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Multi-valued superpositions

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Introduction

Let Ω be an arbitrary set, X a space of functions on Ω with values in $\mathbb{R}^m$, Y a space of functions on Ω with values in $\mathbb{R}^n$, and F a multi-valued function which associates to each pair $(t, u) \in \Omega \times \mathbb{R}^m$ a nonempty set $F(t, u) \subseteq \mathbb{R}^n$. Associating then to each (single-valued) function $t \mapsto x(t)$ in X the set of all (single-valued) selections $t \mapsto y(t)$ of the (multi-valued) function $t \mapsto F(t, x(t))$ in Y defines the *superposition operator* N_F generated by F between the spaces X and Y. The purpose of this paper is to give a systematic description of this operator in terms of the generating multi-valued function F and the underlying spaces X and Y.

In the "single-valued version", this operator has been studied very well in the last 40 years; a detailed exposition may be found in the book [ApZa3]. Far less is known, however, in the multi-valued case, although multi-valued superposition operators occur frequently in applications: we just mention the theory of *integral* and *differential inclusions* (i.e. integral and differential equations with multi-valued right-hand sides), and the theory of *hysteresis* and *relay phenomena*.

The present paper consists of 6 chapters. In Chapter 1 we collect the necessary notions and facts on *multi-valued functions* and their *selections*. Except for Michael's selection principle, all results in this chapter are elementary, and therefore we state them without proofs. (All proofs may be found, for example, in the introductory books [Bo-Ob2] or [AuFr].) Instead, we encourage the reader to study the numerous examples and counter-examples, and to draw pictures of the corresponding multi-valued functions in the scalar case. Many general facts on both the theory and applications of multi-valued functions may also be found in the papers [AuCl, AuFr, Bo-Ob1, Dm, Di, EkTm, Fi4, KlTh, LsRo, Lv1] and elsewhere.

As we are mainly interested in multi-valued functions on the Cartesian product $\Omega \times \mathbb{R}^m$, we study *multi-valued functions of two variables* in Chapter 2. Particular emphasis is laid here on functions which satisfy a *Carathéodory condition* or *Scorza Dragoni condition*: the first means that, loosely speaking, F is measurable in the first and continuous in the second argument; the latter means that F is continuous "up to small sets". The importance of such functions for differential inclusions is the same as in the single-valued case for differential equations. However, if we replace the continuity in the second variable by a weaker semicontinuity assumption, many new features occur which are "hidden" in the single-valued theory.

In Chapter 3 we give a systematic account of *continuity* and *boundedness properties* of the superposition operator in the metric space S of measurable functions, in the normed space C of continuous functions, and between ideal spaces (i.e. L_∞-Banach modules) of measurable functions. As in the classical single-valued theory, the Carathéodory condition on F guarantees both the boundedness and continuity of N_F in the space S, and the continuity of F guarantees both the boundedness and continuity of N_F in the space C. On the other hand, it turns out that the boundedness and continuity of N_F between two ideal spaces X and Y relies very much on properties of these spaces, rather than on properties of the generating multi-valued function F.

As a matter of fact, important multi-valued functions arising in applications do not have the necessary properties for applying the results described in the third chapter. This emphasizes the need of passing from a given function F to some *extension* G which either takes values in a "nicer" class of sets, or generates a superposition operator N_G with "nicer" analytical properties. The most important and useful extensions, in this connection, are the *strong closure* $G = \overline{F}$, the *weak closure* $G = \vec{F}$, and the *convexification* $G = F^{\square}$. These extensions, as well as the superposition operators generated by them, are studied in Chapter 4. In particular, we are interested in conditions under which the operations of "taking extensions" and "taking operators" commute, i.e. $\overline{N}_F = N_{\bar{F}}$, $\vec{N}_F = N_{\vec{F}}$, or $N_F^{\square} = N_{F^{\square}}$. Some special results in this spirit may be found in the fifth chapter of the book [KrPk].

Chapter 5 is concerned with *fixed point theorems* and *integral inclusions*. First, we recall some *fixed point principles for multi-valued operators*. Here the basic results are the fixed point theorems of Nadler, Kakutani, and Bohnenblust–Karlin, which may be considered as multi-valued analogues of the classical fixed point theorems of Banach, Brouwer, and Schauder, respectively. These fixed point principles are then applied to operators of the form KN_F, where K is a linear (single-valued) integral operator, and N_F is the nonlinear (multi-valued) superposition operator described above. In this way we obtain existence (and sometimes also uniqueness) theorems for *nonlinear integral inclusions of Hammerstein type*. In the last section we describe a general method which allows us to reduce the study of (multi-valued) integral inclusions for vector functions to the study of (single-valued) integral equations for scalar functions.

The last Chapter 6 is devoted to selected *applications*. First, an application to an *elliptic system* with multi-valued right-hand side is sketched; in the "variational formulation" this also gives existence of *critical points for nonsmooth energy functionals*. Afterwards, we describe *forced periodic oscillations in nonlinear control systems* with "noise". Finally, we discuss a mathematical model for the *theory of heat regulation by thermostats* which leads to Hammerstein integral inclusions for two-dimensional vector functions.

We intentionally do not consider the most important and advanced field of applications, viz. *differential inclusions*, since this would have required us to increase the size of this paper at least two-fold. The reader is referred to the interesting and well-written books [AuCl] and [Dm].

Some words on the bibliography are in order. To the best of our knowledge, we tried to assign to each theorem the author who found it, or who proved it the way we did. We collected a large number of references on both the topics discussed here and those which are closely related; of course, we do not aim for a complete coverage. To help the reader not to get drowned in these references, we have added after (almost) each paper the corresponding review number of Zentralblatt für Mathematik (Zbl).

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1. Multifunctions and selections

In this chapter we collect all the facts on continuous and measurable multifunctions which will be used in what follows. Particular emphasis is put on the problem of finding selections with special properties. The only nontrivial result in this chapter is Michael's celebrated selection principle for lower semicontinuous multifunctions.

1. Multifunctions and selections. Throughout this paper, we use the following notation. Let (X, d) be a metric space. For $x \in X$, $M \subseteq X$, and $\varepsilon > 0$,

$$U_\varepsilon(x) = \{z : z \in X,\ d(z, x) < \varepsilon\} \tag{1.1}$$

denotes the ε-neighbourhood of x, and

$$U_\varepsilon(M) = \{z : z \in X,\ \varrho(z, M) < \varepsilon\} \tag{1.2}$$

the ε-neighbourhood of M; here

$$\varrho(z, M) = \inf\{d(z, x) : x \in M\},$$

as usual. The (right and left) *Hausdorff deviation* and the *Hausdorff distance*, respectively, of two sets M, $N \subseteq X$ are defined by

$$h^+(M, N) = \sup\{\varrho(z, N) : z \in M\} = \inf\{\varepsilon : \varepsilon > 0, M \subseteq U_\varepsilon(N)\}, \tag{1.3}$$

$$h^-(M, N) = h^+(N, M), \tag{1.4}$$

$$h(M, N) = \max\{h^+(M, N), h^-(M, N)\}, \tag{1.5}$$

respectively. Sometimes we write h_X instead of h in order to point out the underlying metric space X.

By $P(X)$ ($\mathrm{Bd}(X)$, $\mathrm{Cl}(X)$, $\mathrm{Cp}(X)$, respectively) we denote the system of all nonempty (nonempty bounded, nonempty closed, nonempty compact, respectively) subsets of X. If X is a linear space, $\mathrm{Cv}(X)$ denotes the system of all nonempty convex subsets of X. When combining these properties, be write $\mathrm{BdCl}(X)$ for $\mathrm{Bd}(X) \cap \mathrm{Cl}(X)$, $\mathrm{CpCv}(X)$ for $\mathrm{Cp}(x) \cap \mathrm{Cv}(X)$, and so on.

Recall that the system $\mathrm{BdCl}(X)$, equipped with the Hausdorff metric (1.5), is a metric space which is complete if X is. The larger system $\mathrm{Cl}(X)$ may also be equipped with a metric, viz.

$$h^*(M,N) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{h(M \cap U_k(z), N \cap U_k(z))}{1 + h(M \cap U_k(z), N \cap U_k(z))},$$

where z is an arbitrary fixed element of X (usually $z = 0$ if X is a linear space).

A *multifunction* (also called *multi-valued function*, *set-valued function*, *multi*, *relation*, or *correspondence*) between two metric spaces X and Y is a map $F : X \to P(Y)$. The *graph* of F is the subset of $X \times Y$ defined by

$$\Gamma(F) = \{(x,y) : x \in X,\ y \in F(x)\}. \tag{1.6}$$

Given $M \subseteq X$, the *image* of M under F is the set

$$F(M) = \bigcup_{x \in M} F(x). \tag{1.7}$$

Likewise, given $N \subseteq Y$, the *small pre-image* of N under F is

$$F_+^{-1}(N) = \{x : x \in X, F(x) \subseteq N\}, \tag{1.8}$$

while the *large* (or *complete*) *pre-image* of N under F is

$$F_-^{-1}(N) = \{x : x \in X, F(x) \cap N \neq \emptyset\}. \tag{1.9}$$

Observe that the small and large pre-image are related by the equalities

$$X \setminus F_+^{-1}(N) = F_-^{-1}(Y \setminus N), \qquad X \setminus F_-^{-1}(N) = F_+^{-1}(Y \setminus N).$$

This enables us to pass from F_+^{-1} to F_-^{-1} when there is a duality between N and $Y \setminus N$ (e.g., between open and closed sets).

There are some natural set-theoretic operations between multifunctions, viz. the *union*

$$(F \cup G)(x) = F(x) \cup G(x), \tag{1.10}$$

the *intersection*

$$(F \cap G)(x) = F(x) \cap G(x), \tag{1.11}$$

and the *product*

$$(F \times G)(x) = F(x) \times G(x) \tag{1.12}$$

of $F, G : X \to P(Y)$, as well as the *composition*

$$(H \circ G)(x) = H(G(x)) = \bigcup_{y \in G(x)} H(y) \tag{1.13}$$

of $G : X \to P(Y)$ and $H : Y \to P(Z)$. We summarize some elementary properties of these operations with the following

LEMMA 1.1. *Let $F : X \to P(Y)$, $G : X \to P(Y)$, and $H : Y \to P(Z)$ be multifunctions, and let $M, N \subseteq Y$ and $Q \subseteq Z$. Then the following holds*:

(a) $(F \cup G)_+^{-1}(N) = F_+^{-1}(N) \cap G_+^{-1}(N)$;
(b) $(F \cup G)_-^{-1}(N) = F_-^{-1}(N) \cup G_-^{-1}(N)$;
(c) $(F \cap G)_+^{-1}(N) \supseteq F_+^{-1}(N) \cap G_+^{-1}(N)$;

(d) $(F \cap G)_{-}^{-1}(N) \subseteq F_{-}^{-1}(N) \cap G_{-}^{-1}(N)$;
(e) $(F \times G)_{+}^{-1}(M \times N) = F_{+}^{-1}(M) \cap G_{+}^{-1}(N)$;
(f) $(F \times G)_{-}^{-1}(M \times N) = F_{-}^{-1}(M) \cap G_{-}^{-1}(N)$;
(g) $(H \circ G)_{+}^{-1}(Q) = G_{+}^{-1}[H_{+}^{-1}(Q)]$;
(h) $(H \circ G)_{-}^{-1}(Q) = G_{-}^{-1}[H_{-}^{-1}(Q)]$.

It is easy to find examples for strict inclusion in (c) or (d).

Let $F : X \to P(Y)$ be a multifunction. Any (single-valued) function $f : X \to Y$ with the property that $f(x) \in F(x)$ for all $x \in X$ is called a *selection* (or *selector* or *section*) of F. In many fields of both the theory and applications of multifunctions it is extremely important to ensure the existence of selections with special additional properties. For instance, large parts of §2 and §3 will be concerned with continuous and measurable selections, respectively.

Given a multifunction $F : X \to P(Y)$, we write

$$\mathrm{Sel}\, F = \{f : f(x) \in F(x) \text{ for all } x \in X\} \tag{1.14}$$

for the set of all selections of F.

If $M(X, Y)$ is some class of maps from X into Y, the operation (1.14) of taking selections may be considered as an operator $\mathrm{Sel} : M(X, P(Y)) \to P(M(X, Y))$. This operator admits a left inverse which associates with each $\Phi \in P(M(X, Y))$ the multifunction $x \mapsto \{\phi(x) : \phi \in \Phi\}$. Unfortunately, the operator $\mathrm{Sel} : M(X, P(Y)) \to P(M(X, Y))$ does not admit a right inverse, and thus is not onto.

There are many good books and monographs on both the theory and applications of multifunctions. In this chapter we follow the introductory treatise [Bo-Ob2]; the reader may also consult [AuFr, Bo-Ob1, CsVl, Di, LsRo]. A detailed study of useful topologies on the system $\mathrm{Cl}(X)$ (X a metric space) or $\mathrm{ClCv}(X)$ (X a normed linear space) may be found in the recent monograph [Be].

2. Continuous multifunctions and selections. We are now going to discuss continuity properties of multifunctions between metric spaces. The usual notion of continuity of a (single-valued) function may be generalized to multifunctions in several ways. A multifunction $F : X \to P(Y)$ is called *upper semicontinuous* at $x \in X$ if, for any open set $V \subseteq Y$ with $F(x) \subseteq V$, one may find an open neighbourhood $U \subseteq X$ of x such that $F(z) \subseteq V$ for all $z \in U$. Similarly, F is called *lower semicontinuous* at $x \in X$ if, for any open set $V \subseteq Y$ with $F(x) \cap V \neq \emptyset$, one may find an open neighbourhood $U \subseteq X$ of x such that $F(z) \cap V \neq \emptyset$ for all $z \in U$. A multifunction which is both upper semicontinuous and lower semicontinuous at x is simply called *continuous* at x.

In terms of sequences, semicontinuity of $F : X \to P(Y)$ may be characterized as follows: F is upper semicontinuous (respectively lower semicontinuous) at $x \in X$ if, for any open set $V \subseteq Y$ and any sequence $(x_n)_n$ converging to x, from $F(x) \subseteq V$ (respectively $F(x) \cap V \neq \emptyset$) it follows that $F(x_n) \subseteq V$ (respectively $F(x_n) \cap V \neq \emptyset$) for sufficiently large n.

As usual, we say that F is upper semicontinuous (lower semicontinuous, continuous, respectively) on X if F is upper semicontinuous (lower semicontinuous, continuous, respectively) at each point $x \in X$.

The following useful characterization of semicontinuity follows immediately from the definition.

LEMMA 2.1. *The upper semicontinuity of* $F : X \to P(Y)$ *is equivalent to each of the following three conditions*:

(a) $F_+^{-1}(V) \subseteq X$ *is open for any open* $V \subseteq Y$;
(b) $F_-^{-1}(W) \subseteq X$ *is closed for any closed* $W \subseteq Y$;
(c) $F_-^{-1}(\overline{N}) \supseteq \overline{F_-^{-1}(N)}$ *for any set* $N \subseteq Y$.

LEMMA 2.2. *The lower semicontinuity of* $F : X \to P(Y)$ *is equivalent to each of the following three conditions*:

(a) $F_-^{-1}(V) \subseteq X$ *is open for any open* $V \subseteq Y$;
(b) $F_+^{-1}(W) \subseteq X$ *is closed for any closed* $W \subseteq Y$;
(c) $F_+^{-1}(\overline{N}) \supseteq \overline{F_+^{-1}(N)}$ *for any set* $N \subseteq Y$.

Let us call a multifunction $F : X \to P(Y)$ ε-δ-*upper semicontinuous* (ε-δ-*lower semicontinuous*, ε-δ-*continuous*, respectively) at $x \in X$ if, for any $\varepsilon > 0$, one may find a $\delta > 0$ such that $h^+(F(z), F(x)) < \varepsilon$ ($h^-(F(z), F(x)) < \varepsilon$, $h(F(z), F(x)) < \varepsilon$, respectively) for all $z \in U_\delta(x)$. The following simple example shows that semicontinuity is in general not equivalent to ε-δ-semicontinuity.

EXAMPLE 2.1. Let $F : \mathbb{R} \to \mathrm{ClCv}(\mathbb{R}^2)$ be defined by $F(\alpha) = \{(x, \alpha x) : x \in \mathbb{R}\}$, i.e. F associates to each $\alpha \in \mathbb{R}$ the straight line with slope α. Then F is lower semicontinuous on $\mathbb{R}$, but not ε-δ-lower semicontinuous, since $F(\alpha) \subseteq U_\varepsilon(F(\beta))$ ($\varepsilon > 0$) only if $\beta = \alpha$.

On the other hand, let $F : \mathbb{R} \to \mathrm{ClCv}(\mathbb{R}^2)$ be defined by $F(x) = \{x\} \times [x, \infty)$. Then F is ε-δ-upper semicontinuous on $\mathbb{R}$, but not upper semicontinuous, since the closed set

$$W = \{(y, 1/y) : y > 0\}$$

has the nonclosed large pre-image $F_-^{-1}(W) = (0, 1]$. ∎

The point in Example 2.1 is that both multifunctions have unbounded values. This follows from the following relations between semicontinuity and ε-δ-semicontinuity which we state for further reference.

LEMMA 2.3. *The following holds*:

(a) *if* $F : X \to P(Y)$ *is upper semicontinuous*, *then* F *is* ε-δ-*upper semicontinuous*;
(b) *if* $F : X \to P(Y)$ *is* ε-δ-*lower semicontinuous*, *then* F *is lower semicontinuous*;
(c) *for* $F : X \to \mathrm{Cp}(Y)$, *upper semicontinuity is equivalent to* ε-δ-*upper semicontinuity*;
(d) *for* $F : X \to \mathrm{Cp}(Y)$, *lower semicontinuity is equivalent to* ε-δ-*lower semicontinuity.*

From Lemma 2.3 it follows, in particular, that a compact-valued multifunction F is continuous if and only if F is continuous with respect to the Hausdorff metric (1.5).

Apart from the semicontinuity criteria given in Lemma 2.1 and Lemma 2.2, the following characterization in terms of the distance function $\varrho : X \to [0, \infty)$ defined by

$$\varrho(x) = \varrho(y, F(x)) \quad (x \in X) \tag{2.1}$$

is useful, where $F : X \to P(Y)$ and $y \in Y$ is fixed:

LEMMA 2.4. *The following holds*:

(a) *if* $F : X \to \mathrm{Cp}(Y)$ *is upper semicontinuous then* ϱ *given by* (2.1) *is lower semicontinuous for all* $y \in Y$; *the converse is true if* $\overline{F(X)}$ *is compact*;

(b) $F : X \to P(Y)$ *is lower semicontinuous if and only if* ϱ *given by* (2.1) *is upper semicontinuous for all* $y \in Y$.

As Lemma 2.4 shows, the semicontinuity behaviour of the multifunction F and the distance function ϱ is not completely symmetric. We illustrate this by the following

EXAMPLE 2.2. Let F be the second multifunction in Example 2.1. As observed there, F is not upper semicontinuous. On the other hand, for fixed $y = (y_1, y_2) \in \mathbb{R}^2$ the distance function (2.1) is here

$$\varrho(x) = \begin{cases} \sqrt{(x - y_1)^2 + (x - y_2)^2} & \text{if } |x| \geq |y_2|, \\ |x - y_1| & \text{if } |x| < |y_2|, \end{cases}$$

and thus ϱ is lower semicontinuous (even continuous) on $\mathbb{R}$. ■

The following lemma shows how the various continuity properties of multifunctions F and G carry over to the multifunctions $F \cup G$, $F \cap G$, etc.:

LEMMA 2.5. *The following holds*:

(a) *if* $F, G : X \to P(Y)$ *are upper semicontinuous*, *then so is* $F \cup G$;
(b) *if* $F, G : X \to P(Y)$ *are lower semicontinuous*, *then so is* $F \cup G$;
(c) *if* $F, G : X \to \mathrm{Cl}(Y)$ *are upper semicontinuous*, *then so is* $F \cap G$;
(d) *if* $F, G : X \to \mathrm{Cp}(Y)$ *are upper semicontinuous*, *then so is* $F \times G$;
(e) *if* $F, G : X \to \mathrm{Cp}(Y)$ *are lower semicontinuous*, *then so is* $F \times G$;
(f) *if* $G : X \to P(Y)$ *and* $H : Y \to P(Z)$ *are upper semicontinuous*, *then so is* $H \circ G$;
(g) *if* $G : X \to P(Y)$ *and* $H : Y \to P(Z)$ *are lower semicontinuous*, *then so is* $H \circ G$.

A somewhat unexpected fact in Lemma 2.5 is that there is no statement on the lower semicontinuity of the intersection $F \cap G$ of two lower semicontinuous multifunctions F and G. Indeed, the analogue of (c) for lower semicontinuous multifunctions is false:

EXAMPLE 2.3 [Bo-Ob2]. Let $F, G : [0, \pi] \to \mathrm{CpCv}(\mathbb{R}^2)$ be defined by

$$F(\alpha) \equiv \{(\xi, \eta) : \xi^2 + \eta^2 \leq 1,\ \xi \geq 0\},$$
$$G(\alpha) = \{(\varrho \cos \alpha, \varrho \sin \alpha) : -1 \leq \varrho \leq 1\}.$$

Then F is constant (hence trivially continuous), and G is lower semicontinuous on $[0, \pi]$. Nevertheless, the intersection

$$(F \cap G)(\alpha) = \{(\varrho \cos \alpha, \varrho \sin \alpha) : 0 \leq \varrho \leq 1\}$$

is not lower semicontinuous at $\alpha = 0$ and $\alpha = \pi$. ■

It turns out that the lower semicontinuity of the intersection $F \cap G$ may be guaranteed only under additional assumptions. We give a sufficient condition in case of Banach spaces X and Y.

Lemma 2.6 [LeSp]. *Let X and Y be Banach spaces and $F, G : X \to \mathrm{ClCv}(Y)$ be two ε-δ-lower semicontinuous multifunctions. Suppose that $(F \cap G)(x) \in \mathrm{Bd}(Y)$ and $(F \cap G)^\circ(x) \neq \emptyset$ for all $x \in X$. Then $F \cap G$ is also ε-δ-lower semicontinuous.*

Lemma 2.6 applies, in particular, to the case when $G(x) = \{y : y \in Y,\ \|y - g(x)\| \leq r\}$, where $r > 0$ is fixed, $g : X \to Y$ is continuous, and $F(x) \cap U_r(g(x)) \neq \emptyset$ for any $x \in X$. We shall use this version of Lemma 2.6 later (see Theorem 10.1).

The above Example 2.2 shows that one must not drop the assumption that $(F \cap G)(x)$ has nonempty interior. The following example shows that one also must not drop the assumption that $F \cap G$ takes convex values.

Example 2.4 [LeSp]. Let $F, G : [0, 1] \to \mathrm{CpCv}(\mathbb{R}^2)$ be defined by

$$F(t) = \{(\xi, \eta) : 0 \leq \xi \leq 1/2,\ t(1 - 2\xi) \leq \eta \leq 1 - \xi\}$$
$$G(t) = \{(\xi, \eta) : 0 \leq \xi \leq 1, -\xi \leq \eta \leq 0\}.$$

Then $F(t)$ is not convex, and $F \cap G$ is not lower semicontinuous at 0. ■

A notion which is similar to the semicontinuity of a multifunction is that of closedness. A multifunction $F : X \to P(Y)$ is called *closed* if its graph (1.6) is a closed subset of $X \times Y$. Other equivalent characterizations are contained in the following

Lemma 2.7. *The closedness of $F : X \to P(Y)$ is equivalent to each of the following two conditions*:

(a) *for any $(x, y) \in X \times Y$ with $y \notin F(x)$ there exist neighbourhoods U of x and V of y, respectively, such that $F(U) \cap V = \emptyset$;*

(b) *for any sequence $(x_n, y_n) \in X \times Y$ with $x_n \to x$ and $y_n \to y$ the relation $y_n \in F(x_n)$ implies that $y \in F(x)$.*

The following is essentially parallel to Lemma 1.5:

Lemma 2.8. *The following holds*:

(a) *if $F, G : X \to \mathrm{Cl}(Y)$ are closed, then so is $F \cup G$;*
(b) *if $F, G : X \to \mathrm{Cl}(Y)$ are closed, then so is $F \cap G$;*
(c) *if $F, G : X \to \mathrm{Cl}(Y)$ are closed, then so is $F \times G$.*

Unfortunately, the composition of two closed multifunctions need not be closed:

Example 2.5. Let $G, H : \mathbb{R} \to \mathrm{CpCv}(\mathbb{R})$ be defined by

$$G(x) = \begin{cases} \{1/x\} & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0, \end{cases} \qquad H(y) = \begin{cases} \{1/y\} & \text{if } y \neq 0, \\ \{1\} & \text{if } y = 0. \end{cases}$$

Then both G and H are closed, while the composition

$$(H \circ G)(x) = \begin{cases} \{x\} & \text{if } x \neq 0, \\ \{1\} & \text{if } x = 0, \end{cases}$$

is not. ■

It follows from Lemma 2.7(b) that every closed multifunction takes only closed values. The converse is not true:

Example 2.6. Let $F:\mathbb{R}\to\mathrm{CpCv}(\mathbb{R})$ be defined by

$$F(x)=\begin{cases}\{0\} & \text{if } x\le 0,\\ [-1/x,1/x] & \text{if } x>0.\end{cases}$$

Then F is not closed, as may be seen by applying Lemma 2.7. ∎

Recall that a (single-valued) bounded real function is continuous if and only if its graph is closed. In contrast to this, the closedness of a multifunction does not even imply its semicontinuity:

Example 2.7. Let $F:[0,\pi]\to\mathrm{CpCv}(\mathbb{R})$ be defined by

$$F(x)=\begin{cases}[\tan x,1+\tan x] & \text{if } x\ne \pi/2,\\ \{0\} & \text{if } x=\pi/2.\end{cases}$$

Then F is neither upper semicontinuous nor lower semicontinuous but closed. ∎

If $F:X\to\mathrm{Cp}(Y)$ is an upper semicontinuous multifunction, then every image (1.7) of a compact set $M\subseteq X$ is again compact; this is completely analogous to single-valued continuous functions. If F is merely closed, the image (1.7) of a compact set $M\subseteq X$ is, in general, only closed, but not necessarily compact. For instance, in Example 2.7 we have $F([0,\pi])=\mathbb{R}$. Moreover, the image (1.7) of a closed set $M\subseteq X$ under a closed multifunction F need not be closed again:

Example 2.8. Let $F:[0,\infty)\to\mathrm{CpCv}(\mathbb{R})$ be defined by $F(x)=[e^{-x},1]$. Then F is closed, and hence maps compact sets into closed sets. However, the image of the closed noncompact set $M=[0,\infty)$ is the nonclosed set $F(M)=(0,1]$. ∎

In spite of the preceding counterexamples, there are some relations between upper semicontinuity and closedness. Recall that a multifunction $F:X\to\mathrm{Cl}(Y)$ is called *locally compact* if each point $x\in X$ has a neighbourhood U such that $\overline{F(U)}$ is compact.

Lemma 2.9. *If $F:X\to\mathrm{Cl}(Y)$ is upper semicontinuous, then F is closed. Conversely, if $F:X\to\mathrm{Cp}(Y)$ is closed and locally compact, then F is upper semicontinuous.*

Observe that the multifunction in Example 2.6 is not locally compact, since any neighbourhood of $x=\pi/2$ has unbounded image.

The following complements Lemma 2.5 and Lemma 2.8.

Lemma 2.10. *The following holds*:

(a) *if $F:X\to\mathrm{Cl}(Y)$ is closed, and $G:X\to\mathrm{Cp}(Y)$ is upper semicontinuous, then $F\cap G:X\to\mathrm{Cp}(Y)$ is upper semicontinuous*;

(b) *if $G:X\to\mathrm{Cp}(Y)$ is upper semicontinuous, and $H:Y\to\mathrm{Cl}(Z)$ is closed, then $H\circ G:X\to\mathrm{Cl}(Z)$ is closed.*

It is illuminating to compare these results with the counterexamples we considered so far.

We shall now consider the problem of finding *continuous selections*. If X and Y are metric spaces and X is locally compact, we denote by $C(X,Y)$ the space of all continuous functions from X into Y, equipped with the topology of uniform convergence on compact subsets of X. If X is even compact, we use the usual metric

$$d_C(\phi, \psi) = \sup\{d_Y(\phi(x), \psi(x)) : x \in X\}. \tag{2.2}$$

Given a multifunction $F : X \to P(Y)$, we write

$$\operatorname{Sel}_C F = \operatorname{Sel} F \cap C(X, Y) \tag{2.3}$$

for the set of all continuous selections of F. The problem of characterizing precisely the multifunctions in a given class with continuous selections is unsolved. One can only give sufficient conditions. First of all, we note that even very "harmless" upper semicontinuous multifunctions need not have continuous selections:

Example 2.9. Let $F : [0, 2] \to \mathrm{CpCv}(\mathbb{R})$ be defined by

$$F(x) = \begin{cases} \{0\} & \text{if } 0 \le x < 1, \\ [0, 1] & \text{if } x = 1, \\ \{1\} & \text{if } 1 < x \le 2. \end{cases}$$

Then F is upper semicontinuous (and also closed, by Lemma 2.9), but obviously has no continuous selection. ∎

The crucial point in this example is that F is not lower semicontinuous. This follows from the following important selection principle which is due to E. Michael [Mc1, Mc2] and has found numerous applications.

Theorem 2.1 [Mc1]. *Let X be a compact metric space, Y a Banach space, and $F : X \to \mathrm{ClCv}(Y)$ a lower semicontinuous multifunction. Then F admits a continuous selection.* ∎

The above Example 2.9 shows that lower semicontinuity must not be replaced with upper semicontinuity in Theorem 2.1. By means of other counterexamples, one may show that also the other hypotheses of Theorem 2.1 may not be dropped. For instance, the assertion fails without the convexity assumption on the images $F(x)$; this follows from the following counterexample, which is the only nontrivial example in this section:

Example 2.10. Let $D = \{x : |x|^2 = x_1^2 + x_2^2 \le 1\}$ be the closed unit disc in the Euclidean space $\mathbb{R}^2$, $S^1 = \partial D$ its boundary, and $F : D \to \mathrm{Cp}(\mathbb{R}^2)$ be defined by

$$F(x) = \begin{cases} S^1 \setminus \{\xi : |\xi - x|x|^{-1}| < |x|\} & \text{if } x \ne 0, \\ S^1 & \text{if } x = 0. \end{cases}$$

Then F is lower semicontinuous (even continuous!) on D, but does not admit a continuous selection. In fact, any continuous function $f : D \to D$ has a fixed point $\widehat{x} \in D$, by Brouwer's fixed point principle. Now, if f were a selection of F, we would have $\widehat{x} = f(\widehat{x}) \in F(\widehat{x}) \subseteq S^1$ and $\widehat{x} \ne 0$, hence $|\widehat{x} - \widehat{x}|\widehat{x}|^{-1}| \ge |\widehat{x}| = 1$, a contradiction. ∎

One may also show, by means of counterexamples, that the assertion of Theorem 2.1 becomes false without the closedness assumption on the images $F(x)$, or without the completeness assumption on the normed space Y (see [Mc1, Mc2]).

By Theorem 2.1, the operation Sel_C defined in (2.3) may be regarded as a map from $C(X, \mathrm{CpCv}(Y))$ into $\mathrm{CpCv}(C(X, Y))$.

Michael's selection principle admits a simple, though useful, refinement. Given a multifunction $F : X \to P(Y)$, let us call a sequence $(f_k)_k$ of (single-valued) functions $f_k : X \to Y$ an *exhaustion* of F if this sequence is dense at each point, i.e.

$$F(x) = \overline{\{f_1(x), f_2(x), \ldots\}} \quad (x \in X). \tag{2.4}$$

If all functions f_k may be chosen in $C(X,Y)$, we call $(f_k)_k$ a *continuous exhaustion.*

THEOREM 2.2 [Mc1]. *Let X be a compact separable metric space, Y a Banach space, and $F : X \to \mathrm{ClCv}(Y)$ a lower semicontinuous multifunction. Then F admits a continuous exhaustion.*

As another "by-product" of Michael's selection principle, we mention the following

LEMMA 2.11. *Let X be a compact metric space, Y a Banach space, and $F, G : X \to \mathrm{ClCv}(Y)$ two lower semicontinuous multifunctions. Then for every selection $g \in \mathrm{Sel}_C\, G$ and every $\delta > 0$ there exists a selection $f \in \mathrm{Sel}_C\, F$ such that*

$$\|f(x) - g(x)\| \le (1+\delta) h^-(F(x), G(x)) \tag{2.5}$$

for all $x \in X$.

If we choose, in particular, $G(x) = \{g(x)\}$ with $g \in C(X,Y)$ in Lemma 2.11, we obtain the existence of a continuous selection f of F satisfying

$$\|f(x) - g(x)\| \le (1+\delta)\varrho(g(x), F(x)). \tag{2.6}$$

After Michael's pioneering paper [Mc1], a large literature on continuous selections of lower semicontinuous multifunctions appeared. Many of these papers were concerned with weaker continuity assumptions on F. For example, the lower semicontinuity of $F : X \to \mathrm{ClCv}(Y)$ (X compact metric space, Y Banach space) may be replaced by the so-called *weak ε-δ-lower semicontinuity* [DeMy1]. Here F is called weakly ε-δ-lower semicontinuous at $x \in X$ if, for any $\varepsilon > 0$ and any neighbourhood U of x, one may find a $\delta > 0$ such that $U_\delta(x) \subseteq U$, and $x' \in U_\delta(x)$ such that $h^-(F(z), F(x')) < \varepsilon$ for all $z \in U_\delta(x)$. Every ε-δ-lower semicontinuous multifunction is weakly ε-δ-lower semicontinuous, as may be seen by putting $x' = x$, but not vice versa.

We point out that Michael's theorem (and all its consequences) are true not only for compact, but also for paracompact metric spaces. This essentially improves the applicability of this theorem; we shall need this, for instance, in §4 below.

It is interesting to note that, if X and Y are topological linear spaces, the assertion of Michael's theorem for lower semicontinuous multifunctions $F : X \to \mathrm{ClCv}(Y)$ is essentially equivalent to the metrizability of the space Y [Li, Mg]. The paper [Bh] contains various characterizations of multifunctions $F : X \to \mathrm{ClCv}(Y)$ (Y Banach space) with continuous selections. For some recent extensions of Michael's theorem see [PlYo] and the bibliography therein.

