widetilde{x}_U)_{U\in\mathcal{U}} \in \widetilde{P}$ and $U_0 \in \mathcal{U}$, there exists $x \in G$ such that $\pi_{U_0}(x) - \widetilde{x}_{U_0} \in \overline{\pi_{U_0}(U_0)}$; so we get $\iota(\Phi(x)) - (\widetilde{x}_U)_{U\in\mathcal{U}} \in (\prod_{U\in\mathcal{U}\setminus U_0} \widetilde{G/H_U} \times \overline{\pi_{U_0}(U_0)}) \cap \widetilde{P}$.

Assume now that G is a locally convex vector group. Changing the notation, we write the above embedding $\Phi : G \to \prod_{U\in\mathcal{U}}(V_U \times F_U)$ where V_U is a metrizable locally convex vector space and F_U is a discrete vector space. According to 1.5 in [74] and 2.7, the completion $\widetilde{V}_U$ of V_U is a complete metrizable locally convex vector space. Let $\iota : \prod_{U\in\mathcal{U}}(V_U \times F_U) \to \prod_{U\in\mathcal{U}}(\widetilde{V}_U \times F_U)$ denote the canonical embedding. Since both Φ

and ι are continuous linear operators, $\iota(\Phi(G))$ is a linear subspace and hence, according to 9.6, dually embedded. If G is complete, so is $\iota(\Phi(G)$. Hence $\iota(\Phi(G))$ is closed and it follows from 9.5 that $\iota(\Phi(G))$ is dually closed in the product.

Combining 10.1 and 10.6 yields:

PROPOSITION 10.7. *Every locally quasi-convex Hausdorff group G is topologically isomorphic to a Hausdorff quotient group of a subgroup of a locally convex vector space and a product of discrete* (*vector*) *spaces.*

COROLLARY 10.8. *For every complete locally convex vector group V, the evaluation mapping α_V is an open isomorphism. In particular*, *every complete metrizable locally convex vector group is reflexive.*

PROOF. According to the above theorem, V can be considered as a dually closed and dually embedded subspace of a product of complete metrizable locally convex vector spaces and discrete spaces. According to 5.4, 8.18, 5.12 and 5.3, the product is a Pontryagin reflexive group. The assertion follows from 5.25 and 5.12.

REMARK 10.9. In the next chapter we give an example of a complete metrizable locally quasi-convex group which is not reflexive. This shows that the linear structure of locally convex vector groups is essential for the validity of 10.8.

NOTES 10.10. 10.1 is the first part of the proof of Theorem (9.6) in [8]. The statement is new. The assertion of 10.8 is Theorem (15.7) in [8] and the proof given here is a modification of the arguments given there.

11. The character groups of $\mathrm{L}^p_{\mathbb{Z}}([0,1])$ and $\mathrm{L}^p([0,1])$

In this chapter we give an example of a closed subgroup H of a completely metrizable (locally quasi-convex) reflexive group G (more precisely, a locally convex space) which is not the whole group but the dual homomorphism $\iota^* : G^* \to H^*$ is a topological isomorphism. This means that H and G have the same character group. In particular, this example shows that α_H need not be surjective if H is a locally quasi-convex and completely metrizable abelian group (cf. 20.38).

Let $H := \mathrm{L}^p_{\mathbb{Z}}([0,1])$ be the subgroup of $G := \mathrm{L}^p([0,1])$ consisting of all almost everywhere integer-valued functions where $p \in \,]1,\infty[$ and let $\mathcal{S} := \langle 1_B : B \in \mathcal{B}([0,1])\rangle_{\mathbb{R}}$ be the vector space spanned by all indicator functions of Borel measurable sets.

REMARKS 11.1. (i) H is closed in G. [Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H converging to $x \in G$. By the Riesz–Fischer theorem, we can choose a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging almost everywhere to x. Hence the values of x are almost everywhere in $\mathbb{Z}$.]

(ii) $H = \overline{\mathcal{S} \cap H}$. ["$\supseteq$" follows from (i). Each $x \in H$ has the form $x = \sum_{k\in\mathbb{Z}} k \cdot 1_{B_k}$ for suitable $B_k \in \mathcal{B}$. It is a consequence of the dominated convergence theorem that $\sum_{|k|\le n} k \cdot 1_{B_k} \to x$ in L^p as $n \to \infty$.]

(iii) $G = \overline{\mathcal{S}}$. [See [71], p. 69, (3.13).]

REMARK 11.2. For positive numbers a, b we have $(a+b)^p \le 2^p\cdot(\max(a,b))^p \le 2^p\cdot(a^p+b^p)$.

Lemma 11.3. *The mapping*

$$\sigma : [0,1] \times G \to G, \quad (t,x) \mapsto 1_{[0,t]} \cdot x,$$

is continuous, $x \mapsto \sigma(t,x)$ *is*, *for fixed* $t \in [0,1]$, *a homomorphism*, *and* $\sigma([0,1] \times H) \subseteq H$.

Proof. Since $\|x \cdot 1_{[0,t]}\|_p \le \|x\|_p$ for all $x \in G$ and $t \in [0,1]$, σ is well defined. Obviously, $\sigma([0,1] \times H) \subseteq H$ and $\sigma(t,\cdot)$ is a homomorphism.

So it suffices to prove the continuity of σ. Let therefore $(t_n, x_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1] \times G$ converging to (t_0, x_0). For all $n \in \mathbb{N}$ such that $t_n > t_0$, we get

$$\begin{aligned} \int_0^1 |x_n \cdot 1_{[0,t_n]} - x_0 \cdot 1_{[0,t_0]}|^p \, d\lambda &= \int_0^{t_0} |x_0 - x_n|^p \, d\lambda + \int_{t_0}^{t_n} |x_n|^p \, d\lambda \\ &\le \int_0^{t_0} |x_0 - x_n|^p \, d\lambda + \int_{t_0}^{t_n} (|x_n - x_0| + |x_0|)^p \, d\lambda \\ &\overset{11.2}{\le} \int_0^1 |x_0 - x_n|^p \, d\lambda + 2^p \int_{t_0}^{t_n} (|x_n - x_0|^p + |x_0|^p) \, d\lambda \end{aligned}$$

and for $n \in \mathbb{N}$ such that $t_n \le t_0$,

$$\int_0^1 |x_n \cdot 1_{[0,t_n]} - x_0 \cdot 1_{[0,t_0]}|^p \, d\lambda = \int_0^{t_n} |x_0 - x_n|^p \, d\lambda + \int_{t_n}^{t_0} |x_0|^p \, d\lambda.$$

Now the continuity of σ is a consequence of the dominated convergence theorem.

Corollary 11.4. *H is pathwise connected but contains only the trivial one-parameter subgroup.*

Proof. It follows immediately from 11.3 that H is pathwise connected.

Let $f : \mathbb{R} \to H$ be a continuous homomorphism. Suppose there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \ne 0$. Since H is torsion-free, this leads to the contradiction $f(tx_0) = tf(x_0) \notin H$ for suitable $t \in \mathbb{Q} \cap \,]0,1[$.

Theorem 11.5. *Let G be a topological space and let $\varphi : [0,1] \times G \to \mathbb{T}$ be a continuous mapping such that $\varphi(0,\cdot) \equiv 1$. Then there exists a continuous lifting $\Phi : [0,1] \times G \to \mathbb{R}$ such that $e^{2\pi i \Phi} = \varphi$ and $\Phi(0,\cdot) \equiv 0$.*

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1] \times G} \arrow[ur, "\Phi"] \arrow[r, "\varphi"'] & \mathbb{T} \end{tikzcd}$$

If G is a topological group and $\varphi(t,\cdot)$ is a homomorphism for each $t \in [0,1]$, then $\Phi(t,\cdot)$ is a homomorphism as well.

Proof. For the existence of Φ, see [78], pp. 66ff.

Assume now that G is a topological group and that $\varphi(t,\cdot)$ is a homomorphism for all $t \in [0,1]$. For $x, y \in G$, put $\psi_{xy} : [0,1] \to \mathbb{R}$, $t \mapsto \Phi(t, x+y) - \Phi(t,x) - \Phi(t,y)$. Obviously, ψ_{xy} is continuous and $\psi_{xy}(0) = 0$. By assumption, we have $e^{2\pi i \psi_{xy}(t)} =$

$\varphi(t, x+y)\varphi(t,x)^{-1}\varphi(t,y)^{-1} = 1$. Since $\psi_{xy}([0,1])$ is connected and contained in $\mathbb{Z}$, it must be trivial. This completes the proof.

Theorem 11.6 (Nickolas). *If G is an abelian topological group and a k-space then the path-component C_{p} of e_{G^*} is the union of all one-parameter subgroups of G^*.*

Proof. Since the union of all one-parameter subgroups of G^* is contained in C_{p}, it has to be shown that for every $\chi \in C_{\mathrm{p}}$, there exists a continuous homomorphism $f : \mathbb{R} \to G^*$ satisfying $f(1) = \chi$. By assumption, there exists a continuous mapping $\widetilde{\varphi} : [0,1] \to G^*$ such that $\widetilde{\varphi}(0) = e_{G^*}$ and $\widetilde{\varphi}(1) = \chi$. This gives rise to the mapping $\varphi : [0,1] \times G \to \mathbb{T}$, $(t,x) \mapsto \widetilde{\varphi}(t)(x)$. Since $[0,1]\times G$ is a k-space (1.4), φ is continuous if and only if $\varphi|_{[0,1]\times K}$ is continuous for every compact subset $K \subseteq G$ (see 1.5). Therefore, fix $t_0 \in [0,1]$, $x \in K$ and $V \in \mathcal{U}_{\mathbb{T}}(1)$. There exist $U \in \mathcal{U}_G(e)$ such that $\widetilde{\varphi}(t_0)(U) \subseteq V$ and $\varepsilon > 0$ such that $\widetilde{\varphi}(t_0)\overline{\widetilde{\varphi}(t)} \in P(K,V)$ for all $t \in [0,1]$ satisfying $|t - t_0| < \varepsilon$. Hence, $\varphi(t_0, x_0)\overline{\varphi(t,x)} = \widetilde{\varphi}(t_0)(x_0 - x)\cdot(\widetilde{\varphi}(t_0)\cdot\overline{\widetilde{\varphi}(t)})(x) \in V\cdot V$ for all $x \in x_0 + U$. This shows that φ is continuous. According to 11.5, there exists $\Phi : [0,1]\times G \to \mathbb{R}$ satisfying $e^{2\pi i\Phi} = \varphi$. Furthermore, $\Phi(t,\cdot)$ is a continuous homomorphism for every $t \in [0,1]$. Put $\Phi_1 := \Phi(1,\cdot)$. The homomorphism $f : \mathbb{R} \to G^*$, $t \mapsto \exp(2\pi i t\cdot\Phi_1)$, satisfies $f(1) = \varphi(1,\cdot) = \chi$. Fix a compact $K \subseteq G$ and $\varepsilon > 0$. Since $\Phi_1(K)$ is compact, there exists $\delta > 0$ such that $\Phi_1(K) \subseteq\,]-\varepsilon/\delta, \varepsilon/\delta[$. This implies $f(]-\delta,\delta[) \subseteq P(K, \{e^{2\pi i t} : |t| \le \varepsilon\})$. So f is continuous and hence it is the desired one-parameter subgroup.

Corollary 11.7. *If G is a reflexive group such that G^* is a k-space, then the path-component of $e \in G$ is the union of all one-parameter subgroups of G.*

Proof. This is an immediate consequence of 11.6.

Corollary 11.8. *The path-component of the unit element of every almost metrizable reflexive group is the union of all one-parameter subgroups.*

Proof. This follows directly from 11.7 and 5.20.

Example 11.9. H is not reflexive. [H is a metrizable and hence an almost metrizable group (1.23(i)). Hence the assertion follows from 11.8 and 11.4.]

Lemma 11.10. *$\varrho : \operatorname{Hom}(H,\mathbb{R}) \to H^*$, $\psi \mapsto e^{2\pi i\psi}$, is a topological isomorphism where the groups are endowed with the compact-open topology.*

Proof. Obviously, ϱ is a continuous homomorphism.

Since H is connected (11.4), the kernel $\ker\varrho = \{\psi \in \operatorname{Hom}(H,\mathbb{R}) : \psi(H) \le \mathbb{Z}\}$ is trivial and hence ϱ is injective.

ϱ is surjective. Let $\chi \in H^*$. The mapping $\varphi : [0,1]\times H \to \mathbb{T}$, $(t,x) \mapsto \chi(\sigma(t,x))$, is a well defined continuous mapping and $\varphi(t,\cdot)$ is a homomorphism (11.3). Since $\varphi(\{0\}\times H) = \{1\}$, there exists a continuous lifting $\Phi : [0,1]\times H \to \mathbb{R}$ of φ such that $\psi : x \mapsto \Phi(1,x)$ belongs to $\operatorname{Hom}(H,\mathbb{R})$ (by 11.5). Since $e^{2\pi i\psi} = \chi$, the surjectivity of ϱ follows.

ϱ is open. Let $K \subseteq H$ be compact and let $\varepsilon > 0$. Because of 11.3, $\widetilde{K} := \sigma([0,1]\times K) \subseteq H$ is compact. For $V := \{e^{2\pi i t} : |t| < \varepsilon\}$ (and $\varepsilon \in\,]0, 1/2[$), we get $\varrho(P(K,]-\varepsilon,\varepsilon[)) \supseteq P(\widetilde{K}, V)$. [Let $e^{2\pi i\psi} \in P(\widetilde{K}, V)$ where $\psi : H \to \mathbb{R}$ is a continuous homomorphism; since ϱ is an epimorphism, each character is of this form. Fix $x \in K$. Since $[0,1] \to \mathbb{R}$,

$t \mapsto \psi(\sigma(t,x))$, is continuous (11.3), the image is connected. By assumption, $\psi(\sigma([0,1] \times \{x\})) \subseteq \,]{-\varepsilon}, \varepsilon[\, + \mathbb{Z}$. This implies $\psi(x) = \psi(\sigma(1,x)) \in \,]{-\varepsilon}, \varepsilon[\,]$.

Lemma 11.11. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers such that $\sum_{n\in\mathbb{N}} a_n = \infty$. For $n_0 := \min\{n \in \mathbb{N} : a_n > 0\}$ and $s_n := \sum_{k=1}^{n} a_k$ we have $\sum_{n\geq n_0} a_n/s_n = \infty$ and, for $p > 1$, $\sum_{n\geq n_0} a_n/s_n^p < \infty$.*

Proof. For $k > m \geq n_0$ we have

$$\sum_{n=m}^{k} \frac{a_n}{s_n} \geq \sum_{n=m}^{k} \frac{a_n}{s_k} = \frac{s_k - s_{m-1}}{s_k} \xrightarrow{k\to\infty} 1,$$

which shows that $(\sum_{n=n_0}^{k} a_n/s_n)_{k\geq n_0}$ cannot be a Cauchy sequence.

Fix $p > 1$. For $0 < a \leq b$ we get by the Mean Value Theorem

$$b^{1-p} - a^{1-p} \leq (b-a)(1-p)b^{-p} \quad \text{or} \quad \frac{b-a}{b^p} \leq \frac{1}{p-1}\left(\frac{1}{a^{p-1}} - \frac{1}{b^{p-1}}\right).$$

This implies

$$\begin{aligned}\sum_{n=n_0}^{k} \frac{a_n}{s_n^p} = \sum_{n=n_0}^{k} \frac{s_n - s_{n-1}}{s_n^p} &\leq a_{n_0}^{1-p} + \frac{1}{p-1} \sum_{n=n_0+1}^{k} \left(\frac{1}{s_{n-1}^{p-1}} - \frac{1}{s_n^{p-1}}\right) \\ &= a_{n_0}^{1-p} + \frac{1}{p-1}\left(\frac{1}{s_{n_0}^{p-1}} - \frac{1}{s_k^{p-1}}\right).\end{aligned}$$

The assertion follows.

Lemma 11.12. *If $y \in \mathrm{L}^q([0,1])$ (where $1/p + 1/q = 1$) satisfies $|\int_0^1 x \cdot y \, d\lambda| \leq 1$ for all $x \in H$ such that $\|x\|_p \leq \varepsilon$, then $\|y\|_q \leq 4(4/\varepsilon)^p + 6$.*

Proof. Put $y_+ := \max(y, 0)$. There exist disjoint measurable sets B_n such that $z := \sum_{n\in\mathbb{N}} n \cdot 1_{B_n} \leq y_+ \leq z+1$. We may assume that $\|z\|_q^q = \sum_{n\in\mathbb{N}} n^q \cdot \lambda(B_n) > 0$. (Otherwise, $\|y_+\|_q \leq 1$.) Observe that $\|z\|_q \leq \|y_+\|_q \leq \|y\|_q < \infty$. For $n \in \mathbb{N}$, there exist $k_n \in \mathbb{N}_0$ such that

$$\left| \frac{n^{q-1}}{\|z\|_q^{q-1}} - k_n \right| \leq \frac{1}{2}.$$

Put $x_+ := \sum_{n\in\mathbb{N}} k_n \cdot 1_{B_n}$. As a consequence of

$$\begin{aligned}\sum_{n\in\mathbb{N}} k_n^p \cdot \lambda(B_n) &\leq \sum_{n\in\mathbb{N}} \left(\frac{n^{q-1}}{\|z\|_q^{q-1}} + \frac{1}{2}\right)^p \cdot \lambda(B_n) \overset{11.2}{\leq} 2^p \sum_{n\in\mathbb{N}} \left(\frac{n^{pq-p}}{\|z\|_q^{pq-p}} + \frac{1}{2^p}\right) \cdot \lambda(B_n) \\ &= 2^p \sum_{n\in\mathbb{N}} \left(\frac{n^q \cdot \lambda(B_n)}{\|z\|_q^q} + \frac{1}{2^p} \cdot \lambda(B_n)\right) \\ &\leq 2^p\left(1 + \frac{1}{2^p}\right) = 2^p + 1 \leq 2^{p+1} \leq 2^{2p},\end{aligned}$$

we deduce that $x_+ \in H$ and $\|x_+\|_p \leq 4$. Furthermore,

$$\begin{aligned}\int_0^1 y\cdot x_+\,d\lambda = \int_0^1 y_+\cdot x_+\,d\lambda \ge \int_0^1 z\cdot x_+\,d\lambda &= \sum_{n\in\mathbb{N}} k_n\cdot n\cdot\lambda(B_n)\\ &\ge \sum_{n\in\mathbb{N}}\left(\frac{n^{q-1}}{\|z\|_q^{q-1}}-\frac{1}{2}\right)\cdot n\cdot\lambda(B_n) = \sum_{n\in\mathbb{N}}\frac{n^q}{\|z\|_q^{q-1}}\lambda(B_n)-\frac{1}{2}\|z\|_1\\ &\ge \|z\|_q-\frac{1}{2}\|z\|_q=\frac{1}{2}\|z\|_q.\end{aligned}$$

(The last inequality is a consequence of the Hölder inequality.)

On the other hand, according to 11.3, there exists a partition $0=t_0<t_1<\ldots<t_n=1$ such that $\|x_+\cdot 1_{[t_j,t_{j+1}[}\|_p\le\varepsilon$. Observing that $\|x_+\|_p^p=\sum_{j=0}^{n-1}\|x_+\cdot 1_{[t_j,t_{j+1}[}\|_p^p$, we may assume that $n\le(4/\varepsilon)^p+1$. Hence

$$\Big|\int_0^1 x_+\cdot y\,d\lambda\Big|\le\sum_{j=0}^{n-1}\Big|\int_0^1 x_+\cdot 1_{[t_j,t_{j+1}[}\cdot y\,d\lambda\Big|\le n$$

(by assumption). Combining these estimates yields

$$\frac{1}{2}\|z\|_q\le\Big|\int_0^1 x_+\cdot y\,d\lambda\Big|\le n\le\left(\frac{4}{\varepsilon}\right)^p+1$$

and hence $\|y_+\|_q\le\|z\|_q+1\le 2(4/\varepsilon)^p+3$.

The analogous estimate for $y_-:=-\min(0,y)$ implies the assertion.

LEMMA 11.13. *For* $p>1$, $\gamma:(\mathrm{L}^p([0,1]))'\to\mathrm{Hom}(H,\mathbb{R})$, $\Psi\mapsto\Psi|_H$, *is a topological isomorphism.*

PROOF. Obviously, γ is a continuous homomorphism.

γ *is injective.* Let $\Psi\in\mathrm{L}^p([0,1])'$ be such that $\Psi|_H\equiv 0$. Then $\Psi(1_B)=0$ for all $B\in\mathcal{B}([0,1])$ and since Ψ is linear, we get $\Psi(\mathcal{S})=\{0\}$. Since $\mathcal{S}$ is dense (11.1), $\Psi\equiv 0$ and hence γ is injective.

γ *is surjective.* For $\psi\in\mathrm{Hom}(H,\mathbb{R})$, the mapping $\mu:\mathcal{B}([0,1])\to\mathbb{R}$, $B\mapsto\psi(1_B)$, defines a signed measure which is absolutely continuous with respect to the Lebesgue measure $\lambda|_{[0,1]}$. By the Radon–Nikodym Theorem (cf. [71], p. 121), there exists $y\in\mathrm{L}^1([0,1])$ such that $\mu=y\cdot\lambda$. This means $\psi(1_B)=\mu(B)=\int_0^1 1_B\cdot y\,d\lambda$ for all $B\in\mathcal{B}$. Since both sides are additive, we get $\psi(x)=\int_0^1 x\cdot y\,d\lambda$ for all $x\in H\cap\mathcal{S}$. We now show that $y\in\mathrm{L}^q([0,1])$ where q satisfies $1/p+1/q=1$. This is equivalent to $y_+:=\max(0,y)$ and $y_-:=-\min(0,y)$ belong to $\mathrm{L}^q([0,1])$. There exist $B_n\in\mathcal{B}$ such that $z:=\sum_{n\in\mathbb{N}}n\cdot 1_{B_n}\le y_+\le z+1$. Put $s_n:=\sum_{k=1}^n k^q\cdot\lambda(B_\mathrm{k})$ and $k_n:=[n^{q-1}/s_n]$ for those n where s_n is positive and $k_n:=0$ otherwise. (For $t>0$ the largest integer smaller than or equal to t is denoted by $[t]$.) Suppose $(s_n)_{n\in\mathbb{N}}$ is not bounded. Let $n_0:=\min\{n\in\mathbb{N}:s_n>0\}$. According to 11.11, we get

$$\sum_{n\in\mathbb{N}}k_n^p\cdot\lambda(B_n)\le\sum_{n\ge n_0}\frac{n^{pq-p}}{s_n^p}\cdot\lambda(B_n)=\sum_{n\ge n_0}\frac{n^q}{s_n^p}\cdot\lambda(B_n)<\infty$$

and

$$\sum_{n=1}^{N} n\cdot k_n\cdot\lambda(B_n)\geq -\sum_{n\in\mathbb{N}} n\cdot\lambda(B_n)+\sum_{n=n_0}^{N}\frac{n^q}{s_n}\lambda(B_n)\xrightarrow{N\to\infty}\infty,$$

since

$$\sum_{n\in\mathbb{N}} n\cdot\lambda(B_n)=\|z\|_1\leq\|y_+\|_1\leq\|y\|_1<\infty.$$

This shows that $x_N:=\sum_{n=1}^{N}k_n\cdot 1_{B_n}\in H$ tends to $\sum_{n\in\mathbb{N}}k_n\cdot 1_{B_n}\in H$ but

$$\psi(x_N)=\int_0^1 x_N\cdot y\,d\lambda=\int_0^1 x_N\cdot y_+\,d\lambda\geq\sum_{n=1}^{N}k_n\cdot n\cdot\lambda(B_n)$$

does not converge, in contradiction to the continuity of ψ. Hence $(s_n)_{n\in\mathbb{N}}$ is bounded, which implies $y_+\in \mathrm{L}^q([0,1])$. Analogously, y_- and hence $y=y_+-y_-$ belong to $\mathrm{L}^q([0,1])$. By the Hölder inequality, $\Psi:G\to\mathbb{R}$, $x\mapsto\int_0^1 x\cdot y\,d\lambda$, is continuous and hence it is the desired extension of ψ.

γ is open. Lemma 11.12 implies that, for $\varepsilon>0$, there exists $\delta>0$ such that $(\varepsilon B\cap H)^0\subseteq(\delta B)^0$ (where B denotes the unit ball of G). According to 1.3 and 1.32, every compact subset of $\mathrm{Hom}(H,\mathbb{R})$ is compact with respect to $\mathrm{Hom}(G,\mathbb{R})$. Since both are k-spaces (4.7 and 11.4) and since γ is continuous, the topologies of $\mathrm{Hom}(H,\mathbb{R})$ and $\mathrm{Hom}(G,\mathbb{R})$ coincide, which completes the proof.

THEOREM 11.14. *Let $\iota:\mathrm{L}^p_{\mathbb{Z}}([0,1])\to\mathrm{L}^p([0,1])$ denote the embedding and let $p>1$. Then ι^* is a topological isomorphism.*

PROOF. This follows immediately from 11.10, 11.13 and 5.5.

COROLLARY 11.15. *For $p>1$, $H:=\mathrm{L}^p_{\mathbb{Z}}([0,1])$ is a complete metrizable locally quasi-convex group such that α_H is an embedding which is not surjective; in particular, $\mathrm{L}^p_{\mathbb{Z}}([0,1])$ is not reflexive.*

PROOF. Trivially, H is a complete metrizable group and, being a subgroup of the reflexive (8.18) group $\mathrm{L}^p([0,1])$, it is locally quasi-convex (6.8(i)). According to 5.12 and 6.10, α_H is an embedding. Since G is reflexive, we deduce from 11.14 that $H^{**}\cong G^{**}\cong G$. It follows from 11.4 that H is not topologically isomorphic to G. This shows that α_H cannot be onto.

NOTES 11.16. The statement that $\mathrm{L}^p_{\mathbb{Z}}([0,1])$ is not reflexive was pointed out to me by W. Banaszczyk. Helpful discussions with him and G. Turnwald led to 11.3 and 11.11. The results 11.5, 11.6 and 11.7 are taken from [57].

12. Free abelian topological groups

THEOREM 12.1. *Let X be a completely regular space. There exists an abelian Hausdorff group $A(X)$ (resp. $A_2(X)$) which is algebraically the free abelian group generated by X such that the canonical mapping $\eta_X:X\to A(X)$ (resp. $\eta_X:X\to A_2(X)$) is an embedding with closed image and every homomorphism $f':A(X)\to G$ (resp. $f':A_2(X)\to G$) into an abelian (resp. abelian Hausdorff) topological group G is continuous if and only if $f:=f'\circ\eta_X$ is continuous.*

In other words, $A(X)$ is the topological analog of the free abelian group. This means that every continuous mapping $f : X \to G$ can be extended to a continuous homomorphism $f' : A(X) \to G$ and $A(X)$ contains X as a closed subspace.

Proof. We prove both cases simultaneously and write $A_{(2)}(X)$ instead of $A(X)$ and $A_2(X)$.

Let $\mathcal{G}$ be the set of topological isomorphism classes of all abelian (Hausdorff) topological groups of cardinality smaller than or equal to $\max(|X|, |\mathbb{R}|)$, let Φ be the set of continuous mappings $\varphi : X \to G$ (where $G \in \mathcal{G}$), and let Φ' be the set of homomorphisms $\varphi' : A(X) \to G$ ($G \in \mathcal{G}$) satisfying $\varphi' \circ \eta_X = \varphi$ for suitable $\varphi \in \Phi$. If $\mathcal{O}$ denotes the initial topology on $A_{(2)}(X)$ generated by Φ', then $(A_{(2)}(X), \mathcal{O})$ is a topological group. By construction, η_X is continuous. For an arbitrary homomorphism f' from $A_{(2)}(X)$ into an abelian (Hausdorff) topological group we have $|f'(A_{(2)}(X))| \leq \max(|X|, |\mathbb{R}|)$ and hence $f'(A_{(2)}(X)) \cong G$ for suitable $G \in \mathcal{G}$. So, by construction, f' is continuous if and only if $f' \circ \eta_X$ is continuous.

For $0 \neq \sum_{j=1}^{n} k_j \eta_X(x_j) \in A_{(2)}(X)$ where $n \in \mathbb{N}$, $k_j \in \mathbb{Z}\backslash\{0\}$, and $x_j \in X$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_j) = \delta_{1j}$. The (continuous!) homomorphic extension f' separates 0 and $\sum_{j=1}^{n} k_j \eta_X(x_j)$, namely $f'(0) = 0$ and $f'(\sum_{j=1}^{n} k_j \eta_X(x_j)) = k_1 \neq 0$. This shows that $A_{(2)}(X)$ is a Hausdorff space.

Since every Hausdorff group is completely regular, so is $\eta_X(X)$. In order to prove that η is an embedding, it is therefore sufficient to show that X and $\eta(X)$ have the same continuous real-valued functions. Let $f : X \to \mathbb{R}$ be a continuous function. By construction, $f'|_{\eta_X(X)}$ is continuous. The other inclusion is clear. Hence, η is an embedding.

Let $\sum_{j=1}^{n} k_j \eta_X(x_j) \in A_{(2)}(X) \setminus \eta_X(X)$. Suppose first that $n = 1$ and $|k_1| > 1$. Let $f : X \to \mathbb{R}$ be a continuous function such that $f(x_1) = 1$ and $f(X) \subseteq [0, 1]$. Then $f'^{-1}(]-\infty, -3/2[\cup]3/2, \infty[)$ is a neighbourhood of $k_1 \eta_X(x_1)$ missing $\eta_X(X)$. Now suppose $n \geq 1$. There exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_j) = \operatorname{sign}(k_j)$ (for $j \in \{1, \ldots, n\}$) and $f(X) \subseteq [-1, 1]$. Again, $f'^{-1}(]-\infty, -3/2[\cup]3/2, \infty[)$ is a neighbourhood of $\sum_{j=1}^{n} k_j \eta_X(x_j)$ having empty intersection with $\eta_X(X)$. Hence, the image of η_X is closed and the theorem is proved.

Corollary 12.2. *For every completely regular space X we have $A(X) = A_2(X)$.*

Proof. Let $f : X \to A(X)$ and $f_2 : X \to A_2(X)$ be the canonical mappings. Since $A(X)$ and $A_2(X)$ are abelian Hausdorff groups, we get continuous homomorphisms $f' : A_2(X) \to A(X)$ and $f_2' : A(X) \to A_2(X)$. Obviously, f' and f_2' are the identity mappings and hence the assertion follows.

Definition 12.3. $A(X)$ is called the *free abelian topological group* over X.

Theorem 12.4 (Raĭkov). *Let $\vartheta_X : X \to L(X)$ be a mapping of a completely regular space X into a vector space $L(X)$ such that $\vartheta_X(X)$ is a basis of $L(X)$. There exists a locally convex Hausdorff vector space topology on $L(X)$ such that for every continuous mapping f of X into a real locally convex vector space F, there exists a continuous linear functional $\widetilde{f} : L(X) \to F$ satisfying $\widetilde{f} \circ \vartheta_X = f$.*

Let $\mathcal{S}$ *denote the set of subsets of* $\mathcal{C}(X,\mathbb{R})$ *which are equicontinuous and pointwise bounded and let* $\widetilde{\mathcal{S}}$ *be the associated linear forms. Then a neighbourhood basis of* $0 \in L(X)$ *is given by the sets* $\{\widetilde{x} \in L(X) : |\widetilde{s}(\widetilde{x})| \le 1\ \forall \widetilde{s} \in \widetilde{S}\}$ *where* $\widetilde{S}$ *runs through* $\widetilde{\mathcal{S}}$.

Proof. It is a consequence of 8.4 that the above sets form a neighbourhood basis of a (real) locally convex vector space topology $\mathcal{O}$ on $L(X)$. For $\widetilde{x} := \sum_{j=1}^{n} \lambda_j \vartheta_X(x_j) \neq 0$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_j) = \operatorname{sign}(\lambda_j)$. The associated linear functional $\widetilde{f}$ satisfies $\widetilde{f}(\widetilde{x}) = \sum_{j=1}^{n} |\lambda_j|$ and $\widetilde{f}(0) = 0$. This implies that $(L(X),\mathcal{O})$ is a Hausdorff space.

$\vartheta_X : X \to L(X)$ is continuous. [Fix $x_0 \in X$, $S \in \mathcal{S}$ and $\varepsilon > 0$. Since S is equicontinuous, there exists $U \in \mathcal{U}_X(x_0)$ such that $|s(x) - s(x_0)| \le 1$ for all $s \in S$ and $x \in U$. Now, $\widetilde{s}(\vartheta_X(x) - \vartheta_X(x_0)) = s(x) - s(x_0)$ implies $\vartheta_X(U) \subseteq \vartheta_X(x_0) + \{\widetilde{x} \in L(X) : |\widetilde{s}(\widetilde{x})| \le 1\ \forall \widetilde{s} \in \widetilde{S}\}$ and hence the continuity of ϑ_X.]

ϑ_X is injective, since $\vartheta_X(X)$ is linearly independent.

For $x_0 \in X$ and $U \in \mathcal{U}_X(x_0)$, there exists a continuous function $f : X \to \mathbb{R}$ such that $|f(x)| \le 1$ implies $x \in U$. Hence we get $\vartheta_X(U) \supseteq \vartheta_X(X) \cap \{\widetilde{x} \in L(X) : |\widetilde{f}(\widetilde{x})| \le 1\}$. This shows that ϑ_X is an embedding.

Every continuous mapping $f : X \to F$ (where F is a real locally convex vector space) gives rise to a continuous mapping $f^* : F' \to \mathcal{C}(X,\mathbb{R})$, $\varphi \mapsto \varphi \circ f$, where the topological dual F' and $\mathcal{C}(X,\mathbb{R})$ are endowed with the compact-open topology. Let U be a closed symmetric convex neighbourhood of $0 \in F$. Then $U^0 := \{\varphi \in F' : \varphi(U) \subseteq [-1,1]\}$ is an equicontinuous pointwise bounded subset of F', and so is $S := f^*(U^0)$. For $\widetilde{x} \in L(X)$ such that $|\widetilde{s}(\widetilde{x})| \le 1$ for all $\widetilde{s} \in \widetilde{S}$ we get $\widetilde{f}(\widetilde{x}) \in U$, since otherwise there exists (by the Hahn–Banach theorem) a linear functional $\varphi \in U^0$ such that $\varphi(\widetilde{f}(\widetilde{x})) > 1$. Since φ is linear, we get $\varphi \circ \widetilde{f} = \widetilde{f^*(\varphi)}$ and hence $\varphi \circ \widetilde{f} \in \widetilde{S}$, contradicting our assumption. Hence $\widetilde{f}$ is continuous and the assertion follows.

Theorem 12.5 (Tkachenko). $\vartheta'_X : A(X) \to L(X)$ *is an embedding and* $\vartheta'_X(A(X))$ *is closed in the locally convex vector space* $L(X)$.

Proof. This is Theorem 3 in [80].

Corollary 12.6. *Let* $\mathcal{S}$ *denote all equicontinuous pointwise bounded subsets of* $\mathcal{C}(X,\mathbb{R})$ *and, for* $S \in \mathcal{S}$, *we write* S' *for the associated continuous homomorphisms. Then the sets of the form* $\{x' \in A(X) : |s'(x')| \le 1\ \forall s' \in S'\}$ *form a neighbourhood basis of* $0 \in A(X)$.

Proof. For every continuous function $f : X \to \mathbb{R}$ we have $\widetilde{f} \circ \vartheta'_X = f'$. Now the assertion is an easy consequence of 12.4 and 12.5.

Proposition 12.7. *For every abelian Hausdorff group* G, *the mapping* $\mathrm{id}' : A(G) \to G$ *is a continuous open epimorphism.*

Proof. Since every Hausdorff group is completely regular, id' is well defined and, clearly, a continuous epimorphism. Let $S \subseteq \mathcal{C}(G,\mathbb{R})$ be equicontinuous and pointwise bounded. From $\mathrm{id}'(\eta_G(x) - \eta_G(e)) = x$ we get $\mathrm{id}'(\{x' \in A(G) : |s'(x')| \le 1\ \forall s' \in S'\} \supseteq \{x \in G : |s(x) - s(e)| \le 1\ \forall s \in S\}$. Since S is equicontinuous, the latter set is a neighbourhood of $e \in G$. The assertion follows from 12.6.

