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The generalized equations of Riccati and their applications to the theory of linear differential equations

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Introduction

Because of its relations to second order equations, the equation of Riccati plays a special role in the theory of equations. In the present paper we shall define a differential equation of the n-th order which will be called the *Riccati equation* (or shortly: the *R equation*). This name is justified for the following two reasons: 1) in the case where the order of the equation is $n = 1$, we obtain the classical equation of Riccati; 2) these equations and the linear differential equations are related in the same way as the equation of Riccati of the first order and the linear equation of the second order.

The decomposition of the left-hand side of a linear differential equation (an l. d. equation) into the product of symbolic factors leads to the Riccati equation. The method of decomposition of an l. d. expression into symbolic factors has been introduced in the theory of equations by G. Floquet (1). It has also found application in the papers of Grünfeld [4], and later on, independently of that, in the papers of G. Mammana [7] and [9] and M. Nicolesco [8]. In the latter papers the decomposition of the left-hand side of an l. d. equation is determined by linearly independent particular solutions of that equation. In the present paper this decomposition is determined by the particular solution of the generalized non-linear Riccati equation. Far-reaching differences and consequences follow from this approach.

Thus, for instance, one might expect that the method of decomposition of the left-hand side of an equation, in algebra as well as in the theory of differential equations, should lead to the lowering of the order of that equation. But in the classical approach, in order to perform an effective decomposition of the left-hand side of an l. d. equation, one has to know its full solution beforehand. It follows that the method of decomposition fails as a method of effective solution of differential equations. It is for this reason that this method has not played an insignificant part in the theory of equations.

The problem is different if the decomposition of the left-hand side of an equation does not depend upon the solution of that equation. This aim is achieved by introducing Riccati equations, since we find that

(1) Quoted after [1], § 24.

each particular solution of an equation of Riccati determines completely the decomposition of the corresponding linear equation.

Although a Riccati equation is non-linear, and hence, in general, more difficult to solve than a linear equation, nevertheless the reduction of the problem of decomposition of a linear equation to the problem of finding a solution of a non-linear equation, proves to be useful.

This results from the following facts:

1) In order to perform the decomposition of the left-hand side of a linear equation, one does not need to know the general solution of the equation of Riccati; it is enough to know any one of its particular solutions.

2) For a certain category of linear equations the Riccati equations "degenerate" to algebraic equations.

3) It proves possible to specify the class of linear differential equations decomposable in the elementary way, i. e. equations which can be decomposed into factors without solving a differential equation.

4) The theory of linear equations based upon the Riccati equations has certain theoretical advantages. The formulas for the general solutions of l. d. equations in which the solutions of Riccati equations appear (of which we know only that they exist) simplify in some cases the proofs of the theorems in the theory of equations.

In the discussion of the applications of the Riccati equations to the theory of linear equations, it is also necessary to mention the opposite relations, which enable us to use the theory of linear equations for non-linear Riccati equations. It will be seen that the solution of linear equation is equivalent – in a certain sense – to the solution of a non-linear equation of Riccati. This fact is well known in the case of the classical Riccati equation (of the first order). It gives the reduction of a more difficult problem to an easier one.

From this point of view one can also consider some partial differential equations. Without attempting a full treatment of the subject, we shall give some examples of application of the method of symbolic decomposition of a differential expression. There are partial differential equations which can be reduced in this way to partial differential equations of a lower order, in turn, can be reduced to ordinary ones.

The contents of the paper are the following: In paragraph 1 it is shown that the decomposition of the left-hand side of an l. d. equation of $(n+1)$-st order (1.1) into the symbolic product of operator factors (1.2) is determined by every one of the particular solutions of the R equation of n-th order (theorem 1). We give the operator form (1.6) of this equation, and also the recursive method of obtaining these equations in an explicit form (1.8).

Paragraph 2 contains two theorems concerning the existence of the solutions of R equations. Because of the relations of non-linear R equations and linear equations, it has been possible to show that the solutions of R equations exist in the whole interval in which its coefficients satisfy certain assumptions (theorems 2 and 3). From the same assumptions it follows that, having the general solution of an l. d. equation of the $(n+1)$-st order, one can determine $n+1$ particular solutions of the corresponding R equation.

Paragraph 3, conversely, presents the relations between the solutions of R equations and the solutions of the connected linear equations (theorems 4 and 6). According to those theorems, each k "significantly different" solutions of an R equation (definition 2) allows us to lower the order of the corresponding linear equation by the number k. It is also shown that the general solution of the linear equation can be built up from particular solutions of some system of R equations, which can be assigned to the given linear equation (theorem 5). Based upon these theorems, two methods of solving linear equations are presented (methods A and B).

In paragraph 4, the theory of R equations is applied to some problems which are already solved, in order to show the advantages and the methodological simplifications. In addition, it is proved that the only equations of the fourth order whose solution can be reduced to solving the characteristic (algebraic) equation are the generalized Euler equation and the equation with constant coefficients (theorem 7).

In paragraph 5, the definition of so-called elementarily decomposable (class E, definition 3) linear equations is given. In order to perform decomposition (1.2) for such an equation (and the following lowering of its order) one does not need to solve a differential equation. This class comprises linear equations, and equations with coefficients of the Euler type, i. e. equations which are completely solvable, but this class is much more general. The criterion of elementary decomposibility (5.1) and an example of an equation from class E (Example 1) are given. Theorem 8 shows that a linear equation which does not belong to class E can be made elementarily decomposable by changing only one of its coefficients. Finally it is shown by examples how the notion of elementary decomposability can be used in the theory of linear equations.

The last paragraph (paragraph 6) provides an example of the application of the method of decomposition of a linear differential expression into the product of operator factors for some special partial differential equations.

1. Definition of the Riccati equation of the n-th order

We shall introduce certain symbols and assumptions. Suppose we are given a function $a_i(x)$ defined in the interval $a < x < b$ and differentiable as many times as is necessary for our considerations. We shall consider the linear operator:

$$l_i = \frac{d}{dx} + a_i(x).$$

If the function $f(x)$ is defined in the same interval and differentiable, then the symbolic product $l_i f(x)$ will denote the function

$$l_i f(x) = \frac{df}{dx} + a_i f.$$

The expression $l_i^m f$, where m is a positive integer (or zero) and $f(x)$ is defined in the interval (a, b) and belongs to the class C^m, is defined recursively in the following way:

$$l_i^0 f(x) = f(x), \quad l_i f(x) = \frac{df}{dx} + a_i f, \quad l_i^m f(x) = l_i[l_i^{m-1} f(x)].$$

The symbol $l_i^m f(x)$ defines a function from the class C^0 provided that the functions $a_i(x)$ and $f(x)$ belong to the classes C^{m-1} and C^m respectively.

Let us consider the following linear differential equation of the $(n+1)$-st order:

$$(1.1) \qquad L_{n+1}[y] \equiv y^{(n+1)} + a_{n+1,n} y^{(n)} + a_{n+1,n-1} y^{(n-1)} + \ldots + a_{n+1,0} y = b_{n+1}.$$

We shall assume that the coefficients $a_{n+1,i}$ $(i = 0, 1, \ldots, n)$ are functions of x defined in the same interval $a < x < b$ and that the coefficient $a_{n+1,i}$ belongs to the class C^i $(i = 0, 1, \ldots, n)$. The function b_{n+1} will be assumed to be continuous in the interval under consideration. In the sequel these assumptions will be called shortly assumptions (a).

We want to represent the left-hand side of equation (1.1) in the form of the symbolic product of two factors, thus reducing equation (1.1) to the following form:

$$(1.2) \qquad \left(\frac{d}{dx} + a_n\right)(y^{(n)} + a_{n,n-1} y^{(n-1)} + a_{n,n-2} y^{(n-2)} + \ldots + a_{n_0} y) = b_{n+1}.$$

Let us assume that the function a_n belongs to the class C^n. In this case the product which appears in (1.2) should be equal to the left-hand side of equation (1.1), which easily leads to the following system of equations, in which the unknowns are the coefficients of decomposition $a_{ni}(x)$ and $a_n(x)$ $(i = 0, 1, \ldots, n-1)$:

$$(1.3)\qquad \begin{aligned} a_n \qquad\quad &+ a_{n,n-1} = a_{n+1,n}, \\ a'_{n,n-1} + a_n a_{n,n-1} &+ a_{n,n-2} = a_{n+1,n-1}, \\ a'_{n,n-2} + a_n a_{n,n-2} &+ a_{n,n-3} = a_{n+1,n-2}, \\ &\ldots\ldots\ldots\ldots \\ a'_{n1} \quad + a_n a_{n1} \quad &+ a_{n0} \quad = a_{n+1,0}, \\ a'_{n0} \quad + a_n a_{n0} \quad & \qquad\quad = a_{n+1,0}. \end{aligned}$$

Using the notation introduced at the beginning of this paragraph, we may write equations (1.3) in the form

$$(1.4)\qquad \begin{aligned} a_{n,n-1} &= a_{n+1,n} \quad - a_n, \\ a_{n,n-2} &= a_{n+1,n-1} - l_n a_{n,n-1}, \\ a_{n,n-3} &= a_{n+1,n-2} - l_n a_{n,n-2}, \\ &\ldots\ldots\ldots\ldots \\ a_{n0} \quad &= a_{n+1,1} \quad - l_n a_{n1}, \\ l_n a_{n0} \quad &= a_{n+1,0}, \end{aligned}$$

The solution of system (1.3) may be reduced to the solution of one equation of the n-th order, where the unknown is the function a_n. This equation can be obtained after eliminating all unknowns a_{ni} $(i = 0, 1, \ldots, n-1)$ from (1.3). To perform this elimination let us introduce the following notation:

$$A_{n+1,j} = \begin{cases} a_{n+1,n} - a_n, & \text{if} \quad j = n, \\ a_{n+1,j}, & \text{if} \quad j \neq n. \end{cases}$$

As a result of $i-1$ successive eliminations of functions $a_{n,j}$ $(j = n-1, n-2, \ldots, n-i+1)$ we get the relation

$$a_{n,n-i} = \sum_{k=0}^{i-1} (-l_n)^k A_{n+1,n-i+k+1}.$$

We omit here the inductive proof. If we put $i = n$ we obtain

$$a_{n0} = \sum_{k=0}^{n-1} (-l_n)^k A_{n+1,k+1},$$

and using the last equation of system (1.4) we find:

$$\sum_{k=0}^{n}(-l_n)^k A_{n+1,k} = 0. \tag{1.5}$$

If we come back to the previous notation, we obtain, as a result of the elimination considered, the equation:

$$\begin{aligned}R_n[a_n] \equiv l_n^n(a_{n+1,n}-a_n) - l_n^{n-1}a_{n+1,n-1} + l_n^{n-2}a_{n+1,n-2} - \\ -\ldots + (-1)^{n-2}l_n^2 a_{n+1,2} + (-1)^{n-1}l_n a_{n+1,1} + (-1)^n a_{n+1,0} = 0.\end{aligned} \tag{1.6}$$

By assumptions (a) concerning the differentiability of the coefficients of equation (1.1) and by the specification of the expression $l_n^i a_{n+1,i}$ $(i = 0, 1, \ldots, n)$, the left-hand side of (1.6) is a well-defined differential expression; it is easily seen that this expression is non-linear and of the n-th order with respect to the unknown a_n, defined in the interval (a, b).

Equation (1.6) will be called the *R equation of the n-th order* related to the l. d. equation of the $(n+1)$-st order, or, shortly, the *R equation*.

Now we shall present a rule which allows us to write the successive equations (1.6) without using operator symbols.

Equation (1.6) for the index $n+1$ may be written in the form:

$$R_{n+1}[a_{n+1}] = l_{n+1}\Big[\sum_{k=0}^{n}(-1)^{2n-k}l_{n+1}^{n-k}A_{n+2,n+1-k}\Big] + (-1)^{n+1}A_{n+2,0} = 0. \tag{1.7}$$

Hence the rule: In order to write the differential expression $R_{n+1}[a_{n+1}]$ one should: 1) increase by one all the lower indices in the expression R_n; 2) "multiply" the expression obtained in 1) by the operator l_{n+1}; 3) add the term $(-1)^{n+1}a_{n+2,0}$.

The equations below are examples of R equations which correspond to linear equations of the orders $n = 1, 2, 3$:

$$\begin{aligned}
a_1' &= -a_1^2 + a_{21}a_1 + a_{21}' - a_{20},\\
a_2'' + 3a_2a_2' - a_{32}a_2' &= -a_2^3 + a_{32}a_2^2 + (2a_{32}' - a_{31})a_2 + a_{32}'' - a_{31}' + a_{30},\\
&a_3''' + 4a_3a_3'' + 6a_3^2a_3' + 3a_3'^2 - a_{43}a_3'' - 3a_{43}a_3a_3' - (3a_{43}' - a_{42})a_3'\\
&= -a_3^4 + a_{43}a_3^3 + (3a_{43}' - a_{42})a_3^2 + (3a_{43}'' - 2a_{42}' + a_{41})a_3 +\\
&\quad + a_{43}''' - a_{42}'' + a_{41}' - a_{40}.
\end{aligned} \tag{1.8}$$

. . . .

If $n = 1$, i. e. if a linear equation is of the second order, the corresponding R equation is a Riccati equation. In this sense the R equations are generalizations of the Riccati equation.

