POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CLXXX

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Foundations of Vietoris homology theory with applications to non-compact spaces

WARSZAWA 1980
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-01116-5 ISSN 0012-3862

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Preface

An underlying structure is designed for Vietoris homology theory in such a way that the theory applies to metric spaces which are not necessarily compact. The organizational scheme of the work parallels that of an outline given by Borsuk for the compact case. The extended theory is employed to reformulate and prove some of the classical theorems of Vietoris homology theory under hypotheses weaker than the compactness originally required.

Abstract n-dimensional *simple chains* in a set S over an abelian group D are defined as linear combinations, with coefficients in D, of abstract n-dimensional simplexes having vertices in S. A boundary homomorphism ∂ is defined on the resulting chain groups $C_n^{\#}(S)$ giving rise to groups of cycles $Z_n^{\#}(S)$ and boundaries $B_n^{\#}(S)$. In case S is the point set of a metric space X, an *ε-chain* ($\varepsilon > 0$) is a simple chain each of whose simplexes is an ε-simplex, that is, a simplex whose vertex set has diameter less than ε. The ε-chains form a group $C_n^{\varepsilon}(X)$ in each dimension and $\partial\big(C_n^{\varepsilon}(X)\big) \subseteq C_{n-1}^{\varepsilon}(X)$ so there are groups $Z_n^{\varepsilon}(X)$ of ε-cycles and $B_n^{\varepsilon}(X)$ of ε-boundaries. An equivalence relation of *η-homology* ($\eta > 0$) on $Z_n^{\varepsilon}(X)$ is defined by the condition $\gamma \underset{\eta}{\sim} \gamma'$ in X if $\gamma-\gamma' \in B_n^{\eta}(X)$. Abstract n-dimensional *sequential chains* in a set S are sequences $\underline{\gamma} = \langle \gamma_i \rangle$ of elements of $C_n^{\#}(S)$. An addition and a boundary operator ∂, compatible with those for simple chains, are defined for the sequential chains so that these form an abelian group $C_n^{*}(S)$ in each dimension and ∂ is a homomorphism from $C_n^{*}(S)$ to $C_{n-1}^{*}(S)$. Groups of sequential cycles $Z_n^{*}(S)$ and boundaries $B_n^{*}(S)$ are defined in the natural way. A sequential chain $\underline{\gamma} = \langle \gamma_i \rangle$ in a metric space X is an *infinite chain* if there is a compact set X_0, called a *carrier* of $\underline{\gamma}$, and a sequence $\langle \varepsilon_i \rangle$ of positive numbers converging to zero, such that each $\gamma_i \in C_n^{\varepsilon_i}(X_0)$. The infinite chains form a subgroup $C_n^{\infty}(X)$ of $C_n^{*}(X)$ and $\partial\big(C_n^{\infty}(X)\big) \subseteq C_{n-1}^{\infty}(X)$ so that there are groups of infinite cycles $Z_n^{\infty}(X)$ and boundaries $B_n^{\infty}(X)$, with $B_n^{\infty}(X) \subseteq Z_n^{\infty}(X)$. The quotient group $Z_n^{\infty}(X)/B_n^{\infty}(X)$ is the *general homology group* $H_n^{\infty}(X)$, and two cycles $\underline{\gamma}, \underline{\gamma}' \in Z_n^{\infty}(X)$ are *homologous*, $\underline{\gamma} \sim \underline{\gamma}'$ in X, if $\underline{\gamma}-\underline{\gamma}' \in B_n^{\infty}(X)$. An infinite cycle is *essential* if it has a carrier in which it is not homologous to zero. An infinite cycle $\underline{\gamma} = \langle \gamma_i \rangle$ is a *true cycle* if the infinite cycle $\underline{\gamma}' = \langle \gamma_{i+1} - \gamma_i \rangle$ is homologous to zero. The true cycles form a subgroup $Z_n^{t}(X)$ of $Z_n^{\infty}(X)$ and it turns out that $B_n^{\infty}(X) \subseteq Z_n^{t}(X)$. The quotient group $Z_n^{t}(X)/B_n^{\infty}(X)$ is the *Vietoris homology group* $H_n^{t}(X)$. Theorems on subsequences of infinite chains are proved including the result that each subsequence of a true cycle is a true cycle. If $\underline{\gamma}, \underline{\gamma}' \in Z_n^{\infty}(X)$, then $\gamma \sim \gamma'$ in X provided there is a compact set $X_0 \subseteq X$ such that for each $\eta > 0$, $\gamma_i \underset{\eta}{\sim} \gamma_i'$ in X_0 holds eventually. If S and T are sets each function from S to T induces a homomorphism from $C_n^{\#}(S)$ to $C_n^{\#}(T)$ which in turn induces a homomorphism from $C_n^{*}(S)$ to $C_n^{*}(T)$, and each of these homomorphisms preserves cycles and boundaries. If X and Y are metric spaces and f is a function (not necessarily continuous) from X to Y such that for some $\varepsilon, \eta > 0$, $\operatorname{dist}(a, b) < \varepsilon$ implies $\operatorname{dist}(f(a), f(b)) < \eta$, then f induces a homomorphism from $C_n^{\varepsilon}(X)$ to $C_n^{\eta}(Y)$. If f is a mapping from X to Y, then f induces a homomorphism f from $C_n^{\infty}(X)$ to $C_n^{\infty}(Y)$. This last homomorphism carries $Z_n^{\infty}(X)$ to $Z_n^{\infty}(Y)$, $B_n^{\infty}(X)$ to $B_n^{\infty}(Y)$, and $Z_n^{\prime}(X)$ to $Z_n^{\prime}(Y)$, and induces homomorphisms from $H_n^{\infty}(X)$ to $H_n^{\infty}(Y)$ and from $H_n^{\prime}(X)$ to $H_n^{\prime}(Y)$. Topological invariance of the groups $H_n^{\infty}(X)$ and $H_n^{\prime}(X)$ is proved. Examples are given to show that $H_n^{\infty}(X)$ is not in general isomorphic to $H_n^{\prime}(X)$. A generalization of a lemma due to Borsuk is used to prove the Homotopy Theorem which states that if f and g are mappings of X to Y, $\underline{\gamma} \in Z_n^{\infty}(X)$, and $f|X_0$ is homotopic to $g|X_0$ in Y for some carrier X_0 of $\underline{\gamma}$, then $f(\underline{\gamma}) \sim g(\underline{\gamma})$ in Y. A sequence of functions $\underline{f} = \langle f_i \rangle$ from X to X is called a *null translation* provided there is a sequence of positive numbers $\langle \eta_i \rangle$ converging to zero such that for each $a \in X$, $\operatorname{dist}(a, f_i(a)) < \eta_i$. If $\underline{\varkappa} \in C_n^{\infty}(X)$, then $\underline{f}(\underline{\varkappa}) = \langle f_i(\varkappa_i) \rangle \in C_n^{\infty}(X)$, and this correspondence is a homomorphism of $C_n^{\infty}(X)$ which preserves cycles and boundaries. Basic properties of null translations are established and it is proved that if $\underline{f}$ is a null translation of X and $\underline{\gamma} \in Z_n^{\prime}(X)$, then $\underline{f}(\underline{\gamma}) \sim \underline{\gamma}$ in X.

The following extension of the classical Phragmen-Brouwer theorem is proved: If X and Y are closed subsets of a metric space and there exists $\underline{\gamma} \in Z_n^{\prime}(X \cap Y)$ such that $\underline{\gamma} \notin B_n^{\prime}(X \cap Y)$ but $\underline{\gamma} \in B_n^{\prime}(X) \cap B_n^{\prime}(Y)$, then there exists $\underline{\delta} \in Z_{n+1}^{\prime}(X \cup Y)$ such that $\underline{\delta} \notin B_{n+1}^{\prime}(X \cup Y)$. An example is given to show that the assumption that X and Y are closed is essential. A metric space X satisfies the *Alexandroff equivalence* provided $\dim X > k$ if and only if there is an essential cycle $\underline{\gamma} \in B_k^{\infty}(X)$ (where dim = covering dimension). The classical Alexandroff dimension theorem may now be stated as follows: If X is a finite dimensional compact metric space then each closed subspace of X satisfies the Alexandroff equivalence. A metric space X is *compactly dimensioned* if there is a compact set $X_0 \subseteq X$ such that $\dim X_0 = \dim X$. A finite dimensional metric space satisfies the Alexandroff equivalence if and only if the space is compactly dimensioned. A finite dimensional metric space may fail to be compactly dimensioned or may be compactly dimensioned and yet have closed subspaces which are not compactly dimensioned. The following extension of the Alexandroff theorem is proved: If X is a finite dimensional metric space which is a locally countable union of locally compact subspaces, then each closed subspace of X satisfies the Alexandroff equivalence.

Chapter I

Introduction

The subject of Vietoris homology theory is usually regarded as having begun with the publication by Vietoris in 1927 of his classic paper *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen* [28]. In this work he carried the ideas of combinatorial topology over to compact metric spaces by introducing a type of cycle called *fundamental sequence.* Such cycles were sequences of abstract simplicial cycles whose vertices were points of the compact metric space. They were required to satisfy two conditions: first, that the mesh (maximum simplex diameter) of the abstract cycles should converge to zero, and second, that eventually the differences of successive pairs of the abstract cycles should bound complexes of arbitrarily small mesh. Vietoris also defined various homology relations among these cycles and discussed the resulting homology groups. His main purpose in applying combinatorial ideas to compact metric spaces was to establish a proposition now known as the Vietoris mapping theorem [13], p. 347, [3], [25]. This theorem gives a criterion for the isomorphism of the homology groups of two spaces based on properties of a mapping of one of the spaces onto the other.

Although Vietoris' primary interest was in the mapping theorem, his methods attracted much attention. Lefschetz discussed the Vietoris theory in his 1930 Colloquium Publication [18], relating it to a homology theory of his own and to an early one of Alexandroff [1]. The Vietoris theory was soon applied to other problems by such mathematicians as Whyburn [29], [30] and Eilenberg [10]. Perhaps the most important application was that of Alexandroff who in 1932 published his celebrated paper *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen* [2]. In this paper Alexandroff constructed a homological theory of dimension for compact metric spaces and related the homological dimension to the covering dimension of such spaces. He was then able to characterize the covering dimension by means of a homological condition (see Chapter VI of the present work). In the process of developing the homological dimension theory, Alexandroff also developed the homology theory. His paper became and has remained to the present time the standard reference for Vietoris homology theory.

Since the publication of the Alexandroff paper there have been only a few discussions of the theory itself. In his 1942 Colloquium Publication [19], Lefschetz described a general theory of algebraic topology which included Vietoris theory as well as the more recent homology theory due to Čech [8]. He was able to prove that the Vietoris homology groups are isomorphic to the corresponding Čech groups for the case of a compact metric space. Some additional discussion of the Vietoris theory occurs in Lefschetz' companion volume, *Topics in topology* [20]. Slightly earlier Steenrod in constructing a homology theory for compact metric spaces [27] gave a brief comparison of his own theory to that of Vietoris. In 1950 Begle proved a Vietoris mapping theorem for topological spaces which are compact but not necessarily metrizable [3]. In doing so he used a generalized form of Vietoris cycle, first defined by Spanier [26], for which the original metric requirement was replaced by a condition described in terms of coverings of the space. In their 1952 book *Foundations of algebraic topology* [11], Eilenberg and Steenrod gave a thorough treatment of Čech homology theory but mentioned Vietoris theory only in a note on the development of the Čech theory.

Although Vietoris homology theory has not been included in the general reorganization of algebraic topology which has taken place in recent years, applications of this theory have shown a remarkable persistence. Floyd [12] and Newman [22] published papers in 1949 and 1950 (respectively) using Vietoris theory, long after Čech theory was well established and its relation to Vietoris theory was known. Further applications occurred in 1955 papers by Kosiński [17] and by Borsuk and Kosiński [7]. A paper by Bing and Borsuk [4] published in 1965 makes considerable use of Vietoris theory and includes a preliminary brief outline of some of the main theorems of the subject. Vietoris theory is also the fundamental homology theory used in Borsuk's 1967 book *Theory of retracts* [5], and several pages of the book are devoted to statements of the basic definitions and theorems. A somewhat more extensive outline was given by Borsuk in his 1968 lectures at Rutgers University on the *Topology of compacta* [6]. In these lectures, which provided the foundation for his new theory of shape, Borsuk proved many of the basic theorems of Vietoris theory for the case of compacta embedded in the Hilbert cube. Another important application of Vietoris theory was made recently by Rolfsen [23], [24], who employed it in the proof of a lemma essential to his characterization of the 3-cell as a compact 3-dimensional space having a metric which is strongly convex and without ramifications. It is thus evident that topologists have continued to use Vietoris homology theory up to the present time.

One of the two principal objectives of the present work is to provide an underlying structure for Vietoris homology theory as it is used by topologists today. It is hoped that the existence of this structure will give formal status to the theory and enable results to be formulated and proved with a degree of precision not previously attainable. The second objective is to extend the classical theorems of Vietoris homology theory to a broader class of metric spaces by removing the restriction of compactness and replacing it by some weaker requirement. For this purpose it is necessary that the structure provided for the theory be designed in such a way that compactness is not assumed in defining the basic concepts or in establishing relationships among them.

The organizational scheme of this work parallels the outline given by Borsuk in his 1968 Rutgers lectures [6]. However, at each step of the development the assumption that the metric spaces are compacta embedded in the Hilbert cube is eliminated. In most cases results similar to those for the compact case are obtained, but at the cost of adding considerable complexity to the structure. The next paragraphs give a brief outline of the work which appears in the ensuing chapters.

In Chapter II abstract n-dimensional simplexes with vertices in a set S are defined as equivalence classes of ordered $(n+1)$-tuples of elements of the set S. Chains are then defined as linear combinations of simplexes with coefficients in a given abelian group, and a homomorphism of order two, the boundary operator, is defined on the resulting chain groups. Elements of the kernel of the boundary homomorphism are called cycles, elements of the image are called boundaries. In case the set S is the point set of a metric topological space X, further definitions are given. An abstract n-dimensional simplex is called an *ε-simplex* ($\varepsilon > 0$) if the diameter of its vertex set is less than ε. An *ε-chain* is an abstract chain of whose simplexes is an ε-simplex. It is observed that the ε-chains form a group in each dimension and that the boundary operator carries ε-chains to ε-chains, so that we may speak of ε-cycles and ε-boundaries. An equivalence relation of *η-homology* ($\eta > 0$) on the class of ε-cycles is introduced and elementary properties of the ε-chain groups and of the relation of η-homology are discussed. The chains described in Chapter II are referred to as *simple chains* in order to distinguish them from the sequential chains introduced in the following chapter.

Chapter III begins by establishing a terminology for the discussion of sequences. Abstract n-dimensional *sequential chains* in a set S are then defined as sequences of abstract n-dimensional simple chains with vertices in S. An addition is defined for these sequential chains in terms of the addition of simple chains, and a boundary operator on the sequential chains is defined in terms of the boundary operator for simple chains. It is observed that under these definitions the sequential n-dimensional chains form an abelian group in each dimension n, and the boundary operator is a homomorphism of order two from the group of sequential chains of dimension n to that of dimension $n-1$. Sequential cycles and boundaries are then defined in the natural way. If X is a metric space, a sequential chain in X is called an *infinite chain* provided there is a compact set, a *carrier* of the chain, which contains all vertices of each of the simple chains in the sequence, and provided the mesh of the simple chains converges to zero. It is proved that the infinite chains form a subgroup of the sequential chains and that the restriction of the boundary operator gives a homomorphism of order two from each infinite chain group to that of one lower dimension. Thus we may speak of infinite cycles and boundaries, with the infinite boundaries forming a subgroup of the infinite cycles. The resulting quotient group is the *general homology group* of the metric space, and two infinite cycles are *homologous* if they represent the same element of the general homology group. Elementary properties of the infinite chain groups and the homology relation are established. An infinite cycle is called *essential* if it has a carrier in which it is not homologous to the zero infinite cycle, and it is proved that this notion is independent of the subspace which is regarded as containing the cycle. An infinite cycle γ is called a *true cycle* if the infinite cycle $\underline{\gamma}'$ formed by taking the differences of successive pairs of terms of $\underline{\gamma}$ is homologous to the zero cycle. The true cycles are shown to form a subgroup of the infinite cycles. True cycles which are infinite boundaries are called *true boundaries* and these form a subgroup of the true cycles. The quotient group of true cycles modulo true boundaries is called the *Vietoris homology group* of the metric space. Elementary properties of true cycles and boundaries are discussed. Several theorems concerning subsequences of infinite chains are proved. It is shown that infinite boundaries and true boundaries are identical. The main result on subsequences is that each subsequence of a true cycle is itself a true cycle. A theorem is proved stating that two infinite cycles are homologous provided there is a compact set carrying both of them in which for each $\eta > 0$, corresponding terms of the cycles are eventually η-homologous.