The following result generalizes 10.7.

Theorem 12.8. *Every abelian Hausdorff group is topologically isomorphic to a Hausdorff quotient of a closed subgroup of a locally convex vector space.*

Proof. This follows from 12.5 and 12.7.

Proposition 12.9. *For every completely regular space X, the free abelian topological group $A(X)$ is locally quasi-convex. Every abelian Hausdorff group is a quotient of a locally quasi-convex Hausdorff group.*

Proof. The first assertion is a direct consequence of 6.5, 6.8(i) and 12.5 or an easy consequence of 12.6. The second one follows from 12.7.

Proposition 12.10. *If X is a compact space then $A(X)$ is a hemicompact k-space. The sets $X^{(n)} := \{\sum_{x\in X} k_x\eta_X(x) : (k_x) \in \mathbb{Z}^{(X)}, \sum_{x\in X}|k_x| \le n\}$ form a cobasis for the compact subsets of $A(X)$.*

Proof. This is Corollary 1 of [52].

Corollary 12.11. *For every compact subset K' of $A(X)$ where X is a completely regular space*, $\sup\{\sum_{x\in X}|k_x| : (k_x) \in \mathbb{Z}^{(X)}, \sum_{x\in X} k_x\eta_X(x) \in K'\} < \infty$.

Proof. The embedding of X into the Stone–Čech compactification $\beta : X \to \beta X$ gives rise to a continuous homomorphism $\check{\beta} : A(X) \to A(\beta X)$, $\sum_{x\in X} k_x\eta_X(x) \mapsto \sum_{x\in X} k_x\eta_{\beta X}(\beta(x))$, and hence $\check{\beta}(K')$ is a compact subset of $A(\beta X)$. By 12.10, $\check{\beta}(K')$ is contained in $(\beta X)^{(n)}$ for suitable $n \in \mathbb{N}$. This implies $\sup\{\sum_{x\in X}|k_x| : \sum_{x\in X} k_x\eta_X(x) \in K'\} \le n$.

Notation 12.12. For $x' = \sum_{x\in X} k_x \cdot \eta_X(x)$, let $\mathrm{l}(x') := \sum_{x\in X}|k_x|$ be the *length* of x' and put $\mathrm{S}(x') := \{x \in X : k_x \neq 0\}$. For $Y' \subseteq A(X)$ the union $\mathrm{S}(Y') := \overline{\bigcup_{y'\in Y'} \mathrm{S}(y')}$ is called the *support* of Y'. (This is well defined, since every finite set is closed.)

Lemma 12.13. *For every compact subset $K' \subseteq A(X)$ where X is a completely regular space*, $\mathrm{S}(K')$ *is a functionally bounded subset of X.*

Proof. Suppose the contrary. According to 1.16(iii), there exists an infinite discrete family $(U_i)_{i\in I}$ of open sets in X such that $\mathrm{S}(K') \cap U_i \neq \emptyset$ for all $i \in I$. Inductively, we now construct sequences $(i_n)_{n\in\mathbb{N}}$ in I, $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}(X,\mathbb{R})$, $(x'_n)_{n\in\mathbb{N}}$ in K', and a sequence of open sets (W_n) with the following properties:

(i) For every $n \in \mathbb{N}$, there exist $x_n \in \mathrm{S}(x'_n)$ and an open neighbourhood $W_n \in \mathcal{U}(x_n)$ such that $\overline{W}_n \subseteq U_{i_n}$, $\mathrm{S}(x'_n) \cap W_n = \{x_n\}$ and $\bigcup_{k<n} \mathrm{S}(x'_k) \cap W_n = \emptyset$, and

(ii) $\{x \in X : f_n(x) \neq 0\} \subseteq W_n$ and $|f'_n(x'_n)| \ge n + |\sum_{k<n} f_k(x'_n)|$.

Assume we have already constructed the tuples $(i_1,\dots,i_n)$, $(f_1,\dots,f_n)$, $(x'_1,\dots,x'_n)$, and $(W_1,\dots,W_n)$ for $n \in \mathbb{N}_0$ satisfying the above conditions. Since $\bigcup_{k\le n} \mathrm{S}(x'_k)$ is finite, since $\bigcup_{k\le n} \overline{W}_k$ intersects only finitely many U_i non-trivially, and since $(U_i)_{i\in I}$ is infinite and discrete, there exists $i_{n+1} \in I$ such that

$$U_{i_{n+1}} \cap \Big(\bigcup_{k\le n} \mathrm{S}(x'_k) \cup \bigcup_{k\le n} \overline{W}_k\Big) = \emptyset.$$

Furthermore, there exists $x'_{n+1} \in K'$ such that $\mathrm{S}(x'_{n+1}) \cap U_{i_{n+1}} \neq \emptyset$. Choose $x_{n+1} \in$

$\mathrm{S}(x'_{n+1}) \cap U_{i_{n+1}}$ and an open set $W_{n+1} \in \mathcal{U}(x_{n+1})$ which satisfies condition (i). Since X is completely regular, there exist f_{n+1} which satisfies (ii).

Since $f_n(X \setminus W_n) = \{0\}$ and $(W_n)_{n\in\mathbb{N}}$ is a discrete family, $f := \sum_{n\in\mathbb{N}} f_n$ is continuous (1.12).

Taking into consideration that both sums have only finitely many entries $\neq 0$, we get

$$\sum_{n\in\mathbb{N}} f'_n\Big(\sum_{x\in X} k_x \eta_X(x)\Big) = \sum_{n\in\mathbb{N}} \sum_{x\in X} k_x f_n(x) = \sum_{x\in X} k_x f(x),$$

which implies $f' = \sum_{n\in\mathbb{N}} f'_n$. For $n > k$, we have $f'_n(x'_k) = 0$ (by construction) and hence

$$|f'(x'_k)| = \Big|\sum_{n=1}^{k} f'_n(x'_k)\Big| \geq |f'_k(x'_k)| - \Big|\sum_{n=1}^{k-1} f'_n(x'_k)\Big| \geq k$$

(by (ii)), which contradicts the compactness of K'.

Corollary 12.14. *If X is a Nachbin–Shirota space then the support of every compact set K' of $A(X)$ is compact.*

Proof. This follows immediately from 12.13.

Example 12.15. There exists a locally compact space X and a compact subset $K' \subseteq A(X)$ such that $\mathrm{S}(K')$ is not compact. [In the notation of 1.20 we put $K' := \{0\} \cup \{\eta_X(\omega,\omega_1) - \eta_X(\omega_1,\omega) : \omega < \omega_1\}$. We prove that the bijection $g : \Omega_1 \to K'$ defined by $g(\omega) := \eta_X(\omega,\omega_1) - \eta_X(\omega_1,\omega)$ for $\omega < \omega_1$ and $g(\omega_1) := 0$ is continuous. One easily verifies that $g|_{\Omega_1\setminus\{\omega_1\}}$ is continuous. Let $C \subseteq \mathcal{C}(X,\mathbb{R})$ be an equicontinuous pointwise bounded family. According to 1.20, C is the restriction of an equicontinuous pointwise bounded family $\widetilde{C}$ in $\mathcal{C}(\Omega_1 \times \Omega_1, \mathbb{R})$. Hence

$$\begin{aligned}\{\omega \leq \omega_1 : |f'(g(\omega))| \leq 1\ \forall f' \in C'\} &= \{\omega < \omega_1 : |f(\omega,\omega_1) - f(\omega_1,\omega)| \leq 1\ \forall f \in C\} \cup \{\omega_1\}\\ &= \{\omega < \omega_1 : |\widetilde{f}(\omega,\omega_1) - \widetilde{f}(\omega_1,\omega)| \leq 1\ \forall \widetilde{f} \in \widetilde{C}\} \cup \{\omega_1\}\\ &= \{\omega \leq \omega_1 : |\widetilde{f}(\omega,\omega_1) - \widetilde{f}(\omega_1,\omega)| \leq 1\ \forall \widetilde{f} \in \widetilde{C}\}\end{aligned}$$

is a neighbourhood of ω_1 in Ω_1. This shows that g is continuous (12.6). Hence, K' being a Hausdorff space, it is compact. But $\mathrm{S}(K') = \{(\omega,\omega_1) : \omega < \omega_1\} \cup \{(\omega_1,\omega) : \omega < \omega_1\}$ is not compact.]

Notes 12.16. Free (abelian) topological groups were introduced by Markov in 1945 ([53]) and generalized by Graev in 1948 ([31]). Theorem 12.1 was first proved by Markov ([53]). The proof given here is a mixture of those of (1.4) in [79] and of (8.8) in [34]. 12.4 is due to Raĭkov. The proof is similar to that of Theorem 1′ in [68]. The assertion of 12.6 was formulated in the proof of Theorem 2 in [63]. Proposition 12.7 was proved by Markov ([53]); the proof given here is similar to that of Remark II on p. 596 in [68]. 12.11 is Proposition 2.2 in [27]. Example 12.15 was motivated by K. Yamada. 12.13 is a specialization of Proposition 2 in [4]. I am indebted to D. Elsenhans for translating some relevant parts of [68] from Russian into German.

13. $\mathcal{C}(K,\mathbb{T})$ for compact K

We now prove that the group $\mathcal{C}(K,\mathbb{T})$ endowed with the compact-open topology is reflexive if K is compact.

Since $\mathcal{C}(K,\mathbb{T})$ is metrizable (2.8), $\alpha_{\mathcal{C}(K,\mathbb{T})}$ is continuous. It is shown in 13.13 that $\mathcal{C}(K,\mathbb{T})$ is locally quasi-convex. The crucial point is the surjectivity of $\alpha_{\mathcal{C}(K,\mathbb{T})}$. Several lemmas precede the main theorem 13.15.

Notation 13.1.

K	a non-empty compact space
$(K_j)_{j\in J}$	the family of components of K
$\mathcal{F}$	the set of finite subsets of J
$\mathcal{W}$	the set of closed-and-open subsets of K
$\mathcal{W}_F$	$\{W\in\mathcal{W} : W\cap(\bigcup_{j\in F}K_j)=\emptyset\}$ for $F\in\mathcal{F}$
G	$\mathcal{C}(K,\mathbb{R})$ endowed with the compact-open topology
H	$\{f\in G : f(K)\subseteq\mathbb{Z}\}$
C	$\mathcal{C}(K,\mathbb{T})$
G_F	$\overline{\langle 1_W : W\in\mathcal{W}_F\rangle_{\mathbb{R}}}$ $(F\in\mathcal{F})$
H_F	$H+G_F$ $(F\in\mathcal{F})$
π	$G\to G/H$ the canonical projection
π_F	$G\to G/H_F$ the canonical projection
V_ε	$\{e^{2\pi it} : \lvert t\rvert\le\varepsilon\}$

Remark 13.2. Observe that G, G_F and G/G_F are Banach spaces and that H, C, and G/H are topological groups.

Lemma 13.3. *For every neighbourhood* $U\in\mathcal{U}(K_j)$, *there exists* $W\in\mathcal{W}$ *such that* $K_j\subseteq W\subseteq U$.

Proof. See [73], p. 371.

Lemma 13.4. *Let* K *be a compact subset of the completely regular space* X *and let* $f : K\to\mathbb{R}$ *be a continuous function. For every* $\varepsilon>0$, *there exist* $\eta>0$ *and a finite set* $\mathcal{H}\subseteq\mathcal{C}(X,\mathbb{R})$ *such that* $\lvert f(x)-f(y)\rvert<\varepsilon$ *for all* $x,y\in K$ *for which* $\lvert h(x)-h(y)\rvert<\eta$ *for all* $h\in\mathcal{H}$.

Proof. Since X is completely regular, the sets $U(x,\mathcal{H},\delta):=\{y\in X : \lvert h(x)-h(y)\rvert<\delta\ \forall h\in\mathcal{H}\}$, where $\mathcal{H}$ is a finite subset of $\mathcal{C}(X,\mathbb{R})$ and $\delta>0$, form a neighbourhood basis of $x\in X$.

Let $\varepsilon>0$. For $x\in K$, there exists a finite set $\mathcal{H}_x\subset\mathcal{C}(X,\mathbb{R})$ and $\delta_x>0$ such that $\lvert f(x)-f(y)\rvert<\varepsilon/2$ for all $y\in K\cap U(x,\mathcal{H}_x,\delta_x)$. Since K is compact, we can find a finite subset $K_0\subseteq K$ such that $K\subseteq\bigcup_{x\in K_0}U(x,\mathcal{H}_x,\delta_x)$. Put $\mathcal{H}:=\bigcup_{x\in K_0}\mathcal{H}_x$. For every $x\in K$, there exist $\eta_x>0$ and $x_0\in K_0$ such that $U(x,\mathcal{H},\eta_x)\subseteq U(x_0,\mathcal{H}_{x_0},\delta_{x_0})$. Furthermore, there exists a finite subset K_1 of K such that $K\subseteq\bigcup_{x\in K_1}U(x,\mathcal{H},\eta_x/2)$. Put $\eta:=\frac{1}{2}\min(\eta_x : x\in K_1)$. Let $x,y\in K$ be such that $\lvert h(x)-h(y)\rvert<\eta$ for all $h\in\mathcal{H}$; we can find $x_1\in K_1$ such that $x\in U(x_1,\mathcal{H},\eta_{x_1}/2)$, which implies $x,y\in U(x_1,\mathcal{H},\eta_{x_1}/2+\eta)\subseteq U(x_1,\mathcal{H},\eta_{x_1})\subseteq U(x_0,\mathcal{H}_{x_0},\delta_{x_0})$ for suitable $x_0\in K_0$. The assertion follows.

Proposition 13.5. *Let K be a compact subset of the completely regular space X and let $g : X \to \mathbb{T}$ and $f : K \to \mathbb{R}$ be continuous mappings such that $e^{2\pi i f} = g|_K$. There exist a neighbourhood U of K and a continuous extension $f_U : U \to \mathbb{R}$ of f such that $e^{2\pi i f_U} = g|_U$.*

Proof. Let $0 < \varepsilon < 1/3$. Since g is continuous and K is compact, we can find finite subsets $\mathcal{H}_x \subseteq \mathcal{C}(X, \mathbb{R})$, $\delta_x > 0$ and a finite subset $K_0 \subseteq K$ such that $K \subseteq \bigcup_{x \in K_0} U_x$, where $U_x := U(x, \mathcal{H}_x, \delta_x)$, and $g(U_x)\overline{g(x)} \subseteq V_\varepsilon$. In addition, we may assume that $\delta_x < \eta/2$ and $\mathcal{H} \subseteq \mathcal{H}_x$ (for all $x \in K_0$) where η and $\mathcal{H}$ are chosen as in the above lemma. There exist continuous liftings $f_x : U_x \to \mathbb{R}$ of $g|_{U_x}$ which extend f such that $|f_x(x) - f_x(y)| \leq \varepsilon$ for all $y \in U_x$. [Since $\overline{g(x)} \notin g(U_x)$, there exists a continuous lifting $f_x : U_x \to \mathbb{R}$ such that $f_x(x) = f(x)$ and $|f_x(y) - f_x(x)| \leq \varepsilon$ for all $y \in U_x$. For $y \in K \cap U_x \subseteq U(x, \mathcal{H}, \eta)$ we have $|f(y) - f(x)| < \varepsilon$. It follows from $f_x(y) - f(y) \in \mathbb{Z} \cap [-2\varepsilon, 2\varepsilon] = \{0\}$ that f_x is the desired extension.]

For $y, z \in K_0$ and $x \in U_y \cap U_z$, we have $f_y(x) = f_z(x)$. [Since f_y and f_z are liftings, we have $f_y(x) - f_z(x) \in \mathbb{Z}$ and $x \in U(y, \mathcal{H}, \eta/2) \cap U(z, \mathcal{H}, \eta/2)$ implies $y \in U(z, \mathcal{H}, \eta)$. Hence, $|f(y) - f(z)| < \varepsilon$. Since $f_y(y) = f(y)$ and $f_z(z) = f(z)$, we get $|f_y(x) - f_z(x)| \leq |f_y(x) - f(y)| + |f(y) - f(z)| + |f(z) - f_z(x)| \leq 3\varepsilon < 1$.] Hence, the mapping $f : \bigcup_{x \in K_0} U_x \to \mathbb{R}$ given by $y \mapsto f_x(y)$ for $y \in U_x$ is well defined, continuous, and has the desired lifting property.

Corollary 13.6. *Let K be a compact space and let $g : K \to \mathbb{T}$ be a continuous mapping with the property that the restriction of g to every component K_j can be lifted to $\mathbb{R}$. Then g can be lifted to $\mathbb{R}$. In particular, every continuous mapping of a totally disconnected compact space into $\mathbb{T}$ can be lifted to $\mathbb{R}$.*

Proof. According to 13.5 and 13.3, for every $j \in J$, there exist $W_j \in \mathcal{W}$ such that $K_j \subseteq W_j$ and a continuous lifting $f_j : W_j \to \mathbb{R}$ of $g|_{W_j}$. Let $\bigcup_{k=1}^n W_{j_k}$ be a finite subcover of K by members of $(W_j : j \in J)$. Then $f : x \mapsto f_{j_k}(x)$, where $k \in \{1, \ldots, n\}$ is minimal with $x \in W_{j_k}$, defines a continuous mapping $K \to \mathbb{R}$ (observe that the W_{j_k} are closed-and-open) which is (by construction) the desired lifting.

Remark 13.7. (i) $\mathcal{W}$ is closed with respect to forming finite unions and intersections and complements.

(ii) Every $W \in \mathcal{W}$ is a union of components.

(iii) For every $F \in \mathcal{F}$, there exist (by the normality of K and 13.3) pairwise disjoint $W_j \in \mathcal{W}$ such that $K_j \subseteq W_j$ for all $j \in F$.

Lemma 13.8. (i) $H = \langle 1_W : W \in \mathcal{W} \rangle_{\mathbb{Z}}$.

(ii) *Every $f \in H$ satisfying $f(\bigcup_{j \in F} K_j) = \{0\}$ belongs to G_F.*

(iii) $G_F = \{f \in G : f|_{K_j} = \text{const}$ *for all $j \in J$ and* $f(\bigcup_{j \in F} K_j) = \{0\}\}$.

(iv) $H_F = \{f \in G : f|_{K_j} = \text{const}$ *for all $j \in J$ and* $f(\bigcup_{j \in F} K_j) \subseteq \mathbb{Z}\}$.

Proof. (i) This is obvious, since $f(K)$ is a finite subset of $\mathbb{Z}$ for every $f \in H$.

(ii) is an easy consequence of (i) and 13.7.

(iii) and (iv). It is easy to see that that the right-hand sides are closed. Because of 13.7(ii), "$\subseteq$" follows directly in (iii). In (iv), this inclusion is a consequence of this part in (iii) and (i).

Conversely, let f be an element of the right-hand side and fix $\varepsilon > 0$. Choose $x_j \in K_j$ arbitrarily. Because of 13.3, there exist $W_j \in \mathcal{W}$ $(j \in J)$ such that $K_j \subseteq W_j$ and $|f(x) - f(x_j)| < \varepsilon$ for all $x \in W_j$. For $j \in F$, we may assume that the sets W_j are pairwise disjoint (13.7(iii)). Let $\bigcup_{j\in F} W_j \cup \bigcup_{i=1}^{n} W_{j_i}$ be a finite subcover of $\bigcup_{j\in J} W_j$ of K. Let $W_1', \ldots, W_n'$ be recursively defined by $W_i' := W_{j_i} \setminus (\bigcup_{j\in F} W_j \cup \bigcup_{k<i} W_k')$. Then $W_i' \in \mathcal{W}_F$ (13.7(i)) and K is the disjoint union of $W_1', \ldots, W_n'$, $\bigcup_{j\in F} W_j$. For $j \in F$, we have $f(x_j) = 0$, respectively $f(x_j) \in \mathbb{Z}$, in case (iii), respectively (iv). Hence the function

$$f_\varepsilon := \underbrace{\sum_{j\in F} f(x_j) 1_{W_j}}_{\in H} + \underbrace{\sum_{i=1}^{n} f(x_{j_i}) 1_{W_i'}}_{\in G_F}$$

belongs to the left-hand side and satisfies $|f(x) - f_\varepsilon(x)| < \varepsilon$ for all $x \in K$. It remains to observe that G_F and H_F are closed. [This is true for G_F by definition. For sequences $(h_n)_{n\in\mathbb{N}}$ in H and $(f_n)_{n\in\mathbb{N}}$ in G_F such that $(h_n + f_n)_{n\in\mathbb{N}}$ converges in G, there exists $n_0 \in \mathbb{N}$ such that $h_n(\bigcup_{j\in F} K_j) = (h_n + f_n)(\bigcup_{j\in F} K_j)$ is independent of n for all $n \geq n_0$. (Recall that $f(\bigcup_{j\in F} K_j) = \{0\}$ for $f \in G_F$.) Hence $h_n - h_{n_0} \in G_F$ (by (ii)) and the limit of $h_n + f_n = h_{n_0} + (h_n - h_{n_0} + f_n)$ is a member of $H + G_F$. The assertion follows since G is metrizable (2.8).]

Lemma 13.9. *G/H_F is reflexive.*

Proof. Consider the canonical mapping $\psi : G/G_F \to G/H_F$. For $j \in F$ and disjoint $W_j \in \mathcal{W}$ containing K_j (13.7(iii)), we have

$$(*) \qquad \ker \psi = \langle 1_{W_j} + G_F : j \in F \rangle_{\mathbb{Z}}.$$

[Let $h \in H$. For $k_j \in \mathbb{Z}$ such that $h(K_j) = \{k_j\}$ the function $h - \sum_{j\in F} k_j 1_{W_j}$ belongs to H and is 0 on $\bigcup_{j\in F}(K_j)$; hence, it belongs to G_F (13.8(ii)). This shows that every $f \in H_F$ can be written in the form $\sum_{j\in F} k_j 1_{W_j} + \widetilde{f}$ for $k_j \in \mathbb{Z}$ and $\widetilde{f} \in G_F$, which implies "$\subseteq$". The other inclusion follows immediately.]

Observe that ψ is an open continuous homomorphism. Since $(1_{W_j} + G_F)_{j\in F}$ is linearly independent and $G/H_F \cong (G/G_F)/\ker\psi$, the assertion follows from 8.21 (and 13.2).

Lemma 13.10. *For every infinite subset $J_0 \subseteq J$, there exists a sequence $(j_n)_{n\in\mathbb{N}}$ in J_0 and pairwise disjoint $W_n \in \mathcal{W}$ such that $K_{j_n} \subseteq W_n$ for all $n \in \mathbb{N}$.*

Proof. (We tacitly use 13.3 and 13.7(iii).) Assume first that there exists $j_0 \in J_0$ such that for every $W \in \mathcal{W}$ containing K_{j_0} the set $\{j \in J_0 : K_j \subseteq W\}$ is infinite. For $j_1 \in J_0 \setminus \{j_0\}$, there exist disjoint $V_1, W_1 \in \mathcal{W}$ such that $K_{j_0} \subseteq V_1$ and $K_{j_1} \subseteq W_1$. Suppose we have $j_1, \ldots, j_n \in J_0$ and $V_1, \ldots, V_n, W_1, \ldots, W_n \in \mathcal{W}$ such that $K_{j_0} \subseteq V_n \subseteq \ldots \subseteq V_1$, $K_{j_k} \subseteq W_k$, $V_k \cap W_k = \emptyset$ for $k \in \{1, \ldots, n\}$ and $W_k \subseteq V_{k-1}$ for $k \in \{2, \ldots, n\}$. By assumption, we can find $j_{n+1} \in J_0$ such that $K_{j_{n+1}} \subseteq V_n$ and disjoint $V_{n+1}, W_{n+1} \in \mathcal{W}$ contained in V_n such that $K_{j_0} \subseteq V_{n+1}$ and $K_{j_{n+1}} \subseteq W_{n+1}$. Inductively, we are able to construct sequences $(K_{j_n})_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ having the desired properties.

Otherwise, for every $j_0 \in J_0$, there exists $W \in \mathcal{W}$ containing K_{j_0} such that $K_j \cap W = \emptyset$ for all $j \in J_0 \setminus \{j_0\}$. Inductively, we can find a sequence $(j_n)_{n\in\mathbb{N}}$ in J_0 and pairwise disjoint W_n in $\mathcal{W}$ such that $K_{j_n} \subseteq W_n$. Hence the assertion follows.

Lemma 13.11. *For every continuous linear form $\psi : G \to \mathbb{R}$ satisfying $\psi(H) \le \mathbb{Z}$, there exists an $F \in \mathcal{F}$ such that $G_F \le \ker \psi$.*

Proof. We first show that

$$(*) \qquad F := \{j \in J : \forall U \in \mathcal{U}(K_j)\ \exists W \in \mathcal{W} : K_j \subseteq W \subseteq U \text{ and } \psi(1_W) \ne 0\} \in \mathcal{F}.$$

[Suppose the contrary. Because of 13.10, it is possible to find components $(K_{j_n})_{n\in\mathbb{N}}$ and pairwise disjoint $W_n \in \mathcal{W}$ such that $K_{j_n} \subseteq W_n$ and $\psi(1_{W_n}) \ne 0$. Since

$$\sum_{n=1}^{k} \frac{1}{n \cdot \psi(1_{W_n})} 1_{W_n} \in G$$

converges uniformly to

$$\sum_{n\in\mathbb{N}} \frac{1}{n \cdot \psi(1_{W_n})} 1_{W_n} \in G$$

but

$$\psi\Big(\sum_{n=1}^{k} \frac{1}{n \cdot \psi(1_{W_n})} 1_{W_n}\Big) = \sum_{n=1}^{k} \frac{1}{n},$$

the set F must be finite.]

$(*)$ and 13.3 imply

$$(**) \qquad \forall j \in J \setminus F\ \exists W_j \in \mathcal{W} : K_j \subseteq W_j \text{ and } \psi(1_W) = 0\ \forall W \in \mathcal{W} : K_j \subseteq W \subseteq W_j.$$

Since ψ is a continuous linear form, it is sufficient to prove that

$$(***) \qquad \psi(1_W) = 0 \quad \forall W \in \mathcal{W}_F.$$

[Let $W \in \mathcal{W}_F$. For $j \in J$ such that $K_j \subseteq W$ we have $j \notin F$. According to $(**)$, there exists $W_j \in \mathcal{W}$ such that $K_j \subseteq W_j \subseteq W$ and $\psi(1_{W'}) = 0$ for all $W' \in \mathcal{W}$ such that $K_j \subseteq W' \subseteq W_j$. Since W is compact, there are $j_1, \dots, j_n \in J$ such that $W = \bigcup_{k=1}^n W_{j_k}$ (13.7(ii)). Furthermore, there are pairwise disjoint $V_k \in \mathcal{W}$ satisfying $K_{j_k} \subseteq V_k \subseteq W_{j_k}$ (13.7(iii) and (i)). For $k \in \{1, \dots, n\}$, we define recursively

$$W_k' := W_{j_k} \setminus \Big(\bigcup_{i<k} W_i' \cup \bigcup_{i>k} V_i\Big).$$

Then $V_k \subseteq W_k'$ and, for $k > i$, we have $W_k' \cap W_i' \subseteq (W_{j_k} \setminus W_i') \cap W_i' = \emptyset$. Hence the (W_k') are pairwise disjoint. Moreover, $\bigcup_{k=1}^n W_k' = \bigcup_{k=1}^n W_{j_k} = W$ since $V_k \subseteq W_k'$. This implies $1_W = \sum_{k=1}^n 1_{W_k'}$ and $\psi(1_W) = \sum_{k=1}^n \psi(1_{W_k'}) = 0$ (since $K_{j_k} \subseteq V_k \subseteq W_k' \subseteq W_{j_k}$ and $W_k' \in \mathcal{W}$).]

Corollary 13.12. *For every $\chi \in (G/H)^*$, there exist $F \in \mathcal{F}$ and $\chi_F \in (G/H_F)^*$ such that $\chi(f + H) = \chi_F(f + H_F)$ for all $f \in G$.*

Proof. χ is induced by a character χ' of G which is trivial on H. According to 5.5, there exists a continuous linear form $\psi : G \to \mathbb{R}$ such that $\chi' = e^{2\pi i\psi}$. Obviously, we have $\psi(H) \le \mathbb{Z}$. Hence the above lemma implies that $H_F \le \ker \chi'$ and so $\chi(f + H) = \chi'(f) = \chi_F(f + H_F)$ for suitable $F \in \mathcal{F}$ and $\chi_F \in (G/H_F)^*$.

Lemma 13.13. *$\varrho : G/H \to C$, $f + H \mapsto (x \mapsto \exp(2\pi i f(x)))$, is a continuous open monomorphism. There exists a discrete group $D \leq C$ such that $C = \varrho(G/H) \odot D$. Both G/H and C are locally quasi-convex.*

Proof. It is clear that ϱ is a well defined continuous monomorphism. For $0 < \varepsilon < 1/2$ we have $\varrho(\{f + H \in G/H : f(K) \subseteq [-\varepsilon, \varepsilon]\}) = \{g \in C : g(K) \subseteq V_\varepsilon\}$. (Observe that every continuous non-surjective $g : K \to \mathbb{T}$ can be lifted to $\mathbb{R}$.) Hence ϱ is open and G/H is topologically isomorphic to an open subgroup of C. Next, G and hence $\varrho(G/H)$ are divisible. Thus $\varrho(G/H)$, being an open subgroup of C, is an inner direct factor of C. Hence there exists a discrete subgroup $D \leq C$ such that $D \odot \varrho(G/H) = C$.

For every $n \in \mathbb{N}$, the sets $U_n := \{g \in C : g(K) \subseteq V_n\}$ are quasi-convex. [For $g_0 \notin U_n$, there exist $x \in K$ and $k \in \mathbb{Z}$ such that $g_0(x) \notin V_n$ and $g_0(x)^k \notin R$ but $(V_n)^k \subseteq R$ (by 6.4). The character $\chi : C \to \mathbb{T}$, $g \mapsto g(x)^k$, belongs to U_n^0 and $\chi(g_0) \notin R$.]

Hence C is locally quasi-convex, and so is G/H (6.8(i)).

Corollary 13.14. *If K is a totally disconnected compact space, then the characters of $\mathcal{C}(K, T)$ are exactly products of point evaluations. More precisely: For every $\chi \in \mathcal{C}(K, \mathbb{T})^*$, there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in K$, and $k_1, \ldots, k_n \in \mathbb{Z}$ such that $\chi(g) = \prod_{j=1}^n g(x_j)^{k_j}$.*

Proof. According to 13.6 and 13.13, we may consider a character χ of $(G/H)^*$. Let $\psi : G \to \mathbb{R}$ be a continuous linear form such that $\exp(2\pi i\psi) = \chi \circ \pi$ (see 5.5). Since $\psi(H) \leq \mathbb{Z}$, there exists $F \in \mathcal{F}$ such that $G_F \leq \ker \psi$ (by 13.11). By assumption, $\bigcup_{j \in F} K_j = \{x_1, \ldots, x_n\}$ for suitable $x_j \in K$, and hence $G_F = \{f \in \mathcal{C}(K, \mathbb{R}) : f(\{x_1, \ldots, x_n\}) = \{0\}\}$ (13.8(iii)). There exist continuous functions $h_j : K \to \mathbb{Z}$ such that $h_j(x_i) = \delta_{ij}$ ($i, j \in \{1, \ldots, n\}$) (by 13.3). Since every $f \in \mathcal{C}(K, \mathbb{R})$ can be written in the form $f = \sum_{j=1}^n f(x_j) \cdot h_j + \widetilde{f}$ where $\widetilde{f} \in G_F$, we get $\psi(f) = \sum_{j=1}^n f(x_j)\psi(h_j)$. Since $\psi(h_j) \in \mathbb{Z}$, the assertion follows.

Theorem 13.15. *The groups G/H and C are reflexive.*

Proof. Since $C = \varrho(G/H) \odot D$ for a discrete group D (see 13.13), it suffices to prove that G/H is reflexive (5.3 and 5.4).

Since G is metrizable, so is G/H (by 2.8), and hence $\alpha_{G/H}$ is continuous (5.12). According to 13.13, G/H is locally quasi-convex and, because of 6.10, only the surjectivity of $\alpha_{G/H}$ remains to be proved. Therefore, consider

$$\varphi : G/H \to G_{\mathcal{F}}, \quad f + H \mapsto (f + H_F)_{F \in \mathcal{F}},$$

where $G_{\mathcal{F}} := \prod_{F \in \mathcal{F}} G/H_F$. Obviously, φ is a continuous homomorphism.

For $F_1 \subseteq F_2 \in \mathcal{F}$, we get $H_{F_2} \leq H_{F_1}$. Hence, the canonical projection $\Pi_{F_1,F_2} : G/H_{F_2} \to G/H_{F_1}$ is well defined; let G_0 denote the projective limit $\varprojlim\{G/H_F, \Pi_{F_1,F_2}, \mathcal{F}\}$.

For $\kappa \in (G/H)^{**}$, consider the following diagram:

$$\begin{tikzcd}
G_{\mathcal{F}}^* \arrow[r] \arrow[d, "\varphi^*"'] & \mathbb{T} \\
(G/H)^* \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

(1) *There exists $(f_F + H_F)_{F \in \mathcal{F}} \in G_0$ such that $\alpha_{G_{\mathcal{F}}}((f_F + H_F)_{F \in \mathcal{F}}) = \kappa \circ \varphi^*$.*

[Since $G_{\mathcal{F}}$ is reflexive (13.9 and 5.4) and $\kappa \circ \varphi^* \in (G_{\mathcal{F}})^{**}$, there exists $(f_F + H_F)_{F\in\mathcal{F}} \in G_{\mathcal{F}}$ such that $\alpha_{G_{\mathcal{F}}}((f_F + H_F)_{F\in\mathcal{F}}) = \kappa \circ \varphi^*$. Suppose $(f_F + H_F)_{F\in\mathcal{F}} \notin G_0$. According to 5.28 (and 13.9 and 6.10), there exists $\widetilde{\chi} := \chi_{F_1} \circ (\mathit{\Pi}_{F_1,F_2} \circ \mathit{\Pi}_{F_2} - \mathit{\Pi}_{F_1}) \in G_0^{\perp}$ (where $F_1 \subseteq F_2 \in \mathcal{F}$, $\mathit{\Pi}_{F_j}$ denotes the canonical projection, and $\chi_{F_1} \in (G_{H_{F_1}})^*$) such that $\widetilde{\chi}((f_F + H_F)_{F\in\mathcal{F}}) \neq 1$. We have $\varphi^*(\widetilde{\chi})(f + H) = \widetilde{\chi}(\varphi(f + H)) = \widetilde{\chi}((f + H_F)_{F\in\mathcal{F}}) = 1$ (for all $f \in G$) and hence $\kappa(\varphi^*(\widetilde{\chi})) = 1$, but $\alpha_{G_{\mathcal{F}}}((f_F + H_F)_{F\in\mathcal{F}})(\widetilde{\chi}) = \widetilde{\chi}((f_F + H_F)_{F\in\mathcal{F}}) \neq 1$, contradicting our assumption.]

(2) *The mapping* $g : K \to \mathbb{T}$, $x \mapsto \exp(2\pi i f_F(x))$, *for* $x \in \bigcup_{j\in F} K_j$, *is well defined and continuous.*

[For $x \in (\bigcup_{j\in F_1} K_j) \cap (\bigcup_{j\in F_2} K_j)$ we get $f_{F_1}(x) - f_{F_2}(x) \in \mathbb{Z}$, as $(f_F + H_F)_{F\in\mathcal{F}} \in G_0$ (13.8(iv)). Hence g is well defined.

According to 1.31, every compact subset of G/H has the form $\pi(S)$ where S is a compact subset of G.

Since κ is continuous, we can find for each $\varepsilon > 0$ a compact subset $S \subseteq G$ such that $\kappa(P(\pi(S), R)) \subseteq V_\varepsilon$ (3.4(i)). It is a consequence of (1) that

$$\alpha_{G_{\mathcal{F}}}((f_F + H_F)_{F\in\mathcal{F}}) \in P\Big(P\Big(\prod_F \pi_F(S), R\Big), V_\varepsilon\Big).$$

Indeed, note that $\varphi^*(P(\prod_{F\in\mathcal{F}} \pi_F(S), R)) \subseteq P(\pi(S), R)$, as for $\widetilde{\chi} \in P(\prod_{F\in\mathcal{F}} \pi_F(S), R)$ and $s \in S$, we have $\varphi^*(\widetilde{\chi})(s + H) = \widetilde{\chi}((s + H_F)_{F\in\mathcal{F}}) \in R$.