By the term *R equations* (not related to linear equations) we shall understand the equations formed from (1.6) by replacing the coefficients with unknown a_n and its derivatives by symbols which are not related to the coefficients of the differential equation $L_{n+1}[y] = 0$. If $n = 1, 2, 3$ and we put $a_n = y$, then the R equations take the form:

$$
\begin{aligned}
y' &= -y^2 + f_1(x)y + f_0(x),\\
y'' + 3yy' - f_2(x)y' &= -y^3 + f_2(x)y^2 + f_1(x)y + f_0(x),\\
y''' + 4yy'' + 6y^2y' + 3y'^2 - f_3(x)y'' - 3f_3(x)yy' - f_2(x)y' &\\
&= -y^4 + f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x),\\
&\dots\dots\dots
\end{aligned}
\tag{1.9}
$$

It is easy to verify using formulas (1.8) that in the particular case $n = 1, 2, 3, \ldots$ the converse is also true: the R equations have corresponding linear differential equations. The following linear equations correspond to (1.9):

$$
\begin{aligned}
y'' + f_1y' + (f_1' - f_0)y &= 0,\\
y''' + f_2y'' + (2f_2' - f_1)y' + (f_0 - f_1' + f_2'')y &= 0,\\
y^{\mathrm{IV}} + f_3y''' + (3f_3' - f_2)y'' + (3f_3'' - 2f_2' + f_1)y' + (f_3''' - f_2'' + f_1' - f_0)y &= 0,\\
\dots\dots\dots
\end{aligned}
\tag{1.10}
$$

These formulas may be obtained by comparing the coefficients in the corresponding equations (1.8) and (1.9). One has to assume, however, that the function $f_i(x)$ belongs to the class C^i $(i = 0, 1, \ldots, n)$. Thus, in this case we also have an assumption of the same character as the assumption about equation (1.1), i. e. that the regularity of the coefficient in an R equation depends upon its place in the equation.

Thus, linear equations with coefficients satisfying assumptions (a) have corresponding non-linear equations and conversely. In the sequel we shall show how we can make use of this mutual relation.

R equations (1.6) can also be written in the form of a determinant, by means of another method of eliminating the functions $a_{ni}^{(i)}$ from system (1.3). In fact, let us differentiate n times the first equation of this system, $n-1$ times the second equation, and so on; finally, let us differentiate once the equation which is last but one (the very last should not be differentiated at all). In this way we get a system of $k = (n+1)(n+2)/2$ equations with $k-1$ unknowns to be eliminated, namely the functions $a_{n,n-1}^{(i_1)}$ $(i_1 = 0, 1, \ldots, n)$, $a_{n,n-2}^{(i_2)}$ $(i_2 = 0, 1, \ldots, n-1)$, $\ldots$, $a_{n,1}^{(i_{n-1})}$ $(i_{n-1} = 0, 1, 2)$, $a_{n,0}^{(i_n)}$ $(i_n = 0, 1)$.

These unknowns appear linearly in the system of equations obtained in the way described. It follows that the result of elimination is a determinant of the k-th order. By giving a suitable rule one may define all the terms of this determinant; it should only be remembered that while differentiating equations (1.3) we use the Leibniz formula for the derivative of product, whence binomial coefficients will appear in the determinant according to a rule which is easy to guess. Here we shall merely present examples of equations of the second and third order:

for $n = 1$

$$\begin{vmatrix} 0 & 1 & a_1 - a_{21} \\ 1 & 0 & a_1' - a_{21}' \\ 1 & a_1 & -a_{20} \end{vmatrix} = 0;$$

for $n = 2$

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & a_2 - a_{32} \\ 0 & 1 & 0 & 0 & 0 & a_2' - a_{32}' \\ 1 & 0 & 0 & 0 & 0 & a_2'' - a_{32}'' \\ 0 & 1 & a_2 & 0 & 1 & -a_{31} \\ 1 & a_2 & a_2' & 1 & 0 & -a_{31}' \\ 0 & 0 & 0 & 1 & a_2 & -a_{30} \end{vmatrix} = 0.$$

The following theorem results from the above considerations:

THEOREM 1. *An arbitrary particular solution* a_n *of the differential R equation which corresponds to a given linear equation determines the decomposition of the left-hand side of that equation into symbolic factors.*

This decomposition is defined in the domain of existence of the solution a_n.

In fact, in order that the decomposition under consideration be determined, one has to determine besides a_n also the functions a_{ni} $(i = 0, 1, \ldots, n-1)$. If we have a_n, then the functions a_{ni} can be determined from system (1.3) by means of differentiation and rational operations. The theorem is proved.

2. Theorems on the existence of solutions of R equations. Relations between the solutions of linear differential equations and the solutions of the corresponding R equations

In the sequel, while proving the theorems on existence, we shall use the following theorems and lemmas:

LEMMA 1 (Cancellation lemma). *Suppose we are given a matrix with* $n+2$ *rows and* n *columns*:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+2,1} & a_{n+2,2} & \dots & a_{n+2,n} \end{bmatrix}.$$

If we denote by $D(r, s)$ *the determinant of the above matrix with the rows numbered* r *and* s *crossed out, then the following relation is true*:

$$D(l, m)D(p, q)-D(m, q)D(l, p)+D(l, q)D(m, p) = 0, \tag{2.1}$$

provided $l < m < p < q$.

We omit here the inductive proof.

LEMMA 2 (Cancellation lemma). *Suppose we are given a matrix with* $n+1$ *rows and* n *columns. If we denote by* $D(r)$ *the determinant of this matrix with the* r*-th row crossed out, and by* $D(r, s)$ *the determinant of this matrix with the rows numbered* r *and* s *and the last column crossed out, then the following relation holds*:

$$D(l)D(m, p)-D(m)D(p, l)+D(p)D(l, m) = 0. \tag{2.2}$$

The proof is based upon Lemma 1. To obtain it one has to expand the determinants $D(r)$ of n-th order with respect to the last column. After a suitable arrangement we get (2.2) using (2.1).

THEOREM OF MAMMANA (see [7], p. 200). *There exist an infinite number of systems of* $n-1$ *solutions of the linear differential equation* $L_n[y] = 0$ *such that* $n-2$ *of them are real and linearly independent and one is of the form* $u(x)+iv(x)$, *and the Vronsky determinant of such a group does not vanish at any point of the interval* (a, b). *The functions* u *and* v *are solutions of the linear equation under consideration and, together with the remaining* $n-2$ *solutions, form a linearly independent system.*

The interval which appears in the theorem is the interval in which the coefficients of the linear equation are defined. To have the solutions y in the whole interval (a, b) one has to assume the continuity of the coefficients of the equation under consideration.

Coming back to the problem of existence of solutions of our R equation, it is easy to verify that the general theorem of existence of solution in a sufficiently small neighbourhood of each interior point of the interval (a, b) can be applied to the R equation which corresponds to the linear equation with coefficients satisfying assumptions (a).

In fact, equation (1.6) satisfies the assumptions of the Peano theorem:

1) It can be solved with respect to the derivative $a_n^{(n)}$ since the coefficient with this derivative equals -1:

$$a_n^{(n)} = R_{n1}(a_{n+1,j}^{(k)}, a_n^{(i)}) = R_{n2}(x, a_n^{(i)})$$
$$(i = 0, 1, \ldots, n-1; \quad j = 0, 1, \ldots, n; \quad k \leqslant j).$$

2) The right-hand side of the last equation, as a function of variables x, unknown a_n and its derivatives $a_n^{(i)}$, is a continuous function: the function R_{n1} is an algebraic sum of the term $l_n^n a_n$ and $l_n^i a_{n+1,i}$ $(i = 0, 1, \ldots, n)$, whence it is a polynomial of the unknown a_n, its derivatives, the coefficients of equation (1.1) and their derivatives $a_{n+1,j}^{(k)}$; the last by assumptions (a) are functions from the classes C^{j-k} $(k \leqslant j)$.

Now we shall present two theorems which are more advanced.

LEMMA 3. *If the R equation which corresponds to a linear differential equation has a solution in some interval, then the coefficients of decomposition into symbolic factors of the differential expression $L_{n+1}[y]$, denoted by the symbols a_{ni} $(i = 0, 1, \ldots, n-1)$, belong to the class C^{i+1}.*

Proof (by induction). By the regularity of solutions of the differential equation (see for instance [3], p. 110) the function a_n, as a solution of an R equation of n-th order, is a function from the class C^n in the interval of its existence. Hence, by the first of relations (1.3) we have $a_{n,n-1} = a_{n+1,n} - a_n$, and the function $a_{n,n-1}$ is from the class C^n. Suppose now that the function $a_{n,n-j}$ belongs to the class C^{n+1-j}. By (1.3) we get:

$$a_{n,n-(j+1)} = a_{n+1,n-j} - a'_{n,n-j} - a_n a_{n,n-j}.$$

It follows that $a_{n,n-(j+1)}$ belongs to the same class as the function $a'_{n,n-j}$, which, by the inductive assumption is from the class C^{n-j}; this proves the lemma.

LEMMA 4. *Each solution of an R equation of n-th order corresponding to a linear equation of $n+1$-st order defined in the interval (α, β) is of the form*

$$a_n = \frac{W'(y_1, y_2, \ldots, y_n)}{W(y_1, y_2, \ldots, y_n)} + a_{n+1,n} \tag{2.3.1}$$

or

$$(2.3.2)\qquad a_n = \frac{W'(y_1, y_2, \ldots, y_n)}{W(y_1, y_2, \ldots, y_n)} - \frac{W'(y_1, y_2, \ldots, y_n; y_{n+1})}{W(y_1, y_2, \ldots, y_n; y_{n+1})},$$

where the symbols W denote the Vronsky determinants, and the system of functions y_i $(i = 1, 2, \ldots, n+1)$ *is a linearly independent system of particular solutions of the linear equation* $L_{n+1}[y] = 0$, *which corresponds to the R equation considered.*

Fomula (2.3.2) may be rewritten in an equivalent shorter form

$$(2.4)\qquad a_n = \frac{d}{dx} \ln \left| \frac{W(y_1, \ldots, y_n)}{W(y_1, \ldots, y_n; y_{n+1})} \right|.$$

Proof. Consider the interval (α, β) in which the solution a_n of equation (1.6) exists by assumption. By (1.3) the coefficients a_{ni} exist in the interval (α, β), and by assumption (a) they are functions of classes C^{i+1} $(i = 0, 1, \ldots, n-1)$ (lemma 3). Hence in the interval considered the function a_n determines the l. d. equation of n-th order of the form

$$L_n[y] = y^{(n)} + a_{n,n-1} y^{(n-1)} + \ldots + a_{n0} y = 0.$$

Let us consider an arbitrary system of linearly independent solutions of this equation:

$$(2.5)\qquad y_1, y_2, \ldots, y_n.$$

From the form (1.2) of equation (1.1) it follows directly that system (2.5) is a system of solutions of the homogeneous equation corresponding to (1.1), and $W(y_1; \ldots, y_n) \neq 0$ in the interval (α, β).

Let us denote by y_{n+1} the integral of the homogeneous equation which corresponds to (1.1), and which forms, together with functions (2.5), a linearly independent system. Using the first relation (1.3): $a_n + a_{n,n-1} = a_{n+1,n}$, and expressing the functions $a_{n+1,n}$ and $a_{n,n-1}$ by the solution y_i $(i = 0, 1, \ldots, n, n+1)$, we get the formulas (2.3.1) and (2.3.2) using the Liouville theorem.

The determinant $W(y_1, \ldots, y_n)$ does not vanish in the interval (α, β), which proves our theorem.

LEMMA 5. *Each function* a_n *of the form* (2.3), *where*

$$(2.6)\qquad y_1, y_2, \ldots, y_n; y_{n+1},$$

is an arbitrary system of linearly independent solutions of homogeneous equations of the $(n+1)$*-st order which satisfies assumption* (a), *is a solution of the corresponding R equation of n-th order.*

The function (2.3) is not determined at those points of the interval (a, b) at which $W(y_1, \ldots, y_n) = 0$.

It follows from the above theorem that a given system of solutions of a linear equation determines $n+1$ different solutions of the corresponding non-linear R equation. In fact, one particular solution, namely y_{n+1}, is distinguished in (2.3.2); if we distinguish other solutions y_i $(i = 1, 2, \ldots, n)$, we obtain $n+1$ different solutions a_n.

Proof. Let (2.6) denote an arbitrary system of linearly independent solutions of (1.1). Let us form the function a_n according to formula (2.3). This function is determined in the interval (a, b), except at those points where $W(y_1, \ldots, y_n) = 0$. Since the differential expression $L_n[y]$ is uniquely determined by functions y_i $(i = 1, \ldots, n)$, we have a certain decomposition of $L_{n+1}[y]$ determined by a_n. If we write the equation $L_n[y] = 0$ in the form of a determinant and use the symbols of lemmas 1 and 2, we get

$$a_{n,n-1} = (-1)^i \cdot \frac{W(n-i, n+1)}{W(n, n+1)} \qquad (i = 1, 2, \ldots, n).$$

The determinants $W(i, j)$ in the last formula and the determinant $W(k)$ which appears later are formed from the matrix with $n+2$ rows and $n+1$ columns:

$$\begin{bmatrix} y_1 & y_2 & \cdots y_{n+1} \\ y_1' & y_2' & \cdots y_{n+1}' \\ \cdot & \cdot & \cdot \\ y_1^{(n+1)} & y_2^{(n+1)} & \cdots y_{n+1}^{(n+1)} \end{bmatrix}$$

in the manner described in Lemmas 1 and 2.

For the proof, it suffices to show that the functions of decomposition a_n and a_{nj} $(j = 0, 1, \ldots, n-1)$ satisfy system (1.3).

1) The first equation of system (1.3) is satisfied by the definition of function a_n.

2) Let us consider an equation from the second group (1.3):

$$a_{n,n-i}' + a_n a_{n,n-i} + a_{n,n-(i+1)} - a_{n+1,n-i} = 0,$$

where i is an arbitrary number from among $1, 2, \ldots, n-1$. We shall show that this equation is satisfied by the functions a_n and $a_{n,n-i}$ defined above.

Indeed, replacing $a_{n+1,n-i}$ by the ratio of determinants according to the formula

$$a_{n+1,n-i} = (-1)^{i+1} \frac{W(n-i)}{W(n+1)} \qquad (i = 0, 1, 2, \ldots, n)$$

and putting a_n and $a_{n,n-i}$ into the equation of the second group, we represent it in the form:

$$W(n+1)W(n-i, n) - W(n)W(n-i, n+1) + W(n, n+1)W(n-i) = 0,$$

where $n-i < n < n+1$. The last relation is identically satisfied for every $i = 1, 2, \ldots, n-1$ by lemma 2. Thus, the second group of equations (1.3) is satisfied by functions a_n and $a_{n,n-i}$.