Chapter IV is devoted to a discussion of the connection between functions from one space to another and resulting homomorphisms of the various chain groups associated with the spaces. It is first shown that if S and T are sets, each function from S to T induces a homomorphism from the abstract simple chains in S to those in T, and that this homomorphism commutes with the boundary operator so that it preserves cycles and boundaries. The next step is to show that a function from S to T induces a homomorphism from the abstract sequential chains in S to those in T, again commuting with the boundary operator. Suppose that X and Y are metric spaces and that f is a function from X to Y (not necessarily continuous). If for some positive numbers ε and η it is true that points a, b of X within a distance ε must have images $f(a)$, $f(b)$ in Y within a distance η, then f induces a homomorphism from the ε-chains in X to the η-chains in Y, and in fact this homomorphism is defined by the homomorphism of abstract simple chains mentioned above. If f is a mapping (a continuous function) from X to Y

it is proved that f induces a homomorphism from the infinite chains in X to those in Y, and in fact this homomorphism is defined by the homomorphism of abstract sequential chains mentioned above. The proof is based on a lemma which uses the fact f is uniformly continuous on compact sets. Elementary properties of the homomorphism of infinite chains are discussed and it is proved that this homomorphism carries true cycles in X to those in Y. Moreover, this homomorphism induces a homomorphism from the general homology group of X to that of Y and another from the Vietoris homology group of X to that of Y. The question of topological invariance of the general and Vietoris homology groups is considered next. It is first proved that for a metrizable topological space these groups do not depend on the choice of the metric yielding the given topology. Then it is shown that if h is a homeomorphism from X to Y, the homomorphism of infinite chains induced by h is an isomorphism and in turn induces isomorphisms of the general homology groups and of the Vietoris homology groups. That the general homology group is not in general isomorphic to the Vietoris homology group is shown by means of examples, after two theorems are proved in preparation. Next the homotopy theorem is proved which states that if f and g are mappings of X to Y and γ is an infinite cycle in X such that the restrictions of f and g to some carrier of γ are homotopic in Y, then $f(\gamma)$ and $g(\gamma)$ are homologous in Y. The proof is based on a generalization of a lemma due to Borsuk. A corollary to the homotopy theorem is stated to the effect that a contractible space has trivial general homology groups. The final part of Chapter IV is concerned with *null translations*. A sequence of functions $\langle f_i \rangle$ from X to X is called a null translation provided there is a sequence of positive numbers $\langle \eta_i \rangle$ converging to zero such that for each point a of X the distance from a to $f_i(a)$ is always less than η_i. After some preliminary discussion it is proved that a null translation carries each infinite chain to an infinite chain and in fact induces a homomorphism of the infinite chains in X which commutes with the boundary operator. Basic properties of null translations are discussed. The generalized Borsuk lemma is again used, this time to prove that a null translation carries each infinite cycle into an infinite cycle homologous to it.

In Chapter V the classical Phragmen–Brouwer theorem for compact metric spaces is first proved [2], [6]. The theorem states that if X and Y are compact subsets of a metric space and if there is an infinite n-dimensional cycle in $X \cap Y$ which is a boundary in X and in Y but not in $X \cap Y$, then there is an infinite $(n+1)$-dimensional cycle in $X \cup Y$ which is not a boundary in $X \cup Y$. A proof is given of an extension of the theorem in which the restriction that X and Y be compact is replaced by the condition that X and Y be closed subsets of the metric space. An example is then given to show that it is not possible to drop the requirement that X and Y be closed.

The central topic of Chapter VI is the Alexandroff dimension theorem [2]. The classical version of this theorem may be stated as follows (where *dimension* means covering dimension). If X is a finite dimensional compact metric space, then the dimension of X is greater than k if and only if there is an essential k-dimensional infinite cycle in X which is homologous in X to the zero cycle. A metric space X satisfies the *Alexandroff equivalence* if it is true that the dimension of X is greater than k if and only if there is an essential k-dimensional infinite cycle in X which is homologous in X to the zero cycle. It is easy to see that each closed subspace of a finite dimensional compact metric space satisfies the Alexandroff equivalence. A finite dimensional metric space is *compactly dimensioned* if some compact subspace has the same dimension as the space. It is proven that a finite dimensional metric space satisfies the Alexandroff equivalence if and only if the space is compactly dimensioned. Two examples are then given, the first showing that not every finite dimensional metric space is compactly dimensioned, and the second showing that such a space may be compactly dimensioned and yet have closed subspaces which are not compactly dimensioned. The remainder of Chapter VI is devoted to a proof of the following generalization of the classical Alexandroff dimension theorem: If X is a finite dimensional metric space which is a locally countable union of locally compact subspaces, then each closed subspace of X satisfies the Alexandroff equivalence. The proof is based on five lemmas: (1) A locally compact metric space has a locally finite covering by compact sets. (2) A finite dimensional locally compact metric space is compactly dimensioned. (3) A locally compact subset of a metric space is a countable union of closed locally compact sets. (4) If a metric space is a locally countable union of locally compact subspaces, then it is a locally countable union of closed locally compact subspaces. (5) If a finite dimensional metric space is a locally countable union of locally compact subspaces, then it is compactly dimensioned.

Chapter II

Simple chains

2.1. Simplexes. Let S be a set. In the discussion that follows, the elements of S will be referred to as *vertices*. Let $\tilde{Z}$ denote the ring of integers and let $n \in \tilde{Z}$. For $n \geqslant 0$ an *ordered n-simplex* s with vertices in S is an ordered $(n+1)$-tuple $(a_0, a_1, \ldots, a_n)$, where the a_i are (not necessarily distinct) elements of S. That is, s is a function from $\{0, 1, \ldots, n\}$ into S. For $n = -1$ the empty set $\emptyset$ is the only *ordered* (-1)-simplex.

Oriented simplexes with vertices in S are defined as follows. For $n \geqslant 1$ the permutations of $\{0, 1, \ldots, n\}$ may be classified as even or odd. Let $\sigma = [a_0 a_1 \ldots a_n]$, where $a_i \in S$, denote the set of ordered n-simplexes $(a_{\pi(0)}, a_{\pi(1)}, \ldots, a_{\pi(n)})$ for which π is an even permutation of $\{0, 1, \ldots, n\}$ and let $-\sigma$ denote the set of those for which π is odd. Then σ and $-\sigma$ are called *oriented n-simplexes.* For $n = 0$, if $s = (a_0)$ is an ordered 0-simplex the two symbols $\sigma = [a_0]$ and $-\sigma = -[a_0]$ are associated with s and are called *oriented* 0*-simplexes.* For $n = -1$ the two symbols $\sigma = +\emptyset$ and $-\sigma = -\emptyset$ are associated with $\emptyset$ and are called *oriented* (-1)*-simplexes.* Let $\Sigma_n(S)$ denote the set of oriented n-simplexes with vertices in S and for $\sigma \in \Sigma_n(S)$ let $V(\sigma)$ denote the set of vertices of σ. The term "simplex" without further qualification will always refer to an oriented simplex. For $n > 0$ a simplex $[a_0 a_1 \ldots a_n]$ is said to be *degenerate* if $a_j = a_k$ for some j, k with $j \neq k$. In this case $[a_0 a_1 \ldots a_n] = -[a_0 a_1 \ldots a_n]$. Given the set S, for each simplex σ let one member of the pair σ, $-\sigma$ be chosen and called *positively oriented.* This choice remains fixed throughout the entire discussion.

We note that if T is a subset of S, then for each $n \in \tilde{Z}$, $\Sigma_n(T) \subseteq \Sigma_n(S)$.

2.2. Chains. Let D be an abelian group. For $n \geqslant -1$ the collection of all functions $\varkappa\colon \Sigma_n(S) \to D$ such that $\varkappa(\sigma) = 0$ for all but a finite number of $\sigma \in \Sigma_n(S)$ forms an abelian group $C_n(S, D)$ under the addition given for any two such functions $\varkappa_1$, $\varkappa_2$, by the rule

$$(\varkappa_1+\varkappa_2)(\sigma) = \varkappa_1(\sigma)+\varkappa_2(\sigma) \quad \text{for each } \sigma \in \Sigma_n(S).$$

It follows that each element $\varkappa \in C_n(S, D)$ may be regarded as a linear combination $\varkappa = d_1\sigma_1 + \ldots + d_k\sigma_k$, where $d_i \in D$, $\sigma_i \in \Sigma_n(S)$, and $d_i = \varkappa(\sigma_i)$ for $i = 1, \ldots, k$. Let $N_n(S, D)$ denote the subgroup of $C_n(S, D)$ generated by all elements either of the form $d\sigma$, where σ is degenerate or of the form $d\sigma + d(-\sigma)$. The quotient group $C_n(S, D)/N_n(S, D)$ will be denoted $C_n^{\#}(S, D)$ and its elements will be called *n-dimensional chains in S over D* or simply *n-chains.* For $\varkappa \in C_n(S, D)$ let $[\varkappa]$ denote the coset of $N_n(S, D)$ containing $\varkappa$. The notation $\varkappa \in C_n^{\#}(S, D)$ will be understood to mean $[\varkappa] \in C_n^{\#}(S, D)$, and similarly the notation $\varkappa_1 = \varkappa_2$ will mean $[\varkappa_1] = [\varkappa_2]$.

It follows from the above definitions and conventions that the linear combinations which represent the n-dimensional chains satisfy the following rules of arithmetic:

(1) If π is a permutation of $\{1, \ldots, k\}$, then

$$d_{\pi(1)}\sigma_{\pi(1)} + \ldots + d_{\pi(k)}\sigma_{\pi(k)} = d_1\sigma_1 + \ldots + d_k\sigma_k$$
$$(\text{for } d_i \in D,\ \sigma_i \in \Sigma_n(S),\ i = 1, \ldots, k).$$

(2) $$d_1\sigma + d_2\sigma = (d_1+d_2)\sigma \quad (\text{for } d_1, d_2 \in D,\ \sigma \in \Sigma_n(S)).$$

(3) $$0 \cdot \sigma = 0 \quad (\text{for } \sigma \in \Sigma_n(S)).$$

(4) $$-(d\sigma) = (-d)\sigma \quad (\text{for } d \in D,\ \sigma \in \Sigma_n(S)).$$

(5) $$d\sigma = (-d)(-\sigma) \quad (\text{for } d \in D,\ \sigma \in \Sigma_n(S)).$$

(6) If σ is degenerate, $d\sigma = 0$ (for $d \in D$, $\sigma \in \Sigma_n(S)$).

(7) $$t\cdot(d_1\sigma_1 + \ldots + d_k\sigma_k) = (td_1)\sigma_1 + \ldots + (td_k)\sigma_k$$
$$(\text{for } t \in \tilde{Z},\ d_i \in D,\ \sigma_i \in \Sigma_n(S),\ i = 1, \ldots, k).$$

According to these rules it is possible to obtain a representation for each chain which is unique (except for the order of terms) by using only positively oriented simplexes and deleting terms having a zero coefficient or a degenerate simplex. By the set of *simplexes of a chain* $\varkappa$, denoted $\Sigma(\varkappa)$, is meant the collection of all simplexes which occur in the unique representation of $\varkappa$. The set of *vertices of a chain* $\varkappa$, denoted $V(\varkappa)$, is the collection of all vertices of all simplexes in $\Sigma(\varkappa)$; that is, $V(\varkappa) = \bigcup\{V(\sigma): \sigma \in \Sigma(\varkappa)\}$.

In the case $n = -1$, if $+\emptyset$ is chosen to be positively oriented, each (-1)-chain is of the form $d\cdot\emptyset$, where $d \in D$. Thus $C^{\#}_{-1}(S, D)$ may be identified with the group D. For convenience we define $C^{\#}_n(S, D)$ to be the trivial group $\{0\}$ for $n < -1$.

THEOREM 1. *If T is a subset of S, then for each $n \in \tilde{Z}$, $C^{\#}_n(T, D)$ may be regarded as a subgroup of $C^{\#}_n(S, D)$. Moreover, if $\varkappa \in C^{\#}_n(S, D)$ and $V(\varkappa) \subseteq T$, then $\varkappa \in C^{\#}_n(T, D)$.*

Proof. Let $C(S)$, $N(S)$, $C(T)$, $N(T)$ be abbreviations for $C_n(S, D)$, $N_n(S, D)$, $C_n(T, D)$, $N_n(T, D)$ respectively. Thus $C^{\#}_n(S, D) = C(S)/N(S)$ and $C^{\#}_n(T, D) = C(T)/N(T)$. We have noted previously that $T \subseteq S$ implies $\Sigma_n(T) \subseteq \Sigma_n(S)$. It follows that $C(T)$ is isomorphic to a subgroup $\hat{C}(T)$ of $C(S)$. We may describe $\hat{C}(T)$ as the set of functions $\varkappa \in C(S)$ such that $\varkappa(\sigma) = 0$ for each $\sigma \in \Sigma_n(S)\setminus\Sigma_n(T)$. The same isomorphism carries $N(T)$ onto a subgroup $\hat{N}(T)$ of $\hat{C}(T)$. Thus $C^{\#}_n(T, D)$ is isomorphic to $\hat{C}(T)/\hat{N}(T)$. It is clear that $\hat{N}(T) \subseteq N(S)$ and in fact $\hat{N}(T) = \hat{C}(T) \cap N(S)$. Therefore the Noether isomorphism theorem implies that $\hat{C}(T)/\hat{N}(T)$ is isomorphic to the group $(\hat{C}(T)+N(S))/N(S)$, where $\hat{C}(T)+N(S)$ denotes the subgroup of $C(S)$ generated by $\hat{C}(T) \cup N(S)$. Since $(\hat{C}(T)+N(S))/N(S)$ is a subgroup of $C(S)/N(S)$, this proves that $C^{\#}_n(T, D)$ is isomorphic to a subgroup of $C^{\#}_n(S, D)$. For convenience we identify $C^{\#}_n(T, D)$ with its isomorphic image in $C^{\#}_n(S, D)$.

The second assertion of the theorem now follows from this identification. The hypothesis $\varkappa \in C^{\#}_n(S, D)$ actually means $\varkappa \in C(S)$ and $[\varkappa] = \varkappa + N(S) \in C(S)/N(S)$. The condition $V(\varkappa) \subseteq T$ implies that $\varkappa \in \hat{C}(T)$, and consequently $\varkappa \in \hat{C}(T)+N(S)$ since $\hat{C}(T) \subseteq \hat{C}(T) \cup N(S) \subseteq \hat{C}(T)+N(S)$. Thus $[\varkappa] = \varkappa + N(S) \in (\hat{C}(T)+N(S))/N(S)$; that is, $[\varkappa]$ is an element of the subgroup of

$C_n^{\#}(S, D)$ which we have identified with $C_n^{\#}(T, D)$. Therefore according to our conventions we may write $\varkappa \in C_n^{\#}(T, D)$.

2.3. Boundary operator. Cycles and boundaries. Let S be a set, D an abelian group, and let $n \in \tilde{Z}$. We now define a function ∂: $C_n^{\#}(S, D) \to C_{n-1}^{\#}(S, D)$ called the *boundary operator*. It is first defined on "elementary chains" of the form $d\sigma$. For $n \geqslant 1$, let $\sigma = [a_0 \dots a_n] \in \Sigma_n(S)$ and let $d \in D$ so that $d\sigma \in C_n^{\#}(S, D)$. The *boundary* of $d\sigma$, denoted $\partial(d\sigma)$, is defined to be the chain

$$\sum_{i=0}^{n} (-1)^i d\,[a_0 \dots a_{i-1}\, \hat{a}_i\, a_{i+1} \dots a_n] \in C_{n-1}^{\#}(S, D),$$

where the notation $\hat{a}_i$ indicates that a_i is deleted. It may be shown that the resulting chain $\partial(d\sigma)$ is independent of the representation of σ. Moreover, $\partial(d(-\sigma)) = -\partial(d\sigma)$ and consequently if σ is degenerate, $\partial(d\sigma) = 0$. For $n = 0$, let $\sigma = [a_0]$ or $\sigma = -[a_0]$ and let $d \in D$ so that $d\sigma \in C_0^{\#}(S, D)$. The *boundary* of $d\sigma$ is defined by $\partial(d[a_0]) = d \cdot \emptyset$ or $\partial(d(-[a_0])) = d(-\emptyset)$. Thus $\partial(d\sigma) \in C_{-1}^{\#}(S, D)$. For $n = -1$, let $\sigma = \emptyset$ or $\sigma = -\emptyset$ and let $d \in D$ so that $d\sigma \in C_{-1}^{\#}(S, D)$. The *boundary* of $d\sigma$ is defined in either case by $\partial(d\sigma) = 0$. Thus $\partial(d\sigma) \in C_{-2}^{\#}(S, D)$. Now for $n \geqslant -1$ we extend ∂ to all of $C_n^{\#}(S, D)$ by the formula

$$\partial(d_1\sigma_1 + \dots + d_k\sigma_k) = \partial(d_1\sigma_1) + \dots + \partial(d_k\sigma_k).$$

For $n < -1$, we define ∂: $C_n^{\#}(S, D) \to C_{n-1}^{\#}(S, D)$ by $\partial(0) = 0$.