Due to the Arzelà–Ascoli Theorem (1.32), S is equicontinuous. This means that for each $x \in K$, there exists $U \in \mathcal{U}_K(x)$ such that $|s(x) - s(y)| \leq 1/4$ for all $y \in U$ and $s \in S$. For $y \in U$ and $F_0 \in \mathcal{F}$ such that $\{x, y\} \subseteq \bigcup_{j\in F_0} K_j$, $\chi_{F_0} : G/H_{F_0} \to \mathbb{T}$, $f + H_{F_0} \mapsto \exp(2\pi i(f(x) - f(y)))$, is a character and $\widetilde{\chi} := \chi_{F_0} \circ \mathit{\Pi}_{F_0}$ belongs to $P(\prod_F \pi_F(S), R)$. This implies $g(x)\overline{g(y)} = \exp(2\pi i(f_{F_0}(x) - f_{F_0}(y))) = \widetilde{\chi}((f_F + H_F)_{F\in\mathcal{F}}) = \alpha_{G_{\mathcal{F}}}((f_F + H_F))(\widetilde{\chi}) \in V_\varepsilon$. Since ε and y were arbitrary, g is continuous.]

(3) $g = e^{2\pi i f}$ *for a suitable* $f \in G$ *and* $f - f_F \in H_F$ *for all* $F \in \mathcal{F}$.

[Since $g|_{K_j}$ can be lifted to $\mathbb{R}$ for every $j \in J$ $(g|_{K_j} = \exp(2\pi i f_{\{j\}}|_{K_j}))$, it follows from 13.6 that $g = \exp(2\pi i f)$ for suitable $f \in G$.

Fix $F \in \mathcal{F}$. For $j \in F$ and $x \in K_j$ we have

$$e^{2\pi i(f(x) - f_F(x))} = g(x)e^{-2\pi i f_F(x)} = e^{2\pi i f_F(x)}e^{-2\pi i f_F(x)} = 1.$$

Hence $(f - f_F)(\bigcup_{j\in F} K_j) \subseteq \mathbb{Z}$. It follows that, for arbitrary $j \in J$, the functions $f|_{K_j}$ and $f_{F\cup\{j\}}|_{K_j}$ differ only by an integer. Since $f_{F\cup\{j\}} - f_F \in H_F$, we have $(f - f_F)|_{K_j} \equiv \text{const}$ (13.8(iv)); it follows from 13.8(iv) that $f - f_F \in H_F$.]

Part (3) means $(f + H_F)_{F\in\mathcal{F}} = (f_F + H_F)_{F\in\mathcal{F}}$. For arbitrary $\chi \in (G/H)^*$, we can find $F_0 \in \mathcal{F}$ and $\chi_{F_0} \in (G/H_{F_0})^*$ such that $\chi(f' + H) = \chi_{F_0}(f' + H_{F_0})$ for all $f' \in G$ (13.12); this implies $\chi = \varphi^*(\chi_{F_0} \circ \mathit{\Pi}_{F_0})$. Hence

$$\begin{aligned}\kappa(\chi) &= \kappa(\varphi^*(\chi_{F_0} \circ \mathit{\Pi}_{F_0})) = \alpha_{G_{\mathcal{F}}}((f_F + H_F)_{F\in\mathcal{F}})(\chi_{F_0} \circ \mathit{\Pi}_{F_0})\\ &= (\chi_{F_0} \circ \mathit{\Pi}_{F_0})((f_F + H_F)_{F\in\mathcal{F}}) = (\chi_{F_0} \circ \mathit{\Pi}_{F_0})((f + H_F)_{F\in\mathcal{F}})\\ &= \chi_{F_0}(f + H_{F_0}) = \chi(f + H) = \alpha_{G/H}(f + H)(\chi)\end{aligned}$$

and so $\alpha_{G/H}(f + H) = \kappa$, which completes the proof.

Notes 13.16. It is a consequence of results of V. Pestov (cf. [63]) that $\mathcal{C}(K,\mathbb{T})$ is a reflexive group if K is a totally disconnected compact space. The whole chapter is based on ideas mentioned in [63]. 13.4 and the proof of 13.5 were suggested by G. Turnwald. The assertion of 13.10 was stated in [63] and a generalization of 13.14 was proved there by other methods.

14. The group $\mathcal{C}(X,\mathbb{T})$

The aim of this chapter is to generalize the previous result to more general spaces X.

Proposition 14.1. *For each completely regular space X and every compact subset K of X, the restriction mapping $\varphi_K : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(K,\mathbb{R})$ is a continuous epimorphism.*

Proof. Obviously, φ_K is a continuous homomorphism. Since, by Tietze's Theorem, every $f \in \mathcal{C}(K,\mathbb{R})$ can be extended to a continuous function $F : \beta X \to \mathbb{R}$, we get $\varphi_K(F|_X) = f$.

Notation 14.2. For $K \subseteq X$ and $\varepsilon > 0$ we put $(K, V_\varepsilon) := \{g \in \mathcal{C}(X,\mathbb{T}) : g(K) \subseteq V_\varepsilon\}$ (cf. 2.6).

Proposition 14.3. *For every completely regular space X and every character $\chi : \mathcal{C}(X,\mathbb{T}) \to \mathbb{T}$, there exists a compact subset K of X and a character $\chi_K : \mathcal{C}(K,\mathbb{T}) \to \mathbb{T}$ such that $\chi = \chi_K \circ \psi_K$, where $\psi_K : \mathcal{C}(X,\mathbb{T}) \to \mathcal{C}(K,\mathbb{T})$ is the restriction mapping.*

Proof. (Observe that ψ_K is a continuous homomorphism.) Since χ is continuous, there exist a compact subset $K \subseteq X$ and $\varepsilon > 0$ such that $\chi((K,V_\varepsilon)) \subseteq R$; in particular, $\ker\psi_K \le \ker\chi$. This implies that $\chi_0 : \psi_K(\mathcal{C}(X,\mathbb{T})) \to \mathbb{T}$, $\psi_K(g) \mapsto \chi(g)$, is well defined. We conclude from $\chi_0(\psi_K(\mathcal{C}(X,\mathbb{T})) \cap \{g \in \mathcal{C}(K,\mathbb{T}) : g(K) \subseteq V_\varepsilon\}) \subseteq R$ that χ_0 is continuous. It is a consequence of 14.1 that $\{e^{2\pi i f_K} : f_K \in \mathcal{C}(K,\mathbb{R})\} \le \psi_K(\mathcal{C}(X,\mathbb{T}))$, and therefore $\psi_K(\mathcal{C}(X,\mathbb{T}))$ is an open subgroup of $\mathcal{C}(K,\mathbb{T})$ (see 13.13). Any homomorphism $\chi_K : \mathcal{C}(K,\mathbb{T}) \to \mathbb{T}$ extending χ_0 (cf. 2.3) is continuous and satisfies $\chi = \chi_K \circ \psi_K$. This completes the proof.

Lemma 14.4. *For every compact subset K of a completely regular space X, there exists a continuous open epimorphism $\varphi : \mathcal{C}(X,\mathbb{T}) \to \mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})$.*

Proof. Consider

$$\varphi : \mathcal{C}(X,\mathbb{T}) \xrightarrow{\psi_K} \mathcal{C}(K,\mathbb{T}) \xrightarrow{p} \varrho(\mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})) \xrightarrow{\varrho^{-1}} \mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})$$

where ψ_K is the restriction mapping, p denotes a projection onto the inner direct factor (13.13) and $\varrho^{-1}(\exp(2\pi i f)) := f + \mathcal{C}(K,\mathbb{Z})$ (cf. 13.13). It is clear that φ is a continuous homomorphism. Because of 14.1, φ is surjective. For every compact subset $S \subseteq X$ which contains K and for every $\varepsilon \in\,]0,1/2[$, we get $\varphi(\{g \in \mathcal{C}(X,\mathbb{T}) : g(S) \subseteq V_\varepsilon\}) \supseteq \{f + \mathcal{C}(K,\mathbb{Z}) \in \mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z}) : f(K) \subseteq [-\varepsilon,\varepsilon]\}$, according to Tietze's Theorem (and 14.1). This shows that φ is open.

Definition 14.5. A Hausdorff space X is called *punctiform* if the cardinality of every compact connected set is smaller than or equal to 1.

Remark 14.6. There do exist connected metrizable spaces which are punctiform. [Cf. [76] or [54] or Problem (6.3.24) in [26], pp. 380ff.]

Corollary 14.7. *If X is a punctiform completely regular space X, then the characters of $\mathcal{C}(X,\mathbb{T})$ are products of point evaluations.*

Proof. Since every compact subset K of X is totally disconnected, this is a consequence of 13.14 and 14.3.

Theorem 14.8. *Let X be a completely regular k-space. Then $\alpha_{\mathcal{C}(X,\mathbb{T})}$ is an open isomorphism.*

Proof. Let $\mathcal{K}$ denote the set of all compact subsets of X and let $\psi_K:\mathcal{C}(X,\mathbb{T})\to\mathcal{C}(K,\mathbb{T})$, for $K\in\mathcal{K}$, be the restriction mapping. Then

$$\psi:\mathcal{C}(X,\mathbb{T})\to\prod_{K\in\mathcal{K}}\mathcal{C}(K,\mathbb{T}),\quad g\mapsto(\psi_K(g))_{K\in\mathcal{K}},$$

is a continuous monomorphism. For $K_0\in\mathcal{K}$ and $\varepsilon>0$, we have $\psi((K_0,V_\varepsilon))=\psi(\mathcal{C}(X,\mathbb{T}))\cap(\prod_{K\neq K_0}\mathcal{C}(K,\mathbb{T})\times\{g\in\mathcal{C}(K_0,\mathbb{T}):g(K_0)\subseteq V_\varepsilon\})$, which implies that ψ is an embedding.

For $K,K'\in\mathcal{K}$ such that $K\subseteq K'$, let $\psi_{K,K'}:\mathcal{C}(K',\mathbb{T})\to\mathcal{C}(K,\mathbb{T})$ denote the restriction mapping. Since X is a k-space and according to 1.5, it is easily checked that $\psi(\mathcal{C}(X,\mathbb{T}))=\varprojlim\{\mathcal{C}(K,\mathbb{T}),\psi_{K,K'},\mathcal{K}\}$. Since $\alpha_{\mathcal{C}(K,\mathbb{T})}$ is injective for all $K\in\mathcal{K}$ (cf. 13.15), the above description implies that $\psi(\mathcal{C}(X,\mathbb{T}))$ is dually closed in $\prod_{K\in\mathcal{K}}\mathcal{C}(K,\mathbb{T})$ (see 5.28). The group $\prod_{K\in\mathcal{K}}\mathcal{C}(K,\mathbb{T})$ is reflexive (13.15 and 5.4).

Because of 5.25, it suffices to show that $\psi(\mathcal{C}(X,\mathbb{T}))$ is dually embedded. Let $\chi\in(\psi(\mathcal{C}(X,\mathbb{T})))^*$. It follows easily from 14.3 that there exist $K\in\mathcal{K}$ and $\chi_K\in\mathcal{C}(K,\mathbb{T})^*$ such that $\chi\circ\psi=\chi_K\circ\psi_K$ and hence $\chi=\chi_K\circ p_K|_{\psi(\mathcal{C}(X,\mathbb{T}))}$ where $p_K:\prod_{K'\in\mathcal{K}}\mathcal{C}(K',\mathbb{T})\to\mathcal{C}(K,\mathbb{T})$ denotes the canonical projection. This shows that $\psi(\mathcal{C}(X,\mathbb{T}))$ is dually embedded and the assertion follows.

Theorem 14.9. *For every hemicompact k-space X, the group $\mathcal{C}(X,\mathbb{T})$ is reflexive.*

Proof. Since X is hemicompact, $\mathcal{C}(X,\mathbb{T})$ is metrizable (2.8) and hence $\alpha_{\mathcal{C}(X,\mathbb{T})}$ is continuous (5.12). The assertion follows from 1.7 and 14.8.

Notation 14.10. Let X' denote the k-refinement of the completely regular space X. We put $G:=\mathcal{C}(X,\mathbb{T})$, $G':=\mathcal{C}(X',\mathbb{T})$, and $G_0:=\{g'\in G':g'|_K\in\psi_K(G)$ for all compact subsets $K\subseteq X\}$.

Lemma 14.11. (i) *The canonical mapping $\varphi:G\to G'$ is an embedding.*

(ii) *For every compact subset $K\subseteq X$, the image $\psi_K(G)$ is an open subgroup of $\mathcal{C}(K,\mathbb{T})$.*

(iii) $\overline{\varphi(G)}=G_0$.

(iv) *G_0 is dually closed and dually embedded in G'.*

Proof. (i) follows directly from the fact that the compact subsets of X and X' coincide (1.2(ii)).

(ii) $\psi_K(G)$ includes the open subgroup $\{e^{2\pi i f}:f\in\mathcal{C}(K,\mathbb{R})\}$ (13.13 and 14.1).

(iii) Let $\psi'_K:G'\to\mathcal{C}(K,\mathbb{T})$ denote the restriction mapping. The assertion follows from

$$\begin{aligned}\overline{\varphi(G)} &= \{g' \in G' : \forall \text{ compact } K \subseteq X \text{ and } \forall \varepsilon > 0,\ \psi'_K(g') \in \psi'_K(\varphi(G)) \cdot (K, V_\varepsilon)\}\\ &\overset{\text{(ii)}}{=} \{g' \in G' : \psi'_K(g') \in \psi_K(G) \text{ for all compact } K \subseteq X\}.\end{aligned}$$

(iv) 14.3 ensures that every $\chi \in G^*$ has the form $\chi = \chi_K \circ \psi_K$ (for a suitable compact $K \subseteq X$ and $\chi_K \in \mathcal{C}(K, \mathbb{T})^*$). This implies that $\chi' := \chi_K \circ \psi'_K \in (G')^*$ extends $\chi \circ \varphi$. According to (iii) and 4.1, G_0 is dually embedded as well.

For $g' \in G' \setminus G_0$, there exists a compact subset $K \subseteq X$ such that $\psi'_K(g')$ does not belong to the open ((ii)) and hence dually closed (5.15) subgroup $\psi_K(G)$. Hence there exists a character $\chi_K \in \mathcal{C}(K, \mathbb{T})^*$ such that $\chi_K(\psi_K(G)) = \{1\}$ and $\chi_K(\psi'_K(g')) \neq 1$. It is a consequence of (iii) that $\chi' := \chi_K \circ \psi'_K \in G_0^\perp$ and that $\chi'(g') \neq 1$. The assertion follows.

Corollary 14.12. *With the above notation, for every hemicompact completely regular space X, the space X' is a hemicompact k-space and the groups $\mathcal{C}(X, \mathbb{T})$, $\mathcal{C}(X', \mathbb{T})$ and G_0 are metrizable. Furthermore, G_0 is reflexive and $G_0^* \cong \mathcal{C}(X, \mathbb{T})^*$.*

Proof. According to 1.2(i), X' is a hemicompact k-space and hence completely regular (1.7). It follows from 2.8 that $\mathcal{C}(X, \mathbb{T})$, $\mathcal{C}(X', \mathbb{T})$ and G_0 are metrizable groups. Due to 5.12, α_{G_0} is continuous. According to 5.25 and 14.9, G_0 is reflexive. The assertion follows from 14.11(i), (iii), and 4.10.

Theorem 14.13. *For every hemicompact completely regular space X, the character group of $\mathcal{C}(X, \mathbb{T})$ is reflexive.*

Proof. (We use the above notation.) According to 14.12, $G_0^* \cong \mathcal{C}(X, \mathbb{T})^*$. Since G_0 is reflexive (by 14.12), so is G_0^* (5.9). Hence the assertion follows.

Example 14.14. Let H be the group of integers endowed with its weak topology. (Observe that H is completely regular.) As a consequence of the Glicksberg Theorem, the compact subsets of H are finite and hence H is hemicompact. The k-refinement of H is $\mathbb{Z}$ with the discrete topology and it follows from 14.1 (in the above notation) that $G_0 = G'$. Since H and $\mathbb{Z}$ cannot have the same $\mathbb{T}$-valued continuous functions, $G \neq G_0$ but $G_0^* \cong G^*$ (see 14.12). This shows that $\mathcal{C}(H, \mathbb{T})$ is not reflexive. Hence the k-space assumption cannot be dropped in 14.9.

Notes 14.15. 14.1 is a standard result and 14.7 generalizes Theorem 5 in [63].

15. Duality theory for free abelian topological groups

Throughout this chapter, X denotes a completely regular space and $A(X)$ the free abelian topological group over X.

Proposition 15.1. *The restriction mapping $\psi : A(X)^* \to \mathcal{C}(X, \mathbb{T})$, $\chi \mapsto \chi \circ \eta_X$, is a continuous isomorphism. If X is a Nachbin–Shirota space then ψ is a topological isomorphism.*

Proof. Obviously, ψ is well defined and, by the definition of the free abelian topological group, it is an isomorphism. Let K be a compact subset of X and let $\varepsilon > 0$. The continuity of ψ is a consequence of $\psi(\{\chi \in A(X)^* : \chi(\eta_X(K)) \subseteq V_\varepsilon\}) \subseteq (K, V_\varepsilon)$.

Now, let X be a Nachbin–Shirota space and let $K' \subseteq A(X)$ be compact. According to 12.14, $\mathrm{S}(K')$ is a compact subset of X and it follows from 12.11 that K' is contained in $\{\sum_{x\in X} \eta_X(x) : (k_x) \in \mathbb{Z}^{(\mathrm{S}(K'))} \sum_{x\in X} |k_x| \leq n\}$ for suitable $n \in \mathbb{N}$. Hence, $\psi((K')^0) \supseteq \{g \in \mathcal{C}(X,\mathbb{T}) : g(\mathrm{S}(K')) \subseteq V_{1/(4n)}\}$, which shows that ψ is open.

Proposition 15.2. *If X is a k-space then $\alpha_{A(X)}$ is continuous.*

Proof. By the definition of $A(X)$, it suffices to prove that $\alpha_{A(X)} \circ \eta_X$ is continuous. Since X is a k-space, we only have to show that $\alpha_{A(X)} \circ \eta_X|_K$ is continuous for every compact subset $K \subseteq X$ (cf. 1.5). But this is a consequence of 5.11.

Lemma 15.3. *Let K be a compact and connected space of cardinality > 1. Then the character group of $\mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})$ contains a non-trivial one-parameter subgroup.*

Proof. For $x_1 \neq x_2 \in K$, there exists a continuous function $f_0 : K \to \mathbb{R}$ such that $f_0(x_1) = 0$ and $f_0(x_2) = 1/2$. Further, f_0 does not belong to $\overline{\langle \mathcal{C}(K,\mathbb{Z})\rangle_{\mathbb{R}}}$, since this set consists exactly of all constant functions. By the Hahn–Banach Theorem, there exists a continuous linear form $\psi : \mathcal{C}(K,\mathbb{R}) \to \mathbb{R}$ such that $\psi(f_0) \neq 0$ and $\psi(\overline{\langle \mathcal{C}(K,\mathbb{Z})\rangle_{\mathbb{R}}}) = \{0\}$. Hence, for all $\lambda \in \mathbb{R}$, the mapping $\chi_\lambda : \mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z}) \to \mathbb{T}$, $f + \mathcal{C}(K,\mathbb{Z}) \mapsto \exp(2\pi i\lambda\psi(f))$, defines a character. It remains to show that $\gamma : \mathbb{R} \to (\mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z}))^*$, $\lambda \mapsto \chi_\lambda$, is a continuous homomorphism. Obviously, γ is a homomorphism. According to 1.31, every compact subset of $\mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})$ has the form $\{s + \mathcal{C}(K,\mathbb{Z}) : s \in S\}$ where S is a compact subset of $\mathcal{C}(K,\mathbb{R})$. For every compact subset $S \subseteq \mathcal{C}(K,\mathbb{R})$, there exists $M \in \mathbb{R}$ such that $\psi(S) \subseteq [-M,M]$. The continuity of γ is a consequence of $\{\lambda \in \mathbb{R} : \gamma(\lambda)(s + \mathcal{C}(K,\mathbb{Z})) \in R \text{ for all } s \in S\} \supseteq \{\lambda \in \mathbb{R} : |\lambda\psi(s)| \leq 1/4 \text{ for all } s \in S\} \supseteq [-1/(4M), 1/(4M)]$. Since $\gamma(1)(f_0) \neq 1$, the assertion follows.

Theorem 15.4. *For every punctiform Nachbin–Shirota space X which is a k-space, $A(X)$ is reflexive. Conversely, if $A(X)$ is reflexive, then X must be punctiform.*

Proof. Assume first that X is a punctiform Nachbin–Shirota space and a k-space. Because of 15.2, $\alpha_{A(X)}$ is continuous. Since $A(X)$ is locally quasi-convex (12.9), it is a consequence of 6.10 that $\alpha_{A(X)}$ is an embedding. It follows from $\alpha_{A(X)}(\sum_{x\in X} k_x\eta_X(x))(\chi) = \prod_{x\in X}((\chi \circ \eta_X)(x))^{k_x}$, 15.1, and 14.7 that $\alpha_{A(X)}$ is surjective.

Conversely, suppose that $A(X)$ is reflexive and that there exists a compact connected subset $K \subseteq X$ of cardinality > 1. According to 15.1 and 14.4, the composition $A(X)^* \xrightarrow{\psi} \mathcal{C}(X,\mathbb{T}) \xrightarrow{\varphi} \mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})$ is a continuous epimorphism. The dual homomorphism of $\varphi\circ\psi$ is injective (5.16) and 15.3 implies that $A(X)^{**}$ contains a non-trivial one-parameter subgroup, which is obviously divisible. But $A(X)$, being algebraically a free abelian group, contains no non-trivial divisible subgroup and, in particular, no non-trivial one-parameter subgroup. This shows that $A(X)$ and $A(X)^{**}$ cannot be algebraically isomorphic, which completes the proof.

Notes 15.5. V. Pestov [63] has proved results analogous to 15.1, 15.2, and 15.4. But he endowed the character group of $A(X)$ with the topology of uniform convergence on all functionally bounded subsets. 15.3 is a generalization of Corollary 4 in [57].

16. A short survey of the theory of nuclear groups

In this introductory chapter we give a short report on the theory of nuclear groups. They were defined by W. Banaszczyk in [8], and several deep results which stress the importance of this class were proved there.

Roughly speaking, a nuclear group is an abelian Hausdorff group which has the property that for each neighbourhood of the unit element, there exists another neighbourhood which is considerably smaller (cf. 20.4). (If the group is a vector space these neighbourhoods can be compared by the Kolmogorov diameter.) The class of nuclear groups contains all nuclear locally convex vector spaces (20.6(i)) and all locally compact abelian groups (20.8). It is closed under forming (arbitrary) products, countable sums, subgroups, and Hausdorff quotient groups. In particular, every Hausdorff quotient of a subgroup of a nuclear vector space is a nuclear group. We prove that the completion of a nuclear group is nuclear (21.4). (The metrizable case was treated in [8], Theorem (9.8).) Every group which is locally isomorphic to a nuclear group G is nuclear ((7.15) in [8]). Moreover, every nuclear group G is locally quasi-convex ((8.5) in [8] and 20.15). This implies that α_G is injective and that every closed subgroup of a nuclear group is dually closed. Furthermore, every subgroup of a nuclear group is dually embedded ((8.3) in [8] and 20.13).

This is a possibility to characterize nuclear vector spaces: Let E be a metrizable locally convex space. If E is not nuclear, then it contains a discrete subgroup K such that $\langle K\rangle_{\mathbb{R}}/K$ admits only the trivial character ((6.1) in [8]; cf. also [5] and [6]).

Another result which elucidates the dichotomy between infinite-dimensional Banach spaces and nuclear groups is the following generalization of the Lévy–Steinitz Theorem: For a convergent series $\sum_{n\in\mathbb{N}} g_n$ in a metrizable group G, let $S((g_n)_{n\in\mathbb{N}})$ be the set of sums of rearrangements of $\sum_{n\in\mathbb{N}} g_n$. If G is a metrizable nuclear group then $S((g_n)_{n\in\mathbb{N}}) - \sum_{n\in\mathbb{N}} g_n$ is a (not necessarily closed) subgroup of G. If G is complete, metrizable and nuclear, then the above subgroup is the image of a nuclear Fréchet space under a continuous homomorphism ((10.3) in [8]). On the other hand, it was proved by Kadets (cf. [38] and also [10]) that every Banach space contains a convergent series $\sum_{n\in\mathbb{N}} g_n$ such that $S((g_n)_{n\in\mathbb{N}}) - \sum_{n\in\mathbb{N}} g_n$ is not a linear subspace.

The character group of each metrizable nuclear group is again nuclear ((16.1) in [8] and 20.35). (This is not correct without the assumption of metrizability: e.g., $(\mathbb{R}^I)^* \cong \mathbb{R}^{(I)}$ is not nuclear (20.27).) Furthermore, every complete metrizable nuclear group is strongly reflexive ((17.3) in [8]) and so is every countable product of locally compact abelian groups ([7]). We extend the last two results to Čech-complete nuclear groups.

For locally compact abelian groups many analytic results have been proved. Since each locally compact abelian group is nuclear, it is natural to ask which of these results extend to nuclear groups. We give a proof of a version of the Bochner Theorem for nuclear groups which simplifies the proof of (12.1) in [8].

17. Ellipsoids

In this chapter we establish some basic facts on ellipsoids. Let B_n denote the closed unit ball in $\mathbb{R}^n$ (with respect to the metric induced by the standard inner product $\langle\cdot,\cdot\rangle$). If the dimension is clear we write B instead of B_n. We denote by $(e_1,\dots,e_n)$ the standard basis of $\mathbb{R}^n$.

DEFINITION 17.1. $D\subseteq\mathbb{R}^n$ is called an *ellipsoid* if there exists an invertible matrix $A\in\mathrm{Gl}(n,\mathbb{R})$ such that $D=AB$ $(=\{Ax\in\mathbb{R}^n : x\in B\})$.

LEMMA 17.2. *For $A\in\mathrm{Gl}(n,\mathbb{R})$, there exist orthogonal matrices O_1,O_2 and a diagonal matrix C with positive entries such that $A=O_1CO_2$. The family of diagonal entries of C is unique.*

PROOF. The existence is proved in [28], p. 14. It follows from $({}^{\mathrm{t}}A)A=({}^{\mathrm{t}}O_2)C^2O_2$ that $(({}^{\mathrm{t}}O_2)e_1,\dots,({}^{\mathrm{t}}O_2)e_n)$ is a basis of eigenvectors for $({}^{\mathrm{t}}A)A$. Hence the diagonal elements of C are exactly the positive square roots of the eigenvalues of the symmetric positive definite matrix $({}^{\mathrm{t}}A)A$.

LEMMA 17.3. *For every ellipsoid D, there exists an orthonormal basis $(\xi_1,\dots,\xi_n)$ and, if the basis is fixed, a unique tuple $(d_1,\dots,d_n)$ of positive numbers such that*

$$D=\left\{\sum_{k=1}^n x_k\xi_k : \sum_{k=1}^n\left(\frac{x_k}{d_k}\right)^2\le 1\right\}=O(d_k\delta_{kl})_{k,l}B\quad\text{where } O=(\xi_1\dots\xi_n).$$

$d_1,\dots,d_n$ are called the principal semiaxes *of the ellipsoid D.*

PROOF. Let $A\in\mathrm{Gl}(n,\mathbb{R})$ such that $D=AB$ and let O_1,O_2 and $C=(d_k\delta_{kl})_{k,l}$ be as in 17.2. Since $D=O_1CB$, the existence is clear. For orthogonal matrices $O,\widetilde{O}$ and diagonal matrices $C,\widetilde{C}$ satisfying $\widetilde{O}\widetilde{C}B=OCB$, we get $(OC)^{-1}\widetilde{O}\widetilde{C}B=B$. Hence, there exists an orthogonal matrix O' such that $\widetilde{O}\widetilde{C}=OCO'$. The assertion follows from 17.2.

PROPOSITION 17.4. (i) *For every ellipsoid D, there exists an inner product p on $\mathbb{R}^n$ such that $D=B_p:=\{x\in\mathbb{R}^n : p(x,x)\le 1\}$.*

(ii) *Conversely, for every inner product p, the closed unit ball B_p is an ellipsoid.*

PROOF. (i) Let $(d_1,\dots,d_n)$ and $(\xi_1,\dots,\xi_n)$ be as in 17.3. Then $O:={}^{\mathrm{t}}(\xi_1\dots\xi_n)$ is an orthogonal matrix and $C=((1/d_k^2)\delta_{kl})_{k,l}$ is a diagonal matrix. Further, $p:(x,y)\mapsto {}^{\mathrm{t}}x({}^{\mathrm{t}}OCO)y$ defines a symmetric bilinear form. It follows from $p(\xi_k,\xi_k)=1/d_k^2>0$ that p is an inner product. Hence (i) is a consequence of

$$B_p=\left\{\sum_{k=1}^n x_k\xi_k : p\Big(\sum_{k=1}^n x_k\xi_k,\sum_{l=1}^n x_l\xi_l\Big)\le 1\right\}=\left\{\sum_{k=1}^n x_k\xi_k : \sum_{k=1}^n\left(\frac{x_k}{d_k}\right)^2\le 1\right\}$$

and 17.3.

(ii) Since $A:=(p(e_k,e_l))_{k,l}$ is a symmetric positive definite matrix, there exists an orthonormal basis $(\xi_1,\dots,\xi_n)$ of eigenvectors of A (cf. [48], p. 585). Let $(d_1,\dots,d_n)$ denote the corresponding eigenvalues. Hence

$$\begin{aligned}B_p&=\Big\{\sum_{k=1}^n x_k\xi_k : \sum_{k,l=1}^n x_kx_lp(\xi_k,\xi_l)\le 1\Big\}\\&=\Big\{\sum_{k=1}^n x_k\xi_k : \sum_{k,l=1}^n x_kx_l({}^{\mathrm{t}}\xi_k)A\xi_l=\sum_{k=1}^n x_k^2d_k\le 1\Big\}\end{aligned}$$

is an ellipsoid with principal semiaxes $(1/\sqrt{d_1},\dots,1/\sqrt{d_n})$.

Proposition 17.5. *For every positive semidefinite symmetric bilinear form p in $\mathbb{R}^n$, there exists a matrix A such that $p(x,y) = \langle Ax, Ay\rangle$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Let $S := (p(e_k, e_l))_{k,l}$ be the positive semidefinite symmetric matrix associated with p. There exist an orthogonal matrix O and a diagonal matrix $C = (d_k\delta_{kl})_{k,l}$ (where $d_k \geq 0$) such that $S = ({}^{\mathrm{t}}O)CO$ (cf. [48], p. 585). The matrix $A := (\sqrt{d_k}\delta_{kl})_{k,l}O$ has the desired properties.

Proposition 17.6. *Let D be an ellipsoid in $\mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to M$ be the orthogonal projection onto the linear subspace M. Then $\pi(D)$ and $D \cap M$ are ellipsoids in M.*

Proof. Let $A \in \mathrm{Gl}(n,\mathbb{R})$ be such that $D = AB$. Choose orthonormal bases $(\xi'_1,\dots,\xi'_m)$ and $(\xi_1,\dots,\xi_m)$ $(m := \dim M)$ of $A^{-1}M$ and M, respectively. There exists a linear mapping $\varrho : \mathbb{R}^n \to M$ such that $\varrho(\xi'_k) = \xi_k$ for $k \in \{1,\dots,m\}$ and $\varrho(A^{-1}(M^{\perp})) = \{0\}$. Furthermore, the isomorphism $C : M \to M$ determined by $C(\xi_k) = A(\xi'_k)$ satisfies $C \circ \varrho = \pi \circ A$. It follows from $\varrho(B) = B \cap M$ that $\pi(D)$ is an ellipsoid in M.

Let p be an inner product on $\mathbb{R}^n$ such that $D = B_p$ (17.4(i)). Since $p|_{M\times M}$ is an inner product on M and $D \cap M = B_{p|_{M\times M}}$, the assertion follows from 17.4(ii).

Note 17.7. The material of this chapter is standard.

18. Properties of the Kolmogorov diameter

In this chapter we collect some properties of the Kolmogorov diameter. They are important for the treatment of nuclear groups.

Definition 18.1. For symmetric convex subsets X, Y of a vector space E and $k \in \mathbb{N}$, we put $\mathrm{d}_k(X,Y) := \inf\{c > 0 :$ there exists a subspace $L_c \leq E$ such that $\dim L_c < k$ and $X \subseteq cY + L_c\} \in [0,\infty]$. Then $\mathrm{d}_k(X,Y)$ is called the k-th *Kolmogorov diameter* of X and Y in E.

If $T : E_1 \to E_2$ is a linear operator between the normed spaces E_1, E_2, we put $\mathrm{d}_k(T) := \mathrm{d}_k(T(B_{E_1}), B_{E_2})$ where B_{E_j} (for $j \in \{1,2\}$) denotes the closed unit ball.

Remark 18.2. (i) For each convex subset C of a vector space E and all $a, b > 0$ we have $(a+b)C = aC + bC$.

(ii) For symmetric convex subsets X, Y, Z, W of a vector space such that $Z \subseteq X$ and $Y \subseteq W$, for all $k, l \in \mathbb{N}$, and $\lambda > 0$ we have

$$\mathrm{d}_k(Z,Y) \leq \mathrm{d}_k(X,Y) \quad \text{and} \quad \mathrm{d}_k(X,W) \leq \mathrm{d}_k(X,Y),$$
$$\mathrm{d}_k(\lambda X,Y) = \lambda \mathrm{d}_k(X,Y) \quad \text{and} \quad \mathrm{d}_k(X,\lambda Y) = \frac{1}{\lambda}\mathrm{d}_k(X,Y).$$

Proposition 18.3. *Let X, Y, Z be symmetric convex subsets of the vector space E.*

(i) $\mathrm{d}_{k+l-1}(X,Z) \leq \mathrm{d}_k(X,Y) \cdot \mathrm{d}_l(Y,Z)$.
(ii) $\mathrm{d}_{k+l-1}(X+Y,Z) \leq \mathrm{d}_k(X,Z) + \mathrm{d}_l(Y,Z)$.

The proof is straightforward.

Remark 18.4. Let p and q be seminorms on E such that $\mathrm{d}_1(B_p, B_q) < \infty$ (where B_p and B_q denote the closed unit balls). Then:

(i) $q \leq \mathrm{d}_1(B_p, B_q) \cdot p$.

(ii) Each q-open set (i.e., open with respect to the topology $\mathcal{O}_q$ induced by q) is p-open. [This follows directly from (i).]

(iii) The identity mapping $(E, \mathcal{O}_p) \to (E, \mathcal{O}_q)$ is continuous. [This is a direct consequence of (ii).]

Lemma 18.5. (i) *For each linear operator* $\varphi : E_1 \to E_2$, *for all symmetric convex subsets* $X, Y \subseteq E_1$, *and for all* $k \in \mathbb{N}$, *we have* $\mathrm{d}_k(\varphi(X), \varphi(Y)) \leq \mathrm{d}_k(X, Y)$.

(ii) *If* $\varphi : E_1 \to E_2$ *is a homomorphism then for symmetric convex subsets* $X, Y \subseteq E_2$ *and* $k \in \mathbb{N}$ *we have* $\mathrm{d}_k(\varphi^{-1}(X), \varphi^{-1}(Y)) = \mathrm{d}_k(X \cap \varphi(E_1), Y \cap \varphi(E_1))$.

(iii) *Let* $\varphi_j : E_j \to E_{j+1}$ $(j \in \{1, 2, 3\})$ *be continuous linear operators between normed spaces. For* $k \in \mathbb{N}$ *we have* $\mathrm{d}_k(\varphi_3 \circ \varphi_2 \circ \varphi_1) \leq \|\varphi_3\| \cdot \mathrm{d}_k(\varphi_2) \cdot \|\varphi_1\|$.

The proof follows directly from the definitions.