3) The last equation (1.3): $a'_{n0}+a_n a_{n0}-a_{n+1,0} = 0$, after similar reductions, leads to the relation

$$W(n+1)W(0,n)-W(n)W(0,n+1)+W(0)W(n,n+1) = 0,$$

which is identically satisfied by lemma 2.

Lemma 5 is proved.

From the above considerations follows

THEOREM 2. *If the coefficients of linear equation* (1.1) *satisfy assumption* (a), *then the corresponding non-linear R differential equation has real solutions determined in the whole interval* (a, b), *except perhaps at some points. Those solutions are given by formulas* (2.3).

On the other hand, the Mammana theorem implies:

THEOREM 3. *Under the conditions of the preceding theorem, the R equation which corresponds to a given linear equation has a solution in the domain of complex functions of real variable determined in the whole interval* (a, b).

In fact, it follows from the Mammana theorem that one can find a system of n linearly independent solutions such that the Vronsky determinant formed from those solutions does not vanish in the interval (a, b).

It follows that under our assumptions the decomposition of the differential expression $L_{n+1}[y]$ in the interval (a, b) is always possible, but, in general, this decomposition holds in the complex domain, even if the coefficients are real (thus we have here a full analogy with algebraic equations).

Because of the relations of non-linear R equations with linear equations, we have obtained the theorem of existence of solutions in the whole domain (a, b) whence a theorem analogous to that for linear equations. The assumptions of theorem 3 require some explanations. These assumptions are expressed in an indirect way: they concern the coefficients of linear equation (1.1) which corresponds to the R equation discussed in theorem 3. Nevertheless, it is easy to notice that we require from the equation more than the continuity of functions $R_{n2}(x, a_n^{(i)})$. We shall explain that by discussing equations (1.9). In order to relate equations (1.9) to linear equations—and, next, to apply theorem 3—one has, as has already been shown above, to assume that the functions $f_i(x)$, which are the coefficients of an R equation, belong to the class C^i $(i = 0, 1, \ldots)$. Then the linear equations corresponding to the R equation satisfy assumption (a) and theorem 3 can be applied. Thus we require that the R equations in theorem 3 should have the continuous function

$f_0(x)$, function $f_1(x)$ differentiable once, function $f_2(x)$ differentiable twice, and so on. Thus we have here the assumptions of the same nature as the assumptions for linear equations (1.1) formulated at the beginning.

Formula (2.4) is derived in papers devoted to the problem of the decomposition of linear differential expression $L_{n+1}[y]$ by other methods. Since the theory of R equations is not introduced in those papers, the decomposition discussed there is possible only after finding the general solution of a homogeneous linear equation (we have already mentioned it in the introduction). Thus, the method of decomposition has not helped in solving linear equations. Probably because of that, this method has played an insignificant part in the theory of equations, although it has been known for a long time.

On the other hand, the introduction of R equations which correspond to linear equations has made it possible to decompose the expression $L_{n+1}[y]$ without knowing the solution of the linear equation, and consequently, it often helps to lower the order of the equation. We shall show this in the next paragraph.

3. Relations between the solutions of R equations and the solutions of linear equations. Two methods of solving linear equations

In the last paragraph we have proved that the solution of a non-linear R equation which corresponds to a linear equation can be – in the sense defined above – reduced to the solution of a linear equation. In the present paragraph we shall show that the converse is also true: the solution of an R equation, or finding any of its particular integrals enables us to solve or lower the order of the corresponding linear equation.

We shall use the following definition:

Suppose we are given a system of functions

$$a_1(x),\, a_2(x),\, \ldots,\, a_k(x) \tag{3.1}$$

defined in the interval (a, b) and belonging to the class C^{k-2}. The symbol T will denote the determinant of system (3.1):

$$T(a_1, \ldots, a_k) = \begin{vmatrix} 1 & a_1 & l_1 a_1 & l_1^2 a_1 & \ldots & l_1^{k-2} a_1 \\ 1 & a_2 & l_2 a_2 & l_2^2 a_2 & \ldots & l_2^{k-2} a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_k & l_k a_k & l_k^2 a_k & \ldots & l_k^{k-2} a_k \end{vmatrix} \tag{3.2}$$

where l_i is the differential operator defined above.

Thus, for a system of two, three and four functions, we have

$$T(a_1, a_2) = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix},$$

$$T(a_1, a_2, a_3) = \begin{vmatrix} 1 & a_1 & a_1' + a_1^2 \\ 1 & a_2 & a_2' + a_2^2 \\ 1 & a_3 & a_3' + a_3^2 \end{vmatrix},$$

$$T(a_1, a_2, a_3, a_4) = \begin{vmatrix} 1 & a_1 & a_1' + a_1^2 & a_1'' + 3a_1 a_1' + a_1^3 \\ 1 & a_2 & a_2' + a_2^2 & a_2'' + 3a_2 a_2' + a_2^3 \\ 1 & a_3 & a_3' + a_3^2 & a_3'' + 3a_3 a_3' + a_3^3 \\ 1 & a_4 & a_4' + a_4^2 & a_4'' + 3a_4 a_4' + a_4^3 \end{vmatrix}$$

Definition 2. Functions (3.1) will be called *significantly different in* (a, b) if the determinant $T(a_1, a_2, \ldots, a_k)$ does not vanish in this interval.

From the form of the determinant T for two functions it follows that they are significantly different in (a, b) if they do not take equal values in this interval. If all functions (3.1) are constant in (a, b), then the determinant $T(a_1, \ldots, a_k)$ becomes the Vandermonde determinant, and instead of "significantly different functions" we have "pairwise different numbers".

THEOREM 4. *If the coefficients of the l. d. equation* $L_{n+1}[y] = b_{n+1}$ *satisfy assumption* (a), *then each solution of the corresponding R equation reduces the integration of this linear equation to the integration of a linear equation of n-th order, i. e. of an order less by* 1.

Proof. Consider equation (1.1) and the corresponding equation (1.6). Let a_n be a solution of the latter equation defined in the whole interval (a, b). This solution exists by Theorem 3.

It follows from (1.3) that there exist coefficients a_{ni} $(i = n-1, n-2, \ldots, 0)$, and we have decomposition (1.2). Equation (1.2) may be replaced by the system of equations

$$(3.3)\qquad \begin{aligned} L_n[y] \equiv y^{(n)} + a_{n,n-1}y^{(n-1)} + a_{n,n-2}y^{(n-2)} + \ldots + a_{n1}y' + a_{n0}y &= b_n, \\ \frac{db_n}{dx} + a_n b_n &= b_{n+1}, \end{aligned}$$

where $b_n(x)$ is an auxiliary function from the class C^1 defined in (a,b).

Since

$$(3.4)\qquad b_n = C_{n+1}e^{-A_n} + e^{-A_n}\int b_{n+1}e^{A_n}dx,$$

where

$$A_n(x) = \int a_n(x)\,dx,$$

we have by (3.3):

$$(3.5)\qquad \begin{aligned} &y^{(n)} + a_{n,n-1}y^{(n-1)} + a_{n,n-2}y^{(n-2)} + \ldots + a_{n1}y' + a_{n0}y \\ &\qquad = C_{n+1}e^{-A_n} + e^{-A_n}\int b_{n+1}e^{A_n}dx \qquad (C_{n+1} = \text{const}). \end{aligned}$$

It is easy to verify that equation (3.5) is a first ([2]) integral of equation (1.1).

When eliminating the constant C_{n+1} one has to use relations (1.3). The first integral, to which equation (1.1) reduces, is a linear equation of n-th order. This completes the proof of the theorem.

([2]) We use the terminology of [2].

By Lemma 3 we see that if the left-hand side of equation (1.1) is decomposable into symbolic terms, i. e. satisfies assumptions (a), then the left-hand side of the first integral (3.5) is also decomposable.

Thus we conclude that under assumptions (a) we can perform the second decomposition of a linear equation of n-th order, and the result of this decomposition is a linear equation of $(n-1)$-st order, which is the first integral of (3.5) and the second integral of (1.1). Indeed, as a result of the new decomposition we get

$$y^{(n-1)}+a_{n-1,n-2}y^{(n-2)}+\ldots+a_{n-1,0}y = b_{n-1}, \tag{3.6}$$

where b_{n-1} is a solution of the linear equation

$$\frac{db_{n-1}}{dx}+a_{n-1}b_{n-1} = b_n. \tag{3.7}$$

This new decomposition (and the resulting further lowering of the order of the differential equation) is based upon any of the solutions a_{n-1} of the new R equation of $(n-1)$-st order, which corresponds to the equation $L_n[y]=0$, i. e. the equation

$$R_{n-1}[a_{n-1}] = 0. \tag{3.8}$$

From equation (3.7) we obtain

$$b_{n-1} = C_n e^{-A_{n-1}}+e^{-A_{n-1}}\int b_n e^{A_{n-1}}dx$$

or, after considering (3.4):

$$\begin{aligned} b_{n-1} &= C_n e^{-A_{n-1}}+C_{n+1}e^{-A_{n-1}}\int e^{A_{n-1}-A_n}dx+ \\ &+e^{-A_{n-1}}\int e^{A_{n-1}-A_n}\int b_{n+1}e^{A_n}dx^2. \end{aligned} \tag{3.9}$$

Applying in the same way the basic theorem on existence of decomposition, we get a system of n differential equations, which are the intermediate integrals of equation (1.1):

$$y^{(n-k)}+a_{n-k,n-k-1}y^{(n-k-1)}+\ldots+a_{n-k,0}y = b_{n-k} \qquad (k = 0,1,\ldots,n-1), \tag{3.10}$$

where

$$\begin{aligned} b_{n-k} &= C_{n-k+1}\,e^{-A_{n-k}}+C_{n-k+2}\,e^{-A_{n-k}}\int e^{A_{n-k}-A_{n-k+1}}dx+ \\ &+C_{n-k+3}\,e^{-A_{n-k}}\int e^{A_{n-k}-A_{n-k+1}}\int e^{A_{n-k+1}-A_{n-k+2}}dx^2+ \\ &+\ldots+C_{n+1}e^{-A_{n-k}}\int e^{A_{n-k}-A_{n-k+1}}\int\ldots\int e^{A_{n-1}-A_n}dx^k+ \\ &+e^{-A_{n-k}}\int e^{A_{n-k}-A_{n-k+1}}\int\ldots\int e^{A_{n-1}-A_n}\int b_{n+1}e^{A_n}dx^{k+1} \\ &\qquad (k = 0,1,\ldots,n-1), \end{aligned} \tag{3.11}$$

and $A_i = \int a_i dx$ $(i = n, n-1, \ldots, 1)$. The functions a_i are the particular solutions of n differential equations of the R type which correspond to equations (3.10):

$$R_i[a_i] = 0 \quad (i = n, n-1, \ldots, 1).$$

Thus we have obtained

THEOREM 5. *The function*

$$(3.12) \qquad y = b_0 = C_1 e^{-A_{10}} + C_2 e^{-A_{10}} \int e^{A_{10}-A_1} dx + \\ + C_3 e^{-A_{10}} \int e^{A_{10}-A_1} \int e^{A_1-A_2} dx^2 + \ldots + \\ + C_{n+1} e^{-A_{10}} \int e^{A_{10}-A_1} \int e^{A_1-A_2} \int \ldots \int e^{A_{n-1}-A_n} dx^n + \\ + e^{-A_{10}} \int e^{A_{10}-A_1} \int e^{A_1-A_2} \int \ldots \int e^{A_{n-1}-A_n} \int b_{n+1} e^{A_n} dx^{n+1}$$

where $A_{10} = \int a_{10} dx$ *and* a_{10} *is defined by the relation* $a_1 + a_{10} = a_{21}$ *or* $a_{10} = a_{n+1,n} - \sum_{i=1}^{n} a_i$ *is the general solution of a non-homogeneous linear differential equation. The functions* a_i $(i = 1, 2, \ldots, n)$ *are the particular solutions of a system of R differential equations which correspond to the given linear equation and to the system of its intermediate integrals obtained by the method of decomposition of the linear differential expression into symbolic factors.*

In fact, the last equation in system (3.10) is the intermediate integral of equation (1.1) of $(n+1)$-st order. Since this is the linear equation of the first order, its solution is known and has the form

$$y = b_0 = C_1 e^{-A_{10}} + e^{-A_{10}} \int b_1 e^{A_{10}} dx$$

or the form of function (3.12), which was to be proved.

Thus, the system $a_n, a_{n-1}, \ldots, a_1$ of solutions of equations $R_n[a_n] = 0$, $R_{n-1}[a_{n-1}] = 0, \ldots, R_1[a_1] = 0$ determines completely the general solution of l. d. equation (1.1).

The solutions of the system of differential equations of the R type discussed in the previous theorem determine the linearly independent particular solutions of the homogeneous l. d. equation corresponding to (1.1), and

$$(3.13) \qquad \begin{aligned} y_1 &= e^{-A_{10}}, \\ y_2 &= e^{-A_{10}} \int e^{A_{10}-A_1} dx, \\ &\cdots\cdots\cdots\cdots\cdots \\ y_{n+1} &= e^{-A_{10}} \int e^{A_{10}-A_1} \int e^{A_1-A_2} \ldots \int e^{A_{n-1}-A_n} dx^n; \end{aligned}$$

these solutions determine also a particular integral of l. d. equation (1.1):

$$(3.14)\quad Y_{n+1} = e^{-A_{10}}\int e^{A_{10}-A_1}\int e^{A_1-A_2}\int \dots \int e^{A_{n-1}-A_n}\int b_{n+1}e^{A_n}dx^{n+1}.$$

Relations (3.13) and (3.14) easily follow from the above reasonings.

The following method (method A) of solving linear equations may be based upon the last theorem:

1) For a given linear equation which satisfies assumptions (a) we write the equation $R_n[a_n] = 0$.

2) We find (by guessing, or in any other way) any of the particular solutions of equation $R_n[a_n] = 0$.

3) Using (1.3) we find the first integral of linear equation (1.1).

Next, we apply the same procedure to the first integral just found, and so on, until we obtain either the intermediate integral of equation (1.1), whose general solution is known, or until we obtain an equation of the first order.