It is clear from the definition that ∂ is a homomorphism of $C_n^{\#}(S, D)$ into $C_{n-1}^{\#}(S, D)$ for each $n \in \tilde{Z}$. Moreover, a standard argument yields the result that the boundary operator is of order two; that is, $\partial\partial\varkappa = 0$ for $\varkappa \in C_n^{\#}(S, D)$.

For $n \in \tilde{Z}$ we define the group of *n-dimensional cycles in S over D*, denoted $Z_n^{\#}(S, D)$, to be the kernel of ∂: $C_n^{\#}(S, D) \to C_{n-1}^{\#}(S, D)$, and we define the group of *n-dimensional boundaries in S over D*, denoted $B_n^{\#}(S, D)$, to be the image of ∂: $C_{n+1}^{\#}(S, D) \to C_n^{\#}(S, D)$. Thus $Z_n^{\#}(S, D)$ and $B_n^{\#}(S, D)$ are subgroups of $C_n^{\#}(S, D)$. Moreover, since $\partial\partial = 0$ it follows that $B_n^{\#}(S, D) \subseteq Z_n^{\#}(S, D)$.

It is a consequence of Theorem 1 that if T is a subset of S, then for each $n \in \tilde{Z}$, $Z_n^{\#}(T, D)$ is a subgroup of $Z_n^{\#}(S, D)$ and $B_n^{\#}(T, D)$ is a subgroup of $B_n^{\#}(S, D)$.

2.4. Join operator. Let S be a set, D an abelian group, and let $n \in \tilde{Z}$ with $n \geqslant 0$. If $b \in S$ and $\sigma = [a_0 \dots a_n] \in \Sigma_n(S)$ we define the *join of b and σ*, denoted $b \cdot \sigma$, to be the simplex $[ba_0 \dots a_n] \in \Sigma_{n+1}(S)$. If $b \in S$ and $\varkappa = d_1\sigma_1 + \dots + d_k\sigma_k \in C_n^{\#}(S, D)$ we define the *join of b and $\varkappa$*, denoted $b \cdot \varkappa$, to be the chain $d_1(b \cdot \sigma_1) + \dots + d_k(b \cdot \sigma_k) \in C_{n+1}^{\#}(S, D)$ provided $\varkappa \neq 0$ or to be $0 \in C_{n+1}^{\#}(S, D)$ in case $\varkappa = 0$. For $n = -1$ we define the join by the formulas $b\cdot 0 = 0$, $b\cdot\emptyset = [b]$, and $b\cdot(d\emptyset) = d[b]$. For $n < -1$ we define $b\cdot 0 = 0$. Now for each $b\in S$ and $n\in\tilde{Z}$ we define a function $J_b\colon C_n^{\#}(S, D)\to C_{n+1}^{\#}(S, D)$, called the *join operator for b*, by means of the formula $J_b(\varkappa) = b\cdot\varkappa\,(\varkappa\in C_n^{\#}(S, D))$. Then J_b is a homomorphism and it is related to the boundary operator ∂ by the following formulas.

(1) $$\partial(b\cdot d\sigma) = d\sigma - b\cdot\partial(d\sigma) \quad \text{for } b\in S,\ \sigma\in\Sigma_n(S),\ d\in D.$$

(2) $$\partial(b\cdot\varkappa) = \varkappa - b\cdot\partial\varkappa \quad \text{for } b\in S,\ \varkappa\in C_n^{\#}(S, D).$$

2.5. ε-simplexes and ε-chains. Let D be an abelian group and suppose that (X,ϱ) is a metric space; that is, X is a topological space, ϱ is a metric function for X, and the topology of X is that induced by the metric ϱ. Let $\tilde{R}$ denote the field of real numbers and $\tilde{R}_+$ the subset of $\tilde{R}$ consisting of all strictly positive real numbers. If S is a non-empty subset of X we define the *ϱ-diameter of S*, $\Delta_\varrho(S)$, to be $\sup\{\varrho(a,b)\colon a,b\in S\}$ in case the set $\{\varrho(a,b)\colon a,b\in S\}$ is bounded and to be infinite otherwise.

For $\varepsilon\in\tilde{R}_+$ we say that $\sigma\in\Sigma_n(X)$ is an *n-dimensional ε-simplex in* (X,ϱ) if $\Delta_\varrho(V(\sigma)) < \varepsilon$. We denote the set of n-dimensional ε-simplexes in (X,ϱ) by $\Sigma_n^\varepsilon(X,\varrho)$. Thus $\Sigma_n^\varepsilon(X,\varrho)\subseteq\Sigma_n(X)$. A chain $\varkappa\in C_n^{\#}(X, D)$ is said to be an *n-dimensional ε-chain in* (X,ϱ) *over D* if $\Sigma(\varkappa)\subseteq\Sigma_n^\varepsilon(X,\varrho)$. The collection of all n-dimensional ε-chains in (X,ϱ) over D forms a subgroup of $C_n^{\#}(X, D)$ which we denote by $C_n^\varepsilon(X,\varrho,D)$. For $\varkappa\in C_n^{\#}(X,D)$ we know $\varkappa\in C_n^\varepsilon(X,\varrho,D)$ if and only if $\Delta_\varrho(V(\sigma)) < \varepsilon$ for each $\sigma\in\Sigma(\varkappa)$. Moreover, the boundary operator $\partial\colon C_n^{\#}(X,D)\to C_{n-1}^{\#}(X,D)$ defines a function, also called ∂, on $C_n^\varepsilon(X,\varrho,D)$. It is an easy consequence of these statements that if $\sigma\in\Sigma_n^\varepsilon(X,\varrho)$ and $d\in D$, then $\partial(d\sigma)\in C_{n-1}^\varepsilon(X,\varrho,D)$. Therefore ∂ is a homomorphism of $C_n^\varepsilon(X,\varrho,D)$ into $C_{n-1}^\varepsilon(X,\varrho,D)$.

We now define the group of *n-dimensional ε-cycles in* (X,ϱ) *over D*, denoted $Z_n^\varepsilon(X,\varrho,D)$, to be the kernel of $\partial\colon C_n^\varepsilon(X,\varrho,D)\to C_{n-1}^\varepsilon(X,\varrho,D)$, and the group of *n-dimensional ε-boundaries in* (X,ϱ) *over D*, denoted $B_n^\varepsilon(X,\varrho,D)$, to be the image of $\partial\colon C_{n+1}^\varepsilon(X,\varrho,D)\to C_n^\varepsilon(X,\varrho,D)$. Evidently $Z_n^\varepsilon(X,\varrho,D)$ and $B_n^\varepsilon(X,\varrho,D)$ are subgroups of $C_n^\varepsilon(X,\varrho,D)$, and again since $\partial\partial = 0$ it follows that $B_n^\varepsilon(X,\varrho,D)\subseteq Z_n^\varepsilon(X,\varrho,D)$. The quotient group $Z_n^\varepsilon(X,\varrho,D)/B_n^\varepsilon(X,\varrho,D)$ is called the *n-dimensional ε-homology group of* (X,ϱ) *with coefficients in D* and is denoted $H_n^\varepsilon(X,\varrho,D)$.

For $\gamma_1,\gamma_2\in Z_n^\varepsilon(X,\varrho,D)$ and $\eta\in\tilde{R}_+$, we say that γ_1 and γ_2 are *η-homologous in* (X,ϱ), denoted $\gamma_1 \underset{\eta}{\sim} \gamma_2$ in (X,ϱ), if $\gamma_1-\gamma_2\in B_n^\eta(X,\varrho,D)$. Thus $\gamma_1 \underset{\eta}{\sim} \gamma_2$ in (X,ϱ) if and only if there exists $\varkappa\in C_{n+1}^\eta(X,\varrho,D)$ such that $\partial\varkappa = \gamma_1-\gamma_2$. It is easy to show that the relation $\mathscr{R}$ defined on $Z_n^\varepsilon(X,\varrho,D)$ by the rule $\gamma_1\mathscr{R}\gamma_2$ if and only if $\gamma_1 \underset{\eta}{\sim} \gamma_2$ in (X,ϱ) is an equivalence relation.

Suppose now that (Y,ϱ') is a metric subspace of (X,ϱ); that is, Y is a subset of X with topology given by the metric ϱ' which is the restriction of ϱ to $Y \times Y$. In this case ϱ' is called the *metric for Y inherited from* (X, ϱ). Let $\varepsilon \in \tilde{R}_+$ and $n \in \tilde{Z}$. The following facts are easily established:

(1) $\Sigma_n^\varepsilon(Y, \varrho')$ is a subset of $\Sigma_n^\varepsilon(X, \varrho)$.

(2) $C_n^\varepsilon(Y, \varrho', D)$ is a subgroup of $C_n^\varepsilon(X, \varrho, D)$.

(3) $Z_n^\varepsilon(Y, \varrho', D)$ is a subgroup of $Z_n^\varepsilon(X, \varrho, D)$.

(4) $B_n^\varepsilon(Y, \varrho', D)$ is a subgroup of $B_n^\varepsilon(X, \varrho, D)$.

(5) If $\gamma_1, \gamma_2 \in Z_n^\varepsilon(Y, \varrho', D)$ and $\gamma_1 \underset{\eta}{\sim} \gamma_2$ in (Y, ϱ') (where $\eta \in \tilde{R}_+$), then $\gamma_1 \underset{\eta}{\sim} \gamma_2$ in (X, ϱ).

In each of the above definitions of metric concepts the name of the metric has been included as part of the notation. This explicit mention of the metric will usually be dropped in cases where no confusion is likely to result. Thus the groups defined above will ordinarily be denoted $C_n^\varepsilon(X, D)$, $Z_n^\varepsilon(X, D)$, etc. The same agreement applies to definitions of metric concepts which are given in subsequent sections of this work.

Chapter III

Sequential chains

3.1. Sequences and subsequences. Let S be a set and let $\tilde{N}$ denote the system of natural numbers. A function $\underline{s}$: $\tilde{N} \to S$ is called a *sequence into S*. For $k \in \tilde{N}$, $\underline{s}(k)$ is usually written s_k, and the sequence $\underline{s}$ is also denoted $\langle s_1, s_2, \ldots \rangle$ or $\langle s_i \rangle$. The collection of all sequences into S is denoted $S^{\tilde{N}}$. The notation $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$ indicates that $\underline{m}$ is a strictly increasing sequence into $\tilde{N}$. If $\underline{s} \in S^{\tilde{N}}$ and $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$, the sequence $\underline{s} \cdot \underline{m}$ into S defined by $(\underline{s} \cdot \underline{m})(k) = \underline{s}(\underline{m}(k))$ for $k \in \tilde{N}$ is called a *subsequence* of $\underline{s}$. Each $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$ may be regarded as a *subsequence-forming operator* in the sense that the function $F_{\underline{m}}$: $S^{\tilde{N}} \to S^{\tilde{N}}$ defined by $F_{\underline{m}}(\underline{s}) = \underline{s} \cdot \underline{m}$ for $\underline{s} \in S^{\tilde{N}}$ assigns to each sequence into S a subsequence of itself. Moreover, each subsequence of a given sequence is formed in this way for suitable choice of $\underline{m}$. If $\underline{s}, \underline{t} \in S^{\tilde{N}}$ and $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$, then $\underline{s} \cdot \underline{m}$ and $\underline{t} \cdot \underline{m}$ are called *corresponding subsequences* of $\underline{s}$ and $\underline{t}$.

Let $\underline{s} = \langle s_1, s_2, \ldots \rangle$ be a sequence into S. For $n = 0, 1, 2, \ldots$ the sequence $t_n(\underline{s}) = \langle s_{n+1}, s_{n+2}, \ldots \rangle$ is called the *n-th tail* of $\underline{s}$; that is, $t_n(\underline{s})$ is the subsequence $\underline{s} \cdot \underline{m}$ of $\underline{s}$, where $m_k = n+k$, and in particular $t_0(\underline{s}) = \underline{s}$. The first tail of $\underline{s}$, $t_1(\underline{s}) = \langle s_2, s_3, \ldots \rangle$, is called simply the *tail* of $\underline{s}$.

If X is a topological space, $a \in X$, and $\underline{x} \in X^{\tilde{N}}$, the notation $\underline{x} \to a$ indicates that the sequence $\underline{x}$ converges to the point a.

3.2. Sequential chains. Let S be a set, D an abelian group, and let $n \in \tilde{Z}$. A sequence $\underline{\varkappa} = \langle \varkappa_i \rangle$ such that each $\varkappa_i \in C_n^{\#}(S, D)$ is called a *sequential n-dimensional chain in S over D.* The collection of all such sequential n-chains is denoted $C_n^*(S, D)$. Thus $C_n^*(S, D) = (C_n^{\#}(S, D))^{\tilde{N}}$. It is clear that $C_n^*(S, D)$ forms an abelian group under the addition defined by

$$\langle \varkappa_i \rangle + \langle \lambda_i \rangle = \langle \varkappa_i + \lambda_i \rangle, \quad \text{where} \quad \langle \varkappa_i \rangle, \langle \lambda_i \rangle \in C_n^*(S, D).$$

The zero element of $C_n^*(S, D)$ is the sequence $\langle 0, 0, \ldots \rangle$, denoted $\underline{0}$.

We define a *boundary operator* ∂: $C_n^*(S, D) \to C_{n-1}^*(S, D)$ by the formula $\partial \langle \varkappa_i \rangle = \langle \partial \varkappa_i \rangle$ where $\partial \varkappa_i$ has been defined previously in Section 2.3. Evidently ∂ is a homomorphism of $C_n^*(S, D)$ into $C_{n-1}^*(S, D)$ and $\partial(\partial \underline{\varkappa}) = \underline{0}$ for $\underline{\varkappa} \in C_n^*(S, D)$. The kernel of ∂: $C_n^*(S, D) \to C_{n-1}^*(S, D)$ is called the group of *sequential n-dimensional cycles in S over D*, $Z_n^*(S, D)$, and the image of ∂: $C_{n+1}^*(S, D) \to C_n^*(S, D)$ is called the group of *sequential n-dimensional boundaries in S over D*, $B_n^*(S, D)$. Again it follows from $\partial\partial = 0$ that $B_n^*(S, D) \subseteq Z_n^*(S, D)$.

The set of *simplexes of a sequential chain* $\underline{\varkappa} \in C_n^*(S, D)$, denoted by $\Sigma(\underline{\varkappa})$, means the set $\bigcup \{\Sigma(\varkappa_i)\colon i \in \tilde{N}\}$. The set of *vertices of a sequential chain* $\underline{\varkappa} \in C_n^*(S, D)$, denoted $V(\underline{\varkappa})$, means the set $\bigcup \{V(\varkappa_i)\colon i \in \tilde{N}\}$. The set $V(\underline{\varkappa})$ may also be described as $\bigcup \{V(\sigma)\colon \sigma \in \Sigma(\underline{\varkappa})\}$.

If T is a subset of S, it follows from Theorem 1 that for each $n \in \tilde{Z}$, $C_n^*(T, D)$, $Z_n^*(T, D)$, and $B_n^*(T, D)$ are subgroups of $C_n^*(S, D)$, $Z_n^*(S, D)$, and $B_n^*(S, D)$ respectively.

3.3. Infinite chains. General homology groups. Let (X, ϱ) be a metric space, D an abelian group, and let $n \in \tilde{Z}$. A sequential chain $\underline{\varkappa} \in C_n^*(X, D)$ is called an *infinite n-dimensional chain in* (X, ϱ) *over* D provided the following two conditions are satisfied:

(1) There is a sequence $\underline{\varepsilon}$ into $\tilde{R}_+$ with $\underline{\varepsilon} \to 0$ such that for each $i \in \tilde{N}$, $\varkappa_i \in C_n^{\varepsilon_i}(X, \varrho, D)$.

(2) There is a compact subset X_0 of X such that $V(\underline{\varkappa}) \subseteq X_0$.

Any sequence $\underline{\varepsilon}$ satisfying (1) is called a *majorant* of $\underline{\varkappa}$ in (X, ϱ), and any compact set X_0 satisfying (2) is called a *carrier* of $\underline{\varkappa}$ in (X, ϱ). It is easy to verify the following statements:

(3) If $\underline{\varepsilon}$ is a majorant of $\underline{\varkappa}$ in (X, ϱ), and $\underline{\varepsilon}'$ is a sequence into $\tilde{R}_+$ with $\underline{\varepsilon}' \to 0$ such that $\varepsilon_i' \geqslant \varepsilon_i$ for each $i \in \tilde{N}$, then $\underline{\varepsilon}'$ is a majorant of $\underline{\varkappa}$ in (X, ϱ).