Definition 18.6. A seminorm $p : E \to \mathbb{R}$ on a vector space E is called a *pre-Hilbert seminorm* if the parallelogram law holds:

$$p(x+y)^2 + p(x-y)^2 = 2p(x)^2 + 2p(y)^2 \quad \forall x, y \in E.$$

Notation 18.7. For a seminorm p on a vector space E, the set $N_p := \{x \in E : p(x) = 0\}$ is a subspace of E. Further, p induces a norm on the quotient $E_p := E/N_p \to \mathbb{R}$, $x + N_p \mapsto p(x)$. It is denoted by p again. Let $\psi_p : E \to E_p$ be the canonical projection.

The seminorm p satisfies the parallelogram law if and only if the induced norm does.

If q is another seminorm on E such that $\mathrm{d}_1(B_p, B_q) < \infty$ then $N_p \leq N_q$, and hence $\varLambda_{p,q} : E_p \to E_q$, $\psi_p(x) \mapsto \psi_q(x)$, is well defined and continuous (18.4(iii)). The following diagram commutes:

$$\begin{tikzcd}
(E, \mathcal{O}_p) \arrow[r, "\mathrm{id}"] \arrow[d, "\psi_p"'] & (E, \mathcal{O}_q) \arrow[d, "\psi_q"] \\
E_p \arrow[r, "\varLambda_{p,q}"'] & E_q
\end{tikzcd}$$

Moreover, $\mathrm{d}_k(\varLambda_{p,q}) = \mathrm{d}_k(B_p, B_q)$. [Since $B_{E_p} = \{x + N_p : x \in B_p\} = \psi_p(B_p)$ (and $B_{E_q} = \psi_q(B_q)$), we get

$$\mathrm{d}_k(\varLambda_{p,q}) = \mathrm{d}_k(\psi_q(B_{E_p}), \psi_q(B_{E_q})) \overset{18.5(\mathrm{ii})}{=} \mathrm{d}_k(B_p + N_q, B_q) \overset{18.2(\mathrm{ii})}{\geq} \mathrm{d}_k(B_p, B_q);$$

next $N_q \subseteq c \cdot B_q$ for every $c > 0$ implies $\mathrm{d}_k(B_p + N_q, B_q) \overset{18.3(\mathrm{ii})}{\leq} \mathrm{d}_k(B_p, B_q) + \mathrm{d}_1(N_q, B_q) = \mathrm{d}_k(B_p B_q)$.]

Remark 18.8. For every pre-Hilbert seminorm p on E, there exists a positive semidefinite symmetric bilinear form $\widetilde{p}$ such that $p(x)^2 = \widetilde{p}(x, x)$ (for all $x \in E$). [This is an easy consequence of Lemma 20.6 in [36], p. 85, and 18.7.]

If E is a topological vector space then p is continuous if and only if $\widetilde{p}$ is continuous. [This follows immediately from the polar identity.]

Recall that a subset A of a seminormed space (E, p) is called *bounded* if $p(A)$ is bounded in $\mathbb{R}$.

Lemma 18.9. *Let p be a seminorm on the vector space E. Every closed subset $A \subseteq E$ satisfies $A + N_p = A$. Furthermore, if A is closed and bounded, then $A + L$ is closed (in (E, p)) for every finite-dimensional subspace $L \leq E$.*

Proof. First, let $a \in A$ and $x \in N_p$. Since $a \in a + x + \varepsilon B_p$ for every $\varepsilon > 0$, "$\subseteq$" follows. The other inclusion is trivial.

Hence, in order to prove the second assertion, we may assume that $L \cap N_p = \{0\}$. Let $a_n \in A$, $l_n \in L$ be such that $(a_n + l_n)_{n\in\mathbb{N}}$ converges to $x \in E$ (with respect to p). Since $(a_n + l_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ are bounded sequences, so is $(l_n)_{n\in\mathbb{N}}$. Moreover, $(l_n)_{n\in\mathbb{N}}$ belongs to the finite-dimensional normed space $(L, p|_L)$. By the equivalence of norms, there exists a convergent subsequence $(l_{n_k})_{k\in\mathbb{N}}$ which tends to $l \in L$. This implies that $(a_{n_k})_{k\in\mathbb{N}}$ converges to $x - l \in A$ (since A is closed). The assertion follows.

Lemma 18.10 (Auerbach). *For every finite-dimensional normed space $(E, \|\cdot\|)$ of dimension d, there exists a basis $(x_1, \ldots, x_d)$ of normalized vectors such that $B_E \subseteq \{\sum_{k=1}^{d} \alpha_k x_k : |\alpha_k| \leq 1\}$ and there are linear forms $\varphi_1, \ldots, \varphi_d$ of norm 1 satisfying $\varphi_k(x_l) = \delta_{kl}$ for all $k, l \in \{1, \ldots, d\}$.*

Proof. See [70], p. 173 or [64], p. 135, (8.4.1).

Corollary 18.11. *Let L be a finite-dimensional subspace of the normed space E. There exists a projection $\pi : E \to L$ of norm $\leq \dim L$.*

Proof. Let $(x_1, \ldots, x_d)$ ($d := \dim L$) be a basis in L and let $\varphi_1, \ldots, \varphi_d$ be linear forms in L as in the above lemma. According to the Hahn–Banach Theorem, there exist normalized linear functionals $\varphi'_1, \ldots, \varphi'_d$ on E which extend the corresponding φ_k. Hence $\pi : E \to L$, $x \mapsto \sum_{k=1}^{d} \varphi'_k(x) x_k$, is the identity on L and $\|\pi\| \leq d$.

Remark 18.12. It was proved by M. J. Kadets and G. Snobar (cf. 28.2.6 in [65], and [37]) that for every normed space E and every finite-dimensional subspace $L \leq E$, there exists a projection $\pi : E \to L$ such that $\|\pi\| \leq \sqrt{\dim L}$. This constant is optimal.

Lemma 18.13. *Let E be a linear subspace of the vector space $\widetilde{E}$, let q be a seminorm on $\widetilde{E}$ and let $\widetilde{X}$ be a symmetric convex subset of $\widetilde{E}$. Put $\widetilde{B}_q := \{x \in \widetilde{E} : q(x) \leq 1\}$, $B_q := E \cap \widetilde{B}_q$, and $X := \widetilde{X} \cap E$.*

(i) *If $\overline{B}_q^{(q)} \supseteq \widetilde{B}_q$ and $\mathrm{d}_1(\widetilde{X}, \widetilde{B}_q) < \infty$, then $\mathrm{d}_k(X, B_q) \leq \mathrm{d}_k(\widetilde{X}, \widetilde{B}_q)$ for all $k \in \mathbb{N}$.*

(ii) *Let p be a seminorm on $\widetilde{E}$ and let Y be a symmetric convex subset of $\widetilde{E}$ such that $\widetilde{X} \subseteq \overline{Y}^{(p)}$. If $\mathcal{O}_p$ is finer than $\mathcal{O}_q$, then $\mathrm{d}_k(\widetilde{X}, \widetilde{B}_q) \leq \mathrm{d}_k(Y, B_q)$.*

Proof. (i) By assumption, $\mathrm{d}_k(\widetilde{X}, \widetilde{B}_q) < \infty$ for all $k \in \mathbb{N}$. Let $\widetilde{L} \leq \widetilde{E}$ be a finite-dimensional subspace and $c > 0$ such that $\widetilde{X} \subseteq c \cdot \widetilde{B}_q + \widetilde{L}$. We may assume that $\widetilde{L} \cap N_q = \{0\}$. (Otherwise, we replace $\widetilde{L}$ by a suitable subspace L' satisfying $\widetilde{L} = L' \oplus (\widetilde{L} \cap N_q)$ algebraically.) If $\widetilde{L} = \{0\}$ then $X \subseteq c \cdot B_q$. So, let $\widetilde{L} \neq \{0\}$. Since $q|_{\widetilde{L}}$ is a norm, there exist, according to 18.10 and 8.19, a basis $(\widetilde{l}_1, \ldots, \widetilde{l}_m)$ of $\widetilde{L}$ such that $q(\widetilde{l}_k) = 1$ and linear forms $\varphi_1, \ldots \varphi_m \in \widetilde{E}_q$ (where $\widetilde{E}_q$ denotes the space $\widetilde{E}$ endowed with the topology induced by q) such that $\varphi_k(\widetilde{l}_j) = \delta_{jk}$ and $\sup\{|\varphi_k(x)| : x \in \widetilde{B}_q\} = 1$ (for all $j, k \in \{1, \ldots, m\}$).

By assumption, for $\varepsilon > 0$ and $d > \mathrm{d}_1(\widetilde{X}, \widetilde{B}_q)$, there exist $l_1, \ldots, l_m \in B_q$ such that

$$q(l_k - \widetilde{l}_k) < \frac{c}{2(d+c)m}\varepsilon.$$

Put $L := \langle l_1, \ldots, l_m \rangle_{\mathbb{R}}$. For $x \in X$, there exist $\widetilde{y} \in \widetilde{B}_q$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that $x = c\widetilde{y} + \sum_{k=1}^m \alpha_k \widetilde{l}_k$. We get $|\alpha_k| = |\varphi_k(x - c\widetilde{y})| \leq d + c$. For $y \in B_q$ such that $q(\widetilde{y} - y) < \varepsilon/2$ and

$$z := (\widetilde{y} - y) + \frac{1}{c}\sum_{k=1}^m \alpha_k(\widetilde{l}_k - l_k),$$

we get $x = c(y+z) + \sum_{k=1}^m \alpha_k l_k$ and $z \in E$. Moreover,

$$q(z) \leq \frac{\varepsilon}{2} + \frac{1}{c}m(d+c)\frac{c}{2(d+c)m}\varepsilon = \varepsilon$$

implies $X \subseteq c(1+\varepsilon)B_q + L$ and hence (i).

For the proof of (ii), we may assume that $\mathrm{d}_k(Y, B_q) < \infty$. Let $c > 0$ and let L be a subspace of E of dimension $< k$ such that $Y \subseteq cB_q + L$. This implies $\widetilde{X} \subseteq \overline{Y}^{(p)} \subseteq \overline{c\widetilde{B}_q + L}^{(p)}$. If $\mathcal{O}_p$ is finer than $\mathcal{O}_q$, then $\overline{c\widetilde{B}_q + L}^{(p)} \subseteq \overline{c\widetilde{B}_q + L}^{(q)} \overset{18.9}{=} c\widetilde{B}_q + L$. This completes the proof.

Remark 18.14. Let p be a seminorm on a vector space $\widetilde{E}$ and let E be a dense subspace of $\widetilde{E}$ with respect to p. Using the notation of the above lemma, we have $\widetilde{B}_p = \overline{B}_p^{(p)}$.

["$\supseteq$" is trivial. Conversely, let $x \in \widetilde{B}_p$ and $\varepsilon > 0$. We are to show that $(x + \varepsilon\widetilde{B}_p) \cap B_p \neq \emptyset$. There exist $\delta > 0$ and $x' \in \widetilde{E}$ such that $p(x') < 1$ and

$$(*) \qquad x' + \delta\widetilde{B}_p \subseteq (x + \varepsilon\widetilde{B}_p) \cap \widetilde{B}_p.$$

This is trivial if $p(x) \in [0, 1[$. If $p(x) = 1$ then $p|_{\langle x \rangle}$ is a norm and hence there exists $x' \in \langle x \rangle \cap (x + \varepsilon\widetilde{B}_p)$ such that $1 - \varepsilon < p(x') < 1$; now $(*)$ follows easily.

It remains to observe that $x' + \delta\widetilde{B}_p$ has non-empty interior.]

Corollary 18.15. *Let p and q be seminorms on a vector space $\widetilde{E}$ satisfying* $\mathrm{d}_1(\widetilde{B}_p, \widetilde{B}_q) < \infty$. *For every subspace E of $\widetilde{E}$ which is dense with respect to $\mathcal{O}_p$, we have* $\mathrm{d}_k(B_p, B_q) = \mathrm{d}_k(\widetilde{B}_p, \widetilde{B}_q)$.

Proof. According to 18.4, E is also dense in $\widetilde{E}$ with respect to $\mathcal{O}_q$. Putting $\widetilde{X} := \widetilde{B}_p$ in 18.13(i), we get "$\leq$" from 18.14 (and 18.13(i)).

For the proof of "$\geq$", we put $\widetilde{X} = \widetilde{B}_p$ and $Y = B_p$ in 18.13(ii). The assertion follows from 18.14 (and 18.13(ii)).

Corollary 18.16. *Let $\widetilde{E}$ and $\widetilde{F}$ be normed spaces and let $\widetilde{T} : \widetilde{E} \to \widetilde{F}$ be a continuous linear mapping. For all dense subspaces E of $\widetilde{E}$ and and all dense subspaces F of $\widetilde{F}$ containing $\widetilde{T}(E)$, we have* $\mathrm{d}_k(\widetilde{T}) = \mathrm{d}_k(T)$ *where $T : E \to F$ denotes the induced mapping.*

Proof. Using the notation of 18.13, we put $\widetilde{X} := \widetilde{T}(B_{\widetilde{E}})$ and $q = p$ the norm on $\widetilde{F}$. Then we get $\mathrm{d}_k(T) = \mathrm{d}_k(T(B_E), B_F) \overset{18.2(\mathrm{ii})}{\leq} \mathrm{d}_k(\widetilde{T}(B_{\widetilde{E}}) \cap F, B_F) \overset{18.13(\mathrm{i})}{\leq} \mathrm{d}_k(\widetilde{T}(B_{\widetilde{E}}), B_{\widetilde{F}}) = \mathrm{d}_k(\widetilde{T})$. Because of $\widetilde{T}(B_{\widetilde{E}}) \overset{18.14}{=} \widetilde{T}(\overline{B_E}) \subseteq \overline{T(B_E)}$, the other inequality is a consequence of 18.13(ii) (for $\widetilde{X} = \widetilde{T}(B_{\widetilde{E}})$ and $Y = T(B_E)$).

LEMMA 18.17. *Let $\varphi : E_1 \to E_2$ be a continuous linear operator between the normed spaces E_1 and E_2. If E_2 is a subspace of an inner product space H such that E_2 has an orthogonal complement in H, then $\mathrm{d}_k(\varphi) = \mathrm{d}_k(\varphi_H)$ where φ_H denotes the induced mapping $E_1 \to H$.*

PROOF. "$\geq$" follows from 18.2(ii). Conversely, let $\varphi(B_{E_1}) \subseteq c \cdot B_H + L$ ($c > 0$, $L \leq H$ such that $\dim L < \infty$). If $\pi : H \to E_2$ denotes the orthogonal projection then $\varphi(B_{E_1}) = \pi(\varphi(B_{E_1})) \subseteq c \cdot \pi(B_H) + \pi(L) = c \cdot B_{E_2} + \pi(L)$, which implies "$\leq$".

LEMMA 18.18. *Let p and q be seminorms on the vector space E such that $\mathrm{d}_1(B_p, B_q) < \infty$. Assume further that q is a pre-Hilbert seminorm. For every subspace M of E and every $k \in \mathbb{N}$, we have $\mathrm{d}_k(M \cap B_p, M \cap B_q) \leq \mathrm{d}_k(B_p, B_q)$.*

PROOF. We use the notation of 18.7. We denote by r and s the restrictions of p and q onto M. Let $\widehat{\Lambda}_{r,s} : \widehat{M_r} \to \widehat{M_s}$ denote the continuous extension of $\Lambda_{r,s}$ onto the completions $\widehat{M_r}$ and $\widehat{M_s}$ of the spaces M_r and M_s (2.17). ($\widehat{M_r}$ and $\widehat{M_s}$ can be embedded in the completions $\widehat{E}_p$ and $\widehat{E}_q$ of E_p and E_q, respectively.) Since $\widehat{E}_q$ is a Hilbert space and $\widehat{M_s}$ is a complete and hence closed subspace of $\widehat{E}_q$, we get

$$\begin{aligned}\mathrm{d}_k(M \cap B_p, M \cap B_q) &= \mathrm{d}_k(B_r, B_s) \overset{18.7}{=} \mathrm{d}_k(\Lambda_{r,s}) \overset{18.16}{=} \mathrm{d}_k(\widehat{\Lambda}_{r,s}) \\ &\overset{18.17}{=} \mathrm{d}_k(\widehat{\Lambda}_{r,s} : \widehat{M_r} \to \widehat{E}_q) \overset{18.5(\mathrm{iii})}{\leq} \mathrm{d}_k(\widehat{\Lambda}_{p,q} : \widehat{E}_p \to \widehat{E}_q) \\ &\overset{18.16}{=} \mathrm{d}_k(\Lambda_{p,q}) \overset{18.7}{=} \mathrm{d}_k(B_p, B_q).\end{aligned}$$

PROPOSITION 18.19. *Let D be an ellipsoid in $\mathbb{R}^n$ with principal semiaxes $d_1 \geq \ldots \geq d_n$ and let B be the closed unit ball. Then $\mathrm{d}_k(D, B) = d_k$ and $\mathrm{d}_k(B, D) = d_{n-k+1}^{-1}$.*

PROOF. By 17.3 and 18.5(ii), we may assume that $D = \{(x_1, \ldots, x_n) : \sum_{i=1}^n (x_i/d_i)^2 \leq 1\}$. For $L_k := \langle e_1, \ldots, e_{k-1}\rangle_{\mathbb{R}}$ (and $L_1 := \{0\}$) we have $D \subseteq d_k \cdot B + L_k$, which implies $\mathrm{d}_k(D, B) \leq d_k$. Conversely, let $D \subseteq c \cdot B + L$ ($c > 0$, $\dim L < k$). Then

$$\left\{(x_1, \ldots, x_n) : \sum_{i=1}^{k} \left(\frac{x_i}{d_k}\right)^2 + \sum_{i=k+1}^{n} \left(\frac{x_i}{d_i}\right)^2 \leq 1\right\} \subseteq D \subseteq c \cdot B + L.$$

For dimensional reasons, there exists an $x \in L_{k+1} \cap L^{\perp}$ of norm ≤ 1. Hence $d_k x \in D$, which implies that $d_k x = cb + l$ for suitable $b \in B$ and $l \in L$. From this we get $c^2 \geq \|cb\|^2 = \|d_k \cdot x - l\|^2 = d_k^2 + \|l\|^2 \geq d_k^2$, which implies "$\geq$".

For $A := (d_k^{-1}\delta_{kl})_{k,l} \in \mathrm{Gl}(n, \mathbb{R})$ from 18.5(ii) we get $\mathrm{d}_k(B, D) = \mathrm{d}_k(AB, AD) = \mathrm{d}_k(AB, B)$. Since the principal semiaxes of the ellipsoid AB are $d_n^{-1} \geq \ldots \geq d_1^{-1}$, the assertion follows from the first part.

COROLLARY 18.20. *Let M be an m-dimensional subspace of $\mathbb{R}^n$. If $\pi : \mathbb{R}^n \to M$ denotes the orthogonal projection then we have, for every ellipsoid D in $\mathbb{R}^n$ and every $k \in \{1, \ldots, m\}$,*

$$\begin{gathered}\mathrm{d}_k(D, B_n) \geq \mathrm{d}_k(\pi(D), \pi(B_n)) \geq \mathrm{d}_{n-m+k}(D, B_n), \\ \mathrm{d}_k(D, B_n) \geq \mathrm{d}_k(D \cap M, B_n \cap M) \geq \mathrm{d}_{n-m+k}(D, B_n).\end{gathered}$$

PROOF. First consider the projection of D; the first inequality is a consequence of 18.5(i). Since $\pi(D)$ is an ellipsoid in M (see 17.6), for $k \in \{1, \ldots, m\}$ (and $B := B_n$) we get

$$\frac{1}{\mathrm{d}_k(\pi(D),\pi(B))} \overset{18.19}{=} \mathrm{d}_{m-k+1}(\pi(B),\pi(D)) \overset{18.5(\mathrm{i})}{\leq} \mathrm{d}_{m-k+1}(B,D) \overset{18.19}{=} \frac{1}{\mathrm{d}_{n-m+k}(D,B)}.$$

As to the intersection: the first inequality is a consequence of 18.18. For $k \in \{1,\dots,m\}$, we get

$$\frac{1}{\mathrm{d}_k(D\cap M, B\cap M)} \overset{18.19}{=} \mathrm{d}_{m-k+1}(B\cap M, D\cap M) \leq \mathrm{d}_{m-k+1}(B,D) \overset{18.19}{=} \frac{1}{\mathrm{d}_{n-m+k}(D,B)}.$$

(The inequality is a consequence of 17.4(i) and 18.18.)

REMARK 18.21. Observe that in the second part of the above proof only an application of 18.18 to the finite-dimensional case, which is easier to prove, was used.

LEMMA 18.22. *Let* (H,p) *be a Hilbert space. For every closed subspace* H_0, $H_1 := H_0^{\perp}$, *and all pre-Hilbert seminorms* r *and* q *satisfying* $r(H_0) = q(H_0) = \{0\}$ *and* $\mathrm{d}_1(B_p,B_r) < \infty$ *and* $\mathrm{d}_1(B_r,B_q) < \infty$, *we have*:

(i) $\mathrm{d}_k(B_p,B_r) = \mathrm{d}_k(B_p\cap H_1, B_r\cap H_1)$, *and*
(ii) $\mathrm{d}_k(B_r,B_q) = \mathrm{d}_k(B_r\cap H_1, B_q\cap H_1)$.

PROOF. In both cases "$\geq$" is a consequence of 18.18. We have $B_r = (B_r\cap H_1)+H_0$ (and analogously for q).

(i) Since H_1 is the orthogonal complement of H_0 with respect to p, we get $B_p \subseteq (B_p\cap H_1)+H_0$. Hence, $B_p\cap H_1 \subseteq c\cdot(B_r\cap H_1)+L$ ($c>0$ and $L\leq H_1$) implies $B_p \subseteq c\cdot((B_r\cap H_1)+H_0)+L = c\cdot B_r + L$, which shows (i).

(ii) $B_r\cap H_1 \subseteq c\cdot(B_q\cap H_1)+L$ ($c>0$ and $L\leq H_1$) implies $B_r\subseteq c\cdot B_q + L$ and hence (ii).

LEMMA 18.23. *Let* p *be an inner product and let* q *be a pre-Hilbert seminorm on* $\mathbb{R}^n$. *For every* $\varepsilon>0$, *there exists an ellipsoid* $D\subseteq B_q$ *such that* $\mathrm{d}_k(B_p,D)\leq \mathrm{d}_k(B_p,B_q)+\varepsilon$ *for* $k\in\{1,\dots,n\}$.

PROOF. Because of 18.5(ii), we may assume that p is the standard inner product. There exists an orthonormal basis $(\xi_1,\dots,\xi_n)$ of $\mathbb{R}^n$ with respect to p with $B_q = \{\sum_{k=1}^n t_k\xi_k : \sum_{k=1}^m (t_k/d_k)^2 \leq 1\}$ for suitable $m\leq n$ and positive numbers $d_1\geq\ldots\geq d_m$ (18.7 and 17.3). It follows from 18.19 and 18.22 that $\mathrm{d}_k(B_p,B_q) = \mathrm{d}_k(B_p\cap M, B_q\cap M)$ for $M = \langle \xi_1,\dots,\xi_m\rangle_{\mathbb{R}}$, and hence $\mathrm{d}_k(B_p,B_q) = d_{m-k+1}^{-1}$ for $k\in\{1,\dots,m\}$ and 0 for $k>m$. The assertion follows for $D := \{\sum_{k=1}^n t_k\xi_k : \sum_{k=1}^n (t_k/d_k)^2 \leq 1\}$ and sufficiently large $d_{m+1},\dots,d_n$ (from 18.19).

COROLLARY 18.24. *Let* p *and* q *be pre-Hilbert seminorms on* $\mathbb{R}^n$ *satisfying* $\mathrm{d}_1(B_p,B_q) < \infty$. *There exist sequences of pre-Hilbert norms* $(p_n)_{n\in\mathbb{N}}$ *and* $(q_n)_{n\in\mathbb{N}}$ *on* $\mathbb{R}$ *such that*

(i) $q_n\geq q$, $(p_n)_{n\in\mathbb{N}}$ *decreases*,
(ii) $\eta\cdot B_p \subseteq \bigcup_{n\in\mathbb{N}} B_{p_n}$ *for every* $\eta\in\,]0,1[$, *and*
(iii) $\inf\{\mathrm{d}_k(B_{p_n},B_{q_n}) : n\in\mathbb{N}\} = \mathrm{d}_k(B_p,B_q)$ (*for all* $k\in\mathbb{N}$).

PROOF. For $M := N_p^{\perp}$ we deduce from 18.22(ii) that $\mathrm{d}_k(B_p\cap M, B_q\cap M) = \mathrm{d}_k(B_p,B_q)$. For $n\in\mathbb{N}$, there exists a pre-Hilbert norm $\widetilde{q}_n$ on M satisfying $\mathrm{d}_k(B_p\cap M, B_{\widetilde{q}_n}) \leq \mathrm{d}_k(B_p,B_q)+1/n$ and $B_{\widetilde{q}_n}\subseteq B_q\cap M$ (18.23, 17.4). For an arbitrary decreasing sequence $(p_n')_{n\in\mathbb{N}}$ of pre-Hilbert norms on N_p satisfying $\bigcup_{n\in\mathbb{N}} B_{p_n'} = N_p$, there exists a sequence $(q'_n)_{n\in\mathbb{N}}$ of pre-Hilbert norms on N_p with $\mathrm{d}_1(B_{p'_n}, B_{q'_n}) < 1/n$. For $x \in M$ and $y \in N_p$ we put $p_n(x+y) := \sqrt{p(x)^2 + p'_n(y)^2}$ and $q_n(x+y) := \sqrt{\widetilde{q}_n(x)^2 + q'_n(y)^2}$. One easily checks that p_n and q_n define pre-Hilbert norms on $\mathbb{R}^n$. Furthermore

$$\begin{aligned}\mathrm{d}_k(B_{p_n}, B_{q_n}) \quad &\leq \quad \mathrm{d}_k(B_p \cap M + B_{p'_n}, B_{q_n}) \overset{18.3\text{(ii)}}{\leq} \mathrm{d}_k(B_p \cap M, B_{q_n}) + \mathrm{d}_1(B_{p'_n}, B_{q_n}) \\ &\overset{18.2\text{(ii)}}{\leq} \mathrm{d}_k(B_p \cap M, B_{\widetilde{q}_n}) + \mathrm{d}_1(B_{p'_n}, B_{q'_n}) \leq \mathrm{d}_k(B_p, B_q) + \frac{1}{n} + \frac{1}{n}.\end{aligned}$$

Let $\eta < 1$, $x \in M$ and $y \in N_p$ be such that $p(x+y) = p(x) \leq \eta$. There exists $n \in \mathbb{N}$ such that $p'_n(y) \leq \sqrt{1-\eta^2}$ and hence $p_n(x+y) \leq 1$. Since $B_{q_n} \subseteq B_{\widetilde{q}_n} + N_p \subseteq B_q$, the assertion follows.

Theorem 18.25. (i) *Let* $(H, \langle\cdot|\cdot\rangle)$ *be a Hilbert space. For every self-adjoint compact operator* $T : H \to H$, *there exist an orthonormal system* $(x_k)_{k\in N}$ *and a family* $(\lambda_k)_{k\in N}$ *of positive numbers where* $N = \{1, \ldots, n\}$ *for suitable* $n \in \mathbb{N}$ *or* $N = \mathbb{N}$ *satisfying* $\lambda_{k+1} \leq \lambda_k$ *for all* $k, k+1 \in N$ (*and* $\lambda_k \downarrow 0$ *if* $N = \mathbb{N}$) *and such that* $T(x) = \sum_{k\in N} \lambda_k \langle x|x_k\rangle x_k$ *for all* $x \in H$.

(ii) *Each continuous linear operator* $S : H \to H$ *which belongs to the closure* (*with respect to the usual operator norm*) *of the continuous operators having finite rank is compact.*

Proof. (i) This is the assertion of Satz 30.1 in [33], p. 202.

(ii) is a consequence of part (a) and (c) of Theorem 4.18 in [73], p. 104.

Lemma 18.26. (i) *Let* $(H, \langle\cdot|\cdot\rangle)$ *be a real Hilbert space and let* q *be a continuous pre-Hilbert seminorm on* H. *Assume that* $\mathrm{d}_k(B_H, B_q) \to 0$. *Then there exists a compact self-adjoint operator* $Q : H \to H$ *such that* $\langle Q(x)|x\rangle = q(x)^2$.

(ii) *Let* $(H, \langle\cdot|\cdot\rangle)$ *be a real inner product space which has an orthogonal basis* $(x_k, y_i : k \in N,\ i \in I)$ ($N = \{1, \ldots, n\}$ *or* $N = \mathbb{N}$ *and* I *is a suitable* (*possibly empty*) *index set*) *and let* $(\lambda_k)_{k\in N}$ *be a decreasing sequence of positive numbers. Then* $Q : H \to H$, $x \mapsto \sum_{k\in N} \lambda_k \langle x|x_k\rangle x_k$, *is a continuous linear operator. Further,* $q : x \mapsto \sqrt{\langle Q(x)|x\rangle}$ *defines a pre-Hilbert seminorm on* H *for which we have* $\mathrm{d}_k(B, B_q) = \sqrt{\lambda_k}$ *for* $k \in N$ *and* 0 *for* $k \in \mathbb{N} \setminus N$.

Proof. Put $B := B_H$.

(i) Let $\widetilde{q}$ be the continuous positive semidefinite bilinear form associated with q (see 18.8). For $x \in H$, the mapping $y \mapsto \widetilde{q}(x,y)$ is a continuous linear form. Hence, there exists a unique $Q(x) \in H$ such that $\langle Q(x)|y\rangle = \widetilde{q}(x,y)$ for all $y \in H$. Since q is bilinear, Q is linear. For $d > \mathrm{d}_1(B, B_q)$ we have $\sup\{|\widetilde{q}(x,y)| : x, y \in B\} \leq d^2 \sup\{|\widetilde{q}(x,y)| : x, y \in B_q\} \leq d^2$ (by the Cauchy–Schwarz inequality; cf. Satz 20.3 in [36], p. 84). This shows that Q is continuous. It is self-adjoint since $\widetilde{q}$ and $\langle\cdot|\cdot\rangle$ are symmetric.

Moreover, Q is a compact operator. [For $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\mathrm{d}_k(B, B_q) < \varepsilon/(2d)$ (where $d > \mathrm{d}_1(B, B_q)$). This implies $B \subseteq (\varepsilon/(2d))B_q + L$ for a suitable finite-dimensional subspace L of H. Since $Q(L)$ is also finite-dimensional and hence closed, there exists an orthogonal projection $\pi : H \to Q(L)$ of norm ≤ 1. Hence $\widetilde{Q} := \pi \circ Q$ is continuous and has finite rank. For $x \in B$, there exist $\widetilde{x} \in B_q$ and $l \in L$ such that

$x = (\varepsilon/(2d))\widetilde{x} + l$, which implies

$$\begin{aligned}\|Q(x) - \widetilde{Q}(x)\| &= \left\| Q\Big(\frac{\varepsilon}{2d}\widetilde{x}\Big) - \pi\Big(Q\Big(\frac{\varepsilon}{2d}\widetilde{x}\Big)\Big)\right\| \\ &\le \frac{\varepsilon}{2d}\cdot 2\cdot \|Q(\widetilde{x})\| = \frac{\varepsilon}{d}\sup\{\widetilde{q}(\widetilde{x},y) : y \in B\} \le \frac{\varepsilon}{d}\cdot d.\end{aligned}$$

It follows from 18.25(ii) that Q is compact.

(ii) It is easy to see that Q is a self-adjoint linear operator and that $\widetilde{q} : H \times H \to \mathbb{R}$, $(x,y) \mapsto \langle Q(x)|y\rangle$, is a continuous positive semidefinite bilinear form. Hence q defined as above is a pre-Hilbert seminorm.

We may assume that $N = \mathbb{N}$. (Otherwise we put $\lambda_k = 0$ for $k \in \mathbb{N} \setminus N$ and choose arbitrary $x_k \in H$.) It remains to prove that $\mathrm{d}_k(B, B_q) = \sqrt{\lambda_k}$. Fix $k \in \mathbb{N}$ and put $L_k := \langle x_1, \ldots, x_{k-1}\rangle$. For $x \in B$ we have

$$\begin{aligned}\widetilde{q}\Big(x - \sum_{j=1}^{k-1}\langle x|x_j\rangle x_j, x - \sum_{j=1}^{k-1}\langle x|x_j\rangle x_j\Big) &= \Big\langle \sum_{j\ge k}\lambda_j\langle x|x_j\rangle x_j, \sum_{j\ge k}\langle x|x_j\rangle x_j + \sum_{i\in I}\langle x|y_i\rangle y_i\Big\rangle \\ &= \Big\langle \sum_{j\ge k}\lambda_j\langle x|x_j\rangle x_j \Big| \sum_{j\ge k}\langle x|x_j\rangle x_j\Big\rangle = \sum_{j\ge k}\lambda_j\langle x|x_j\rangle^2 \\ &\le \lambda_k \sum_{j\ge k}\langle x|x_j\rangle^2 \le \lambda_k\|x\|^2 \le \lambda_k,\end{aligned}$$

which implies $B \subseteq \sqrt{\lambda_k}\cdot B_q + L_k$. To show "$\ge$", we may assume that $\lambda_k > 0$. According to 18.18, we have $\mathrm{d}_k(B, B_q) \ge \mathrm{d}_k(B \cap L_{k+1}, B_q \cap L_{k+1})$. Since the principal semiaxes of the ellipsoid $B_q \cap L_{k+1} = \{x \in L_{k+1} : \langle Q(x)|x\rangle \le 1\} = \{\sum_{j=1}^k \alpha_j x_j : \sum_{j=1}^k \lambda_j\alpha_j^2 \le 1\}$ are $1/\sqrt{\lambda_1} \le \ldots \le 1/\sqrt{\lambda_k}$, the assertion follows from 18.19.

Lemma 18.27. *Let $(a_k)_{k\in N}$, $(b_k)_{k\in N}$ and $(d_k)_{k\in N}$ be decreasing sequences of positive numbers such that $d_k \le a_k\cdot b_k$ for all $k \in N$ and $N = \{1,\ldots,n\}$ or $N = \mathbb{N}$. Then there exists a decreasing sequence $(c_k)_{k\in N}$ of positive numbers such that $(d_k/c_k)_{k\in N}$ is also decreasing, $c_k \le a_k$ and $d_k/c_k \le b_k$ for all $k \in N$.*

Proof. (i) Let $N = \{1,\ldots,n\}$. Put $c_n := a_n$ and suppose that $c_n,\ldots,c_{k+1}$ $(1 \le k \le n-1)$ have already been constructed. Define

$$c_k := \begin{cases}(d_k/d_{k+1})c_{k+1} & \text{if } d_k \le a_k(d_{k+1}/c_{k+1}),\\ a_k & \text{if } d_k > a_k(d_{k+1}/c_{k+1}).\end{cases}$$

It is easy to see that $d_k/c_k \ge d_{k+1}/c_{k+1}$. From $(d_k/d_{k+1})c_{k+1} \ge c_{k+1}$ and $a_k \ge a_{k+1} \ge c_{k+1}$ we get $c_k \ge c_{k+1}$. In the first case we have $d_k/c_k = d_{k+1}/c_{k+1} \le b_{k+1} \le b_k$ and in the second $d_k/c_k = d_k/a_k \le b_k$. This implies that the sequence $(c_k)_{k\in N}$ can be defined recursively.

(ii) Now, let $N = \mathbb{N}$. For each $n \in \mathbb{N}$, there exists, according to (i), a sequence $(c_k^n)_{k\in\mathbb{N}}$ such that $c_k^n = 0$ for $k > n$ and $(c_k^n)_{k\in\{1,\ldots,n\}}$ has the desired properties for the truncated sequences. The sequence $((c_k^n)_k)_n$ (in the compact metric space $[0,a_1]^{\mathbb{N}}$) has a convergent subsequence $((c_k^{n_m})_k)_{m\in\mathbb{N}}$. Let $c_k := \lim_{m\in\mathbb{N}} c_k^{n_m}$. Obviously, $c_k \le a_k$ and $d_k/c_k \le b_k$ $(\Leftrightarrow c_k \ge d_k/b_k > 0)$. Since $c_k^{n_m} \ge c_{k+1}^{n_m}$ for all $m \in \mathbb{N}$, the sequence $(c_k)_{k\in\mathbb{N}}$ is decreasing and hence the assertion follows.