Another method of solving a linear equation (method B) is based upon the following theorem:

THEOREM 6. *Suppose we are given l. d. equation* (1.1) *of* $(n+1)$-st *order and k significantly different solutions of the R equation of n-th order corresponding to linear equation* (1.1). *Then we can find an l. d. equation of* $(n-k+1)$-*st order which is an intermediate integral of equation* (1.1).

Proof. Since we know k significantly different solutions of the differential equation of type R which corresponds to linear equation (1.1), we can perform k different decompositions of this equation and find the following k equations of n-th order (first integrals of equation (1.1)):

$$(3.15)\quad y^{(n)} + a_{n,n-1(i)}y^{(n-1)} + a_{n,n-2(i)}y^{(n-2)} + \dots + a_{n_0(i)}y$$
$$= C_{n+1(i)}e^{-A_{n(i)}} + e^{-A_{n(i)}}\int b_{n+1}e^{A_{n(i)}}dx \qquad (i = 1, 2, \dots, k).$$

Thus we have a system of k algebraic equations, linear with respect to functions $y^{(n)}, y^{(n-1)}, \dots, y', y$. Hence we can express the first k of them: $y^{(n)}, y^{(n-1)}, \dots, y^{(n+1-k)}$ by the remaining ones (and by the coefficients of the system), provided the determinant formed of the first k columns of the coefficients of the system does not vanish in the considered interval (a, b). In particular, we can determine the function $y^{(n+1-k)}$. From the linearity of relations (3.15) it follows that $y^{(n+1-k)}$ can be expressed linearly by $y^{(n-k)}$, $y^{(n-k-1)}, \dots, y', y$. We shall also have arbitrary constants $C_{n+1(i)}$ $(i = 1, 2, \dots, k)$ in this equation, and it will be a linear equation of the order $n+1-k$. This will be the intermediate integral of equation (1.1).

It remains to show that the determinant of the system under consideration does not vanish in the interval (a, b), i. e. that

$$(3.16)\qquad D = \begin{vmatrix} 1 & a_{n,n-1(1)} & a_{n,n-2(1)} & \cdots & a_{n,n+1-k(1)} \\ 1 & a_{n,n-1(2)} & a_{n,n-2(2)} & \cdots & a_{n,n+1-k(2)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & a_{n,n-1(k)} & a_{n,n-2(k)} & \cdots & a_{n,n+1-k(k)} \end{vmatrix} \neq 0.$$

As we know, the coefficients of equations (3.15), whence the coefficients of determinant (3.16), can be determined by the solutions of the R equations and the coefficients of equations (1.1) by the use of formulas (1.3). These formulas are, in this case, of the form:

$$(3.17)\qquad \begin{aligned} a_{n,n-1(i)} &= a_{n+1,n} - a_{n(i)}, \\ a_{n,n-2(i)} &= a_{n+1,n-1} - a'_{n,n-1(i)} - a_{n(i)} a_{n,n-1(i)}, \\ a_{n,n-3(i)} &= a_{n+1,n-2} - a'_{n,n-2(i)} - a_{n(i)} a_{n,n-2(i)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n,n+1-k(i)} &= a_{n+1,n-k+2} - a'_{n,n-k+2(i)} - a_{n(i)} a_{n,n-k+2(i)} \\ & \qquad\qquad (i = 1, 2, \ldots, k). \end{aligned}$$

The first column of the determinant D consists of 1's and will not be transformed. The second column is defined by the first of relations (3.17), whence it consists of the differences of two functions, the first of which, namely $a_{n+1,n}$ is repeated in all the terms of this column. Since the first column is constant, this first term may be omitted; then the second column will consist of the system of solutions $a_{n(i)}$ $(i = 1, 2, \ldots, k)$ multiplied by -1.

The third and the next columns are the algebraic sums of three terms which appear on the right-hand side of relations (3.17), starting from the second term. For the same reasons as before we may omit the first term in all columns. Thus the third column will be of the form:

$$-a'_{n,n-1(i)} - a_{n(i)} a_{n,n-1(i)}$$

and in view of the first of functional relations (3.17) it can be transformed as follows:

$$-a'_{n,n-1(i)} - a_{n(i)} a_{n,n-1(i)} = -a'_{n+1,n} + a'_{n(i)} - a_{n(i)} a_{n+1,n} + a^2_{n(i)}.$$

In this column, as in the preceding one, we may omit the repeated term $-a'_{n+1,n}$, and the repeated term $a_{n(i)} a_{n+1,n}$, since the second column is formed by functions $a_{n(i)}$ and the second factor $a_{n+1,n}$ of the product $a_{n(i)} a_{n+1,n}$ appears in every row. Thus we come to the conclusion that the third column can be replaced by the sum $a'_{n(i)} + a^2_{n(i)}$.

Let us notice that this column may be obtained from the preceding one by the following rule:

$$a'_{n(i)} + a^2_{n(i)} = -l_{n(i)}(-a_{n(i)}) \qquad (i = 1, 2, \ldots, k).$$

This is a general rule: each column is obtained from the preceding one according to the same rule, i. e. it is the product of the operator $-l_{n(i)}$ and the preceding column. In fact, let us consider any column:

$$K_j = -a'_{n,n-j(i)} - a_{n(i)} a_{n,n-j(i)} \quad (j = 2, 3, \ldots, k-1) \tag{3.18}$$

and the next column

$$K_{j+1} = -a'_{n,n-1-j(i)} - a_{n(i)} a_{n,n-1-j(1)}. \tag{3.19}$$

By the recursive formulas (3.17) we may heighten the indices in the formula (3.19). Then we get

$$K_{j+1} = -a'_{n+1,n-j} + a''_{n,n-j(i)} + a'_{n(i)} a_{n,n-j(i)} + a_{n(i)} a'_{n,n-j(i)} -$$
$$- a_{n(i)} a_{n+1,n-j} + a_{n(i)} a'_{n,n-j(i)} + a^2_{n(i)} a_{n,n-j(i)}.$$

Considering the form of the first column of the determinant D we are entitled to omit in this column the terms $-a'_{n+1,n-j}$ since they appear in each row; considering the form of the second column we may omit terms $-a_{n(i)}\ a_{n+1,n-j}$. After rearrangement we get

$$\begin{aligned} K^1_{j+1} &= a_{n(i)}(a'_{n,n-j(i)} + a_{n(i)} a_{n,n-j(i)}) + \\ &\quad + \frac{d}{dx}(a'_{n,n-j(i)} + a_{n(i)} a_{n,n-j(i)}), \\ K^1_{j+1} &= -l_{n(i)} K_j, \end{aligned} \tag{3.20}$$

which was to be shown.

Let us notice that this general rule of forming the determinant D may also be applied to the second column since

$$-a_{n(i)} = -l_{n(i)} \cdot 1 = -\left(\frac{d}{dx} + a_{n(i)}\right) \cdot 1.$$

Let us also mention that the terms of the determinant do not depend upon the coefficients of the linear equations but only upon the system of solutions of R equation. In fact, by rule (3.20), the columns depend only upon the operator $l_{n(i)}$, whence upon the functions $a_{n(i)}$. Thus we have found

$$\begin{aligned} D &= \det\|1, a_{n,n-1(i)}, a_{n,n-2(i)}, \ldots, a_{n,n+1-k(i)}\| \\ &= \det\|1, -a_{n(i)}, l_{n(i)} a_{n(i)}, -l^2_{n(i)} a_{n(i)}, \ldots, (-1)^{k-1} l^{k-2}_{n(i)} a_{n(i)}\| \end{aligned} \tag{3.21}$$

or

$$D = (-1)^p T(a_{n(1)}, a_{n(2)}, \ldots, a_{n(k)}),$$

where $p = k(k-1)/2$. Thus the determinant of system (3.15) is the determinant T of the system of solutions of the R equation multiplied by 1 or -1. Thus if the system $a_{n(i)}$ is a system of solutions or the R equation which corresponds to a given linear equation (1.1), and, at the same time, a system of significantly different functions, then system (3.15) may be solved, and equation (1.1) may be replaced by the equivalent equation of the order $n+1-k$. The theorem is proved.

Let us notice a certain analogy between the Vronsky determinant in the theory of linear equations, and the determinant T in the theory of R equations.

Theorem 6 allows us to reduce the problem of solution of a linear differential equation to the solution of a linear equation of lower order, provided we know some of the solutions of the corresponding R equation, and these solutions are significantly different in the interval under consideration.

If we know n significantly different solutions of the R type differential equation of n-th order, then we can solve completely the corresponding system of linear differential equation of $(n+1)$-st order. In fact, a system of n significantly different solutions allows to reduce the linear equation of $(n+1)$-st order to the equation of the first order, which can be already solved. Thus the solving of a differential equation of type R, or, more precisely, the knowledge of a system of significantly different solutions of this equation, where the number of solutions is equal to the order of the equation, means the solution of the corresponding linear equation.

If, however, we know $n+1$ significantly different solutions of the R equation which correspomds to a given linear equation of $(n+1)$-st order, then problem of the solution of that equation becomes simply an algebraic problem, i. e. it requires the solving of a linear system of algebraic equations (3.15) for $k = n+1$.

The second method (method B) of solving linear equations will make use of some significantly different solutions of the R equation which corresponds to the given linear equation.

4. Applications of the method of decomposition into operator factors

We shall now apply the preceding results to some problems from the domain of linear equations. We shall do this not in order to present new results, but in order to present new and simplified proofs of theorems and formulas based upon the method of decomposition and upon the theory of R equations.

It will follow from our considerations that completely solvable linear equations (equations with constant coefficients and Euler-type equations), are elementarily decomposable equations, i. e. the R equations corresponding to them "degenerate" to algebraic equations. We shall also present an example of a converse proposition for linear equations of the fourth order, namely that the generalized Euler equation is the only one whose solution can be reduced to the solution of a characteristic algebraic equation.

4.1. Application of method B to l. d. equations with constant coefficients. If the coefficients $a_{n+1,i}$ $(i = 0, 1, \ldots, n)$ of (1.1) are constant (b_{n+1} is assumed to be a function of the class C^1), then it is a natural idea to look for the decomposition of the differential expression $L_{n+1}[y]$ into symbolic components in the domain of constant functions. We have then $a_n = \text{const}$. If we consider in addition that $l_n^m C = a_n^m C$ for $C = \text{const}$, then the R equation (1.6) which corresponds to (1.1) will take the form

$$(4.1)\qquad a_n^{n+1} - a_{n+1,n}a_n^n + a_{n+1,n-1}a_n^{n-1} - \ldots + \\ + (-1)^n a_{n+1,1}a_n + (-1)^{n+1}a_{n+1,0} = 0.$$

Thus, in this case the R equation degenerates to an algebraic equation. This is the classical characteristic equation with the inessential change of sign of unknown a_n. Suppose that the roots of the characteristic equation are all distinct. Then we have, as we know, not one but $n+1$ different decompositions of the differential expression $L_{n+1}[y]$:

$$(4.2)\qquad y^n + a_{n,n-1(i)}y^{(n-1)} + \ldots + a_{n1(i)}y' + a_{n0(i)}y = C_{n+1(i)}e^{-a_{n(i)}x} + B_{n+1(i)},$$

where

$$B_{n+1(i)} = e^{-a_{n(i)}x}\int b_{n+1}(x)e^{a_{n(i)}x}dx \qquad (i = 1, 2, \ldots, n+1).$$

From this system we may determine the unknown function y, provided that the determinant of this system is different from zero.

By theorem 6 we know that this condition holds if we have significantly different solutions, which is equivalent to $T(a_{n(1)}, \ldots, a_{n(n+1)}) \neq 0$. But in our case the determinant T is the Vandermonde determinant of $(n+1)$-st order formed of the roots of the characteristic equation. These roots are distinct by assumption, whence system (4.2) has a solution.

There is no need to solve this system. It suffices to note that the unknown y is a ratio of two determinants; all the elements of the numerator except one column formed of right-hand sides of (4.2) are constants, and the denominator is also a constant. Thus the expansion of this ratio is of the form

$$y = C_1 e^{-a_{(1)}x} + C_2 e^{-a_{n(2)}x} + \ldots + C_{n+1} e^{-a_{n(n+1)}x} + Y_{n+1} \qquad (C_i = \text{const}),$$

where Y_{n+1} is a particular solution of a non-homogeneous equation.

From (3.21) we find

$$Y_{n+1} = (-1)^n \frac{\det\|1, a_{n(i)}, a_{n(i)}^2, \ldots, a_{n(i)}^{n-1}, e^{-a_{n(i)}x} \int b_{n+1} e^{a_{n(i)}x} dx\|}{\det\|1, a_{n(i)}, a_{n(i)}^2, \ldots, a_{n(i)}^{n-1}, a_{n(i)}^n\|}.$$

The denominator is the Vandermonde determinant V^{n+1} of numbers $a_{n(1)}, \ldots, a_{n(n+1)}$. Expanding the numerator with respect to the last column we find the coefficients of the terms $(-1)^{n+i+1} e^{-a_{n(i)}x} \int b_{n+1} e^{a_{n(i)}x} dx$. These coefficients are the determinants of the same type formed from the numbers $a_{n(1)}, \ldots, a_{n(i-1)}, a_{n(i+1)}, \ldots, a_{n(n+1)}$ $(i = 1, \ldots, n+1)$. Hence, after some reductions, we find

$$(4.3) \qquad Y_{n+1} = \sum_{i=1}^{n+1} \Big[\prod_{\substack{s=1 \\ s\neq i}}^{n+1} (a_{n(s)} - a_{n(i)})\Big]^{-1} e^{-a_{n(i)}x} \int b_{n+1} e^{a_{n(i)}x} dx.$$

Thus, method B gives in a simple way the general solution of a non-homogeneous equation; there is no need to predict in advance the functional form of the solution, or to apply the method of variation of constants for the determination of the integral Y_{n+1}.