(4) If X_0 is a carrier of $\underline{\varkappa}$ in (X, ϱ), and X_0' is a compact subset of X such that $X_0' \supseteq X_0$, then X_0' is a carrier of $\underline{\varkappa}$ in (X, ϱ).

In the definition of an infinite n-chain given above we may replace condition (1) by the following requirement:

(5) There is a sequence $\underline{\varepsilon}$ into $\tilde{R}_+$ with $\underline{\varepsilon} \to 0$ such that for each $i \in \tilde{N}$, if $\sigma \in \Sigma(\varkappa_i)$, then $\Delta_\varrho(V(\sigma)) < \varepsilon_i$.

The collection of all infinite n-dimensional chains in (X, ϱ) over D is denoted $C_n^\infty(X, \varrho, D)$. Since $C_n^\infty(X, \varrho, D) \subseteq C_n^*(X, D)$, the boundary operator ∂: $C_n^*(X, D) \to C_{n-1}^*(X, D)$ defines a function, again called ∂, on $C_n^\infty(X, \varrho, D)$.

THEOREM 2. *For each $n \in \tilde{Z}$, $C_n^\infty(X, \varrho, D)$ is a subgroup of $C_n^*(X, D)$. The boundary operator ∂ is a homomorphism of $C_n^\infty(X, \varrho, D)$ into $C_{n-1}^\infty(X, \varrho, D)$ and satisfies the condition $\partial\partial = 0$.*

Proof. Suppose $\underline{\varkappa} \in C_n^\infty(X, \varrho, D)$ with majorant $\underline{\eta}$, carrier X_0, and $\underline{\varkappa}' \in C_n^\infty(X, \varrho, D)$ with majorant $\underline{\eta}'$, carrier X_0'. Then for each $i \in \tilde{N}$, $\varkappa_i$, $\varkappa_i' \in C_n^{\varepsilon_i}(X, \varrho, D)$, where $\varepsilon_i = \max\{\eta_i, \eta_i'\}$, and hence $\varkappa_i + \varkappa_i' \in C_n^{\varepsilon_i}(X, \varrho, D)$. Clearly, $\underline{\varepsilon} = \langle\varepsilon_i\rangle \to 0$. Also since $V(\varkappa_i) \subseteq X_0$ and $V(\varkappa_i') \subseteq X_0'$ it follows that $V(\varkappa_i + \varkappa_i') \subseteq X_0 \cup X_0'$. Thus $\underline{\varkappa} + \underline{\varkappa}' \in C_n^\infty(X, \varrho, D)$ with majorant $\underline{\varepsilon}$, carrier $X_0 \cup X_0'$. If $\underline{\varkappa} \in C_n^\infty(X, \varrho, D)$ with majorant $\underline{\eta}$, carrier X_0, then since $\varkappa_i \in C_n^{\eta_i}(X, \varrho, D)$ implies $-\varkappa_i \in C_n^{\eta_i}(X, \varrho, D)$ and since $V(-\varkappa_i) = V(\varkappa_i)$, it follows that $-\underline{\varkappa} = \langle -\varkappa_i\rangle \in C_n^\infty(X, \varrho, D)$ with majorant $\underline{\eta}$, carrier X_0. This proves that $C_n^\infty(X, \varrho, D)$ is a subgroup of $C_n^*(X, D)$.

That ∂ is a homomorphism satisfying $\partial\partial = 0$ follows from the corresponding fact about ∂: $C_n^*(X, D) \to C_{n-1}^*(X, D)$. It remains only to show that $\partial(C_n^\infty(X, \varrho, D)) \subseteq C_{n-1}^\infty(X, \varrho, D)$. Suppose $\underline{\varkappa} \in C_n^\infty(X, \varrho, D)$ with majorant $\underline{\eta}$, carrier X_0. Then since $\varkappa_i \in C_n^{\eta_i}(X, \varrho, D)$ implies $\partial\varkappa_i \in C_{n-1}^{\eta_i}(X, \varrho, D)$ with $V(\partial\varkappa_i) \subseteq V(\varkappa_i)$, we see that $\partial\underline{\varkappa} = \langle\partial\varkappa_i\rangle \in C_{n-1}^\infty(X, \varrho, D)$ with majorant $\underline{\eta}$, carrier X_0.

We may now define the group of *infinite n-dimensional cycles in (X, ϱ) over D*, denoted $Z_n^\infty(X, \varrho, D)$, to be the kernel of ∂: $C_n^\infty(X, \varrho, D) \to C_{n-1}^\infty(X, \varrho, D)$, and the group of *infinite n-dimensional boundaries in (X, ϱ) over D*, denoted $B_n^\infty(X, \varrho, D)$, to be the image of ∂: $C_{n+1}^\infty(X, \varrho, D) \to C_n^\infty(X, \varrho, D)$. Again $B_n^\infty(X, \varrho, D) \subseteq Z_n^\infty(X, \varrho, D)$ since $\partial\partial = 0$. The quotient group $Z_n^\infty(X, \varrho, D)/B_n^\infty(X, \varrho, D)$ is called the *n-dimensional general homology group of (X, ϱ) with coefficients in D* and is denoted $H_n^\infty(X, \varrho, D)$. We remark that if $\underline{\gamma} \in C_n^\infty(X, \varrho, D)$, then $\underline{\gamma} \in Z_n^\infty(X, \varrho, D)$ if and only if $\gamma_i \in Z_n^{\#}(X, D)$ for each $i \in \tilde{N}$.

For $\underline{\gamma}, \underline{\gamma}' \in Z_n^\infty(X, \varrho, D)$ we say that $\underline{\gamma}$ and $\underline{\gamma}'$ are *homologous in (X, ϱ)*, denoted $\underline{\gamma} \sim \underline{\gamma}'$ in (X, ϱ), if $\underline{\gamma} - \underline{\gamma}' \in B_n^\infty(X, \varrho, D)$. Clearly the statement that $\underline{\gamma} \sim \underline{\gamma}'$ in (X, ϱ) is equivalent to the statement that $[\underline{\gamma}] = [\underline{\gamma}']$ in $H_n^\infty(X, \varrho, D)$, which in turn is equivalent to the statement that there exists $\underline{\varkappa} \in C_{n+1}^\infty(X, \varrho, D)$ such that $\partial\underline{\varkappa} = \underline{\gamma} - \underline{\gamma}'$. It is also clear that the relation $\mathscr{R}$ defined on $Z_n^\infty(X, \varrho, D)$ by the rule $\underline{\gamma}\mathscr{R}\underline{\gamma}'$ if and only if $\underline{\gamma} \sim \underline{\gamma}'$ in (X, ϱ) is an equivalence relation.

3.4. Infinite chains in subspaces.

THEOREM 3. *Suppose that (Y, ϱ') is a metric subspace of (X, ϱ). Then for each $n \in \tilde{Z}$:*

(1) *$C_n^\infty(Y, \varrho', D)$ is a subgroup of $C_n^\infty(X, \varrho, D)$.*

(2) *$Z_n^\infty(Y, \varrho', D)$ is a subgroup of $Z_n^\infty(X, \varrho, D)$.*

(3) *$B_n^\infty(Y, \varrho', D)$ is a subgroup of $B_n^\infty(X, \varrho, D)$.*

(4) *If $\underline{\gamma}, \underline{\gamma}' \in Z_n^\infty(Y, \varrho', D)$ and $\underline{\gamma} \sim \underline{\gamma}'$ in (Y, ϱ'), then $\underline{\gamma} \sim \underline{\gamma}'$ in (X, ϱ).*

Proof. (1) Suppose $\underline{\gamma} \in C_n^\infty(Y, \varrho', D)$ with majorant $\underline{\varepsilon}$, carrier Y_0. Then $\underline{\gamma} \in C_n^*(X, D)$ since $C_n^\infty(Y, \varrho', D) \subseteq C_n^*(Y, D) \subseteq (X, D)$ by previously stated results. For each $i \in \tilde{N}$, $\gamma_i \in C_n^{\varepsilon_i}(Y, \varrho', D)$ and hence $\gamma_i \in C_n^{\varepsilon_i}(X, \varrho, D)$, since $C_n^{\varepsilon_i}(Y, \varrho', D) \subseteq C_n^{\varepsilon_i}(X, \varrho, D)$ by another previous result. Thus $\underline{\varepsilon}$ is a majorant for $\underline{\gamma}$ relative to the space (X, ϱ). Also Y_0 is a carrier for $\underline{\gamma}$ relative to the space (X, ϱ) since Y_0 is a compact subset of X. This proves that $\underline{\gamma} \in C_n^\infty(X, \varrho, D)$. (2) If $\underline{\gamma} \in Z_n^\infty(Y, \varrho', D)$, then by (1), $\underline{\gamma} \in C_n^\infty(X, \varrho, D)$. Therefore $\underline{\gamma} \in Z_n^\infty(X, \varrho, D)$ since $\partial\underline{\gamma} = \underline{0}$. (3) If $\underline{\gamma} \in B_n^\infty(Y, \varrho', D)$, then $\underline{\gamma} = \partial\underline{\varkappa}$ for some $\underline{\varkappa} \in C_{n+1}^\infty(Y, \varrho', D)$. But $C_{n+1}^\infty(Y, \varrho', D) \subseteq C_{n+1}^\infty(X, \varrho, D)$ by (1) and so $\underline{\gamma} = \partial\underline{\varkappa}$, where $\underline{\varkappa} \in C_{n+1}^\infty(X, \varrho, D)$. Hence $\underline{\gamma} \in B_n^\infty(X, \varrho, D)$. (4) Suppose $\underline{\gamma}, \underline{\gamma}' \in Z_n^\infty(Y, \varrho', D)$ and $\underline{\gamma} \sim \underline{\gamma}'$ in (Y, ϱ') so that $\underline{\gamma} - \underline{\gamma}' \in B_n^\infty(Y, \varrho', D)$. Then by (2), $\underline{\gamma}, \underline{\gamma}' \in Z_n^\infty(X, \varrho, D)$, and by (3), $\underline{\gamma} - \underline{\gamma}' \in B_n^\infty(x, \varrho, D)$. Therefore $\underline{\gamma} \sim \underline{\gamma}'$ in (X, ϱ).

THEOREM 4. *Let $\underline{\varkappa} \in C_n^\infty(X, \varrho, D)$ and suppose that X_0 is a carrier for $\underline{\varkappa}$ in (X, ϱ). Let ϱ' be the metric for X_0 inherited from (X, ϱ). Then $\underline{\varkappa} \in C_n^\infty(X_0, \varrho', D)$. Moreover, if $\underline{\varkappa} \in Z_n^\infty(X, \varrho, D)$, then $\underline{\varkappa} \in Z_n^\infty(X_0, \varrho', D)$.*

Proof. Let $\underline{\varepsilon}$ be a majorant for $\underline{\varkappa}$ in (X, ϱ) so that for each $i \in \tilde{N}$, $\varkappa_i \in C_n^{\varepsilon_i}(X, \varrho, D)$. Since X_0 is a carrier for $\underline{\varkappa}$ in (X, ϱ), $V(\underline{\varkappa}) \subseteq X_0$, and hence, for each $i \in \tilde{N}$, $V(\varkappa_i) \subseteq X_0$. Now the condition $\varkappa_i \in C_n^{\varepsilon_i}(X, \varrho, D)$ implies that $\varkappa_i \in C_n^\#(X, D)$ and $\Delta_\varrho(V(\sigma)) < \varepsilon_i$ for each $\sigma \in \Sigma(\varkappa_i)$. Then $\varkappa_i \in C_n^\#(X_0, D)$ by Theorem 1 since $\varkappa_i \in C_n^\#(X, D)$ and $V(\varkappa_i) \subseteq X_0$, and $\Delta_{\varrho'}(V(\sigma)) < \varepsilon_i$ for each $\sigma \in \Sigma(\varkappa_i)$ since $\varrho' = \varrho$ on X_0. Thus $\underline{\varkappa} \in C_n^*(X_0, D)$ and $\varkappa_i \in C_n^{\varepsilon_i}(X_0, \varrho', D)$ for each $i \in \tilde{N}$. Therefore $\underline{\varkappa} \in C_n^\infty(X_0, \varrho', D)$ with majorant $\underline{\varepsilon}$, carrier X_0. The second assertion of the theorem now follows from the first, since if $\underline{\varkappa} \in Z_n^\infty(X, \varrho, D)$, then $\underline{\varkappa} \in C_n^\infty(X_0, \varrho', D)$ and $\partial\underline{\varkappa} = \underline{0}$ so that $\underline{\varkappa} \in Z_n^\infty(X_0, \varrho', D)$.

An infinite cycle $\underline{\gamma} \in Z_n^\infty(X, \varrho, D)$ is called *essential in* (X, ϱ) if $\underline{\gamma}$ has a carrier X_0 such that $\underline{\gamma} \nsim \underline{0}$ in (X_0, ϱ'), where ϱ' is the metric for X_0 inherited from (X, ϱ). The condition $\underline{\gamma} \nsim \underline{0}$ in (X_0, ϱ') may be restated in the form $\underline{\gamma} \notin B_n^\infty(X_0, \varrho', D)$.

THEOREM 5. *Let (Y, ϱ') be a metric subspace of (X, ϱ), and let $n \in \tilde{Z}$. Suppose that $\underline{\gamma} \in Z_n^\infty(Y, \varrho', D)$. Then $\underline{\gamma}$ is essential in (Y, ϱ') if and only if $\underline{\gamma}$ is essential in (X, ϱ).*

Proof. Assume $\underline{\gamma} \in Z_n^\infty(Y, \varrho', D)$ with carrier Y_0, and the $\underline{\gamma}$ is essential in (X, ϱ). Then $\underline{\gamma}$ has a carrier X_0 in (X, ϱ) such that $\underline{\gamma} \nsim \underline{0}$ in (X_0, ϱ''), where ϱ'' is the metric for X_0 inherited from (X, ϱ). Now $X_0 \cap Y_0$ is a compact subset of Y and hence is a carrier of $\underline{\gamma}$ in (Y, ϱ'). Assume $\underline{\gamma} \sim \underline{0}$ in $(X_0 \cap Y_0, \hat{\varrho})$, where $\hat{\varrho}$ is the metric for $X_0 \cap Y_0$ inherited from (Y, ϱ'). Since $X_0 \cap Y_0$ is a subset of X_0 and $\hat{\varrho} = \varrho' = \varrho = \varrho''$ on $X_0 \cap Y_0$, it follows that $(X_0 \cap Y_0, \hat{\varrho})$ is a metric subspace of (X_0, ϱ''). Thus by part (4) of Theorem 3, $\underline{\gamma} \sim \underline{0}$ in $(X_0 \cap Y_0, \hat{\varrho})$ implies $\underline{\gamma} \sim \underline{0}$ in (X_0, ϱ''). This contradiction proves that γ is essential in (Y, ϱ').

Conversely, if $\underline{\gamma}$ is essential in (Y, ϱ') then $\underline{\gamma}$ has a carrier Y_0 in (Y, ϱ') such that $\underline{\gamma} \nsim \underline{0}$ in (Y_0, ϱ''), where ϱ'' is the metric for Y_0 inherited from (Y, ϱ'). Now Y_0 is a compact subset of X and hence is a carrier of $\underline{\gamma}$ in (X, ϱ); and $\varrho'' = \varrho' = \varrho$ on Y_0 so that ϱ'' is also the metric for Y_0 inherited from (X, ϱ). Therefore the condition $\underline{\gamma} \nsim \underline{0}$ in (Y_0, ϱ'') implies that $\underline{\gamma}$ is essential in (X, ϱ).