Theorem 18.28. *Let p and q be pre-Hilbert seminorms on the vector space E such that* $\mathrm{d}_1(B_p, B_q) < \infty$ *and* $\mathrm{d}_k(B_p, B_q) \to 0$. *For all decreasing sequences* $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ *of positive numbers such that* $\mathrm{d}_k(B_p, B_q) \leq a_k b_k$ $(k \in \mathbb{N})$, *there exists a pre-Hilbert seminorm r on E such that* $\mathrm{d}_k(B_p, B_r) \leq a_k$ *and* $\mathrm{d}_k(B_r, B_q) \leq b_k$.

Proof. We use the notation of 18.7. Therefore, it suffices to show that there exists a pre-Hilbert seminorm r on E_p having the desired properties. Hence we may assume that p is a norm. Let $\widehat{p}$ denote the norm on the completion $\widehat{E}_p$ which extends p. Since q is continuous with respect to $\widehat{p}$ (18.4(ii)), there exists a continuous norm $\widehat{q}$ on $\widehat{E}_p$ extending q. Obviously, $\widehat{p}$ and $\widehat{q}$ satisfy the parallelogram law. This means that $(\widehat{E}_p, \widehat{p})$ is a Hilbert space. Since $\widehat{q}$ is continuous with respect to $\widehat{p}$, we get $\mathrm{d}_1(B_{\widehat{p}}, B_{\widehat{q}}) < \infty$ and hence, according to 18.15, $\mathrm{d}_k(B_p, B_q) = \mathrm{d}_k(B_{\widehat{p}}, B_{\widehat{q}})$. If $\widetilde{r}$ is the desired pre-Hilbert seminorm on $\widehat{E}_p$, then $r := \widetilde{r}|_{E_p}$ satisfies the required conditions on E_p because of 18.18. This means that we may assume that (E_p, p) is a Hilbert space.

For q, there exists, according to 18.26(i), a compact self-adjoint operator $Q : H \to H$ satisfying $q(x)^2 = p(Q(x), x)$ (for all $x \in H$). Let $(x_k)_{k\in N}$ and $(\lambda_k)_{k\in N}$ be as in 18.25 and let $(c_k)_{k\in N}$ be as in 18.27 with $d_k := \mathrm{d}_k(B_p, B_q)$. If $\widetilde{p}$ denotes the inner product associated with p, then $R : E_p \to E_p$, $x \mapsto \sum_{k\in N} c_k^2 \widetilde{p}(x, x_k) x_k$, is a continuous self-adjoint operator which gives rise to an inner product $\widetilde{r} : (x, y) \mapsto \widetilde{p}(R(x)|y) = \sum_{k\in N} c_k^2 \widetilde{p}(x, x_k)\widetilde{p}(y, x_k)$. Put $H_1 := \overline{\langle x_k : k \in N\rangle}$ and $H_0 := H_1^{\perp}$. Then $r : x \mapsto \sqrt{\widetilde{r}(x, x)}$ defines a continuous pre-Hilbert seminorm on E_p. Hence we get $\mathrm{d}_k(B_p, B_r) \overset{18.26(\mathrm{ii})}{=} c_k \leq a_k$ for $k \in N$, and 0 otherwise. Because of 18.4(ii), $(x_k/c_k)_{k\in N}$ is an orthonormal basis of $(H_1, r|_{H_1})$. The linear operator $Q : H \to H$, $x \mapsto \sum_{k\in N} d_k^2 p(x, x_k) x_k$, satisfies $Q(x_k/c_k) = (d_k^2/c_k) x_k = (d_k^2/c_k^2) R(x_k/c_k)$. Since $d_k^2/c_k^2 \leq b_k^2$ and $(b_k^2)_{k\in N}$ is bounded, it follows that Q is continuous with respect to the topology induced by r. Hence we get $\mathrm{d}_k(B_r, B_q) \overset{18.22(\mathrm{ii})}{=} \mathrm{d}_k(B_r \cap H_1, B_q \cap H_1) \overset{18.26(\mathrm{ii})}{=} d_k/c_k \leq b_k$ for $k \in N$ and 0 otherwise. This completes the proof.

Definition 18.29. Let E_1, E_2 be normed spaces and let $T : E_1 \to E_2$ be a continuous linear operator. Then $\mathrm{a}_k(T) := \inf\{\|S - T\| : S : E_1 \to E_2$ is a continuous linear operator such that $\dim S(E_1) < k\}$ is called the k-th *approximation number* of T.

Proposition 18.30. *Let* $T : E_1 \to E_2$ *be a continuous linear operator between the normed spaces E_1 and E_2; then* $\mathrm{d}_k(T) \leq \mathrm{a}_k(T) \leq k \cdot \mathrm{d}_k(T)$.

Proof. For every continuous operator $S : E_1 \to E_2$ we get

$$T(B_{E_1}) \subseteq \|S - T\| B_{E_2} + S(E_1),$$

which shows the first inequality.

Now, let $T(B_{E_1}) \subseteq c \cdot B_{E_2} + L$ (for $c > 0$ and a finite-dimensional subspace $L \leq E_2$). According to 18.11, there exists a projection $\pi : E_2 \to L$ such that $\|\pi\| \leq \dim L$. For $S := \pi \circ T$, $x \in B_{E_1}$, $y \in B_{E_2}$ and $l \in L$ such that $T(x) = c \cdot y + l$ we get $\|(S - T)(x)\| = c \cdot \|y - \pi(y)\| \leq c \cdot (1 + \|\pi\|) \leq c \cdot (1 + \dim L)$, which implies the assertion.

Theorem 18.31. *Let* $T : E_1 \to E_2$ *be a continuous linear operator between the normed spaces E_1 and E_2. Suppose there exists* $r \in\,]0, 1]$ *such that*

$$\Big(\sum_{k\in\mathbb{N}} \mathrm{a}_k(T)^r\Big)^{1/r} < \infty.$$

Then we can find $\varphi_k \in E_1'$, $y_k \in E_2$ *and* $\lambda_k \in \mathbb{R}$ *such that* $\|\varphi_k\|, \|y_k\| \leq 1$ (*for all* $k \in \mathbb{N}$), $T(x) = \sum_{k\in\mathbb{N}} \lambda_k \varphi_k(x) y_k$, *and* $(\sum_{k\in\mathbb{N}} |\lambda_k|^r)^{1/r} \leq 2^{2+3/r}(\sum_{k\in\mathbb{N}} \mathrm{a}_k(T)^r)^{1/r}$.

Proof. This is Proposition (8.4.2) in [64].

Lemma 18.32. *Let* $m \geq 3$ *be a natural number. There exists* $c = c(m) > 0$ *with the following property: for every vector space* E *and all symmetric convex subsets* $X, Y \subseteq E$ *and all* $\lambda > 0$ *such that* $\mathrm{d}_k(X, Y) \leq \lambda \cdot k^{-m}$ (*for all* $k \in \mathbb{N}$), *there exist pre-Hilbert seminorms* p *and* q *on* $\langle X\rangle_{\mathbb{R}}$ *such that* $X \subseteq B_p$, $B_q \subseteq Y$ *and* $\mathrm{d}_k(B_p, B_q) \leq c \cdot \lambda \cdot k^{-m+3}$.

Proof. Fix $m \geq 3$. We may assume $\lambda = 1$ (18.2(ii)). Let $N_X := \{x \in X : tx \in X \ \forall t \in \mathbb{R}\}$ be the largest subspace contained in X. Endow $E_X := \langle X\rangle/N_X$ with the norm which is induced by the Minkowski functional of X and let $\psi_X : \langle X\rangle_{\mathbb{R}} \to E_X$ denote the canonical projection. (We use analogous notation with respect to Y.)

$\Lambda : E_X \to E_Y$ is well defined since $\mathrm{d}_1(X, Y) < \infty$ and hence $N_X \leq N_Y$. As in 18.7 we get $\mathrm{d}_k(\Lambda) = \mathrm{d}_k(X, Y)$ and it follows from 18.30 that $\mathrm{a}_k(\Lambda) \leq k \cdot \mathrm{d}_k(\Lambda) \leq k^{-m+1}$. For $r := 1/(m - 3/2) < 1$, the above estimate yields $(\sum_{k\in\mathbb{N}} \mathrm{a}_k(\Lambda)^r)^{1/r} < \infty$. According to 18.31, there exist $\varphi_k \in E_X'$, $y_k \in E_Y$ and $\lambda_k \in \mathbb{R}$ such that $\|\varphi_k\| \leq 1$, $\|y_k\| \leq 1$, and

$$\begin{aligned}\Big(\sum_{k\in\mathbb{N}} |\lambda_k|^r\Big)^{1/r} &\leq 2^{2+3/r}\Big(\sum_{k\in\mathbb{N}} \mathrm{a}_k(\Lambda)^r\Big)^{1/r}\\ &\leq 4 \cdot 2^{3\cdot(m-3/2)}\Big(\sum_{k\in\mathbb{N}} k^{-(m-1)/(m-3/2)}\Big)^{m-3/2} =: c_m',\end{aligned}$$

and $\Lambda(u) = \sum_{k\in\mathbb{N}} \lambda_k \varphi_k(u) y_k$. We may assume that $(|\lambda_k|)_{k\in\mathbb{N}}$ decreases. Then $k|\lambda_k|^r \leq \sum_{j=1}^k |\lambda_j|^r \leq (c_m')^r$, which implies

$$(*) \qquad |\lambda_k| \leq k^{-1/r} c_m' = c_m' k^{-m+3/2} \quad (k \in \mathbb{N}).$$

Let $(e_k)_{k\in\mathbb{N}}$ denote the canonical orthonormal basis of ℓ^2; we put

$$c_\Phi := \frac{1}{\sqrt{\sum_{k\in\mathbb{N}} k^{-3/2}}}, \quad c_\Psi := \frac{1}{2} c_\Phi, \quad c_\Delta := \frac{1}{c_\Phi \cdot c_\Phi};$$

$$\Phi : E_X \to \ell^2, \qquad u \mapsto c_\Phi \sum_{k\in\mathbb{N}} k^{-3/4} \varphi_k(u) e_k,$$

$$\Delta : \ell^2 \to \ell^2, \qquad \sum_{k\in\mathbb{N}} \mu_k e_k \mapsto c_\Delta \sum_{k\in\mathbb{N}} \mu_k k^{3/2} \lambda_k e_k,$$

$$\Psi : \ell^2 \to E_Y, \qquad \sum_{k\in\mathbb{N}} \mu_k e_k \mapsto c_\Psi \sum_{k\in\mathbb{N}} \mu_k k^{-3/4} y_k.$$

We have

$$\begin{tikzcd} E_X \arrow[r, "\Lambda"] \arrow[d, "\Phi"'] & E_Y \\ \ell^2 \arrow[r, "\Delta"'] & \ell^2 \arrow[u, "\Psi"'] \end{tikzcd}$$

Hence we get $\|\Phi\| \le c_\Phi\sqrt{\sum_{k\in\mathbb{N}} k^{-3/2}} = 1$ and the Cauchy–Schwarz inequality implies $\|\Psi\| \le c_\Psi\sqrt{\sum_{k\in\mathbb{N}} k^{-3/2}} = 1/2$. It follows from $(*)$ that $\sup\{k^{3/2}|\lambda_k| : k\in\mathbb{N}\} \le c'_m \sup\{k^{-m+3} : k\in\mathbb{N}\} \le c'_m$, which shows that Δ is continuous. Moreover, we get $\Psi\circ\Delta\circ\Phi = \Lambda$: let $u\in E_X$; then

$$\begin{aligned}\Psi(\Delta(\Phi(u))) &= c_\Phi\Psi\Big(\Delta\Big(\sum_{k\in\mathbb{N}} k^{-3/4}\varphi_k(u)e_k\Big)\Big) = c_\Phi c_\Delta\Psi\Big(\sum_{k\in\mathbb{N}} k^{3/4}\varphi_k(u)\lambda_k e_k\Big)\\ &= c_\Phi c_\Delta c_\Psi \sum_{k\in\mathbb{N}} \varphi_k(u)\lambda_k y_k = \Lambda(u).\end{aligned}$$

Obviously, $p(x) := \|\Phi(\psi_X(x))\|$ and $q(x) := \|\Delta(\Phi(\psi_X(x)))\|$ define pre-Hilbert seminorms on $\langle X\rangle$. For $x\in X$ we have $p(x)\le 1$ and hence $X\subseteq B_p$ and for $y\in\langle X\rangle_{\mathbb{R}}$ we get $\|\psi_Y(y)\| = \|\Lambda\psi_X(y)\| = \|\Psi\Delta\Phi\psi_X(y)\| \le \|\Psi\|\cdot q(y) \le \frac{1}{2}q(y)$, which shows that $B_q\subseteq Y(\cap\langle X\rangle)$. Furthermore,

$$(**)\qquad \mathrm{d}_k(B_p,B_q) \le \mathrm{d}_k(\Delta).$$

Indeed,

$$\begin{aligned}\mathrm{d}_k(B_p,B_q) &= \mathrm{d}_k(\psi_X^{-1}(\Phi^{-1}(B)),\psi_X^{-1}(\Phi^{-1}(\Delta^{-1}(B))))\\ &\overset{18.5(\mathrm{ii})}{=} \mathrm{d}_k(\Phi^{-1}(B),\Phi^{-1}(\Delta^{-1}(B)))\\ &\overset{18.5(\mathrm{ii})}{=} \mathrm{d}_k(B\cap\Phi(E_X),\Delta^{-1}(B)\cap\Phi(E_X))\\ &\overset{18.18,\,(\diamond)}{\le} \mathrm{d}_k(B,\Delta^{-1}(B)) \overset{18.2(\mathrm{ii})}{\le} \mathrm{d}_k(\Delta^{-1}(\Delta(B)),\Delta^{-1}(B))\\ &\overset{18.5(\mathrm{ii})}{=} \mathrm{d}_k(\Delta(B),B\cap\Delta(\ell^2)) \overset{18.18}{\le} \mathrm{d}_k(\Delta(B),B) = \mathrm{d}_k(\Delta).\end{aligned}$$

(($\diamond$): Observe that $\Delta^{-1}(B) = B_r$ for the pre-Hilbert seminorm $r:\ell^2\to\mathbb{R}$, $x\mapsto\|\Delta(x)\|$.)

Since $\mathrm{d}_k(\Delta)\le c'_m c_\Delta k^{3-m}$ (observe that $\Delta(B)\subseteq|\lambda_k|k^{3/2}\cdot B + \langle e_1,\ldots,e_{k-1}\rangle_{\mathbb{R}}$ and that $|\lambda_k|k^{3/2}\le c'_m k^{-m+3}$), the assertion follows for $c_m := c'_m c_\Delta$.

Remark 18.33. Applying 18.12 instead of 18.11 would have yielded a better estimate, namely $\mathrm{d}_k(B_p,B_q)\le\lambda k^{-m+2}$. This is the formulation of Lemma (2.14) given in [8].

Theorem 18.34. *Let p, q be pre-Hilbert seminorms on E with $d := \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p,B_q)^2 < 1/4$. For every subgroup K of E we have*

$$\mathrm{d}_k(\mathrm{conv}(K\cap B_p),\mathrm{conv}(K\cap B_q)) \le 2\mathrm{d}_k(B_p,B_q).$$

Proof. This is Theorem (3.2) in [8].

Notes 18.35. The assertions of 18.1 to 18.8 are either simple or taken from Section 2 in [8]. 18.9, 18.13, 18.14, 18.15, and 18.16 have been established in order to prove 18.18 ((2.13) in [8]).

The assertions of 18.19 and 18.20 can be found in [8] on p. 26 and p. 27 (Lemma (3.3)).

18.27 was proved by G. Turnwald. 18.26 and 18.27 were necessary to give a proof of 18.28, which is (2.15) in [8]. 18.30 is Lemma (9.1.6) in [64].

I am very grateful to W. Banaszczyk for sending me a proof of 18.32 by e-mail.

19. Gaussian-like measures

In this chapter elementary properties of the measure σ_L (defined below) are proved. It can be considered as a Gaussian measure on the lattice L. The main result is Lemma 19.14, which will enable us to prove in the following chapter that closed subgroups of nuclear groups are dually closed and dually embedded.

Let n be a fixed natural number; for $x, y \in \mathbb{R}^n$, we put $x \cdot y := \langle x, y\rangle$ and $x^2 := x \cdot x$.

Notation 19.1.

$\varrho(A)$	$\sum\limits_{x\in A} e^{-\pi x^2}$ $(A \subseteq \mathbb{R}^n)$
$\mathcal{L}$	the set of all lattices in $\mathbb{R}^n$
$\mathcal{C}$	the set of all non-empty compact symmetric convex subsets of $\mathbb{R}^n$
σ_L	$\frac{1}{\varrho(L)} \sum\limits_{x\in L} e^{-\pi x^2}\varepsilon_x$ (where $L \in \mathcal{L}$ and ε_x is the Dirac measure)
$\varphi_L(u)$	$\frac{1}{\varrho(L)} \sum\limits_{x\in L+u} e^{-\pi x^2}$ $(L \in \mathcal{L},\ u \in \mathbb{R}^n)$
$\widehat{L}$	$\{x \in \mathbb{R}^n : x \cdot y \in \mathbb{Z}$ for all $y \in L\}$ $(L \in \mathcal{L})$, the *dual lattice* of L
p_U	the Minkowski functional of U $(U \in \mathcal{C})$
d_U	the metric generated by p_U
d	the standard metric of $\mathbb{R}^n$
$\alpha(U)$	$\sup\limits_{L\in\mathcal{L}} \varrho(L \setminus U)/\varrho(L) = \sup\limits_{L\in\mathcal{L}} \sigma_L(L \setminus U)$ $(U \in \mathcal{C})$
$\beta(U)$	$\sup\limits_{L\in\mathcal{L}} \sup\limits_{u\in\mathbb{R}^n} \varrho((L + u) \setminus U)/\varrho(L)$ $(U \in \mathcal{C})$
$\zeta(U, V)$	$\sup\limits_{L\in\mathcal{L}} \sup\limits_{u\notin L} \inf\limits_{v\in\widehat{L}:\, u\cdot v\notin\mathbb{Z}} p_V(v) \cdot d_U(u, L)/d(u \cdot v, \mathbb{Z})$ $(U, V \in \mathcal{C})$

For each lattice L in $\mathbb{R}^n$ and $u \in \mathbb{R}^n$, we have $0 < \varrho(L + u) < \infty$. [Clearly, we have $0 < \varrho(L + u)$. Let $A \in \mathrm{Gl}(n, \mathbb{R})$ be a matrix such that $A\mathbb{Z}^n = L$ and $u' := A^{-1}u$. Then

$$\sum_{x\in L+u} e^{-\pi x^2} = \sum_{y\in\mathbb{Z}^n} e^{-\pi(A(y+u'))^2} \le \sum_{y\in\mathbb{Z}^n} e^{-c\cdot\pi(y+u')^2} < \infty$$

for suitable $c > 0$.]

Theorem 19.2 (Poisson summation formula). *Let L be a lattice in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuous integrable function satisfying*:

(i) $u \mapsto \sum_{x\in L} |f(x + u)|$ *converges uniformly on compact subsets*, *and*

(ii) $\sum_{y\in\widehat{L}} |\widehat{f}(y)| < \infty$ *where* $\widehat{f}(y) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot y} f(x)\, dx$. *Then*

$$\sum_{x\in L} f(x) = \frac{1}{\mathrm{m}(L)} \sum_{y\in\widehat{L}} \widehat{f}(y)$$

where $\mathrm{m}(L) := \det(x_1 \ldots, x_n)$, *the mesh of* L, *is the determinant of a basis* $(x_1, \ldots, x_n)$ *of* L.

Proof. See [25], p. 40.

In the next proof, we make use of the following two integrals:

$$\underset{\mathbb{R}^n}{T} e^{-2\pi i a x} e^{-\pi x^2} dx = e^{-\pi a^2}, \quad \underset{\mathbb{R}^n}{T} e^{-2\pi i a x} x_k^2 e^{-\pi x^2} dx = e^{-\pi a^2}\left(\frac{1}{2\pi} - a_k^2\right)$$

(they can be computed by the residue theorem).

Corollary 19.3. *Let $L \in \mathcal{L}$, $k \in \{1, \dots, n\}$, and $v \in \mathbb{R}^n$. Then*

(i) $\displaystyle\sum_{x \in L} e^{2\pi i x \cdot v} e^{-\pi x^2} = \frac{1}{\mathrm{m}(L)} \sum_{y \in \widehat{L}} e^{-\pi(y-v)^2}$,

(ii) $\displaystyle\sum_{x \in L} e^{-2\pi i x v} x_k^2 e^{-\pi x^2} = \frac{1}{\mathrm{m}(L)} \sum_{y \in \widehat{L}} e^{(-\pi(y+v)^2)}\left(\frac{1}{2\pi} - (y_k + v_k)^2\right)$.

(iii) $\displaystyle\frac{1}{\varrho(L)} \sum_{x \in L} e^{2\pi i x \cdot v} e^{-\pi x^2} = \varphi_{\widehat{L}}(v)$.

(iv) *φ_L is a positive definite function and hence $\varphi_L(x) \le \varphi_L(0) = 1$.*

Proof. (i) and (ii). Fix $v \in \mathbb{R}^n$ and put $f_1 : \mathbb{R}^n \to \mathbb{C}$, $x \mapsto e^{2\pi i x \cdot v} e^{-\pi x^2}$, and $f_2 : x \mapsto e^{-2\pi i x v} x_k^2 e^{-\pi x^2}$. Obviously, both functions are continuous and integrable. Since every lattice is the homomorphic image of $\mathbb{Z}^n$ under a suitable matrix $A \in \mathrm{Gl}(n, \mathbb{R})$, it is easy to verify that $u \mapsto \sum_{x \in L} |f_j(x+u)|$ converges uniformly on compact subsets for $j \in \{1, 2\}$. Hence condition (i) of 19.2 is satisfied. Using the above integrals, we see that $\widehat{f_1}(y) = e^{-\pi(y-v)^2}$ and $\widehat{f_2}(y) = e^{-\pi(y+v)^2}(1/(2\pi) - (y_k + v_k)^2)$. Hence the sums $\sum_{y \in \widehat{L}} |\widehat{f_j}(y)|$ are convergent. This enables us to apply 19.2, which yields

$$\sum_{x \in L} e^{2\pi i x \cdot v} e^{-\pi x^2} = \sum_{x \in L} f_1(x) \overset{19.2}{=} \frac{1}{\mathrm{m}(L)} \sum_{y \in \widehat{L}} \widehat{f_1}(y) = \frac{1}{\mathrm{m}(L)} \sum_{y \in \widehat{L}} e^{-\pi(y-v)^2},$$

and analogously,

$$\sum_{x \in L} e^{-2\pi i x v} x_k^2 e^{-\pi x^2} = \frac{1}{\mathrm{m}(L)} \sum_{y \in \widehat{L}} e^{-\pi(y+v)^2}\left(\frac{1}{2\pi} - (y_k + v_k)^2\right).$$

(iii) From (i) we get

$$\sum_{x \in L} e^{2\pi i x \cdot v} e^{-\pi x^2} = \frac{1}{\mathrm{m}(L)} \sum_{-y \in \widehat{L}} e^{-\pi(y+v)^2} = \frac{1}{\mathrm{m}(L)} \cdot \varrho(\widehat{L}) \cdot \varphi_{\widehat{L}}(v).$$

For $v = 0$, we obtain $\varrho(L) = \frac{1}{\mathrm{m}(L)} \cdot \varrho(\widehat{L})$, which implies (iii).

(iv) The following functions are positive definite:

1. $x \mapsto e^{2\pi i x \cdot v}$ ([84], p. 189, Example 1),
2. $x \mapsto e^{-\pi x^2}$ ([84], p. 189, Example 3),

and so are sums (this is trivial) and products ([84], p. 188, Proposition 1.1(d)) of positive definite functions. Hence the assertion follows from (iii) and the fact that $\widehat{\widehat{L}} = L$.

Corollary 19.4. *The function $\mathbb{R}^n \to \mathbb{C}$, $v \mapsto \sum_{y \in L} e^{-\pi(y+v)^2}(1/(2\pi) - (y_k + v_k)^2)$, is positive definite for every lattice L.*

Proof. Consider the positive measure $\mu := \sum_{x\in\widehat{L}} x_k^2 e^{-\pi x^2}$. It is an easy consequence of 19.3(ii) that the above function equals $\widehat{\mu}$ (for the definition, see 22.1) composed with the projection $\mathbb{R}^n \to \widehat{L}^*$ up to a positive constant. Hence the assertion follows.

Lemma 19.5. (i) *ϱ is invariant under orthogonal transformations.*

(ii) *α and β are invariant under orthogonal transformations.*

(iii) *$\widehat{AL} = ({}^{\mathrm{t}}A)^{-1} \cdot \widehat{L}$ (for $A \in \mathrm{Gl}(n,\mathbb{R})$ and $L \in \mathcal{L}$).*

(iv) *$\zeta(A^{-1}U, ({}^{\mathrm{t}}A)\cdot V) = \zeta(U,V)$ (for $U, V \in \mathcal{C}$ and $A \in \mathrm{Gl}(n,\mathbb{R})$).*

(v) *$\zeta(tU, tU) = t^{-2}\zeta(U,U)$ for all $t > 0$ and all $U \in \mathcal{C}$.*

Proof. (i) is trivial and (ii) is a direct consequence of (i).

(iii) follows from $\widehat{AL} = \{v \in \mathbb{R}^n : v \cdot (Au) \in \mathbb{Z}\ \forall u \in L\} = \{v \in \mathbb{R}^n : ({}^{\mathrm{t}}Av)\cdot u \in \mathbb{Z}\ \forall u \in L\} = ({}^{\mathrm{t}}A)^{-1}\widehat{L}$.

(iv) From $u \cdot v = (Au)\cdot(({}^{\mathrm{t}}A)^{-1}v)$ and $p_{A^{-1}U}(v) = p_U(Av)$ and (iii) we get

$$\begin{aligned}
\zeta(A^{-1}U, ({}^{\mathrm{t}}A)V) &= \sup_{L\in\mathcal{L}}\ \sup_{u\in\mathbb{R}^n\setminus L}\ \inf_{v\in\widehat{L}:\, u\cdot v\notin\mathbb{Z}} \frac{p_{{}^{\mathrm{t}}AV}(v)\cdot d_{A^{-1}U}(u,L)}{d(u\cdot v,\mathbb{Z})}\\
&= \sup_{L\in\mathcal{L}}\ \sup_{u\in\mathbb{R}^n\setminus L}\ \inf_{v\in\widehat{L}:\, u\cdot v\notin\mathbb{Z}} \frac{p_V(({}^{\mathrm{t}}A)^{-1}v)\cdot d_U(Au,AL)}{d((Au)\cdot {}^{\mathrm{t}}(A^{-1}v),\mathbb{Z})}\\
&= \sup_{AL:L\in\mathcal{L}}\ \sup_{Au\notin AL}\ \inf_{{}^{\mathrm{t}}A^{-1}v\in{}^{\mathrm{t}}A^{-1}\widehat{L}:\, (Au)\cdot({}^{\mathrm{t}}A^{-1}v)\notin\mathbb{Z}} \frac{p_V(({}^{\mathrm{t}}A)^{-1}v)\cdot d_U(Au,AL)}{d((Au)\cdot {}^{\mathrm{t}}(A^{-1}v),\mathbb{Z})}\\
&= \zeta(U,V).
\end{aligned}$$

(v) We calculate

$$\begin{aligned}
\zeta(tU,tU) &= \sup_{L\in\mathcal{L}}\sup_{u\notin L}\ \inf_{v\in\widehat{L}:u\cdot v\notin\mathbb{Z}} \frac{p_{tU}(v)\cdot d_{tU}(u,L)}{d(u\cdot v,\mathbb{Z})}\\
&= \sup_{L\in\mathcal{L}}\sup_{u\notin L}\ \inf_{v\in\widehat{L}:u\cdot v\notin\mathbb{Z}} \frac{t^{-2}p_U(tv)\cdot d_U(t^{-1}u,t^{-1}L)}{d((t^{-1}u)\cdot(tv),\mathbb{Z})}\\
&= t^{-2}\cdot \sup_{t^{-1}L\in\mathcal{L}}\ \sup_{t^{-1}u\notin t^{-1}L}\ \inf_{tv\in=t\widehat{L}:(t^{-1}u)\cdot tv\notin\mathbb{Z}} \frac{p_U(tv)\cdot d_U(t^{-1}u,t^{-1}L)}{d((t^{-1}u)\cdot(tv),\mathbb{Z})}\\
&= t^{-2}\zeta(U,U).
\end{aligned}$$

Lemma 19.6. *For $U, V \in \mathcal{C}$ such that $2\beta(U) + 3\alpha(V) \le 1$ we have $\zeta(U,V) \le 6$.*

Proof. Let $L \in \mathcal{L}$, $u \in \mathbb{R}^n \setminus L$, and $\varepsilon > 0$ be arbitrary. We have to find $v \in \widehat{L}$ such that $u \cdot v \notin \mathbb{Z}$ and $p_V(v)\cdot d_U(u,L) \le 6\cdot(1+\varepsilon)\cdot d(u\cdot v,\mathbb{Z})$. It suffices to consider the case of $d_U(u,L) = 1+\varepsilon$. Otherwise, we replace u by tu, L by tL and v by $(1/t)v$ for suitable $t > 0$. (Observe that $d_U(t\cdot u, t\cdot L) = t\cdot d_U(u,L)$.) So we have to find $v \in \widehat{L}$ such that $u\cdot v \notin \mathbb{Z}$ and $p_V(v) \le 6\cdot d(u\cdot v,\mathbb{Z})$. Notice that

$$(*)\qquad u \in \mathbb{R}^n \setminus (L+U) \quad\text{implies}\quad \varphi_L(u) \le \beta(U).$$

[For $L + u = (L+u)\setminus U$ we get $\varphi_L(u) = \varrho(L+u)/\varrho(L) = \varrho((L+u)\setminus U)/\varrho(L) \le \beta(U)$.]

Hence we get

$$\begin{aligned}\beta(U) \geq \varphi_L(u) &\overset{19.3}{=} \frac{1}{\varrho(\widehat{L})}\sum_{v\in\widehat{L}} \exp(2\pi i u\cdot v)\exp(-\pi v^2)\\ &= \frac{1}{\varrho(\widehat{L})}\sum_{v\in\widehat{L}} \cos(2\pi u\cdot v)\exp(-\pi v^2) \quad (\widehat{L} \text{ is symmetric})\\ &= \frac{1}{\varrho(\widehat{L})}\Big(\sum_{v\in V\cap\widehat{L}} \cos(2\pi u\cdot v)\exp(-\pi v^2)\\ &\quad + \sum_{v\in\widehat{L}\setminus V} \cos(2\pi u\cdot v)\exp(-\pi v^2)\Big)\\ &\geq s\cdot\sigma_{\widehat{L}}(\widehat{L}\cap V) - \sigma_{\widehat{L}}(\widehat{L}\setminus V)\\ &= s-(1+s)\sigma_{\widehat{L}}(\widehat{L}\setminus V) \geq s-(1+s)\alpha(V)\end{aligned}$$

for $s := \min_{v\in\widehat{L}\cap V}\cos(2\pi u\cdot v)$. This implies $s(1-\alpha(V)) \leq \alpha(V)+\beta(U) \leq \frac{1}{2}(1-\alpha(V))$ (by assumption). Furthermore, we have $\alpha(V) \leq 1/3 < 1$ and hence $s \leq 1/2$. So, there exists $v \in \widehat{L}\cap V$ such that $\cos(2\pi(u\cdot v)) \leq 1/2$, which means $d(u\cdot v,\mathbb{Z}) \geq 1/6$. The assertion follows.

Lemma 19.7. *For $L \in \mathcal{L}$, $u \in \mathbb{R}^n$ and $x_k := e_k\cdot x$ $(x\in\mathbb{R}^n)$ we have*:

(i) $\sum_{x\in L} x_k^2 e^{-\pi x^2} \leq \varrho(L)/(2\pi)$, *and*
(ii) $\sum_{x\in L+u} x_k^2 e^{-\pi x^2} \leq \varrho(L)/\pi$.

Proof. Recall the function

$$\varphi_L : \mathbb{R}^n \to \mathbb{R}, \quad u \mapsto \frac{1}{\varrho(L)}\sum_{x\in L+u} e^{-\pi x^2}.$$

According to 19.3(iv), φ_L is positive definite and hence

$$|\varphi_L(u)| \leq \varphi_L(0) = 1 \quad \forall u\in\mathbb{R}^n. \tag{1}$$

Observing that the sums of the derivatives converge uniformly on compact subsets, the second derivative of φ_L in the k-th direction is

$$\begin{aligned}D_{kk}\varphi_L(u) &= \frac{1}{\varrho(L)}\sum_{x\in L+u} e^{-\pi x^2}((-2\pi x_k)^2-2\pi)\\ &= -2\pi\varphi_L(u) + \frac{4\pi^2}{\varrho(L)}\sum_{x\in L+u} x_k^2 e^{-\pi x^2} \geq -2\pi\varphi_L(u)\end{aligned} \tag{2}$$

(where x_k denotes the k-th coordinate of x). According to 19.4, $-D_{kk}\varphi_L$ is positive definite (and not the zero function), hence

$$D_{kk}\varphi_L(u) \leq -D_{kk}\varphi_L(0) \quad \forall u\in\mathbb{R}^n \tag{3}$$

and $D_{kk}\varphi_L(0) < 0$. For

$$\psi(u) := \frac{1}{\varrho(L)}\sum_{x\in L+u} x_k^2 e^{-\pi x^2},$$

we get

$$\begin{aligned}\psi(u) &= \frac{1}{4\pi^2}(D_{kk}\varphi_L(u)+2\pi\varphi_L(u)) \overset{(1),(3)}{\le} \frac{1}{4\pi^2}(-D_{kk}\varphi_L(0)+2\pi\varphi_L(0))\\ &\overset{(2)}{\le} \frac{1}{4\pi^2}(2\pi\varphi_L(0)+2\pi\varphi_L(0)) \overset{(1)}{=} \frac{1}{\pi}.\end{aligned}$$

Since $D_{kk}\varphi_L(0) < 0$, for $u = 0$ we get

$$\psi(0) = \frac{1}{4\pi^2}(D_{kk}\varphi_L(0)+2\pi\varphi_L(0)) < \frac{1}{4\pi^2}2\pi\varphi_L(0) \overset{(1)}{=} \frac{1}{2\pi},$$

which was to be shown.

Lemma 19.8. *Let D be an ellipsoid in $\mathbb{R}^n$ with principal semiaxes $d_1,\dots,d_n$. Then*

(i) $\alpha(D) \le \frac{1}{2\pi}\sum_{k=1}^{n} d_k^{-2}$, *and*

(ii) $\beta(D) \le \frac{1}{\pi}\sum_{k=1}^{n} d_k^{-2}$.

Proof. We give the proof for (ii). Using 19.7(i) instead of 19.7(ii) and putting $u = 0$, we get (i).