4.2. Application of method A to linear equations with constant coefficients. In the preceding case the appearance of multiple roots would complicate the derivation of formulas. This difficulty disappears if we apply method A. The answer is given by formulas (3.13) and (3.14), which take the form

$$y_{i+1} = e^{-a_{10}x} \int e^{(a_{10}-a_1)x} \int e^{(a_1-a_2)x} \ldots \int e^{(a_{i-1}-a_i)x} dx^i,$$

$$(4.4) \qquad Y_{n+1} = e^{-a_{10}x} \int e^{(a_{10}-a_1)x} \int e^{(a_1-a_2)x} \ldots$$

$$\ldots \int e^{(a_{n-1}-a_n)x} \int b_{n+1} e^{a_n x} dx^{n+1} \qquad (i = 0, 1, \ldots, n).$$

The proper use of formulas (4.4) requires, however, some additional remarks. By the preceding theorem all linearly independent integrals are obtained from the solution of a single algebraic equation (the characteristic equation). Formulas (4.4) make use of one of the roots of this equation, namely a_n. Other numbers which appear in (4.4) are, as we know, the roots of the system of characteristic equations which correspond to the system of l. d. equations which we obtain from (1.1) by successive lowering of its order. It seems to follow that for using the formulas under consideration one would have to write down and solve the system of all algebraic equations which can be obtained from equation (1.1) by successive lowering of its order; hence it would require complicated computations. We shall show that this is not the case.

LEMMA 6. *The roots of the characteristic equation*

$$r^k - a_{k,k-1}r^{k-1} + a_{k,k-2}r^{k-2} + \ldots \mp a_{k1}r \pm a_{k0} = 0 \tag{4.5}$$

which corresponds to a certain l. d. equation of k-th order, obtained in the process of successive lowering of its order, are the roots of the characteristic equation

$$r^{k+1} - a_{k+1,k}r^k + a_{k+1,k-1}r^{k-1} + \ldots \pm a_{k+1,1}r \mp a_{k+1,0} = 0, \tag{4.6}$$

which corresponds to the preceding differential equation of $(k+1)$-st order.

Proof. The coefficients of both characteristic equations are connected by formulas (1.3), if we rewrite these formulas for $n = k$ and omit the terms with derivatives. These formulas determine the coefficients of equation (4.6) by means of parameter a_k and the coefficients of (4.5). Thus we can rewrite (4.6) in the form

$$r^{k+1} - (a_k + a_{k,k-1})r^k + (a_k a_{k,k-1} + a_{k,k-2})r^{k-1} -$$
$$- (a_k a_{k,k-2} + a_{k,k-3})r^{k-2} + \ldots \pm (a_k a_{k1} + a_{k0})r \mp a_k a_{k0} = 0$$

or, after some rearranging,

$$\begin{aligned} -a_k(r^k - a_{k,k-1}r^{k-1} + a_{k,k-2}r^{k-2} - \ldots \mp a_{k1}r \pm a_{k0}) + \\ + r(r^k - a_{k,k-1}r^{k-1} + a_{k,k-2}r^{k-2} - a_{k,k-3}r^{k-3} + \ldots \pm a_{k0}) = 0. \end{aligned} \tag{4.7}$$

Equation (4.6) reduced to the form (4.7) is satisfied by every root r_i $(i = 1, \ldots, k)$ of equation (4.5), which was to be proved.

It follows that the sequence of numbers in formulas (4.4) is a system of solutions of characteristic equation (4.1):

$$a_{10} = a_{n(1)}, \quad a_i = a_{n(i+1)} \quad (i = 1, \ldots, n).$$

A desirable simplification of the theory presented above consists in the fact that the general formulas (4.4) determine the solutions of linear equation with constant coefficients even in the case where the characteristic equation has multiple roots. In fact, these formulas determine the general solution of linear equation (1.1) according to theorem 5, and the integrals which appear in these formulas remain meaningful in the case of multiple roots of the characteristic equation.

Elementary integration leads to well-known formulas. To find all the linearly independent solutions in the case of multiple roots of the characteristic equation, one should number them in such a way that the multiple ones should appear at the last places. For instance, if we have $k+3$ different roots and two roots, a_{k+1} and a_{l+1}, are multiple, we should arrange the roots in the following order:

$$a_{10} \neq a_1 \neq \ldots \neq a_k = a_{k+1} = \ldots = a_l \neq a_{l+1} = \ldots = a_n. \tag{4.8}$$

The arrangement (4.8) is essential. If they were arranged in a different way, we should not be able to get all the linearly independent integrals, as can be easily shown.

Let us also observe that the formula for the particular solution of non-homogeneous equation (4.4) is more general than formula (4.3), which can be applied only to the case of all roots being distinct.

4.3. Application of method B **to Euler equations.** Suppose we are given the Euler equation

$$\begin{gathered} y^{(n+1)} + \tilde{a}_{n+1,n} x^{-1} y^{(n)} + \tilde{a}_{n+1,n-1} x^{-2} y^{(n-1)} + \ldots + \\ + \tilde{a}_{n+1,1} x^{-n} y' + \tilde{a}_{n+1,0} x^{-(n+1)} y = b_{n+1} \\ (\tilde{a}_{n+1,i} = \text{const};\ i = n, n-1, \ldots, 0). \end{gathered} \tag{4.9}$$

If we try to find the decomposition under the natural assumption

$$a_n = \tilde{a}_n x^{-1}, \qquad a_{n,n-1} = \tilde{a}_{n,n-1} x^{-1}, \tag{4.10}$$

where the symbols with $\sim$ denote constant numbers (this will be the usual notation in the subsequent parts of this paragraph), then the differential equations (1.3) will become algebraic equations, and the R type equation—will become an algebraic equation of order n with respect to $\tilde{a}_n$. Indeed, it can easily be shown that system (1.3), after dividing the first equation by x^{-1}, the second by x^{-2}, the third by x^{-3}, and so on, will be transformed into the following system of algebraic equations:

$$\left\{\begin{aligned} &\tilde{a}_{n,n-1} + \tilde{a}_n - \tilde{a}_{n+1,n} = 0, \\ &\tilde{a}_{n,n-2} + \tilde{a}_{n,n-1}(\tilde{a}_n - 1) - \tilde{a}_{n+1,n-1} = 0, \\ &\tilde{a}_{n,n-3} + \tilde{a}_{n,n-2}(\tilde{a}_n - 2) - \tilde{a}_{n+1,n-2} = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\tilde{a}_{n,0} + \tilde{a}_{n1}(\tilde{a}_n - n + 1) - \tilde{a}_{n+1,1} = 0, \\ &\tilde{a}_{n0}(\tilde{a}_n - n) - \tilde{a}_{n+1,0} = 0. \end{aligned}\right. \tag{4.11}$$

After eliminating $\tilde{a}_{ni}$ $(i = 0, 1, \ldots, n-1)$ from this system, or introducing (4.10) to (1.6), we get

$$(4.12)\qquad P_{n+1}(\tilde{a}_n) \equiv [\tilde{a}_n]_n^{n+1} - \tilde{a}_{n+1,n}[\tilde{a}_n]_n^n + \tilde{a}_{n+1,n-1}[\tilde{a}_n]_n^{n-1} - \\ - \ldots \pm \tilde{a}_{n+1,1}[\tilde{a}_n]_n \mp \tilde{a}_{n+1,0} = 0,$$

where [ ] denotes the factorial product

$$(4.13)\qquad [\tilde{a}_n]_n^i = (\tilde{a}_n - n)(\tilde{a}_n - n + 1)(\tilde{a}_n - n + 2)\ldots\big(\tilde{a}_n - (n - i + 1)\big).$$

Equation (4.12) is the characteristic equation of a given Euler differential equation (4.9). Let us mention that this equation differs from that which can be derived in the classical way; the classical equation may be obtained from equation (4.12) by linear transformation $r = -\tilde{a}_n + n$. In our case the characteristic equation (4.12) is also a degenerate form of a differential equation of the R type which corresponds to the Euler equation under assumption (4.10).

If the characteristic equation (4.12) has all roots distinct, then these roots determine the general solution of the differential equation (4.9), and the solution of the homogeneous equation which corresponds to (4.9) is of the form

$$(4.14)\qquad y = \sum_{i=1}^{n+1} \tilde{C}_i x^{n - \tilde{a}_{n(i)}},$$

where $\tilde{a}_{n(i)}$ $(i = 1, 2, \ldots, n+1)$ are the roots of the characteristic equation.

This fact follows from the general theorem 6 on the possibility of using $n+1$ significantly different solutions of the R type equation which corresponds to the given linear equation. Indeed, if we have distinct solutions of the characteristic equation, we also have $n+1$ solutions of the R differential equation, namely $\tilde{a}_{n(i)}x^{-1}$ $(i = 1, \ldots, n+1)$. It can easily be shown that they all are significantly different. Indeed, from (3.21) we get

$$D = \det\|1, -\tilde{a}_{n(i)}x^{-1}, (-\tilde{a}_{n(i)} + \tilde{a}_{n(i)}^2)x^{-2}, (-2\tilde{a}_{n(i)} + 3\tilde{a}_{n(i)}^2 - \tilde{a}_{n(i)}^3)x^{-3}, \ldots\| \\ (i = 1, \ldots, n+1)$$

or

$$D = \det\|1, -\tilde{a}_{n(i)}x^{-1}, \tilde{a}_{n(i)}^2 x^{-2}, -\tilde{a}_{n(i)}^3 x^{-3}, \ldots, (-1)^n \tilde{a}_{n(i)}^n x^{-n}\|.$$

This determinant is the ratio of entire function x^{-p} and the Vandermonde determinant formed from the roots of the characteristic equation. Thus $D \neq 0$, whence $T \neq 0$, if the roots of this equation are distinct; in this case the system of functions $\tilde{a}_{n(i)}x^{-1}$ is a system of significantly different functions.

Thus, applying theorem 6 we may compute y from equations (3.15). It is easily seen that

(4.15)
$$y = (-1)^n \frac{\det \| 1, \tilde{a}_{n,n-1(i)} x^{-1}, \tilde{a}_{n,n-2(i)} x^{-2}, \ldots, \tilde{a}_{n1(i)}^{(n-1)} x^{-(n-1)}, C_{n+1(i)} x^{-\tilde{a}_{n(i)}} \|}{\det \| 1, \tilde{a}_{n(i)} x^{-1}, \tilde{a}_{n(i)}^2 x^{-2}, \ldots, \tilde{a}_{n(i)}^{(n-1)} x^{-(n-1)}, \tilde{a}_{n(i)}^n x^{-n} \|}.$$

In this case we do not need to compute the values of the determinants which appear in (4.15). The powers of the variable x cancel out in all columns but the last one in the denominator. Expanding the determinant with respect to the last column we obtain formula (4.14), and the values of constant coefficients are of no interest to us. The particular solution Y_{n+1} of non-homogeneous equation (4.9) can also be found from (4.15), if we replace the last column of the terms $C_{n+1(i)} x^{-\tilde{a}_{n(i)}}$ by $x^{-\tilde{a}_{n(i)}} \int x^{\tilde{a}_{n(i)}} b_{n+1} dx$. Then we find

$$Y_{n+1} = x^n \sum_{i=1}^{n+1} \Big[\prod_{\substack{s=1 \\ s \neq i}}^{n+1} (a_{n(s)} - a_{n(i)}) \Big]^{-1} x^{-a_{n(i)}} \int b_{n+1} x^{a_{n(i)}} dx.$$

By formulas (4.10) and (1.3) we easily find that the first integral (3.5) is an Euler linear equation of the n-th order.

4.4. Application of method A to Euler equations. As in 4.2 the application of method A does not require separate consideration of multiple roots. The corresponding formulas can be obtained from the solutions (3.13) and (3.14), which requires, however, the following lemma:

LEMMA 7. *Suppose we are given a system of characteristic equations* (4.12) $P_k(r) = 0$ $(k = n+1, n, \ldots, 2)$ *which correspond to a given Euler equation* (4.9) *and is obtained as a result of successive lowering of order of this equation. If* r_{ki} $(i = 1, 2, \ldots, k)$ *are the roots of the equation* $P_k(r) = 0$, *i. e.*

(4.16) $$P_k(r_{ki}) = 0,$$

then the preceding characteristic equation $P_{k+1}(r) = 0$, *has* k *roots* $r_{k+1,i}$ $(i = 1, \ldots, k)$ *such that each of them is the sum of the corresponding root of the "next" equation* $P_k(r) = 0$ *and unity*:

(4.17) $$P_{k+1}(r_{ki} + 1) = 0.$$

Thus we have the following relation between the roots of two successive characteristic equations:

(4.18) $$r_{k+1,i} = r_{ki} + 1 \quad (i = 1, 2, \ldots, k).$$

Proof. We shall use the common letter r for denoting the independent variable in all the equations considered.

As in (4.13) we write

$$[r]_n^i = (r-n)(r-n+1)(r-n+2)\dots[r-(n-i+1)]. \tag{4.19}$$

We shall use the following obvious formulas:

$$[r+1]_n^i = [r]_{n-1}^i, \tag{4.20}$$

$$[r]_{n-1}^{n-i} + (i+1)[r]_{n-1}^{n-(i+1)} = (r+1)[r]_{n-1}^{n-(i+1)}. \tag{4.21}$$

To prove our lemma let us consider two successive characteristic equations:

$$P_{k+1}(r) \equiv [r]_k^{k+1} - \tilde{a}_{k+1,k}[r]_k^k + \tilde{a}_{k+1,k-1}[r]_k^{k-1} - \\ - \dots \pm \tilde{a}_{k+1,1}[r]_k^1 \mp \tilde{a}_{k+1,0} = 0 \tag{4.22}$$

and

$$P_k(r) \equiv [r]_{k-1}^k - \tilde{a}_{k,k-1}[r]_{k-1}^{k-1} + \tilde{a}_{k,k-2}[r]_{k-1}^{k-2} - \\ - \dots \mp \tilde{a}_{k1}[r]_{k-1}^1 \pm \tilde{a}_{k0} = 0. \tag{4.23}$$

Let r_ν denote a root of equation (4.23) and substitute $r = r_\nu + 1$ in equation (4.22). By (4.20) we get

$$P_{k+1}(r_\nu + 1) \equiv [r_\nu]_{k-1}^{k+1} - \tilde{a}_{k+1,k}[r_\nu]_{k-1}^k + \\ + \tilde{a}_{k+1,k-1}[r_\nu]_{k-1}^{k-1} - \dots \pm \tilde{a}_{k+1,1}[r_\nu]_{k-1}^1 \mp \tilde{a}_{k+1,0}. \tag{4.24}$$

Next, relations (4.11) written for the index $n = k$ hold for the coefficients of equations (4.22) and (4.23). Let us replace the coefficients of equation (4.24) by those of equation (4.23) according to formulas (4.11). We find

$$\begin{aligned} P_{k+1}(r_\nu+1) \equiv\ & [r_\nu]_{k-1}^{k+1} - (\tilde{a}_k + \tilde{a}_{k,k-1})[r_\nu]_{k-1}^k + \\ & + \{\tilde{a}_{k,k-1}(\tilde{a}_k - 1) + \tilde{a}_{k,k-2}\}[r_\nu]_{k-1}^{k-1} - \{\tilde{a}_{k,k-2}(\tilde{a}_k - 2) + \tilde{a}_{k,k-3}\}[r_\nu]_{k-1}^{k-2} + \\ & + \dots \mp \{\tilde{a}_{k2}[\tilde{a}_k - (k-2)] + \tilde{a}_{k1}\}[r_\nu]_{k-1}^2 \pm \\ & \pm \{\tilde{a}_{k1}[\tilde{a}_k - (k-1)] + \tilde{a}_{k0}\}[r_\nu]_{k-1}^1 \mp \tilde{a}_{k0}(\tilde{a}_n - k). \end{aligned}$$

After grouping and using (4.21) we easily derive

$$P_{k+1}(r_\nu + 1) = -\tilde{a}_k P_k(r_\nu) + (r_\nu + 1) P_k(r_\nu).$$

Since, by assumption, r_ν is a root of (4.23), we have $P_{k+1}(r_\nu + 1) = 0$ for $\nu = 1, 2, \dots, k$ and $k = n, n-1, \dots, 2$. Lemma 7 is proved.