3.5. True cycles. Vietoris homology groups. Suppose that $\underline{\gamma} \in Z_n^{\infty}(X, \varrho, D)$ with majorant $\underline{\varepsilon}$, carrier X_0, so that for $i \in \tilde{N}$, $\gamma_i \in C_n^{\varepsilon_i}(X, \varrho, D)$ and $\partial\gamma_i = 0$. Then for each $i \in \tilde{N}$, $\gamma_i, \gamma_{i+1} \in C_n^{\eta_i}(X, \varrho, D)$, where $\eta_i = \max\{\varepsilon_i, \varepsilon_{i+1}\}$, and hence $\gamma_{i+1} - \gamma_i \in C_n^{\eta_i}(X, \varrho, D)$ also. It is clear that $\underline{\eta} \to 0$ and that $V(\gamma_{i+1} - -\gamma_i) \subseteq X_0$, for $i \in \tilde{N}$. Therefore the sequence $\underline{\gamma}' = \langle\gamma_i'\rangle = \langle\gamma_{i+1} - \gamma_i\rangle$ is an element of $C_n^{\infty}(X, \varrho, D)$ with majorant $\underline{\eta}$, carrier X_0. Moreover, $\underline{\gamma}' \in Z_n^{\infty}(X, \varrho, D)$ since for $i \in \tilde{N}$, $\partial\gamma_i' = \partial\gamma_{i+1} - \partial\gamma_i = 0$. We say that $\underline{\gamma} = \langle\gamma_i\rangle$ is a *true n-dimensional cycle in* (X, ϱ) *over* D if $\underline{\gamma}' = \langle\gamma_{i+1} - \gamma_i\rangle \sim \underline{0}$ in (X, ϱ). We denote by $Z_n^t(X, \varrho, D)$ the collection of all such true cycles in $Z_n^{\infty}(X, \varrho, D)$. The collection of all those true cycles $\underline{\gamma} \in Z_n^t(X, \varrho, D)$ such that $\underline{\gamma} \sim \underline{0}$ in (X, ϱ) is denoted $B_n^t(X, \varrho, D)$ and its elements are called *true n-dimensional boundaries in* (X, ϱ) *over* D.

THEOREM 6. *For each* $n \in \tilde{Z}$, $Z_n^t(X, \varrho, D)$ *is a subgroup of* $Z_n^{\infty}(X, \varrho, D)$, *and* $B_n^t(X, \varrho, D)$ *is a subgroup of* $Z_n^t(X, \varrho, D)$.

Proof. Suppose $\underline{\gamma}, \underline{\delta} \in Z_n^t(X, \varrho, D)$ so that each of the infinite cycles $\langle\gamma_{i+1} - \gamma_i\rangle$ and $\langle\delta_{i+1} - \delta_i\rangle$ is an element of $B_n^{\infty}(X, \varrho, D)$. Since $\underline{\gamma}, \underline{\delta} \in Z_n^{\infty}(X, \varrho, D)$ we also have $\underline{\beta} = \underline{\gamma} + \underline{\delta} \in Z_n^{\infty}(X, \varrho, D)$. Furthermore

$$\langle\beta_{i+1} - \beta_i\rangle = \langle(\gamma_{i+1} + \delta_{i+1}) - (\gamma_i + \delta_i)\rangle = \langle\gamma_{i+1} - \gamma_i\rangle + \langle\delta_{i+1} - \delta_i\rangle$$

and hence $\langle\beta_{i+1} - \beta_i\rangle \in B_n^{\infty}(X, \varrho, D)$ since it is the sum of two elements of this group. Thus $\underline{\beta} \in Z_n^t(X, \varrho, D)$. Therefore $Z_n^t(X, \varrho, D)$ is closed under addition. Now if $\underline{\gamma} \in Z_n^t(X, \varrho, D)$ so that $\langle\gamma_{i+1} - \gamma_i\rangle \in B_n^{\infty}(X, \varrho, D)$, it follows that

$$\langle(-\gamma_{i+1}) - (-\gamma_i)\rangle = -\langle\gamma_{i+1} - \gamma_i\rangle \in B_n^{\infty}(X, \varrho, D)$$

and hence $-\underline{\gamma} \in Z_n^t(X, \varrho, D)$. This proves that $Z_n^t(X, \varrho, D)$ is a subgroup of $Z_n^{\infty}(X, \varrho, D)$.

The definition of $B_n^t(X, \varrho, D)$ implies that it is the intersection of the two subgroups $Z_n^t(X, \varrho, D)$ and $B_n^{\infty}(X, \varrho, D)$ of $Z_n^{\infty}(X, \varrho, D)$. Thus $B_n^t(X, \varrho, D)$ is a subgroup of each of them.

The quotient group $Z_n^t(X, \varrho, D)/B_n^t(X, \varrho, D)$ is called the *n-dimensional Vietoris homology group of* (X, ϱ) *with coefficients in* D and is denoted $H_n^t(X, \varrho, D)$.

THEOREM 7. *Suppose that (Y, ϱ') is a metric subspace of (X, ϱ). Then for each $n \in \tilde{Z}$:*

(1) $Z_n^{\iota}(Y, \varrho', D)$ is a subgroup of $Z_n^{\iota}(X, \varrho, D)$.

(2) $B_n^{\iota}(Y, \varrho', D)$ is a subgroup of $B_n^{\iota}(X, \varrho, D)$.

Proof. (1) Suppose $\underline{\gamma} \in Z_n^{\iota}(Y, \varrho', D)$ so that $\underline{\gamma} \in Z_n^{\infty}(Y, \varrho', D)$ and $\underline{\gamma}' = \langle \gamma_{i+1} - \gamma_i \rangle \in B_n^{\infty}(Y, \varrho', D)$. Then parts (2) and (3) of Theorem 3 imply that $\underline{\gamma} \in Z_n^{\infty}(X, \varrho, D)$ and $\underline{\gamma}' \in B_n^{\infty}(X, \varrho, D)$. Thus $\underline{\gamma} \in Z_n^{\iota}(X, \varrho, D)$. (2) Since $Z_n^{\iota}(Y, \varrho', D) \subseteq Z_n^{\iota}(X, \varrho, D)$ by part (1), and $B_n^{\infty}(Y, \varrho', D) \subseteq B_n^{\infty}(X, \varrho, D)$ by part (3) of Theorem 3, it follows that $B_n^{\iota}(Y, \varrho', D) = Z_n^{\iota}(Y, \varrho', D) \cap B_n^{\infty}(Y, \varrho', D)$ is contained in $B_n^{\iota}(X, \varrho, D) = Z_n^{\iota}(X, \varrho, D) \cap B_n^{\infty}(X, \varrho, D)$.

3.6. Subsequences of infinite chains. Let (X, ϱ) be a metric space, D an abelian group, and let $n \in \tilde{Z}$.

THEOREM 8. *A subsequence of an infinite chain is an infinite chain. More precisely, if $\underline{\varkappa} \in C_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon}$, carrier X_0, and $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$, then $\underline{\varkappa} \cdot \underline{m} \in C_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon} \cdot \underline{m}$, carrier X_0.*

Proof. Evidently $\underline{\lambda} = \underline{\varkappa} \cdot \underline{m} \in C_n^{*}(X, D)$ and $\lambda_i = \varkappa_{m(i)} \in C_n^{\varepsilon(m(i))}(X, D)$ for $i \in \tilde{N}$. Since $\underline{\varepsilon} \cdot \underline{m} \to 0$, $\underline{\varepsilon} \cdot \underline{m}$ is a majorant for $\underline{\lambda}$. And X_0 is a carrier for $\underline{\lambda}$ since $V(\underline{\lambda}) \subseteq V(\underline{\varkappa})$.

THEOREM 9. *Each subsequence-forming operator is a homomorphism of the infinite chain group. More precisely, if $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$ and $\underline{\varkappa}, \underline{\varkappa}' \in C_n^{\infty}(X, D)$, then $(\underline{\varkappa} + \underline{\varkappa}') \cdot \underline{m} = (\underline{\varkappa} \cdot \underline{m}) + (\underline{\varkappa}' \cdot \underline{m})$.*

Proof. Let $i \in \tilde{N}$. Then $((\underline{\varkappa} + \underline{\varkappa}') \cdot \underline{m})(i) = (\underline{\varkappa} + \underline{\varkappa}')(m_i)$ while

$$((\underline{\varkappa} \cdot \underline{m}) + (\underline{\varkappa}' \cdot \underline{m}))(i) = (\underline{\varkappa} \cdot \underline{m})(i) + (\underline{\varkappa}' \cdot \underline{m})(i) = \underline{\varkappa}(m_i) + \underline{\varkappa}'(m_i),$$

and these two expressions are equal.

THEOREM 10. *The boundary operator commutes with each subsequence-forming operator. More precisely, if $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$ and $\underline{\varkappa} \in C_n^{\infty}(X, D)$, then $\partial(\underline{\varkappa} \cdot \underline{m}) = (\partial \underline{\varkappa}) \cdot \underline{m}$.*

Proof. For $i \in \tilde{N}$, $(\partial(\underline{\varkappa} \cdot \underline{m}))(i) = \partial((\underline{\varkappa} \cdot \underline{m})(i)) = \partial(\varkappa_{\underline{m}(i)})$, and $((\partial \underline{\varkappa}) \cdot \underline{m})(i) = (\partial \underline{\varkappa})(\underline{m}(i)) = \partial(\varkappa_{\underline{m}(i)})$.

THEOREM 11. *A subsequence of an infinite cycle is an infinite cycle. More precisely, if $\underline{\gamma} \in Z_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon}$, carrier X_0, and $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$, then $\underline{\gamma} \cdot \underline{m} \in Z_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon} \cdot \underline{m}$, carrier X_0.*

Proof. By Theorem 8, $\underline{\gamma} \cdot \underline{m} \in C_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon} \cdot \underline{m}$, carrier X_0. By Theorem 10, $\partial(\underline{\gamma} \cdot \underline{m}) = (\partial \underline{\gamma}) \cdot \underline{m} = \underline{0}$.

THEOREM 12. *A subsequence of an infinite boundary is an infinite boundary. More precisely, suppose $\underline{\gamma} \in B_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon}$, carrier X_0, and $\underline{\gamma} = \partial \underline{\varkappa}$, where $\underline{\varkappa} \in C_{n+1}^{\infty}(X, D)$ with majorant $\underline{\eta}$, carrier Y_0, and suppose $\underline{m}$: $\tilde{N} \nearrow \tilde{N}$. Then $\underline{\gamma} \cdot \underline{m} \in B_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon} \cdot \underline{m}$, carrier X_0, and $\underline{\gamma} \cdot \underline{m} = \partial(\underline{\varkappa} \cdot \underline{m})$, where $\underline{\varkappa} \cdot \underline{m} \in C_{n+1}^{\infty}(X, D)$ with majorant $\underline{\eta} \cdot \underline{m}$, carrier Y_0.*

Proof. By Theorem 8, $\underline{\gamma}\cdot\underline{m}\in C_n^{\infty}(X, D)$ with majorant $\underline{\varepsilon}\cdot\underline{m}$, carrier X_0, and $\underline{\varkappa}\cdot\underline{m}\in C_{n+1}^{\infty}(X, D)$ with majorant $\underline{\eta}\cdot\underline{m}$, carrier Y_0. By Theorem 10, $\partial(\underline{\varkappa}\cdot\underline{m}) = (\partial\underline{\varkappa})\cdot\underline{m} = \underline{\gamma}\cdot\underline{m}$ so that $\underline{\gamma}\cdot\underline{m}\in B_n^{\infty}(X, D)$.

THEOREM 13. *Corresponding subsequences of homologous cycles are homologous. More precisely, if* $\underline{\gamma}\sim\underline{\delta}$ *in* X, *where* $\underline{\gamma}, \underline{\delta}\in Z_n^{\infty}(X, D)$, *and if* $\underline{m}$: $\tilde{N}\nearrow\tilde{N}$, *then* $\underline{\gamma}\cdot\underline{m}\sim\underline{\delta}\cdot\underline{m}$ *in* X.

Proof. Theorem 11 implies that $\underline{\gamma}\cdot\underline{m}$, $\underline{\delta}\cdot\underline{m}\in Z_n^{\infty}(X, D)$. Now if $\underline{\gamma}\sim\underline{\delta}$ in X, then $\underline{\gamma}-\underline{\delta}\in B_n^{\infty}(X, D)$, and thus by Theorem 12, $(\underline{\gamma}-\underline{\delta})\cdot\underline{m}\in B_n^{\infty}(X, D)$. But by Theorem 9, $(\underline{\gamma}-\underline{\delta})\cdot\underline{m} = (\underline{\gamma}\cdot\underline{m})-(\underline{\delta}\cdot\underline{m})$, and therefore $\underline{\gamma}\cdot\underline{m}\sim\underline{\delta}\cdot\underline{m}$ in X.

THEOREM 14. *An infinite cycle is true if and only if it is homologous to its tail. More precisely, if* $\underline{\gamma}\in Z_n^{\infty}(X, D)$, *then* $\underline{\gamma}\in Z_n^{t}(X, D)$ *if and only if* $\underline{\gamma}\sim t_1(\underline{\gamma})$ *in* X.

Proof. The definition of true cycle implies $\underline{\gamma}\in Z_n^{t}(X, D)$ if and only if $\underline{\gamma}' = \langle\gamma_{i+1}-\gamma_i\rangle\in B_n^{\infty}(X, D)$. But since $\underline{\gamma}' = t_1(\underline{\gamma})-\underline{\gamma}$ the latter condition is satisfied if and only if $\underline{\gamma}\sim t_1(\underline{\gamma})$ in X.

THEOREM 15. *Each infinite boundary is a true cycle; thus infinite boundaries are the same as true boundaries. More precisely,* $B_n^{\infty}(X, D)\subseteq Z_n^{t}(X, D)$; *thus* $B_n^{\infty}(X, D) = B_n^{t}(X, D)$.

Proof. If $\underline{\gamma}\in B_n^{\infty}(X, D)$, then by Theorem 12, $t_1(\underline{\gamma})\in B_n^{\infty}(X, D)$ also. Hence $\underline{\gamma}-t_1(\underline{\gamma})\in B_n^{\infty}(X, D)$ so that $\underline{\gamma}\sim t_1(\underline{\gamma})$ in X, and consequently by Theorem 14, $\underline{\gamma}\in Z_n^{t}(X, D)$. The equality of $B_n^{\infty}(X, D)$ and $B_n^{t}(X, D)$ follows immediately since $B_n^{t}(X, D)\cap B_n^{\infty}(X, D)$ by definition.

THEOREM 16. *An infinite cycle homologous to a true cycle is true. More precisely, if* $\underline{\gamma}\in Z_n^{t}(X, D)$, $\underline{\delta}\in Z_n^{\infty}(X, D)$, *and* $\underline{\delta}\sim\underline{\gamma}$ *in* X, *then* $\underline{\delta}\in Z_n^{t}(X, D)$.

Proof. Since $\underline{\delta}\sim\underline{\gamma}$ in X, Theorem 13 implies that $t_1(\underline{\delta})\sim t_1(\underline{\gamma})$ in X. Theorem 14 implies that $\underline{\gamma}\sim t_1(\underline{\gamma})$ in X. Thus we have $\underline{\delta}\sim\underline{\gamma}$ in X, $\underline{\gamma}\sim t_1(\underline{\gamma})$ in X, and $t_1(\underline{\gamma})\sim t_1(\underline{\delta})$ in X. Transitivity of the relation "$\sim$ in X" implies $\underline{\delta}\sim t_1(\underline{\delta})$ in X, and consequently $\underline{\delta}\in Z_n^{t}(X, D)$ by Theorem 14.

THEOREM 17. *A subsequence of a true cycle is an infinite cycle homologous to it. More precisely, if* $\underline{\gamma}\in Z_n^{t}(X, D)$ *and* $\underline{m}$: $\tilde{N}\nearrow\tilde{N}$, *then* $\underline{\gamma}\cdot\underline{m}\in Z_n^{\infty}(X, D)$ *and* $\underline{\gamma}\cdot\underline{m}\sim\underline{\gamma}$ *in* X.