Because of 19.5(ii) we may assume that $D = \{(x_k) \in \mathbb{R}^n : \sum_{k=1}^{n}(x_k/d_k)^2 \le 1\}$. The assertion follows from

$$\begin{aligned}\varrho((L+u)\setminus D) &= \sum_{x\in(L+u)\setminus D}\exp(-\pi x^2) \le \sum_{x\in(L+u)\setminus D}\sum_{k=1}^{n}\left(\exp(-\pi x^2)\cdot\left(\frac{x_k}{d_k}\right)^2\right)\\ &\le \sum_{k=1}^{n} d_k^{-2}\sum_{x\in L+u} x_k^2\exp(-\pi x^2) \overset{19.7(\mathrm{ii})}{\le} \varrho(L)\frac{1}{\pi}\sum_{k=1}^{n} d_k^{-2}.\end{aligned}$$

Theorem 19.9. *For every ellipsoid $D \subseteq \mathbb{R}^n$ with principal semiaxes $d_1,\dots,d_n$ we have $\zeta(D,B) = \zeta(B,D) \le (21/\pi)\cdot\sum_{k=1}^{n} d_k^{-1}$.*

Proof. Put $d := (\sum_{k=1}^{n} 1/d_k)^{-1}$. By 19.5(iv), we may assume that $D = \{(x_k) \in \mathbb{R}^n : \sum_{k=1}^{n}(x_k/d_k)^2 \le 1\}$. For $A := (\delta_{kl}\sqrt{d_k})_{k,l} \in \mathrm{Gl}(n,\mathbb{R})$ we get $AB = A^{-1}D = \{(x_k) : \sum_{k=1}^{n} x_k^2/d_k \le 1\} =: C$. Since A is a diagonal matrix we get $\zeta(B,D) \overset{19.5(\mathrm{iv})}{=} \zeta(C,C) \overset{19.5(\mathrm{iv})}{=} \zeta(AC,A^{-1}C) = \zeta(D,B)$. For $t := \sqrt{7/(2\pi d)}$ we deduce from 19.6 that $\zeta(C,C) \overset{19.5(\mathrm{v})}{=} t^2\zeta(tC,tC) \le 6t^2 = 21/(\pi d)$, since

$$3\alpha(tC) + 2\beta(tC) \le \frac{3}{2\pi d t^2} + \frac{2}{\pi d t^2} = 1.$$

This completes the proof.

Remark 19.10. For $U, V \in \mathcal{C}$, $L \in \mathcal{L}$ and $u \notin L$, it is possible to find $v \in \widehat{L}$ such that $u\cdot v \notin \mathbb{Z}$ and $p_V(v)\cdot d_U(u,L) \le \zeta(U,V)\cdot d(u\cdot v,\mathbb{Z})$. [We may assume $\zeta(U,V) < \infty$. There exists a sequence $(v_k)_{k\in\mathbb{N}}$ in $\widehat{L}$ such that $u\cdot v_k \notin \mathbb{Z}$ and $p_V(v_k)\cdot d_U(u,L) \le (\zeta(U,V)+1/k)\cdot d(u\cdot v_k,\mathbb{Z})$. This implies that the sequence $(v_k)_{k\in\mathbb{N}}$ is bounded; since it is contained in the discrete set $\widehat{L}$, it is eventually constant. The assertion follows.]

Remark 19.11. For $x \in \mathbb{R} \setminus \mathbb{Z}$, let k be the smallest natural number satisfying $d(kx, \mathbb{Z}) > 1/4$. Then $d(kx, \mathbb{Z}) = k \cdot d(x, \mathbb{Z})$. [Since $j \in \mathbb{Z}$ implies $d(x+j, \mathbb{Z}) = d(x, \mathbb{Z})$ we may assume $0 < |x| \le 1/2$. If $1/4 < |x|$ then $k = 1$ and the assertion is trivial. So, let $0 < |x| \le 1/4$. Then k satisfies $(k-1)|x| \le 1/4 < k|x| \le 1/2$, which implies the assertion.]

Lemma 19.12. *For each $u \in \mathbb{R}^n$, every subgroup K of $\mathbb{R}^n$, and every ellipsoid D with principal semiaxes $d_1, \dots, d_n$ such that $K \cap (u + D) = \emptyset$, there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(K) \le \mathbb{Z}$, $d(f(u), \mathbb{Z}) \ge 1/4$, and $\|f\| \le (21/(2\pi)) \cdot \sum_{k=1}^n d_k^{-1}$.*

Proof. (1) Suppose first that $K = L$ is a lattice. Since $u \notin L$, there exists $v \in \widehat{L}$ such that $u \cdot v \notin \mathbb{Z}$ and

$$\frac{\|v\| \cdot d_D(u, L)}{d(u \cdot v, \mathbb{Z})} \le \zeta(D, B) \le \frac{21}{\pi} \cdot \sum_{k=1}^{n} d_k^{-1}$$

(according to 19.10 and 19.9). For $k \in \mathbb{N}$ which is minimal with the property that $k(u \cdot v) \in]1/4, 3/4[+ \mathbb{Z}$, we get $kd(u \cdot v, \mathbb{Z}) \overset{19.11}{=} d(u \cdot (kv), \mathbb{Z}) > 1/4$. By assumption, $d_D(u, L) \ge 1$. Hence,

$$\|kv\| \le \frac{21}{\pi} \cdot \frac{d(u \cdot (kv), \mathbb{Z})}{d_D(u, L)} \cdot \sum d_k^{-1} \le \frac{21}{2\pi} \cdot \sum d_k^{-1}.$$

It follows easily that the linear mapping $f : x \mapsto x \cdot (kv)$ has the desired properties.

(2) If K is a discrete group then it is contained in a lattice L which misses $u + D$. Hence the assertion follows from (1).

(3) Let K be a closed subgroup of $\mathbb{R}^n$. According to 2.2, there exist a linear subspace $V \le \mathbb{R}^n$ and a discrete subgroup $L \le V^\perp$ such that $K = V \oplus L$. If $\pi : \mathbb{R}^n \to V^\perp$ denotes the orthogonal projection then we get $\pi(u + D) \cap \pi(K) = \pi(u + D) \cap L = \emptyset$ since $l \in \pi(u + D) \cap L$ implies $l + v \in u + D$ (for suitable $v \in V$) contradicting the assumption. According to (2), there exists a linear form $f' : V^\perp \to \mathbb{R}$ satisfying $f'(L) \le \mathbb{Z}$, $d(f'(\pi(u)), \mathbb{Z}) > 1/4$ and

$$\|f'\| \overset{18.19}{\le} \frac{21}{2\pi} \sum_{k=1}^{\dim V^\perp} \mathrm{d}_k(B_{V^\perp}, \pi(D)) \overset{18.20}{\le} \frac{21}{2\pi} \sum_{k=1}^{n} d_k^{-1}.$$

Hence, $f := f' \circ \pi$ has the desired properties.

(4) Now, let K be an arbitrary subgroup of $\mathbb{R}^n$. We may assume $\langle K \rangle_{\mathbb{R}} = \mathbb{R}^n$. (Otherwise, we may replace D by $D \cap \langle K \rangle_{\mathbb{R}}$, taking into consideration 18.18.) Since $\overline{K} \cap (u + \lambda D) = \emptyset$ for all $\lambda \in]1/2, 1[$, there exist (according to (3)) linear forms f_λ satisfying $f_\lambda(\overline{K}) \le \mathbb{Z}$, $d(f_\lambda(u), \mathbb{Z}) \ge 1/4$, and $\|f_\lambda\| \le \frac{21}{2\pi} \cdot \frac{1}{\lambda} \cdot \sum d_k^{-1}$. (In the last step we made use of 18.19 and 18.2(ii).) For each increasing sequence $\lambda_n \to 1$, the norms of the corresponding f_{λ_n} are bounded. Hence the sequence $(f_{\lambda_n})_{n \in \mathbb{N}}$ has an accumulation point which has the desired properties.

Lemma 19.13. *For each subgroup $K \le \mathbb{R}^n$ and each ellipsoid D with principal semiaxes $(d_1, \dots, d_n)$ and each character $\chi : K \to \mathbb{T}$ satisfying $\chi(K \cap D) \subseteq R$, there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\exp(2\pi i f|_K) = \chi$ and $\|f\| \le (21/(2\pi)) \sum_{k=1}^n d_k^{-1}$.*

Proof. (1) Assume first that $K = L$ is a lattice. There exists $x_0 \in \mathbb{R}^n$ such that $\chi(y) = \exp(2\pi i(x_0 \cdot y))$. It suffices to show that $d(x_0, \widehat{L}) \le (21/(2\pi)) \cdot \sum_{k=1}^n d_k^{-1}$. [For

$x \in \widehat{L}$ such that $\|x - x_0\| = d(x_0, \widehat{L})$, the linear function $f : y \mapsto (x_0 - x) \cdot y$ has the desired properties.] We may assume $x_0 \notin \widehat{L}$. According to 19.9 and 19.10, there exists $y_0 \in L$ such that $x_0 \cdot y_0 \notin \mathbb{Z}$ and

$$\frac{p_D(y_0) \cdot d(x_0, \widehat{L})}{d(x_0 \cdot y_0, \mathbb{Z})} \leq \frac{21}{\pi} \cdot \sum_{k=1}^{n} d_k^{-1}.$$

If $y_0 \in D$ then $d(x_0 \cdot y_0, \mathbb{Z}) \leq 1/4$ (by assumption). So, let $k \in \mathbb{N}$ be minimal with the property that $d(x_0 \cdot (ky_0), \mathbb{Z}) > 1/4$. This implies $ky_0 \notin D$. Using 19.11, we get

$$d(x_0, \widehat{L}) \leq \frac{21}{\pi} \cdot \frac{d(x_0 \cdot y_0, \mathbb{Z})}{p_D(y_0)} \cdot \sum_{k=1}^{n} d_k^{-1} = \frac{21}{\pi} \cdot \frac{d(x_0 \cdot (ky_0), \mathbb{Z})}{p_D(ky_0)} \cdot \sum_{k=1}^{n} d_k^{-1} \leq \frac{21}{2\pi} \cdot \sum_{k=1}^{n} d_k^{-1}.$$

(2) An arbitrary discrete subgroup K is contained in a lattice L such that $K \cap D = L \cap D$. Applying (1) to L and an arbitrary homomorphism extending χ (2.3), we obtain the assertion for K.

(3) Every closed subgroup $K \leq \mathbb{R}^n$ has the form $K = V \oplus L$ where V is a linear subspace of $\mathbb{R}^n$ and L is a discrete subgroup of $V^\perp$ (see 2.2). There is an increasing sequence $(L_m)_{m\in\mathbb{N}}$ of discrete subgroups of V such that their union is dense. Since $L_m \oplus L$ is a discrete subgroup of $\mathbb{R}^n$, there are, according to (2), linear functionals $f_m : \mathbb{R}^n \to \mathbb{R}$ satisfying $\exp(2\pi i f_m(x)) = \chi(x)$ and $\|f_m\| \leq (21/(2\pi)) \sum_{k=1}^{n} d_k^{-1}$. Hence the sequence $(f_m)_{m\in\mathbb{N}}$ has an accumulation point with the desired properties.

(4) Let K be an arbitrary subgroup of $\mathbb{R}^n$. For every $\lambda \in]0,1[$, we have $\overline{K} \cap \lambda D \subseteq \overline{K} \cap \operatorname{int}(D) \subseteq \overline{K \cap D}$. Since χ is continuous, there exists a continuous homomorphism $\chi' : \overline{K} \to \mathbb{R}$ extending χ (cf. 2.17). Hence $\chi'(\overline{K} \cap \lambda D) \subseteq R$ and, according to (4), there exists a linear form f_λ satisfying $\exp(2\pi i f_\lambda|_{\overline{K}}) = \chi'$ and $\|f_\lambda\| \leq (21/(2\pi))\lambda^{-1} \sum d_k^{-1}$ (18.19 and 18.2(ii)). The estimate of the norm shows that the sequence $(f_{(n-1)/n})_{n\geq 2}$ has an accumulation point with the desired properties.

Lemma 19.14. *Let p and q be pre-Hilbert seminorms on the vector space E such that $\sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q) < \infty$ and let K be a subgroup of E.*

(i) *For every $u \in E \setminus (K + B_q)$, there exists a linear form $f : E \to \mathbb{R}$ such that $f(u) \in [1/4, 3/4] + \mathbb{Z}$ and $\sup\{|f(x)| : x \in B_p\} \leq (21/(2\pi)) \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q)$.*

(ii) *For every homomorphism $\chi : K \to \mathbb{T}$ satisfying $\chi(K \cap B_q) \subseteq R$, there exists a linear form $f : E \to \mathbb{R}$ such that $\exp(2\pi i f|_K) = \chi$ and $\sup\{|f(x)| : x \in B_p\} \leq (21/(2\pi)) \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q)$.*

Proof. We use the notation of 18.7.

Because of 18.2(ii), we may assume that $\sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q) = 1$. (Otherwise, we replace p by a suitable multiple.)

For each $x \in K$, the set $A_x := \{h \in E_p' : \exp(2\pi i h(\psi_p(x))) = \chi(x)\}$ is closed with respect to the weak* topology on E_p.

For $x_1, \ldots, x_n \in K$, we put $M_0 := \langle \psi_q(x_1), \ldots, \psi_q(x_n)\rangle_{\mathbb{R}}$ and $M_1 := \langle M_0, \psi_q(u)\rangle_{\mathbb{R}}$. According to 18.7, the set $\psi_q(B_q)$ is absorbing and the corresponding Minkowski functional satisfies the parallelogram law. Hence $M_j \cap \psi_q(B_q)$ are $\dim M_j$-dimensional ellipsoids for $j \in \{0, 1\}$ (17.4(ii)).

(i) Observe that $\ker\psi_q \subseteq B_q$ implies $\psi_q(u) \notin \psi_q(K) + \psi_q(B_q)$. According to 19.12, there exists a (continuous) linear form $f_1 : M_1 \to \mathbb{R}$ such that $f_1(\psi_q(K)\cap M_1) \le \mathbb{Z}$, $f_1(\psi_q(u)) \in [1/4, 3/4] + \mathbb{Z}$ and

$$\|f_1\| \overset{18.20}{\le} \frac{21}{2\pi} \sum_{k=1}^{\dim M_1} \mathrm{d}_k(\psi_q(B_p)\cap M_1, \psi_q(B_q)\cap M_1) \overset{18.18}{\le} \frac{21}{2\pi}.$$

Next, $\widetilde{f_1} : \Lambda_{p,q}^{-1}(M_1) \to \mathbb{R}$, $\psi_p(x) \mapsto f_1(\psi_q(x))$ ($\Lambda_{p,q}$ as in 18.7), is a well defined continuous linear form which has a continuous extension F_1 to E_p such that $\|F_1\| = \sup\{|\widetilde{f_1}(\psi_p(x))| : \psi_p(x) \in \Lambda_{pq}^{-1}(M_1) \cap B_p\} \le 21/(2\pi)$ (8.19) and, of course, $F_1(\psi_p(u)) \in [1/4, 3/4] + \mathbb{Z}$. Hence the sets of the form $(21/(2\pi)) \cdot B(E_p^*) \cap P \cap \bigcap_{x\in F} A_x$, where F is a finite subset of K and $P := \{h \in E_p^* : h(\psi_p(u)) \in [1/4, 3/4] + \mathbb{Z}\}$, are weak*-closed and not empty. According to Alaoglu–Bourbaki (cf. [73], p. 68, Theorem (3.15)), the set $(21/(2\pi))\cdot B(E_p^*)$ is weak*-compact. Hence the intersection of the above sets is not empty. This shows (i).

(ii) Since $\chi(K\cap B_q) \subseteq R$ implies $K\cap N_q \le \ker\chi$ (3.3(ii)), the mapping $\kappa : \psi_q(K) \to \mathbb{T}$, $\psi_q(x) \mapsto \chi(x)$, is a well defined homomorphism satisfying $\kappa(\psi_q(K)\cap B_{E_q}) = \chi(K\cap B_q) \subseteq R$. Due to 19.13, we get a linear form $f_0 : M_0 \to \mathbb{R}$ such that $\exp(2\pi i f_0|_{\psi_q(K)}) = \kappa$ and

$$\sup\{|f_0(x)| : x \in M_0 \cap B_p\} \le \frac{21}{2\pi} \sum_{k=1}^{\dim M_0} \mathrm{d}_k(B_{E_p}\cap M_0, B_{E_q}\cap M_0) \overset{18.18}{\le} \frac{21}{2\pi}.$$

Hence $\widetilde{f_0} : \Lambda_{pq}^{-1}(M_0) \to \mathbb{R}$, $\psi_p(x) \mapsto f_0(\psi_q(x))$, is a continuous linear form which has a norm preserving extension $F_0 : E_p \to \mathbb{R}$; this means $\|F_0\| = \sup\{|\widetilde{f_0}(x)| : x \in \Lambda_{pq}^{-1}(M_0) \cap B_{E_p}\} \le 21/(2\pi)$. For every finite subset F of K, the sets of the form $\bigcap_{x\in F} A_x \cap (21/(2\pi)) \cdot B(E_p^*)$ are not empty and weak*-compact (since $B_{E_p^*}$ is weak*-compact due to Alaoglu-Bourbaki). Hence $\bigcap_{x\in K} A_x \cap (21/(2\pi)) \cdot B(E_p^*) \ne \emptyset$, which implies (ii).

NOTES 19.15. 19.1 to 19.9 establish some results of [11] and [9] more explicitly. 19.11 was pointed out to me by G. Turnwald. W. Banaszczyk informed me by e-mail that 19.9 enables one to get improved versions of (3.11) and (3.14) in [8] (19.12 and 19.13). Besides that, I am indebted to the referee for improving the constant in 19.13 and for providing me with a proof of 19.7. The result 19.14 is (8.1) and (8.3) of [8] where the improved version of Lemma (3.11) and (3.14) in [8] have been used.

20. Nuclear groups

Nuclear vector spaces have been introduced by Grothendieck in [32] by means of the tensor product. Recall first that a locally convex space V is called *nuclear* if for every symmetric convex neighbourhood $U \in \mathcal{U}_V(0)$, there exists a symmetric convex neighbourhood $U' \in \mathcal{U}_V(0)$ such that $\mathrm{d}_k(U', U) \le 1/k$ for all $k \in \mathbb{N}$. This enables us to define nuclear vector groups and, afterwards, nuclear groups. We prove some permanence properties (20.7) and show that every subgroup of a nuclear group is dually embedded (20.13) and that every nuclear group is locally quasi-convex (20.15). We give a proof of the following fact: a locally convex vector space is a nuclear group if and only if it is a nuclear vector space

(20.20). At the end of the chapter we show some kind of structure theorem for nuclear groups (20.32).

Definition 20.1. A locally convex vector group V is called a *nuclear vector group* if for every $U \in \mathcal{U}_V(0)$, there exists a neighbourhood $U' \in \mathcal{U}_V(0)$ such that $\mathrm{d}_k(U', U) \leq 1/k$ for all $k \in \mathbb{N}$.

It is clear that every nuclear vector space is a nuclear vector group. For examples of nuclear vector spaces see [64], Chapter 6.

Lemma 20.2. *Let V be a nuclear vector group. For $U \in \mathcal{U}_V(0)$, $m \in \mathbb{N}$ and $c > 0$, there exists a neighbourhood $U' \in \mathcal{U}_V(0)$ such that $\mathrm{d}_k(U', U) \leq ck^{-m}$ (for all $k \in \mathbb{N}$).*

Proof. Inductively, we can find a sequence $(U_n)_{n\in\mathbb{N}_0}$ of symmetric convex neighbourhoods with $U_0 := U$ and $\mathrm{d}_k(U_n, U_{n-1}) \leq 1/k$ for all $k \in \mathbb{N}$ (and $n \in \mathbb{N}$). Hence, for $m \in \mathbb{N}$, we get

$$\begin{aligned} \mathrm{d}_{mk-(m-1)}(U_m, U_0) &\overset{18.3(\mathrm{i})}{\leq} \mathrm{d}_k(U_m, U_{m-1})\mathrm{d}_{(m-1)k-(m-2)}(U_{m-1}, U_0) \\ &\overset{18.3(\mathrm{i})}{\leq} \dots \overset{18.3(\mathrm{i})}{\leq} \prod_{l=1}^{m} \mathrm{d}_k(U_l, U_{l-1}) \leq k^{-m}. \end{aligned}$$

For $l \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $m(k-1)+1 \leq l \leq mk$ $(< mk+1)$. By the monotonicity of $(\mathrm{d}_l(U_m, U_0))_{l\in\mathbb{N}}$, we get $\mathrm{d}_l(U_m, U_0) \leq \mathrm{d}_{m(k-1)+1}(U_m, U_0) \leq k^{-m} \leq m^m l^{-m}$ since $l \leq km$. This shows that $(1/(cm^m))U_m$ has the desired properties (cf. 18.2(ii) and 9.3(i)).

In order to define something like the Kolmogorov diameter for abelian groups we compare the group with a vector space; more precisely:

Notation 20.3. Let G be an abelian topological group. For $U, V \in \mathcal{U}_G(e)$ we use the symbol $(\mathrm{d}_k(V, U))_{k\in\mathbb{N}} \leq (c_k)_{k\in\mathbb{N}}$ if there exists a tuple (E, H, φ, X, Y) where E is a vector space, H is a subgroup of E, φ is a homomorphism $H \to G$, and X and Y are symmetric convex subsets of E such that $\mathrm{d}_k(X, Y) \leq c_k$ for all $k \in \mathbb{N}$ (where d_k is the usual Kolmogorov diameter), $\varphi(H \cap X) \supseteq V$ and $\varphi(H \cap Y) \subseteq U$.

Definition 20.4. An abelian Hausdorff group G is called *nuclear* if for every $U \in \mathcal{U}_G(e)$, $m \in \mathbb{N}$, and $c > 0$, there exists $V \in \mathcal{U}_G(e)$ such that $(\mathrm{d}_k(V, U))_{k\in\mathbb{N}} \leq (c \cdot k^{-m})_{k\in\mathbb{N}}$.

An abelian Hausdorff-group G is called *quasi-nuclear* if for every $U \in \mathcal{U}_G(e)$, there exists $V \in \mathcal{U}_G(e)$ such that

$$(\mathrm{d}_k(V, U))_{k\in\mathbb{N}} \leq \left(\frac{1}{12\pi^2 \cdot c(6)} k^{-6} \right)_{k\in\mathbb{N}}$$

(with $c(6)$ as in 18.32).

Remark 20.5. It will be proved in 20.33 that every quasi-nuclear group is nuclear.

Example 20.6. (i) Every nuclear vector group and, in particular, every nuclear vector space is a nuclear group. [This follows immediately from 20.2.]

(ii) Every discrete group is nuclear. [Put $V = \{e\}$ and $H = E = X = Y = \{0\}$ and let φ be the trivial homomorphism.]

(iii) $\mathbb{R}^n$ is a nuclear group. [Let $U = Y$ be a compact symmetric convex neighbourhood of $\{0\}$. Fix $c > 0$ and $m \in \mathbb{N}$. Put $H = E = \mathbb{R}^n$ and $\varphi = \mathrm{id}$. It follows easily from 18.2(ii) that $X := ck^{-n}Y$ satisfies the above conditions.] Besides that, $\mathbb{R}^n$ is a nuclear vector space and hence, according to (i), a nuclear group.

Proposition 20.7. (i) *Every subgroup of a nuclear group is nuclear.*

(ii) *Every Hausdorff quotient of a nuclear group is nuclear.*

(iii) *Every product of nuclear groups is nuclear.*

(iv) *Every countable direct sum of nuclear groups is nuclear.*

(v) *Every abelian group which has an open nuclear subgroup is nuclear.*

Proof. Let G be a nuclear group.

(i) Let $H \leq G$. For $U, V \in \mathcal{U}_G(e)$ such that $(\mathrm{d}_k(V, U))_{k\in\mathbb{N}} \leq (c_k)_{k\in\mathbb{N}}$ we get $(\mathrm{d}_k(V \cap H, U \cap H))_{k\in\mathbb{N}} \leq (c_k)_{k\in\mathbb{N}}$. [Let E, K, φ, X, Y be as in 20.3. If we put $K_H := \varphi^{-1}(H)$ and $\varphi_H := \varphi|_{K_H}$, then the tuple $(E, K_H, \varphi_H, X, Y)$ satisfies the conditions for $V \cap H$ and $U \cap H$.] This implies (i).

(ii) Let H be a closed subgroup of the nuclear group G and let $\pi : G \to G/H$ denote the canonical projection. If we replace the tuple (E, K, φ, X, Y) for $U, V \in \mathcal{U}_G(e)$ by $(E, K, \pi \circ \varphi, X, Y)$ for $\pi(U), \pi(V) \in \mathcal{U}_{G/H}(e)$ we get the desired properties.

(iii) (and (iv)) Let G_i be nuclear groups and let I be a non-empty (countable) index set. $G := \prod_{i\in I} G_i$ ($G := \sum_{i\in I} G_i$) is an abelian Hausdorff group. Choose $U_i \in \mathcal{U}_{G_i}(e)$ such that $U_i = G_i$ for all $i \in I \setminus F$ where F is a finite subset of I ($U_i \in \mathcal{U}_{G_i}(e)$ arbitrarily) and put $U := \prod_{i\in I} U_i$ ($U := \sum_{i\in I} U_i$). For $i \in F$ ($i \in I$), choose $k_i \in \mathbb{N}$ such that $\sum_{i\in F} 1/k_i < 1$ ($\sum_{i\in I} 1/k_i < 1$). By assumption, for every $i \in F$ ($i \in I$), we can find $V_i \in \mathcal{U}_{G_i}(e)$ such that $(\mathrm{d}_k(V_i, U_i))_{k\in\mathbb{N}} \leq ((c/k_i^m)k^{-m})_{k\in\mathbb{N}}$ with a corresponding tuple $(E_i, H_i, \varphi_i, X_i, Y_i)$. For $i \in I \setminus F$ we choose a vector space E_i, a subgroup K_i and an epimorphism $\varphi_i : K_i \to G_i$. [One may take K_i as a free abelian group and E_i the corresponding direct sum of copies of $\mathbb{R}$.] For $i \in I \setminus F$ we put $X_i = Y_i = E_i$ and set $E := \prod_{i\in I} E_i$ ($E = \sum_{i\in I} E_i$), $K := \prod_{i\in I} K_i$ ($K = \sum_{i\in I} K_i$), $X := \prod_{i\in I} X_i$ ($X := \sum_{i\in I} X_i$), $Y := \prod_{i\in I} Y_i$ ($Y := \sum_{i\in I} Y_i$) and $\varphi = (\varphi_i)_{i\in I}$. It is clear that E is a vector space, K is a subgroup of E, and $\varphi : K \to G$ is a homomorphism satisfying $\varphi(K \cap X) \supseteq V$ where $V = \prod_{i\in F} V_i \times \prod_{i\in I\setminus F} G_i$ ($V := \sum_{i\in I} V_i$) and $\varphi(K \cap Y) \subseteq U$.

Hence it suffices to show that $\mathrm{d}_k(X, Y) \leq c \cdot k^{-m}$. Fix $k \in \mathbb{N}$. For $i \in F$ ($i \in I$), let $l_i := [k/k_i]$ denote the integer part of k/k_i. Then $\mathrm{d}_{l_i+1}(X_i, Y_i) < c \cdot (1/k_i^m) \cdot (k_i/k)^m = c \cdot k^{-m}$. Hence, there exists a subspace $L_i \leq E_i$ such that $X_i \subseteq ck^{-m}Y_i + L_i$ where $\dim L_i \leq l_i \leq k/k_i$. Put $L_i := \{0\}$ for $i \in I \setminus F$ and let $L = \prod L_i$ ($L = \sum L_i$). Then $X \subseteq ck^{-m}Y + L$ and $\dim L = \sum_{i\in I} \dim L_i < k$, which implies (iii) (and (iv)).

(v) is trivial.

Theorem 20.8. *Every locally compact abelian group is nuclear.*

Proof. (1) Since $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ we find as a consequence of 20.6(iii) and 20.7(ii) that $\mathbb{T}$ is nuclear.

(2) Since every compact abelian group K can be embedded into a product of tori: $K \to \mathbb{T}^{K^*}$, $x \mapsto (\chi(x))_{\chi\in K^*}$, it follows from (1) and 20.7(i) and (iii) that K is a nuclear group.

(3) Since every locally compact abelian group G has an open subgroup which is topologically isomorphic to $\mathbb{R}^n \times K$ where K is a compact group (e.g. Theorem (24.30) in [34], p. 389), it is a consequence of 20.6(iii), (2) and 20.7(iii) and (v) that G is nuclear.

Remark 20.9. We will show in 20.27 that $\mathbb{R}^{(I)}$ is not nuclear if I is an uncountable index set.

Lemma 20.10. *An abelian Hausdorff group G which is (quasi-) nuclear satisfies the following condition: for every $U \in \mathcal{U}_G(e)$, every $m \in \mathbb{N}$ and every $c > 0$, there exist a vector space E, two pre-Hilbert seminorms p and q on E such that $\mathrm{d}_k(B_p, B_q) \le ck^{-m}$ $(\mathrm{d}_k(B_p, B_q) \le (1/(12\pi^2))k^{-3})$, a subgroup $H \le E$ such that $E = \langle H\rangle_{\mathbb{R}}$ and a homomorphism $\varphi : H \to G$ satisfying $\varphi(H \cap B_p) \in \mathcal{U}_G(e)$ and $\varphi(H \cap B_q) \subseteq U$.*

Proof. Let G be a (quasi-) nuclear group, let $U \in \mathcal{U}_G(e)$, $m \in \mathbb{N}$ and $c > 0$. By the definition, there exist: a vector space E', a subgroup $H' \le E'$ and two symmetric convex sets $X, Y \subseteq E$ such that $\mathrm{d}_k(X, Y) \le (c/c(m+3))k^{-(m+3)}$ $(\mathrm{d}_k(X, Y) \le (1/(12\pi^2 \cdot c(6))k^{-6})$ with $c(m)$ as in 18.32; furthermore, there exists a homomorphism $\varphi' : H' \to G$ such that $\varphi'(H' \cap X) \in \mathcal{U}_G(e)$ and $\varphi'(H' \cap Y) \subseteq U$. According to 18.32, we can find pre-Hilbert seminorms p' and q' on $\langle X\rangle$ such that $\mathrm{d}_k(B_{p'}, B_{q'}) \le c(m+3) \times (c/(c(m+3)))k^{-m} = ck^{-m}$ $(\mathrm{d}_k(B_{p'}, B_{q'}) \le c(6)(1/(12\pi^2 \cdot c(6)))k^{-3} = (1/(12\pi^2))k^{-3})$ and $X \subseteq B_{p'}$ and $B_{q'} \subseteq Y$. For $E := \langle H' \cap B_{p'}\rangle_{\mathbb{R}}$ and $H := \langle H' \cap B_{p'}\rangle_{\mathbb{Z}}$, we get $\langle H\rangle_{\mathbb{R}} = E$. If we put $p := p'|_E$, $q := q'|_E$ and $\varphi := \varphi'|_H$ then $\varphi(H \cap B_q) \subseteq \varphi(H' \cap B_{q'}) \subseteq U$. Furthermore, $H \cap B_p = \langle H' \cap B_{p'}\rangle_{\mathbb{Z}} \cap B_p = \langle H' \cap B_{p'}\rangle_{\mathbb{Z}} \cap B_{p'} = H' \cap B_{p'}$ implies $\varphi(H \cap B_p) = \varphi'(H' \cap B_{p'}) \in \mathcal{U}_G(e)$. The assertion follows from 18.18.

Remark 20.11. Observe that

$$\sum_{k\in\mathbb{N}} \frac{1}{12\pi^2}k^{-3} \le \frac{1}{12\pi^2}\sum_{k\in\mathbb{N}} k^{-2} = \frac{1}{12\pi^2}\frac{\pi^2}{6} = \frac{1}{72}.$$

Theorem 20.12. *Let H be a subgroup of the quasi-nuclear group G. Every equicontinuous subset $S \subseteq H^*$ is the image of an equicontinuous subset of G^* under ι^* where $\iota : H \to G$ denotes the canonical embedding.*

Proof. Let $W \in \mathcal{U}_H(e)$ and $U \in \mathcal{U}_G(e)$ be open neighbourhoods such that $S \subseteq W^0$ and $W = H \cap U$. According to 20.10, there exist: a vector space E, a subgroup $K \le E$, two pre-Hilbert seminorms p and q on E such that $\sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q) \le 1/15$ and a homomorphism $\varphi : K \to G$ such that $\varphi(K \cap B_p) \in \mathcal{U}_G(e)$ and $\varphi(K \cap B_q) \subseteq U$ (cf. 20.11). Obviously, $\widetilde{S} := (\varphi(K \cap B_p))^0$ is an equicontinuous subset of G^*. So it suffices to show that $\iota^*(\widetilde{S}) \supseteq S$. For $\chi \in S$ and $K' := \varphi^{-1}(H) \le K$, the mapping $x \mapsto \chi(\varphi(x))$ defines a homomorphism $K' \to \mathbb{T}$ which satisfies $\chi(\varphi(K' \cap B_q)) \subseteq R$. It is a consequence of 19.14(ii) that there exists a linear function $f : E \to \mathbb{R}$ such that $e^{2\pi i f|_{K'}} = (\chi \circ \varphi)|_{K'}$ and $\sup\{|f(x)| : x \in B_p\} \le (21/(2\pi)) \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q) \le 1/4$. Since $\ker\varphi \le K'$, the homomorphism $\widetilde{\chi} : \varphi(K) \to \mathbb{T}$, $\varphi(x) \mapsto \exp(2\pi i f(x))$, is well defined. By construction, $\widetilde{\chi}(\varphi(K \cap B_p)) \subseteq R$, which shows that $\widetilde{\chi}$ is continuous (3.4(ii)). Next, $\chi' : \varphi(K) + H \to \mathbb{T}$, $\varphi(x) + h \mapsto \widetilde{\chi}(\varphi(x)) + \chi(h)$, is a well defined homomorphism (2.4) which satisfies $\chi'|_{\varphi(K)} = \widetilde{\chi}$ and $\chi'|_H = \chi$. Let χ'' be an arbitrary homomorphism $G \to \mathbb{T}$ extending χ' (cf. 2.3).

Since $\varphi(K)$ is an open subgroup of G, we see that χ'' is continuous and, by construction, it is an element of $\widetilde{S}$.

Corollary 20.13. *Every subgroup H of a nuclear group is dually embedded.*

Proof. The assertion is equivalent to the surjectivity of the dual homomorphism of the embedding. Since every one-point set of H^* is equicontinuous, this is a direct consequence of 20.12.

Remark 20.14. Leptin constructed a closed subgroup H of a group G which is a product of discrete groups such that α_H is not continuous ([49]). Because of 20.6(ii) and 20.7(iii) and (i), G and hence H is nuclear. According to 5.4, G is reflexive and, in particular, α_G is continuous. This means that every compact subset of G^* is equicontinuous (5.10). If ι^* were a compact-covering then every compact subset of H^* would be equicontinuous. But this is equivalent to the continuity of α_H (cf. 5.10). Hence, in 20.12 "equicontinuous" cannot be replaced by "compact".

Theorem 20.15. *Every quasi-nuclear group is locally quasi-convex.*

Proof. For $U \in \mathcal{U}_G(e)$, there exist, according to 20.10 and 20.11: a vector space E, a subgroup $H \leq E$, two pre-Hilbert seminorms p and q such that $\sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q) \leq 1/30$, and a homomorphism $\varphi : H \to G$ such that $\varphi(H \cap B_p) \in \mathcal{U}_G(e)$ and $\varphi(H \cap B_q) \subseteq U$. It suffices to show that the quasi-convex hull U_0 of $\varphi(H \cap B_p)$ is contained in U. This is equivalent to: For every $x \notin U$, there exists $\chi \in G^*$ such that $\chi(\varphi(K \cap B_p)) \subseteq R$ and $\chi(x) \notin R$. If x is not a member of the open subgroup $\varphi(H)$, then the assertion is trivial. Let therefore $x \in \varphi(H)$ and let $u \in H \setminus (\ker\varphi + B_q)$ satisfy $\varphi(u) = x$. According to 19.14(i), there exists a linear function $f : E \to \mathbb{R}$ such that $f(\ker\varphi) \leq \mathbb{Z}$, $f(u) \in [1/4, 3/4] + \mathbb{Z}$ and $\sup\{|f(y)| : y \in B_p\} \leq (21/(2\pi)) \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q) \leq 1/8$. The homomorphism $\chi' : \varphi(H) \to \mathbb{T}$, $\varphi(y) \mapsto e^{2\pi i f(y)}$, is continuous since $\chi'(\varphi(H \cap B_p)) \subseteq V_2$ (3.4(ii)). Let χ'' be a character of G extending χ' (observe that $\varphi(H)$ is an open subgroup of G and use 2.3). If $\chi''(x) \notin R$ then we put $\chi := \chi''$, otherwise we set $\chi := (\chi'')^2$. Then $\chi(x) \notin R$ and $\chi(\varphi(H \cap B_p)) \subseteq R$, which implies the assertion.

Corollary 20.16. *For every nuclear group G, the evaluation mapping α_G is injective.*

Proof. This is an immediate consequence of 20.15 and 6.10.