Apart from the selection problem for semicontinuous multifunctions, the following *approximation problem* is of importance in multi-valued analysis: given a multifunction $F : X \to \mathrm{ClCv}(Y)$ and $\varepsilon > 0$, find a function $f \in C(X,Y)$ such that

$$h^+(\Gamma(f), \Gamma(F)) \le \varepsilon, \tag{2.7}$$

where h^+ denotes the Hausdorff deviation (1.3) in $X \times Y$, and Γ is the graph (1.6). One of the first papers on such problems for compact upper semicontinuous multifunctions F goes back to J. von Neumann. Important extensions were made in [Cl1, LaOp]; a detailed survey may be found in [DeMy2].

As a sample result which has important applications in fixed point theory and elsewhere, let us cite the following

LEMMA 2.12 [Cl2]. *Let $X \subset \mathbb{R}^m$ be compact and $F : X \to \mathrm{CpCv}(\mathbb{R}^n)$ be an upper semicontinuous multifunction. For $\delta > 0$, define $F_\delta : X \to \mathrm{ClCv}(\mathbb{R}^n)$ by*

$$F_\delta(x) = \mathrm{co}\{y : y \in F(z),\ |z - x| \le \delta\}.$$

Then for all $\varepsilon > 0$ the estimate

$$h^+(\Gamma(F_\delta), \Gamma(F)) \le \varepsilon \tag{2.8}$$

holds for $\delta > 0$ small enough.

3. Measurable multifunctions and selections. Apart from semicontinuous multifunctions, measurable multifunctions will be of great importance in the sequel. In this section we assume throughout that Y is a separable metric space, and $(\Omega, \mathfrak{A}, \mu)$ is a measure space, i.e. a set Ω equipped with a σ-algebra $\mathfrak{A}$ of subsets and a countably additive measure μ on $\mathfrak{A}$. A typical example is when Ω is a bounded domain in the Euclidean space $\mathbb{R}^k$, equipped with the Lebesgue measure.

A multifunction $F : \Omega \to P(Y)$ is called *measurable* if $F_+^{-1}(V) \subseteq \Omega$ is measurable for each open $V \subseteq Y$ or, equivalently, $F_-^{-1}(W) \subseteq \Omega$ is measurable for each closed $W \subseteq Y$. Similarly, F is called *weakly measurable* if $F_+^{-1}(W) \subseteq \Omega$ is measurable for each closed $W \subseteq Y$ or, equivalently, $F_-^{-1}(V) \subseteq \Omega$ is measurable for each open $V \subseteq Y$.

Other ways of defining measurability consist in requiring the measurability of the graph (1.6) in the product $\Omega \times Y$, equipped with the minimal σ-algebra $\mathfrak{A} \otimes \mathfrak{B}(Y)$ generated by the sets $A \times B$ with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}(Y)$ (the Borel subsets of Y), or requiring the measurability of the distance function (2.1) for every $y \in Y$.

For further reference, we collect some relations between these definitions in the following

LEMMA 3.1 [Hm2]. *Let $F : \Omega \to \mathrm{Cl}(Y)$ be a multifunction. Then the following holds*:

(a) *if F is measurable*, *then F is also weakly measurable*;

(b) *if F takes compact values*, *measurability and weak measurability of F are equivalent*;

(c) *F is weakly measurable if and only if the distance function $\varrho(y, F(\cdot))$ is measurable for all $y \in Y$*;

(d) *if F is weakly measurable*, *the graph $\Gamma(F)$ is product-measurable*;

(e) *if Y is σ-compact* (*i.e. a countable union of compact sets*), *measurability of F*, *weak measurability of F*, *measurability of the distance function $\varrho(y, F(\cdot))$ for each $y \in Y$*, *and product-measurability of the graph $\Gamma(F)$ are all equivalent.*

Since we do not want to deal with different measurability concepts in what follows, we shall assume throughout that Y is σ-compact, and hence the conclusion (e) holds. Thus, the term "measurable" means from now on any of the four measurability properties considered above.

The following is parallel to Lemma 2.5.

Lemma 3.2. *The following holds*:

(a) *if* $F, G : \Omega \to \mathrm{Cl}(Y)$ *are measurable*, *then so is* $F \cup G$;
(b) *if* $F, G : \Omega \to \mathrm{Cl}(Y)$ *are measurable*, *then so is* $F \cap G$;
(c) *if* $F, G : \Omega \to \mathrm{Cl}(Y)$ *are measurable*, *then so is* $F \times G$.

We point out that the σ-compactness of the space Y is essential in Lemma 3.2(b). In fact, from [Hm2, Corollary 4.2] and [Cr, Theorem 3.10] it follows that the σ-compactness of Y is even *equivalent* to the following property of Y: for any measure space $(\Omega, \mathfrak{A}, \mu)$ and every sequence $(F_n)_n$ of weakly measurable multifunctions $F_n : \Omega \to \mathrm{Cl}(Y)$, the intersection $\bigcap_{n\in\mathbb{N}} F_n : \Omega \to \mathrm{Cl}(Y)$ is also weakly measurable.

Of course, the composition ot two measurable multifunctions need not be measurable; this may be shown by simple examples as in the single-valued case, e.g. by the following

Example 3.1. Let $\Omega = [0,1]$ be equipped with the Lebesgue measure, and let $g : \Omega \to \mathbb{R}$ be a strictly increasing Cantor function (see e.g. [Ha]). It is well known that one may find a measurable set $D \subset \mathbb{R}$ such that $g^{-1}(D)$ is not measurable. If we define $G : \Omega \to \mathrm{CpCv}(\mathbb{R})$ and $H : \mathbb{R} \to \mathrm{CpCv}(\mathbb{R})$ by

$$G(t) = \{g(t)\} \quad (t \in \Omega), \qquad H(u) = \begin{cases} \{1\} & \text{if } u \in D, \\ \{0\} & \text{if } u \notin D, \end{cases}$$

then both G and H are measurable, but $H \circ G$ is not. ∎

For further reference, we collect the results and counterexamples given so far on the conservation of semicontinuity, closedness, or measurability properties in the following table:

$F,\ G$	upper semicontinuous	lower semicontinuous	closed	measurable
$F \cup G$	yes (L.2.5)	yes (L.2.5)	yes* (L.2.8)	yes* (L.3.2)
$F \cap G$	yes* (L.2.5)	no (E.2.3)	yes* (L.2.8)	yes* (L.3.2)
$F \times G$	yes** (L.2.5)	yes** (L.2.5)	yes* (L.2.8)	yes* (L.3.2)
$F \circ G$	yes (L.2.5)	yes (L.2.5)	no (E.2.5)	no (E.3.1)

* if F and G have closed values   ** if F and G have compact values

A famous relation between measurability and continuity of single-valued functions is established by *Luzin's theorem*, which states, roughly speaking, that $f : \Omega \to Y$ is measurable if and only if f is continuous "up to subsets of Ω of arbitrarily small measure". It is not surprising that this result has an analogue for multifunctions. We shall say that a multifunction $F : \Omega \to P(Y)$ has the *Luzin property* if, given $\delta > 0$, one may find a closed subset Ω_δ of Ω such that $\mu(\Omega \setminus \Omega_\delta) \le \delta$, and the restriction of F to Ω_δ is continuous.

Theorem 3.1 [Ja2]. *A multifunction* $F : \Omega \to \mathrm{Cl}(Y)$ *is measurable if and only if* F *has the Luzin property.*

In what follows, by $S(\Omega, Y)$ we denote the space of all (classes of) measurable functions from Ω into Y, equipped with the metric

$$d_S(\phi, \psi) = \inf\{\tau + \mu(\phi - \psi, \tau) : 0 < \tau < \infty\}, \tag{3.1}$$

where

$$\mu(z, \tau) = \mu(\{t : t \in \Omega,\ |z(t)| > \tau\}). \tag{3.2}$$

Convergence $d_S(\phi_n, \phi) \to 0$ with respect to this metric is then equivalent to convergence in measure, i.e.

$$\mu(\{t : d(\phi_n(t), \phi(t)) > \tau\}) \to 0,$$

as $n \to \infty$, for any $\tau > 0$.

We are now going to study the problem of finding *measurable selections*. To this end, we consider again the selection set (1.14), where now $f(t) \in F(t)$ is required, of course, only for almost all $t \in \Omega$, and write

$$\mathrm{Sel}_S F = \mathrm{Sel}\, F \cap S(\Omega, Y). \tag{3.3}$$

It turns out that, in contrast to continuous selections, the problem of finding measurable selections is nearly trivial.

THEOREM 3.2 [KwRy]. *Every measurable multifunction $F : \Omega \to \mathrm{Cl}(Y)$ admits a measurable selection.*

If we define a *measurable exhaustion* as a sequence $(f_k)_k$ of (single-valued) measurable functions satisfying (2.4), we get the following analogue to Theorem 2.2:

THEOREM 3.3 [Cs2]. *Every measurable multifunction $F : \Omega \to \mathrm{Cl}(Y)$ admits a measurable exhaustion.*

Because of Theorem 3.3, measurable exhaustions are often called *Castaing representations* in the literature. Obviously, the converse of Theorem 3.3 is also true: *if $F : \Omega \to \mathrm{Cl}(Y)$ admits a measurable exhaustion, then F is measurable.*

Since just measurability is a rather weak property, one is sometimes interested in measurable exhaustions with additional properties. For instance, in [CeCo] it is shown that, given a measurable multifunction $F : \Omega \to \mathrm{Cl}(\mathbb{R}^n)$ which is L_p-dominated, i.e.

$$\sup\{|u| : u \in F(t)\} \le \varphi(t) \qquad (\varphi \in L_p),$$

one may always find an exhaustion (2.4) which is precompact in L_p (see also [Bt]).

We close with the following corollary which is parallel to Lemma 2.11:

LEMMA 3.3. *Let $F, G : \Omega \to \mathrm{Cl}(Y)$ be two measurable multifunctions. Then for every selection $g \in \mathrm{Sel}_S G$ and every $\delta > 0$ there exists a selection $f \in \mathrm{Sel}_S F$ such that*

$$d(f(t), g(t)) \le (1 + \delta) h^-(F(t), G(t)) \tag{3.4}$$

for almost all $t \in \Omega$. In particular, for any $g \in S(\Omega, Y)$ one can find a measurable selection $f \in \mathrm{Sel}_S F$ such that

$$d(f(t), g(t)) \le (1 + \delta) \varrho(g(t), F(t)) \tag{3.5}$$

for almost all $t \in \Omega$.

A vast literature is devoted to the problem of finding measurable selections of measurable multifunctions between general spaces. A standard reference is [CsVl]; moreover, without pretending to completeness, we refer to the survey articles [De, Gf, Lv2, Wg1, Wg2].

2. Multifunctions of two variables

In this chapter we shall be concerned only with multifunctions which are defined on the topological product of some measurable set with the Euclidean space $\mathbb{R}^m$ and take as values closed (sometimes compact) subsets of the Euclidean space $\mathbb{R}^n$. We are particularly interested in Carathéodory multifunctions, Scorza Dragoni multifunctions, and their various generalizations. Apart from their fundamental importance in all fields of multi-valued analysis, such multifunctions are useful for obtaining implicit function theorems of Filippov type which will also be discussed below.

4. Carathéodory multifunctions and selections. Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space as in the preceding section, and let X and Y be two metric spaces. Recall that a single-valued function $f : \Omega \times X \to Y$ is called a *Carathéodory function* if $f(\cdot, u)$ is measurable on Ω for all $u \in X$, and $f(t, \cdot)$ is continuous on X for all (or almost all) $t \in \Omega$. We may generalize this definition to multifunctions in rather the same way as we generalized the continuity in Section 2. Since we do not need Carathéodory multifunctions in the most general form, we shall restrict ourselves to the case $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ in what follows. A multifunction $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ is called *upper Carathéodory* (respectively *lower Carathéodory*) if $F(t, \cdot) : \mathbb{R}^m \to P(\mathbb{R}^n)$ is upper semicontinuous (respectively lower semicontinuous) for (almost) all $t \in \Omega$, and $F(\cdot, u) : \Omega \to P(\mathbb{R}^n)$ is measurable for all $u \in \mathbb{R}^m$. If F is both upper and lower Carathéodory, we call F simply a *Carathéodory multifunction*.

As before, by $\mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$ we denote the minimal σ-algebra generated by the sets $A \in \mathfrak{A}$ and the Borel subsets of $\mathbb{R}^m$, and the term “product-measurable” means measurability with respect to $\mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$.

An important property of Carathéodory multifunctions is given in the following

LEMMA 4.1. *Let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ be a Carathéodory multifunction. Then F is product-measurable.*

Proof. Consider the countable dense subset $\mathbb{Q}^m \subset \mathbb{R}^m$. For closed $W \subseteq \mathbb{R}^n$, $a \in \mathbb{Q}^m$, and $k \in \mathbb{N}$, the set

$$G_k(W; a) = \{t : t \in \Omega,\ F(t, a) \cap U_{1/k}(W) \neq \emptyset\} \times \{u : u \in \mathbb{R}^m,\ |u - a| \leq 1/k\}$$

belongs to $\mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$. By the lower semicontinuity of F in the second variable, we have

$$F_-^{-1}(W) \subseteq \bigcap_{k\in\mathbb{N}} \bigcup_{a\in\mathbb{Q}^m} G_k(W; a),$$

while the upper semicontinuity implies the reverse inclusion. ∎

As we shall see later (Example 4.1), an upper or lower Carathéodory multifunction need not be product-measurable.

Let $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ be a fixed multifunction. We are interested in the existence of *Carathéodory selections*, i.e. Carathéodory functions $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ such that $f(t,u) \in F(t,u)$ for almost all $t \in \Omega$ and all $u \in \mathbb{R}^m$. It is evident that, in case F is upper Carathéodory, this selection problem does not have a solution, in general. (The counterexamples are the same as in Michael's selection principle.) For lower Carathéodory multifunctions F, however, this is an interesting problem.

There are two ways, essentially, to attack this problem. On the one hand, we may show that the multifunction $\Phi : \Omega \to \mathrm{ClCv}(C(\mathbb{R}^m, \mathbb{R}^n))$ defined by

$$\Phi(t) = \mathrm{Sel}_C\, F(t,\cdot) \tag{4.1}$$

is measurable (note that $\Phi(t) \neq \emptyset$ by Theorem 2.1!). Every measurable selection of Φ will then give rise to a Carathéodory selection of F. In this connection, one of the most important and useful tools is the Aumann–Sainte-Beuve selection theorem [Sn1, Sn2, Sn3].

On the other hand, we may show that the multifunction $\Psi : \mathbb{R}^m \to \mathrm{ClCv}(S(\Omega, \mathbb{R}^n))$ defined by

$$\Psi(u) = \mathrm{Sel}_S\, F(\cdot, u) \tag{4.2}$$

is lower semicontinuous (note that $\Psi(u) \neq \emptyset$ by Theorem 3.1!). Every continuous selection of Ψ will then give rise to a Carathéodory selection of F.

The first way was followed, for example, in [Cs5, Fr, Ku1], the second way, for example, in [Ri2]. We shall employ the multifunction (4.1) in the sequel.

Lemma 4.2 given below establishes a necessary and sufficient condition, in terms of the multifunction (4.1), for the existence of a *Carathéodory exhaustion*, i.e. a sequence $(f_k)_k$ of Carathéodory functions $f_k : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ such that

$$F(t,u) = \overline{\{f_1(t,u), f_2(t,u), \ldots\}}. \tag{4.3}$$

We remark that a Carathéodory multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{ClCv}(\mathbb{R}^n)$ always admits a Carathéodory exhaustion, but the converse is not true. To prove the first statement in the autonomous case $F = F(u)$, say, fix $u_0 \in \mathbb{Q}^m$ and $v_0 \in F(u_0) \cap \mathbb{Q}^n$, and replace F by the multifunction

$$F(u; u_0, v_0) = \begin{cases} \{v_0\} & \text{if } u = u_0, \\ F(u) & \text{otherwise.} \end{cases}$$

Then $F(\cdot; u_0, v_0)$ is lower semicontinuous and so admits a continuous selection $f(\cdot; u_0, v_0) : \mathbb{R}^m \to \mathbb{R}^n$ with $f(u_0; u_0, v_0) = v_0$, by Theorem 2.1. The family of all these selections gives then rise to a continuous exhaustion of F.

A multifunction which is not Carathéodory, but admits a Carathéodory exhaustion, is given in the following

Example 4.1. Let $\Omega = [0,1]$ be equipped with the Lebesgue measure, and $F : \Omega \times \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ defined by

$$F(t,u) = \begin{cases} \{0\} & \text{if } u = 0, \\ [0,1] & \text{otherwise.} \end{cases}$$

Then F is lower Carathéodory, but not upper Carathéodory, and hence not Carathéodory. Nevertheless, F admits many Carathéodory exhaustions. ■

LEMMA 4.2 [Ku1]. *Let* $F : \Omega\times\mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ *be a multifunction such that* $F(t,\cdot)$ *is lower semicontinuous for* (*almost*) *all* $t \in \Omega$. *Then* F *admits a Carathéodory exhaustion if and only if the multifunction* Φ *defined in* (4.1) *is measurable.*

Proof. Suppose that the multifunction (4.1) is measurable, and hence there exists a measurable exhaustion $(\phi_k)_k$ of Φ, by Theorem 3.3. Define $f_k : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ ($k = 1, 2, \ldots$) by $f_k(t, u) = \phi_k(t)(u)$; we claim that each f_k is a Carathéodory function. The continuity of $f_k(t,\cdot)$, for fixed $t \in \Omega$, follows from the fact that ϕ_k maps Ω into $C(\mathbb{R}^m,\mathbb{R}^n)$. Further, by the separability of $\mathbb{R}^n$ and Lemma 3.1(c), for proving the measurability of $f_k(\cdot, u)$ for fixed $u \in \mathbb{R}^m$, it suffices to show that the distance function $\varrho(t) = |v - f_k(t, u)|$ is measurable for any $v \in \mathbb{R}^n$. But this follows from the equality

$$\varrho^{-1}((-\infty,\tau]) = \{t : t \in \Omega, \varrho(t) \le \tau\} = \Phi^{-1}(\{g : g \in C(\mathbb{R}^m,\mathbb{R}^n), |v - g(u)| \le \tau\}).$$

The density of $\{f_1(t, u), f_2(t, u), \ldots\}$ in $F(t, u)$, for almost all $t \in \Omega$ and all $u \in \mathbb{R}^m$, follows from the fact that for every $v \in F(t, u)$ we may choose a continuous function $\phi \in \Phi(t)$ such that $\phi(u) = v$. This shows, altogether, that $(f_k)_k$ is a Carathéodory exhaustion of F.

Conversely, suppose that F admits a Carathéodory exhaustion $(f_k)_k$ and define $\phi_k : \Omega \to C(\mathbb{R}^m,\mathbb{R}^n)$ ($n = 1, 2, \ldots$) by $\phi_k(t) = f_k(t,\cdot)$. It is clear that every ϕ_k is a measurable selection of the multifunction (4.1). However, it may happen that the set $\{\phi_1(t), \phi_2(t), \ldots\}$ is not dense in $\Phi(t)$ for t in a subset of Ω of positive measure. In this case we take, for each $k \in \mathbb{N}$, a finite partition of unity $\{g^k_1, g^k_2, \ldots, g^k_{p(k)}\}$ such that the diameter of $(g^k_j)^{-1}((0, 1])$ is less than $1/k$. The family of all Carathéodory selections of the form

$$\widetilde{f}_{k,m}(t, u) = g^k_1(u)f_{m_1}(t, u) + \ldots + g^k_{p(k)}(u)f_{m_{p(k)}}(t, u)$$

($k = 1, 2, \ldots$; $m_j = 1, 2, \ldots$) is still countable, and the corresponding family $\widetilde{\phi}_{k,m}(t) = \widetilde{f}_{k,m}(t,\cdot)$ is dense in $\Phi(t)$ for almost all $t \in \Omega$. The measurability of the multifunction (4.1) follows now from the remark after Theorem 3.3. ■

Lemma 4.2 reduces the problem of finding Carathéodory exhaustions for F to that of proving the measurability of the multifunction (4.1).

To characterize the multifunctions with Carathéodory exhaustions, another definition is in order. We call $(\Omega,\mathfrak{A},\mu)$ *m-projective* if, for any $D \in \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$, the projection $P_\Omega(D)$ of D onto Ω belongs to the σ-algebra $\mathfrak{A}$, possibly up to some nullset. There are three important cases in which $(\Omega,\mathfrak{A},\mu)$ is m-projective, viz. if the measure μ is σ-finite on Ω, if μ has the "direct sum property" (see [CsVl] or [Lv1]), or if μ is a Radon measure over a locally compact topological space Ω.

THEOREM 4.1 [Fr]. *Let* $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ *be a multifunction such that* $F(t,\cdot)$ *is lower semicontinuous for* (*almost*) *all* $t \in \Omega$. *Assume that* $(\Omega,\mathfrak{A},\mu)$ *is* m*-projective. Then* F *admits a Carathéodory exhaustion if and only if* F *is product-measurable.*

Proof. Let $(f_k)_k$ be a Carathéodory exhaustion for F. By Lemma 4.1, every f_k is product-measurable, and hence so is F, by the remark after Theorem 3.3.

Conversely, suppose that F is product-measurable. For any function $g \in C(\mathbb{R}^m, \mathbb{R}^n)$, the function $\varphi_g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\varphi_g(t,u) = \varrho(g(u), F(t,u)) = \inf\{|g(u) - v| : v \in F(t,u)\} \tag{4.4}$$

is then also product-measurable. Define $\chi_g : \Omega \to [0,\infty]$ by

$$\chi_g(t) = \sup\{\varphi_g(t,u) : u \in \mathbb{R}^m\}\,. \tag{4.5}$$

For each $\tau \in \mathbb{R}$ we have

$$\begin{aligned}\chi_g^{-1}([\tau,\infty)) &= \{t : t \in \Omega, \chi_g(t) \geq \tau\}\\ &= \bigcap_{k\in\mathbb{N}} \{t : t \in \Omega, \varphi_g(t,u) \geq \tau - 1/k \text{ for some } u \in \mathbb{R}^m\}\\ &= \bigcap_{k\in\mathbb{N}} P_\Omega(\{(t,u) : (t,u) \in \Omega \times \mathbb{R}^m,\ \varphi_g(t,u) \geq \tau - 1/k\}),\end{aligned}$$

and thus $\chi_g^{-1}([\tau,\infty)) \in \mathfrak{A}$, since the function (4.4) is product-measurable and $(\Omega, \mathfrak{A}, \mu)$ is m-projective. This shows that the function (4.5) is measurable for every $g \in C(\mathbb{R}^m, \mathbb{R}^n)$.

Consider now again the multifunction Φ given in (4.1). As already observed, for every $v \in F(t,u)$ we may choose a function $\phi \in \Phi(t)$ ($\subseteq C(\mathbb{R}^m, \mathbb{R}^n)$) such that $\phi(u) = v$. Consequently, we have

$$\sup_{u\in\mathbb{R}^m} \inf_{\phi\in\Phi(t)} |g(u) - \phi(u)| = \sup_{u\in\mathbb{R}^m} \inf_{v\in F(t,u)} |g(u) - v| = \sup_{u\in\mathbb{R}^m} \varphi_g(t,u) = \chi_g(t). \tag{4.6}$$

Now, the following reasoning shows that we may reverse the sup over $u \in \mathbb{R}^m$ and the inf over $\phi \in \Phi(t)$ in the first term of (4.6). On the one hand, the inequality

$$\sup_{u\in\mathbb{R}^m} \inf_{\phi\in\Phi(t)} |g(u) - \phi(u)| \leq \inf_{\phi\in\Phi(t)} \sup_{u\in\mathbb{R}^m} |g(u) - \phi(u)|$$

is always true. On the other hand, given $\varepsilon > 0$ and $u_0 \in \mathbb{R}^m$, choose $\phi_0 \in \Phi(t)$ such that

$$|g(u_0) - \phi_0(u_0)| < \sup_{u\in\mathbb{R}^m} \inf_{\phi\in\Phi(t)} |g(u) - \phi(u)| + \varepsilon. \tag{4.7}$$

Denote the right-hand side of (4.7) by r. Since $\phi_0(u_0) \in F(t,u_0)$, by Lemma 2.11 we may find $\phi \in \Phi(t)$ such that $|g(u_0) - \phi(u_0)| \leq r$. Since $u_0 \in \mathbb{R}^m$ was arbitrary, we have proved that also

$$\inf_{\phi\in\Phi(t)} \sup_{u\in\mathbb{R}^m} |g(u) - \phi(u)| \leq \sup_{u\in\mathbb{R}^m} \inf_{\phi\in\Phi(t)} |g(u) - \phi(u)|.$$

From (4.6) we get now

$$\chi_g(t) = \inf_{\phi\in\Phi(t)} \sup_{u\in\mathbb{R}^m} |g(u) - \phi(u)| = \inf\{\|g - \phi\|_C : \phi \in \Phi(t)\} = \varrho(g, \Phi(t)).$$

We conclude that the distance function $\varrho(t) = \varrho(g, \Phi(t))$ is measurable, for each $g \in C(\mathbb{R}^m, \mathbb{R}^n)$, and thus the multifunction Φ is measurable, by Lemma 3.1(c). The assertion follows now from Lemma 4.2. ∎

Observe that in Theorem 4.1 one cannot replace the phrase "F admits a Carathéodory exhaustion" by "F is a Carathéodory multifunction". In fact, in Example 4.1 the multifunction F is product-measurable, and the space $(\Omega, \mathfrak{A}, \mu)$ is 1-projective.

A crucial point in Theorem 4.1 is the product-measurability of F, which is of course stronger than the measurability of $F(\cdot, u)$ for each $u \in \mathbb{R}^m$. From a naive point of view, one could expect to get the existence of Carathéodory selections simply by combining the selection principles for lower semicontinuous multifunctions (Theorem 2.1) and for measurable multifunctions (Theorem 3.2), i.e. by requiring that $F : \Omega\times\mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ is lower Carathéodory. The following example shows that this is false.

EXAMPLE 4.2 [Ku1]. Let $\Omega = [0,1]$, $\mathfrak{A}$ the σ-algebra generated by all singletons, and μ the counting measure on $\mathfrak{A}$. Define a multifunction $F : \Omega \times \mathbb{R} \to \mathrm{CpCv}(\mathbb{R})$ by

$$F(t,u) = \begin{cases} \{t\} & \text{if } t = u \text{ or } |t-u|^{-1} \in \mathbb{N}, \\ [0,1] & \text{otherwise.} \end{cases}$$

For fixed $t \in \Omega$ and $V \subseteq \mathbb{R}$, $F(t,\cdot)_-^{-1}(V)$ is equal to Ω or $\Omega\setminus\{t, t\pm1, t\pm2, \ldots\}$, and hence $F(t,\cdot)$ is lower semicontinuous. For fixed $u \in \Omega$ and $V \subseteq \mathbb{R}$, in turn, $F(\cdot,u)_-^{-1}(V)$ is equal to Ω or $\Omega \setminus C$, where C is some subset of the countable set $\{t, t\pm 1/u, t\pm 2/u, \ldots\}$, and hence $F(\cdot,u)$ is measurable. Nevertheless, a straightforward but somewhat cumbersome computation shows that F does not admit a Carathéodory selection. ∎

We point out that $(\Omega, \mathfrak{A}, \mu)$ is in fact 1-projective in Example 4.2, since μ has the "direct sum property". The nonexistence of Carathéodory selections of F is therefore due, according to Theorem 4.1, to the fact that F is not product-measurable. Indeed, the large pre-image $F_-^{-1}(W)$ of the set $W = [0,1/2]$ cannot belong to $\mathfrak{A}\otimes\mathfrak{B}(\mathbb{R})$, since $P_\Omega(F_-^{-1}(W)) = [0,1]$ does not belong to $\mathfrak{A}$.

Observe that in this example the functions (4.4) and (4.5) are not measurable for each $g \in C(\mathbb{R},\mathbb{R})$. In fact, for $g(u) \equiv 0$, say, we get

$$\varphi_0(t,u) = \inf\{|v| : v \in F(t,u)\} = \begin{cases} \{t\} & \text{if } t = u \text{ or } |t-u|^{-1} \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$
$$\chi_0(t) = \sup\{\varphi_0(t,u) : u \in \mathbb{R}\} = t.$$

But the function $\chi_0(t) = t$ is *not* measurable with respect to the σ-algebra $\mathfrak{A}$, since χ_0 is not constant outside a countable set $C \subset \Omega$.

Example 4.2 shows that a lower Carathéodory multifunction need not be product-measurable; compare this with Lemma 4.1.

There is a vast literature on Carathéodory selections for multifunctions. Actually, Lemma 4.2 and Theorem 4.1 above have been proved in [Ku1] and [Fr], respectively, in the more general setting of metric and Banach spaces. The papers [Cs4, Cs5, Cs6, Ry] deal with the existence of Carathéodory selections of lower Carathéodory multifunctions under additional "regularity" assumptions. The papers [DeMy3] and [Ku2] are concerned with the problem of extending a Carathéodory multifunction from $\Omega\times A$ (A a closed subset of a metric space X) to $\Omega\times X$; in [Ku2] one may also find a counterexample which shows that this is not always possible. In [Cl3] it is shown that $F : \Omega\times X \to \mathrm{CpCv}(Y)$ admits a Carathéodory selection if $F(t,\cdot)$ is lower semicontinuous for all $t \in \Omega$, and $F(\cdot,u)$ is upper semicontinuous for all $u \in X$. An interesting geometrical argument based on metric projections in Hilbert spaces is used in [KuNo1] to construct many Carathéodory selections f of a given Carathéodory multifunction $F : \Omega\times X \to \mathrm{ClCv}(H)$ (H a Hilbert space) explicitly, viz. $f(t,u) = P(g(t); F(t,u))$, where $g : \Omega \to H$ is an arbitrary measurable (single-valued) function, and $P(h;C)$ is the point of best approximation of h in $C \in \mathrm{ClCv}(H)$. For further results on Carathéodory multifunctions, see [KuNo3].

The fact that a product-measurable lower Carathéodory multifunction admits a Carathéodory exhaustion was generalized in various directions. For instance, [Jn] gives essentially a generalization of Theorem 4.1 to product-measurable multifunctions F with the property that $F(t,\cdot)$ is weakly ε-δ-lower semicontinuous (see the end of §2) for each $t \in \Omega$. A "direct" proof of (a generalization of) Theorem 4.1 (i.e. without using the auxiliary multifunctions (4.1) or (4.2)) is given in [KiPrYa1, KiPrYa2].

In [Ri2] it is shown that Carathéodory selections exist also if the underlying space X is not necessarily separable. Another additional property (the so-called "Michael property") of a lower Carathéodory multifunction which ensures the existence of a Carathéodory selection is discussed in [ArPr]. The existence of Carathéodory selections in more general spaces is discussed in [Sl1, Sl2, Sl3]. Moreover, in [Io1] it is shown that measurable multifunctions with uncountable values may be represented as "slices" of Carathéodory multifunctions. Finally, an extension theorem of Tietze–Urysohn–Dugundji type for Carathéodory multifunctions may be found in [DeMy4].

Sometimes it is useful to study multifunctions F on the graph of some other multifunction $G : \Omega \to \mathrm{Cl}(\mathbb{R}^m)$, rather than on the whole "rectangle" $\Omega \times \mathbb{R}^m$. Given such a multifunction G, we say that $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ is *Carathéodory on* $\Gamma(G)$ if $F(t,\cdot) : G(t) \to \mathrm{Cl}(\mathbb{R}^n)$ is continuous for (almost) all $t \in \Omega$, and $F(\cdot,u) : \Omega \to \mathrm{Cl}(\mathbb{R}^n)$ is measurable for all $u \in \mathbb{R}^m$. The following theorem provides Carathéodory selections for such multifunctions; the proof is taken from [AuFr, Theorem 9.5.2].

THEOREM 4.2. *Let* $F : \Omega\times\mathbb{R}^m \to \mathrm{ClCv}(\mathbb{R}^n)$ *be a Carathéodory multifunction on* $\Gamma(G)$, *where* $G : \Omega \to \mathrm{Cl}(\mathbb{R}^m)$ *is a fixed multifunction. Let* $x \in S(\Omega,\mathbb{R}^m)$ *and* $y \in \mathrm{Sel}_S F(\cdot,x(\cdot))$ *be given*, *i.e.*

$$y(t) \in F(t,x(t)) \tag{4.8}$$

for almost all $t \in \Omega$. *Then there exists a Carathéodory selection* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ *of* F *such that*

$$y(t) = f(t,x(t)) \tag{4.9}$$

for almost all $t \in \Omega$.

Proof. For fixed $(t,u) \in \Omega \times \mathbb{R}^m$, denote by $f(t,u)$ the point of best approximation to $y(t)$ in the closed convex set $F(t,u)$, i.e.

$$|y(t) - f(t,u)| = \varrho(y(t),F(t,u)).$$

It is clear that (4.8) implies (4.9). The measurability of $f(\cdot,u)$ is obvious, while the continuity of $f(t,\cdot)$ follows from the fact that, for a continuous multifunction $\Phi : \mathbb{R}^m \to \mathrm{ClCv}(\mathbb{R}^n)$, a *minimal selection* $\phi : \mathbb{R}^m \to \mathbb{R}^n$ which may be characterized by the relation

$$|\phi(u)| = \min\{|v| : v \in \Phi(u)\}$$

is single-valued and continuous. ∎

We remark that an analogous result may be proved in case the continuity in the second argument is replaced by Lipschitz continuity [Lo2,Or]: if $F(t,\cdot) : G(t) \to \mathrm{ClCv}(\mathbb{R}^n)$

is Lipschitz continuous, then there exists a Carathéodory selection f of F such that $f(t,\cdot):\mathbb{R}^m\to\mathbb{R}^n$ is Lipschitz continuous.