Thus, lemma 7 allows us to use formulas (4.13) for the solution of Euler differential equation in the general case, i. e. no matter whether the characteristic equation $P_{n+1}(\tilde{a}_n) = 0$ has multiple roots or not.

In fact, using the previous notation, we infer from our recursive formulas

$$\tilde{a}_{k-1} = \tilde{a}_k - 1 \qquad (k = n, n-1, \ldots, 2) \tag{4.25}$$

that each root $\tilde{a}_i$ may be related to a root of the first characteristic equation (4.12), namely

$$\tilde{a}_i = \tilde{a}_{n(i+1)} - n + i \qquad (i = 0, 1, \ldots, n;\ \tilde{a}_0 = \tilde{a}_{10}). \tag{4.26}$$

In the case of multiple roots we retain formula (4.26), which means that multiple roots are used several times.

Since, in our case, $a_i(x) = \tilde{a}_i x^{-1}$, we have

$$e^{A_i} = x^{\tilde{a}_i} \quad \text{and} \quad e^{A_i - A_{i+1}} = x^{\tilde{a}_i - \tilde{a}_{i+1}} = x^{a_{n(i+1)} - a_{n(i+2)} - 1}.$$

Hence formulas (3.13) and (3.14) for Euler differential equation take the form

$$y_i = x^{n-\tilde{a}_{n(1)}} \int x^{\tilde{a}_{n(1)} - \tilde{a}_{n(2)} \ 1} \int \ldots \int x^{\tilde{a}_{n(i-1)} - \tilde{a}_{n(i)} - 1} dx^{i-1} \tag{4.27}$$

$$(i = 1, 2, \ldots, n+1)$$

and

$$Y_{n+1} = x^{n-\tilde{a}_{n(1)}} \int x^{\tilde{a}_{n(1)} - \tilde{a}_{n(2)} - 1} \int \ldots \int x^{\tilde{a}_{n(n)} - \tilde{a}_{n(n+1)} - 1} \int b_{n+1} x^{\tilde{a}_{n(n+1)}} dx^{n+1}$$

where the numbers $\tilde{a}_{n(i)}$ $(i = 1, 2, \ldots, n+1)$ are the roots of characteristic equation (4.12) in the case of all roots being distinct as well as in the case of there being multiple roots.

Computation of elementary integrals leads to well-known formulas. If the characteristic equation has all its roots distinct, then it is easy to note that all exponents in successive integrals are different from -1; hence logarithms do not appear in integrals (4.27).

In the opposite case, if the characteristic equation has multiple roots, logarithms appear in the solution.

In order to obtain all linearly independent integrals one has to arrange the roots as before, i. e. the multiple ones should be placed last.

4.5. The class of differential equations whose solutions reduces to solutions of algebraic equations. As can easily be verified the generalized Euler equation

$$y^{(n+1)} + \tilde{a}_{n+1,n}(\tilde{c}x + \tilde{b})^{-1} y^{(n)} + \\ + \tilde{a}_{n+1,n-1}(\tilde{c}x + \tilde{b})^{-2} y^{(n-1)} + \ldots + \tilde{a}_{n+1,0}(\tilde{c}x + \tilde{b})^{-(n+1)} y = b_{n+1}$$

has a corresponding R equation which is also an algebraic equation.

Thus, the following question arises: do the examples of linear equations considered so far exhaust the class of linear equations for which the corresponding R equations degenerate to algebraic (characteristic) equations?

First, however, we should specify what will be called the characteristic equation.

Definition 3. If the substitution $a_n = \tilde{a}_n a_{n+1,n}$ reduces the equation $R_n[a_n] = 0$, which corresponds to a given linear equation, to an algebraic equation with respect to $\tilde{a}_n$, then such an algebraic equation will be called the *characteristic equation* of an l. d. equation (here $\tilde{a}_n$ is a number).

As we have seen, linear differential equations with constant coefficients and the Euler equation have the characteristic equations satisfying the above definition.

In particular cases, when we have an explicit form of R equation, it is easy to answer the above problem using formulas (1.8).

THEOREM 7. *The generalized Euler equation is the only linear equation which satisfies the condition* $a_{n+1,n} \neq 0$ *and which has the characteristic equation* $(n = 1, 2, 3)$.

Proof. We have already proved that if an equation is a generalized Euler equation, then it has a characteristic equation. Now we shall prove that if a linear equation has a characteristic equation, then it is a generalized Euler equation.

We shall prove it for the linear equation of fourth order $(n = 3)$. Consider the corresponding R equation (1.8), and suppose that our linear equation has a characteristic equation. Then, after substituting $a_3 = \tilde{a}_3 a_{43}$ in (1.8) and dividing both sides by a_{43}, this equation should become an algebraic equation in $\tilde{a}_3$ (by assumption a_{43}, does not vanish in the interval considered). It is easy to find that the characteristic equation should be

$$\begin{aligned}\tilde{a}_3^4 + \tilde{a}_3^3(6a_{43}^{-2}a_{43}' - 1) + \tilde{a}_3^2(4a_{43}''a_{43}^{-3} + 3a_{43}'^2 a_{43}^{-4} - 6a_{43}'a_{43}^{-2} +\\ + a_{42}a_{43}^{-2}) + \tilde{a}_3(a_{43}'''a_{43}^{-4} - 4a_{43}''a_{43}^{-3} - 3a_{43}'^2 a_{43}^{-4} + a_{42}a_{43}'a_{43}^{-4} +\\ + 2a_{42}'a_{43}^{-3} - a_{41}a_{43}^{-3}) - (a_{43}''' - a_{42}'' + a_{41}' - a_{40})a_{43}^{-4} = 0.\end{aligned}$$

Hence the coefficients in $\tilde{a}_3^i$ should be numbers, and the following conditions should be satisfied:

$$(4.28)\quad \begin{cases} 6a_{43}^{-2}a_{43}' - 1 = \text{const}, \\ a_{42}a_{43}^{-2} = \text{const} - 4a_{43}''a_{43}^{-3} - 3a_{43}'^2 a_{43}^{-4} + 6a_{43}'a_{43}^{-2}, \\ a_{41}a_{43}^{-3} = \text{const} + a_{43}'''a_{43}^{-4} - 4a_{43}''a_{43}^{-3} - 3a_{43}'^2 a_{43}^{-4} + a_{42}a_{43}'a_{43}^{-4} + 2a_{42}'a_{43}^{-3}, \\ a_{40}a_{43}^{-4} = \text{const} + a_{43}^{-4}(a_{43}''' - a_{42}'' + a_{41}'). \end{cases}$$

The first condition (4.28) is a differential equation of the first order whose solution is

$$a_{43} = \tilde{a}_{43}(\tilde{c}x+\tilde{b})^{-1}.$$

The next functional relations allow us to find the coefficients a_{42}, a_{41} and a_{40} without solving differential equations; having a_{43} we can find a_{42}; having these two functions we can find a_{41} and in the same way a_{40}. It is easy to see that the right-hand sides of these relations become constants

$$a_{42}a_{43}^{-2} = \text{const}, \quad a_{41}a_{43}^{-3} = \text{const}, \quad a_{40}a_{43}^{-4} = \text{const},$$

which proves our theorem, since the coefficients of our equation are of the form

$$a_{42} = \tilde{a}_{42}(\tilde{c}x+\tilde{b})^{-2}, \quad a_{41} = \tilde{a}_{41}(\tilde{c}x+\tilde{b})^{-3},$$
$$a_{40} = \tilde{a}_{40}(\tilde{c}x+\tilde{b})^{-4}.$$

The proofs for equations of other orders are analogous.

4.6. Examples of proofs based upon the theory of decomposition. We shall present two examples, which will illustrate the advantages of the theory presented for the derivation of formulas of the theory of linear equations.

Example 1. It is well known that if we have n linearly independent solutions y_i $(i = 1, \ldots, n)$ of equation $L_{n+1}[y] = 0$, then the last particular solution of this equation, forming together with y_i's a linearly independent system, is the function

(4.29)
$$y_{n+1} = y_1 \int \frac{W(y_1, y_2)}{y_1^2} \int \frac{y_1 W(y_1, y_2, y_3)}{W^2(y_1, y_2)} \int \frac{W(y_1, y_2) W(y_1, y_2, y_3, y_4)}{W^2(y_1, y_2, y_3)} \int \cdots$$
$$\cdots \int \frac{W(y_1, \ldots, y_{n-2}) W(y_1, \ldots, y_n)}{W^2(y_1, \ldots, y_{n-1})} \int \frac{W(y_1, \ldots, y_{n-1}) e^{-\int a_{n+1,n} dx}}{W^2(y_1, \ldots, y_n)} dx^n$$

and the particular solution of non-homogeneous equation (1.1), $L_{n+1}[y] = b_{n+1}$, is the function

$$(4.30) \quad Y_{n+1} = y_1 \int \frac{W(y_1, y_2)}{y_1^2} \int \cdots \int \frac{W(y_1, \ldots, y_{n-2}) W(y_1, \ldots, y_n)}{W^2(y_1, \ldots, y_{n-1})} \times$$
$$\times \int \frac{W(y_1, \ldots, y_{n-1}) e^{-\int a_{n+1,n} dx}}{W^2(y_1, \ldots, y_n)} \int W(y_1, \ldots, y_n) e^{\int a_{n+1,n} dx} b_{n+1} dx^{n+1}.$$

We assume that the Vronsky determinant formed of given solutions does not vanish at any point of the interval considered (a, b). The theorem of Mammana guarantees the existence of such solutions in the domain of complex functions of real variable.

In the theory considered, the proofs of formulas (4.29) and (4.30) are very simple.

From the first relation (1.3),

$$a_n + a_{n,n-1} = a_{n+1,n}, \tag{4.31}$$

it follows that

$$e^{-A_n} = \tilde{K}_n \frac{W(y_1, \dots, y_n, y_{n+1})}{W(y_1, \dots, y_n)} \qquad (\tilde{K}_n = \text{const}), \tag{4.32}$$

where $A_n = \int a_n dx$. In fact, relation (4.31) is equivalent to

$$e^{-A_n} = \exp\Big[-\int a_{n+1,n} dx + \int a_{n,n-1} dx + \text{const}\Big].$$

If we remark that the functions $a_{n+1,n}$ and $a_{n,n-1}$ are the coefficients in equations (1.1), $L_{n+1}[y] = 0$ and $L_n[y] = 0$, respectively, then we find (4.32) using the Liouville theorem.

If we write (4.32) for indices $i = n, n-1, \dots, 1$, which appear in the process of successive lowering of order of (1.1), as described in § 3, we get

$$e^{A_{j-1} - A_j} = \frac{\tilde{K}_j}{\tilde{K}_{j-1}} \frac{W(y_1, \dots, y_{j-1}) W(y_1, \dots, y_{j+1})}{W^2(y_1, \dots, y_j)} \tag{4.33}$$

$$(\tilde{K}_j = \text{const},\ j = 2, 3, \dots, n,\ W(y_1) = y_1).$$

Let us write the Liouville formula for equation (1.1) in the following form:

$$W(y_1, \dots, y_{n+1}) = \tilde{K} e^{-\int a_{n+1,n} dx} \qquad (\tilde{K} = \text{const}). \tag{4.34}$$

Now we get (4.29) and (4.30) by replacing the exponential functions in (3.13) and (3.14) by the ratios of determinants according to formulas (4.32), (4.33) and (4.34).

Example 2. We shall prove the following theorem:

If the system of functions $y_1, y_2, \dots, y_k$ $(k < n+1)$ forms a linearly independent system of homogeneous linear equations which corresponds to (1.1), *then the substitution*

(4.35)

$$y = y_1 \int \frac{W(y_1, y_2)}{y_1^2} \int \frac{y_1 W(y_1, y_2, y_3)}{W^2(y_1, y_2)} \int \frac{W(y_1, y_2) W(y_1, y_2, y_3, y_4)}{W^2(y_1, y_2, y_3)} \int \cdots$$

$$\cdots \int \frac{W(y_1, \dots, y_{k-2}) W(y_1, \dots, y_k)}{W^2(y_1, \dots, y_{k-1})} \int \frac{W(y_1, \dots, y_{k-1}) z(x)}{W(y_1, \dots, y_k)} dx^k$$

reduces the integration of equation (1.1) *to the integration of an equation of* $(n+1-k)$*-th order with respect to new unknown* z. *The inverse substitution is of the form*

$$z = \frac{W(y_1, \dots, y_k, y)}{W(y_1, \dots, y_k)}. \tag{4.36}$$

We assume that the equation $L_{n+1}(y) = 0$ *is decomposable.*

We shall prove the second part of the theorem first. There is no need to assume that the functions y_i $(i = 1, 2, \dots, k)$ are solutions of a linear differential equation. It is enough to assume that they are functions of the class C^k (so that all the determinants which appear in formula (4.35) exist), and assume that in the interval considered the Vronsky determinants which appear in the denominators of formulas (4.35) do not vanish.