Proof. By Theorem 11, $\underline{\gamma}\cdot\underline{m}\in Z_n^{\infty}(X, D)$. Since $\underline{\gamma}\in Z_n^{t}(X, D)$ there exists $\underline{\varkappa}\in C_{n+1}^{\infty}(X, D)$ with majorant $\underline{\eta}$, carrier Y_0, such that $\partial\underline{\varkappa} = \langle\gamma_{i+1}-\gamma_i\rangle$. We must show $\underline{\gamma}\cdot\underline{m}-\underline{\gamma}\in B_n^{\infty}(X, D)$. Now $\underline{\gamma}\cdot\underline{m}-\underline{\gamma} = \langle\gamma_{\underline{m}(j)}-\gamma_j\rangle$ and

$$\gamma_{\underline{m}(j)}-\gamma_j = (\gamma_{j+1}-\gamma_j)+(\gamma_{j+2}-\gamma_{j+1})+\dots+(\gamma_{\underline{m}(j)}-\gamma_{\underline{m}(j)-1})$$

which in turn may be written

$$\partial\varkappa_j+\partial\varkappa_{j+1}+\dots+\partial\varkappa_{\underline{m}(j)-1},$$

where $\varkappa_j \in C^{\eta_j}_{n+1}(X, D)$,

$$\varkappa_{j+1} \in C^{\eta_{j+1}}_{n+1}(X, D), \ldots, \varkappa_{\underline{m}(j)-1} \in C^{\eta_{\underline{m}(j)-1}}_{n+1}(X, D).$$

If we let

$$\tilde{\eta}_j = \max\{\eta_j, \eta_{j+1}, \ldots, \eta_{\underline{m}(j)-1}\},$$

then

$$\varkappa_j, \varkappa_{j+1}, \ldots, \varkappa_{\underline{m}(j)-1} \in C^{\tilde{\eta}_j}_{n+1}(X, D),$$

and hence

$$\tilde{\varkappa}_j = \varkappa_j + \varkappa_{j+1} + \ldots + \varkappa_{\underline{m}(j)-1} \in C^{\tilde{\eta}_j}_{n+1}(X, D)$$

also. Now $\underline{\tilde{\varkappa}} = \langle\tilde{\varkappa}_j\rangle \in C^*_{n+1}(X, D)$ and $\partial\underline{\tilde{\varkappa}} = \langle\partial\tilde{\varkappa}_j\rangle = \langle\gamma_{\underline{m}(j)} - \gamma_j\rangle = \underline{\gamma}\cdot\underline{m} - \underline{\gamma}$. It remains only to show that $\underline{\tilde{\varkappa}} \in C^{\infty}_{n+1}(X, D)$. Clearly Y_0 is a carrier for $\underline{\tilde{\varkappa}}$ since $V(\underline{\tilde{\varkappa}}) \subseteq V(\underline{\varkappa}) \subseteq Y_0$. We show $\underline{\tilde{\eta}} = \langle\tilde{\eta}_j\rangle$ is a majorant for $\underline{\tilde{\varkappa}}$ by proving $\underline{\tilde{\eta}} \to 0$. Let $\delta \in \tilde{R}_+$ be given. Since $\underline{\eta} \to 0$ there exists $i_0 \in \tilde{N}$ such that $i \geqslant i_0$ implies $0 < \eta_i < \delta$. Thus $j \geqslant i_0$ implies $\eta_j, \eta_{j+1}, \ldots, \eta_{\underline{m}(j)-1} < \delta$ and so $\tilde{\eta}_j < \delta$ also. This proves that $\underline{\tilde{\varkappa}} \in C^{\infty}_{n+1}(X, D)$ and therefore that $\underline{\gamma}\cdot\underline{m} - \underline{\gamma} = \partial\underline{\tilde{\varkappa}} \in B^{\infty}_n(X, D)$.

THEOREM 18. *A subsequence of a true cycle is a true cycle. More precisely, if* $\underline{\gamma} \in Z^t_n(X, D)$ *with majorant* $\underline{\varepsilon}$, *carrier* X_0, *and* $\underline{m}\colon \tilde{N} \nearrow \tilde{N}$, *then* $\underline{\gamma}\cdot\underline{m} \in Z^t_n(X, D)$ *with majorant* $\underline{\varepsilon}\cdot\underline{m}$, *carrier* X_0.

Proof. By Theorem 11, $\underline{\gamma}\cdot\underline{m} \in Z^{\infty}_n(X, D)$ with majorant $\underline{\varepsilon}\cdot\underline{m}$, carrier X_0. By Theorem 17, $\underline{\gamma}\cdot\underline{m} \sim \underline{\gamma}$ in X, and consequently by Theorem 16, $\underline{\gamma}\cdot\underline{m} \in Z^t_n(X, D)$.

3.7. A condition for homology of infinite cycles. Let (X, ϱ) be a metric space, D an abelian group, and let $n \in \tilde{Z}$.

THEOREM 19. *Let* $\underline{\gamma}, \underline{\gamma}'$ *be elements of* $Z^{\infty}_n(X, D)$. *Assume there exists a compact set* $X_0 \subseteq X$ *such that the following two conditions are satisfied*:

(1) $V(\underline{\gamma}) \cup V(\underline{\gamma}') \subseteq X_0$.

(2) *For each* $\eta \in \tilde{R}_+$ *there exists* $i_0 \in \tilde{N}$ *such that* $i \geqslant i_0$ *implies* $\gamma_i \underset{\eta}{\sim} \gamma'_i$ *in* X_0.

Then $\underline{\gamma} \sim \underline{\gamma}'$ *in* X_0.

Proof. For each $\eta \in \tilde{R}_+$, let $i(\eta)$ denote the least natural number i_0 such that $i \geqslant i_0$ implies $\gamma_i \underset{\eta}{\sim} \gamma'_i$ in X_0. Then $i \geqslant i(\eta)$ implies $\gamma_i \underset{\eta}{\sim} \gamma'_i$ in X_0, or equivalently, $i \geqslant i(\eta)$ implies there exists $\varkappa_i \in C^{\eta}_{n+1}(X_0, D)$ such that $\partial\varkappa_i = \gamma_i - \gamma'_i$. It is clear from the definition of $i(\eta)$ that $\eta_1 < \eta_2$ implies $i(\eta_1) \geqslant i(\eta_2)$. We now apply these remarks to the case where the numbers η are reciprocals of natural numbers. For each $k \in \tilde{N}$, $i(1/k) \leqslant i(1/(k+1))$ so that $k + i(1/k) < (k+1) + i(1/(k+1))$. If $i \geqslant k + i(1/k)$, then $i \geqslant i(1/k)$ and thus there exists $\varkappa_i \in C^{1/k}_{n+1}(X_0, D)$ such that $\partial\varkappa_i = \gamma_i - \gamma'_i$. We define a sequential chain $\underline{\varkappa} = \langle\varkappa_i\rangle \in C^*_{n+1}(X, D)$ and a sequence $\zeta = \langle\zeta_i\rangle$ into $\tilde{R}_+$ as follows.

Let b be an arbitrary point of X_0. For $1 \leqslant i < 1+i(1)$, define $\varkappa_i = b\cdot(\gamma_i - \gamma_i')$ and let $\zeta_i = 1+\max\{\Delta(V(\sigma)): \sigma\in\Sigma(\varkappa_i)\}$. Let $k\in\tilde{N}$. For

$$k+i(1/k) \leqslant i < (k+1)+i(1/(k+1)),$$

define $\varkappa_i$ to be an element of $C_{n+1}^{1/k}(X_0, D)$ such that $\partial\varkappa_i = \gamma_i - \gamma_i'$, and let $\zeta_i = 1/k$. Thus for each $i\in\tilde{N}$, $\varkappa_i\in C_{n+1}^{\zeta_i}(X_0, D)$ and $\partial\varkappa_i = \gamma_i-\gamma_i'$; moreover, $\underline{\zeta} = \langle\zeta_i\rangle \to 0$. It follows that $\underline{\varkappa} = \langle\varkappa_i\rangle \in C_{n+1}^{\infty}(X_0, D)$ and $\partial\underline{\varkappa} = \underline{\gamma}-\underline{\gamma}'$. Therefore $\underline{\gamma} \sim \underline{\gamma}'$ in X_0.

Chapter IV

Functions, mappings, and null translations

4.1. Homomorphisms of simple chains induced by functions. Let S and T be sets and let $f\colon S\to T$ be a function. For each $\sigma = [a_0 \dots a_n]\in\Sigma_n(S)$ we define $f'(\sigma)$ to be the simplex $[f(a_0)\dots f(a_n)]\in\Sigma_n(T)$. Thus f induces a function $f'\colon \Sigma_n(S)\to\Sigma_n(T)$. Now let D be an abelian group. For each $\varkappa = d_1\sigma_1+\dots+d_k\sigma_k\in C_n^{\#}(S, D)$ we define $f''(\varkappa)$ to be the chain $d_1 f'(\sigma_1)+\dots+d_k f'(\sigma_k)\in C_n^{\#}(T, D)$. It is clear that $f''(\varkappa+\varkappa') = f''(\varkappa)+f''(\varkappa')$ for all $\varkappa, \varkappa'\in C_n^{\#}(S, D)$. An easily proved theorem summarizes properties of f''.

THEOREM 20. *Let S and T be sets and let D be an abelian group. Each function $f\colon S\to T$ induces a homomorphism $f''\colon C_n^{\#}(S, D)\to C_n^{\#}(T, D)$ such that the following conditions are satisfied:*

(1) *f'' commutes with the boundary operator; that is, $f''\partial = \partial f''\colon C_n^{\#}(S, D)\to C_{n-1}^{\#}(T, D)$.*

(2) *f'' carries cycles into cycles; that is, $f''(Z_n^{\#}(S, D)) \subseteq Z_n^{\#}(T, D)$.*

(3) *f'' carries boundaries into boundaries; that is, $f''(B_n^{\#}(S, D)) \subseteq B_n^{\#}(T, D)$.*

(4) id$''$ = id; *that is, if $f\colon S\to S$ is the identity function, then $f''\colon C_n^{\#}(S, D)\to C_n^{\#}(S, D)$ is the identity homomorphism.*

(5) $(gf)'' = g''f''$; *that is, if $f\colon R\to S$ and $g\colon S\to T$ are functions (where R is a set), then $(gf)'' = g''f''\colon C_n^{\#}(R, D)\to C_n^{\#}(T, D)$.*

(6) *f'' interacts with the join operator J_b (where $b\in S$) according to the rule $f''J_b = J_{f(b)}f''\colon C_n^{\#}(S, D)\to C_{n+1}^{\#}(T, D)$.*

4.2. Homomorphisms of sequential chains induced by functions. Let S and T be sets and let D be an abelian group. For each function $f\colon S\to T$ we have defined a homomorphism $f''\colon C_n^{\#}(S, D)\to C_n^{\#}(T, D)$. We now define a function $f'''\colon C_n^{*}(S, D)\to C_n^{*}(T, D)$ by the rule $f'''(\langle\varkappa_i\rangle) = \langle f''(\varkappa_i)\rangle$ for each $\underline{\varkappa} = \langle \varkappa_i \rangle \in C_n^*(S, D)$. It is easy to see that f''' is a homomorphism. This and other elementary properties of f''' are stated in the next theorem.

THEOREM 21. *Let S and T be sets and let D be an abelian group. Each function $f\colon S \to T$ induces a homomorphism $f'''\colon C_n^*(S, D) \to C_n^*(T, D)$ such that the following conditions are satisfied.*

(1) *f''' commutes with the boundary operator; that is, $f'''\partial = \partial f'''\colon C_n^*(S, D) \to C_{n-1}^*(T, D)$.*

(2) *f''' carries cycles into cycles; that is, $f'''(Z_n^*(S, D)) \subseteq Z_n^*(T, D)$.*

(3) *f''' carries boundaries into boundaries; that is, $f'''(B_n^*(S, D)) \subseteq B_n^*(T, D)$.*

(4) *$\mathrm{id}''' = \mathrm{id}$; that is, if $f\colon S \to S$ is the identity function, then $f'''\colon C_n^*(S, D) \to C_n^*(S, D)$ is the identity homomorphism.*

(5) *$(gf)''' = g'''f'''$; that is, if $f\colon R \to S$ and $g\colon S \to T$ are functions (where R is a set), then $(gf)''' = g'''f'''\colon C_n^*(R, D) \to C_n^*(T, D)$.*

(6) *f''' commutes with each subsequence-forming operator; that is, if $\underline{m}\colon \tilde{N} \nearrow \tilde{N}$ and $\underline{\varkappa} \in C_n^*(S, D)$, then $f'''(\underline{\varkappa} \cdot \underline{m}) = (f'''(\underline{\varkappa})) \cdot \underline{m}$.*

4.3. Homomorphisms of ε-chains induced by functions. Let (X, ϱ) and (Y, ϱ') be metric spaces and let D be an abelian group. Suppose that $f\colon X \to Y$ is a function (not necessarily continuous). For each $\varepsilon \in \tilde{R}_+$ we know that $\Sigma_n^\varepsilon(X, \varrho) \subseteq \Sigma_n(X)$. Thus the induced function $f'\colon \Sigma_n(X) \to \Sigma_n(Y)$ defines a function $f'\colon \Sigma_n^\varepsilon(X, \varrho) \to \Sigma_n(Y)$. We also know that $C_n^\varepsilon(X, \varrho, D) \subseteq C_n^{\#}(X, D)$ so that the homomorphism $f''\colon C_n^{\#}(X, D) \to C_n^{\#}(Y, D)$ defines a homomorphism $f''\colon C_n^\varepsilon(X, \varrho, D) \to C_n^{\#}(Y, D)$.

Now suppose that $\varepsilon \in \tilde{R}_+$ and $\varkappa = d_1\sigma_1 + \ldots + d_k\sigma_k \in C_n^\varepsilon(X, \varrho, D)$. Let $\eta(\varkappa) \in \tilde{R}_+$ be such that $\eta(\varkappa) > \max\{\Delta_{\varrho'}(V(f'\sigma_i))\colon i = 1, \ldots, k\}$. Then $f''(\varkappa) = d_1 f'(\sigma_1) + \ldots + d_k f'(\sigma_k) \in C_n^{\eta(\varkappa)}(Y, \varrho', D)$. However, it may be false, even if f is continuous, that there exists $\eta \in \tilde{R}_+$ such that $f''(C_n^\varepsilon(X, \varrho, D)) \subseteq C_n^\eta(Y, \varrho', D)$. To prove this statement a simple may be constructed for the case $X = Y = \tilde{R}_+$ by defining $f(x) = 1/x$ for each $x \in X$. On the other hand let $\varepsilon, \eta \in \tilde{R}_+$ and suppose that $\varrho(a, b) < \varepsilon$ implies $\varrho'(f(a), f(b)) < \eta$, for all $a, b \in X$. Then $\sigma \in \Sigma_n^\varepsilon(X, \varrho)$ implies $f'(\sigma) \in \Sigma_n^\eta(Y, \varrho')$ so that f' defines a function $f'\colon \Sigma_n^\varepsilon(X, \varrho) \to \Sigma_n^\eta(Y, \varrho')$. It follows that if $\varkappa \in C_n^\varepsilon(X, \varrho, D)$, then $f''(\varkappa) \in C_n^\eta(Y, \varrho', D)$ and hence f'' defines a homomorphism $f''\colon C_n^\varepsilon(X, \varrho, D) \to C_n^\eta(Y, \varrho', D)$. Because f'' commutes with the boundary operator (by Theorem 20) we see that f'' carries ε-cycles in X into η-cycles in Y and carries ε-boundaries in X into η-boundaries in Y. Thus f'' induces a homomorphism $\hat{f}''\colon H_n^\varepsilon(X, \varrho, D) \to H_n^\eta(Y, \varrho', D)$ defined by the rule $\hat{f}''([\gamma]) = [f''(\gamma)]$ for each $[\gamma] \in H_n^\varepsilon(X, \varrho, D)$. The following theorem summarizes properties of f''.

THEOREM 22. *Let (X, ϱ) and (Y, ϱ') be metric spaces and let D be an abelian group. Suppose that for some $\varepsilon, \eta \in \tilde{R}_+$ a function $f\colon X \to Y$ satisfies* *the condition that* $\varrho(a,b) < \varepsilon$ *implies* $\varrho'(f(a), f(b)) < \eta$, *for all* $a, b \in X$. *Then the homomorphism* f'': $C_n^{\#}(X, D) \to C_n^{\#}(Y, D)$ *defines a homomorphism* f'': $C_n^{\varepsilon}(X, \varrho, D) \to C_n^{\eta}(Y, \varrho', D)$ *such that the following statements hold*:

(1) $f''\partial = \partial f''$: $C_n^{\varepsilon}(X, \varrho, D) \to C_{n-1}^{\eta}(Y, \varrho', D)$.

(2) $f''(Z_n^{\varepsilon}(X, \varrho, D)) \subseteq Z_n^{\eta}(Y, \varrho', D)$.

(3) $f''(B_n^{\varepsilon}(X, \varrho, D)) \subseteq B_n^{\eta}(Y, \varrho', D)$.

(4) f'' *induces a homomorphism* $\hat{f}''$: $H_n^{\varepsilon}(X, \varrho, D) \to H_n^{\eta}(Y, \varrho', D)$.

4.4. Homomorphisms of infinite chains induced by maps. Let (X, ϱ) and (Y, ϱ') be metric spaces and let D be an abelian group. Suppose that f: $X \to Y$ is a map; that is, f is a continuous function. Since $C_n^{\infty}(X, \varrho, D) \subseteq C_n^{*}(X, D)$ the homomorphism f''': $C_n^{*}(X, D) \to C_n^{*}(Y, D)$ defines a homomorphism f''': $C_n^{\infty}(X, \varrho, D) \to C_n^{*}(Y, D)$. We will show that $f'''(C_n^{\infty}(X, \varrho, D)) \subseteq C_n^{\infty}(Y, \varrho', D)$. It will then follow that f''' defines a homomorphism f''': $C_n^{\infty}(X, \varrho, D) \to C_n^{\infty}(Y, \varrho', D)$ which commutes with the boundary operator and which in turn induces a homomorphism $\hat{f}'''$: $H_n^{\infty}(X, \varrho, D) \to H_n^{\infty}(Y, \varrho', D)$. We first prove a lemma.