5. The Scorza Dragoni property. In this section we assume throughout that Ω is also a metric space and the σ-algebra $\mathfrak{A}$ contains the Borel subsets of Ω. We say that a multifunction $F:\Omega\times\mathbb{R}^m\to P(\mathbb{R}^n)$ has the *upper Scorza Dragoni property* (respectively *lower Scorza Dragoni property*) if, given $\delta>0$, one may find a closed subset Ω_δ of Ω such that $\mu(\Omega\setminus\Omega_\delta)\le\delta$ and the restriction of F to $\Omega_\delta\times\mathbb{R}^m$ is upper semicontinuous (respectively lower semicontinuous). If F has both the upper and lower Scorza Dragoni property, we say that F has the *Scorza Dragoni property.* Thus, the Scorza Dragoni property plays the same role for multifunctions of two variables as the Luzin property for multifunctions of one variable (see Theorem 3.1). The Scorza Dragoni property of single-valued functions has been introduced in [Sc]; the first who studied this property for multifunctions seems to be Kikuchi [Ki1, Ki2].

There is a close connection between Carathéodory multifunctions and multifunctions having the Scorza Dragoni property. For the sake of simplicity, we assume now that Ω is σ-compact, i.e. a countable union of compact sets.

THEOREM 5.1 [Ki2]. *A multifunction $F:\Omega\times\mathbb{R}^m\to\mathrm{Cp}(\mathbb{R}^n)$ is Carathéodory if and only if F has the Scorza Dragoni property.*

Proof. The fact that every multifunction F having the Scorza Dragoni property is Carathéodory is obvious.

Suppose, conversely, that F is Carathéodory and that, without loss of generality, Ω is compact. For a fixed natural number N, denote by K_N the cube

$$K_N=\{u:u=(u_1,\ldots,u_m)\in\mathbb{R}^m,\ |u_1|,\ldots,|u_m|\le N\}$$

of volume 2^mN^m in $\mathbb{R}^m$. For $p=1,2,\ldots,$ we decompose K_N into 2^{mp} equal subcubes $K_{p,q}$ $(q=1,\ldots,2^{mp})$ of volume $2^{-m(p-1)}N^m$. Define $\sigma_{p,q}:\Omega\to\mathbb{R}$ by

$$\sigma_{p,q}(t)=\sup\{h(F(t,u),F(t,v)):u,v\in K_{p,q}\}\tag{5.1}$$

$(p=1,2,\ldots;\ q=1,\ldots,2^{mp})$, where h is the Hausdorff metric (1.5) on $\mathrm{Cp}(\mathbb{R}^n)$. The function (5.1) is obviously measurable, hence also the functions

$$\sigma_p(t)=\max\{\sigma_{p,q}(t):q=1,\ldots,2^{mp}\}.\tag{5.2}$$

By assumption, $F(t,\cdot)$ is (uniformly) continuous on K_N for almost all $t\in\Omega$; consequently, the functions (5.2) converge a.e. on Ω to 0, as $p\to\infty$.

Fix $\delta>0$, and let $\delta_j=\delta 2^{-(N+j+1)}$ for $j\in\mathbb{N}$. Denoting by $z_{p,q}$ the centre of the subcube $K_{p,q}$, by Theorem 3.1 we may find a compact subset $\widetilde{\Omega}_1$ of Ω such that $\mu(\Omega\setminus\widetilde{\Omega}_1)\le\delta_1$, and the restriction of $F(\cdot,z_{1,q})$ $(q=1,\ldots,2^m)$ to $\widetilde{\Omega}_1$ is continuous. Afterwards, we may find a compact subset $\widetilde{\Omega}_2$ of $\widetilde{\Omega}_1$ such that $\mu(\Omega\setminus\widetilde{\Omega}_2)\le\delta_1+\delta_2$ and the restriction of $F(\cdot,z_{p,q})$ $(p=1,2;\ q=1,\ldots,2^{mp})$ to $\widetilde{\Omega}_2$ is continuous. Proceeding this way, we construct a decreasing sequence $(\widetilde{\Omega}_j)_j$ of compact subsets of Ω such that $\mu(\Omega\setminus\widetilde{\Omega}_j)\le\delta_1+\ldots+\delta_j$, and the restriction of $F(\cdot,z_{p,q})$ $(p=1,\ldots,j;\ q=1,\ldots,2^{mp})$

to $\widetilde{\Omega}_j$ is continuous. Putting now $\widetilde{\Omega}_\infty = \widetilde{\Omega}_1 \cap \widetilde{\Omega}_2 \cap \ldots$, we have

$$\mu(\Omega \setminus \widetilde{\Omega}_\infty) \leq \sum_{j=1}^{\infty} \delta_j = \delta 2^{-(N+1)}, \tag{5.3}$$

and the restriction of $F(\cdot, z_{p,q})$ $(p = 1, 2, \ldots; q = 1, \ldots, 2^{mp})$ to $\widetilde{\Omega}_\infty$ is continuous.

Since the functions (5.2) converge to 0 a.e. on $\widetilde{\Omega}_\infty$, by Egorov's theorem we may find a compact set $\Omega_N \subseteq \widetilde{\Omega}_\infty$ such that $\mu(\widetilde{\Omega}_\infty \setminus \Omega_N) \leq \delta 2^{-(N+1)}$ (hence $\mu(\Omega \setminus \Omega_N) \leq \delta 2^{-N}$, by (5.3)) and (5.2) converges to 0 uniformly on Ω_N.

Now fix $(t_0, u_0) \in \Omega_N \times K_N$, and let $\varepsilon > 0$. Choose $p_0 \in \mathbb{N}$ such that $\sigma_{p_0}(t) \leq \varepsilon$ for all $t \in \Omega_N$. Denote by $C(u_0)$ the union of all cubes $K_{p_0,q}$ $(q = 1, \ldots, 2^{mp_0})$ which contain points u with $|u - u_0| \leq N 2^{-(p_0+1)}$, and denote by $D(u_0)$ the set of all corresponding centres $z_{p_0,q}$ of such cubes. For $t \in \Omega_N$ and $u_1, u_2 \in C(u_0)$ we have

$$h(F(t, u_1), F(t, u_2)) \leq 2\sigma_{p_0}(t) \leq 2\varepsilon,$$

hence also

$$h(F(t, u), F(t, u_0)) \leq 2\varepsilon \tag{5.4}$$

for any $(t, u) \in \Omega_N \times C(u_0)$. Since $F(\cdot, z_{p,q})$ is continuous on Ω_N, by construction, we may find $\eta > 0$ such that $|t - t_0| \leq \eta$ implies

$$h(F(t, D(u_0)), F(t_0, D(u_0))) \leq \varepsilon. \tag{5.5}$$

Combining (5.4) and (5.5) yields

$$\begin{aligned} h(F(t, u), F(t_0, u_0)) \leq{}& h(F(t, u), F(t, D(u_0))) \\ &+ h(F(t, D(u_0)), F(t_0, D(u_0))) \\ &+ h(F(t_0, D(u_0)), F(t_0, u_0))) \leq 5\varepsilon \end{aligned}$$

whenever $(t, u) \in \Omega_N \times K_N$, $|t - t_0| \leq \eta$, and $|u - u_0| \leq N \cdot 2^{-(p_0+1)}$.

This shows that the restriction of F to $\Omega_N \times K_N$ is continuous. Now it remains to put

$$\Omega_\delta = \bigcap_{N \in \mathbb{N}} \Omega_N,$$

and to observe that $\mu(\Omega \setminus \Omega_\delta) \leq \delta$. ∎

The above proof is somewhat technical but elementary. There are other proofs which require some knowledge on dual spaces and measure theory but are quite elegant. The following proof was kindly communicated to the authors by C. Castaing [Cs7]. Let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ be a Carathéodory multifunction. We may consider F as a (single-valued!) function from $\Omega \times \mathbb{R}^m$ into the separable metric space $(\mathrm{Cp}(\mathbb{R}^n), h)$ with Hausdorff metric (1.5). Since any separable metric space can be imbedded into a countable product of $[0, 1]$ and the space $\mathbb{R}^m$ is a countable union of compact sets, everything reduces to the case of a Carathéodory function $f : \Omega \times K \to [0, 1]$, where $K \subset \mathbb{R}^m$ is compact. Now, since f is product-measurable (see Lemma 4.1) and the map $t \mapsto \int_K f(t, u)\, \mu(du)$ is measurable on Ω for any fixed Borel measure μ, we conclude that also the map $\varphi : \Omega \to C(K, \mathbb{R})$ defined by $\varphi(t) = f(t, \cdot)$ is measurable. By Luzin's theorem (for functions with values in a separable Banach space) this implies that the map φ has the Luzin property, i.e. for

$\delta > 0$ we may find a closed subset Ω_δ of Ω such that $\mu(\Omega \backslash \Omega_\delta) \le \delta$, and the resriction of φ to Ω_δ is continuous. It follows that f is continuous on $\Omega_\delta \times K$, and so we are done.

Theorem 5.1 has been generalized to multifunctions with closed values in [Br]. Later on, analogous results have been proved for more general spaces Ω, X and Y. For instance, [Cs3] treats the case of a complete metric space X and a separable Banach space Y, while [HmVl2] considers multifunctions with closed values in a locally compact separable metric space Y and reduces the problem to the compact-valued case by means of the Aleksandrov one point compactification of Y. In [Hm1] it is shown that, for Y a separable metric space, a compact-valued Carathéodory multifunction has the Scorza Dragoni property, while a closed-valued Carathéodory multifunction has only the lower Scorza Dragoni property, in general (see also [HmVl1]).

One should expect that an analogue to Theorem 5.1 of the following form is true: a multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ is upper (respectively lower) Carathéodory if and only if F has the upper (respectively lower) Scorza Dragoni property. Surprisingly, this is true only in one direction. In the following theorem we suppose that the measure μ is complete on Ω.

Theorem 5.2. *If $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ has the upper Scorza Dragoni property, then F is upper Carathéodory. Similarly, if $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ has the lower Scorza Dragoni property, then F is lower Carathéodory.*

Proof. Suppose that F has the upper Scorza Dragoni property, and let $(\Omega_k)_k$ be a sequence of closed subsets of Ω such that $\mu(\Omega \setminus \Omega_k) \le 1/k$ and the restriction F_k of F to $\Omega_k \times \mathbb{R}^m$ is upper semicontinuous. Fix $u \in \mathbb{R}^m$ and put $\Phi(t) = F(t, u)$ and $\Phi_k(t) = F_k(t, u)$. The set $(\Phi_k)_+^{-1}(V)$ is then open for any open $V \subseteq \mathbb{R}^n$, and hence measurable. This implies that

$$\Phi_+^{-1}(V) = \bigcup_{k \in \mathbb{N}} (\Phi_k)_+^{-1}(V) \cup \{t : t \in \Omega \setminus (\Omega_1 \cup \Omega_2 \cup \ldots),\ \Phi(t) \subseteq V\}$$

is also measurable, since $\Omega \setminus (\Omega_1 \cup \Omega_2 \cup \ldots)$ is a nullset and μ is complete. This shows that $F(\cdot, u) : \Omega \to \mathrm{Cl}(\mathbb{R}^n)$ is measurable for each $u \in \mathbb{R}^m$.

The upper semicontinuity of $F(t, \cdot) : \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ for almost all $t \in \Omega$ is easy to prove. In fact, if we assume that the set of all $t \in \Omega$ such that $F(t, \cdot)$ is *not* upper semicontinuous has positive measure, we arrive at a contradiction by considering F_k for sufficiently large k. Finally, the fact that a multifunction with the lower Scorza Dragoni property is lower Carathéodory is proved analogously. ∎

We remark that Theorem 5.2 is also true for multifunctions $F : \Omega \times X \to Y$, where X and Y are metric spaces and $(\Omega, \mathfrak{A}, \mu)$ is a Radon measure space [Zy10]. This follows from the fact that the Luzin theorem (see Theorem 3.1) is true for functions between a Radon measure space into a metric space [Fm].

The following example shows that a lower Carathéodory multifunction need not have the lower Scorza Dragoni property:

Example 5.1 [Ob2]. Let $\Omega = [0, 1]$ be equipped with the Lebesgue measure, $D \subset \Omega$ a nonmeasurable subset, and $F : \Omega \times \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ defined by

$$F(t,u)=\begin{cases}\{0\} & \text{if } u=t \text{ and } t\in \Omega\setminus D,\\ \{1\} & \text{if } u=t \text{ and } t\in D,\\ [0,1] & \text{otherwise.}\end{cases}$$

Then F is lower Carathéodory, but does not have the lower Scorza Dragoni property. In fact, if the restriction of F to $\Omega_\delta\times\mathbb{R}$ ($\mu(\Omega\setminus\Omega_\delta)\le\delta$) were lower semicontinuous, the same would be true for the restriction of F to the set $\{(t,t):t\in\Omega_\delta\}$, which is impossible. ∎

As communicated in the book [Dm], a similar counterexample was given by Bothe for proving a nonexistence result for the differential inclusion $x'(t)\in F(t,x(t))$ with initial condition $x(0)=0$ in $\mathbb{R}^2$.

It is easy to see that the multifunction F given in Example 5.1 is not product-measurable; this is also a consequence of the following

THEOREM 5.3 [HmVl2, ArPr]. *If $F:\Omega\times\mathbb{R}^m\to \mathrm{Cl}(\mathbb{R}^n)$ is a product-measurable lower Carathéodory multifunction, then F has the lower Scorza Dragoni property.*

Proof. Let $\delta>0$, and let $\{v_1,v_2,\ldots\}$ be a dense subset in $\mathbb{R}^n$. Define functions $s_k:\Omega\times\mathbb{R}^m\to\mathbb{R}$ and multifunctions $S_k:\Omega\to \mathrm{Cl}(\mathbb{R}^m\times\mathbb{R})$ by

$$s_k(t,u)=\varrho(v_k,F(t,u)), \tag{5.6}$$

$$S_k(t)=\{(u,r):0\le r\le s_k(t,u)\}. \tag{5.7}$$

Since the multifunctions (5.7) are measurable, by Theorem 3.1 we find a closed set $\Omega_k\subseteq\Omega$ such that $\mu(\Omega\setminus\Omega_k)\le\delta 2^{-k}$, and the restriction of S_k to Ω_k is continuous. By Lemma 2.4 and the lower semicontinuity of $F(t,\cdot)$, we know in turn that the functions (5.6) are upper semicontinuous on $\Omega_k\times\mathbb{R}^m$. Putting $\Omega_\delta=\Omega_1\cap\Omega_2\cap\ldots$, we have $\mu(\Omega\setminus\Omega_k)\le\delta$, and the functions (5.6) are upper semicontinuous on $\Omega_\delta\times\mathbb{R}^m$ for all $k\in\mathbb{N}$. Since $(v_k)_k$ is a dense sequence it follows again from Lemma 2.4 that F is lower semicontinuous on $\Omega_\delta\times\mathbb{R}^m$, and so F has the lower Scorza Dragoni property as claimed. ∎

We show now by means of another counterexample (which is a modification of a counterexample by Brunovský [Br]) that also an upper Carathéodory multifunction need not have the upper Scorza Dragoni property:

EXAMPLE 5.2. Let $\Omega=[0,1]$ be equipped with the Lebesgue measure, $D\subset\Omega$ a nonmeasurable set, and $F:\Omega\times\mathbb{R}\to \mathrm{Cp}(\mathbb{R})$ defined by

$$F(t,u)=\begin{cases}[0,1] & \text{if } u=t \text{ and } t\in D,\\ \{0\} & \text{otherwise.}\end{cases} \tag{5.8}$$

It is not hard to see that F is an upper Carathéodory multifunction. On the other hand, suppose that $\Omega_k\subseteq\Omega$ $(k=1,2,\ldots)$ is closed such that $\mu(\Omega\setminus\Omega_k)\le 1/k$, and the restriction F_k of F to $\Omega_k\times\mathbb{R}$ is upper semicontinuous. By Lemma 2.1(b), the set $(F_k)_-^{-1}(\{1\})=\{(t,t):t\in D\cap\Omega_k\}$ is then closed. Consequently, the set

$$D_+=D\cap\Big(\bigcup_{k\in\mathbb{N}}\Omega_k\Big)=\bigcup_{k\in\mathbb{N}}(D\cap\Omega_k)$$

is measurable. On the other hand, since

$$\mu\Big(D\setminus\bigcup_{k\in\mathbb{N}}\Omega_k\Big)\le\mu\Big(\Omega\setminus\bigcup_{k\in\mathbb{N}}\Omega_k\Big)=0,$$

the set

$$D_{-} = D \setminus \Big(\bigcup_{k\in\mathbb{N}} \Omega_k \Big)$$

is also measurable, contradicting our choice of $D = D_{+} \cup D_{-}$. ■

We remark that a similar reasoning shows that the multifunction (5.8) does not have the lower Scorza Dragoni property either. If we exchange the sets $\{0\}$ and $[0,1]$ in Example 5.2 we get the lower Carathéodory multifunction

$$F(t,u) = \begin{cases} \{0\} & \text{if } u = t \text{ and } t \in D, \\ [0,1] & \text{otherwise.} \end{cases}$$

In view of Example 5.2, the problem arises to characterize those upper Carathéodory multifunctions which have the upper Scorza Dragoni property. Surprisingly enough, such a characterization is in fact possible for compact-valued multifunctions. Let us say that a multifunction $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ satisfies the *Filippov condition* if, for any open sets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, the set

$$\Omega[U,V] = \{t : t \in \Omega,\ F(t,U) \subseteq V\} \tag{5.9}$$

is measurable, i.e. belongs to $\mathfrak{A}$.

LEMMA 5.1 [Fv1, Fv2]. *Let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ be an upper Carathéodory multifunction. Then F has the upper Scorza Dragoni property if and only if F satisfies the Filippov condition.*

Proof. The fact that the Filippov condition implies the upper Scorza Dragoni property is obvious; therefore we prove only the converse. Suppose that F has the upper Scorza Dragoni property, and let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. Up to a homeomorphism, we may represent U as set of sequences $u = (u_1, u_2, \ldots)$ of natural numbers, equipped with the discrete topology (see e.g. [Kw]). Choose a basis in U consisting of sets of the form $U(m_1, \ldots, m_k) = \{(u_1, u_2, \ldots) : u_j = m_j\ (j = 1, \ldots, k)\}$. Moreover, let $\mathfrak{B}$ be a countable basis in $\mathbb{R}^m$ consisting of convex sets with the property that, whenever $K \subset \mathbb{R}^n$ is convex and compact, and $L \supset K$ is open, we have $K \subseteq B \subseteq L$ for some $B \in \mathfrak{B}$. Finally, arrange all sets $B \in \mathfrak{B}$ with $B \subseteq V$ in a sequence $\{B_1, B_2, \ldots\}$. By construction, the sets $\Omega[U(m_1, \ldots, m_k), B_j]$ $(k, j = 1, 2 \ldots)$ are measurable, hence also the sets

$$G(m_1, \ldots, m_k) = \bigcup_{j=1}^{\infty} \Omega[U(m_1, \ldots, m_k), B_j], \qquad F(m_1, \ldots, m_k) = \Omega \setminus G(m_1, \ldots, m_k).$$

Now, the sets $F(m_1, \ldots, m_k)$ form a regular system (see again [Kw]), and thus the set T of all $t \in \Omega$ for which there exists some sequence $u = (u_1, u_2, \ldots) \in U$ such that $t \in F(m_1, \ldots, m_k)$ for all $k \in \mathbb{N}$, is measurable. By construction, $t \in \Omega$ belongs to T if and only if t has no neighbourhood $U(m_1, \ldots, m_k) \subseteq U$ such that $t \in \Omega[U(m_1, \ldots, m_k), B_j]$ for some $j \in \mathbb{N}$. But this means precisely that $\Omega \setminus T = \Omega[U,V]$, and the measurability of $\Omega[U,V]$ follows from that of T. Consequently, F satisfies the Filippov condition. ■

Another proof of Lemma 5.1 may be found in the recent book [Fv3]. The following counterexample shows that the "if" part of Lemma 5.1 is not true any more if we assume F merely to be closed-valued.

EXAMPLE 5.3 [Zy10]. Let $\Omega = [0,1]$ be equipped with the Lebesgue measure, and $F : \Omega \times \mathbb{R} \to \mathrm{Cl}(\mathbb{R}^2)$ defined by

$$F(t,u) = \{(\xi, t\xi) : \xi \in \mathbb{R}\}.$$

Then F is Carathéodory, since $F(t,\cdot)$ is constant for all $t \in [0,1]$, and $F(\cdot,u)$ is measurable for all $u \in \mathbb{R}$, the graph $\Gamma(F(\cdot,u))$ being closed in $[0,1]\times\mathbb{R}^2$. Moreover, it is easily checked that F satisfies the Filippov condition; in fact, for any open set $V \subseteq \mathbb{R}^2$ the set $\Omega[V]$ (see (5.9)) consists of all $t \in [0,1]$ such that the straight line through the origin with slope t is entirely contained in V.

Nevertheless, F cannot have the upper Scorza Dragoni property, since $F(\cdot,u)$ is not upper semicontinuous on any subset $\Omega_\delta \subset \Omega$. ∎

As a consequence of Lemma 5.1, we may in turn give the following simple characterization which is in some sense similar to Theorem 4.1:

THEOREM 5.4 [Zy5]. *Let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ be an upper Carathéodory multifunction. Assume that $(\Omega, \mathfrak{A}, \mu)$ is m-projective. Then F has the upper Scorza Dragoni property if and only if F is product-measurable.*

Proof. We use Lemma 5.1. Suppose first that F is product-measurable, let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, and put $W = \mathbb{R}^n \setminus V$. Since $F_-^{-1}(W) \cap (\Omega \times U)$ belongs to $\mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$ and $(\Omega, \mathfrak{A}, \mu)$ is m-projective, we have

$$\{t : t \in \Omega, F(t,U) \cap W \neq \emptyset\} = P_\Omega(F_-^{-1}(W) \cap (\Omega \times U)) \in \mathfrak{A},$$

and hence (see (5.9))

$$\Omega[U,V] = \Omega \setminus \{t : t \in \Omega, F(t,U) \cap W \neq \emptyset\} \in \mathfrak{A}.$$

Conversely, suppose that F satisfies the Filippov condition. For each closed set $W \subseteq \mathbb{R}^n$, define a multifunction $\Phi_W : \Omega \to \mathrm{Cl}(\mathbb{R}^m)$ by

$$\Phi_W(t) = F(t,\cdot)_-^{-1}(W). \tag{5.10}$$

(Note that Φ_W takes closed values, since $F(t,\cdot)$ is upper semicontinuous.) We claim that Φ_W is measurable. In fact, for any open set $U \subseteq \mathbb{R}^m$ we have

$$\begin{aligned}(\Phi_W)_-^{-1}(U) &= \{t : t \in \Omega,\ \Phi_W(t) \cap U \neq \emptyset\} \\ &= \{t : t \in \Omega,\ F(t,U) \cap W \neq \emptyset\} = \Omega \setminus \Omega[U, \mathbb{R}^n \setminus W],\end{aligned}$$

and the last set belongs to $\mathfrak{A}$, since F satisfies the Filippov condition. By Lemma 3.1(d), the graph $\Gamma(\Phi_W)$ of Φ_W is measurable, i.e. $\Gamma(\Phi_W) \in \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$. But

$$\Gamma(\Phi_W) = \{(t,u) : u \in \Phi_W(t)\} = \{(t,u) : F(t,U) \cap W \neq \emptyset\} = F_-^{-1}(W).$$

This shows that $F_-^{-1}(W) \in \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$ for any closed set $W \subseteq \mathbb{R}^n$ and thus F is product-measurable. ∎

It is interesting to consider again Example 5.2 in view of Lemma 5.1 and Theorem 5.4. First of all, the multifunction (5.8) does not satisfy the Filippov condition, since for $U = \mathbb{R}$ and $V = (-1/2, 1/2)$ one gets $\Omega[U,V] = \Omega \setminus D \notin \mathfrak{A}$.

Moreover, although $(\Omega, \mathfrak{A}, \mu)$ is 1-projective in Example 5.2, the assertion of Theorem 5.4 fails, since F is not product-measurable.

Of course, Theorem 5.4 has also been generalized to the case of metric spaces X and Y and multifunctions $F : \Omega \times X \to \mathrm{Cl}(Y)$. Some results in this spirit may be found, for example, in [AvFa, Ri1, RiVi]. Moreover, various "mixed" properties of the multifunction F guarantee the lower or upper Scorza Dragoni property of F. As a sample result, we mention the following [Bn]: if $F(t,\cdot)$ is lower semicontinuous for almost all $t \in \Omega$, and $F(\cdot,u)$ has the "upper Luzin property" for all $u \in X$ (i.e. for $\delta > 0$ there exists $\Omega_\delta \subseteq \Omega$ such that $\mu(\Omega \setminus \Omega_\delta) \le \delta$, and the restriction of $F(\cdot,u)$ to Ω_δ is upper semicontinuous), then F has the lower Scorza Dragoni property. Related more general results may be found in [Zy1, Zy2, Zy3].

In [Rz] it is shown that every upper Carathéodory multifunction $F : \Omega \times X \to \mathrm{Cl}(Y)$ ($\Omega = [0,1]$) contains a multifunction $G : \Omega \times X \to \mathrm{Cl}(Y)$ which has the Scorza Dragoni property and satisfies

$$\mathrm{Sel}_S\, G(\cdot,\phi(.)) = \mathrm{Sel}_S\, F(\cdot,\phi(.)) \tag{5.11}$$

for all $\phi \in S(\Omega,X)$ (see also [JrKz1, JrKz2]).

The most general result of this type is a Scorza Dragoni type theorem for multifunctions with closed graph given in [CsMa]. Although this theorem is proved in a very general setting of multifunctions between metric spaces, we recall here only the special case $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ which we considered throughout this section.

THEOREM 5.5 [CsMa]. *Let $\Omega = [0,T]$ be equipped with the Lebesgue measure μ, and let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ be a multifunction such that the graph $\Gamma(F(t,\cdot))$ of $F(t,\cdot)$ is closed in $\mathbb{R}^m \times \mathbb{R}^n$ for all $t \in \Omega$, and $\mathrm{Sel}_S\, F(\cdot,u) \ne \emptyset$ for all $u \in \mathbb{R}^m$. Then there exists a multifunction $G : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n) \cup \{\emptyset\}$ with the following three properties*:

(a) *there exists a nullset $N \subset \Omega$, independent of $(t,u) \in \Omega \times \mathbb{R}^m$, such that*

$$G(t,u) \subseteq F(t,u) \tag{5.12}$$

for $t \in \Omega \setminus N$ and $u \in \mathbb{R}^m$;

(b) *the relation* (5.11) *holds for all $\phi \in S(\Omega,\mathbb{R}^m)$*;

(c) *for each $\delta > 0$ there exists a closed subset Ω_δ of Ω such that $\mu(\Omega \setminus \Omega_\delta) \le \delta$ and the restriction of G to $\Omega_\delta \times \mathbb{R}^m$ is closed*, *has nonempty values*, *and satisfies* (5.12) *for all $(t,u) \in \Omega_\delta \times \mathbb{R}^m$.*

Proof. Define a multifunction $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^m \times \mathbb{R}^n)$ by

$$\Phi(t) = \{(u,v) : u \in \mathbb{R}^m, v \in \mathbb{R}^n, v \in F(t,u)\}.$$

By [Va, Proposition 14] we can find a maximal (with respect to inclusion) measurable multifunction $\Psi : \Omega \to \mathrm{Cl}(\mathbb{R}^m \times \mathbb{R}^n)$ such that

$$\Psi(t) \subseteq \Phi(t) \tag{5.13}$$

for almost all $t \in \Omega$, and

$$\mathrm{Sel}_S\, \Psi = \mathrm{Sel}_S\, \Phi. \tag{5.14}$$

Let $G : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n) \cup \{\emptyset\}$ be defined by

$$G(t,u) = \{v : v \in \mathbb{R}^n,\ (u,v) \in \Psi(t)\} \qquad (t \in \Omega,\ u \in \mathbb{R}^m).$$

The properties (a) and (b) follow from (5.13) and (5.14), respectively. Moreover, since Ψ is measurable, we conclude that also G is measurable (in the sense that $\Gamma(G) \in \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m) \otimes \mathfrak{B}(\mathbb{R}^n)$, see [CsVl, Proposition III-13]). By Theorem 3.1(e), this implies that the scalar function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(t,u,v) = \varrho((u,v),\Psi(t)) = \inf\{|u-\widetilde{u}| + |v-\widetilde{v}| : (\widetilde{u},\widetilde{v}) \in \Psi(t)\}$$

is a Carathéodory function. Applying the classical Scorza Dragoni theorem to this function we conclude that there exists a closed subset $\Omega_\delta \subseteq \Omega$ such that $\mu(\Omega \backslash \Omega_\delta) \le \delta$, and the restriction of f to $\Omega_\delta \times \mathbb{R}^m \times \mathbb{R}^n$ is continuous. But this implies that the restriction of the multifunction Ψ to Ω_δ is closed, and hence the restriction of G to $\Omega_\delta \times \mathbb{R}^m$ is closed as well.

It remains to show that $G(t,u) \neq \emptyset$ for $t \in \Omega_\delta$ and $u \in \mathbb{R}^m$. This is technical but straightforward (see [CsMa, Theorem 2.4]). ∎

We still mention the following interesting characterization of the upper Scorza Dragoni property [Zy4]: an upper Carathéodory multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ has the upper Scorza Dragoni property if and only if there exists a sequence $(F_k)_k$ of Carathéodory multifunctions $F_1 \supseteq \ldots \supseteq F_k \supseteq \ldots \supseteq F$ such that $F(t,u) = F_1(t,u) \cap F_2(t,u) \cap \ldots$ for almost all $t \in \Omega$ and all $u \in \mathbb{R}^m$.

There are also some recent papers [Ku3, To2], where Scorza Dragoni type properties are studied for multifunctions F which are defined on the graph $\Gamma(G) \subseteq \Omega \times X$ of some other fixed multifunction $G : \Omega \to \mathrm{Cl}(X)$, rather than on the "rectangle" $\Omega \times X$. The most complete and advanced presentation of Carathéodory type multifunctions, multifunctions having Scorza Dragoni type properties, and relations between such multifunctions is the thesis [Zy7]. Theorems of Scorza Dragoni type with applications to differential inclusions may be found, for example, in [Co, Lo1, My, Os, To3].

6. Implicit function theorems. In this section we shall be concerned with a special property of Carathéodory multifunctions which is usually referred to as *Filippov's implicit function theorem.* We suppose again that Ω is compact. First, we need the following technical

LEMMA 6.1. *Let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ be a Carathéodory multifunction. For open $V \subseteq \mathbb{R}^n$, define $\Psi_V : \Omega \to \mathrm{Cp}(\mathbb{R}^n)$ by*

$$\Psi_V(t) = F(t,\cdot)_+^{-1}(V) = \{u : u \in \mathbb{R}^m,\ F(t,u) \subseteq V\}; \tag{6.1}$$

similarly, for closed $W \subseteq \mathbb{R}^n$, define $\Phi_W : \Omega \to \mathrm{Cp}(\mathbb{R}^n)$ by

$$\Phi_W(t) = F(t,\cdot)_-^{-1}(W) = \{u : u \in \mathbb{R}^m,\ F(t,u) \cap W \neq \emptyset\}. \tag{6.2}$$

Then both multifunctions (6.1) *and* (6.2) *are measurable.*

P r o o f. The proof follows from the fact that F is product-measurable, by Lemma 4.1, and that $\Gamma(\Psi_V) = F_+^{-1}(V)$ and $\Gamma(\Phi_W) = F_-^{-1}(W)$. ∎

THEOREM 6.1 [Fil]. *Let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ be a Carathéodory multifunction, and let $\Gamma : \Omega \to \mathrm{Cp}(\mathbb{R}^m)$ be a measurable multifunction. Suppose that $g \in S(\Omega, \mathbb{R}^n)$ satisfies*

$$g(t) \in F(t,\Gamma(t)) \tag{6.3}$$

for almost all $t \in \Omega$. *Then there exists a function* $\gamma \in \mathrm{Sel}_S(\Gamma)$ *such that*

$$g(t) \in F(t, \gamma(t)) \tag{6.4}$$

for almost all $t \in \Omega$.

P r o o f. Define multifunctions $G, G_k : \Omega \to P(\mathbb{R}^m)$ by

$$G(t) = \{u : u \in \mathbb{R}^m,\ \varrho(g(t),\ F(t,u)) = 0\}, \tag{6.5}$$

$$G_k(t) = \{u : u \in \mathbb{R}^m,\ \varrho(g(t),\ F(t,u)) < 1/k\} \tag{6.6}$$

$(k = 1, 2, \ldots)$. By (6.3), $G(t) \cap \Gamma(t)$ is nonempty and compact for all $t \in \Omega$. Moreover, G_k is measurable, by Lemma 6.1.

Thus, the multifunction

$$G(t) \cap \Gamma(t) = \bigcap_{k \in \mathbb{N}} G_k(t) \cap \Gamma(t)$$

is measurable as well. By Theorem 3.2, the multifunction $G \cap \Gamma$ admits a measurable selection γ; obviously, this selection γ satisfies (6.4). ■

We remark that Theorem 6.1 holds also for closed-valued Carathéodory multifunctions, as Theorem 3.2 shows.

Recall that the vector space $\mathbb{R}^k$ may be ordered by defining $(\xi_1, \ldots, \xi_k) \le (\eta_1, \ldots, \eta_k)$ as $\xi_j \le \eta_j$ $(j = 1, \ldots, k)$. If M is a compact subset of $\mathbb{R}^k$, we define $\max M$ as k-tuple $(m_1, \ldots, m_k)$, where $m_j = p_j(M)$ $(j = 1, \ldots, k)$.

The next theorem is a typical application of Theorem 6.1 and will be used in subsequent sections (e.g. in Theorem 7.2).

THEOREM 6.2 [Fil]. *Let* $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ *be a Carathéodory multifunction, and let* $v, w \in S(\Omega, \mathbb{R}^m)$ *be fixed with* $v(t) \le w(t)$. *Moreover, define* $g : \Omega \to \mathbb{R}^n$ *by*

$$g(t) = \max \bigcup_{v(t) \le u \le w(t)} F(t,u). \tag{6.7}$$

Then there exists a measurable function $\gamma : \Omega \to \mathbb{R}^m$ *such that* (6.4) *holds.*

P r o o f. Observe that the function (6.7) is well-defined, since the multifunction $F(t, \cdot)$ maps compact sets into compact sets (see the remark after Example 2.6). The assertion follows now by putting

$$\Gamma(t) = \{u : u \in \mathbb{R}^m,\ v(t) \le u \le w(t)\}$$

and using Theorem 6.1. ■

Theorem 6.1 has been generalized to separable metric spaces in [McWf]; see also [Jal] and [HmJaVl]. In the meantime so called implicit function theorems of Filippov type have found a great deal of attention in the literature (e.g. [Cs2, Dl, EkVl, Ho, Io2, Io3, KuNo2, Mo1, Mo2]). An interesting application to random operator equations may be found in [KuNo2].