Corollary 20.17. *Every closed subgroup H of a nuclear group G is dually closed.*

Proof. According to 20.7(ii), G/H is a nuclear group and it follows from 20.16 that $\alpha_{G/H}$ is injective.

Lemma 20.18. *Let K be a subgroup of $\mathbb{R}^n$, let V be a locally convex Hausdorff space and let $\varphi : K \to V$ be a continuous (group) homomorphism. Then there exists a continuous linear operator $\Phi : \mathbb{R}^n \to V$ which extends φ. If $\langle K\rangle_{\mathbb{R}} = \mathbb{R}^n$, then Φ is unique.*

Proof. We may assume that $\langle K\rangle_{\mathbb{R}} = \mathbb{R}^n$ (see 8.19). Let $\overline{V}$ denote the completion of V and let $\overline{\varphi}$ be the composition of φ with the canonical embedding of V into $\overline{V}$. Because of 2.17, $\overline{\varphi}$ can be extended to a group homomorphism $\varphi' : \overline{K} \to \overline{V}$ where $\overline{K}$ denotes the closure of K in $\mathbb{R}^n$. Suppose we have already proved that φ' has an extension to a linear functional $\varPhi' : \mathbb{R}^n \to \overline{V}$. Then $\varphi(K) \subseteq \varphi'(\overline{K}) \subseteq \varPhi'(\mathbb{R}^n)$ is contained in a finite-dimensional subspace of $\overline{V}$. Hence there exists a finite-dimensional (and hence complete) subspace of V which contains $\varphi(K)$. This shows that $\varPhi'(\mathbb{R}^n) = \varPhi'(\langle K\rangle_{\mathbb{R}}) = \langle \varphi(K)\rangle_{\mathbb{R}}$ and hence $\varPhi'(\mathbb{R}^n)$ lies in V. So we may assume that K is a closed subgroup of $\mathbb{R}^n$.

According to 2.2, there exists a basis $(x_1, \ldots, x_m, y_{m+1}, \ldots, y_n)$ of $\mathbb{R}^n$ (where $m \le n$) such that $K = \langle x_1, \ldots, x_m\rangle_{\mathbb{R}} \oplus \langle y_{m+1}, \ldots, y_n\rangle_{\mathbb{Z}}$. Since V is torsion-free, we have $\varphi(\lambda x) = \lambda\varphi(x)$ for all $x \in \langle x_1, \ldots, x_m\rangle_{\mathbb{R}} =: V_0$ and $\lambda \in \mathbb{Q}$. By the continuity of φ, the restriction $\varphi|_{V_0}$ is a continuous linear mapping. Now we may suppose that $m = 0$ and that K is a lattice. $\varPhi : \sum_{j=1}^n \lambda_j y_j \mapsto \sum_{j=1}^n \lambda_j \varphi(y_j)$ is obviously a well defined linear mapping which is continuous, since $\dim \varPhi(\mathbb{R}^n) \le n$. The uniqueness of φ is clear.

Lemma 20.19. *Let* (E, s) *be a seminormed space and let* V *be a locally convex Hausdorff space. For every subgroup* $K \le E$ *and every continuous* (*with respect to the topology induced by* s) *homomorphism* $\varphi : K \to V$, *there exists a linear mapping* $\varPhi : \langle K\rangle_{\mathbb{R}} \to V$ *extending* φ.

Proof. It must be proved that for arbitrary $x_1, \ldots, x_n \in K$ (and $n \in \mathbb{N}$), there exists a unique linear mapping $\varphi' : E_2 := \langle x_1, \ldots, x_n\rangle_{\mathbb{R}} \to V$ extending $\varphi|_{E_2 \cap K}$. Let $E_0 := \{x \in E_2 : s(x) = 0\}$ and let E_1 be a subspace of E_2 such that $E_2 = E_0 \oplus E_1$ (algebraically). For any norm r on E_0, the mapping $S : E_0 \oplus E_1 \to \mathbb{R}$, $x_0 + x_1 \mapsto r(x_0) + s(x_1)$ (where $x_j \in E_j$ for $j \in \{0, 1\}$), defines a norm on E_2. It is clear that $\varphi|_{K\cap E_2}$ is also continuous with respect to the topology induced by S. The assertion follows from 20.18.

Theorem 20.20. *Every topological vector space* (*locally convex vector group*) V *which is also a nuclear group is a nuclear vector space* (*nuclear vector group*).

Proof. Assume first that V is a topological vector space and a nuclear group. It follows from 20.15 and 6.5 that V is a (Hausdorff) locally convex vector space.

Hence, according to 9.7, we have to show in both cases that for each weakly closed symmetric convex neighbourhood $U \in \mathcal{U}_V(0)$, there exists a symmetric convex neighbourhood W such that $\mathrm{d}_k(W, U) \le 1/k$ for all $k \in \mathbb{N}$ (see 9.7). Fix a weakly closed symmetric convex neighbourhood U and let s denote the Minkowski functional associated with U which is defined on $\langle U\rangle_{\mathbb{R}}$ (cf. the proof of 10.5). Let $\psi_s : \langle U\rangle_{\mathbb{R}} \to \langle U\rangle_{\mathbb{R}}/\{x \in \langle U\rangle_{\mathbb{R}} : s(x) = 0\} =: V_s$ denote the canonical projection and let V_s be endowed with the topology determined by the norm induced by s. According to 20.10 and 20.7(i), there exist: a vector space E, a subgroup $H \le E$ such that $\langle H\rangle_{\mathbb{R}} = E$, a homomorphism $\varphi : H \to \langle U\rangle_{\mathbb{R}}$, and two pre-Hilbert seminorms p and q on E such that $\mathrm{d}_k(B_p, B_q) \le \frac{1}{3}k^{-1}$, $\varphi(H \cap B_p) \in \mathcal{U}_{\langle U\rangle_{\mathbb{R}}}(0)$ and $\varphi(H \cap B_q) \subseteq U$.

Since $\psi_s \circ \varphi$ is continuous with respect to the topology induced by q, there exists, according to 20.19, a linear operator $\varPhi : E \to V_s$ which extends $\psi_s \circ \varphi$.

For $X := \psi_s^{-1}(\mathrm{co}(\varPhi(H \cap B_p)))$ and $Y := \psi_s^{-1}(\mathrm{co}(\varPhi(H \cap B_q)))$, we get

$$\begin{aligned}\mathrm{d}_k(X, Y) &\overset{18.5(\mathrm{ii})}{=} \mathrm{d}_k(\varPhi(\mathrm{co}(H \cap B_p)), \varPhi(\mathrm{co}(H \cap B_q)))\\ &\overset{18.5(\mathrm{i})}{\le} \mathrm{d}_k(\mathrm{co}(H \cap B_p), \mathrm{co}(H \cap B_q)) \overset{18.34}{\le} 2\mathrm{d}_k(B_p, B_q) \le k^{-1}.\end{aligned}$$

(Observe that $\sum_{n\in\mathbb{N}} \mathrm{d}_n^2(B_p, B_q) \le \pi^2/54 < 1/4$.)

The symmetric convex sets X and Y satisfy $X \supseteq \varphi(H \cap B_p) \in \mathcal{U}_V(0)$ and $Y = \psi_s^{-1}(\mathrm{co}(\psi_s \circ \varphi(H \cap B_q))) = \psi_s^{-1}(\psi_s(\mathrm{co}(\varphi(H \cap B_q)))) = \mathrm{co}(\varphi(H \cap B_q)) + s^{-1}(\{0\}) \subseteq \{x \in \langle U \rangle_{\mathbb{R}} : s(x) \leq 1\} = U$; hence the assertion follows.

Corollary 20.21. *A (real) Banach space V is a nuclear group if and only if it is finite-dimensional.*

Proof. If V is of finite dimension, then it is topologically isomorphic to $\mathbb{R}^n$ for a suitable $n \in \mathbb{N}_0$ and hence, according to 20.6(iii), nuclear.

Conversely, assume that the normed space V is a nuclear group. As a consequence of 20.20 we deduce that V is a nuclear vector space. Exercise 2 in [88], p. 291, implies that V has finite dimension.

Lemma 20.22. *Every infinite-dimensional Banach space E contains a compact set K having the following property: for every sequence $(E_k)_{k \in \mathbb{N}}$ of subspaces satisfying* $\dim(E/E_k) < k$, *the set $\bigcup_{k \in \mathbb{N}}(E_k \cap k \cdot K)$ is not bounded.*

Proof. Let $(F_n)_{n \in \mathbb{N}}$ be an increasing sequence of subspaces of E satisfying $\dim F_n = n$. Let B denote the closed unit ball of E. Then

$$K := \overline{\bigcup_{n \in \mathbb{N}} \left(\frac{1}{\sqrt{n}} B \cap F_n \right)}$$

is compact. [It is sufficient to show that $\bigcup_{n \in \mathbb{N}}((1/\sqrt{n})B \cap F_n)$ is totally bounded. For $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $1/\sqrt{k} \leq \varepsilon$. Hence $\bigcup_{n \geq k}((1/\sqrt{n})B \cap F_n) \subseteq \varepsilon B$ and $\bigcup_{n \in \mathbb{N}}((1/\sqrt{n})B \cap F_n) \subseteq (\varepsilon B \cap F_n) \cup (F_k \cap B)$. Since $F_k \cap B$ can be covered by finitely many translates of $\varepsilon \cdot B$, the same is true for $\bigcup_{n \in \mathbb{N}}((1/\sqrt{n})B \cap F_n)$.]

Now, let $(E_k)_{k \in \mathbb{N}}$ be a sequence of subspaces satisfying $\dim(E/E_k) < k$. For dimensional reasons, there exist $0 \neq x_k \in E_k \cap F_k$ (for all $k \in \mathbb{N}$). We may assume $\|x_k\| = 1/\sqrt{k}$, which implies $x_k \in K$. But $\bigcup_{k \in \mathbb{N}}(E_k \cap k \cdot K) \supseteq \{k \cdot x_k : k \in \mathbb{N}\}$ is not bounded, which completes the proof.

Proposition 20.23. *The dual space of an infinite-dimensional Banach space E endowed with the compact-open topology is not a nuclear group.*

Proof. Since E'_{c} is a locally convex vector space, it is sufficient to show that it is not a nuclear vector space. Being a nuclear vector space is equivalent to: for every compact subset $K \subseteq E$, there exists a compact subset $S \subseteq E$ such that $\mathrm{d}_k(S^0, K^0) < 1/k$ (see 20.2) where $K^0 = \{f \in E' : |f(x)| \leq 1 \ \forall x \in K\}$. Assuming this condition holds, we can find subspaces $L_k \leq E'$ satisfying $\dim L_k < k$ and $S^0 \subseteq (1/k)K^0 + L_k = (k \cdot K)^0 + L_k$. Putting $E_k := \{x \in E : f(x) = 0 \ \forall f \in L_k\}$, we get

$$E_k \cap k \cdot K \subseteq \bigcap_{f \in (k \cdot K)^0 + L_k} \{x \in E : |f(x)| \leq 1\} \subseteq \bigcap_{f \in S^0} \{x \in E : |f(x)| \leq 1\} = \overline{\mathrm{co}(S \cup -S)}.$$

The last equality holds by the Hahn–Banach Theorem. Hence $\bigcup_{k \in \mathbb{N}}(E_k \cap k \cdot K) \subseteq \overline{\mathrm{co}(S \cup -S)}$. For K as in 20.22, this set is not bounded (observe that $\dim(E/E_k) < k$) and hence it cannot be contained in the closed convex hull of the compact (and hence bounded) set $S \cup -S$. This gives the desired contradiction.

Lemma 20.24. *For all subsets* X, Y *of an abelian Hausdorff group* G *with* $(\mathrm{d}_k(X,Y))_{k\in\mathbb{N}} \leq (c\cdot k^{-m})_{k\in\mathbb{N}}$ *where* $m \geq 6$ *is a natural number and* $c > 0$, *we have* $(\mathrm{d}_k(Y^0, X^0))_{k\in\mathbb{N}} \leq (100\cdot c\cdot c(m)\cdot k^{6-m})_{k\in\mathbb{N}}$ *where* $c(m)$ *is the constant from* 18.32.

Proof. This is Lemma (16.4) in [8].

Proposition 20.25. *For every nuclear group* G *such that* α_G *is continuous*, *the second character group* G^{**} *is also a nuclear group.*

Proof. For every neighbourhood $W \in \mathcal{U}_{G^{**}}(e)$, there exists (according to 3.4, 5.10 and by assumption) a neighbourhood $U \in \mathcal{U}_G(e)$ such that $U^{00} \subseteq W$. Fix $m \in \mathbb{N}$ and $c > 0$. Since G is nuclear, there exists $V \in \mathcal{U}_G(e)$ such that $(\mathrm{d}_k(V,U))_{k\in\mathbb{N}} \leq (c\cdot 10^{-4}\cdot c(m+12)^{-1}\cdot c(m+6)^{-1}\cdot k^{-m-12})_{k\in\mathbb{N}}$. Applying 20.24 twice, we obtain

$$\begin{aligned}(\mathrm{d}_k(U^0,V^0))_{k\in\mathbb{N}} &\leq (c\cdot 10^{-2}\cdot c(m+6)^{-1}\cdot k^{-m-6})_{k\in\mathbb{N}},\\ (\mathrm{d}_k(V^{00},U^{00}))_{k\in\mathbb{N}} &\leq (c\cdot k^{-m})_{k\in\mathbb{N}},\end{aligned}$$

which implies the assertion.

Example 20.26. Let E_{w} be an infinite-dimensional reflexive Banach space endowed with its weak vector space topology. According to 8.24, $\alpha_{E_{\mathrm{w}}}$ is not continuous. Since E_{w} can be considered as a subspace of a product of the reals, it is a nuclear group and, according to 20.20, even a nuclear vector space. It is a consequence of 5.5 and 8.23 that $(E_{\mathrm{w}})^*$ is topologically isomorphic to the infinite-dimensional Banach space $(E_{\mathrm{n}})'_{\mathrm{b}}$. Hence, according to 20.23, the group $(E_{\mathrm{w}})^{**}$ is not a nuclear group.

This example shows that the condition "α_G is continuous" cannot be omitted in 20.25.

Example 20.27. For each uncountable index set I, the topological vector space $\mathbb{R}^{(I)}$ (endowed with the asterisk topology) is not a nuclear vector space and, according to 20.20, not a nuclear group. [Since $\mathbb{R}^{(I)}$ is a locally convex vector space (2.10), it suffices to prove that $\mathbb{R}^{(I)}$ is not a nuclear vector space. Let $(\varepsilon_i)_{i\in I}$ be a family of positive numbers and let U be the asterisk neighbourhood associated with $(]-\varepsilon_i,\varepsilon_i[)_{i\in I}$. We are to show that for arbitrary $\delta_i > 0$ (and $i \in I$) the asterisk neighbourhood V associated with $(]-\delta_i,\delta_i[)_{i\in I}$ does not satisfy $\mathrm{d}_k(V,U) \leq 1/k$ for all $k\in\mathbb{N}$. Therefore, put $I_n := \{i\in I : \delta_i/\varepsilon_i \in\,]1/n, 1/(n-1)]$ for $n\geq 2$ and $I_1 := \{i \in I : \delta_i > \varepsilon_i\}$. Since $I = \bigcup_{n\in\mathbb{N}} I_n$, there exists $n_0\in\mathbb{N}$ such that I_{n_0} is uncountable. We now show that for any finite-dimensional subspace L of $\mathbb{R}^{(I)}$,

$$V \not\subseteq \frac{1}{2n_0}\cdot U + L.$$

So, let L be a finite-dimensional subspace. There exists an index $i_0 \in I_{n_0}$ such that $E_{i_0} := \{(x_i)_{i\in I} \in \mathbb{R}^{(I)} : x_i = 0\ \forall i \neq i_0\} \cap L = \{(0)_{i\in I}\}$. Let $x_i = 0$ for $i \neq i_0$ and $x_{i_0} = \delta_{i_0}/2$. Then $x = (x_i)_{i\in I} \in V$. But $x \in (1/(2n_0))\cdot U + L$ would imply $\delta_{i_0}/2 = x_{i_0} \in\,]-\varepsilon_{i_0}/(2n_0), \varepsilon_{i_0}/(2n_0)[$ contradicting $\delta_{i_0}/\varepsilon_{i_0} > 1/n_0$.]

Definition 20.28. Topological groups G and H are called *locally isomorphic* if there exist open neighbourhoods $U \in \mathcal{U}_G(e)$ and $V \in \mathcal{U}_H(e)$ and a homeomorphism $\varphi : U \to V$ which satisfies $\varphi(x+y) = \varphi(x)+\varphi(y)$ whenever $x, y, x+y \in U$.

Example 20.29. Let G be an abelian Hausdorff group and let H be a discrete subgroup of G. Then G and G/H are locally isomorphic. [One may take an open neighbourhood $U \in \mathcal{U}_G(e)$ which satisfies $(U + (-U)) \cap H = \{e\}$ and define φ to be the canonical projection restricted to U.]

Theorem 20.30. *If G and H are locally isomorphic abelian topological groups then G is nuclear if and only if H is nuclear.*

Proof. This is Proposition (7.15) in [8].

Corollary 20.31. *For a completely regular space X, the group $\mathcal{C}(X,\mathbb{T})$ is nuclear if and only if all compact subsets of X are finite.*

Proof. If all compact subsets of X are finite then $\mathcal{C}(X,\mathbb{T})$ can be considered as a subgroup of $\mathbb{T}^X$. Hence the assertion in this case follows from 20.7(i) and 20.8.

Conversely, suppose that $\mathcal{C}(X,\mathbb{T})$ is a nuclear group. Let K be an arbitrary compact subset of X. According to 14.4, there exists a continuous open epimorphism $\varphi : \mathcal{C}(X,\mathbb{T}) \to \mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})$. Hence, $\mathcal{C}(X,\mathbb{T})/\ker\varphi \cong \mathcal{C}(K,\mathbb{R})/\mathcal{C}(K,\mathbb{Z})$. This implies that the nuclear group $\mathcal{C}(X,\mathbb{T})/\ker\varphi$ (20.7(ii)) is locally isomorphic to the Banach space $\mathcal{C}(K,\mathbb{R})$ (20.29). So 20.30 and 20.21 imply that $\mathcal{C}(K,\mathbb{R})$ must be finite-dimensional. This is equivalent to the fact that K is finite. Since K was arbitrary, the assertion follows.

Theorem 20.32. *For every quasi-nuclear group G, there exist: a nuclear vector group V, a* (*not necessarily closed*) *subgroup $H \leq V$ and a closed subgroup K of H such that $G \cong H/K$. The group V can be chosen so that the cardinalities of a neighbourhood basis of G and V coincide.*

Proof. We make use of the notation and construction established in the proof of 10.1.

According to 20.15, G is locally quasi-convex. As a consequence of 10.1, there exist a locally convex vector group V, a subgroup H of E and a closed subgroup K of H such that $G \cong H/K$ and the assertion concerning the cardinality of the neighbourhood bases is satisfied.

So, it suffices to show that V is a nuclear vector group. Let U be a fixed quasi-convex neighbourhood of e in G. Since G is a quasi-nuclear group, there exist a vector space E, two pre-Hilbert seminorms p and q on E such that $\mathrm{d}_k(B_p, B_q) \leq (1/(12\pi^2))k^{-3}$, a subgroup $G_0 \leq E$ and a homomorphism $\varphi : G_0 \to G$ satisfying $\varphi(G_0 \cap B_p) \in \mathcal{U}_G(e)$ and $\varphi(G_0 \cap B_q) \subseteq U$ (cf. 20.10). According to 18.28, there exists a pre-Hilbert seminorm r on E such that

$$\mathrm{d}_k(B_p, B_r) < \frac{1}{3}k^{-1} \quad \text{and} \quad \mathrm{d}_k(B_r, B_q) < \frac{1}{4\pi^2}k^{-2}.$$

For $\chi \in U^0$, we have $(\chi \circ \varphi)(G_0 \cap B_q) \subseteq R$. As a consequence of 19.14(ii), there exists a linear mapping $f_\chi : E \to \mathbb{R}$ such that $\exp(2\pi i f_\chi)|_{G_0} = \chi \circ \varphi$ and

$$\sup\{|f_\chi(x)| : x \in B_r\} \leq \frac{21}{2\pi}\sum_{k\in\mathbb{N}} \mathrm{d}_k(B_r, B_q) < 6 \cdot \frac{1}{4\pi^2}\frac{\pi^2}{6} = \frac{1}{4}.$$

For $\chi \in U^0$, let f_χ be an arbitrary but fixed linear functional as above; for $\chi \notin U^0$, we put $f_\chi \equiv 0$. Then

$$\Phi : E \to \mathbb{R}^{G^*}, \quad u \mapsto (f_\chi(u))_{\chi\in G^*},$$

defines a linear mapping. Furthermore,

$$P : \mathbb{R}^{G^*} \to \mathbb{R}^{G^*}, \quad (x_\chi)_{\chi\in G^*} \mapsto (1_{U^0}(\chi) \cdot x_\chi)_{\chi\in G^*},$$

is a projection which satisfies

$$P(H) \cap \Phi(B_r) \subseteq P(X_U). \tag{1}$$

[For $\widetilde{\kappa} = \Phi(u) = P(\kappa)$ (where $\kappa \in H$ and $u \in B_r$), there exists $x \in G$ such that $\alpha_G(x) = \widetilde{\varrho}(\kappa)$. Hence, for $\chi \in U^0$, we get $\chi(x) = \alpha_G(x)(\chi) = \varrho(\kappa(\chi))$ and $\kappa(\chi) = P(\kappa)(\chi) = \Phi(u)(\chi) = f_\chi(u) \in [-1/4, 1/4]$ since $u \in B_r$. This shows that $\chi(x) \in R$ for all $\chi \in U^0$ and hence x belongs to U, since U is quasi-convex. Hence κ belongs to X_U and (1) follows.]

According to 20.15, there exists a quasi-convex neighbourhood $W \in \mathcal{U}_G(e)$ such that $W \subseteq \varphi(G_0 \cap B_p)$. Then

$$P(X_W) \subseteq P(H) \cap \Phi(B_p). \tag{2}$$

[For $\kappa \in X_W$, there exists $x \in W$ such that $|\kappa(\chi)| \le 1/4$ and $\exp(2\pi i\kappa(\chi)) = \chi(x)(\in R)$ for all $\chi \in W^0$. By assumption, there exists $u \in G_0 \cap B_p$ such that $\varphi(u) = x$. Since $W \subseteq \varphi(G_0 \cap B_p) \subseteq \varphi(G_0 \cap B_q) \subseteq U$ we get $U^0 \subseteq W^0$. For an arbitrary $\chi \in U^0$, we get $|\kappa(\chi)| \le 1/4$ and $\exp(2\pi i P(\kappa)(\chi)) = \exp(2\pi i\kappa(\chi)) = \chi(x) = \chi(\varphi(u)) = \exp(2\pi i f_\chi(u)) = \exp(2\pi i\Phi(u)(\chi))$. Since both $|P(\kappa)(\chi)| \le 1/4$ and $|\Phi(u)(\chi)| = |f_\chi(u)| \le 1/4$ (the second inequality holds since $u \in B_p \subseteq B_r$), we get $P(\kappa)(\chi) = \Phi(u)(\chi)$ for all $\chi \in U^0$. For $\chi \in G^* \setminus U^0$ we have $P(\kappa)(\chi) = 0 = \Phi(u)(\chi)$ and hence $P(\kappa) = \Phi(u)$. This shows that $P(X_W) \subseteq \Phi(B_p)$.

Since $\widetilde{\varrho}$ is surjective, there exists $(\eta_\chi)_{\chi\in G^*} \in H$ such that $\widetilde{\varrho}((\eta_\chi)_{\chi\in G^*}) = \alpha_G(x)$. This means $\exp(2\pi i\eta_\chi) = \chi(x)$ (for all $\chi \in G^*$). In addition, we may assume that $|\eta_\chi| \le 1/4$ for all $\chi \in U^0(\subseteq W^0)$. Since we have $|P(\kappa)(\chi)| \le 1/4$ and $\exp(2\pi i P(\kappa)(\chi)) = \chi(x)$ (for $\chi \in U^0$), we get $P(\kappa) = P((\eta_\chi)_{\chi\in G^*})$ and hence (2) follows.]

Taking into consideration that

$$\sum_{k\in\mathbb{N}} \mathrm{d}_k(\Phi(B_p), \Phi(B_r))^2 \overset{18.2(\mathrm{i})}{\le} \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_r)^2 = \frac{\pi^2}{54} < \frac{1}{4}$$

and applying 18.34, we get

$$\mathrm{d}_k(\mathrm{co}(P(H) \cap \Phi(B_p)), \mathrm{co}(P(H) \cap \Phi(B_r))) \le 2\mathrm{d}_k(\Phi(B_p), \Phi(B_r)) \overset{18.5(\mathrm{i})}{\le} 2\mathrm{d}_k(B_p, B_r) < k^{-1}.$$

Furthermore, we have $X_W \subseteq P^{-1}(P(X_W))$ and $X_U = P^{-1}(P(X_U))$. ["$\subseteq$" is trivial in both cases. For $\kappa \in \mathbb{R}^{G^*}$ such that $P(\kappa) \in P(X_U)$, we have $P(\kappa)(\chi) = \kappa(\chi)$ for all $\chi \in U^0$, which yields "$\supseteq$".] This implies $Y_W \subseteq P^{-1}(P(Y_W))$ and $Y_U = P^{-1}(P(Y_U))$, and hence

$$\begin{aligned}
\mathrm{d}_k(Y_W, Y_U) &\overset{18.2(\mathrm{ii})}{\le} \mathrm{d}_k(P^{-1}(P(Y_W)), P^{-1}(P(Y_U))) \\
&\overset{18.5(\mathrm{ii})}{=} \mathrm{d}_k(P(Y_W), P(Y_U)) = \mathrm{d}_k(\mathrm{co}(P(X_W)), \mathrm{co}(P(X_U))) \\
&\overset{18.2(\mathrm{ii}),(1),(2)}{\le} \mathrm{d}_k(\mathrm{co}(P(H) \cap \Phi(B_p)), \mathrm{co}(P(H) \cap \Phi(B_r))) \le k^{-1}.
\end{aligned}$$

This completes the proof.

Corollary 20.33. *Every quasi-nuclear group is nuclear.*

Proof. This follows from 20.32, 20.7 and 20.6(i).

In [8], Theorem 20.32 was used to prove

Theorem 20.34. *The completion of every metrizable nuclear group is a* (*metrizable*) *nuclear group.*

Proof. This is Theorem (9.8) in [8].

Theorem 20.35. *The character group of a metrizable nuclear group G is again nuclear.*

Proof. Theorem (16.1) in [8] states that G^*_{tb} endowed with the topology of uniform convergence on totally bounded sets is nuclear. According to 4.10, we have $G^* = G^*_{\mathrm{tb}}$. This implies the assertion.

Corollary 20.36. *The character group of every Čech-complete nuclear group G is nuclear.*

Proof. According to 2.21, there exists a compact subgroup H of G such that G/H is complete and metrizable. Since G/H is also nuclear (20.7(ii)), the above theorem states that $(G/H)^*$ is nuclear. H being compact implies that $(G/H)^* \to G^*$, $\chi \mapsto (x \mapsto \chi(x+H))$, is a continuous open monomorphism (5.16, 5.18). So, G^* has an open subgroup which is nuclear. The assertion follows from 20.7(v).

Definition 20.37. A topological group G is named *strongly reflexive* if every closed subgroup and every Hausdorff quotient group of G and G^* is a reflexive group.

Theorem 20.38. *Every complete metrizable nuclear group is strongly reflexive.*

Proof. This is a special case of Corollary (17.3) in [8].

Theorem 20.39. *Let H be a compact subgroup of a Hausdorff group G which admits sufficiently many characters. If G/H is strongly reflexive*, *so is G.*

Proof. This is Corollary (3.4) in [13].

Combining the last two results we are able to improve Corollary (17.3) in [8]. (Observe that every countable product of Čech-complete spaces is again Čech-complete (1.25(iii)).)

Theorem 20.40. *Every Čech-complete nuclear group G is strongly reflexive.*

Proof. According to 2.21, there exists a compact subgroup $H \leq G$ such that G/H is complete and metrizable and, because of 20.7(ii), nuclear. Since G/H is strongly reflexive (20.38) and α_G is injective (20.16), the assertion follows from 20.39.

Notes 20.41. 20.2 is similar to Proposition (7.1.1) of [70] (cf. also (2.17) of [8]). 20.6, 20.7, and 20.8 can be found in Section 7 of [8] (cf. also [81]). 20.10 is (7.2) in [8]. 20.12 to 20.17 are taken from Section 8 of [8]. 20.18 and 20.19 have been established in order to give a more detailed proof of 20.20. The part concerning topological vector spaces was formulated in (8.9) in [8], the assertion referring to locally convex vector groups is an answer to a question I was asked by H. Glöckner. 20.27 has been mentioned in Remark (16.2) in [8] and 20.32 is Theorem (9.6) in [8].

21. An embedding theorem for nuclear groups

Remark 21.1. For $U, V \in \mathcal{U}_G(e)$ such that $(\mathrm{d}_k(V, U))_{k\in\mathbb{N}} \leq (c_k)_{k\in\mathbb{N}}$ where $c_1 < 1/2$, we have $V + V \subseteq U$. [Let E, H, φ, X, Y be as in 20.3. Since X is convex we get $X + X = 2X \subseteq Y$ and hence $V + V \subseteq \varphi(H \cap X) + \varphi(H \cap X) \subseteq \varphi(H \cap Y) \subseteq U$.]

Proposition 21.2. *Let G be a nuclear group and let U be a quasi-convex neighbourhood of e. There exists a group topology $\mathcal{N}_U$ on G/H_U (H_U as in 10.3) which is metrizable and nuclear and such that the canonical projection $\pi_U : G \to (G/H_U, \mathcal{N}_U)$ is continuous and $\pi_U(U)$ is a neighbourhood of $\pi_U(e)$.*

Proof. Since H_U is a closed subgroup of G (10.3), the quotient group G/H_U (endowed with the quotient topology) is nuclear (20.7(ii)). According to 20.15, there exists for $\widetilde{U}_2 := \pi_U(U)$ a quasi-convex neighbourhood $\widetilde{U}_3$ of $\pi_U(e)$ (in the quotient topology) such that $(\mathrm{d}_k(\widetilde{U}_3, \widetilde{U}_2))_{k\in\mathbb{N}} \leq \left(\frac{1}{2}k^{-2}\right)_{k\in\mathbb{N}}$. Inductively, we can find quasi-convex neighbourhoods $\widetilde{U}_{n+1}$ satisfying $\mathrm{d}_k(\widetilde{U}_{n+1}, \widetilde{U}_n) \leq (1/n)k^{-n}$. It is a consequence of 21.1 that $\widetilde{U}_{n+1} + \widetilde{U}_{n+1} \subseteq \widetilde{U}_n$ and it follows from 2.19 that $\bigcap_{n\in\mathbb{N}_0} \widetilde{U}_n \subseteq \widetilde{U}_0$ is a subgroup of $\pi_U(U)$ which must be trivial (10.3). According to 2.5, the sets $(\widetilde{U}_n : n \in \mathbb{N}_0)$ form a countable neighbourhood base of a Hausdorff group topology $\mathcal{N}_U$ on G/H_U. One easily verifies that $\mathcal{N}_U$ is nuclear. According to 2.7, it is metrizable. Since $\mathcal{N}_U$ is coarser than the quotient topology, the assertion follows.

Theorem 21.3. *Every nuclear group G can be embedded into a product of complete metrizable nuclear groups such that the image of G is dually embedded. If G is complete then the image is dually closed.*

Proof. Let $\mathcal{U}$ be a neighbourhood base of $e \in G$ consisting of quasi-convex sets (20.15), let H_U be as in 10.3, and let $\mathcal{N}_U$ be a metrizable group topology as in 21.2. Then G_U, the completion of $(G/H_U, \mathcal{N}_U)$, is nuclear and metrizable (20.34). It is a consequence of 10.2 that

$$\varPhi : G \to \prod_{U\in\mathcal{U}} G_U, \quad x \mapsto (\iota_U(x + H_U))$$

(where $\iota_U : G/H_U \to G_U$ denotes the canonical embedding), is an embedding. Since each G_U is a complete and nuclear group, so is the product $\prod_{U\in\mathcal{U}} G_U$ (20.7(iii)). Hence, according to 20.13, $\varPhi(G)$ is dually embedded in the product. If G is complete then $\varPhi(G)$ is a closed subgroup of the product and hence dually closed (20.17). The assertion follows.

Corollary 21.4. *The completion of a nuclear group is nuclear.*

Proof. Let G be a nuclear group and let $\varPhi : G \to \prod G_U$ be an embedding into a product of complete metrizable nuclear groups (21.3). Obviously, $\overline{\varPhi(G)}$ is a closed subgroup of the complete nuclear group $\prod_U G_U$ and hence complete and nuclear (20.7).

Corollary 21.5. *If G is a complete nuclear group then α_G is an open isomorphism.*

Proof. According to 20.38, every complete metrizable nuclear group is reflexive and so is every product of reflexive groups (5.4). Hence G is a dually closed and dually embedded subgroup of a reflexive group. The assertion follows from 5.25.

Notes 21.6. The material of this chapter is new.

22. The Bochner Theorem for nuclear groups

The aim of this chapter is to prove a version of the Bochner Theorem for nuclear groups which generalizes the Bochner Theorem for locally compact abelian groups and the Minlos Lemma for nuclear vector spaces.

We use the notation introduced in 19.1.

Definition 22.1. Let G be an abelian topological group and let ν be a finite measure on a σ-algebra on G which makes all characters measurable. Then

$$\widehat{\nu} : G^* \to \mathbb{C}, \quad \chi \mapsto \int_G \overline{\chi}(x)\, \nu(dx),$$

is called the *Fourier transform* of ν.

For a finite measure μ on a σ-algebra of G^* which makes the point evaluations measurable,

$$\check{\mu} : G \to \mathbb{C}, \quad x \mapsto \int_{G^*} \chi(x)\, \mu(d\chi),$$

is called the *inverse Fourier transform* of μ.

Lemma 22.2. *Let L be a lattice in $\mathbb{R}^n$.*

(i) *For all $u, z \in \mathbb{R}^n$ such that $\|z\| \le 1$ and for all $t > 0$ we have $\varrho(\{x \in L+u : |x\cdot z| \ge t\}) < 2e^{-\pi t^2}\varrho(L)$.*

(ii) *If $\chi \in L^* \setminus \left(\frac{1}{4}B \cap L\right)^0$ then $\widehat{\sigma}_L(\chi) < 2e^{-\pi}$.*

Proof. (i) Let $x_0 := x \cdot z$. We deduce from

$$\begin{aligned}
\sum_{x\in L+u} e^{-\pi x^2}\frac{1}{2}(e^{2\pi t x_0} + e^{-2\pi t x_0}) &\le \frac{1}{2}e^{\pi t^2}\sum_{x\in L+u}(e^{-\pi(x-tz)^2} + e^{-\pi(x+tz)^2})\\
&= \frac{1}{2}e^{\pi t^2}\Big(\sum_{x\in L+u-tz} e^{-\pi x^2} + \sum_{x\in L+u+tz} e^{-\pi x^2}\Big)\\
&\overset{19.3(\mathrm{iii})}{\le} e^{\pi t^2}\varrho(L)
\end{aligned}$$

and

$$\sum_{x\in L+u} e^{-\pi x^2}\cosh(2\pi t x_0) \ge \sum_{x\in L+u:\,|x_0|\ge t} e^{-\pi x^2}\cosh(2\pi t x_0) \ge \cosh(2\pi t^2)\sum_{x\in L+u:\,|x_0|\ge t} e^{-\pi x^2}$$

that

$$\frac{\varrho(\{x\in L+u : |x\cdot z| \ge t\})}{\varrho(L)} \le \frac{e^{\pi t^2}}{\frac{1}{2}(e^{2\pi t^2} + e^{-2\pi t^2})} < 2e^{-\pi t^2}.$$

This proves (i).