THEOREM 23. *Let* (X, ϱ) *and* (Y, ϱ') *be metric spaces and assume that* X *is compact. Then for each map* f: $X \to Y$ *and for each* $\varepsilon \in \tilde{R}_+$ *there exists* $\hat{\eta}(f, \varepsilon) \in \tilde{R}_+$ *such that the following conditions hold*:

(1) $\varrho(a, b) < \varepsilon$ *implies* $\varrho'(f(a), f(b)) < \hat{\eta}(f, \varepsilon)$ *for all* $a, b \in X$.

(2) $\hat{\eta}(f, \varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Let f be a map of X into Y. For each $\varepsilon \in \tilde{R}_+$ we define $\eta(\varepsilon)$ to be the greatest lower bound of the non-empty set $S(\varepsilon) = \{\eta \in \tilde{R}_+ : \varrho(a, b) < \varepsilon$ implies $\varrho'(f(a), f(b)) < \eta$ for all $a, b \in X\}$. That $S(\varepsilon)$ is non-empty is a consequence of the compactness of X, for this in turn implies that $f(X)$ is a compact subset of Y and hence has finite diameter $d = \Delta\varrho'(f(X))$. Then for all $a, b \in X$, $\varrho'(f(a), f(b)) < d+1$ so that $d+1 \in S(\varepsilon)$. It is clear from the definition of $\eta(\varepsilon)$ that if $a, b \in X$ and $\varrho(a, b) < \varepsilon$, then $\varrho'(f(a), f(b)) \leqslant \eta(\varepsilon)$. We prove now that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $s \in \tilde{R}_+$ be given. We must show there exists $t \in \tilde{R}_+$ such that $0 < \varepsilon < t$ implies $\eta(\varepsilon) < s$. Since f is continuous on the compact set X, it follows that f is uniformly continuous. Thus there exists $t \in \tilde{R}_+$ such that $\varrho(a, b) < t$ implies $\varrho'(f(a), f(b)) < s/2$. Now suppose that $0 < \varepsilon < t$. If $\varrho(a, b) < \varepsilon$, then $\varrho(a, b) < t$ and so $\varrho'(f(a), f(b)) < s/2$. Then $s/2 \in S(\varepsilon)$ and consequently $\eta(\varepsilon) = glb\ S(\varepsilon) \leqslant s/2 < s$. This proves $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. We now define $\hat{\eta}(f, \varepsilon) = \eta(\varepsilon) + \varepsilon$, for each $\varepsilon \in \tilde{R}_+$. Then condition (1) is satisfied since if $a, b \in X$ and $\varrho(a, b) < \varepsilon$, then $\varrho'(f(a), f(b)) \leqslant \eta(\varepsilon) < \eta(\varepsilon) + \varepsilon = \hat{\eta}(f, \varepsilon)$, and condition (2) is satisfied since as $\varepsilon \to 0$, $\eta(\varepsilon) \to 0$, and hence $\eta(\varepsilon) + \varepsilon = \hat{\eta}(f, \varepsilon) \to 0$.

Theorem 23 enables us to prove the remark made earlier that $f'''(C_n^{\infty}(X, \varrho, D)) \subseteq C_n^{\infty}(Y, \varrho', D)$.

THEOREM 24. *Let* (X, ϱ) *and* (Y, ϱ') *be metric spaces and let* D *be an* *abelian group. Suppose that* $f: X \to Y$ *is a map. Then the homomorphism* $f''': C_n^{\infty}(X, \varrho, D) \to C_n^*(Y, D)$ *satisfies the condition* $f'''(C_n^{\infty}(X, \varrho, D)) \subseteq C_n^{\infty}(Y, \varrho', D)$.

Proof. Let $\underline{\varkappa} \in C_n^{\infty}(X, \varrho, D)$ with majorant $\underline{\varepsilon}$, carrier X_0. Then by Theorem 4, $\underline{\varkappa} \in C_n^{\infty}(X_0, \varrho, D)$ and hence for each $i \in \tilde{N}$, $\varkappa_i \in C_n^{\varepsilon_i}(X_0, \varrho, D)$. We must show $f'''(\underline{\varkappa}) = \langle f''(\varkappa_i)\rangle$ is an element of $C_n^{\infty}(Y, \varrho', D)$. To do this it is sufficient to show that (1) there is a sequence $\underline{\eta}$ into $\tilde{R}_+$ with $\underline{\eta} \to 0$ such that for each $i \in \tilde{N}$, $f''(\varkappa_i)$ $C_n^{\eta_i}(Y, \varrho', D)$, and (2) there exists a compact set $Y_0 \subseteq Y$ such that $V(f'''(\underline{\varkappa})) \subseteq Y_0$. (1) Since X_0 is compact and the restriction of f is a map of X_0 into Y, Theorem 23 may be applied. Thus for each $i \in \tilde{N}$ there exists $\eta_i = \hat{\eta}(f, \varepsilon_i) \in \tilde{R}_+$ such that for all $a, b \in X_0$, $\varrho(a, b) < \varepsilon_i$ implies $\varrho'(f(a), f(b)) < \eta_i$. Moreover, $\underline{\eta} \to 0$ since $\underline{\varepsilon} \to 0$. Since the hypotheses of Theorem 22 are satisfied (for $X = X_0$, $\mathfrak{k} = \varepsilon_i$, $\eta = \eta_i$) we conclude that for each $i \in \tilde{N}$, f'' carries $C_n^{\varepsilon_i}(X_0, \varrho, D)$ into $C_n^{\eta_i}(Y, \varrho', D)$ and hence $f''(\varkappa_i) \in C_n^{\eta_i}(Y, \varrho', D)$. (2) Let $Y_0 = f(X_0)$. Then Y_0 is compact since X_0 is compact and f is continuous. And $V(f'''(\underline{\varkappa})) \subseteq Y_0$ since for each $i \in \tilde{N}$ the condition $V(\varkappa_i) \subseteq X_0$ implies that $V(f''(\varkappa_i)) \subseteq f(X_0) = Y_0$.

It follows from Theorem 24 that f''' defines a homomorphism of $C_n^{\infty}(X, \varrho, D)$ into $C_n^{\infty}(Y, \varrho', D)$. We summarize properties of this homomorphism in the following theorem:

THEOREM 25. *Let* (X, ϱ) *and* (Y, ϱ') *be metric spaces and let* D *be an abelian group. Suppose that* $f: X \to Y$ *is a map. Then the homomorphism* $f''': C_n^{\infty}(X, \varrho, D) \to C_n^*(Y, D)$ *defines a homomorphism* $f''': C_n^{\infty}(X, \varrho, D) \to$ $\to C_n^{\infty}(Y, \varrho', D)$ *such that the following conditions are satisfied*:

(1) $f'''\partial = \partial f''': C_n^{\infty}(X, \varrho, D) \to C_{n-1}^{\infty}(Y, \varrho', D)$.

(2) $f'''(Z_n^{\infty}(X, \varrho, D)) \subseteq Z_n^{\infty}(Y, \varrho', D)$.

(3) $f'''(B_n^{\infty}(X, \varrho, D)) \subseteq B_n^{\infty}(Y, \varrho', D)$.

(4) f''' *induces a homomorphism* $\hat{f}''': H_n^{\infty}(X, \varrho, D) \to H_n^{\infty}(Y, \varrho', D)$.

(5) $f'''(Z_n^t(X, \varrho, D)) \subseteq Z_n^t(Y, \varrho', D)$.

(6) $f'''(B_n^t(X, \varrho, D)) \subseteq B_n^t(Y, \varrho', D)$.

(7) f''' *induces a homomorphism* $\hat{f}''': H_n^t(X, \varrho, D) \to H_n^t(Y, \varrho', D)$.

Proof. Only (5) requires further discussion. Suppose $\underline{\gamma} \in Z_n^t(X, \varrho, D)$. Then $t_1(\underline{\gamma}) - \underline{\gamma} \in B_n^{\infty}(X, \varrho, D)$ and (3) implies that $f'''(t_1(\underline{\gamma}) - \underline{\gamma}) \in B_n^{\infty}(Y, \varrho', D)$. But $f'''(t_1(\underline{\gamma}) - \underline{\gamma}) = f'''(t_1(\underline{\gamma})) - f'''(\underline{\gamma}) = t_1(f'''(\underline{\gamma})) - f'''(\underline{\gamma})$, where the last equality is obtained from part (6) of Theorem 21. Then $t_1(f'''(\underline{\gamma})) -$ $- f'''(\underline{\gamma}) \in B_n^{\infty}(Y, \varrho', D)$ and this is equivalent to the statement $f'''(\underline{\gamma}) \in$ $\in Z_n^t(Y, \varrho', D)$.

From this point on we shall usually delete the notations f', f'', f''', $\hat{f}''$, $\hat{f}'''$ and refer to all of these functions simply as f.

4.5. Topological invariance of the ceneral and Vietoris homology groups. Let X be a metrizable topological space and let D be an abelian group.

We show first that the general and Vietoris homology groups of the metric space (X, ϱ) are independent of the choice of the metric ϱ, provided ϱ is a metric which induces the given topology for X. To prove this we suppose that ϱ and ϱ' are metrics for X both inducing the given topology. We know that $C_n^\infty(X, \varrho, D)$ and $C_n^\infty(X, \varrho', D)$ are subsets of $C_n^*(X, D)$. The identity function φ: $(X, \varrho) \to (X, \varrho')$ is continuous and thus by Theorem 25 induces a homomorphism φ''': $C_n^\infty(X, \varrho, D) \to C_n^\infty(X, \varrho', D)$. Moreover, φ''' is the identity homomorphism by Theorem 21. It follows that $C_n^\infty(X, \varrho, D)$ and $C_n^\infty(X, \varrho', D)$ are the same subset of $C_n^*(X, D)$. If $\underline{\gamma} \in Z_n^\infty(X, \varrho, D)$, then $\underline{\gamma} \in C_n^\infty(X, \varrho, D) = C_n^\infty(X, \varrho', D)$ and $\partial\gamma_i = 0$ for each $i \in \tilde{N}$, so that $\underline{\gamma} \in Z_n^\infty(X, \varrho', D)$. Conversely $Z_n^\infty(X, \varrho', D) \subseteq Z_n^\infty(X, \varrho, D)$. If $\underline{\gamma} \in B_n^\infty(X, \varrho, D)$, then $\underline{\gamma} \in Z_n^\infty(X, \varrho, D) = Z_n^\infty(X, \varrho', D)$ and $\underline{\gamma} = \partial\underline{\varkappa}$ for some $\underline{\varkappa} \in C_{n+1}^\infty(X, \varrho, D)$ $= C_{n+1}^\infty(X, \varrho', D)$, so that $\underline{\gamma} \in B_n^\infty(X, \varrho', D)$. Conversely $B_n^\infty(X, \varrho', D)$ $\subseteq B_n^\infty(X, \varrho, D)$. It is an immediate consequence of these statements that $H_n^\infty(X, \varrho, D) = H_n^\infty(X, \varrho', D)$. Suppose now that $\underline{\gamma} \in Z_n^t(X, \varrho, D)$. Then $\underline{\gamma} \in Z_n^\infty(X, \varrho, D) = Z_n^\infty(X, \varrho', D)$ and $t_1(\underline{\gamma}) - \underline{\gamma} \in B_n^\infty(X, \varrho, D) = B_n^\infty(X, \varrho', D)$, so that $\underline{\gamma} \in Z_n^t(X, \varrho', D)$. Conversely $Z_n^t(X, \varrho', D) \subseteq Z_n^t(X, \varrho, D)$. Moreover, $B_n^t(X, \varrho, D) = B_n^\infty(X, \varrho, D)$ and $B_n^t(X, \varrho', D) = B_n^\infty(X, \varrho', D)$ by Theorem 15, so that $B_n^t(X, \varrho, D) = B_n^t(X, \varrho', D)$. Consequently $H_n^t(X, \varrho, D) = H_n^t(X, \varrho', D)$. Thus we have the following theorem:

THEOREM 26. *Let X be a metrizable topological space and let D be an abelian group. Suppose that ϱ and ϱ' are metrics for X both of which induce the given topology for X. Then the following statements hold:*

(1) $C_n^\infty(X, \varrho, D) = C_n^\infty(X, \varrho', D)$.

(2) $Z_n^\infty(X, \varrho, D) = Z_n^\infty(X, \varrho', D)$.

(3) $B_n^\infty(X, \varrho, D) = B_n^\infty(X, \varrho', D)$.

(4) $H_n^\infty(X, \varrho, D) = H_n^\infty(X, \varrho', D)$.

(5) $Z_n^t(X, \varrho, D) = Z_n^t(X, \varrho', D)$.

(6) $B_n^t(X, \varrho, D) = B_n^t(X, \varrho', D)$.

(7) $H_n^t(X, \varrho, D) = H_n^t(X, \varrho', D)$.

Now the topological invariance of the general and Vietoris homology groups can be established. Let (X, ϱ) and (Y, ϱ') be metric spaces and let D be an abelian group. Suppose that h is a homeomorphism of (X, ϱ) onto (Y, ϱ'). It will first be shown that $C_n^\infty(X, \varrho, D)$ is isomorphic to $C_n^\infty(Y, \varrho', D)$. We use the homeomorphism h to carry the metric ϱ' for Y back to a new metric $\hat{\varrho}$ for X. That is, we define $\hat{\varrho}(x_1, x_2) = \varrho'(h(x_1), h(x_2))$ for all $x_1, x_2 \in X$. Then $\hat{\varrho}$ is a metric for X and since h is a homeomorphism, $\hat{\varrho}$ and ϱ induce the same topology for X. Thus by Theorem 26, $C_n^\infty(X, \varrho, D)$ $= C_n^\infty(X, \hat{\varrho}, D)$. It is easy to see that $C_n^\infty(X, \hat{\varrho}, D)$ is isomorphic to $C_n^\infty(Y, \varrho', D)$. This follows from the fact that h is an isometry (that is, a distance-preserving homeomorphism) of $(X, \hat{\varrho})$ onto (Y, ϱ'). Then the composition of the identity isomorphism of $C_n^\infty(X,\varrho,D)$ onto $C_n^\infty(X,\hat{\varrho},D)$ and the isomorphism of $C_n^\infty(X,\hat{\varrho},D)$ onto $C_n^\infty(Y,\varrho',D)$ induced by the isometry h is an isomorphism of $C_n^\infty(X,\varrho,D)$ onto $C_n^\infty(Y,\varrho',D)$. We note that the last-mentioned isomorphism is in fact the homomorphism h''': $C_n^\infty(X,\varrho,D)\to C_n^\infty(Y,\varrho',D)$ described in Theorem 25. Then h''' carries $Z_n^\infty(X,\varrho,D)$ onto $Z_n^\infty(Y,\varrho',D)$ and $B_n^\infty(X,\varrho,D)$ onto $B_n^\infty(Y,\varrho',D)$ thus inducing an isomorphism of the quotient groups, that is, of $H_n^\infty(X,\varrho,D)$ onto $H_n^\infty(Y,\varrho',D)$. Moreover, h''' carries $Z_n^t(X,\varrho,D)$ onto $Z_n^t(Y,\varrho',D)$ and $B_n^t(X,\varrho,D)$ onto $B_n^t(Y,\varrho',D)$ similarly inducing an isomorphism of $H_n^t(X,\varrho,D)$ onto $H_n^t(Y,\varrho',D)$. These facts are summarized in the following theorem:

THEOREM 27. *Let (X,ϱ) and (Y,ϱ') be metric spaces and let D be an abelian group. Suppose that h is a homeomorphism of (X,ϱ) onto (Y,ϱ'). Then the homomorphism h''': $C_n^\infty(X,\varrho,D)\to C_n^\infty(Y,\varrho',D)$ is an isomorphism of $C_n^\infty(X,\varrho,D)$ onto $C_n^\infty(Y,\varrho',D)$. Moreover, h''' induces isomorphisms of $H_n^\infty(X,\varrho,D)$ onto $H_n^\infty(Y,\varrho',D)$ and of $H_n^t(X,\varrho,D)$ onto $H_n^t(Y,\varrho',D)$.*

4.6. Non-equivalence of the general and Vietoris homology groups. Let (X,ϱ) be a metric space and let D be an abelian group. We shall prove that if the general homology group $H_n^\infty(X,D)$ is not trivial and if the Vietoris homology group $H_n^t(X,D)$ is finite, then these two groups are not isomorphic. This result will enable us to mention some specific examples of spaces and coefficient groups for which the general homology groups differ from the Vietoris homology groups. We begin by proving a lemma.