3. The superposition operator

In this chapter we give a systematic account of some important properties of the superposition operator generated by a vector-valued multifunction. This operator will be studied in the metric space S of (classes of) measurable functions, in the normed space C of continuous functions, and in various function spaces which are important in applications (e.g., Lebesgue and Orlicz spaces). The theory is most complete and satisfactory for Carathéodory multifunctions; however, many results carry over as well to larger classes of multifunctions.

7. The superposition operator in the space S. Let $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ be a fixed multifunction. Applying F to a (single-valued) function $x : \Omega \to \mathbb{R}^m$, we get a multifunction

$$Y(t) = F(t, x(t)) \tag{7.1}$$

on Ω. If the function x is measurable, we put

$$N_F(x) = \mathrm{Sel}_S\, Y \tag{7.2}$$

i.e. $N_F(x)$ consists of all measurable selections y of the multifunction (7.1). In this way, we have defined the (multi-valued) *superposition operator* (also called *composition operator* or *Nemytskiĭ operator*) N_F from $S(\Omega, \mathbb{R}^m)$ into $P(S(\Omega, \mathbb{R}^n))$. Observe, however, that it is not clear a priori that we end up in fact in $P(S(\Omega, \mathbb{R}^n))$, i.e. that the multifunction (7.1) admits a measurable selection at all.

Let us call a multifunction $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ *superpositionally measurable* (or *sup-measurable*, for short) if, for any $x \in S(\Omega, \mathbb{R}^m)$, the multifunction (7.1) is measurable. If, for any $x \in S(\Omega, \mathbb{R}^m)$, the multifunction (7.1) admits at least a measurable (single-valued) selection y, we call F *weakly sup-measurable*. In either case, the superposition operator N_F is then well-defined, since $S(\Omega, \mathbb{R}^n) \cap N_F(x) \neq \emptyset$ for all $x \in S(\Omega, \mathbb{R}^m)$.

The problem of characterizing the class of all (weakly) sup-measurable multifunctions F is unsolved. Nevertheless, one may easily give some sufficient conditions. We start with the following

THEOREM 7.1. *If $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ has the Scorza Dragoni property, then F is sup-measurable.*

Proof. Let $x \in S(\Omega, \mathbb{R}^m)$. For $\delta > 0$, choose $\Omega_\delta \subseteq \Omega$ such that $\mu(\Omega \setminus \Omega_\delta) \le \delta$, and the restriction of F to $\Omega_\delta \times \mathbb{R}^m$ is continuous; without loss of generality, we may assume that also the restriction of x to Ω_δ is continuous. By Lemma 2.5, the multifunction (7.1) is then also continuous on Ω_δ. We have shown that (7.1) has the Luzin property, and the assertion follows from Theorem 3.1. ■

By Theorem 5.1, any Carathéodory multifunction F is sup-measurable. The following example shows that this is false for upper Carathéodory multifunctions.

EXAMPLE 7.1 [Ob1]. Let $\Omega = [0, 1]$ be equipped with the Lebesgue measure, $D \subset \Omega$ a nonmeasurable subset and $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R})$ defined by

$$F(t, u) = \begin{cases} [0,1] & \text{if } u = t \text{ and } t \in \Omega \setminus D, \\ [0,1] & \text{if } u = t+1 \text{ and } t \in D, \\ \{1\} & \text{otherwise.} \end{cases} \tag{7.3}$$

Then F is upper Carathéodory, but not sup-measurable, since F maps the function $x(t) = t$ into the multifunction

$$Y(t) = \begin{cases} [0,1] & \text{if } t \in \Omega \setminus D, \\ \{1\} & \text{if } t \in D, \end{cases} \tag{7.4}$$

which is not measurable. ∎

One could ask whether or not an upper Carathéodory multifunction is at least *weakly* sup-measurable; for instance, the multifunction (7.4) admits the selection $y(t) \equiv 1$. In fact, the following is true:

LEMMA 7.1 [Cs1]. *If* $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ *is upper Carathéodory, then* F *is weakly sup-measurable.*

Proof. Let $x : \Omega \to \mathbb{R}^m$ be measurable, and let $(x_k)_k$ be a sequence of simple functions such that $x_k(t) \to x(t)$ a.e. on Ω. Obviously, all multifunctions $Z_k(t) = F(t, x_k(t))$ are measurable, and hence also the multifunction

$$Z(t) = \bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} Z_j(t)}. \tag{7.5}$$

We claim that $Z(t) \subseteq Y(t) = F(t, x(t))$ for almost all $t \in \Omega$. In fact, if $v \notin Y(t)$ for some $v \in \mathbb{R}^n$, then

$$v \notin \overline{Z_k(t) \cup Z_{k+1}(t) \cup \ldots}$$

for $k \in \mathbb{N}$ large enough, and hence $v \notin Z(t)$. To prove the statement, it suffices now to choose any measurable selection of the multifunction $Z : \Omega \to \mathrm{Cl}(\mathbb{R}^n)$. ∎

We turn now to the analogous problem for lower Carathéodory multifunctions. Surprisingly, a lower Carathéodory multifunction need not even be weakly sup-measurable.

EXAMPLE 7.2 [Ob2]. Let $\Omega = [0,1]$, $D \subset \Omega$ a nonmeasurable subset, and let $F : \Omega \times \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ be defined as in Example 5.1. Then F is lower Carathéodory, but not weakly sup-measurable, since F maps the function $x(t) = t$ into the multifunction

$$Y(t) = \begin{cases} \{0\} & \text{if } t \in \Omega \setminus D, \\ \{1\} & \text{if } t \in D, \end{cases} \tag{7.6}$$

which of course does not admit a measurable selection. ∎

So far we discussed sufficient conditions for the weak sup-measurability of a multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$. The importance of weakly sup-measurable multifunctions follows, for example, from Theorem 4.2: In fact, if $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ is weakly sup-measurable, for all $x \in S(\Omega, \mathbb{R}^m)$ we may choose a measurable function $y \in N_F x$; by Theorem 4.2, we have then $y = N_f x$ for some function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$. In this way we may reduce the study of weakly sup-measurable superpositions to single-valued superpositions.

Suppose that $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ is a product-measurable multifunction. For any $x \in S(\Omega, \mathbb{R}^m)$ define $\widehat{x} : \Omega \to \Omega \times \mathbb{R}^n$ by $\widehat{x}(t) = (t, x(t))$. We claim that $\widehat{x}(M) \in \mathfrak{A}$ for any $M \in \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$. In fact, for $A \in \mathfrak{A}$ and $B \in \mathfrak{B}(\mathbb{R}^m)$ we have

$$\widehat{x}^{-1}(A \times B) = \{t : t \in A, x(t) \in B\} = A \cap x^{-1}(B) \in \mathfrak{A}.$$

So, if $V \subseteq \mathbb{R}^n$ is open we have for the multifunction (7.1),

$$Y_-^{-1}(V) = \{t : t \in \Omega, F(t, x(t)) \cap V \neq \emptyset\} = \widehat{x}^{-1}(F_-^{-1}(V)) \in \mathfrak{A},$$

since $F_-^{-1}(V) \in \mathfrak{A} \otimes \mathfrak{B}(\mathbb{R}^m)$. Thus, we have proved the following

LEMMA 7.2. *If $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ is product-measurable, then F is sup-measurable.*

Lemma 7.2 is contained in [Ts], but the above short proof has been kindly communicated to the authors by W. Zygmunt [Zy9]; see also [Zy11].

If we suppose, apart from the product-measurability of F, that $F(t,\cdot)$ is lower semicontinuous for all $t \in \Omega$, then Lemma 7.2 becomes trivial. In fact, from Theorem 4.1 it follows then that F admits a Carathéodory exhaustion $(f_k)_k$, and thus all functions $y_k(t) = f_k(t, x(t))$ are measurable.

Observe that, by Lemma 4.1, every Carathéodory multifunction is sup-measurable; this is essentially our Theorem 7.1 above.

Lemma 7.2 is proved in [Ts] in the metric space setting. Further generalizations may be found in [Sp, Zy6, Zy8], and the Appendix of [Mo3]. For example, in some work on multi-valued superpositions the operator (7.2) is supposed to act on *multifunctions* $X : \Omega \to P(\mathbb{R}^m)$ rather than single-valued functions $x : \Omega \to \mathbb{R}^m$, i.e. (7.1) is replaced by

$$Y(t) = F(t, X(t)) = \bigcup_{x \in X(t)} F(t,x). \tag{7.7}$$

In this case Lemma 7.2 is true only under the additional hypothesis that $(\Omega, \mathfrak{A}, \mu)$ is m-projective [Zy8].

The paper [Sp] contains also further examples of non-sup-measurable multifunctions which are either upper but not lower Carathéodory, or lower but not upper Carathéodory. Moreover, in [Mo1, Mo2, No, Pa1, Zy2] one can find various sufficient conditions for the product-measurability of a multifunction $F : \Omega \times X \to \mathrm{Cl}(Y)$ (X, Y normed or metric spaces) which make it possible to prove the superpositional measurability of F by means of Lemma 7.2 and its generalizations.

We point out that the converse of Lemma 7.2 is false: even in the single-valued case there exist rather exotic functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ (called "monsters" in the literature, see [KrPk]) such that f is not product-measurable, but $f(t, x(t)) = 0$ for any measurable function $x : [0,1] \to \mathbb{R}$. It is interesting to note, however, that a certain converse of Lemma 7.2 *is* true for upper Carathéodory multifunctions:

LEMMA 7.3 [Zy8]. *If $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ is upper Carathéodory and sup-measurable, and $(\Omega, \mathfrak{A}, \mu)$ is m-projective, then F is product-measurable.*

Proof. From the sup-measurability of F it follows that F satisfies the Filippov condition (5.9) and hence, by the same reasoning as in the second part of the proof of Theorem 5.4, we obtain the product-measurability of F. ∎

Again, the assertion of Lemma 7.3 becomes false if we replace "upper" by "lower" Carathéodory. In fact, the multifunction F given in Example 4.2 is lower Carathéodory and (as was shown in [Zy8]) sup-measurable, but *not* product-measurable.

In the following table we collect the properties of the most relevant counterexamples of this and the preceding sections:

F as given in	upper Car.	lower Car.	Filip-pov	upper SD	lower SD	product-meas.	sup-meas.	weakly sup-meas.
Ex. 4.1	no	yes	yes	no	yes	yes	yes	yes
Ex. 4.2	no	yes	no	—	—	no	yes	yes
Ex. 5.1	no	yes	no	no	no	no	no	no
Ex. 5.2	yes	no	no	no	no	no	yes	yes
Ex. 5.3	yes	yes	yes	no	yes	yes	yes	yes
Ex. 7.1	yes	no	no	no	no	no	no	yes

So far we considered only conditions for the superposition operator (7.2) to act from $S(\Omega,\mathbb{R}^m)$ into $P(S(\Omega,\mathbb{R}^n))$. Now we shall be interested in analytic properties of N_F, viz. boundedness and continuity.

Recall that a set M in a metric linear space S is called *bounded* if, for any sequence $(z_k)_k$ in M and any real sequence $(\delta_k)_k$ converging to zero, the product sequence $(\delta_k z_k)_k$ converges to zero in S.

THEOREM 7.2 [ApDeZa]. *Let $F:\Omega\times\mathbb{R}^m\to\mathrm{Cp}(\mathbb{R}^n)$ be a Carathéodory multifunction. Then the superposition operator* (7.2) *generated by F is bounded*, *i.e. maps any bounded set $M\subset S(\Omega,\mathbb{R}^m)$ into a bounded set $N_F(M)\subset S(\Omega,\mathbb{R}^n)$.*

Proof. Let $M\subset S(\Omega,\mathbb{R}^m)$ be bounded, $(x_k)_k$ an arbitrary sequence in M, and $(\delta_k)_k$ a real sequence converging to zero; we have to show that any sequence $y_k\in N_F(x_k)$ satisfies

$$\mu(\delta_k y_k,\tau)\to 0\qquad(h\to\infty)\tag{7.8}$$

for $0<\tau<\infty$, where $\mu(z,\tau)$ is defined in (3.2). Given $\varepsilon>0$, we may choose $\tau_\varepsilon>0$ such that $\mu(x_k,\tau_\varepsilon)\le\varepsilon$, uniformly in $k\in\mathbb{N}$, since M is bounded in $S(\Omega,\mathbb{R}^m)$. Put

$$\widetilde{x}_k(t)=\begin{cases}x_k(t) & \text{if } |x_k(t)|\le\tau_\varepsilon,\\ \tau_\varepsilon\operatorname{sign}x_k(t) & \text{if } |x_k(t)|>\tau_\varepsilon,\end{cases}$$

and observe that $|\widetilde{x}_k(t)|\le\tau_\varepsilon$ for all $t\in\Omega$. By Theorem 6.2, the function

$$g_\varepsilon(t)=\max\bigcup_{|u|\le\tau_\varepsilon}F(t,u)\qquad(t\in\Omega)\tag{7.9}$$

is well-defined and measurable. Denote by D_k the set of all $t\in\Omega$ for which $F(t,x_k(t))\ne F(t,\widetilde{x}_k(t))$; by construction, $\mu(D_k)\le\varepsilon$. But for any $t\in\Omega\setminus D_k$ we have $y_k(t)\in F(t,x_k(t))=F(t,\widetilde{x}_k(t))$, hence $|y_k(t)|\le g_\varepsilon(t)$. We conclude that $|\delta_k y_k(t)|\le\delta_k g_\varepsilon(t)\to 0$ $(k\to\infty)$ for $t\in\Omega\setminus D_k$, and thus (7.8) holds, since $\varepsilon>0$ is arbitrary. ∎

THEOREM 7.3 [ApDeZa]. *Let $F:\Omega\times\mathbb{R}^m\to\mathrm{Cp}(\mathbb{R}^n)$ be a Carathéodory multifunction. Then the superposition operator* (7.2) *generated by F is continuous*, *i.e. maps any convergent sequence $(x_k)_k$ in $S(\Omega,\mathbb{R}^m)$ into a convergent sequence $(N_F(x_k))_k$ in $P(S(\Omega,\mathbb{R}^n))$.*

Proof. Let $(x_k)_k$ be a sequence in $S(\Omega,\mathbb{R}^m)$ converging with respect to the metric (3.1) to $x_*\in S(\Omega,\mathbb{R}^m)$; without loss of generality, we may assume that $(x_k)_k$ converges to x_* a.e. on Ω. Since F is Carathéodory, we have

$$Y_k(t)=F(t,x_k(t))\to F(t,x_*(t))=Y_*(t)\qquad(k\to\infty)$$

(convergence with respect to the metric (1.5)) a.e. on Ω. By Egorov's theorem, for any $\varepsilon > 0$ we may find a set $D \in \mathfrak{A}$ such that $\mu(D) \le \varepsilon$ and

$$h(Y_k(t), Y_*(t)) \le \varepsilon \tag{7.10}$$

for $t \in \Omega \setminus D$ and large $k \in \mathbb{N}$. But this implies that also

$$h_S(N_F(x_k), N_F(x_*)) \le 2\varepsilon. \tag{7.11}$$

In fact, given $y \in N_F(x_k)$, by (7.10) we may choose $z \in N_F(x_*) = \mathrm{Sel}_S\, Y_*$ such that $|y(t) - z(t)| \le \varepsilon$ on $\Omega \setminus D$; consequently,

$$\begin{aligned} d_S(y,z) &= \inf\{\tau + \mu(y - z, \tau) : \tau > 0\} \\ &\le \varepsilon + \mu(P_D(y-z), \varepsilon) \le \varepsilon + \mu(D) \le 2\varepsilon, \end{aligned} \tag{7.12}$$

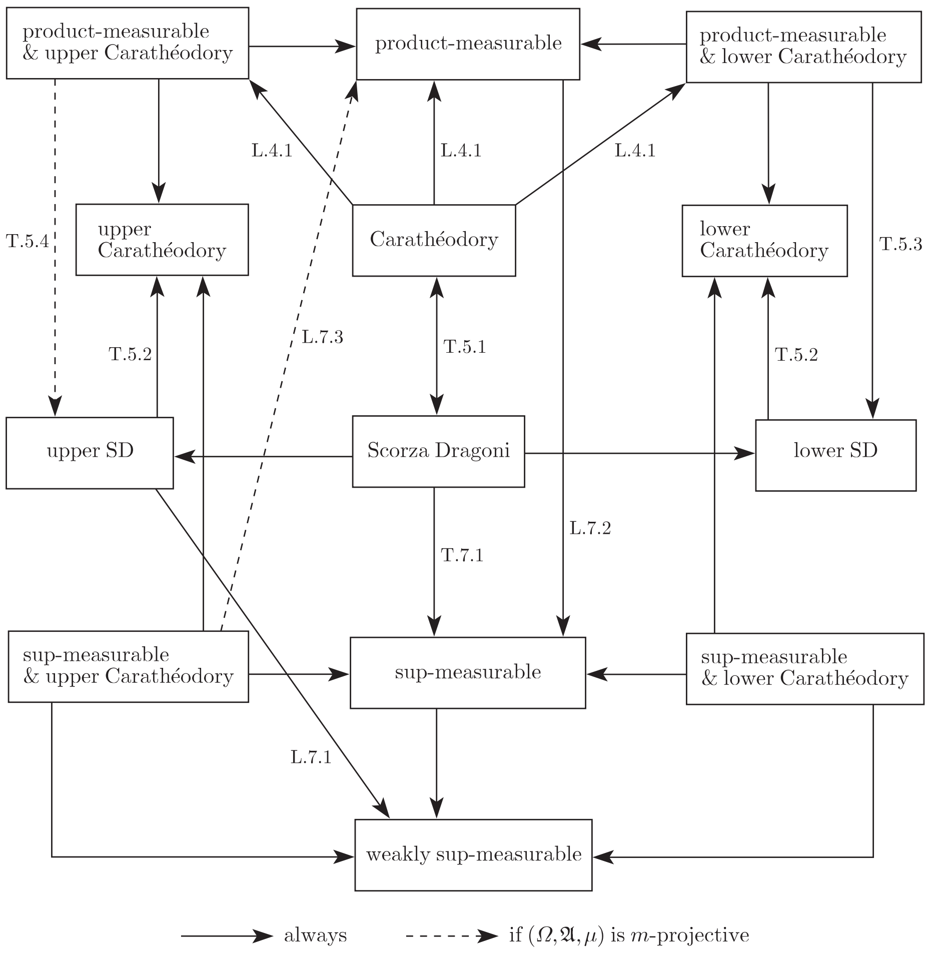

Fig. 1

where P_D denotes the restriction operator of the set D, i.e.

$$P_D x(t) = \begin{cases} x(t) & \text{if } t \in D, \\ 0 & \text{if } t \in \Omega \setminus D. \end{cases} \tag{7.13}$$

From (7.12) we conclude that $h_S^+(N_F(x_k), N_F(x_*)) \le 2\varepsilon$; the relation $h_S^-(N_F(x_k), N_F(x_*)) \le 2\varepsilon$ is proved similarly. ■

We close this section by stating a remarkable property of the superposition operator (7.2), viz. its *disjoint additivity*. This means that, whenever $x_1, \ldots, x_n$ are measurable functions with mutually disjoint supports (i.e. $x_i(t)x_j(t) = 0$ for $i \ne j$), one has

$$N_F(x_1 + \ldots + x_n) + (n-1)N_F(0) = N_F(x_1) + \ldots + N_F(x_n), \tag{7.14}$$

where $N_F(0) = \mathrm{Sel}_S\, F(\cdot, 0)$, of course. In terms of the restriction operator (7.13), this may be stated as

$$P_D N_F(x) = P_D N_F(P_D x) \quad (D \in \mathfrak{A},\ x \in S(\Omega, \mathbb{R}^m)). \tag{7.15}$$

In Figure 1 opposite ([1]), we compare various properties of multifunctions $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ which we discussed so far. The implications indicated by continuous arrows are always true, while the implications indicated by dotted arrows are true under the additional hypothesis that $(\Omega, \mathfrak{A}, \mu)$ be m-projective.

8. The superposition operator in ideal spaces. Let $S(\Omega, \mathbb{R}^m)$ be the space of all (classes of) measurable functions with the metric (3.1). A Banach space $X \subset S(\Omega, \mathbb{R}^m)$ with norm $\|\cdot\|_X$ is called an *ideal space* if the relations $x \in X$ and $\theta \in L_\infty(\Omega, \mathbb{R})$ (the space of all essentially bounded real functions on Ω) imply that $\theta x \in X$ and $\|\theta x\|_X \le \|\theta\|_{L_\infty} \|x\|_X$. In the scalar case $m = 1$ this simply means that X contains, together with a function x, also all measurable functions z with $|z| \le |x|$ (i.e. $|z(t)| \le |x(t)|$ for almost all $t \in \Omega$); these functions z satisfy then the estimate $\|z\| \le \|x\|$.

The simplest examples of ideal spaces are the *Lebesgue space* $L_p = L_p(\Omega, \mathbb{R}^m)$ with norm

$$\|x\|_{L_p} = \begin{cases} (\int_\Omega |x(t)|^p\, dt)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}\{|x(t)| : t \in \Omega\} & \text{if } p = \infty, \end{cases} \tag{8.1}$$

or, more generally, the *weighted Lebesgue space* $L_{p,w} = L_{p,w}(\Omega, \mathbb{R}^m)$ with norm

$$\|x\|_{L_{p,w}} = \begin{cases} (\int_\Omega |x(t)|^p w(t)\, dt)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}\{|x(t)|w(t) : t \in \Omega\} & \text{if } p = \infty, \end{cases} \tag{8.2}$$

where $w : \Omega \to (0, \infty)$ is a fixed measurable function. Another important class of ideal spaces is that of the *Orlicz spaces* L_ϕ which are, however, more difficult to describe. Let $\phi : \Omega \times \mathbb{R}^m \to [0, \infty]$ be a *Young function*, i.e. $\phi(t, \cdot)$ is even, convex, and upper semicontinuous on $\mathbb{R}^m$ for almost all $t \in \Omega$, $\phi(\cdot, u)$ is measurable on Ω for all $u \in \mathbb{R}^m$, and the equality $\phi(t, \lambda u) = 0$ for all $\lambda \in (0, \infty)$ implies that $u = 0$. The Orlicz space $L_\phi = L_\phi(\Omega, \mathbb{R}^m)$ consists, by definition, of all functions $x \in S(\Omega, \mathbb{R}^m)$ for which the

([1]) We are greatly indebted to Mauszwei for drawing the nice Figure 1.

(Luxemburg) norm

$$\|x\|_{L_\phi} = \inf\Big\{\lambda : \lambda > 0,\ \int_\Omega \phi[t, x(t)/\lambda]\, dt \le 1\Big\} \tag{8.3}$$

is finite (see e.g. [KrRu, RaRe] for the case $m = 1$ and [Ng1, ZaNg1] for the case $m > 1$). In particular, let $\phi : \Omega \times \mathbb{R}^m \to [0,\infty)$ be a Young function with the additional property that

$$\phi(t, \lambda u) = \lambda\phi(t, u) \qquad (0 < \lambda < \infty). \tag{8.4}$$

We associate with ϕ two spaces $L(\phi)$ and $M(\phi)$ consisting of all functions $x \in S(\Omega, \mathbb{R}^m)$ for which the norms

$$\|x\|_{L(\phi)} = \int_\Omega \phi(t, x(t))\, dt \tag{8.5}$$

$$\|x\|_{M(\phi)} = \operatorname{ess\,sup}\{\phi(t, x(t)) : t \in \Omega\}, \tag{8.6}$$

respectively, are finite. Both spaces, $L(\phi)$ and $M(\phi)$ are Orlicz spaces, which we shall need below.

Finally, other important examples of ideal spaces are Lorentz and Marcinkiewicz spaces arising in interpolation theory [KnPeSe]. We briefly recall the definition of these spaces. Let Ω be some domain in Euclidean space and $\varphi : [0,\infty) \to [0,\infty)$ an increasing concave function with $\varphi(0) = 0$. The *Lorentz space* $\Lambda_\varphi = \Lambda_\varphi(\Omega, \mathbb{R}^m)$ consists, by definition, of all functions $x \in S(\Omega, \mathbb{R}^m)$ for which the norm

$$\|x\|_{\Lambda_\varphi} = \int_0^1 x^*(t)\, d\varphi(t) \tag{8.7}$$

is finite, where x^* denotes the decreasing rearrangement of $|x|$ (see e.g. [KnPeSe]). Similarly, the *Marcinkiewicz space* $M_\varphi = M_\varphi(\Omega, \mathbb{R}^m)$ is defined by the norm

$$\|x\|_{M_\varphi} = \sup \frac{\varphi(\tau)}{\tau} \int_0^\tau x^*(t)\, dt. \tag{8.8}$$

An important special case is $\varphi(t) = \varphi_\alpha(t) = t^\alpha$ for some $\alpha \in (0, 1]$. Here the spaces Λ_{φ_α} and M_{φ_α} are closely related to the Lebesgue space L_p inasmuch as

$$\Lambda_{\varphi_\alpha} \subseteq L_{1/\alpha} \subseteq M_{\varphi_\alpha}, \qquad \|x\|_{M_{\varphi_\alpha}} \le \|x\|_{L_{1/\alpha}} \le \|x\|_{\Lambda_{\varphi_\alpha}} \quad (x \in \Lambda_{\varphi_\alpha}).$$

The classical interpolation theorems by Marcinkiewicz [Mr] and Stein–Weiss [StWe] for operators between Lebesgue spaces essentially use the spaces Λ_{φ_α} and M_{φ_α}; for details see, for example, [BgLo].

We do not go further into the abstract theory of ideal spaces; one may find much material in the scalar case $m = 1$ in [LuZn, Zn, Za2], and in the vector case $m > 1$ in [Za3]. Let us just point out one of the main difficulties which one encounters when passing from scalar functions to vector functions. First of all, there is *no natural ordering* on ideal spaces of vector functions, but one may introduce a certain ordering via multifunctions in the following way. Let us call a measurable multifunction $\Phi : \Omega \to \mathrm{CpCv}(\mathbb{R}^m)$ an *m-unit* if its values $\Phi(t)$ are symmetric absolutely convex compact subsets of $\mathbb{R}^m$. Moreover, we say that Φ is an *m-unit in X* ($X \subset S(\Omega, \mathbb{R}^m)$ an ideal space) if $\mathrm{Sel}_S\,\Phi \subseteq X$. The family of all m-units in X is ordered by inclusion.

In the scalar case $m = 1$, any 1-unit has the form

$$\Phi(t) = [-\phi(t), \phi(t)], \tag{8.9}$$

where ϕ is a nonnegative element in X. This establishes a 1-1 correspondence between the family of all nonnegative elements in X and the family of all 1-units in X; in particular, this induces precisely the natural ordering "almost everywhere" for functions in $S(\Omega, \mathbb{R})$. In the vector case $m > 1$ every m-unit in X may be regarded as unit ball in a suitable $M(\phi)$-space (see (8.6)) which is continuously imbedded in X. This means that m-units in an ideal space of vector functions may be described by means of Young functions with the special homogeneity property (8.4).

The superposition operator (7.2) has been studied so far only in special ideal spaces of scalar or vector functions. For instance, the papers [Ca, CeSu, ClFrRz, NsRi] consider various properties of multi-valued superposition operators in Lebesgue spaces, the papers [Se1, Se2] in Orlicz spaces, and the papers [RoSo1, RoSo2] in more general spaces. In this section we shall be concerned with various analytical properties of the superposition operator (7.2) between two general ideal spaces X and Y. Suppose that $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ is a sup-measurable multifunction, and the superposition operator N_F generated by F maps X into $P(Y)$ (X, Y ideal spaces). It is clear that many properties of the images $F(t, u) \subseteq \mathbb{R}^n$ of the multifunction F carry over to corresponding properties of the images $N_F(x) \subseteq Y$ of the superposition operator N_F. For instance, if $F(t, u) \in \mathrm{Cv}(\mathbb{R}^n)$ ($\mathrm{Bd}(\mathbb{R}^n)$, $\mathrm{Cl}(\mathbb{R}^n)$, respectively) then also $N_F(x) \in \mathrm{Cv}(Y)(\mathrm{Bd}(Y), \mathrm{Cl}(Y)$, respectively) since the imbeddings $L_\infty(\Omega, \mathbb{R}^n) \subseteq Y \subseteq S(\Omega, \mathbb{R}^n)$ hold for any ideal space Y. On the other hand, it is obvious that $N_F(x)$ need not be compact, even if $F(t, u)$ is.

As in the preceding section, we are interested in boundedness and continuity of the operator (7.2). Throughout the following, we write

$$\|M\|^* = \sup\{\|x\| : x \in M\}, \tag{8.10}$$

$$\|M\|_* = \inf\{\|x\| : x \in M\} \tag{8.11}$$

for M being a bounded subset of a Banach space X. Moreover, we denote by $B_r(X)$ the closed ball $\{x : x \in X, \|x\|_X \le r\}$.

LEMMA 8.1. *If $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ is a sup-measurable multifunction, then the corresponding superposition operator $N_F : X \to \mathrm{Cl}(Y)$ is bounded if and only if, for each $r > 0$, the scalar function*

$$\phi_r(t) = \sup\{\|N_F(x)\|^*(t) : x \in \Gamma(r)\} \tag{8.12}$$

is finite and measurable on Ω; here $\Gamma(r)$ denotes the set of all functions $x \in X$ whose graphs are contained (almost everywhere) in $\Omega \times B_r(\mathbb{R}^m)$. Likewise, if $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ is a Carathéodory multifunction, then the corresponding superposition operator $N_F : X \to \mathrm{Cl}(Y)$ is bounded if and only if, for each $r > 0$, the scalar function

$$\psi_r(t) = \sup\{|F(t, u)|^* : |u| \le r\} \tag{8.13}$$

is finite and measurable on Ω.

The proof is elementary, and therefore we drop it.

We point out that, in contrast to Theorem 7.2, the superposition operator N_F generated by a Carathéodory multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ need not be bounded between two ideal spaces:

Example 8.1 [KrRu]. Let $\Omega = [0,1]$, $m = n = 1$, $\phi(u) = e^{|u|} - |u| - 1$ and $F : \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ be defined by

$$F(u) = \{\phi(u)\}. \tag{8.14}$$

We consider the superposition operator N_F generated by (8.14) from the Orlicz space $L_\phi = L_\phi(\Omega, \mathbb{R})$ into the Lebesgue space $L_1 = L_1(\Omega, \mathbb{R})$. Fix a function $x \in L_\phi$ such that

$$\int_0^1 \phi[x(t)]\,dt = \infty, \qquad \int_0^1 \phi[x(t)/2]\,dt < \infty \tag{8.15}$$

(for example, $x(t) = -\log t$), and put

$$x_k(t) = \begin{cases} x(t) & \text{if } |x(t)| \le k, \\ 0 & \text{if } |x(t)| > k. \end{cases} \tag{8.16}$$

We have then $\|x_k\|_{L_\phi} \le 2$, on the one hand, but

$$\sup_k \|N_F(x_k)\|^*_{L_1} = \sup_k \int_0^1 \phi[x_k(t)]\,dt = \infty,$$

on the other. This shows that the operator N_F is unbounded, for example, on the ball $B_2(L_\phi)$. ∎

Before stating our main boundedness result, we need a technical definition. An ideal space X is called a *split space* if one can find a sequence $\sigma = (\sigma(1), \sigma(2), \ldots)$ of natural numbers, depending only on X, with the following property: given a sequence $(x_k)_k$ of functions $x_k \in B_1(X)$ with disjoint supports, one can decompose each x_k in the form

$$x_k = x_{k,1} + \ldots + x_{k,\sigma(k)} \tag{8.17}$$

such that the functions $x_{k,j}$ $(j = 1, \ldots, \sigma(k))$ have also disjoint supports, and for each choice $s = (s(1), s(2), \ldots)$ of natural numbers $s(k) \in \{1, \ldots, \sigma(k)\}$ the function $x_s = x_{1,s(1)} + x_{2,s(2)} + \ldots$ belongs also to $B_1(X)$. Although this definition seems rather restrictive and technical, "almost all" ideal spaces arising in applications are split spaces. For example, every Orlicz space L_ϕ is a split space, as may be seen by putting

$$\sigma(k) \ge \sup\Big\{ \int_\Omega \phi[x(t)]\,dt : \|x\|_{L_\phi} \le 2^k \Big\};$$

in particular, the Lebesgue space L_p $(1 \le p < \infty)$ is a split space with $\sigma(k) = 2^{kp}$.

We remark that, if the measure μ over Ω has atoms, there exist only trivial split spaces over Ω [ApZa1]; therefore, we assume that μ is atom-free in the next theorem.