(ii) Let $\chi \in L^* \setminus \left(\frac{1}{4}B \cap L\right)^0$. There exist $v \in \mathbb{R}^n$ such that $\chi(x) = e^{2\pi i x\cdot v}$ for all $x \in L$ (cf. [34], p. 366, (23.27)(b) and p. 362, (23.18)) and $y_0 \in \frac{1}{4}B \cap L$ such that $v \cdot y_0 \in \,]1/4, 3/4[\, + \mathbb{Z}$. So, for each $y \in \widehat{L} + v$, we have $|y_0 \cdot y| > 1/4$, which implies

$$\widehat{\sigma}_L(\chi) \overset{19.3\text{(ii)}}{=} \varphi_{\widehat{L}}(v) = \frac{1}{\varrho(\widehat{L})} \cdot \varrho(\{y \in \widehat{L} + v : |y \cdot (4y_0)| > 1\}) \overset{\text{(i)}}{<} 2e^{-\pi}.$$

This completes the proof.

Lemma 22.3. *Let D be an ellipsoid and let L be a lattice in $\mathbb{R}^n$. For each Borel probability measure μ on L^* and each $\varepsilon \in \,]0,1[$ such that $\operatorname{Re}\check{\mu}(x) \geq 1-\varepsilon$ for all $x \in L \cap D$, we have*

$$\mu\left(\left(\frac{1}{4}B \cap L\right)^0\right) \geq \frac{1 - 2e^{-\pi} - \varepsilon - 2\alpha(D)}{1 - 2e^{-\pi}}.$$

Proof. Put $Z := \left(\frac{1}{4}B \cap L\right)^0$. Since Z is symmetric, we may assume that μ is symmetric as well. This implies that $\check{\mu}$ is real-valued. We

$$\begin{aligned}\int_{L^*} \widehat{\sigma}_L(\chi)\,\mu(d\chi) &\overset{\text{Fubini}}{=} \int_L \check{\mu}(x)\,\sigma_L(dx) = \int_{L\cap D} \check{\mu}(x)\,\sigma_L(dx) + \int_{L\setminus D} \check{\mu}(x)\,\sigma_L(dx)\\ &\geq (1-\varepsilon)\sigma_L(D) - \sigma_L(L\setminus D) = (1-\varepsilon) - \sigma_L(L\setminus D)(2-\varepsilon)\\ &\geq 1 - \varepsilon - 2\alpha(D).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{L^*} \widehat{\sigma}_L(\chi)\,\mu(d\chi) &= \int_Z \widehat{\sigma}_L(\chi)\,\mu(d\chi) + \int_{L^*\setminus Z} \widehat{\sigma}_L(\chi)\,\mu(d\chi)\\ &\overset{22.2\text{(ii)}}{\leq} \mu(Z\cap L^*) + 2e^{-\pi}(1 - \mu(Z \cap L^*)) = \mu(Z\cap L^*)(1 - 2e^{-\pi}) + 2e^{-\pi}.\end{aligned}$$

The assertion follows.

Lemma 22.4 (Minlos Lemma for lattices). *Let p and q be pre-Hilbert seminorms on $\mathbb{R}^n$, let L be a lattice in $\mathbb{R}^n$, let μ be a Borel probability measure on L^*, and let $\varepsilon \in \,]0,1[$ be such that $\operatorname{Re}\check{\mu}(x) \geq 1-\varepsilon$ for all $x \in L \cap B_q$. Then*

$$\mu\left(\left(\frac{1}{4}B_p \cap L\right)^0\right) \geq 1 - 2\varepsilon - \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q)^2.$$

Proof. Put $Z := \left(\frac{1}{4}B_p \cap L\right)^0$. In the case when p is the Euclidean norm and q is an arbitrary pre-Hilbert norm, we have

$$\begin{aligned}\mu(Z) &\overset{22.3}{\geq} \frac{1 - 2e^{-\pi} - \varepsilon - 2\alpha(B_q)}{1 - 2e^{-\pi}} \overset{19.8,\,18.19}{\geq} 1 - \frac{\varepsilon + (1/\pi)\sum_{k=1}^n \mathrm{d}_k(B_p, B_q)^2}{1 - 2e^{-\pi}}\\ &> 1 - 2\varepsilon - \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p, B_q)^2,\end{aligned}$$

since $e^{-\pi} < 1/4$.

If both p and q are pre-Hilbert norms, there exists $A \in \mathrm{Gl}(n, \mathbb{R})$ such that $AB_p = B$; if we replace B_q by AB_q, L by $A\cdot L$ and μ by $(\varphi^*)^{-1}(\mu)$ where φ^* is the dual homomorphism of the topological isomorphism $\varphi : L \to A\cdot L$, $x \mapsto Ax$, then the assertion follows from the first case. (Observe that the inverse Fourier transform of $(\varphi^*)^{-1}(\mu)$ equals $\check{\mu} \circ \varphi^{-1}$.)

Now, suppose p and q are arbitrary. Let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ have the properties stated in 18.24. Since $B_{q_n} \subseteq B_q$, for each $\eta \in \,]0,1[$ we get

$$\begin{aligned}
\mu(Z) &\ge \mu\Big(\Big\{\chi \in L^* : \operatorname{Re}\chi(x) \ge 0 \ \forall x \in \bigcup_{n\in\mathbb{N}} B_{\eta\cdot p_n} \cap L\Big\}\Big)\\
&\ge \mu\Big(\bigcap_{n\in\mathbb{N}} \Big\{\chi \in L^* : \operatorname{Re}\chi(x) \ge 0 \ \forall x \in B_{\eta\cdot p_n}\Big\}\Big)\\
&= \lim_{n\in\mathbb{N}} \mu(\{\chi \in L^* : \operatorname{Re}\chi(x) \ge 0 \ \forall x \in B_{\eta\cdot p_n}\})\\
&\ge \sup_{n\in\mathbb{N}} \Big(1 - 2\varepsilon - \sum_{k=1}^{n} \mathrm{d}_k(B_{\eta\cdot p_n}, B_{q_n})\Big) \ge 1 - 2\varepsilon - \frac{1}{\eta}\sum_{k=1}^{n} \mathrm{d}_k(B_p, B_q).
\end{aligned}$$

This completes the proof, since η was arbitrary.

Corollary 22.5. *Let K be a finitely generated subgroup of a vector space E and let p and q be pre-Hilbert seminorms on E. For each Borel probability measure μ on the character group of the discrete group $(K,\mathcal{D})$ and each $\varepsilon \in \,]0,1[$ such that* $\operatorname{Re}\check{\mu}(x) \ge 1-\varepsilon$ *for all $x \in K \cap B_q$, we have*

$$\mu\bigg(\bigg\{\chi \in (K,\mathcal{D})^* : \operatorname{Re}\chi(x) \ge 0 \ \forall x \in K \cap \frac{1}{4}B_p\bigg\}\bigg) \ge 1 - 2\varepsilon - \sum_{k\in\mathbb{N}} \mathrm{d}_k^2(B_p, B_q).$$

Proof. We may assume that $\mathrm{d}_1(B_p,B_q) < \infty$. Let $(x_1,\dots,x_n)$ be a generating system of K. We define a homomorphism $f : \mathbb{R}^n \to E$ by $f(e_j) := x_j$ (where $(e_1,\dots,e_n)$ is the standard basis of $\mathbb{R}^n$). Obviously, $f(\mathbb{R}^n) = \langle K\rangle_{\mathbb{R}}$. Let $f_K : \mathbb{Z}^n \to K$ denote the induced epimorphism. Then $\widetilde{p} := p\circ f$ and $\widetilde{q} := q \circ f$ are pre-Hilbert seminorms on $\mathbb{R}^n$ satisfying $B_{\widetilde p} = f^{-1}(B_p)$, $B_{\widetilde q} = f^{-1}(B_q)$ and $\mathrm{d}_k(B_{\widetilde p}, B_{\widetilde q}) \overset{18.5(\mathrm{ii})}{=} \mathrm{d}_k(B_p \cap \langle K\rangle_{\mathbb{R}}, B_q \cap \langle K\rangle_{\mathbb{R}}) \overset{18.18}{\le} \mathrm{d}_k(B_p,B_q)$. Put $\nu := f_K^*(\mu)$. An easy computation shows that $\check{\nu} = \check{\mu}\circ f_K$ (cf. (11.2) in [8]) and hence $\operatorname{Re}\check{\nu}(x) \ge 1-\varepsilon$ for all $x \in \mathbb{Z}^n \cap B_{\widetilde q}$. Lemma 22.4 implies that $\nu\big(\{\chi \in (\mathbb{Z}^n)^* : \operatorname{Re}\chi(x) \ge 0 \ \forall x \in \mathbb{Z}^n \cap \frac14 B_{\widetilde p}\}\big) \ge 1-2\varepsilon - \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_{\widetilde p}, B_{\widetilde q})^2 \ge 1-2\varepsilon-\sum_{k\in\mathbb{N}}\mathrm{d}_k(B_p,B_q)^2$. The assertion follows from

$$\begin{aligned}
(f_K^*)^{-1}\big(\{\chi \in (\mathbb{Z}^n)^* : \operatorname{Re}\chi(x) \ge 0 \ \forall x \in \mathbb{Z}^n \cap \tfrac14 B_{\widetilde p}\}\big)&\\
&= \big\{\psi \in K_{\mathrm{d}}^* : \operatorname{Re}(\psi\circ f_K)(x) \ge 0 \ \forall x \in \mathbb{Z}^n \cap \tfrac14 B_{\widetilde p}\big\}\\
&= \big\{\psi \in K_{\mathrm{d}}^* : \operatorname{Re}\psi(y) \ge 0 \ \forall y \in K \cap \tfrac14 B_p\big\}.
\end{aligned}$$

(Note that $f_K\big(\mathbb{Z}^n \cap \frac14 B_{\widetilde p}\big) = K \cap \frac14 B_p$.)

Definition 22.6. Let X be a Hausdorff space and let μ be a finite measure on a σ-algebra $\mathcal{A}$ which contains the Borel σ-algebra.

(i) μ is called *regular* if for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exists a closed set $F \subseteq A$ such that $\mu(A\setminus F) < \varepsilon$.

(ii) μ is called a *Radon measure* if for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exists a compact subset $K \subseteq A$ such that $\mu(A \setminus K) < \varepsilon$.

(iii) If $X = G^*$ is the character group of an (abelian) topological group G then μ is called *strongly regular* if for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exists a compact equicontinuous subset $K \subseteq A$ with $\mu(A \setminus K) < \varepsilon$.

Remark 22.7. (i) If G is an abelian topological group such that α_G is continuous, then every compact subset of G^* is equicontinuous (5.10) and hence every Radon measure on G^* is strongly regular. (The converse is always true.)

(ii) A measure μ on G^* is strongly regular if and only if it is a Radon measure and $\sup\{\mu(K) : K \subseteq G^* \text{ is compact and equicontinuous}\} = \mu(G^*)$.

(iii) Let $f : X \to Y$ be a continuous mapping between the Hausdorff spaces X and Y. For every Radon measure μ on $\mathcal{B}(X)$, the image measure $f(\mu)$ is a Radon measure on $\mathcal{B}(Y)$.

Corollary 22.8. *Let p and q be pre-Hilbert seminorms on a vector space E and let K be an arbitrary subgroup of E. For each Radon probability measure μ on $(K,\mathcal{D})^*$ and every $\varepsilon \in \,]0,1[$ such that $\operatorname{Re}\check{\mu}(x) \geq 1-\varepsilon$ for all $x \in K \cap B_q$, we have $\mu(Z_K) \geq 1 - 2\varepsilon - \sum_{k\in\mathbb{N}} \mathrm{d}_k(B_p,B_q)^2$ for $Z_K := \big(K \cap \frac{1}{4}B_p\big)^0$.*

Proof. Let U be an open set containing Z_K and let $\mathcal{F}$ be the set of all finitely generated subgroups of K. For $Z_L := (K,\mathcal{D})^* \cap \big(L \cap \frac{1}{4}B_p\big)^0$ (and $L \in \mathcal{F}$) we have $\bigcap_{L\in\mathcal{F}} Z_L = Z_K \subseteq U$. Since Z_L is compact (being a closed subset of the compact space $(K,\mathcal{D})^*$) and since $L_1 \leq L_2$ implies $Z_{L_2} \subseteq Z_{L_1}$, there is an $L \in \mathcal{F}$ such that $Z_L \subseteq U$ (1.27). Since μ is a Radon measure, it suffices to show that $\mu(Z_L) \geq 1 - 2\varepsilon - \sum_{k\in\mathbb{N}} \mathrm{d}_k^2(B_p,B_q)$. If $\iota_L : L \to K$ denotes the embedding (where both groups are endowed with the discrete topology) and $\mu_L := \iota_L^*(\mu)$ then μ_L satisfies the hypotheses of 22.5 (22.7(iii)). Hence, $\mu_L\big(\{\chi \in (L,\mathcal{D})^* : \operatorname{Re}\chi(x) \geq 0\ \forall x \in L \cap \frac{1}{4}B_p\}\big) \geq 1 - 2\varepsilon - \sum_{k\in\mathbb{N}} \mathrm{d}_k^2(B_p,B_q)$. Since $Z_L = (\iota_L^*)^{-1}\big(\{\chi \in (L,\mathcal{D})^* : \operatorname{Re}\chi(x) \geq 0\ \forall x \in L \cap \frac{1}{4}B_p\}\big)$ (see 2.3), the proof is complete.

Remark 22.9. Let $\mathcal{E}$ be a generator of a σ-algebra $\mathcal{A}$ on Ω and let Ω_0 be a subset of Ω. Then the σ-algebra $\Omega_0 \cap \mathcal{A}$ is equal to the σ-algebra generated by $\Omega_0 \cap \mathcal{E}$ where the intersections are defined "pointwise".

["$\supseteq$" trivially holds. Since $\mathcal{E} \subseteq \{A \in \mathcal{A} : A \cap \Omega_0 \in \sigma(\Omega_0 \cap \mathcal{E})\}$ and the latter set is a σ-algebra, the other inclusion follows.]

For every (abelian) topological group G we deduce from the definition of the pointwise-open topology that the following Borel σ-algebras coincide: $\mathcal{B}(G_{\mathrm{d}}^*) \cap G_{\mathrm{p}}^* = \mathcal{B}(G_{\mathrm{p}}^*)$.

Lemma 22.10. *For every closed subgroup H of an abelian topological group G we have $(G/H)_{\mathrm{p}}^* \cong H_{\mathrm{p}}^\perp$.*

Proof. Let $\pi : G \to G/H$ denote the canonical projection. We have to prove that $\pi^* : (G/H)_{\mathrm{p}}^* \to G_{\mathrm{p}}^*$ is an embedding with image $H^\perp$. It is clear that π^* is injective (5.16) and that $\pi^*((G/H)^*) = H^\perp$. For $x_1,\ldots,x_n \in G$ and an open subset $V \subseteq \mathbb{T}$, we get $\pi^*(P(\{x_1+H,\ldots,x_n+H\},V)) = \{\chi \in H^\perp : \chi(x_j) \in V \text{ for } j = 1,\ldots,n\}$. This completes the proof.

Lemma 22.11 (Prokhorov). *Let G be an abelian topological group and let $\mathcal{T}$ be a group topology on G^* which is finer than the pointwise-open topology. If μ is a Radon probability measure on G_{p}^* such that for each $\varepsilon > 0$, there is a compact subset of $(G^*,\mathcal{T})$ with $\mu(G^* \setminus K) < \varepsilon$ then there exists a unique extension of μ to a Radon measure on the Borel σ-algebra of $(G,\mathcal{T})$.*

Proof. This is an easy consequence of Theorem 3.4 in [84], p. 39.

Proposition 22.12. *Let G be an abelian topological group, and let $\mathcal{T}$ be a group topology on G^* which is finer than the topology of pointwise convergence. Then the inverse Fourier transform of every strongly regular measure μ on the Borel σ-algebra of $(G^*, \mathcal{T})$ is a continuous positive definite function on G.*

Proof. In ([35], p. 291, (33.1)) it is proved that $\check{\mu}$ is positive definite.

It suffices to show that $\check{\mu}$ is continuous at e. We may assume that $\mu(G^*) = 1$. By hypothesis, for each $\varepsilon > 0$, there exists a compact equicontinuous set $K \subseteq G^*$ such that $\mu(G^* \setminus K) < \varepsilon$. Furthermore, we can find a neighbourhood $U \in \mathcal{U}_G(e)$ such that $|1 - \chi(x)| < \varepsilon$ for all $x \in U$ and $\chi \in K$. The assertion follows from

$$\begin{aligned} |\check{\mu}(e) - \check{\mu}(x)| &= \Big| \int_{G^*} (1 - \chi(x))\, \mu(d\chi) \Big| \\ &\le \int_K |1 - \chi(x)|\, \mu(d\chi) + 2\mu(G^* \setminus K) \le \varepsilon + 2\varepsilon \quad (x \in U). \end{aligned}$$

Lemma 22.13. *Let G be an abelian topological group and let $\mathcal{T}$ be a topology on G^* which makes the point evaluations continuous. Two Radon measures on the Borel σ-algebra of $(G^*, \mathcal{T})$ whose Fourier transforms coincide, are equal.*

Proof. This is Proposition (11.3) in [8].

Definition 22.14. A Hausdorff group topology $\mathcal{T}$ on the character group G^* of an (abelian) topological group G is called *admissible* if $\mathcal{T}$ is finer than the pointwise-open topology and has the property that for each $U \in \mathcal{U}_G(e)$ the polar U^0 is compact in $\mathcal{T}$.

Example 22.15. It is a consequence of 3.5 that, for an abelian Hausdorff group G, the topology of uniform convergence on all finite or compact or totally bounded subset of G is an admissible topology for G^*.

Theorem 22.16 (Bochner Theorem for nuclear groups). *Let G be a nuclear group and let $\mathcal{T}$ be an admissible topology on G^*. Then $\mu \mapsto \check{\mu}$ is a bijection from the set of all strongly regular measures on the Borel σ-algebra of $(G^*, \mathcal{T})$ onto the set of all continuous positive definite functions on G.*

Proof. Because of 22.12 and 22.13, it suffices to prove that every continuous positive definite function φ on G is the inverse Fourier transform of a strongly regular measure on G.

By 20.32, we may assume $G = H/K$ where K is a closed subgroup of H and H is a subgroup of a nuclear vector group F. Without loss of generality, we may assume $\varphi(0) = 1$. Then $\widetilde{\varphi} := \varphi \circ \pi$, where $\pi : H \to G$ is the canonical projection, is a continuous positive definite function on H. By the Bochner Theorem (for discrete groups), there exists a Radon probability measure ν on $(H, \mathcal{D})^*$ such that $\check{\nu} = \widetilde{\varphi}$.

Since $\widetilde{\varphi}$ is continuous, for each $\varepsilon > 0$ there exist: a subspace $F_0 \le F$ and pre-Hilbert seminorms p and q on F_0 satisfying $B_p, B_q \in \mathcal{U}_F(0)$, $\sum_{k \in \mathbb{N}} \mathrm{d}_k^2(B_p, B_q) < \varepsilon/4$, and $\operatorname{Re} \widetilde{\varphi}(x) \ge 1 - \varepsilon$ for all $x \in H \cap B_q$ (18.32 and 20.2). Put $H_0 := H \cap F_0$ and $\varphi_0 := \widetilde{\varphi}|_{H_0}$. Let $\iota : H_0 \to H$ denote the inclusion and let $\iota^* : (H, \mathcal{D})^* \to (H_0, \mathcal{D})^*$ be the dual homomorphism. For $\nu_0 := \iota^*(\nu)$ we get $\check{\nu_0} = \check{\nu}|_{H_0}$. Corollary 22.8 (and 22.7(iii) and

18.2(ii)) imply

$$(*)\qquad \begin{aligned} 1-3\varepsilon &\le \nu_0(\{\psi \in (H_0,\mathcal{D})^* : \operatorname{Re}\psi(x) \ge 0\ \forall x \in H_0 \cap B_p\}) \\ &= \nu(\{\chi \in (H,\mathcal{D})^* : \operatorname{Re}\chi(x) \ge 0\ \forall x \in H \cap B_p\}) \end{aligned}$$

because of $H_0 \cap B_p = H \cap B_p$ and 2.3. Since $B_p^0 := \{\chi \in (H,\mathcal{D})^* : \operatorname{Re}\chi(x) \ge 0\ \forall x \in H \cap B_p\}$ consists of continuous homomorphisms only (observe that $H \cap B_p \in \mathcal{U}_H(0)$ and 3.4(ii)), and $\varepsilon > 0$ was arbitrary, there exists a measurable set $A \subseteq H^*$ with $\nu(A) = 1$. Hence H^* belongs to the completion of $\mathcal{B}((H,\mathcal{D})^*)$ with respect to ν (where H^* is the set of all continuous (!) characters). Let $\widetilde{\mu}$ denote the restriction to $\mathcal{B}(H_{\mathrm{p}}^*) = H^* \cap \mathcal{B}(H_{\mathrm{d}}^*)$ (22.9) of the extension of ν onto the completion. For $B \in \mathcal{B}((H,\mathcal{D})^*)$, there is a compact subset $K \subseteq B$ such that $\nu(B \setminus K) < \varepsilon$. Condition $(*)$ implies $\widetilde{\mu}((B \cap H^*) \setminus (K \cap B_p^0)) \le \nu(B \setminus K) + \nu(H_{\mathrm{d}}^* \setminus B_p^0) \le 4\varepsilon$. Hence $\widetilde{\mu}$ is a Radon measure. Furthermore, $\check{\widetilde{\mu}}(x) = T_{H^*}\, \chi(x)\, \widetilde{\mu}(d\chi) = T_{(H,\mathcal{D})^*}\, \chi(x)\, \nu(d\chi) = \widetilde{\varphi}$.

If $j : K \to H$ denotes the canonical embedding and $j^* : H_{\mathrm{p}}^* \to K_{\mathrm{p}}^*$ the dual homomorphism where the character groups are endowed with the pointwise-open topology, then the Fourier transform of $j^*(\widetilde{\mu})$ equals $\check{\widetilde{\mu}}|_K \equiv 1$. Since $j^*(\widetilde{\mu})$ is a Radon measure (22.7(iii)), 22.13 yields $j^*(\widetilde{\mu}) = \varepsilon_0$ and $\widetilde{\mu}(K^\perp) = \widetilde{\mu}((j^*)^{-1}(\{0\})) = \varepsilon_0(\{0\}) = 1$. Hence $\widetilde{\mu}$ may be considered as a Radon probability measure on $K^\perp$. (Observe that $K^\perp$ is closed.) According to 22.10, there is a Radon measure μ' on $\mathcal{B}((H/K)_{\mathrm{p}}^*)$ such that $\pi^*(\mu') = \widetilde{\mu}$. It follows from $\check{\mu}' \circ \pi = \check{\widetilde{\mu}} = \varphi \circ \pi$ that $\check{\mu}' = \varphi$. Since $(\pi^*)^{-1}((H \cap B_p)^0) = \{\chi \in (H/K)_{\mathrm{p}}^* : \operatorname{Re}\chi(x) \ge 0\ \forall x \in \pi(H \cap B_p)\} =: C$, condition $(*)$ implies

$$(**)\qquad \mu'(C) \ge 1 - 3\varepsilon.$$

Since C is a compact subset of $(G^*, \mathcal{T})$, it is a consequence of 22.11 and $(**)$ that μ' can be extended to a Radon probability measure μ on the Borel σ-algebra of $(G^*, \mathcal{T})$.

The equicontinuity of the set C, $(**)$ and 22.7(ii) imply that μ is strongly regular. It is easy to see that $\check{\mu} = \varphi$, which shows that μ is the desired measure.

COROLLARY 22.17 (Minlos Lemma). *For a nuclear vector space V, the inverse Fourier transform $\mu \mapsto \check{\mu}$ is a bijection of the set of all strongly regular measures on V' (endowed with the compact-open topology) onto the set of all continuous positive definite functions on G.*

PROOF. This is a direct consequence of 5.5, 20.6(i) and the above theorem.

COROLLARY 22.18. *Let G be a reflexive group such that G^* is nuclear. Then $\mu \mapsto \widehat{\mu}$ is a bijection of the set of all Radon measures on G onto the set of all continuous positive definite functions on G^* where G^* is endowed with the compact-open topology.*

PROOF. It follows from 5.9 that α_{G^*} is continuous. Hence, we can conclude from 22.7(i) and 22.16 that the mapping $\nu \mapsto \check{\nu}$ of all Radon measures on G^{**} onto the set of all continuous positive definite functions on G^* is a bijection. An easy computation shows that, for a (Radon) measure μ on G, we get $\check{\alpha}_G(\mu) = \overline{\widehat{\mu}}$. This implies the assertion.

EXAMPLES 22.19. Every Čech-complete nuclear group and, in particular, every locally compact abelian group is a reflexive group having the property that the dual group is nuclear as well (20.36 and 20.40).

Moreover, $\mathbb{R}^{(I)}$ is (for an arbitrary non-empty index set I) a reflexive group (5.4) and its character group, being topologically isomorphic to $\mathbb{R}^I$, is nuclear.

Corollaries 22.17 and 22.18 and Examples 22.19 show that the Bochner Theorem for nuclear groups generalizes the usual Bochner Theorem for locally compact abelian groups and the Minlos Lemma for nuclear vector spaces.

NOTES 22.20. 22.2 to 22.4 are taken from [12]. I am indebted to W. Banaszczyk for stimulating suggestions which enabled me to simplify his proof of the Bochner Theorem ((12.2) in [8], 22.16). This approach no longer uses the technique established in [43].

References

[1] I. Amemiya and Y. Kōmura, *Über nicht-vollständige Montelräume*, Math. Ann. 177 (1968), 273–277.

[2] R. Arens, *Duality in linear spaces*, Duke Math. J. 14 (1947), 787–794.

[3] A. V. Arkhangel'skiĭ, *Open and near open mappings. Connections between spaces*, Trans. Moscow Math. Soc. 15 (1966), 204–250.

[4] —, *Linear homeomorphisms of function spaces*, Soviet Math. Dokl. 25 (1982), 852–855.

[5] M. Banaszczyk and W. Banaszczyk, *Characterization of nuclear spaces by means of additive subgroups*, Math. Z. 186 (1994), 125–133.

[6] W. Banaszczyk, *On the existence of exotic Banach–Lie groups*, Math. Ann. 264 (1983), 485–493.

[7] —, *Countable products of LCA groups: their closed subgroups, quotients and duality properties*, Colloq. Math. 59 (1990), 53–57.

[8] —, *Additive Subgroups of Topological Vector Spaces*, Springer, Berlin, 1991.

[9] —, *New bounds in some transference theorems in the geometry of numbers*, Math. Ann. 296 (1993), 625–635.

[10] —, *Rearrangement of series in nonnuclear spaces*, Studia Math. 107 (1993), 213–222.

[11] —, *Inequalities for convex bodies and polar reciprocal lattices in* $\mathbb{R}^n$, Discrete Comput. Geom. 13 (1995), 217–231.

[12] —, *The Minlos lemma for positive-definite functions on additive subgroups of* $\mathbb{R}^n$, Studia Math. 126 (1997), 13–25.

[13] W. Banaszczyk, M. J. Chasco and E. Martín-Peinador, *Open subgroups and Pontryagin duality*, Math. Z. 215 (1994), 195–204.

[14] W. Banaszczyk and E. Martín-Peinador, *The Glicksberg theorem on weakly compact sets for nuclear groups*, Ann. New York Acad. Sci. 788 (1996), 34–39.

[15] —, —, *Weakly pseudocompact subsets of nuclear groups*, J. Pure Appl. Algebra 138 (1999), 99–106.

[16] N. Bourbaki, *Espaces vectoriels topologiques*, chap. 2, Hermann, Paris, 1953.

[17] —, *General Topology I*, Hermann, Paris, 1966.

[18] —, *General Topology II*, Hermann, Paris, 1966.

[19] L. G. Brown, *Topologically complete groups*, Proc. Amer. Math. Soc. 35 (1972), 593–600.

[20] M. M. Bruguera, *Some properties of locally quasi-convex groups*, Topology Appl. 77 (1997), 87–94.

[21] M. J. Chasco, *Pontryagin duality for metrizable groups*, Arch. Math. (Basel) 70 (1998), 22–28.

[22] J. Dieudonné, *Espaces uniformes complets*, Ann. Sci. Éc. Norm. Sup. 56 (1939), 277–291.

[23] —, *La dualité dans les espaces vectoriels topologiques*, ibid. 59 (1942), 107–139.

[24] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York, 1957.

[25] W. Ebeling, *Lattices and Codes*, Vieweg, Braunschweig, 1994.

[26] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[27] T. H. Fay, E. T. Ordman and B. V. S. Thomas, *The free topological group over the rationals*, Topology Appl. 10 (1979), 33–47.

[28] O. Forster, *Analysis 3*, Vieweg, Braunschweig, ^{3}1984.

[29] A. Frölicher, *Zur Dualitätstheorie kompakt erzeugter und lokalkonvexer Vektorräume*, Comment. Math. Helv. 47 (1972), 289–310.

[30] I. Glicksberg, *Uniform boundedness for groups*, Canad. J. Math. 14 (1962), 269–276.

[31] M. I. Graev, *Free topological groups*, Amer. Math. Soc. Transl. Ser. 1 (8), 305–364.

[32] A. Grothendieck, *Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une classe remarquable d'espaces vectoriels liée à cette notion*, C. R. Acad. Sci. Paris 233 (1951), 1556–1558.

[33] H. Heuser, *Funktionalanalysis*, Teubner, Stuttgart, ^{2}1986.

[34] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis Vol. I*, Springer, Berlin, ^{2}1979.

[35] —, —, *Abstract Harmonic Analysis, Vol. II*, Springer, Berlin, 1970.

[36] F. Hirzebruch and W. Scharlau, *Einführung in die Funktionalanalysis*, BI Wissenschaftsverlag, Mannheim, 1971.

[37] M. J. Kadets and M. G. Snobar, *Certain functionals on the Minkowski compactum*, Mat. Zametki 10 (1971), 453–457 (in Russian).

[38] V. M. Kadets, *A problem of S. Banach* (*problem* 106 *from "The Scottish Book"*), Funktsional. Anal. i Prilozhen. 20 (1986), 74–75 (in Russian); English transl.: Functional Anal. Appl. 20 (1986), 317–319.

[39] E. R. van Kampen, *Locally bicompact abelian groups and their character groups*, Ann. of Math. 36 (1935), 448–463.

[40] S. Kaplan, *Extension of Pontrjagin duality I: infinite products*, Duke Math. J. 15 (1948), 649–658.

[41] P. Kenderov, *Topological vector groups*, Math. USSR-Sb. 10 (1970), 531–546.

[42] M. A. Khan, *Chain conditions on subgroups of LCA groups*, Pacific J. Math. 86 (1980), 517–534.

[43] J. Kisyński, *On the generation of tight measures*, Studia Math. 30 (1968), 141–151.

[44] G. Koethe, *Topologische lineare Räume*, Springer, Berlin, ^{2}1966.

[45] Y. Kōmura, *Some examples on linear topological spaces*, Math. Ann. 153 (1964), 150–162.

[46] R. D. Kulkarni, *On one-parameter subgroups in the dual of a topological group*, J. Indian Math. Soc. 53 (1988), 67–73.

[47] S.-H. Kye, *Pontryagin duality in real linear topological spaces*, Chinese J. Math. 12 (1984), 129–136.

[48] S. Lang, *Algebra*, Addison-Wesley, Reading, Massachusetts, ^{3}1993.

[49] H. Leptin, *Zur Dualitätstheorie projektiver Limites abelscher Gruppen*, Abh. Math. Sem. Univ. Hamburg 19 (1955), 264–268.

[50] P. Lurje, *Über topologische Vektorgruppen*, Dissertation, München, 1972.

[51] —, *Tonneliertheit in lokalkonvexen Vektorgruppen*, Manuscripta Math. 14 (1974), 107–121.

[52] J. Mack, S. A. Morris and E. T. Ordman, *Free topological groups and the projective dimension of a locally compact abelian group*, Proc. Amer. Math. Soc. 40 (1973), 303–308.

[53] M. M. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR 9 (1945), 3–64 (in Russian); English transl.: Amer. Math. Soc. Transl. Ser. 1 (8), 195–272.

[54] S. Mazurkiewicz, *Sur un ensemble G_δ punctiforme, qui n'est pas homéomorphe avec aucun ensemble linéaire*, Fund. Math. 1 (1920), 61–81.

[55] S. A. Morris, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*, Cambridge Univ. Press, Cambridge, 1977.

[56] L. Nachbin, *Topological vector spaces of continuous functions*, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 471–474.

[57] P. Nickolas, *Reflexivity of topological groups*, Proc. Amer. Math. Soc. 65 (1977), 137–141.

[58] N. Noble, *k-groups and duality*, Trans. Amer. Math. Soc. 151 (1970), 551–561.

[59] A. Okuyama and T. Terada, *Function spaces*, in: Topics in General Topology, K. Morita and J. Nagata (eds.), North-Holland, Amsterdam, 1989, 411–458.

[60] B. A. Pasynkov, *Almost metrizable topological groups*, Soviet Math. Dokl. 6 (1965), 404–408.

[61] —, *On topological groups*, ibid. 10 (1969), 1115–1118.

[62] G. K. Pedersen, *Analysis Now*, Springer, New York, 1988.

[63] V. Pestov, *Free abelian topological groups and the Pontryagin–van Kampen duality*, Bull. Austral. Math. Soc. 52 (1995), 297–311.

[64] A. Pietsch, *Nuclear Locally Convex Spaces*, Springer, Berlin, 1972.

[65] —, *Operator Ideals*, North-Holland, Amsterdam, 1980.

[66] L. Pontrjagin, *The theory of topological commutative groups*, Ann. of Math. 35 (1934), 361–388.

[67] —, *Selected Works. Vol.* 2: *Topological Groups*, Gordon and Breach, New York, ^{3}1986.

[68] D. A. Raĭkov, *Free locally convex spaces for uniform spaces*, Mat. Sb. 63 (105) (1964), 582–590 (in Russian).

[69] W. Roelke and S. Dierolf, *Uniform Structures on Topological Groups and their Quotients*, McGraw-Hill, New York, 1981.

[70] S. Rolewicz, *Metric Linear Spaces*, PWN–Polish Scientific Publishers, Warszawa, 1972.

[71] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, ^{3}1987.

[72] —, *Fourier Analysis on Groups*, Wiley, New York, 1990.

[73] —, *Functional Analysis*, McGraw-Hill, New York, ^{2}1991.

[74] H. H. Schaefer, *Topological Vector Spaces*, Macmillan, New York, 1966.

[75] T. Shirota, *On locally convex vector spaces of continuous functions*, Proc. Japan Acad. 30 (1954), 294–298.

[76] W. Sierpiński, *Sur un ensemble punctiforme connexe*, Fund. Math. 1 (1920), 7–10.

[77] M. F. Smith, *The Pontrjagin duality theorem in linear spaces*, Ann. of Math. (2) 56 (1952), 248–253.

[78] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[79] B. V. S. Thomas, *Free topological groups*, Topology Appl. 4 (1974), 51–72.

[80] M. G. Tkachenko, *On completeness of free abelian topological groups*, Soviet Math. Dokl. 27 (1983), 341–345.

[81] G. Turnwald, *Duality theory of abelian topological groups*, in: Infinite Dimensional Harmonic Analysis, Transactions of a German–Japanese Symposium 1995, H. Heyer and T. Hirai (eds.), Gräbner, Hamburg, 1996, 224–232.

[82] —, *On the continuity of the evaluation mapping associated with a group and its character group*, Proc. Amer. Math. Soc. 126 (1998), 3413–3415.

[83] V. V. Uspenskiĭ, *On the topology of a free locally convex space*, Soviet Math. Dokl. 27 (1983), 781–785.

[84] N. N. Vakhania, V. I. Tarieladze and S. A. Chobanyan, *Probability Distributions on Banach Spaces*, Reidel, Dordrecht, 1987.

[85] N. Ya. Vilenkin, *The theory of characters of topological Abelian groups with a given boundedness*, Izv. Akad. Nauk SSSR 15 (1951), 439–462 (in Russian).

[86] S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. 25 (1958), 265–282.

[87] R. Whitley, *The Krein–Smulian theorem*, Proc. Amer. Math. Soc. 97 (1986), 376–377.

[88] K. Yosida, *Functional Analysis*, Springer, Berlin, ^{5}1978.