Under these assumptions the proof of the second part of our theorem is based upon the following differential formulas:

$$\frac{d}{dx}\frac{W(y_1, \dots, y_l, y)}{W(y_1, \dots, y_l, y_{l+1})} = \frac{W(y_1, \dots, y_l)W(y_1, \dots, y_{l+1}, y)}{W^2(y_1, \dots, y_{l+1})}. \tag{4.37}$$

We shall proceed by induction. It is easy to verify the theorem for $k = 1$:

$$\text{if} \quad y = y_1 \int \frac{z}{y_1}\, dx, \quad \text{then} \quad z = \frac{W(y_1, y)}{W(y_1)}.$$

Suppose that (4.36) is a consequence of (4.35). Let us denote the right-hand side of (4.35) by $P_k(z)$. It is easy to see that the transformation (4.35) for the index $k+1$, namely $y = P_{k+1}(t)$, is a superposition of the following two transformations:

$$y = P_k(z), \quad z = P_1(t) = z_{k+1} \int \frac{t(x)\,dx}{z_{k+1}}, \tag{4.38}$$

where

$$z_{k+1} = \frac{W(y_1, \dots, y_{k+1})}{W(y_1, \dots, y_k)}. \tag{4.39}$$

Thus, replacing the transformation $y = P_{k+1}(t)$ by system (4.38) and applying the inductive assumption to the transformation $y = P_k(z)$, we shall find, using (4.39),

$$z = \frac{W(y_1, \dots, y_k, y)}{W(y_1, \dots, y_k)}$$

or

$$\frac{W(y_1, \dots, y_k, y)}{W(y_1, \dots, y_k)} = \frac{W(y_1, \dots, y_{k+1})}{W(y_1, \dots, y_k)} \int \frac{W(y_1, \dots, y_k)t(x)\,dx}{W(y_1, \dots, y_{k+1})}.$$

If we divide both sides of the last formula by the ratio which appears before the integral sign, and then differentiate both sides, we find, using (4.37),

$$t = \frac{W(y_1, \ldots, y_{k+1})}{W(y_1, \ldots, y_k)} \frac{d}{dx} \frac{W(y_1, \ldots, y_k, y)}{W(y_1, \ldots, y_{k+1})} = \frac{W(y_1, \ldots, y_{k+1}, y)}{W(y_1, \ldots, y_{k+1})}$$

which proves the theorem on the inverse transformation.

In the proof of the first part we shall also proceed by induction. As we know, by the assumption concerning the decomposability of equation (1.1), its solutions can be represented in the form (3.12). By assumption we know the solutions $y_1, \ldots, y_k$ of the form (3.13). Since $y_1 = e^{-A_{10}}$, the solution is:

(4.40) $$y = C_1 y_1 + C_2 y_1 \int \frac{e^{-A_1}}{y_1} dx + C_3 y_1 \int \frac{e^{-A_1}}{y_1} \int e^{A_1 - A_2} dx^2 + \ldots + \\ + C_{n+1} y_1 \int \frac{e^{-A_1}}{y_1} \int e^{A_1 - A_2} \int \ldots \int e^{A_{n-1} - A_n} dx^n + \\ + y_1 \int \frac{e^{-A_1}}{y_1} \int e^{A_1 - A_2} \int \ldots \int e^{A_{n-1} - A_n} \int b_{n+1} e^{A_n} dx^{n+1}.$$

If we divide both sides of (4.40) by y_1 and then differentiate and multiply by y_1, we get

(4.41) $$y_1 \frac{d}{dx} \left(\frac{y}{y_1} \right) = C_2 e^{-A_1} + C_2 e^{-A_1} \int e^{A_1 - A_2} dx + \ldots + \\ + C_{n+1} e^{-A_1} \int e^{A_1 - A_2} \int \ldots \int e^{A_{n-1} - A_n} dx^{n-1} + e^{-A_1} \int e^{A_1 - A_2} \int \ldots \\ \ldots \int e^{A_{n-1} - A_n} \int b_{n+1} e^{A_n} dx^n.$$

Let us introduce a new variable by putting

(4.42) $$u = y_1 \frac{d}{dx} \left(\frac{y}{y_1} \right) = \frac{W(y_1, y)}{y_1}.$$

This is transformation (4.36) for $k = 1$, and its inverse transformation, which corresponds to (4.35), is:

(4.43) $$y = y_1 \int \frac{u}{y_1} dx.$$

Using (4.42) we find from (4.41)

(4.44) $$u = C_2 e^{-A_1} + C_3 e^{-A_1} \int e^{A_1 - A_2} dx + C_4 e^{-A_1} \int e^{A_1 - A_2} \int e^{A_2 - A_3} dx^2 + \ldots + \\ + C_{n+1} e^{-A_1} \int e^{A_1 - A_2} \int \ldots \int e^{A_{n-1} - A_n} dx^{n-1} + \\ + e^{-A_1} \int e^{A_1 - A_2} \int \ldots \int e^{A_{n-1} - A_n} \int b_{n+1} e^{A_n} dx^n.$$

From the form of this function it is seen that the new variable u is a solution of an l. d. equation of the n-th order. Hence our theorem is proved for $k=1$.

Let us notice that transformation (4.43) has a different form than the traditional one, namely $y=y_1\int u\,dx$ (see for instance [5], p. 202).

To the solution y_2 corresponds the solution u_2, and, by (4.42),

$$u_2=\frac{W(y_1,y_2)}{y_1}.$$

Suppose that our theorem holds for the case where we know k linearly independent solutions of equation (1.1). Then transformation (4.35) leads to a linear equation of the order $[(n+1)-k]$, whose solution can be represented in the form

$$\begin{aligned}(4.45)\quad z=C_{k+1}e^{-A_k}+C_{k+2}e^{-A_k}\int e^{A_k-A_{k+1}}dx+\ldots+\\ +C_{n+1}e^{-A_k}\int e^{A_k-A_{k+1}}\int\ldots\int e^{A_{n-1}-A_n}dx^{n-k}+\\ +e^{-A_k}\int e^{A_k-A_{k+1}}\int\ldots\int e^{A_{n-1}-A_n}\int b_{n+1}e^{A_n}dx^{n-k+1}.\end{aligned}$$

Let the function y_{k+1} be the $(k+1)$-st solution of the equation $L_{n+1}[y]=0$, which forms, together with the previous solutions, a linearly independent system of solutions. By formula (4.36) to the function y_{k+1} corresponds the function (3.39). This is a solution of an equation of $[(n+1)-k]$-th order, whose solution is (4.45).

Transformation (4.35) for the case $k+1$ and the new variable t is replaced, as before, by the system of transformation (4.38). The first transformation leads by induction to formula (4.45) and lowers the order of equation (1.1) by k. The second transformation $z=P_1(t)$, by the assumption for $k=1$, leads to the formula

$$\begin{aligned}t=C_{k+2}e^{-A_{k+1}}+C_{k+3}e^{-A_{k+1}}\int e^{A_{k+1}-A_{k+2}}dx+\ldots+\\ +C_{n+1}e^{-A_{k+1}}\int e^{A_{k+1}-A_{k+2}}\int\ldots\int e^{A_{n-1}-A_n}dx^{n-k-1}+\\ +e^{-A_{k+1}}\int e^{A_{k+1}-A_{k+2}}\int\ldots\int e^{A_{n-1}-A_n}\int b_{n+1}e^{A_n}dx^{n-k}.\end{aligned}$$

By the last formulas, we see that t is a solution of a linear equation of $(n-k)$-th order. Thus we have proved the first part of our theorem.

5. Elementarily decomposable linear differential equations (class E)

5.1. As we have already seen, the decomposition into symbolic factors of a linear differential equation with constant coefficients and of the Euler equation does not require the solution of a differential equation. We shall say that these equations are elementarily decomposable.

In general, a linear differential equation (1.1) will be called *elementarily decomposable* if the corresponding R equation $R_n[a_n] = 0$ has a solution of the form

$$a_n = \tilde{a}_n a_{n+1,n},$$

where $\tilde{a}_n$ is a number. The class of all equations which are elementarily decomposable will be called the *class E.*

Let us look for the solution of equation (1.6) which corresponds to (1.1) of the form $a_n = \tilde{a}_n a_{n+1,n}$. The solution a_n will have such a form if and only if there exists a number $\tilde{a}_n$ such that the following relation is satisfied:

$$(5.1)\qquad l_n^n[a_{n+1,n}(1-\tilde{a}_n)] - l_n^{n-1} a_{n+1,n-1} + l_n^{n-2} a_{n+1,n-2} + \\ + \ldots + (-1)^{n-2} l_n^2 a_{n+1,2} + (-1)^{n-1} l_n a_{n+1,1} + (-1)^n a_{n+1,0} \underset{x}{\equiv} 0,$$

where

$$(5.2)\qquad l_n = \frac{d}{dx} + \tilde{a}_n a_{n+1,n}.$$

This identity has been obtained from (1.6) by substituting $a_n = \tilde{a}_n a_{n+1,n}$. In the case of equations with constant coefficients (or in the case of Euler equations) identity (5.1) reduces to the characteristic equation. However, in more general cases condition (5.1), which is of the form $F(x, \tilde{a}_n) \equiv 0$, is not an algebraic equation with respect to $\tilde{a}_n$. Nevertheless, there may exist an $\tilde{a}_n$ such that this identity is satisfied and our differential equation will be elementarily decomposable.

For practical purpose, we shall present the decomposability condition (5.1) in an explicit form for $n = 1, 2, 3$:

$$(5.3)\quad \begin{cases} a_{21}^2\tilde{a}_1^2+(a_{21}'-a_{21}^2)\tilde{a}_1-a_{21}'+a_{20} \underset{x}{\equiv} 0, \\ a_{32}^3\tilde{a}_2^3+(3a_{32}a_{32}'-a_{32}^3)\tilde{a}_2^2+ \\ \qquad +(a_{32}''-3a_{32}a_{32}'+a_{31}a_{32})\tilde{a}_2-a_{32}''+a_{31}'-a_{30} \underset{x}{\equiv} 0, \\ a_{43}^4\tilde{a}_3^4+(6a_{43}^2a_{43}'-a_{43}')\tilde{a}_3^3+ \\ \qquad +(4a_{43}''a_{43}+3a_{43}'^2-6a_{43}^2a_{43}'+a_{42}a_{43}^2)\tilde{a}_3^2+ \\ \qquad +(a_{43}'''-4a_{43}a_{43}''-3a_{43}'^2+a_{43}'a_{42}+2a_{42}'a_{43}- \\ \qquad -a_{41}a_{43})\tilde{a}_3-a_{43}'''+a_{42}''-a_{41}'+a_{40} \underset{x}{\equiv} 0. \end{cases}$$

Such a form of the decomposability condition is convenient and it allows us to decide in practice whether a given equation belongs to the class E or not.

We shall present some examples which will illustrate the different possibilities of using condition (5.1).

Example 1. An example of an equation from class E is the equation of large deflections of orthotropic plates [6], [1]:

$$(5.4)\quad y'''+2x^{-1}y''-(k^2x^{-2}+\varkappa^2x^{k-1})y'+ +(k^2x^{-3}-\varkappa^2kx^{k-2})y = \frac{q(x)}{D(x)}$$

$$(k, \varkappa = \text{const},\ D(x)\neq 0).$$

In the case of equation (5.4) we have:

$$a_{32} = 2x^{-1},$$
$$a_{31} = -k^2x^{-2}-\varkappa^2x^{k-1},$$
$$a_{30} = k^2x^{-3}-\varkappa^2kx^{k-2}.$$

Substituting these functions in the second condition (5.3) we get

$$x^{-3}(8\tilde{a}_2^3-20\tilde{a}_2^2+16\tilde{a}_2-4)-k^2x^{-3}(2\tilde{a}_2-1)-(2\tilde{a}_2-1) \equiv 0.$$

This condition is satisfied if $\tilde{a}_2 = 1/2$, and the equation of othotropic plates is elementarily decomposable. Having established this fact, we can find the general solution of this equation.

In fact, applying formulas (2.4) and the above theorem we shall find an equation (the first integral) which is equivalent to equation (5.4):

$$(5.5)\quad y+x^{-1}y-(k^2x^{-2}+\varkappa^2x^{k-1})y = x^{-1}[P(x)+C_3],$$

where

$$P(x) = \int\frac{q(x)}{D(x)}x\,dx.$$

The homogeneous equation which corresponds to (5.5) can be reduced to a Bessel equation, and its general integral, provided that $k \neq -1$ and excluding the trivial case $\varkappa = 0$, is of the form

$$y = Z_\nu(t) \equiv C_1 J_\nu(t) + C_2 N_\nu(t),$$

where J_ν and N_ν denote cylindrical functions of the first and second kind respectively, and

$$\nu = \frac{2k}{k+1}, \qquad t = \frac{2i\varkappa}{k+1} x^{(k+1)/2}.$$

Thus the general solution of (5.4) is

$$y = C_1 J_\nu(t) + C_2 N_\nu(t) + C_3 \left[N_\nu(t) \int J_\nu(t)\,dx - J_\nu(t) \int N_\nu(t)\,dx\right] + \tag{5.6}$$
$$+ \frac{\pi}{k+1} \left[N_\nu(t) \int J_\nu(t) P(x)\,dx - J_\nu(t) \int N_\nu(t) P(x)\,dx\right].$$

Thus the necessary and sufficient condition (5.1) for the elementary decomposability of the equation allows us not only to establish the decomposability of this equation, but also to find the full solution of this equation.