Theorem 8.1 [ApNgZa1]. *Let $X \subset S(\Omega, \mathbb{R}^m)$ and $Y \subset S(\Omega, \mathbb{R}^n)$ be two ideal spaces, where X is a split space. Suppose that $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ is a Carathéodory multifunction, and assume that the interior G of the domain of definition of the corresponding* *superposition operator $N_F : X \to \mathrm{Cl}(Y)$ is nonempty. Then N_F is bounded on each ball $B_r(X) \subseteq G$.*

Proof. Suppose that $B_1(X) \subseteq G$ (without loss of generality), and that N_F is unbounded on $B_1(X)$. Choose sequences $(x_k)_k$ in X and $(y_k)_k$ in Y such that

$$\|x_k\| \leq 1, \quad y_k \in N_F(x_k), \quad \|y_k\| > k\sigma(k) + [\sigma(k) - 1]R, \tag{8.18}$$

where $\sigma(k)$ is the sequence of natural numbers occurring in the definition of the split space X, and

$$R = \|N_F(0)\|^* = \sup\{|y| : y \in Y \cap \mathrm{Sel}_S F(\cdot, 0)\}.$$

By modifying the functions x_k, if necessary, we may assume that they have mutually disjoint supports. Since X is split, we may decompose each x_k in the form (8.17). By (7.14), we find in turn functions $y_{k,j} \in N_F(x_{k,j})$ $(j = 1, 2, \ldots, \sigma(k))$ and $z_k \in N_F(0)$ such that

$$y_k + (\sigma(k) - 1)z_k = y_{k,1} + \ldots + y_{k,\sigma(k)}.$$

For at least one index $s(k) \in \{1, \ldots, \sigma(k)\}$ we have then $\|y_{k,s(k)}\| > k$, by the last condition in (8.18). But then the function $x_* = x_{1,s(1)} + x_{2,s(2)} + \ldots$ belongs to $B_1(X)$ (since X is a split space), while the function $y_* = y_{1,s(1)} + y_{2,s(2)} + \ldots$ does not belong to Y (since $\|y_*\| \geq \|y_{k,s(k)}\| > k$). This contradicts our hypothesis that $N_F(x) \subseteq Y$ for all $x \in G$. ∎

Theorem 8.1 implies, in particular, that *the superposition operator N_F generated by a Carathéodory multifunction F and considered as an operator between two Orlicz spaces L_ϕ and L_ψ, is bounded on each ball which is entirely contained in its domain of definition.* Of course, the point in the above Example 8.1 is that the ball $B_r(L_\phi)$ is contained in the domain of definition of the operator N_F, with F given by (8.14), only if $r \leq 1$.

Observe that we actually did not use the Carathéodory property of the multifunction F in the proof of Theorem 8.1, but only its (weak!) sup-measurability. Some more boundedness results for the superposition operator between ideal spaces may be found in [ApNgZa1]. Boundedness results for multi-valued superposition operators between so called Orlicz–Musielak spaces [Mu] of Banach-space-valued functions are given in [RoSo1].

We turn now to the problem of finding continuity conditions for the superposition operator (7.2). In contrast to Theorem 7.3, the superposition operator N_F generated by a Carathéodory multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ need not be continuous between two ideal spaces:

Example 8.2 [KrRu]. Let Ω and ϕ be as in Example 8.1. We shall consider the superposition operator generated by the multifunction

$$F(u) = \{\phi^{-1}(u)\} \tag{8.19}$$

from the Lebesgue space $L_1 = L_1(\Omega, \mathbb{R})$ into the Orlicz space $L_\phi = L_\phi(\Omega, \mathbb{R})$. We claim that N_F is discontinuous at 0. To see this, let x and x_k be again as in Example 8.1. The functions $z_k(t) = \phi[x(t) - x_k(t)]$ belong then to L_1 and converge to 0, since

$$\lim_{k\to\infty} \|z_k\|_{L_1} = \lim_{k\to\infty} \int_0^1 \phi[x(t) - x_k(t)]\, dt = 0.$$

On the other hand, the sequence $N_F(z_k) = \{x - x_k\}$ cannot converge to 0, by the choice (8.15) of x (see [KrRu]). ■

For the remaining part of this section, we use the terminology of the paper [ApNgZa2], which we recall now. An ideal space $X \subset S(\Omega, \mathbb{R}^m)$ is called *regular* if

$$\lim_{\mu(D)\to 0} \|P_D x\|_X = 0 \tag{8.20}$$

for every $x \in X$, where P_D is the restriction operator (7.13). The relation (8.20) means that all elements $x \in X$ have absolutely continuous norms. For example, the Lebesgue space L_p is regular for $1 \le p < \infty$, and the Orlicz space L_ϕ is regular if and only if its generating Young function ϕ satisfies a Δ_2 condition [KrRu]. Moreover, the Lorentz space Λ_φ is always regular, while the Marcinkiewicz space M_φ is never regular: in fact, a function $x \in M_\varphi$ satisfies (8.20) if and only if

$$\lim_{\tau\to 0} \frac{\varphi(\tau)}{\tau} \int_0^\tau x^*(t)\, dt = 0;$$

compare this with (8.8).

Regular ideal spaces are especially convenient since they admit a simple convergence criterion. In fact, a sequence $(x_k)_k$ converges in the norm of a regular ideal space X if and only if $(x_k)_k$ converges in measure, and all elements x_k have uniformly absolutely continuous norms, i.e.

$$\lim_{\mu(D)\to 0} \sup_{k\in\mathbb{N}} \|P_D x_k\|_X = 0. \tag{8.21}$$

This may be regarded as a generalization of the classical Vitali compactness criterion in the space L_p $(1 \le p < \infty)$.

We are now going to state the main continuity results on the superposition operator N_F in ideal spaces. In the case of a single-valued Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, the continuity of the corresponding superposition operator N_f is mainly a consequence of the important fact that N_f preserves *order-bounded subsets*. Recall that a subset M of an ideal space $X \subset S(\Omega, \mathbb{R})$ is called *order-bounded* if there exists a nonnegative function $\phi \in X$ such that $|x(t)| \le \phi(t)$ for all $x \in M$. Thus, before proving a continuity result for the superposition operator N_F in ideal spaces of vector functions, we have to define some kind of "order-boundedness" for vector functions. But as we have seen above, the role of nonnegative functions is played by the m-units in the m-dimensional vector case. So, let us call a subset M of an ideal space $X \subset S(\Omega, \mathbb{R}^m)$ *U-bounded* [Ng2, Ng3, Ng4] if there exists an m-unit Φ in X such that $M \subseteq \mathrm{Sel}_S \Phi$. Since every 1-unit has the form (8.9), in the scalar case this gives the usual definition of order-boundedness. In the vector case the U-boundedness of a set $M \subset X$ means that M is bounded in a suitable $M(\phi)$-space (see (8.6)) which is continuously imbedded in X.

The following lemma shows that superposition operators generated by Carathéodory multifunctions preserve U-boundedness:

LEMMA 8.2. *Let $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ be a Carathéodory multifunction, and suppose that the corresponding superposition operator N_F acts between two ideal spaces $X \subset$*

$S(\Omega,\mathbb{R}^m)$ and $Y \subset S(\Omega,\mathbb{R}^n)$. Then N_F maps any U-bounded set $M \subset X$ into a U-bounded set $N_F(M) \subset Y$.

Proof. Let $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^n)$ be an m-unit in X such that $M \subseteq \mathrm{Sel}_S\,\Phi$. Since F is Carathéodory, the multifunction $\Sigma : \Omega \to \mathrm{Cp}(\mathbb{R}^n)$ defined by

$$\Sigma(t) = F(t,\Phi(t)) = \bigcup_{u\in\Phi(t)} F(t,u) \tag{8.22}$$

is measurable, and also the multifunction $\Psi : \Omega \to \mathrm{Cp}(\mathbb{R}^n)$ defined by

$$\Psi(t) = \Big\{ \sum_{j=1}^{n+1} \lambda_j\sigma_j(t) : \sigma_j \in \mathrm{Sel}_S\,\Sigma,\ \sum_{j=1}^{n+1} |\lambda_j| = 1\Big\}. \tag{8.23}$$

By Theorem 6.1, for any $\sigma \in \mathrm{Sel}_S\,\Sigma$ we find $\phi \in \mathrm{Sel}_S\,\Phi$ such that $\sigma(t) \in F(t,\phi(t))$; thus, all measurable selections of the multifunction (8.22) belong to Y, by hypothesis. Since Y is an ideal space, all measurable selections of the multifunction (8.23) belong to Y as well. We conclude that $N_F(M) \subseteq \mathrm{Sel}_S\,\Psi$; hence $N_F(M)$ is U-bounded in Y as claimed. ∎

Lemma 8.3 [Ng2]. *Let $X \subset S(\Omega,\mathbb{R}^m)$ be a regular ideal space, Φ an m-unit in X, and $a_k = a_k(t)$ a sequence of positive real functions converging in $S(\Omega,\mathbb{R})$ to 0. Then for every $\varepsilon > 0$ there exists a natural number $N = N_\varepsilon$ such that $\|\phi_k\|_X \le \varepsilon$ $(k \ge N)$ for any sequence of functions $\phi_k \in \mathrm{Sel}_S\,\Phi$ with $|\phi_k| \le a_k$.*

Proof. Let $\varepsilon > 0$. Since X is regular, we have

$$\lim_{\mu(D)\to 0} \|P_D\Phi\|^* = \lim_{\mu(D)\to 0} \sup\{\|P_D\phi\| : \phi \in \mathrm{Sel}_S\,\Phi\} = 0 \tag{8.24}$$

(see [Ng3]). Consequently, we find a $\delta > 0$ such that $\|P_D\phi\| \le \varepsilon/2$ for $\mu(D) \le \delta$, uniformly in $\phi \in \mathrm{Sel}_S\,\Phi$. Now let $(\phi_k)_k$ be any sequence of selections of Φ. Since $|\phi_k| \le a_k$, and $(a_k)_k$ converges in $S(\Omega,\mathbb{R})$ to zero, $(\phi_k)_k$ also converges in $S(\Omega,\mathbb{R}^m)$ to zero. Putting

$$\Psi_k(t) = \inf\{\lambda : \lambda > 0,\ \phi_k(t) \in \lambda\Phi(t)\},$$

we see that the sequence $(\Psi_k)_k$ converges in $S(\Omega,\mathbb{R})$. This implies that, denoting by D_k the set of all $t \in \Omega$ for which $\Psi_k(t) > \varepsilon/(2\|\Phi\|^*)$, we have $\mu(D_k) \le \delta$ for $k \ge N = N_\varepsilon$. Combining this with (8.24) yields

$$\|\phi_k\| \le \|P_{D_k}\phi_k\| + \|P_{\Omega\setminus D_k}\phi_k\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|\Phi\|^*}\|P_{\Omega\setminus D_k}\Phi\|^* \le \varepsilon,$$

as claimed. ∎

We are now in a position to state our main continuity result.

Theorem 8.2 [ApNgZa2]. *Let $X \subset S(\Omega,\mathbb{R}^m)$ and $Y \subset S(\Omega,\mathbb{R}^n)$ be two ideal spaces, where Y is regular. Suppose that $F : \Omega\times\mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ is a Carathéodory multifunction, and assume that the corresponding superposition operator N_F acts between X and Y. Then N_F is continuous.*

Proof. Without loss of generality, we prove the continuity of N_F at 0. Suppose first that N_F is not lower semicontinuous at 0, say. Then we find a function $y_* \in N_F(0)$ and a sequence $(x_k)_k$ such that $\|x_k\| \le 2^{-k}$, but no sequence $y_k \in N_F(x_k)$ converges to y_*. The multifunction

$$\Phi(t) = \sum_{k=1}^{\infty} \mathrm{co}\{-x_k(t), x_k(t)\}$$

is then an m-unit in X, and the set $M = \mathrm{Sel}_S\,\Phi$ is U-bounded in X (by construction), and contained in the unit ball $B_1(X)$ (since $\|\phi\| \le \|x_1\| + \|x_2\| + \ldots \le 1$ for any $\phi \in M$). By Lemma 8.2, the set $N_F(M)$ is U-bounded in Y; this means that $N_F(M) \subseteq \mathrm{Sel}_S\,\Psi$ for some n-unit Ψ in Y.

Since $(x_k)_k$ converges to 0 in the norm of X, $(x_k)_k$ converges to 0 also in measure. By Theorem 7.3, we know that $N_F(x_k) \to N_F(0)$ in measure. Moreover, by Lemma 3.3 we may find $y_k \in N_F(x_k) = \mathrm{Sel}_S\,F(\cdot, x_k(\cdot))$ such that

$$|y_k(t) - y_*(t)| \le \tfrac{3}{2} h^-(F(t, x_k(t)), F(t,0)). \tag{8.25}$$

This shows that the sequence $(y_k)_k$ converges to y_* in measure. On the other hand, we have

$$\lim_{\mu(D)\to 0} \sup_k \|P_D y_k\| \le \lim_{\mu(D)\to 0} \sup\{\|P_D \psi\| : \psi \in \mathrm{Sel}_S\,\Psi\} = 0, \tag{8.26}$$

since $y_k \in N_F(M) \subseteq Y \cap \mathrm{Sel}_S\,\Psi$, and Y is regular. By (8.26), the elements y_k have uniformly absolutely continuous norms in Y (see (8.21)), and hence $(y_k)_k$ converges in the norm of Y to y_*, contradicting our assumption.

Suppose now that N_F is not upper semicontinuous at 0. Then we find sequences $(x_k)_k$ in X and $(y_k)_k$ in Y such that

$$\|x_k\| \le 2^{-k}, \qquad y_k \in N_F(x_k), \qquad \varrho(y_k, N_F(0)) > \delta \tag{8.27}$$

for some $\delta > 0$. By Lemma 3.3, we find in turn $z_k \in N_F(0) = \mathrm{Sel}_F(\cdot, 0)$ such that

$$|y_k(t) - z_k(t)| \le (1+\delta) h^+(F(t, x_k(t)), F(t,0)). \tag{8.28}$$

Denote the scalar function on the right-hand side of (8.28) by $a_k(t)$. Since the sequence $(a_k)_k$ converges in $S(\Omega, \mathbb{R})$ to 0, and $y_k - z_k \in \mathrm{Sel}_S(2\Psi)$ (Ψ as in the first part of the proof), we conclude with Lemma 8.3 that $\|y_k - z_k\|_Y \le \delta$ for k sufficiently large. But this contradicts the last inequality in (8.26), and so we are done. ∎

Theorem 8.2 implies, in particular, that *the superposition operator N_F generated by a Carathéodory multifunction F and considered as an operator between two Orlicz spaces L_ϕ and L_ψ, is continuous on the whole space L_ϕ provided that the Young function ψ satisfies a Δ_2 condition*. Of course, the point in the above Example 8.2 is that $\psi(v) = e^{|u|} - |u| - 1$ does not satisfy a Δ_2 condition.

Some more continuity results for the multi-valued superposition operator between ideal spaces may be found in [ApNgZa2]. Continuity results for multi-valued superposition operators between so called Orlicz–Musielak F-spaces [Mu] of Banach space-valued functions are given in [RoSo2].

In [AnCl] the following is shown (see also [DoSh]): For every compact set $K \subset C([0,1], \mathbb{R}^n)$ and a Carathéodory multifunction $F : [0,1] \times \mathbb{R}^m \to \mathbb{R}^n$, one may find a continuous function $\varphi : K \to L_1([0,1], \mathbb{R}^n)$ such that $\varphi(x) \in N_F(x)$ for all $x \in K$.

9. The superposition operator in the space C. In this section we assume that Ω is a compact domain in Euclidean space without isolated points. In analogy to what we have done in Section 7, we call a multifunction $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ *superpositionally continuous* (or *sup-continuous*, for short) if, for any $x \in C(\Omega, \mathbb{R}^m)$, the multifunction

$$Y(t) = F(t, x(t)) \tag{9.1}$$

is continuous; if, for any $x \in C(\Omega, \mathbb{R}^m)$, the multifunction (9.1) admits at least a continuous (single-valued) selection y, we call F *weakly sup-continuous.* In either case, the superposition operator

$$N_F(x) = \mathrm{Sel}_C\, Y \tag{9.2}$$

is then well-defined, since $C(\Omega, \mathbb{R}^m) \cap N_F(x) \neq \emptyset$ for all $x \in C(\Omega, \mathbb{R}^m)$.

The following theorem gives a (necessary and sufficient) condition for sup-continuity, and a (sufficient) condition for weak sup-continuity of a multifunction F.

THEOREM 9.1. *If $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ is continuous, then F is sup-continuous, and vice versa. If $F : \Omega \times \mathbb{R}^m \to \mathrm{ClCv}(\mathbb{R}^n)$ is lower semicontinuous, then F is weakly sup-continuous.*

Proof. The fact that the continuity of $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ implies its sup-continuity is a simple consequence of Lemma 2.5. Conversely, suppose that F is sup-continuous, and let $(t_k, u_k) \in \Omega \times \mathbb{R}^m$ converge $(t_*, u_*) \in \Omega \times \mathbb{R}^m$. By the classical Tietze–Urysohn theorem, we find $x \in C(\Omega, \mathbb{R}^m)$ such that $x(t_k) = u_k$ and $x(t_*) = u_*$. Since the multifunction $Y(t) = F(t, x(t))$ is continuous in the Hausdorff metric (1.5), we conclude that

$$F(t_k, u_k) = Y(t_k) \to Y(t_*) = F(t_*, u_*),$$

as $k \to \infty$, and thus F is a continuous multifunction.

To prove the second assertion, it suffices to show that the lower semicontinuity of the multifunction F implies the lower semicontinuity of the operator N_F, and to use Theorem 2.1. Let $W \subseteq C(\Omega, \mathbb{R}^n)$ be closed and $(x_k)_k$ a sequence in $N_{F+}^{-1}(W)$; this means that every continuous function y_k satisfying $y_k(t) \in F(t, x_k(t))$ belongs to W. Suppose that $(x_k)_k$ converges uniformly on Ω to x_*; we have to show that $x_* \in N_{F+}^{-1}(W)$. Since the set $W(t) = \{w(t) : w \in W\}$ is closed in $\mathbb{R}^n$ for any $t \in \Omega$, and F is lower semicontinuous, by hypothesis, the set

$$A(t) = F(t, \cdot)_+^{-1}(W(t)) = \{u : u \in \mathbb{R}^m,\ F(t, u) \subseteq W(t)\}$$

is closed. Thus, the fact that $x_k(t) \in A(t)$ implies that also $x_*(t) \in A(t)$, hence $x_* \in N_{F+}^{-1}(W)$. ∎

We remark that the lower semicontinuity of F is not necessary for the weak sup-continuity of the corresponding superposition operator:

EXAMPLE 9.1. Let $F : \mathbb{R} \to \mathrm{CpCv}(\mathbb{R}^n)$ be defined by

$$F(u) = \begin{cases} [0,1] & \text{if } u \neq 0, \\ [0,2] & \text{if } u = 0. \end{cases}$$

Then the function $y(t) \equiv 0$ belongs to $N_F(x)$ for each $x \in C(\Omega, \mathbb{R})$, and thus F is weakly sup-continuous. However, F is not lower semicontinuous at $u = 0$. ∎

The lower semicontinuity is a natural assumption in Theorem 9.1 in order to apply Michael's selection principle. One could ask whether or not an upper semicontinuous multifunction is weakly sup-continuous. As Example 2.9 shows, this is false, since the function $x(t) = t$ leads to the multifunction $Y(t) = F(t)$, which has no continuous selection. Thus, lower semicontinuous multifunctions are weakly sup-continuous, while upper semicontinuous multifunctions are not.

Note that in the corresponding problem for the weak sup-measurability of a multifunction the opposite was true (see Lemma 7.1 and Example 7.2).

Now we briefly study boundedness and continuity properties of the superposition operator (9.2) in the space C of continuous functions. Here it turns out that the continuity of the multifunction F, which ensures the acting of the corresponding operator N_F in the space C, is also sufficient to guarantee its boundedness and continuity. The following two theorems may be regarded as analogues to Theorem 7.2 and Theorem 7.3, respectively.

THEOREM 9.2. *Assume that* $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ *is a continuous multifunction. Then the superposition operator* (9.2) *generated by* F *is bounded from* $C(\Omega, \mathbb{R}^m)$ *into* $\mathrm{BdCl}(C(\Omega, \mathbb{R}^n))$.

Proof. Let $M \subset C(\Omega, \mathbb{R}^m)$ be bounded. The closure $\bar{\Delta}$ of the set

$$\Delta = \bigcup_{x \in M} \{x(t) : t \in \Omega\}$$

is then compact in $\mathbb{R}^m$; by the continuity of F, the set $F(\Omega \times \bar{\Delta})$ is in turn compact in $\mathbb{R}^n$. This implies that

$$\|F(\Omega \times \bar{\Delta})\|^* = \sup \bigcup_{t \in \Omega,\, u \in \bar{\Delta}} \{\|v\| : v \in F(t, u)\} < \infty,$$

and hence $N_F(M)$ is bounded in $C(\Omega, \mathbb{R}^n)$. ∎

THEOREM 9.3. *Assume that* $F : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n)$ *is a continuous multifunction. Then the superposition operator* (9.2) *generated by* F *is continuous from* $C(\Omega, \mathbb{R}^m)$ *into* $\mathrm{BdCl}(C(\Omega, \mathbb{R}^n))$.

Proof. Suppose that $x_k \to x$ in $C(\Omega, \mathbb{R}^m)$. By the continuity of F, we have then

$$F(t, x_k(t)) = Y_k(t) \to Y(t) = F(t, x(t))$$

in $C(\Omega, \mathrm{Cp}(\mathbb{R}^n))$. This means that, given $\varepsilon > 0$, we may find $N = N_\varepsilon \in \mathbb{N}$ such that $h(Y_k(t), Y(t)) \leq \varepsilon$ for $k \geq N$, uniformly in $t \in \Omega$. We conclude that $h(\mathrm{Sel}_C Y_k, \mathrm{Sel}_C Y) \leq \varepsilon$ for $k \geq N$. By the the definition (9.2) of the superposition operator, this proves the continuity of N_F at x. ∎

Apart from the case of ideal spaces and the space C, very little is known on multivalued superposition operators in other function spaces. The only results we are aware of are contained in [Do] for almost periodic functions, in [MeNi, Zw] for functions of bounded variation, in [SmSm] for Lipschitz continuous functions, and in [Kc] for Sobolev space functions.

4. Closures and convexifications

In this chapter we discuss the problem of passing from a given multifunction to some extension which has in a certain sense nicer properties. As a matter of fact, such extensions are an extremely useful tool in both the theory and applications of differential and integral inclusions and give rise to various notions of generalized solutions of such inclusions. The main emphasis will be put on strong closures, weak closures, and convexifications.

10. Strong closures. Let X and Y be two arbitrary normed linear spaces, and let $N : X \to P(Y)$ be a nonlinear (multi-valued) operator. For fixed $x_0 \in X$ we put

$$\overline{N}(x_0) = \bigcap_{\varepsilon,\delta>0} \{y : y \in N(x) + z,\ \|x - x_0\|_X \le \varepsilon,\ \|z\|_Y \le \delta\} \tag{10.1}$$

and call $\overline{N}$ the (*strong*) *closure* of the operator N. The notation $\overline{N}$ is justified by the fact that the graph $\Gamma(\overline{N})$ of $\overline{N}$ (see (1.6)) coincides with the closure $\overline{\Gamma(N)}$ of the graph of N; in particular, $\overline{N} = N$ if and only if N is a closed operator. In order to avoid misunderstandings, we point out that the operator $\overline{N}$ is *not* defined by closing the values of N (i.e. $\overline{N}(x) = \overline{N(x)}$), but by closing its graph.

Suppose now that $X \subset S(\Omega, \mathbb{R}^m)$ and $Y \subset S(\Omega, \mathbb{R}^n)$ are ideal spaces, and $N = N_F$ is the superposition operator (7.2) generated by some sup-measurable multifunction $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$. The question arises whether or not the closure $\overline{N}_F$ of N_F is again a superposition operator N_G and, if so, what is the generating multifunction G. It turns out that this question admits a complete and very natural answer.

For fixed $t_0 \in \Omega$ and $u \in \mathbb{R}^m$ we put

$$\overline{F}(t_0, u_0) = \bigcap_{\varepsilon,\delta>0} \{v : v \in F(t_0, u) + h,\ |u - u_0| \le \varepsilon,\ |h| \le \delta\} \tag{10.2}$$

and call $\overline{F}(t_0, \cdot)$ the (strong) closure of the multifunction $F(t_0, \cdot)$.

The following lemma shows which properties of a multifunction F carry over to its closure $\overline{F}$.

LEMMA 10.1. *The following holds*:

(a) *if* $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ *is upper semicontinuous*, *then so is* $\overline{F}$;
(b) *if* $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ *is closed*, *then so is* $\overline{F}$;
(c) *if* $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ *is measurable*, *then so is* $\overline{F}$;

Proof. (a) Fix $(t_0, u_0) \in \Omega \times \mathbb{R}^m$, and let $V \subseteq \mathbb{R}^n$ be open with $\overline{F}(t_0, u_0) \subseteq V$. Choose $U \supseteq \overline{F}(t_0, u_0)$ open with $\overline{U} \subset V$. Since F is upper semicontinuous at (t_0, u_0), by Lemma 2.3(a) we find a $\delta > 0$ such that $F(t_0, u) \subseteq U$ for $\|u - u_0\| < \delta$. It follows that $\overline{F}(t_0, u) \subseteq \overline{U} \subseteq V$ for these u.

(b) The statement is trivial, since $\overline{F}$ is always closed.

(c) Since the graph $\Gamma(\overline{F})$ is closed, it is a measurable subset of $\Omega \times \mathbb{R}^m \times \mathbb{R}^n$; by Lemma 3.1(e), $\overline{F}$ is measurable. ∎

The following example shows that the lower semicontinuity of a multifunction F does not carry over, in general, to its closure $\overline{F}$:

EXAMPLE 10.1. Let $F : \mathbb{R} \to \mathrm{CpCv}(\mathbb{R})$ be defined by

$$F(u) = \begin{cases} [0,1] & \text{if } u \le 0, \\ [0,2] & \text{if } u > 0. \end{cases}$$

Then F is lower semicontinuous on $\mathbb{R}$. On the other hand, we have

$$\overline{F}(u) = \begin{cases} [0,1] & \text{if } u < 0, \\ [0,2] & \text{if } u \ge 0, \end{cases}$$

which is not lower semicontinuous at $u_0 = 0$. ∎

Now we answer the question how to describe the multifunction G which generates the closure $\overline{N}_F$ of the superposition operator N_F.

THEOREM 10.1. *Suppose that* $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ *is sup-measurable. Then the multifunction* (10.2) *is again sup-measurable*, *the corresponding superposition operator* $N_{\bar{F}}$ *acts between* X *and* Y, *and the equality*

$$N_{\bar{F}} = \overline{N}_F \tag{10.3}$$

holds.

Proof. Let $x_0 \in X$ and $y_0 \in N_{\bar{F}}(x_0)$ be fixed; we show that $y_0 \in \overline{N}_F(x_0)$. Let Φ be an m-unit in X, Ψ an n-unit in Y, and put for $k = 1, 2, \ldots$

$$G_k(t) = \left\{(u,v) : u \in \mathbb{R}^m,\ v \in F(t,u),\ u - x_0(t) \in \frac{1}{k}\Phi(t),\ v - y_0(t) \in \frac{1}{k}\Psi(t)\right\}. \tag{10.4}$$

Since $G_k \in S(\Omega, \mathbb{R}^{m+n})$, by Sainte-Beuve's selection theorem [Sn2, Sn3] we may find sequences $(x_k)_k$ in X and $(y_k)_k$ in Y such that $(x_k(t), y_k(t)) \in G_k(t)$ for almost all $t \in \Omega$. This implies, in particular, that $y_k(t) \in F(t, x_k(t))$, i.e. $y_k \in N_F(x_0)$. Moreover, from

$$x_k(t) - x_0(t) \in \frac{1}{k}\Phi(t), \qquad y_k(t) - y_0(t) \in \frac{1}{k}\Psi(t),$$

and the fact that $M(\Phi) \subseteq X$ and $M(\Psi) \subseteq Y$ (continuous imbeddings, see (8.6)) it follows that

$$\|x_k - x_0\|_X \to 0, \qquad \|y_k - y_0\|_Y \to 0 \qquad (k \to \infty),$$

i.e. $y_0 \in \overline{N}_F(x_0)$.

Conversely, let $y_0 \in \overline{N}_F(x_0)$ be fixed; we claim that $y_0 \in N_{\bar{F}}(x_0)$, i.e. $y_0(t) \in \overline{F}(t, x_0(t))$ for almost all $t \in \Omega$. Choose sequences $(x_k)_k$ in X and $(y_k)_k$ in Y such that $x_k \to x_0$, $y_k \to y_0$, and $y_k \in N_F(x_k)$. By passing to subsequences, if necessary, we may assume that $x_k(t) \to x_0(t)$ and $y_k(t) \to y_0(t)$ a.e. on Ω. Applying now (10.2) to $u_0 = x_0(t)$, $u = x_k(t)$, $v = y_0(t)$, and $h = y_0(t) - y_k(t)$, we conclude that $y_0(t) \in \overline{F}(t, x_0(t))$. ∎

Suppose now that $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ is a continuous (hence sup-continuous, by Theorem 9.1) multifunction. We want to establish an analogue to formula (10.3) for the superposition operator N_F generated by F, and considered as an operator from $C(\Omega, \mathbb{R}^m)$ into $P(C(\Omega, \mathbb{R}^n))$. To this end, we shall adapt the proof of Theorem 10.1; unfortunately, this requires a new technical assumption.

We call a multifunction $F : \Omega \times \mathbb{R}^m \to P(\mathbb{R}^n)$ *quasi-concave* if

$$F(t, (1-\lambda)u_0 + \lambda u_1) \supseteq (1-\lambda)F(t, u_0) + \lambda F(t, u_1) \tag{10.5}$$

for all $t \in \Omega$, $\lambda \in [0,1]$, and $u_0, u_1 \in \mathbb{R}^m$. Choosing $u_0 = u_1$ in (10.5) one sees that a quasi-concave multifunction necessarily takes convex values; the following very simple example shows that the converse is not true.

EXAMPLE 10.2. Let $F : \mathbb{R} \to \mathrm{CpCv}(\mathbb{R})$ be defined by

$$F(u) = [0, u^2]. \tag{10.6}$$

Then (10.5) fails, for example, for $u_0 = 0$, $u_1 = 1$, and $0 < \lambda < 1$. It is of course easy to see that, in general, the multifunction $F(u) = [0, f(u)]$ is quasi-concave if and only if the function f is nonnegative and concave on $\mathbb{R}$. ∎

THEOREM 10.2. *Suppose that* $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ *is continuous and quasi-concave. Then the multifunction* (10.2) *is again continuous*, *the corresponding superposition operator* $N_{\bar{F}}$ *acts in the space* C, *and the equality* (10.3) *holds.*

Proof. The inclusion $\overline{N}_F(x_0) \subseteq N_{\bar{F}}(x_0)$ is proved in the same way as in Theorem 10.1. To prove the converse inclusion, fix $y_0 \in N_{\bar{F}}(x_0)$ and consider again the multifunction (10.4). Since F is quasi-concave, by assumption, the multifunction G_k takes closed convex values in $\mathbb{R}^{m+n}$. In order to apply Michael's selection theorem in the same way as we applied Sainte-Beuve's theorem in the proof of Theorem 10.1, we have to show that G_k is lower semicontinuous for each k.

We write G_k in the form $G_k(t) = H(t) \cap H_k(t)$, where

$$H(t) = \{(u,v) : u \in \mathbb{R}^m,\ v \in F(t,u)\} \tag{10.7}$$

and

$$H_k(t) = \left\{(u,v) : u \in \mathbb{R}^m,\ u - x_0(t) \in \frac{1}{k}\Phi(t),\ v - y_0(t) \in \frac{1}{k}\Psi(t)\right\} \tag{10.8}$$

and apply Lemma 2.6. Let us first show that H is ε-δ-lower semicontinuous. Given $\varepsilon > 0$, $t_0 \in \Omega$, and $u_0 \in \mathbb{R}^m$, by the continuity of F we find a $\delta > 0$ such that

$$h(F(t,u), F(t_0,u_0)) < \varepsilon$$

for $|t - t_0| < \delta$ and $|u - u_0| < \delta$. This implies, in particular, that $h^-(H(t), H(t_0)) < \varepsilon$ for $|t - t_0| < \delta$ and thus the multifunction (10.7) is ε-δ-lower semicontinuous at t_0.

The fact that the multifunction (10.8) is ε-δ-lower semicontinuous for each $k \in \mathbb{N}$ follows from the continuity of the functions x_0 and y_0. From Lemma 2.6 we conclude that the multifunction G_k is ε-δ-lower semicontinuous, and hence lower semicontinuous, by Lemma 2.3(b). ∎

The quasi-concavity condition (10.5) which we only needed for guaranteeing the convexity of the values of the multifunction G_k, is of course very restrictive. It is very likely that Theorem 10.2 is also true without the assumption (10.5). Multifunctions satisfying (10.5) or the related *additivity condition*

$$F(t, u_0 + u_1) \supseteq F(t, u_0) + F(t, u_1)$$

have been studied by several authors. For example, it is shown in [Sm] that such multifunctions always admit an additive selection.

The preceding two theorems show that the equality (10.3) holds if we consider the superposition operator N_F either between two ideal spaces X and Y, or in the space C of continuous functions.

It is interesting to note that (10.3) may fail if we consider N_F, say, from the space C into an ideal space Y.

Example 10.3. Let $F : \mathbb{R} \to \mathrm{CpCv}(\mathbb{R})$ be defined by

$$F(u) = \begin{cases} \{\sin(1/u)\} & \text{if } u \neq 0, \\ \{0\} & \text{if } u = 0. \end{cases} \tag{10.9}$$

Obviously, we then have

$$\overline{F}(u) = \begin{cases} \{\sin(1/u)\} & \text{if } u \neq 0, \\ [-1,1] & \text{if } u = 0. \end{cases} \tag{10.10}$$

Consider the superposition operator N_F generated by the multifunction (10.9) from the space $C[0,1]$ into the space $L_1[0,1]$. On the one hand, the set $N_{\bar{F}}(0)$ consists then of all measurable selections of the multifunction $Y(0) = [-1,1]$, i.e. is the closed unit ball in the space $L_\infty[0,1]$. On the other hand, any function $y \in \overline{N}_F(0)$ may be represented in the form

$$y(t) = \sin\frac{1}{x(t)} + z(t),$$

where x is continuous on $[0,1]$, and z is a measurable function with small L_1-norm. ■

Theorem 10.1 was given for Lebesgue spaces in [KrPk], for ideal spaces of scalar functions in [ApDeZa], and for general ideal spaces in [ApNgZa3].

11. Convexifications. In this section we are concerned with various extensions of multifunctions which may be called convexifications. Let X and Y be two normed linear spaces and $N : X \to P(Y)$ a (multi-valued) operator. Given $x_0 \in X$, the simplest convexifications of N which come to mind are of course

$$(\mathrm{co}\, N)(x_0) = \mathrm{co}\, N(x_0) \tag{11.1}$$

and

$$(\overline{\mathrm{co}}\, N)(x_0) = \overline{\mathrm{co}\, N(x_0)}. \tag{11.2}$$

However, the most useful definition is the operator $N^\square$ defined by

$$N^\square(x_0) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\{y : y \in N(x),\ \|x - x_0\|_X \le \varepsilon\}; \tag{11.3}$$

in what follows, the term *convexification* of the operator N will always refer to (11.3). Likewise, for fixed $t_0 \in \Omega$ and $u_0 \in \mathbb{R}^m$ we put

$$(\mathrm{co}\, F)(t_0,u_0) = \mathrm{co}\, F(t_0,u_0), \tag{11.4}$$

$$(\overline{\mathrm{co}}\, F)(t_0,u_0) = \overline{\mathrm{co}\, F(t_0,u_0)}, \tag{11.5}$$

and

$$F^\square(t_0,u_0) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\{v : v \in F(t_0,u), |u-u_0| \le \varepsilon\}, \tag{11.6}$$

and call $F^{\square}(t_0,\cdot)$ the convexification of the multifunction $F(t_0,\cdot)$. A comparison with (10.1) and (10.2) shows that

$$\overline{\mathrm{co}}\,\overline{F}(t_0,u_0) \subseteq F^{\square}(t_0,u_0), \qquad \overline{\mathrm{co}}\,\overline{N}(x_0) \subseteq N^{\square}(x_0). \tag{11.7}$$

There are in fact multifunctions F such that the inclusion in (11.7) is strict.

EXAMPLE 11.1. Let $F:\mathbb{R}\to \mathrm{CpCv}(\mathbb{R})$ be defined by

$$F(u) = \begin{cases} [1/|u|, 1/|u|+1] & \text{if } u\in\mathbb{Q},\ u\neq 0,\\ [-1/|u|-1, -1/|u|] & \text{if } u\notin\mathbb{Q},\\ \{0\} & \text{if } u=0. \end{cases} \tag{11.8}$$

Then $\overline{F}(0)=\{0\}$, hence $\overline{\mathrm{co}}\,\overline{F}(0)=\{0\}$, but $F^{\square}(0)=\mathbb{R}$. ∎

Note that the multifunction (11.8) in the preceding example is not upper semicontinuous at 0. This follows as well from the following

LEMMA 11.1. *Let $F:\Omega\times\mathbb{R}^m\to \mathrm{ClCv}(\mathbb{R}^n)$ be a given multifunction. If $F(t_0,\cdot)$ is upper semicontinuous at u_0, then*

$$\overline{\mathrm{co}}\,\overline{F}(t_0,u_0) = F^{\square}(t_0,u_0). \tag{11.9}$$

Similarly, if the superposition operator N_F generated by F is upper semicontinuous at some $x_0\in X$, then

$$\overline{\mathrm{co}}\,\overline{N}_F(x_0) = N_F^{\square}(x_0). \tag{11.10}$$

Proof. We prove only the equality (11.9). By (11.7), we have to show that $F^{\square}(t_0,u_0) \subseteq \overline{\mathrm{co}}\,\overline{F}(t_0,u_0)$.

Without loss of generality, let $u_0=0$, and suppose that there is a $v_0\in F^{\square}(t_0,0)$ such that $v_0\notin \overline{\mathrm{co}}\,\overline{F}(t_0,0)$, Choose an open convex set $V\supset \overline{\mathrm{co}}\,\overline{F}(t_0,0)$ in $\mathbb{R}^n$ such that $v_0\notin V$. Since $F(t_0,\cdot)$ is upper semicontinuous at $u_0=0$ (and hence ε-δ-upper semicontinuous, by Lemma 2.3(a)), we may find a $\delta>0$ such that $F(t_0,u)\subseteq V$ for $|u|\le\delta$. Consequently,

$$v_0\in F^{\square}(t_0,0)\subseteq \mathrm{co}\bigcup_{|u|\le\delta} F(t_0,u)\subseteq \mathrm{co}\,V = V,$$

contradicting our choice of V. ∎

At this point we collect again some properties of a multifunction F which carry over to its extensions (11.4), (11.5), or (11.6).

LEMMA 11.2. *The following holds*:

(a) *if $F:\Omega\times\mathbb{R}^m\to \mathrm{Cp}(\mathbb{R}^n)$ is upper semicontinuous, then so is* $\mathrm{co}\,F$;
(b) *if $F:\Omega\times\mathbb{R}^m\to \mathrm{Cp}(\mathbb{R}^n)$ is lower semicontinuous, then so is* $\mathrm{co}\,F$;
(c) *if $F:\Omega\times\mathbb{R}^m\to P(\mathbb{R}^n)$ is measurable, then so is* $\mathrm{co}\,F$;
(d) *if $F:\Omega\times\mathbb{R}^m\to \mathrm{Cp}(\mathbb{R}^n)$ is upper semicontinuous, then so is* $\overline{\mathrm{co}}\,F$;
(e) *if $F:\Omega\times\mathbb{R}^m\to \mathrm{Cp}(\mathbb{R}^n)$ is lower semicontinuous, then so is* $\overline{\mathrm{co}}\,F$;
(f) *if $F:\Omega\times\mathbb{R}^m\to P(\mathbb{R}^n)$ is measurable, then so is* $\overline{\mathrm{co}}\,F$;
(g) *if $F:\Omega\times\mathbb{R}^m\to \mathrm{Cl}(\mathbb{R}^n)$ is upper semicontinuous, then so is* $F^{\square}$;
(h) *if $F:\Omega\times\mathbb{R}^m\to P(\mathbb{R}^n)$ is closed, then so is* $F^{\square}$;
(i) *if $F:\Omega\times\mathbb{R}^m\to P(\mathbb{R}^n)$ is measurable, then so is* $F^{\square}$.

P r o o f. (a) By Lemma 2.3(a), F is ε-δ-upper semicontinuous. This means that for $\varepsilon > 0$ we may find a $\delta > 0$ such that $F(t_0, u) \subseteq U_\varepsilon(F(t_0, u_0))$ for $\|u - u_0\| < \delta$. Consequently,

$$\operatorname{co} F(t_0, u) \subseteq \operatorname{co} U_\varepsilon(F(t_0, u_0)) = U_\varepsilon(\operatorname{co} F(t_0, u_0))$$

for $\|u - u_0\| < \delta$. By Lemma 2.3(c), $\operatorname{co} F$ is upper semicontinuous.

(b) may be proved like (a) by interchanging u_0 and u.

(c), (f), (i): obvious.

(d), (e) follow the same lines as (a), (b).

(g) If F is upper semicontinuous, the equality (11.9) holds, and hence $F^\square$ is the composition of the two upper semicontinuous multifunctions $\overline{F}$ and $\overline{\operatorname{co}}$.

(h) The statement is trivial, since $F^\square$ is always closed. ■

Example 10.1 shows that the lower semicontinuity of a multifunction F does not imply the lower semicontinuity of its convexification $F^\square$. In fact, in Example 10.1 we have $F^\square = \overline{F}$.

The following example shows that the multifunctions $\operatorname{co} F$ and $\overline{\operatorname{co}}\, F$ need not be closed, even if the multifunction F is.

EXAMPLE 11.2. Let $F : \mathbb{R} \to \operatorname{Cp}(\mathbb{R})$ be defined by

$$F(u) = \begin{cases} \{0, 1/u\} & \text{if } u \neq 0, \\ \{0\} & \text{if } u = 0. \end{cases}$$

It is easy to see that F is closed on $\mathbb{R}$. However, the multifunction

$$\operatorname{co} F(u) = \overline{\operatorname{co}}\, F(u) = \begin{cases} [0, 1/u] & \text{if } u \neq 0, \\ \{0\} & \text{if } u = 0 \end{cases}$$

is not closed, as may be seen by taking $(x_n, y_n) = (1/n, 1)$ in Lemma 2.7(b). ■

As in Section 3, we collect the properties which carry over from F to $\overline{F}, \operatorname{co} F, \overline{\operatorname{co}}\, F$, and $F^\square$ in the following table:

F	upper semicontinuous	lower semicontinuous	closed	measurable
$\overline{F}$	yes (L.10.1)	no (E.10.1)	yes (L.10.1)	yes (L.10.1)
$\operatorname{co} F$	yes** (L.11.2)	yes** (L.11.2)	no (E.11.2)	yes (L.11.2)
$\overline{\operatorname{co}}\, F$	yes** (L.11.2)	yes** (L.11.2)	no (E.11.2)	yes (L.11.2)
$F^\square$	yes* (L.11.2)	no (E.10.1)	yes (L.11.2)	yes (L.11.2)

* if F has closed values ** if F has compact values

As pointed out before, the convexification $F^\square$ of a multifunction F may have better properties than F itself; for example, $F^\square$ is always closed and convex-valued. This may be used to get "virtual selections" of F, i.e. functions $f \in \operatorname{Sel} F^\square$. For instance, consider again the continuous multifunction F from Example 2.10 which does not admit continuous selections. Its convexification $F^\square$ is given by $F^\square = \overline{\operatorname{co}}\, F$, by (11.9). It is clear that $F^\square$ has lots of continuous selections.

In this connection, it is interesting to study the problem "how far" these selections $f \in F^\square$ are actually from F, i.e. to estimate the Hausdorff deviation $h^+(\Gamma(f), \Gamma(F))$ (see (2.7)). A partial answer to this problem is contained in the following

LEMMA 11.3. *Let $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ be an upper Carathéodory multifunction. Then for each $\varepsilon > 0$ there exists a Carathéodory function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ such that* (2.7) *holds and*

$$f(t_0, u_0) \in \overline{\mathrm{co}}\{v : v \in F(t_0, u),\ |u - u_0| \le \varepsilon\}$$

for almost all $t_0 \in \Omega$ and all $u_0 \in \mathbb{R}^m$.

Proof. Given $\varepsilon > 0$, we may find for any $u_0 \in \mathbb{R}^m$ a $\delta(u_0) > 0$ such that

$$h^+(F(t_0, u), F(t_0, u_0)) < \varepsilon$$

for $|u - u_0| < \delta(u_0)$, since $F(t_0, \cdot)$ is ε-δ-continuous, by Lemma 2.3(a). The system of all balls $\{u : u \in \mathbb{R}^m,\ |u - u_0| < \delta(u_0)\}$, where u_0 runs over the whole space $\mathbb{R}^m$, is an open covering of $\mathbb{R}^m$. Let $\{U_1, U_2, U_3, \ldots\}$ be a locally finite refinement of this covering and $\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$ a subordinate partition of unity, i.e.

$$\operatorname{supp} \varphi_j \subseteq U_j \quad (j = 1, 2, \ldots), \qquad \sum_{j=1}^{\infty} \varphi_j \equiv 1.$$

Finally, choose $u_j \in U_j$ and $v_j(t_0) \in F(t_0, u_j)$ arbitrarily. Then the function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$f(t, u) = \sum_{j=1}^{\infty} \varphi_j(u) v_j(t)$$

has the required properties. ∎

We turn now to the problem of "interchanging" the superposition operator N_F with the convexifications (11.3) and (11.6). The following is parallel to Theorem 10.1.

THEOREM 11.1. *Suppose that $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ is sup-measurable. Then the multifunction* (11.6) *is again sup-measurable, the corresponding superposition operator $N_{F^\square}$ acts between X and Y, and*

$$N_{F^\square} = N_F^\square. \tag{11.11}$$

Proof. Let $x_0 \in X$ and $y_0 \in N_{F^\square}(x_0)$ be fixed; we show that $y_0 \in N_F^\square(x_0)$. Since $\Phi(t) = F^\square(t, x_0(t))$ is a closed convex subset of $\mathbb{R}^n$, we find an n-unit Ψ in Y such that $\Phi(t) \subseteq \Psi(t)$ for almost all $t \in \Omega$. Moreover, by Carathéodory's parametrization theorem [CsVl] we have

$$y_0(t) = \sum_{j=1}^{n+1} \alpha_j(t) \phi_j(t), \qquad 0 \le \alpha_j(t) \le 1,\ \sum_{j=1}^{n+1} \alpha_j(t) = 1, \tag{11.12}$$

where $\phi_1, \ldots, \phi_{n+1} \in \mathrm{Sel}_S\, \Phi$ have the property that $\phi_j(t)$ is an extremal point in $\Phi(t)$ for almost all $t \in \Omega$. From the classical Kreĭn–Mil'man theorem we conclude that $\phi_j(t) \in \overline{F}(t, x_0(t))$, hence $\phi_j \in N_{\overline{F}}(x_0) \subseteq N_{F^\square} \subseteq Y$, by (11.3). Now fix $\varepsilon > 0$, and let $\beta_j : \Omega \to \mathbb{Q}$ $(j = 1, \ldots, n + 1)$ be simple functions, say

$$\beta_j(t) = \frac{1}{\gamma}[\gamma_{j,1}\chi_{\Omega_1}(t) + \ldots + \gamma_{j,k}\chi_{\Omega_k}(t)]$$

such that $\gamma, \gamma_{j,1}, \ldots, \gamma_{j,k} \in \mathbb{N}$ with $\gamma_{1,i} + \ldots + \gamma_{n+1,i} = \gamma$ for $i = 1, 2, \ldots, k$ (without loss of generality), and

$$\alpha_j(t) - \beta_j(t) \in \frac{\varepsilon}{n+1}\Psi(t) \quad (j = 1, \ldots, n+1). \tag{11.13}$$

Constructing z_0 from $\beta_1, \ldots, \beta_{n+1}$ as y_0 from $\alpha_1, \ldots, \alpha_{n+1}$ we have, in analogy to (11.8),

$$z_0(t) = \sum_{j=1}^{n+1} \beta_j(t)\phi_j(t), \quad 0 \le \beta_j(t) \le 1, \ \sum_{j=1}^{n+1} \beta_j(t) = 1. \tag{11.14}$$

Moreover, from (11.13) and the definition (8.6) of the space $M(\Psi)$ we get $\|y_0 - z_0\|_{M(\Psi)} \le \varepsilon$. Consequently, instead of showing that $y_0 \in N_F^{\Box}(x_0)$ it suffices to show that $z_0 \in N_F^{\Box}(x_0)$, since $N_F^{\Box}(x_0)$ is closed in Y, and $M(\Psi)$ is continuously imbedded in Y. To this end, on each of the sets $\Omega_1, \ldots, \Omega_k \subseteq \Omega$ we define functions $\psi_{1,i}^{(i)}, \ldots, \psi_{\gamma,i}^{(i)}$ by putting

$$\begin{aligned} \psi_1^{(i)} &= \ldots = \psi_{\gamma_{1,i}}^{(i)} = \phi_1, \\ \psi_{\gamma_{1,i}+1}^{(i)} &= \ldots = \psi_{\gamma_{2,i}}^{(i)} = \phi_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_{\gamma_{n,i}+1}^{(i)} &= \ldots = \psi_{\gamma_{n+1,i}}^{(i)} = \phi_{n+1}. \end{aligned}$$

The relation

$$z_0(t) = \sum_{j=1}^{n+1} \beta_j(t)\phi_j(t) = \frac{1}{\gamma} \sum_{j=1}^{\gamma} \sum_{i=1}^{k} \psi_{j,i}^{(i)}(t)$$

implies then that $z_0 \in N_F^{\Box}(x_0)$, by construction.

Now let $y_0 \in N_F^{\Box}(x_0)$ be fixed; we have to show that $y_0 \in N_{F^{\Box}}(x_0)$. Since the set $N_{F^{\Box}}(x_0)$ does not depend on Y, and the set $N_F^{\Box}(x_0)$ may only become larger if we pass to a larger space Y, we assume without loss of generality that $Y = L(\Psi)$ (see (8.5)), where Ψ is some n-unit in Y. But the upper semicontinuity of the closure $\overline{N}_F : X \to \mathrm{Cl}(L(\Psi))$ of N_F implies that $y_0 \in \overline{\mathrm{co}}\, \overline{N}_F(x_0)$. From Theorem 10.1 we conclude that

$$y_0(t) \in \overline{\mathrm{co}}\, \overline{N}_F(x_0)(t) = \overline{\mathrm{co}}\, \overline{F}(t, x_0(t)) \subseteq F^{\Box}(t, x_0(t)),$$

and so we are done. ∎

The proof of the following theorem is essentially the same as that of Theorem 11.1.

THEOREM 11.2. *Suppose that* $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ *is continuous and quasi-concave. Then the multifunction* (11.6) *is again continuous*, *the corresponding superposition operator* $N_{F^{\Box}}$ *acts in the space* C, *and the equality* (11.7) *holds.*

Theorem 11.1 was given for Lebesgue spaces in [KrPk], for ideal spaces of scalar functions in [ApDeZa], and for general ideal spaces in [ApNgZa3].

12. Weak closures. In this section we briefly discuss the problem of passing to the closure of the superposition operator (7.2) with respect to some duality between ideal spaces. We shall assume throughout that the measure μ is atom-free on Ω and that $\mu(\Omega) < \infty$.

Given an ideal space $X \subset S(\Omega,\mathbb{R}^m)$, by X' we denote the *associate space* of X consisting, by definition, of all functions $x' \in S(\Omega,\mathbb{R}^m)$ for which the pairing

$$\langle x, x'\rangle = \int_\Omega (x(t), x'(t))\, d\mu(t) \tag{12.1}$$

is finite for all $x \in X$; here $(\cdot,\cdot)$ denotes the usual scalar product on $\mathbb{R}^m$ [Za2, ZaNg2]. The space X' is a (possibly strict) closed subspace of the *dual space* X^*, and coincides with X^* if and only if X is regular (see (8.20)). The space X' is throughout considered with the natural norm

$$\|x'\|_{X'} = \sup\{\langle x, x'\rangle : \|x\|_X \le 1\}. \tag{12.2}$$

We give some examples. From the classical Hölder inequality it follows that $L_p' = L_p^* = L_{p/(p-1)}$ for $1 \le p < \infty$, but $L_\infty' = L_1 \ne L_\infty^*$. More generally, the spaces $L(\Phi)$ and $M(\Phi)$ (see (8.5) and (8.6)) form a pair of mutually associate spaces. The associate space L_ϕ' to the Orlicz space L_ϕ (see (8.3)) coincides with the Orlicz space $L_{\phi'}$ generated by the Young function

$$\phi'(v) = \sup\{u|v| - \phi(u) : u \ge 0\}. \tag{12.3}$$

Finally, the associate space Λ_φ' to the Lorentz space Λ_φ (see (8.7)) is the Marcinkiewicz space $M_{\varphi'}$ (see (8.8)) generated by the function

$$\varphi'(t) = \frac{t}{\varphi(t)} \qquad (0 < t \le 1).$$

Likewise, the associate space M_φ' to the Marcinkiewicz space M_φ is the Lorentz space $\Lambda_{\varphi'}$ [KnPeSe].

In what follows, the notation $x_k \rightharpoonup x_0$ means that $\langle x_k - x_0, x'\rangle \to 0$ for every $x' \in X'$, where $(x_k)_k$ is a sequence in X, and $x_0 \in X$ is fixed. Given two ideal spaces $X \subset S(\Omega,\mathbb{R}^m)$ and $Y \subset S(\Omega,\mathbb{R}^m)$ and a (multi-valued) operator $N : X \to P(Y)$, denote by $\vec{N}(x_0)$ the set of all $y_0 \in Y$ such that there exist sequences $(x_k)_k$ in X and $(y_k)_k$ in Y satisfying $x_k \rightharpoonup x_0$, $y_k \rightharpoonup y_0$, and $y_k \in N(x_k)$. In this way, we have defined an operator $\vec{N} : X \to P(Y)$, which we call the *weak closure* of the operator N. It follows from standard theorems of functional analysis that

$$N(x_0) \subseteq \overline{N}(x_0) \subseteq \vec{N}(x_0) \subseteq \overline{\mathrm{co}}\, \overline{N}(x_0) \tag{12.4}$$

for any $x_0 \in X$, where $\overline{N}(x_0)$ denotes the strong closure (10.1).

Since closed convex sets in Banach spaces are also weakly closed, it is not surprising that there is a natural relation between the weak closure and the convexification (11.1); we make this precise in the following

LEMMA 12.1. *Let X and Y be two ideal spaces, and let $N : X \to \mathrm{Bd}(Y)$ be a multi-valued operator. Assume that the associate space Y' to Y is separable. Then*

$$\vec{N}(x_0) = N^{\Box}(x_0), \tag{12.5}$$

whenever the set $\vec{N}(x_0)$ is closed and convex.

Proof. By (11.3) and (12.4), we have only to show that $N^{\Box}(x_0) \subseteq \vec{N}(x_0)$. Suppose that there exists a function $y_0 \in N^{\Box}(x_0)$ which does not belong to $\vec{N}(x_0)$. Since by assumption, $\vec{N}(x_0)$ is closed and convex, by the classical Hahn–Banach theorem there exists an element $y_0' \in Y'$ such that

$$\langle y - y_0, y_0' \rangle \geq \eta > 0 \tag{12.6}$$

for all $y \in \vec{N}(x_0)$. Consider the "weak neighbourhood"

$$U_\eta(y_0) = \{y : y \in Y, \langle y, y_0' \rangle > \langle y_0, y_0' \rangle + \eta\}$$

of y_0. By definition (11.1) and the fact that $y_0 \in N^{\Box}(x_0)$ we have

$$U_\eta(y_0) \cap \overline{\mathrm{co}}\{y : y \in N(x), \|x - x_0\| \leq \varepsilon\} \neq \emptyset$$

for all sufficiently small $\varepsilon > 0$. Choose sequences $(x_k)_k$ in X and $(y_k)_k$ in Y such that $x_k \to x_0$ and $y_k \in N(x_k) \cap U_\eta(y_0)$. Since Y' is separable, we may find a subsequence $(y_{k'})_{k'}$ of $(y_k)_k$ such that $y_{k'} \rightharpoonup \widetilde{y} \in \vec{N}(x_0) \cap U_\eta(y_0)$. But for this $\widetilde{y}$ we have

$$\langle \widetilde{y} - y_0, y_0' \rangle = \langle \widetilde{y}, y_0' \rangle - \langle y_0, y_0' \rangle < \langle y_0, y_0' \rangle + \eta - \langle y_0, y_0' \rangle = \eta,$$

contradicting (12.6). ■

Before stating the main theorem of this section, we need another technical result.

LEMMA 12.2 [ApZa2]. *Let $Y \subset S(\Omega, \mathbb{R}^n)$ be an ideal space such that Y' is separable. Suppose that $(y_k)_k$ and $(\widetilde{y}_k)_k$ are sequences in Y such that $y_k \rightharpoonup y$ and $\widetilde{y}_k \rightharpoonup \widetilde{y}$. Then there exists a sequence $(D_k)_k$ of sets $D_k \subseteq \Omega$ such that*

$$\widehat{y}_k = P_{D_k} y_k + P_{\widetilde{D}_k} \widetilde{y}_k \rightharpoonup (y + \widetilde{y})/2, \tag{12.7}$$

where $\widetilde{D}_k = \Omega \setminus D_k$, and P_D denotes the restriction operator (7.13).

Proof. We construct a sequence of partitions on Ω into sets $D(\varepsilon_1, \ldots, \varepsilon_n)$ ($\varepsilon_i \in \{0, 1\}$) as follows. First, let $\{D(0), D(1)\}$ be a partition of Ω such that $\mu(D(0)) = \mu(D(1)) = \mu(\Omega)/2$ (it is here that we use the assumption on μ to be atom-free on Ω). Next, we take $D(0) = D(0,0) \cup D(0,1)$ and $D(1) = D(1,0) \cup D(1,1)$, where $\mu(D(\varepsilon_1, \varepsilon_2)) = \mu(\Omega)/4$. Similarly, if $\{D(\varepsilon_1, \ldots, \varepsilon_n) : \varepsilon_i \in \{0, 1\}\}$ is the nth partition of Ω, we divide each $D(\varepsilon_1, \ldots, \varepsilon_n)$ into two parts $D(\varepsilon_1, \ldots, \varepsilon_n, 0)$ and $D(\varepsilon_1, \ldots, \varepsilon_n, 1)$ such that

$$\mu(D(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1})) = 2^{-(n+1)} \mu(\Omega) \quad (\varepsilon_i \in \{0, 1\}).$$

Now put

$$D_k = \bigcup_{\varepsilon_i \in \{0,1\}} D(\varepsilon_1, \ldots, \varepsilon_n, 0), \qquad \widetilde{D}_k = \bigcup_{\varepsilon_i \in \{0,1\}} D(\varepsilon_1, \ldots, \varepsilon_n, 1). \tag{12.8}$$

Since the functions $\theta_k = \chi_{D_k} - \chi_{\widetilde{D}_k}$ satisfy the orthogonality relation

$$\langle \theta_j, \theta_k \rangle = \begin{cases} \mu(\Omega) & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

we conclude that

$$\lim_{k \to \infty} \langle \theta_k, z \rangle = 0 \tag{12.9}$$

for each $z \in L_1$. Defining now $\widehat{y}_k$ as in (12.7), we get

$$\begin{aligned} 2\widehat{y}_k - (y + \widetilde{y}) &= 2P_{D_k} y_k + 2P_{\widetilde{D}_k} \widetilde{y}_k - y - \widetilde{y} \\ &= 2P_{D_k}(y_k - y) + 2P_{\widetilde{D}_k}(\widetilde{y}_k - \widetilde{y}) + 2P_{D_k}(y - \widetilde{y}) - 2P_{\widetilde{D}_k}(y - \widetilde{y}). \end{aligned}$$

Consequently, for any $y' \in Y'$ we may apply (12.9) to $z = y'(y - \widetilde{y})$ and obtain

$$\begin{aligned} \lim_{k\to\infty} \langle 2\widehat{y}_k - (y + \widetilde{y}), y' \rangle &= 2 \lim_{k\to\infty} \langle P_{D_k}(y_k - y), y' \rangle \\ &\quad + 2 \lim_{k\to\infty} \langle P_{\widetilde{D}_k}(\widetilde{y}_k - \widetilde{y}), y' \rangle + \lim_{k\to\infty} \langle \theta_k, y'(y - \widetilde{y}) \rangle = 0. \end{aligned}$$

This shows that $\widehat{y}_k \rightharpoonup (y + \widetilde{y})/2$ as claimed. ■

THEOREM 12.1. *Suppose that* $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ *is sup-measurable, and the superposition operator* N_F *generated by* F *acts between two ideal spaces* $X \subset S(\Omega, \mathbb{R}^m)$ *and* $Y \subset S(\Omega, \mathbb{R}^m)$, *where* Y' *is separable. Then the weak closure* $\vec{N}_F$ *of* N_F *coincides with the convexification* $N_F^{\Box}$.

Proof. By Lemma 12.1, it suffices to show that $\vec{N}_F(x_0)$ is convex for all $x_0 \in X$. Since $\vec{N}_F(x_0)$ is closed, we have to show that $y, \widetilde{y} \in \vec{N}_F(x_0)$ implies that also $(y+\widetilde{y})/2 \in \vec{N}_F(x_0)$. Choose sequences $(x_k)_k$, $(\widetilde{x}_k)_k$, $(y_k)_k$, and $(\widetilde{y}_k)_k$, such that $x_k \rightharpoonup x_0$, $\widetilde{x}_k \rightharpoonup x_0$, $y_k \rightharpoonup y_0$, $\widetilde{y}_k \rightharpoonup y_0$, $y_k \in N_F(x_k)$, and $\widetilde{y}_k \in N_F(\widetilde{x}_k)$. By the local determination (7.15) of the operator N_F, we have $\widehat{y}_k \in N_F(\widehat{x}_k)$, where $\widehat{y}_k$ is defined as in (12.7), and

$$\widehat{x}_k = P_{D_k} x_k + P_{\widetilde{D}_k} \widetilde{x}_k.$$

But this implies that $\widehat{x}_k \rightharpoonup x_0$, hence $(y + \widetilde{y})/2 \in \vec{N}_F(x_0)$ as claimed. ■

Theorem 12.1 implies, in particular, that *the superposition operator* N_F *generated by a sup-measurable multifunction* F *and considered as an operator between two Orlicz spaces* L_ϕ *and* L_ψ, *has the same weak closure and convexification provided that the associate Young function* (12.3) *of* ψ *satisfies a* Δ_2 *condition.*

We point out that, apart from the closures and convexifications considered in this chapter, there are other useful extensions of multifunctions. Several such extensions for the investigation of differential inclusions may be found in [Fi2, Fi3, To1, Va]; a classical reference is the book [Fi4]. For instance, given a measurable multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$, the convexification

$$F^{\#}(t_0, u_0) = \bigcap_{\varepsilon>0} \bigcap_{N\in\mathfrak{N}} \overline{\mathrm{co}}\{v : v \in F(t_0, u),\ |u - u_0| \le \varepsilon,\ u \notin N\}, \tag{12.10}$$

where N runs over the system $\mathfrak{N}$ of all nullsets in $\mathbb{R}^m$, is called the *Filippov extension* of $F(t_0, \cdot)$. Some interesting results on the Scorza Dragoni property of Filippov extensions may be found in [Vr]. Moreover, the recent paper [BsSp] is concerned with the following question: Suppose that F and G are two multifunctions such that $F^{\#} = G^{\#}$; in what way are F and G then related?

5. Fixed points and integral inclusions

In this chapter we shall first recall some generalizations of well-known fixed point principles of nonlinear analysis to multi-valued operators in metric or normed spaces.

Applying this to integral inclusions of Hammerstein type (i.e. products of single-valued integral operators and multi-valued superposition operators) allows us to obtain several existence and uniqueness results.

13. Fixed point theorems for multi-valued operators. Let (X, d_X) and (Y, d_Y) be two metric spaces. A multi-valued operator $A : X \to \mathrm{ClBd}(Y)$ is said to satisfy a *Lipschitz condition* if

$$h_Y(Ax, Ay) \le L d_X(x, y) \quad (x, y \in X) \tag{13.1}$$

for some $L > 0$, where h_Y denotes the Hausdorff distance (1.5) in Y. The minimal constant L in (13.1) is called *Lipschitz constant* of A and denoted by $\mathrm{Lip}(A)$ in the sequel. In case $\mathrm{Lip}(A) < 1$ the multi-valued operator A is called a *contraction.*

For further reference, we collect some elementary properties of such multi-valued operators in the following

LEMMA 13.1. *Let* $A : X \to \mathrm{ClBd}(Y)$ *and* $B : Y \to \mathrm{ClBd}(Z)$ *be multi-valued operators. Then the following holds*:

(a) $\mathrm{Lip}(BA) \le \mathrm{Lip}(B)\mathrm{Lip}(A)$;
(b) $\mathrm{Lip}(\mathrm{co}\, A) = \mathrm{Lip}(A)$ (*with* $\mathrm{co}\, A$ *as in* (11.1));
(c) $\mathrm{Lip}(\overline{\mathrm{co}}\, A) = \mathrm{Lip}(A)$ (*with* $\overline{\mathrm{co}}\, A$ *as in* (11.2));
(d) $\mathrm{Lip}(\overline{A}) = \mathrm{Lip}(A)$ (*with* $\overline{A}$ *as in* (10.1));
(e) $\mathrm{Lip}(A^{\square}) = \mathrm{Lip}(A)$ (*with* $A^{\square}$) *as in* (11.3)).

Proof. (a) First of all, it is easy to see that

$$h_Z(Bu, Bv) \le L d_Y(u, v) \quad (u, v \in Y)$$

implies

$$h_Z(B(M), B(N)) \le L h_Y(M, N) \quad (M, N \in \mathrm{ClBd}(Y)).$$

Consequently, for $x, y \in X$ we take $M = Ax$, $N = Ay$, and get

$$h_Z(B(Ax), B(Ay)) \le \mathrm{Lip}(B) h_Y(Ax, Ay) \le \mathrm{Lip}(B)\mathrm{Lip}(A) d_X(x, y).$$

(b) follows from (c), since $\overline{\mathrm{co}}\, A$ is an extension of $\mathrm{co}\, A$.

(c) On the one hand, taking $Z = Y$ and $B = \overline{\mathrm{co}}$ we get $\mathrm{Lip}(B) = 1$ and hence $\mathrm{Lip}(\overline{\mathrm{co}}\, A) \le \mathrm{Lip}(A)$ from (a). On the other hand, the reverse inequality is trivial, since $\overline{\mathrm{co}}\, A$ is an extension of A.

(d) follows from (e), since $A^{\square}$ is an extension of $\overline{A}$ (see (11.3)).

(e) Let $x, y \in X$ and $L > \mathrm{Lip}(A)$. For $\varepsilon > 0$ we have

$$\begin{aligned} h_Y(\overline{\mathrm{co}}\, A[\overline{U}_\varepsilon(x)], \overline{\mathrm{co}}\, A[\overline{U}_\varepsilon(y)]) &= h_Y(A[\overline{U}_\varepsilon(x)], A[\overline{U}_\varepsilon(y)]) \\ &\le L h_X(\overline{U}_\varepsilon(x), \overline{U}_\varepsilon(y)) \le L\{d_X(x, y) + \varepsilon\}. \end{aligned}$$

Letting now ε tend to zero we obtain the assertion. ■

Recall that a point $x_* \in X$ is called a *fixed point* of a multi-valued operator $A : X \to P(X)$ if

$$x_* \in A x_*. \tag{13.2}$$

In case of an operator which attains only singletons as values (i.e. is actually single-valued), this definition coincides with the usual definition of a fixed point, of course.

The following is a natural generalization of the classical *Banach–Caccioppoli–Picard fixed point principle* to multi-valued operators.

THEOREM 13.1 [Na]. *Let* (X,d) *be a complete metric space and* $A : X \to \mathrm{ClBd}(X)$ *a* (*multi-valued*) *contraction. Then* A *has a fixed point in* X.