Example 2. The Bessel equation

$$y'' + x^{-1} y' + (1 - n^2 x^{-2}) y = 0 \qquad (x \neq 0) \tag{5.7}$$

does not belong to the class E, since the decomposability condition for this equation is of the form

$$x^{-2}(\tilde{a}_1^2 - 2\tilde{a}_1 + 1 - n^2) + 1 \underset{x}{\equiv} 0$$

and is not satisfied for any number $\tilde{a}_1$.

Example 3. As we know, there are types of differential equations which cannot be solved by means of quadratures. Nevertheless, their particular cases may be solved by elementary methods. We shall show by an example that the notion of elementary decomposability allows us to discover these particular cases.

Let us consider the Weber equation

$$y'' - xy' - ay = 0. \tag{5.8}$$

It is known that this equation may be solved by means of quadratures in the case $a = 1$.

We shall check whether equation (5.8) belongs to the class E. In this case $a_{21} = -x$, $a_{20} = -a$ and the condition for elementary decomposability is:

$$x^2\tilde{a}_1(\tilde{a}-1)-\tilde{a}_1-a+1 \underset{x}{\equiv} 0.$$

Excluding the trivial case $a = 0$ we find that equation (5.8) belongs to the class E if $a = 1$, since in this case $\tilde{a}_1 = 0$ determines the decomposition of (5.8).

Example 4. Condition (5.1) for elementary decomposability of an l. d. equation is a condition of the type of identity between the coefficients of the equation considered and one numerical parameter $\tilde{a}_n$. If we fix the value of this parameter, we can obtain the corresponding criteria for elementary decomposability of a differential equation.

Condition (5.1) will be especially simple if we put $\tilde{a}_n = 0$ or $\tilde{a}_n = 1$.

In the first case $\tilde{a}_n = 0$, the lowering of the order of equation is based upon the relation

$$L_{n+1}[y] = \frac{d}{dx} L_n[y],$$

and the decomposability condition (5.1) takes the form

$$(5.9) \qquad a^{(n)}_{n+1,n} - a^{(n-1)}_{n+1,n-1} + \ldots + (-1)^{n-2}a''_{n+1,2} + (-1)^{n-1}a_{n+1,1} + (-1)^n a_{n+1,0} \underset{x}{\equiv} 0$$

since $l_n = d/dx$.

Thus we have obtained, in a new and very simple way, the condition for the left-hand side of the equation to be a complete derivative. Moreover, if $a_n = 0$, from (1.3) we obtain ready made formulas for the coefficients of the first integral $L_n[y] = \text{const}$.

5.2. The method of obtaining an elementarily decomposable equation by the choice of one coefficient. One can look upon the criterion for the elementary decomposability condition (5.1) from another point of view, namely we can consider that it is a relation between the coefficients of equation (1.1) and the number $\tilde{a}_n$: $G(\tilde{a}_n; a_{n+1,0}; a_{n+1,1}; a'_{n+1,1}; \ldots; a^{(n)}_{n+1,n}) = 0$. If we assume that one of the coefficients $a_{n+1,i}$ $(i = 0, 1, \ldots, n)$ is an unknown function, one can treat condition (5.1) as an equation for this coefficient. It is a differential equation with parameter $\tilde{a}_n$ provided $i \neq 0$ (no derivative of the function $a_{n+1,0}$ appears in (5.1)).

Each solution of the equation formed in the above way may be taken as a required coefficient of the linear equation (1.1). Then we obtain a new equation $L^{(1)}_{n+1}[y] = b_{n+1}$, which differs from equation

(1.1) only by one coefficient, and which is already elementarily decomposable. The following theorem results from the above considerations:

THEOREM 8. *Every elementarily non-decomposable linear differential equation whose coefficients satisfy certain regularity conditions can be made elementarily decomposable by means of the change of only one of its coefficients*

If we consider the explicit form of condition (5.1), we see that the coefficient $a_{n+1,0}$ may be chosen without solving a differential equation. Solving a differential equation of the first order (linear if $1 < n$) we can choose the coefficient $a_{n+1,1}$, solving an equation of the second order (linear if $2 < n$), we can choose the coefficient $a_{n+1,2}$, and so on. In general, an equation of the order i (linear if $i < n$) allows us to choose the coefficient $a_{n+1,i}$.

Thus the problem can always be solved if we restrict ourselves to the change of the coefficient $a_{n+1,0}$ or to those of the remaining coefficients whose change requires the solution of a linear equation of the first order.

However, even the change of only one coefficient may change the equation essentially, together with the character of its solutions. In spite of that, the method of choice of one coefficient may be of some help in the theory of differential equations. We shall show it by some examples.

Example 1. Let us observe that the equations which we are able to solve have the coefficients chosen in a suitable way. It may happen that this choice is too specialized, and if that is the case we may, by means of changing one coefficient, obtain a more general equation, which can also be solved. Thus for instance, the equation

$$y''+a_{21}y'+(1-\tilde{a}_1)(\tilde{a}_1a_{21}^2+a_{21}')y = 0, \tag{5.10}$$

where a_{21} is an arbitrary function from the class C^1 and $\tilde{a}_1$ is an arbitrary number—has the coefficient a_{20} chosen according to the criterion of elementary decomposability (5.1). In spite of its very special character, this equation is much more general that the Euler equation of the second order. In fact, it is easy to verify that if we put $a_{21} = (\tilde{c}x+\tilde{b})^{-1}$ equation (5.10) will become an Euler equation. Equation (5.10) is elementarily decomposable; one can find its first integral and then the general solution.

Example 2. Book [2] contains the solution of the equation

$$y''-(\tilde{a}x+\tilde{b})y'+[(\tilde{a}x+\tilde{b})^2-a]y = 0. \tag{5.11}$$

The solutions of this equation are

$$y_1 = \exp\left(\frac{\tilde{a}x^2}{2}+\tilde{b}x\right), \quad y_2 = y_1'. \tag{5.12}$$

It is easy to see that if we change the coefficient a_{20} of equation (5.11) replacing it by a function chosen according to the condition of elementary decomposability (5.3), we get

$$a_{20} = \tilde{c}(\tilde{a}x+\tilde{b})^2-\tilde{a}(1\pm\sqrt{1-\tilde{c}}),$$

where $\tilde{c}$ is an arbitrary parameter. Hence we obtain an equation which is solvalbe by elementary methods:

$$y''-(\tilde{a}x+\tilde{b})y'+[\tilde{c}(\tilde{a}x+\tilde{b})^2-\tilde{a}(1\pm\sqrt{1-\tilde{c}})]y = 0. \tag{5.13}$$

This is a more general equation, and it becomes equation (5.11) if $\tilde{c} = 1$. Applying the general formulas (1.3) for equations from class E we find

$$\begin{aligned}
\tilde{a}_1 &= \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1-\tilde{c}}\\
a_1 &= \tilde{a}_1 a_{12} = -2\tilde{a}_1(\tilde{a}x+\tilde{b}),\\
A_1 &= -\tilde{a}_1(\tilde{a}x^2+2\tilde{b}x),\\
a_{10} &= a_{21}-a_1 = 2(\tilde{a}_1-1)(\tilde{a}x+\tilde{b}),\\
A_{10} &= (\tilde{a}_1-1)(\tilde{a}x^2+2\tilde{b}x).
\end{aligned}$$

Hence

$$y_1 = \exp[-(\tilde{a}_1-1)(\tilde{a}x^2+2\tilde{b}x)].$$

In the second particular solution we have to consider two cases:

$$\text{(I)}\quad \tilde{a}_1 = \tfrac{1}{2}, \qquad\qquad \text{(II)}\quad \tilde{a}_1 \neq \tfrac{1}{2}.$$

The first case corresponds to equation (5.11) and, after applying the general formulas for the solution y_2 of the second order equation we find $y_2 = xy_1$. This solution does not, in fact, differ from solution (5.12), p. 43.

In the second case we find, for the more general equation (5.13), the solution

$$y_2 = y_1 \int e^{\tilde{a}x^2+2\tilde{b}x}dx.$$

Thus, we have shown that the theory of class E equations allows us to solve an equation whose solution has been known only in a particular case.

5.3. Approximate method of solution of linear differential equations based upon the choice of one coefficient. According to what was said above, the choice of coefficient of a linear equation in order to satisfy the decomposibility condition is not unique. The expression for the unknown coefficient will always contain the arbitrary parameter $\tilde{a}_n$ and also the integral constants (in the case of change of $a_{n+1,i}$, for

$i > 0$). This fact is a very useful one, and it may serve as a basis of a method of approximate solution of equations, especially when these equations are connected with problems of physics or technology. In fact, in such problems the coefficients usually have an explicit physical interpretation, and often they are functions obtained by the approximation of experimental data. Taking this fact into account, we may determine the parameters in the coefficient chosen according to (5.1) in such a way that the equation will be a reasonably good description of the phenomenon under consideration.

6. Final remarks. Application of the method of decomposition into operator factors in the theory of partial linear differential equations

Finally, without trying to present the full treatment of the subject, we shall present an example of application of the method of decomposition in the theory of partial differential equations. Here we obtain the same methodological simplification: equations of higher order can be reduced to equations of lower order, to linear partial differential equations, and the latter — to ordinary linear differential equations.

Suppose, for instance, that we are given an equation of the second order of the form

$$\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial a}{\partial x} + a\frac{\partial a}{\partial y}\right)\frac{\partial u}{\partial y} - a^2\frac{\partial u^2}{\partial y^2} = f. \tag{6.1}$$

The function $u(x, y)$ is unknown, while $a(x, y)$ and $f(x, y)$ are given; they will be assumed to be as regular as is needed for further considerations. If a is a real number, then equation (6.1) is a non-homogeneous equation of vibration. If $a = ib$ (where b is real), we have the Poisson equation.

It is easily seen that equation (6.1) can be represented in the form

$$\left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y}\right) = f. \tag{6.2}$$

Equation (6.1) is equivalent to a system of two equations of the first order:

$$\begin{cases} \dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y} = w, \\[2ex] \dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f. \end{cases} \tag{6.3}$$

In the second equation only one unknown function $w(x, y)$ appears. Thus its solution reduces to the successive solution of two linear equations of the first order, in this case non-homogeneous. Thus the problem of integration of the second order partial differential equation (6.1) has been reduced to the solution of two equations of the first order, which in turn have been reduced to ordinary equations.

This method may, for instance, prove useful for the determination of the general solution of a differential equation, meaning here solution which contains a given number of arbitrary functions.

System (6.3) may be solved effectively in some particular cases. Suppose for instance that $a = \text{const}$. In the sequel we shall use the following theorem:

The general solution of the partial differential equation

$$a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=f \quad (a \neq 0), \tag{6.4}$$

where a and b are constants, and $f(\xi, \eta)$ is a function integrable in the interval under consideration, is of the form:

$$u=\frac{1}{a}\int_{x_0}^{x} f\left(\xi, \frac{b}{a}\xi+y-\frac{b}{a}x\right)d\xi+H(ay-bx), \tag{6.5}$$

where H is an arbitrary function from the class C^1.

In fact, the following system of ordinary differential equations corresponds to equation (6.4):

$$\frac{dx}{a}=\frac{dy}{b}, \quad \frac{dx}{a}=\frac{du}{f} \quad (f(x,y)\neq 0). \tag{6.6}$$

Since the solutions of this system are

$$y=\frac{b}{a}x+C_1 \quad \text{and} \quad u=\frac{1}{a}\int_{x_0}^{x} f\left(\xi, \frac{b}{a}\xi+C_1\right)d\xi+C_2,$$

system (6.6) has the following first integrals:

$$C_1=y-\frac{b}{a}x, \quad C_2=u-\frac{1}{a}\int_{x_0}^{x} f\left(\xi, \frac{b}{a}\xi-\frac{b}{a}x\right)d\xi.$$

Hence the general solution of (6.4) satisfies the equation:

$$H^*\left(ay-bx, u-\frac{1}{a}\int_{x_0}^{x} f\left(\xi, \frac{b}{a}\xi-\frac{b}{a}x\right)d\xi\right)=0.$$

Thus the general solution of (6.4) is (6.5), provided H^* is differentiable in the interval under consideration.

If follows from this theorem that the solution of the second equation (6.3) (in the case $a = \text{const}$) is

$$w(x, y) = \int_{x_0}^{x} f(\xi, a\xi + y - ax)\,d\xi + H_1(y - ax).$$

By the same theorem, the solution of the first equation (6.3) (i. e. the string equation) is

$$\begin{aligned} u(x, y) = \int_{x_0}^{x} d\eta \int_{\eta_0}^{\eta} f(\xi, a\xi - 2a\eta + y + ax)\,d\xi + \\ + H_1(y - ax) + H_2(y + ax), \end{aligned} \tag{6.7}$$

where H_1 and H_2 are arbitrary functions from the class C^2. This is the d'Alembert result for the non-homogeneous equation of the string.

If $a = i$, equation (6.1) becomes a Laplace equation, and, because of symmetry with respect to both independent variables, function (6.6) may be written in the form

$$u(x, y) = \int_{x_0}^{x} d\eta \int_{\eta_0}^{\eta} f(\xi, i\xi - 2i\eta + y - ix)\,d\xi + H_1(z) + H_2(\bar{z}),$$

where $z = x + iy$ and $\bar{z} = x - iy$, and H_1 and H_2 satisfy the assumptions of the preceding example.

In the case of the biharmonical equation,

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f(x, y),$$

one can reduce the problem of its integration, in a manner similar to that described above, either to the problem of four homogeneous partial differential equations of the first order, which can be succesively solved, or to two Poisson equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = w(x, y), \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x, y).$$

This allows us to write easily the general solution of the homogeneous biharmonical equation. Using formula (6.6) twice we get

$$\begin{aligned} u(x, y) = \int_{x_0}^{x} dt \int_{t_0}^{t} dr \int_{r_0}^{r} d\eta \int_{\eta_0}^{\eta} f(\xi, i\xi - 2i\eta + 2ir - 2it + y - ix)\,d\xi + \\ + H_1(z) + H_2(\bar{z}) + xH_3(z) + xH_4(\bar{z}), \end{aligned}$$

where $z = x + iy$ and $\bar{z} = x - iy$, and H_i $(i = 1, 2, 3, 4)$ are arbitrary functions which belong to the class C^4 in the domain under consideration.

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