Proof. Let $\mathrm{Lip}(A) < L < 1$ and $x_0 \in X$ be fixed, and choose any $x_1 \in Ax_0$. By the definition (1.5) of the Hausdorff distance, we find $x_2 \in Ax_1$ such that

$$d(x_1, x_2) \le h(Ax_0, Ax_1) + L.$$

Similarly, we find $x_3 \in Ax_2$ such that

$$d(x_2, x_3) \le h(Ax_1, Ax_2) + L^2.$$

Continuing this way, we find a sequence $(x_n)_n$ in X such that $x_{n+1} \in Ax_n$ and

$$d(x_n, x_{n+1}) \le h(Ax_{n-1}, Ax_n) + L^n.$$

For fixed k we have then

$$\begin{aligned} d(x_k, x_{k+1}) &\le h(Ax_{k-1}, Ax_k) + L^k \le Ld(x_{k-1}, x_k) + L^k \\ &\le L\{h(Ax_{k-2}, Ax_{k-1}) + L^{k-1}\} + L^k \le L^2 d(x_{k-2}, x_{k-1}) + 2L^k \\ &\le \ldots \le L^k d(x_0, x_1) + kL^k. \end{aligned}$$

Consequently,

$$d(x_n, x_{n+p}) \le \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) \le \sum_{k=n}^{n+p-1} L^k d(x_0, x_1) + \sum_{k=n}^{n+p-1} kL^k.$$

This shows that $(x_n)_n$ is a Cauchy sequence, and hence $x_n \to x_*$ for some $x_* \in X$. Since A satisfies a Lipschitz condition, we have $Ax_n \to Ax_*$ in $\mathrm{ClBd}(X)$ as well. But from $x_{n+1} \in Ax_n$ it follows that then $x_* \in Ax_*$ as claimed. ∎

Observe that, in contrast to the classical Banach–Caccioppoli principle, one cannot expect uniqueness of the fixed point in Theorem 13.1.

A typical situation where the preceding fixed point principle applies is the following. Suppose that X is a Banach space, $\overline{B}_r(x_0) = \{x : x \in X,\ \|x - x_0\| \le r\}$ is the closed ball with centre $x_0 \in X$ and radius $r > 0$, and $A : \overline{B}_r(x_0) \to \mathrm{ClBd}(\overline{B}_r(x_0))$ is a multi-valued operator satisfying (13.1) for some $L < 1$. If $\varrho(x_0, Ax_0) < (1 - L)r$, then A has a fixed point in the ball $\overline{B}_\varrho(x_0)$, where ϱ satisfies

$$\frac{\varrho(x_0, Ax_0)}{1 - L} < \varrho \le r.$$

We shall apply this variant of Theorem 13.1 in the following section.

Another classical fixed point principle which is at least as important as the Banach–Caccioppoli theorem is the *Schauder fixed point principle* which states that *a continuous nonlinear operator* T *which leaves a nonempty convex compact subset* C *in a Banach space invariant*, *has a fixed point in* C. The following Theorem 13.2 gives a "multi-valued variant" of this fixed point principle.

THEOREM 13.2 [BbKl]. *Let X be a normed linear space, $C \in \mathrm{CpCv}(X)$, and $A : C \to \mathrm{ClCv}(C)$ an upper semicontinuous multi-valued operator. Then A has a fixed point in C.*

Proof. Since C is a compact subset of X, for $\varepsilon > 0$ we find a finite ε-net $\{z_1^\varepsilon, \ldots, z_{m(\varepsilon)}^\varepsilon\}$ for C, i.e.

$$h^+(C, \{z_1^\varepsilon, z_2^\varepsilon, \ldots, z_{m(\varepsilon)}^\varepsilon\}) < \varepsilon. \tag{13.3}$$

For $j = 1, \ldots, m(\varepsilon)$, define $\phi_j^\varepsilon : C \to [0, \infty)$ by

$$\phi_j^\varepsilon(x) = \begin{cases} \varepsilon - \|x - z_j^\varepsilon\| & \text{if } \|x - z_j^\varepsilon\| < \varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{13.4}$$

The functions $\psi_j^\varepsilon : C \to [0, 1]$ defined by

$$\psi_j^\varepsilon(x) = \frac{\phi_j^\varepsilon(x)}{\phi_1^\varepsilon(x) + \ldots + \phi_{m(\varepsilon)}^\varepsilon(x)}$$

($j = 1, \ldots, m(\varepsilon)$) are then continuous on C and satisfy $\psi_1^\varepsilon + \ldots + \psi_{m(\varepsilon)}^\varepsilon \equiv 1$. Choosing any points $y_1^\varepsilon \in Az_1^\varepsilon, \ldots, y_{m(\varepsilon)}^\varepsilon \in Az_{m(\varepsilon)}^\varepsilon$, by the convexity of C we see that the operator T_ε defined by

$$T_\varepsilon x = \psi_1^\varepsilon(x) y_1^\varepsilon + \ldots + \psi_{m(\varepsilon)}^\varepsilon y_{m(\varepsilon)}^\varepsilon \tag{13.5}$$

maps C into C and is continuous. By the classical Schauder principle, there is a point $x_\varepsilon \in C$ such that $T_\varepsilon x_\varepsilon = x_\varepsilon$. Now we take $\varepsilon = 1/n$ ($n = 1, 2, \ldots$) and write x_n for $x_{1/n}$, T_n for $T_{1/n}$, etc. ($n = 1, 2, \ldots$). Since C is compact, we may assume, without loss of generality, that $x_n \to x_*$ for some $x_* \in C$; we claim that x_* is a fixed point of the multi-valued operator A.

Suppose that $x_* \notin Ax_*$. Since Ax_* is compact, we have then $x_* \notin U_\eta(Ax_*)$ for some $\eta > 0$. By the upper semicontinuity of A at x_*, we find a $\delta > 0$ such that $Az \subseteq U_{\eta/2}(Ax_*)$ for all $z \in U_\delta(x_*) \cap C$; without loss of generality, suppose that $\delta \le \eta$. We claim that

$$T_n x \in U_{\eta/2}(Ax_*) \quad (x \in U_{\delta - 1/n}(x_*) \cap C). \tag{13.6}$$

In fact, for $x \in U_{\delta-1/n}(x_*) \cap C$ we may choose $z_j^n \in \{z_1^n, \ldots, z_{m(n)}^n\}$ such that $\|x - z_j^n\| < 1/n$, hence

$$\|x_* - z_j^n\| \le \|x_* - x\| + \|x - z_j^n\| < \delta - 1/n + 1/n = \delta.$$

This shows that $z_j^n \in U_\delta(x_*) \cap C$, and thus $Az_j^n \subseteq U_{\eta/2}(Ax_*)$. By the convexity of $U_{\eta/2}(Ax_*)$ we conclude that

$$T_n x = \psi_1^n(x) y_1^n + \ldots + \psi_{m(n)}^n(x) y_{m(n)}^n \in U_{\eta/2}(Ax_*), \tag{13.7}$$

i.e. (13.6) holds.

Now choose $N \in \mathbb{N}$ so large that $1/n < \delta/2$ for $n \ge N$, hence $\delta/2 < \delta - 1/n$. For $x_n \in U_{\delta/2}(x_*) \cap C$ we have then $T_n x_n \in U_{\eta/2}(Ax_*)$, by (13.6). Consequently,

$$\varrho(x_*, Ax_*) \le \|x_* - x_n\| + \|x_n - T_n x_n\| + \varrho(T_n x_n, Ax_*) < \delta/2 + 0 + \eta/2 \le \eta,$$

contradicting the fact that $x_* \notin U_\eta(Ax_*)$. ∎

In the proof of Theorem 13.2 we have used the Schauder fixed point principle to obtain the existence of fixed points of the (single-valued) operator (13.5). Alternatively, we could have proved first a multi-valued version of the classical *Brouwer fixed point principle* in $\mathbb{R}^n$ due to Kakutani [Ka], and then pass to infinite-dimensional normed spaces.

In the following two sections, we shall use Theorem 13.2 often in the following *equivalent* form:

THEOREM 13.3. *Let X be a normed linear space, $B \in \mathrm{ClBdCv}(X)$, and $A : B \to \mathrm{ClCv}(B)$ an upper semicontinuous compact multi-valued operator. Then A has a fixed point in B.*

Proof. We show the equivalence of Theorem 13.2 and Theorem 13.3. Setting $C = \overline{\mathrm{co}}\, A(B)$ under the hypotheses of Theorem 13.3, we have $C \in \mathrm{CpCv}(X)$, by the Mazur lemma and the compactness of the multi-valued operator A. By Theorem 13.2, the multi-valued operator A has a fixed point $x_* \in C$; moreover, $C = \overline{\mathrm{co}}\, A(B) \subseteq \overline{\mathrm{co}}\, B = B$, by assumption.

Conversely, let $C \in \mathrm{CpCv}(X)$ and A be as in the hypotheses of Theorem 13.2. Since Ax is a closed subset of C for any $x \in C$, we actually have $A : C \to \mathrm{CpCv}(C)$. By the remark after Example 2.7, the image $A(C)$ of C is compact, and thus Theorem 13.3 applies. ∎

While in Theorem 13.2 the compactness assumption is imposed on the domain of definition C, in Theorem 13.3 it is imposed on the multi-valued operator A. Nevertheless, in both theorems we have to verify the invariance of the set C which sometimes may be difficult. In this situation, one may use the following result which is precisely the multi-valued analogue of the well-known *Schaefer continuation principle*:

THEOREM 13.4. *Let X be a normed linear space and $A : X \to \mathrm{CpCv}(X)$ an upper semicontinuous compact multi-valued operator. Suppose that there exists an $r > 0$ such that the a priori estimate*

$$x \in \lambda Ax \ (0 < \lambda \le 1) \Rightarrow \|x\| \le r \tag{13.8}$$

holds. Then A has a fixed point in the ball $\overline{B}_r(0)$.

Proof. Let $U = B_R(0)$ for some $R > r$. Then U is an open subset of X, and the multi-valued vector field Φ defined by $\Phi x = x - Ax$ is nondegenerate on ∂U, by (13.8). Consequently, the rotation $\gamma(\Phi; \partial U)$ of Φ on ∂U satisfies $\gamma(\Phi; \partial U) = 1$ (see [Bo-Ob1, Bo-Ob2, Ma]), and thus the operator A has a fixed point. ∎

The statement of Theorem 13.4 is rather surprising: just knowing a priori that all possible fixed points of the multi-valued operators λA (if there are any!), are contained in a fixed ball independent of λ we may deduce the existence of a fixed point of the multi-valued operator A. The crucial point in the application of Theorem 13.4 (see e.g. Theorem 14.3 below) is of course the verification of the a priori estimate (13.8).

14. Hammerstein integral inclusions. Let X and Y be two Banach spaces of functions $x : \Omega \to \mathbb{R}^m$ and $y : \Omega \to \mathbb{R}^n$, respectively, and suppose that a multifunction $F : \Omega \times \mathbb{R}^m \to \mathrm{Cl}(\mathbb{R}^n)$ generates a superposition operator N_F between X and Y. Moreover, let $k : \Omega \times \Omega \to \mathbb{R}^{m \times n}$ be a matrix-valued function which generates a (single-valued) linear integral operator

$$Ky(s) = \int_{\Omega} k(s,t)y(t)\,dt \tag{14.1}$$

from Y into X. The present section is concerned with the *integral inclusion of Hammerstein type*

$$x \in KN_F x. \tag{14.2}$$

Here the action of the integral operator K on the set $N_F x$ is meant "selection-wise", i.e.

$$KN_F x = \{Ky : y \in N_F x\}. \tag{14.3}$$

With this definition, the integral of a multifunction has very natural properties. For a general integration theory of multifunctions we refer to the survey articles [An, Db, BlLs, Va] and the book [IoTi]. Some applications to Hammerstein inclusions may be found in [Ly3].

Of course, if the nonlinearity F is single-valued, i.e. $F(t,u) = \{f(t,u)\}$ for some function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ which generates a (single-valued) superposition operator

$$N_f x(t) = f(t, x(t)), \tag{14.4}$$

the inclusion (14.2) reduces to the classical integral equation (system) of Hammerstein type

$$x(s) = \int_{\Omega} k(s,t) f(t, x(t))\,dt. \tag{14.5}$$

By our hypotheses on K and N_F, we may consider the inclusion (14.2) as fixed point problem (13.2) for the multi-valued operator $A = KN_F$. To apply the fixed point principles of the preceding section, we therefore have to verify the hypotheses of those fixed point principles by imposing appropriate conditions on the nonlinearity F and the kernel function k. If these conditions are not satisfied in applications, it is a useful device to pass from the multifunction F to a suitable extension G of F like those studied in Chapter 4; the most common choice is here $G = \overline{F}$ (see (10.2)) or $G = F^{\square}$ (see (11.2)). This leads to various notions of *generalized solutions*, i.e. functions x satisfying instead of (14.2) the inclusion

$$x \in K\overline{N}_F x \tag{14.6}$$

or the inclusion

$$x \in KN_F^{\square} x. \tag{14.7}$$

For example, if $N_F : B \to \mathrm{Cp}(Y)$ is bounded for some $B \in \mathrm{ClBdCv}(X)$, and $K : Y \to X$ is compact with $KN_F(B) \subseteq B$, then the operator $A = (KN_F)^{\square} = KN_F^{\square}$ is closed (by Lemma 11.2(h)) and locally compact, hence upper semicontinuous (by Lemma 2.9). Applying Theorem 13.3 yields the existence of a generalized solution $x \in B$, i.e. $x \in KN_F^{\square}x$. In some cases one may then conclude that this generalized solution belongs actually to the smaller set $KN_F x$, i.e. satisfies (14.2). This procedure is a certain analogue to what is a common device for solving elliptic boundary value problems: first one proves that there exists a generalized solution (existence theory), and then one shows that every generalized solution is actually a classical solution (regularity theory).

Now we are going to apply the fixed point principles of the preceding section. We start with the simplest case described in Theorem 13.1. As before, we use the notation

$$\overline{B}_r(x_0) = \{x : x \in X, \|x - x_0\| \le r\} \tag{14.8}$$

for the closed ball with centre $x_0 \in X$ and radius $r > 0$.

THEOREM 14.1. *Suppose that the superposition operator N_F maps a ball $\overline{B}_r(x_0) \subset X$ into* ClBd(Y), *and the linear integral operator K is bounded from Y into X. Moreover, assume that the Lipschitz condition*

$$h_Y(N_F x_1, N_F x_2) \le L\|x_1 - x_2\| \quad (x_1, x_2 \in \overline{B}_r(x_0)) \tag{14.9}$$

holds, where

$$L\,\|K\| < 1. \tag{14.10}$$

Finally, suppose that the estimate

$$\|x_0 - Ky_0\| < (1 - L)r \tag{14.11}$$

holds for all $y_0 \in N_F x_0$. Then the Hammerstein inclusion (14.2) *has a solution $x_* \in \overline{B}_r(x_0)$.*

Proof. The hypotheses on N_F and K imply that the multi-valued operator $A = KN_F$ maps $\overline{B}_r(x_0)$ into ClBd(X). By (14.10) and Lemma 13.1(a), A is a contraction. Finally, the condition (14.11) guarantees that actually $Ax \in$ ClBd$(\overline{B}_r(x_0))$ for any $x \in \overline{B}_r(x_0)$ (see the remark after Theorem 13.1). Thus, all the hypotheses of Theorem 13.1 are fulfilled, and hence the multi-valued operator A has a fixed point $x_* \in \overline{B}_r(x_0)$. ∎

Apart from the geometrical condition (14.11) on the "initial value" x_0, the Lipschitz condition (14.9) for the superposition operator N_F is the most important hypothesis in Theorem 14.1. It is clear that a sufficient condition for (14.9) is a Lipschitz condition

$$h_{\mathbb{R}^n}(F(t, u_1), F(t, u_2)) \le L\|u_1 - u_2\| \quad (u_1, u_2 \in \mathbb{R}^m) \tag{14.12}$$

for the generating multifunction F in the second variable. Moreover, Lemma 2.11 and Lemma 3.3 suggest that (14.12) is often "close" to being also necessary for (14.9).

Observe that, by Lemma 13.1(d) and (e), one may pass in the Lipschitz condition (14.9) from N_F to either $\overline{N}_F$ or $N_F^{\square}$ without increasing the Lipschitz constant L. This shows that, loosely speaking, it is not necessary to study generalized solutions (in the sense of (14.6) or (14.7)) when applying the contraction principle for multi-valued operators.

The situation is different, however, for the various fixed point theorems which generalize the classical Schauder principle. For example, Theorem 13.2 requires that the multi-valued operator A takes closed convex values and is upper semicontinuous; this may be achieved sometimes only after passing from A to, say, the convexification $A^{\square}$. On the other hand, in most applications it is easier to apply Theorem 13.3, rather than Theorem 13.2, for at least two reasons: First, the domain of definition of A is usually closed and bounded but not compact (e.g. a ball); second, the compactness of A may usually be obtained simply by the fact that the linear integral operator (14.1) is compact.

In what follows we assume that X and Y are ideal spaces over Ω. As a model case, one may always think of the Lebesgue spaces $X = L_p(\Omega, \mathbb{R}^m)$ and $Y = L_q(\Omega, \mathbb{R}^n)$, or the Orlicz spaces $X = L_\phi(\Omega, \mathbb{R}^m)$ and $Y = L_N(\Omega, \mathbb{R}^n)$.

We suppose throughout that the linear integral operator (14.1) is *compact* from Y into X; many sufficient conditions for the compactness of integral operators may be found, for example, for Lebesgue spaces in [Kr-So], for Orlicz spaces in [KrRu], and for general ideal spaces in [Za1]. Since any linear compact operator is also continuous, for the upper semicontinuity of $A = KN_F$ it suffices, by Lemma 2.5(f), to show that the multi-valued operator N_F is upper semicontinuous. To this end, we collect some sufficient conditions in the following

LEMMA 14.1. *Let* $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^n)$ *be a sup-measurable multifunction, and assume that the corresponding superposition operator* N_F *acts between* X *and* Y. *Suppose that one of the following three conditions is satisfied*:

(a) F *is a Carathéodory multifunction, and* Y *is regular*;

(b) $F(t,\cdot)$ *is upper semicontinuous for almost all* $t \in \Omega$, N_F *maps any U-bounded set* $M \subset X$ *into a U-bounded set* $N_F(M) \subset Y$, *and* Y *is regular*;

(c) $F(t,\cdot)$ *has a closed graph* (*in* $\mathbb{R}^m \times \mathbb{R}^n$), *and both* X *and* Y *are regular.*

Then the superposition operator N_F *is upper semicontinuous between* X *and* Y.

Proof. The fact that (a) implies the upper semicontinuity of N_F is essentially contained in the proof of Theorem 8.2. Analyzing the proof of Theorem 8.2 one sees that we actually used condition (b) which is slightly weaker than condition (a), by Lemma 8.2. Finally, to see that (c) implies the upper semicontinuity of N_F is straightforward. ■

The hypotheses (a)–(c) of Lemma 14.1 are easily verified if X and Y are Orlicz spaces (in particular, Lebesgue spaces) [ZaNg3]. A natural growth condition on F under which the upper semicontinuity of the multifunction $F(t,\cdot)$ implies the upper semicontinuity of the operator N_F may be found in [ClFrRz]. Another set of sufficient conditions for the upper semicontinuity of N_F between Lebesgue spaces is contained in [Ly6]. Some new results for Banach space-valued functions in the Bochner–Lebesgue space $L_p(\Omega, X)$ are given in [So].

Using Lemma 14.1, an existence theorem for the Hammerstein inclusion (14.2) may be formulated, for instance, as follows.

THEOREM 14.2. *Suppose that the superposition operator* N_F *maps a ball* $\overline{B}_r(x_0) \subset X$ *into* $\mathrm{ClCv}(Y)$, *and the linear integral operator* K *is compact from* Y *into* X. *Moreover, assume that one of the conditions* (a)–(c) *of Lemma* 14.1 *is satisfied. Finally, suppose that the estimate*

$$\|x_0 - Ky\| \le r \tag{14.13}$$

holds for all $y \in N_F(\overline{B}_r(x_0))$. *Then the Hammerstein inclusion* (14.2) *has a solution* $x_* \in \overline{B}_r(x_0)$.

P r o o f. The hypotheses on N_F and K and the estimate (14.13) imply that the multi-valued operator $A = KN_F$ maps $\overline{B}_r(x_0)$ into $\mathrm{ClCv}(\overline{B}_r(x_0))$. Moreover, A is upper semicontinuous and compact, since N_F is upper semicontinuous and K is compact. Thus, all the hypotheses of Theorem 13.3 are fulfilled, and hence the multi-valued operator A has a fixed point $x_* \in \overline{B}_r(x_0)$. ■

We remark that a special variant of Theorem 14.2 for operators between Lebesgue spaces is given in [Ly7].

We discuss now the applicability of the more sophisticated Theorem 13.4. To this end, we recall a definition which is motivated by some applications of integral operators to elliptic boundary value problems.

Let Y be an ideal space and Y' its associate space (see (12.1)). We call a linear operator $K : Y \to Y'$ *positive* if

$$\langle Ky, Ky\rangle \le \mu\langle y, Ky\rangle \qquad (y \in Y) \tag{14.14}$$

for some $\mu > 0$, where $\langle\cdot,\cdot\rangle$ denotes the pairing (12.1). The smallest μ with this property will be denoted by $\mu(K;Y)$ and called the *positivity constant* of K (in Y).

Throughout the following, X and Y are two ideal spaces of $\mathbb{R}^m$-valued functions.

THEOREM 14.3 [Ap-Za]. *Suppose that the superposition operator N_F maps X into* $\mathrm{ClCv}(Y)$, *and the linear integral operator K is compact from Y into X. Moreover, assume that one of the conditions* (a)–(c) *of Lemma* 14.1 *is satisfied. Let K be positive from Y into Y', and suppose that*

$$\|Ky\|_X^2 \le \delta\langle y, Ky\rangle \qquad (y \in Y) \tag{14.15}$$

for some $\delta > 0$. Finally, assume that the multifunction F satisfies the unilateral estimate

$$(u, F(t,u)) \subseteq (-\infty, a\|u\|^2 + b(t)] \tag{14.16}$$

for some $a \ge 0$ and $b \in L_1(\Omega,\mathbb{R})$. Then the Hammerstein inclusion (14.2) *has a solution $x_* \in X$ if*

$$a\mu(K;Y) < 1. \tag{14.17}$$

P r o o f. We apply Theorem 13.4 to the operator $A = KN_F$ in X. To this end, suppose that $x \in \lambda KN_F x$ for some $\lambda \in (0,1]$ and $x \in X$, i.e. $x = \lambda Ky$ for some $y \in N_F x$. By (14.16) this implies that

$$\langle x, y\rangle \le a\langle x, x\rangle + \|b\|_{L_1}. \tag{14.18}$$

By the positivity condition (14.14) we have in turn

$$\langle x, x\rangle = \lambda^2\langle Ky, Ky\rangle \le \lambda^2\mu(K;Y)\langle y, Ky\rangle = \lambda\mu(K;Y)\langle y, x\rangle. \tag{14.19}$$

Combining (14.18) and (14.19) yields

$$\langle x, y\rangle \le \frac{\|b\|_{L_1}}{1 - a\mu(K;Y)} < \infty.$$

Now, the hypothesis (14.15) implies that

$$\delta(K) = \sup\{\|Ky\|_X^2 : y \in Y,\ \langle y, Ky\rangle \le 1\} < \infty. \tag{14.20}$$

We conclude that the a priori estimate (13.8) holds with

$$r = \left(\frac{\delta(K)\|b\|_{L_1}}{1 - a\mu(K;Y)} \right)^{1/2},$$

and the assertion follows from Theorem 13.4. ∎

We make some remarks on condition (14.14). In the Russian literature, this condition is usually attributed to M. A. Krasnosel'skiĭ [Kr]. However, essentially the same condition, as well as the condition (14.15), has been introduced 3 years before Krasnosel'skiĭ by P. Hess [He]. In [He] it is also proved that every angle-bounded operator in the sense of H. Amann [Am] is positive in the sense of (14.14). The first paper where these conditions are discussed in the setting of general ideal spaces seems to be [ZaNg3].

The question arises how to verify the hypotheses of Theorem 14.3. The condition (14.16) is usually guaranteed by imposing appropriate growth restrictions on the nonlinearity F. The conditions (14.14) and (14.15) hold, for example, if K maps Y into Y' and is normal (in particular, self-adjoint) and positive definite in $L_2(\Omega, \mathbb{R}^m)$. In this case one may put

$$\mu = \|K\|, \qquad \delta = \|K^{1/2}\|^2$$

in (14.14) and (14.15), respectively.

The verification of the inclusion (14.16) depends on the specific problem under consideration. Some examples will be considered in the following chapter.

Apart from the existence and uniqueness problem for Hammerstein inclusions, the *eigenvalue problem* for (14.2) has also found some attention in the literature. Some facts based on topological methods may be found in [PoSv], while more sophisticated results obtained by means of variational methods are given in [Cf].

15. A reduction method. In the preceding section we have applied only some rather elementary fixed point principles for multi-valued operators to the Hammerstein inclusion (14.2). The main contributions to the existence and uniqueness problem for (14.2) are due to Bulgakov, Lyapin, and Ragimkhanov [Bu1, Bu2, BuLy1, BuLy2, GaRg, Ly1, Ly2, Ly3, Ly4, Ly5, Ly6, Ly7, Rg1, Rg2]. In particular, in the papers [Ly3, Ly4, Ly5] the author discusses an interesting approach to reduce the study of (14.2) to a scalar integral equation. This approach is based on the notion of the so-called support function of a bounded set which is defined as follows. Given a bounded set $E \subset \mathbb{R}^k$ and a point $v \in S^{k-1}$ (i.e. $v \in \mathbb{R}^k$ and $|v| = 1$), let

$$E_v = \sup\{(z, v) : z \in E\},$$

where $(\cdot, \cdot)$ denotes the usual scalar product in $\mathbb{R}^k$. Thus, we may associate with each $E \in \mathrm{Bd}(\mathbb{R}^k)$ a map $v \mapsto E_v$ in $C(S^{k-1}, \mathbb{R})$ which is sometimes called the *support function* of E. Some basic properties of the support function are given in the following

Lemma 15.1 [Ly3]. *For $E, \widetilde{E} \in \mathrm{Bd}(\mathbb{R}^k)$, the following holds*:

(a) *the estimate*

$$|E_v - E_{\widetilde{v}}| \le |E|^* |v - \widetilde{v}| \qquad (v, \widetilde{v} \in S^{k-1}) \tag{15.1}$$

is true, where $|E|^*$ is defined by (8.10);

(b) *the equality*

$$h(E,\widetilde{E}) = \sup_{v\in S^{k-1}} |E_v - \widetilde{E}_v| \tag{15.2}$$

holds, *where* h *is the Hausdorff distance* (1.5);

(c) *the following relation holds*:

$$\overline{\mathrm{co}}\, E = \bigcap_{v\in S^{k-1}} \{z : z\in\mathbb{R}^k, (z,v)\le E_v\}. \tag{15.3}$$

Proof. (a) We have

$$|E_v - E_{\tilde v}| = |\sup_{z\in E}(z,v) - \sup_{z\in E}(z,\widetilde{v})| \le \sup_{z\in E}|(z,v-\widetilde{v})| \le \sup_{z\in E}|z|\,|v-\widetilde{v}|.$$

(b) We show that $E_v \le \widetilde{E}_v + \varepsilon$ if and only if $h^+(E,\widetilde{E}) \le \varepsilon$, where h^+ is the Hausdorff deviation (1.3); the assertion follows then by symmetry. Now, $h^+(E,\widetilde{E})\le\varepsilon$ means that for all $z\in E$ we may find a $\widetilde{z}\in\widetilde{E}$ such that $|z-\widetilde{z}|\le\varepsilon$, hence $(z,v)\le(\widetilde{z},v)+\varepsilon$ for any $v\in S^{k-1}$. Conversely, from $h^+(E,\widetilde{E})>\varepsilon$ it follows that there is a $z\in E$ such that $|z-\widetilde{z}|>\varepsilon$ for all $\widetilde{z}\in\widetilde{E}$, hence $E_v > \widetilde{E}_v+\varepsilon$.

(c) Denote the right-hand side of (15.3) by $\widetilde{E}$. Given $z\in\overline{\mathrm{co}}\,E$, we can find $z_\varepsilon = \lambda_1 z_1 + \ldots + \lambda_m z_m$ such that $z_1,\ldots,z_m\in E$, $\lambda_1+\ldots+\lambda_m = 1$, and $|z-z_\varepsilon|\le\varepsilon$. Thus, for any $v\in S^{k-1}$ we get

$$(z,v) = (z-z_\varepsilon,v) + (z_\varepsilon,v) \le |z-z_\varepsilon|\,|v| + \sum_{j=1}^m \lambda_j(z_j,v) \le \varepsilon + \sum_{j=1}^m \lambda_j E_v = \varepsilon + E_v,$$

hence $z\in\widetilde{E}$, since $\varepsilon>0$ is arbitrary.

Now, if $\overline{\mathrm{co}}\,E$ were a strict subset of $\widetilde{E}$, we could find an $\varepsilon>0$ such that $h^+(\widetilde{E},\overline{\mathrm{co}}\,E) > \varepsilon$. By what has been proved in (b), this implies that

$$\widetilde{E}_v > [\overline{\mathrm{co}}\,E]_v + \varepsilon = E_v + \varepsilon,$$

contradicting the fact that $\widetilde{E}_v \le E_v$. ∎

Lemma 15.1(a) shows that the map $v\mapsto E_v$ is Lipschitz continuous for any bounded set E. Moreover, (b) implies that the right-hand side of (15.2) may be regarded as natural metric on $\mathrm{Cp}(\mathbb{R}^k)$. Finally, it follows from (c) that one may recover a set $E\in\mathrm{CpCv}(\mathbb{R}^k)$ from knowing the values E_v for all (or a dense subset of all) $v\in S^{k-1}$.

Suppose now that $Y : \Omega\to\mathrm{CpCv}(\mathbb{R}^n)$ is some multifunction, and $v\in S^{n-1}$ is fixed. Putting

$$Y_v(t) = [Y(t)]_v \tag{15.4}$$

defines a (single-valued!) scalar function $Y_v : \Omega\to\mathbb{R}$. Moreover, given a multi-valued operator $A : C(\Omega,\mathbb{R}^m)\to\mathrm{ClCv}(C(\Omega,\mathbb{R}^n))$, for fixed $v\in S^{n-1}$ we may define a (single-valued) operator $A_v : C(\Omega,\mathbb{R}^m)\to C(\Omega,\mathbb{R})$ by putting

$$A_v x(t) = [Ax(t)]_v \qquad (x\in C(\Omega,\mathbb{R}^m)). \tag{15.5}$$

The following two lemmas show how the regularity properties of Y and A carry over to Y_v and A_v, respectively, and vice versa.

LEMMA 15.2 [Ly2, Ly3]. *The following holds*:

(a) *the multifunction Y is upper semicontinuous at $t_0 \in \Omega$ if and only if the scalar function Y_v is upper semicontinuous at t_0 for all $v \in S^{n-1}$*;

(b) *the multifunction Y is measurable if and only if the scalar function Y_v is measurable for all $v \in S^{n-1}$.*

Proof. (a) Since Y takes compact values, we know from Lemma 2.3(c) that for $\varepsilon > 0$ we can find $\delta > 0$ such that $h^+(Y(t), Y(t_0)) \le \varepsilon$ for $|t - t_0| \le \delta$. By Lemma 15.1(b), this is in turn equivalent to the fact that $[Y(t)]_v \le [Y(t_0)]_v + \varepsilon$ for $v \in S^{n-1}$ and $|t - t_0| \le \delta$, i.e. the upper semicontinuity of the function Y_v.

(b) The fact that the measurability of Y implies the measurability of Y_v for any $v \in S^{n-1}$ is obvious. Suppose that Y_v is measurable for any $v \in S^{n-1}$. Choose a countable dense subset $\{v_1, v_2, \ldots\}$ of S^{n-1}. For fixed $y \in \mathbb{R}^n$ we have then

$$\varrho(y, Y(t)) = \sup_{v \in S^{n-1}} [(y, v) - Y_v(t)] = \sup_{n \in \mathbb{N}} [(y, v) - Y_{v_n}(t)],$$

and hence the distance function $t \mapsto \varrho(y, Y(t))$ is measurable as supremum of countably many measurable functions. The assertion follows now from Lemma 3.1(e). ∎

LEMMA 15.3 [Ly2,Ly3]. *The following holds*:

(a) *the multi-valued operator A is upper semicontinuous at $x_0 \in C(\Omega, \mathbb{R}^m)$ if and only if A is locally bounded at x_0 and the operator A_v is upper semicontinuous at x_0 for all $v \in S^{n-1}$*;

(b) *the multi-valued operator A is compact if and only if the operator A_v is compact for all $v \in S^{n-1}$.*

Proof. (a) The fact that the upper semicontinuity of A implies the upper semicontinuity of A_v for any $v \in S^{n-1}$ is obvious. Suppose that A_v is upper semicontinuous at x_0 for each $v \in S^{n-1}$. This means that for all $\varepsilon > 0$ there exists a $\delta(v) > 0$ such that $A_v x(t) \le A_v x_0(t) + \varepsilon$ for $\|x - x_0\| \le \delta(v)$. If A is not upper semicontinuous at x_0 we may find sequences $(x_k)_k$ in $C(\Omega, \mathbb{R}^m)$, $(v_k)_k$ in S^{n-1}, and $(t_k)_k$ in Ω such that $x_k \to x_0$ and

$$A_{v_k} x_k(t_k) > A_{v_k} x_0(t_k) + \varepsilon_0 \tag{15.6}$$

for some $\varepsilon_0 > 0$. Without loss of generality, assume that $v_k \to v_0$ and choose $\delta(v_0) > 0$ such that

$$A_{v_0} x_k(t_k) \le A_{v_0} x_0(t_k) + \varepsilon_0/2 \tag{15.7}$$

for $\|x_k - x_0\| \le \delta(v_0)$. Since the operator A is locally bounded at x_0, by assumption, Lemma 15.1(a) implies that there exists some $\gamma > 0$ such that

$$|A_v x(t) - A_{\widetilde{v}} x(t)| \le \gamma |v - \widetilde{v}|$$

for $\|x - x_0\| \le \delta(v_0)$ and $t \in \Omega$. Consequently, if we choose $k_0 \in \mathbb{N}$ such that $\|v_k - v_0\| \le \varepsilon_0/(4\gamma)$ for $k \ge k_0$ we obtain

$$|A_{v_k} x(t_k) - A_{v_0} x(t_k)| \le \frac{\varepsilon_0}{4} \tag{15.8}$$

for $\|x - x_0\| \le \delta(v_0)$. Combining (15.7) and (15.8) we get

$$\begin{aligned} A_{v_k} x_k(t_k) - A_{v_k} x_0(t_k) &\le A_{v_k} x_k(t_k) - A_{v_0} x_k(t_k) + A_{v_0} x_k(t_k) \\ &\quad - A_{v_0} x_0(t_k) + A_{v_0} x_0(t_k) - A_{v_k} x_0(t_k) \\ &\le \varepsilon_0/4 + \varepsilon_0/2 + \varepsilon_0/4 = \varepsilon_0 \end{aligned}$$

contradicting (15.6). This proves (a).

(b) The fact that the compactness of A implies the compactness of A_v for any $v \in S^{n-1}$ is again obvious. Suppose that A_v is compact for each $v \in S^{n-1}$, and let $(x_k)_k$ be a bounded sequence in $C(\Omega, \mathbb{R}^m)$. Choose a countable dense subset $V \subset S^{n-1}$. By assumption, we may find subsequences $(x_{k_j})_j$ such that $(A_v x_{k_j})_j$ converges in $C(\Omega, \mathbb{R})$ for each $v \in V$, say $A_v x_{k_j} \to y_v$ $(j \to \infty)$. Putting

$$Y(t) = \bigcap_{v \in V} \{z : z \in \mathbb{R}^n, (z, v) \le y_v(t)\}$$

it follows that $h(Ax_{k_j}, Y) \to 0$ in $\mathrm{ClCv}(C(\Omega, \mathbb{R}^n))$ as $j \to \infty$. This shows that A is compact. ∎

Lemma 15.2(a) gives a comparison between the upper semicontinuity of the multifunction Y and the functions Y_v. We remark that a similar result holds also for the lower semicontinuity of Y and Y_v.

Passing from Y to the functions Y_v one often remains "in the same type of space". We illustrate this for the important class of *ideal spaces of Bochner type* $X \subset S(\Omega, \mathbb{R}^n)$ which may be represented as a direct sum $\widetilde{X} \oplus \ldots \oplus \widetilde{X}$ of n copies of the corresponding ideal space $\widetilde{X}$ of *scalar* functions, equipped with the norm

$$\|x\|_X = \|\, |x| \,\|. \tag{15.9}$$

For example, the Lebesgue space $X = L_p(\Omega, \mathbb{R}^n)$ is of Bochner type, as the definition (8.1) of its norm shows.

LEMMA 15.4. *Let $X \subset S(\Omega, \mathbb{R}^n)$ be an ideal space of Bochner type, and denote by $\widetilde{X} \subset S(\Omega, \mathbb{R})$ the corresponding ideal space of scalar functions. Let $Y : \Omega \to \mathrm{CpCv}(\mathbb{R}^n)$ be a multifunction. Then $\mathrm{Sel}_S Y \subseteq X$ if and only if $Y_v \in \widetilde{X}$ for any $v \in S^{n-1}$.*

Proof. Suppose first that $\mathrm{Sel}_S Y \subseteq X$ and fix $v \in S^{n-1}$. We claim that we can find a function $y \in \mathrm{Sel}_S Y$ such that $(y(t), v) = Y_v(t)$ for almost all $t \in \Omega$. In fact, we may choose y as a *maximal selection* of Y which may be characterized by the relation

$$|y(t)| = \max\{|\eta| : \eta \in Y(t)\}.$$

It follows that

$$|Y_v(t)| = |(y(t), v)| \le \sup_{\widetilde{v} \in S^{n-1}} |(y(t), \widetilde{v})| = |y(t)|,$$

and hence $Y_v \in \widetilde{X}$, by (15.9).

Conversely, suppose that $Y_v \in \widetilde{X}$ for all $v \in S^{n-1}$. Denote by $\{e_1, \ldots, e_n\}$ the canonical basis in $\mathbb{R}^n$ and put

$$\sigma_j(t) = \max\{Y_{e_j}(t), Y_{-e_j}(t)\} \qquad (j = 1, \ldots, n).$$

Then the function $\sigma = (\sigma_1, \ldots, \sigma_n)$ belongs to X, and $|Y(t)|^* \leq \sigma(t)$ for almost all $t \in \Omega$. Since X is an ideal space, we conclude that $\mathrm{Sel}_S\, Y \subseteq X$. ■

Let us now briefly sketch how to apply the reduction method described so far to the study of the integral inclusion (14.2). As indicated in (14.3), the action of the integral operator (14.1) on a multifunction $Y : \Omega \to \mathrm{ClCv}(\mathbb{R}^n)$ is meant "selection-wise", i.e.

$$\int_\Omega Y(t)\, dt = \Big\{ \int_\Omega y(t)\, dt : y \in \mathrm{Sel}_S\, Y \cap L_1(\Omega, \mathbb{R}^n)\Big\}. \tag{15.10}$$

This definition is due to Aumann [An] and Hukuhara [Hu] and therefore called the *Aumann–Hukuhara integral* in the literature. Another definition [Ly3] based on the support function considered above is

$$\int_\Omega Y(t)\, dt = \bigcap_{v \in S^{n-1}} \Big\{ z : z \in \mathbb{R}^n, (z, v) \leq \int_\Omega Y_v(t)\, dt \Big\}. \tag{15.11}$$

In view of the analogy with (15.3), let us call (15.11) the *Lyapin integral* of Y. One of the main results in [Ly3] is that the Aumann–Hukuhara integral and the Lyapin integral coincide for $Y : \Omega \to \mathrm{CpCv}(\mathbb{R}^n)$. Consequently, the action of the operator $A = KN_F$ on a function $x : \Omega \to \mathbb{R}^m$ may be described by the formula

$$Ax(t) = \bigcap_{v \in S^{n-1}} \Big\{ z : z \in \mathbb{R}^n, (z, v) \leq \int_\Omega [k(t,s) N_F x(s)]_v\, ds \Big\}.$$

This may be applied to get existence results for the Hammerstein inclusion (14.2) in a very elegant way; for details see [Ly4, Ly5].

6. Applications

In this chapter we shall apply the existence results obtained in the previous chapter to selected problems in mathematics, mechanics, and physics. More precisely, in the first two sections we discuss applications to elliptic systems with multi-valued right-hand side, forced periodic oscillations in nonlinear control systems with noise, and critical points for nonsmooth energy functionals. In the final section we briefly describe a mathematical model for the problem of heat regulation by thermostats.

16. Applications to elliptic systems. There exist various motivations for studying inclusions of type (14.2); let us mention some of them.

First of all, when investigating boundary value problems in physics, mechanics, or control theory which define the state x of a system by an acting force h, one is led to equations of the form

$$Lx = h, \tag{16.1}$$

where L is a linear operator on an appropriate function space. Now, if the force h is perturbed, i.e. is subject to both the state x and an "undetermined noise", (16.1) has to be replaced by the equation with multi-valued right-hand side

$$Lx \in Nx, \tag{16.2}$$

where N is some multi-valued nonlinear operator (for example, the superposition operator (7.2)). In many cases L is some differential operator which admits a Green's function on a space determined by suitable boundary conditions. In this case the problem (16.2) may be written in the form (14.2) by putting $K = L^{-1}$.

The second motivation is related to "nonsmooth" calculus of variations (see e.g. the monograph [Ck]). Suppose that we are interested in minimizing the energy functional

$$\Psi x = \int_{\Omega} \{h(x(s)) - f(s, x(s))\}\, ds, \tag{16.3}$$

where h denotes the kinetic energy of the system, and f is a potential energy generating a (single-valued) superposition operator (14.4). Assume further that the functional (16.3) is not differentiable in the usual sense, due to some lack of regularity of the operator (14.4), but admits a generalized gradient or subgradient in the sense, for instance, of Clarke's generalized subgradient, Aubin's contingent derivative, Ioffe's fan, etc. (see e.g. [Au, AuCl, AuEk, Ck, Dm, Io4]). Consequently, the problem of minimizing (16.3) leads to the study of boundary value problems for the "Euler–Lagrange inclusion"

$$Lx \in \partial N_f x, \tag{16.4}$$

where ∂N_f is one of the generalized gradients or subgradients mentioned above. The problem (15.4) in turn is in various function spaces equivalent to the Hammerstein inclusion (14.2).

Finally, we mention another typical situation where the inclusion (14.2) arises quite naturally. Suppose that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a (single-valued) function which, however, is so "badly behaved" that one cannot apply the usual solvability criteria to the Hammerstein equation (14.5). In this case one may "improve" the problem by passing from the (single-valued) function f to the convexification $F = f^{\square}$ (see (11.2)). Thus, putting $F(t, u) = f^{\square}(t, u)$ one arrives again at the Hammerstein inclusion (14.2). Moreover, it is then possible to apply the fixed point principles of Section 13 to (14.2), since the operator $N_F = N_f^{\square}$ has nicer properties than the operator N_f (for example, N_F is always closed and "often" upper semicontinuous).

Now we start discussing some specific applications of the Hammerstein inclusion (14.2). Apart from the examples treated below, other applications may be found in [Dc, Pn, Pa1, Pa2, Pa3, Pa4, Pa5, Pa6, Te].

Let Ω be a bounded domain in $\mathbb{R}^m$ ($m \geq 2$) with smooth boundary $\partial\Omega$, $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^m)$ a Carathéodory multifunction, and L a uniformly elliptic linear differential operator of order $2k$ in divergence form, i.e.

$$Lx(s) = \sum_{|\alpha|,|\beta| \leq k} D^{\alpha}(a_{\alpha\beta}(s) D^{\beta}) x(s) \qquad (s \in \Omega) \tag{16.5}$$

with matrix-valued coefficients $a_{\alpha\beta} : \Omega \to \mathbb{R}^{m\times m}$. Consider the system

$$Lx(s) \in F(s, x(s)) \qquad (s \in \Omega), \tag{16.6}$$

subject to the Dirichlet boundary condition

$$D^{\gamma} x(s) = 0 \quad (s \in \partial\Omega,\ |\gamma| \leq k). \tag{16.7}$$

It is well known that the linear problem

$$Lx(s) = y(s) \qquad (s \in \Omega)$$

with boundary condition (16.7) has a unique generalized solution $x = Ky$, where the (integral) operator K maps the Sobolev space $H^{-k} = H^{-k}(\Omega, \mathbb{R}^m)$ into the Sobolev space $H_0^k = H_0^k(\Omega, \mathbb{R}^m)$ and is bounded. Sufficient conditions for the existence and boundedness of the operator K may be found in a vast literature on linear elliptic operators (see e.g. [LdUr]). For our purpose, the classical *Gårding inequality*

$$\langle Lx, x\rangle \geq \alpha \|x\|_{H_0^k}^2 \qquad (x \in H_0^k) \tag{16.8}$$

is sufficient. Define an ideal space Z by

$$Z = \begin{cases} L_{2m/(m-2k)} & \text{if } m > 2k, \\ L_\phi & \text{if } m = 2k, \\ L_\infty & \text{if } m < 2k; \end{cases} \tag{16.9}$$

here L_ϕ is the Orlicz space generated by the Young function

$$\phi(u) = e^{|u|^2} - 1 \qquad (u \in \mathbb{R}^m).$$

By classical imbedding theorems of Sobolev, Pokhozhaev and Trudinger (see e.g. [Ad, GiTr]), the operator K acts then also between the ideal spaces $Y = Z'$ and $Y' = Z$. Moreover, if $X \supseteq Z$ is any ideal space with the property that the unit ball of Z is an absolutely bounded subset of X (for example, $X = L_p$ with $1 \leq p < 2m/(m-2k)$ for $m > 2k$ and $1 \leq p < \infty$ for $m \leq 2k$), K is compact and self-adjoint as an operator from X' into X. From the continuity of the imbeddings $H_0^k \subseteq L_2$ and $H_0^k \subseteq Z \subseteq X$ it follows that

$$\|x\|_{H_0^k} \geq c \max\{\|x\|_{L_2}, \|x\|_X\} \qquad (x \in H_0^k)$$

for some constant $c > 0$. Combining this with Gårding's inequality (16.8) we get

$$\langle Lx, x\rangle \geq \alpha c^2 \max\{\|x\|_{L_2}^2, \|x\|_X^2\} \qquad (x \in H_0^k)$$

which shows that the operator K is positive in the sense of (14.14) and also satisfies (14.15). The inclusion (14.16) leads here to the condition

$$\sup\Big\{\sum_{j=1}^m u_j v_j : v_j \in F_j(s, u_1, \ldots, u_m)\Big\} \leq a \sum_{j=1}^m u_j^2 + b(s) \qquad (b \in L_1(\Omega, \mathbb{R})).$$

If this is satisfied, we may apply Theorem 14.3 and get an existence result for the elliptic system (16.6) with boundary condition (16.7).

We show now how variational problems for nonsmooth energy functionals may also lead to inclusions of the form (16.6). Let $L : H_0^k \to H^{-k}$ be again a uniformly elliptic operator (16.5) which satisfies Gårding's inequality (16.8). Suppose that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a (single-valued) Carathéodory function such that $f(s, \cdot)$ is locally Lipschitz for almost all $s \in \Omega$. Finally, we assume that the corresponding superposition operator is bounded from the ideal space Z defined in (16.9) into $L(\Omega, \mathbb{R})$. Under these assumptions, the *energy functional*

$$\Psi x = \frac{1}{2}\langle Lx, x\rangle - \int_\Omega f(s, x(s))\, ds \tag{16.10}$$

is correctly defined on the space $H_0^k = H_0^k(\Omega, \mathbb{R}^m)$. Moreover, the functional

$$\Gamma x = \int_\Omega g(s, x(s))\, ds \tag{16.11}$$

is locally Lipschitz both from Z into $\mathbb{R}$ and from H_0^k into $\mathbb{R}$. By [Ck, Theorem 2.1.2], the generalized gradient $\partial\Gamma x$ of the functional (16.11) acts both from Z into $\mathrm{Cv}(Z')$ and from H_0^k into $\mathrm{Cv}(H^{-k})$, and hence the same is true for the generalized gradient

$$\partial\Psi x = Lx - \partial\Gamma x \tag{16.12}$$

of the functional (16.10). A *critical point* of (16.10) is, by definition, any element $x \in H_0^k$ such that $0 \in \partial\Psi x$; by (16.12), this may be written equivalently as

$$Lx \in \partial\Gamma x. \tag{16.13}$$

For reducing (16.13) to the form (16.16), we have to find a multi-valued Carathéodory function $F : \Omega \times \mathbb{R}^m \to \mathrm{CpCv}(\mathbb{R}^m)$ such that $\partial\Gamma x = N_F x$. This problem was solved in [AuCk] (see also [Ck, Theorem 2.7.3 and Theorem 2.7.5]). In fact, if we put $F(s, u) = \partial_u f(s, u)$ (the generalized gradient of the function $f(s, \cdot)$) we always have $\partial\Gamma x \subseteq N_F x$, and equality holds if $f(s, \cdot)$ is "regular" in the sense of [Ck]; in particular, $\partial\Gamma x = N_F x$ if the function $f(s, \cdot)$ is convex for almost all $s \in \Omega$.

Combining the previous assumptions and the assumptions made in Section 14, we get an existence result for critical points of the energy functional (16.10) by means of Theorem 14.3. The crucial condition (14.16) in Theorem 14.3 is here nothing else but the *coerciveness* of the functional Ψ (see again [Ck]).

We remark that other existence results for critical points of (16.10) have been obtained by means of "nonsmooth variants" of classical minimax principles (e.g., mountain pass lemmas) in [Ch1,Ch2], and of classical dual variational principles in [AmBd] and [StTo].

17. Applications to nonlinear oscillations. In this section we discuss an application to the theory of forced periodic oscillations. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathrm{CpCv}(\mathbb{R}^n)$ be a Carathéodory multifunction which is 2π-periodic in the first argument. For $j = 1, 2, \ldots, n$ consider the polynomials

$$L_j(\xi) = \xi^{p_j} + a^j_{p_j-1}\xi^{p_j-1} + \ldots + a^j_2\xi^2 + a^j_1\xi + a^j_0, \tag{17.1}$$

$$M_j(\xi) = \xi^{q_j} + b^j_{q_j-1}\xi^{q_j-1} + \ldots + b^j_2\xi^2 + b^j_1\xi + b^j_0, \tag{17.2}$$

where $\deg L_j = p_j > q_j = \deg M_j$. We are interested in finding 2π-periodic solutions (so-called forced periodic oscillations) in the nonlinear control system with multi-valued right-hand side (so-called nonlinearity with white noise) described by

$$\begin{gathered} L_1(d/dt)x_1(t) \in M_1(d/dt)F_1(t, x_1(t), \ldots, x_n(t)), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_n(d/dt)x_n(t) \in M_n(d/dt)F_n(t, x_1(t), \ldots, x_n(t)). \end{gathered} \tag{17.3}$$

Let

$$\alpha(L_j, M_j) = \inf_k \frac{\mathrm{Re}[L_j(-ik)M_j(ik)]}{|M_j(ik)|^2},$$

where the infimum is taken over all indices k such that $M_j(ik) \neq 0$; moreover, fix numbers $\alpha_j \in (-\infty, \alpha(L_j, M_j))$. It is then known [KrLiSo] that the problem (17.3) may be transformed into the system of Hammerstein inclusions

$$\begin{aligned} & x_1(t) \in \int_0^{2\pi} h_1(\alpha_1; t-\tau)\widetilde{F}_1[\alpha_1; \tau, x_1(\tau), \dots, x_n(\tau)]\, d\tau, \\ (17.4) \qquad & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & x_n(t) \in \int_0^{2\pi} h_n(\alpha_n; t-\tau)\widetilde{F}_n[\alpha_n; \tau, x_1(\tau), \dots, x_n(\tau)]\, d\tau, \end{aligned}$$

where

$$(17.5) \qquad \widetilde{F}_j\,(\alpha_j; t, v_1, \dots, v_n) = F_j(t, v_1, \dots, v_n) - \alpha_j v_j,$$

and $h_j(\alpha_j; \cdot)$ is the so-called *impulse-frequency characteristic* of the nonlinear link f_j with respect to the *transfer function*

$$(17.6) \qquad W_j(\alpha_j; \xi) = \frac{M_j(\xi)}{L_j(\xi) - \alpha_j M_j(\xi)} \qquad (j = 1, \dots, n).$$

We remark that the systems (17.3) and (17.5) are not quite equivalent, since the components x_j $(j = 1, \dots, n)$ of (17.3) in general belong to the space C^{m_j} with $m_j = p_j - q_j$, but those of (17.5) do not. To apply the results of Section 14 we consider only solutions of the integral inclusions (17.5) and regard them as "generalized solutions" of (17.3).

Systems of the form (17.3) arise in nonlinear control problems. For example, in case of a control system with a *simple circuit* governed by one (single-valued) nonlinear link f and some set U of admissible controls u (see Figure 2), the system (17.3) turns into a

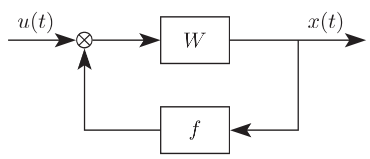

Fig. 2

single integral inclusion:

$$(17.7) \qquad L(d/dt)x(t) \in M(d/dt)F(t, x(t)),$$

where

$$F(t, x(t)) = \{f(x(t)) + u(t) : u \in U\}.$$

More generally, a control system with one circuit governed by several nonlinear links $f_1, \dots, f_n$ and several sets $U, U_1, \dots, U_{n-1}$ of admissible controls $u, u_1, \dots, u_{n-1}$, respectively (see Figure 3), is described by (17.3), where

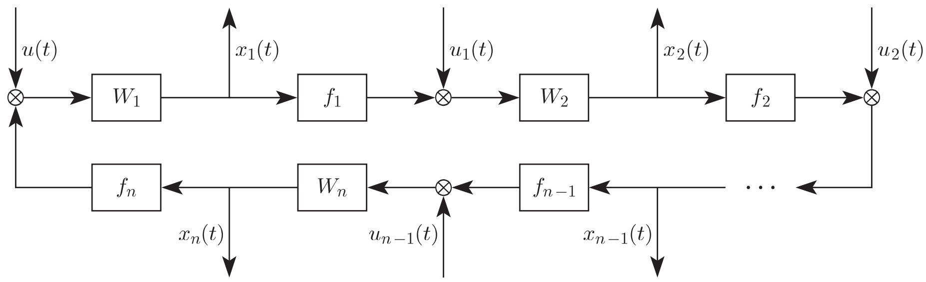

Fig. 3

$$F_1(t, x_1(t), \ldots, x_n(t)) = \{f_n(x_n(t)) + u(t) : u \in U\},$$
$$F_2(t, x_1(t), \ldots, x_n(t)) = \{f_1(x_1(t)) + u_1(t) : u_1 \in U_1\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_n(t, x_1(t), \ldots, x_n(t)) = \{f_{n-1}(x_{n-1}(t)) + u_{n-1}(t) : u_{n-1} \in U_{n-1}\}.$$

As a third example, consider a multiple circuit with *crossed feedback* and a single control set as shown in Figure 4; here the multifunctions in (17.3) have the form

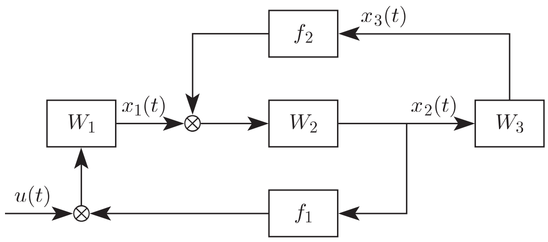

Fig. 4

$$F_1(t, x_1(t), x_2(t), x_3(t)) = \{f_1(x_2(t)) + u(t) : u \in U\},$$
$$F_2(t, x_1(t), x_2(t), x_3(t)) = \{x_1(t) + f_2(x_3(t))\},$$
$$F_3(t, x_1(t), x_2(t), x_3(t)) = \{x_2(t)\}.$$

Now consider the linear integral operator

$$K_j x(t) = \int_0^{2\pi} h_j(\alpha_j; t-\tau) x(\tau)\, d\tau$$

generated by the impulse-frequency characteristic $h_j(\alpha_j; \cdot)$. As in [KrLiSo, Theorem 26.1] one may show that this operator is compact from $Y = L_p([0, 2\pi], \mathbb{R}^n)$ into $Y' = L_{p/(p-1)}([0, 2\pi], \mathbb{R}^n)$ ($1 \le p \le 2$) and satisfies (14.14). Finally, it is easy to see that (14.15) holds for $X = Y'$. The inclusion (14.16) leads here to the condition

$$\sup\Big\{ \sum_{j=1}^n v_j w_j : w_j \in F_j(t, v_1, \ldots, v_n) \Big\} \le \sum_{j=1}^n a_j v_j^2 + b(t) \tag{17.8}$$

with $a_j < \alpha(L_j, M_j)$ and $b \in L_1([0, 2\pi], \mathbb{R})$. If this is satisfied, we may apply Theorem 14.3 and get an existence result for forced 2π-periodic oscillations in the control system

(17.3). For example, in the simple circuit shown in Figure 2 the growth condition (17.8) reads

$$\sup_{u\in U} |x(t)||f(x(t)) + u(t)| \le a|x(t)|^2 + b(t),$$

where U is the set of admissible controls u, x is the input function, and a is any positive real number satisfying

$$a < \alpha(L, M) = \inf_k \frac{\mathrm{Re}[L(-ik)M(ik)]}{|M(ik)|^2}.$$

A detailed discussion of the control problem (17.3) for single-valued nonlinearities may be found in the recent book [BoBuKo].

18. Applications to relay problems. One of the most important motivations for studying multi-valued superpositions comes from the mathematical modelling of *relay* and *hysteresis phenomena*. Let us sketch how such problems lead to multifunctions and multi-valued operators, and how one can employ natural properties of these operators to obtain existence results. We restrict ourselves to the simplest relay regulation problems; the interested reader may find more material in the book [KrPk].

Let $-\infty < \varrho_1 < \varrho_0 < \infty$, and denote by Σ the union of the two rays $(-\infty, \varrho_1) \times \{0\}$ and $(\varrho_0, \infty) \times \{1\}$ in the Euclidean plane $\mathbb{R}^2$. Physically, Σ describes the set of all possible *states* of an *ideal relay* which is assumed to switch off above the upper threshold value ϱ_0, and to switch on below the lower threshold value ϱ_1. Given an initial state $(x_0, y_0) \in \Sigma$, a *feasible input* is given by a continuous function $x : [0, T] \to \mathbb{R}$ such that $x(0) = x_0$, while an *admissible output* is a function $y : [0, T] \to \mathbb{R}$ defined by

$$y(t) = \begin{cases} 0 & \text{if } \begin{cases} x(t) \le \varrho_0 \text{ or} \\ \varrho_1 < x(t) < \varrho_0 \text{ and } x(\tau) = \varrho_0 \text{ for some } \tau \in [0, t), \end{cases} \\ y_0 & \text{if } \varrho_1 < x(\tau) < \varrho_0 \text{ for all } \tau \in [0, t], \\ 1 & \text{if } \begin{cases} x(t) \ge \varrho_1 \text{ or} \\ \varrho_1 < x(t) < \varrho_0 \text{ and } x(\tau) = \varrho_1 \text{ for some } \tau \in [0, t). \end{cases} \end{cases} \tag{18.1}$$

Although the mathematical description of the output y by formula (18.1) is rather clumsy, its physical meaning is clear: the output y is constant on each time interval $[t_1, t_2]$, where either $y(t_1) = 0$ and $x(t) < \varrho_0$ for $t_1 \le t \le t_2$, or $y(t_2) = 1$ and $x(t) > \varrho_1$ for $t_1 \le t \le t_2$. Intuitively speaking, this means that "no superfluous switches occur in the relay".

By means of formula (18.1) we may define an operator $R = R[y_0; \varrho_0, \varrho_1]$ putting

$$y(t) = R[y_0; \varrho_0, \varrho_1]x(t) \qquad (0 \le t \le T). \tag{18.2}$$

This so-called *relay operator* R has some interesting analytical properties. For example, R is always locally compact between the space $C = C([0, T], \mathbb{R})$ of all continuous inputs and the space $L_q = L_q([0, T], \mathbb{R})$ $(1 \le q < \infty)$ of all q-integrable outputs [KrPk, Theorem 28.1]. Consequently, one may pass to the strong closure (see Section 10) or the convexification (see Section 11) of the operator R which will be automatically upper semicontinuous, by Lemma 2.9. It turns out that the strong closure and the convexification of the relay operator (18.2) may be described explicitly. We state this in the following two theorems; the proofs are straightforward and follow directly from the corresponding definitions.

THEOREM 18.1 [KrPk]. *The set of initial states of the* (*strong*) *closure* $\overline{R} = \overline{R}[y_0; \varrho_0, \varrho_1]$ *of the relay operator* (18.2) *is given by*

$$\overline{\Sigma} = \Sigma \cup \{(\varrho_1, 0), (\varrho_0, 1)\}.$$

Given an initial state $(x_0, y_0) \in \overline{\Sigma}$ *and a feasible input* x, *an output* y *belongs to* $\overline{R}[y_0; \varrho_0, \varrho_1]x$ *if*

$$y(0) = R[y_0; \varrho_0, \varrho_1]x_0, \tag{18.3}$$

$(x(t), y(t)) \in \overline{\Sigma}$ *for* $0 \leq t \leq T$, *and the following two conditions are satisfied*:

(a) y *is increasing on each interval where* $x(t) > \varrho_1$;

(b) y *is decreasing on each interval where* $x(t) < \varrho_0$.

THEOREM 18.2 [KrPk]. *The set of initial states of the convexification* $R^{\square} = R^{\square}[y_0; \varrho_0, \varrho_1]$ *of the relay operator* (18.2) *is given by*

$$\Sigma^{\square} = \Sigma \cup ([\varrho_1, \varrho_0] \times [0, 1]).$$

Given an initial state $(x_0, y_0) \in \Sigma^{\square}$ *and a feasible input* x, *an output* y *belongs to* $R^{\square}[y_0; \varrho_0, \varrho_1]x$ *if* (18.3) *holds*, $(x(t), y(t)) \in \Sigma^{\square}$ *for* $0 \leq t \leq T$, *and the conditions* (a) *and* (b) *in Theorem* 18.1 *are satisfied.*

To illustrate these conditions, let us consider the action of the closure and convexification of the relay operator $R = R[0; 1, -1]$ on the continuous input $x(t) = \sin t$ $(0 \leq t \leq \pi)$. Here we have

$$\overline{\Sigma} = \{(x_0, 0) : x_0 \leq 1\} \cup \{(x_0, 1) : x_0 \geq -1\},$$
$$\Sigma^{\square} = \overline{\Sigma} \cup \{(x_0, y_0) : -1 \leq x_0 \leq 1, 0 \leq y_0 \leq 1\}.$$

The output set $\overline{R}[0; 1, -1]x$ contains the three functions

$$y_0(t) \equiv 0, \qquad y_1(t) = \chi_{(\pi/2,\pi]}(t), \qquad \widehat{y}_1(t) = \chi_{[\pi/2,\pi]}(t).$$

On the other hand, the output set $R^{\square}[0; 1, -1]x$ contains the infinitely many functions

$$y_0(t) \equiv 0, \qquad y_c(t) = c\chi_{(\pi/2,\pi]}(t), \qquad \widehat{y}_c(t) = c\chi_{[\pi/2,\pi]}(t),$$

where c is any constant between 0 and 1. Of course, the functions y_c and $\widehat{y}_c$ cannot be distinguished as elements of the space L_q.

To conclude, let us describe a "real-life" application of relay nonlinearities, namely a heat conduction problem arising in the mathematical modelling of *temperature regulation by thermostats* [GlSk1, GlSk2]. The thermostat is assumed to switch off above a threshold value ϱ_0 and to switch on below a smaller threshold value ϱ_1. "Convexifying" the resulting discontinuous behaviour of the temperature $u = x(t)$ and its time derivative $v = \dot{x}(t)$ leads to the nonlinearity $F : \mathbb{R} \times \mathbb{R} \to \mathrm{CpCv}(\mathbb{R})$ given by

$$F(u,v)=\begin{cases}\{1\} & \text{if } \begin{cases} u<\varrho_1 \text{ or} \\ \varrho_1\le u<\varrho_0 \text{ and } v>0,\end{cases}\\ [0,1] & \text{if } \begin{cases} u=\varrho_1 \text{ and } v\le 0 \text{ or} \\ \varrho_1<u<\varrho_0 \text{ and } v=0 \text{ or}\\ u=\varrho_0 \text{ and } v\ge 0,\end{cases}\\ \{0\} & \text{if } \begin{cases} u>\varrho_1 \text{ or} \\ \varrho_1<u\le\varrho_0 \text{ and } v<0.\end{cases}\end{cases} \tag{18.4}$$

We briefly describe how to put the thermostat problem into the form (14.2) with a suitable integral operator (14.1); the details may be found in [GlSk1, GlSk2].

Consider the superposition operator

$$N_F x(t)=\mathrm{Sel}_S\, F(x(t),\dot{x}(t)) \tag{18.5}$$

defined by the multifunction (18.4). This operator is upper semicontinuous from the space $C^1=C^1([0,T],\mathbb{R})$, equipped with the norm

$$\|x\|_{C^1}=\max_{0\le t\le T}|x(t)|+\max_{0\le t\le T}|\dot{x}(t)|, \tag{18.6}$$

into $\mathrm{ClCv}(L_\infty([0,T],\mathbb{R}))$. For $v\in L_\infty=L_\infty([0,T],\mathbb{R})$, let $u=Hv$ be the unique solution of the initial value problem

$$\beta\dot{u}(t)+u(t)=v(t)\quad (0\le t\le T),\qquad u(0)=0 \tag{18.7}$$

($\beta>0$). The operator H is given explicitly by the formula

$$Hv(t)=\frac{e^{-t/\beta}}{\beta}\int_0^t e^{-s/\beta}v(s)\,ds \tag{18.8}$$

and maps the space L_∞ into the Sobolev space $W^1_\infty=W^1_\infty([0,T],\mathbb{R})$. Next consider the linear operator G which assigns to every $u\in W^1_\infty$ the "trace" at $\xi=0$ of the solution y of the initial boundary value problem

$$\begin{aligned} &\frac{\partial y}{\partial t}(\xi,t)=\frac{\partial^2 y}{\partial \xi^2}(\xi,t) && (0<\xi<1,0<t\le T),\\ &\alpha\frac{\partial y}{\partial \xi}(1,t)+y(1,t)=u(t) && (0<t\le T)\\ &\frac{\partial y}{\partial \xi}(0,t)=0 && (0<t\le T),\\ &y(\xi,0)=0 && (0\le\xi\le 1)\end{aligned} \tag{18.9}$$

($\alpha>0$), i.e. let $Gu=y(0,\cdot)$. As is well known, the operator G is given explicitly by

$$Gu(t)=\sum_{k=1}^{\infty}a_k\mu_k^2\int_0^t e^{-\mu_k^2(t-s)}u(s)\,ds, \tag{18.10}$$

where $(\mu_k)_k$ is the sequence of eigenvalues of the Sturm–Liouville problem

$$-w''(\xi)=\mu^2 w(\xi)\quad (0<\xi<1),\qquad w'(0)=w(1)+\alpha w'(1)=0, \tag{18.11}$$

and $(a_k)_k$ is a real sequence defined through the corresponding eigenfunctions $(w_k)_k$. By classical regularity theory, the operator G may be considered either from L_∞ into C or from W^1_∞ into C^1.

Putting now $K = HG$, the solutions $x \in C^1$ of the thermostat problem are precisely the fixed points of the Hammerstein inclusion (14.2) (see [GlSk2]). To apply Theorem 13.3, we put $X = C^1$ with the norm (18.6) and $B = \overline{B}_r(0)$, where $r = 1 + 2/\beta$. From the definition (18.4) of the multifunction F it is clear that $\|v\|_{L_\infty} \le 1$ for any $v \in N_F(B)$. Now, from (18.8) it follows that $u = Hv$ satisfies

$$|u(t)| \le 1 - e^{-t/\beta} \le 1 \qquad (0 \le t \le T),$$
$$|\dot{u}(t)| = \frac{|v(t) - u(t)|}{\beta} \le \frac{2}{\beta} \qquad (0 \le t \le T),$$

hence $\|u\|_{W^1_\infty} \le r$. By the maximum principle, this implies that $\|Gu\|_{C^1} = \|y(0,\cdot)\|_{C^1} \le r$ as well, where y solves (18.9). This shows that the operator $A = KN_F = GHN_F$ leaves the ball $\overline{B}_r(0) \subset X$ invariant.

The upper semicontinuity and compactness of A follow from the upper semicontinuity of N_F and the compactness of $K = GH$. Thus, we may apply Theorem 13.3 and get an existence result for the thermostat problem.

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