THEOREM 28. *Let (X,ϱ) be a metric space and let D be an abelian group. If $Z_n^\infty(X,D)=Z_n^t(X,D)$, then $Z_n^\infty(X,D)=B_n^\infty(X,D)$.*

Proof. Assume $Z_n^\infty(X,D)\neq B_n^\infty(X,D)$. Then there exists an infinite cycle $\underline{\gamma}=\langle\gamma_1,\gamma_2,\ldots\rangle\in Z_n^\infty(X,D)$ such that $\underline{\gamma}$ is not homologous to zero in X. It follows that the infinite cycle $\underline{\gamma}'=\langle\gamma_1,0,\gamma_2,0,\ldots\rangle$ is not true. That is, if we define $\gamma_i'=0$ when i is even and $\gamma_i'=\gamma_{(i+1)/2}$ when i is odd, then $\underline{\gamma}'=\langle\gamma_i'\rangle\notin Z_n^t(X,D)$. For if $\underline{\gamma}'\in Z_n^t(X,D)$, then

$$\underline{\gamma}'-t_1\underline{\gamma}'=\langle\gamma_1,-\gamma_2,\gamma_2,-\gamma_3,\gamma_3,\ldots\rangle\in B_n^\infty(X,D)$$

and consequently $\underline{\gamma}\in B_n^\infty(X,D)$ by Theorem 12 since $\underline{\gamma}$ is a subsequence of $\underline{\gamma}'-t_1\underline{\gamma}'$. This contradiction proves that $Z_n^\infty(X,D)\neq Z_n^t(X,D)$.

We may now prove the result mentioned at the beginning of the section.

THEOREM 29. *Let (X,ϱ) be a metric space and let D be an abelian group. Assume that $H_n^\infty(X,D)\neq 0$ and that the group $H_n^t(X,D)$ is finite. Then $H_n^\infty(X,D)$ is not isomorphic to $H_n^t(X,D)$.*

Proof. Let B denote the group $B_n^\infty(X,D)=B_n^t(X,D)$. Then $B\subseteq Z_n^t(X,D)\subseteq Z_n^\infty(X,D)$. The assumption $H_n^\infty(X,D)\neq 0$ implies that $Z_n^\infty(X,D)\neq B_n^\infty(X,D)$ and hence by Theorem 28 that $Z_n^\infty(X,D)\neq Z_n^t(X,D)$.

Thus $Z_n^\iota(X, D)$ is a proper subgroup of $Z_n^\infty(X, D)$. It follows that $H_n^\iota(X, D) = Z_n^\iota(X, D)/B$ is a proper subgroup of $H_n^\infty(X, D) = Z_n^\infty(X, D)/B$ since $(Z_n^\infty(X, D)/B)/(Z_n^\iota(X, D)/B)$ is isomorphic to $Z_n^\infty(X, D)/Z_n^\iota(X, D)$ by a standard isomorphism theorem. If $H_n^\iota(X, D)$ is finite the conclusion of the theorem now follows since a finite group cannot be isomorphic to a group which properly contains it.

It is a consequence of Theorem 29 that any choice of (X, ϱ), D, and n for which $H_n^\iota(X, D)$ is finite and non-trivial will provide an example where $H_n^\infty(X, D)$ is not isomorphic to $H_n^\iota(X, D)$. We appeal to some classical results to establish the existence of such examples. Lefschetz [19] has proved that for the case of a compact metric space X, each Vietoris homology group $H_n^\iota(X, D)$ is isomorphic to the corresponding Čech homology group $\hat{H}_n(X, D)$. Moreover, for the case of a polyhedron (the point set of a finite simplicial complex) it is well known that each Čech homology group is isomorphic to the corresponding simplicial homology group. Thus for example $H_k^\infty(\tilde{S}^k, F)$ is not isomorphic to $H_k^\iota(\tilde{S}^k, F)$, where $\tilde{S}^k$ is the k-dimensional sphere and F is any non-trivial finite abelian group. That the coefficient group need not be finite in such an example is shown by the fact that $H_1^\infty(\tilde{P}^2, \tilde{Z})$ is not isomorphic to $H_1^\iota(\tilde{P}^2, \tilde{Z})$, where $\tilde{P}^2$ is real projective 2-space and $\tilde{Z}$ is the group of integers. In fact $H_1^\iota(\tilde{P}^2, \tilde{Z})$ is isomorphic to $\tilde{Z}_2$, the group of integers modulo two.

4.7. The homotopy theorem. In this section we prove a version suitable for Vietoris homology theory of the classical proposition that homotopic cycles are homologous. We begin by proving a theorem which is a slight generalization of a lemma due to Borsuk [6].

THEOREM 30. *Let (X, ϱ), (Y, ϱ') be metric spaces, D an abelian group, and let f and g be functions (not necessarily continuous) from X into Y. Assume that for some pair of numbers $\varepsilon, \eta \in \tilde{R}_+$ the following statements hold:* (1) *If $a \in X$, then $\varrho'(f(a), g(a)) < \eta$.* (2) *If $a, b \in X$ and $\varrho(a, b) < \varepsilon$, then $\varrho'(f(a), f(b)) < \eta$ and $\varrho'(g(a), g(b)) < \eta$. Then for each $n \in \tilde{Z}$ and each $\gamma \in Z_n^\varepsilon(X, D)$, $f(\gamma), g(\gamma) \in Z_n^\eta(Y, D)$ and $f(\gamma) \underset{2\eta}{\sim} g(\gamma)$ in Y. Moreover, there exists a chain $\mu \in C_{n+1}^{2\eta}(Y, D)$ such that $\partial\mu = g(\gamma) - f(\gamma)$ and such that $V(\mu) \subseteq V(f(\gamma)) \cup V(g(\gamma))$.*

Proof. Let $\gamma \in Z_n^\varepsilon(X, D)$ and suppose that the unique representation of γ is $\gamma = d_1\sigma_1 + \ldots + d_k\sigma_k$. Assume that the vertices of the simplexes $\sigma_1, \ldots, \sigma_k$ have been ordered in some way so that we may write $V(\gamma) = \{a_1, a_2, \ldots, a_m\}$. For $j = 0, 1, \ldots, m$ we define a function $f_j\colon X \to Y$ by the rule $f_j(x) = g(x)$ if $x = a_1, \ldots, a_j$ and $f_j(x) = f(x)$ otherwise. We note that each f_j induces a function $f_j\colon \Sigma_n^\varepsilon(X) \to \Sigma_n Y$ and a homomorphism $f_j\colon C_n^\varepsilon(X, D) \to C_n^\#(Y, D)$ as described in Section 4.3. Also by condition (2) of the hypothesis f and g induce functions from $\Sigma_n^\varepsilon(X)$ into $\Sigma_n^\eta(Y)$, and Theorem 22 implies that f and g induce homomorphisms from $C_n^\varepsilon(X, D)$ into $C_n^\eta(Y, D)$ such that

$f(\gamma), g(\gamma) \in Z_n^\eta(Y, D)$. We see from the definition of the f_j that $f_0 = f$ so that $f_0(\gamma) = f(\gamma)$, and that $f_m(a_i) = g(a_i)$ for $i = 1, \ldots, m$ so that $f_m(\gamma) = g(\gamma)$. Also if $0 \leqslant j < m$, then $f_j(x) = f_{j+1}(x)$ for each $x \neq a_{j+1}$ and $f_j(a_{j+1}) = f(a_{j+1})$, $f_{j+1}(a_{j+1}) = g(a_{j+1})$. Now for each simplex σ_i of γ we define V_i to be the set $V(f(\sigma_i)) \cup V(g(\sigma_i))$. It follows that $V(f_j(\sigma_i)) \subseteq V_i$ for $j = 0, 1, \ldots, m$, and that $\bigcup \{V_i: i = 1, \ldots, k\} = V(f(\gamma)) \cup V(g(\gamma))$. Conditions (1) and (2) of the hypothesis imply that $\Delta_{\varrho'}(V_i) < 2\eta$ for each i since $\Delta_\varrho(V(\sigma_i)) < \varepsilon$. Consequently $f_j(\sigma_i) \in \Sigma_n^{2\eta}(Y)$ for $j = 0, \ldots, m$ and $i = 1, \ldots, k$ so that $f_j(\gamma) \in C_n^{2\eta}(Y, D)$. Of course $g(\gamma) \in C_n^{2\eta}(Y, D)$ also. Therefore $f(\gamma)$, $g(\gamma)$, and all $f_j(\gamma)$ for $j = 0, \ldots, m$ are element of $Z_n^{2\eta}(Y, D)$. For each j $(0 \leqslant j < m)$ we wish to show that $f_{j+1}(\gamma) - f_j(\gamma)$ is the boundary of a 2η-chain in Y whose vertices lie in the set $V(f(\gamma)) \cup V(g(\gamma))$. We begin by writing γ in the form $\gamma = a_{j+1} \cdot \varkappa + \lambda$, where $\varkappa \in C_{n-1}^\varepsilon(X, D)$, $\lambda \in C_n^\varepsilon(X, D)$, and a_{j+1} is not a vertex of $\varkappa$ or of λ. Now γ is a cycle so that $\partial(a_{j+1} \cdot \varkappa + \lambda) = \partial(\gamma) = 0$. But by statement (2) of Section 2.4, $\partial(a_{j+1} \cdot \varkappa + \lambda) = \varkappa - a_{j+1} \cdot \partial\varkappa + \partial\lambda$. Hence $a_{j+1} \cdot \partial\varkappa = \varkappa + \partial\lambda$. We know a_{j+1} is not a vertex of the chain $\varkappa + \partial\lambda$ nor of the chain $\partial\varkappa$. If $\partial\varkappa \neq 0$ we have the contradiction that a_{j+1} is a vertex of $a_{j+1} \cdot \partial\varkappa$. Thus $\partial\varkappa = 0$ and hence $\varkappa$ is a cycle, that is, $\varkappa \in Z_{n-1}^\varepsilon(X, D)$. It follows immediately that $f_j(\varkappa)$ is a cycle, that is, $f_j(\varkappa) \in Z_{n-1}^{\#}(Y, D)$. To obtain an expression for $f_{j+1}(\gamma) - f_j(\gamma)$ we write

$$\begin{aligned} f_{j+1}(\gamma) - f_j(\gamma) &= f_{j+1}(a_{j+1} \cdot \varkappa + \lambda) - f_j(a_{j+1} \cdot \varkappa + \lambda) \\ &= f_{j+1}(a_{j+1} \cdot \varkappa) + f_{j+1}(\lambda) - f_j(a_{j+1} \cdot \varkappa) - f_j(\lambda) \\ &= f_{j+1}(a_{j+1}) \cdot f_{j+1}(\varkappa) - f_j(a_{j+1}) \cdot f_j(\varkappa) \\ &= f_{j+1}(a_{j+1}) \cdot f_j(\varkappa) - f_j(a_{j+1}) \cdot f_j(\varkappa). \end{aligned}$$

In this computation we have twice used fact that f_j and f_{j+1} must agree on a chain which does not have a_{j+1} as a vertex, and we have also made use of part (6) of Theorem 20. In view of the expression obtained for $f_{j+1}(\gamma) - f_j(\gamma)$ we now define $\mu_j = f_j(a_{j+1}) \cdot f_{j+1}(a_{j+1}) \cdot f_j(\varkappa) \in C_{n+1}^{\#}(Y, D)$. To compute $\partial\mu_j$ we appeal to formula (2) of Section 2.4 and write

$$\begin{aligned} \partial\mu_j &= \partial\big(f_j(a_{j+1}) \cdot f_{j+1}(a_{j+1}) \cdot f_j(\varkappa)\big) \\ &= f_{j+1}(a_{j+1}) \cdot f_j(\varkappa) - f_j(a_{j+1}) \cdot \partial\big(f_{j+1}(a_{j+1}) \cdot f_j(\varkappa)\big) \\ &= f_{j+1}(a_{j+1}) \cdot f_j(\varkappa) - f_j(a_{j+1}) \cdot \big(f_j(\varkappa) - f_{j+1}(a_{j+1}) \cdot \partial f_j(\varkappa)\big) \\ &= f_{j+1}(a_{j+1}) \cdot f_j(\varkappa) - f_j(a_{j+1}) \cdot f_j(\varkappa), \end{aligned}$$

where the last equality follows from the fact that $f_j(\varkappa)$ is a cycle. The previous two computations yield the result $\partial\mu_j = f_{j+1}(\gamma) - f_j(\gamma)$. We show now that $\mu_j \in C_{n+1}^{2\eta}(Y, D)$ and that $V(\mu_j) \subseteq V(f(\gamma)) \cup V(g(\gamma))$. Suppose that $\tau \in \Sigma(\mu_j)$. Then since $\mu_j = f_j(a_{j+1}) \cdot f_{j+1}(a_{j+1}) \cdot f_j(\varkappa)$, τ is of the form $f_j(a_{j+1}) \cdot f_{j+1}(a_{j+1}) \cdot f_j(\sigma)$, where $\sigma \in \Sigma(\varkappa)$. But if $\sigma \in \Sigma(\varkappa)$, then $a_{j+1} \cdot \sigma \in \Sigma(\gamma)$; say $a_{j+1} \cdot \sigma = \sigma_i$. We know V_i contains all vertices of $f_j(\sigma_i)$ and $f_{j+1}(\sigma_i)$

and therefore V_i contains all vertices of $\tau = f_j(a_{j+1})\cdot f_{j+1}(a_{j+1})\cdot f_j(\sigma)$; that is, $V(\tau) \subseteq V_i$. Thus $\Delta_{\varrho'}(V(\tau)) < 2\eta$ since $\Delta_{\varrho'}(V_i) < 2\eta$. Then for each $\tau \in \Sigma(\mu_j)$, $\Delta_{\varrho'}(V(\tau)) < 2\eta$ and $V(\tau) \subseteq \bigcup\{V_i\colon i = 1, \ldots, k\}$. Therefore $\mu_j \in C^{2\eta}_{n+1}(Y, D)$ and $V(\mu_j) \subseteq \bigcup\{V_i\colon i = 1, \ldots, k\} = V(f(\gamma)) \cup V(g(\gamma))$. The final step of the proof consists of combining the chains μ_j to form a chain μ which will satisfy the conclusion of the theorem. For this purpose we define $\mu = \mu_0 + \mu_1 + \ldots \ldots + \mu_{m-1}$. Then $\mu \in C^{2\eta}_{n+1}(Y, D)$ since each $\mu_j \in C^{2\eta}_{n+1}(Y, D)$, and $V(\mu) \subseteq V(f(\gamma)) \cup V(g(\gamma))$ since for each j, $V(\mu_j) \subseteq V(f(\gamma)) \cup V(g(\gamma))$. Moreover,

$$\begin{aligned}\partial\mu &= \partial\mu_0 + \ldots + \partial\mu_{m-1}\\ &= (f_1(\gamma) - f_0(\gamma)) + \ldots + (f_m(\gamma) - f_{m-1}(\gamma))\\ &= f_m(\gamma) - f_0(\gamma) = g(\gamma) - f(\gamma).\end{aligned}$$

It follows immediately that $f(\gamma) \underset{2\eta}{\sim} g(\gamma)$ in Y. This completes the proof of the theorem.

Now we apply Theorem 30 to obtain a proof of the theorem about homotopic cycles mentioned at the beginning of this section. First, if $\varphi\colon S \to T$ is a function and A is a subset of S, the symbol $\varphi|A$ denotes the restriction of φ to A; that is, $\varphi|A$ is the function from A into T defined by the rule $(\varphi|A)(x) = \varphi(x)$ for each $x \in A$.

THEOREM 31 (homotopy theorem). *Let (X, ϱ), (Y, ϱ') be metric spaces, D an abelian group, and let f and g be maps of X into Y. Suppose that $\underline{\gamma} \in Z^{\infty}_n(X, D)$ so that $f(\underline{\gamma}), g(\underline{\gamma}) \in Z^{\infty}_n(Y, D)$. If $f|X_0$ is homotopic to $g|X_0$ in Y for some carrier X_0 of $\underline{\gamma}$, then $f(\underline{\gamma}) \sim g(\underline{\gamma})$ in Y.*

Proof. By hypothesis there is a compact set $X_0 \subseteq X$ such that X_0 is a carrier of $\underline{\gamma}$ and $f|X_0$ is homotopic to $g|X_0$ in Y. Hence there is a map $\varphi\colon X_0 \times \tilde{I} \to Y$ (where $\tilde{I}$ denotes the unit interval $[0, 1] \subseteq \tilde{R}$) such that for each $x \in X_0$, $\varphi(x, 0) = f(x)$ and $\varphi(x, 1) = g(x)$. We note that $X_0 \times \tilde{I}$ is a compact topological space and that its topology is given by the metric $\hat{\varrho}$ defined by the formula $\hat{\varrho}((x, t), (x', t')) = (\varrho(x, x')^2 + |t - t'|^2)^{1/2}$ for $(x, t), (x', t') \in X_0 \times \tilde{I}$. Thus $\varphi(X_0 \times \tilde{I})$ is a compact subset of Y since φ is continuous. Also $\varphi(X_0 \times \tilde{I})$ contains the set $V(f(\underline{\gamma})) \cup V(g(\underline{\gamma}))$ because $f(X_0)$ and $g(X_0)$ are subsets of $\varphi(X_0 \times \tilde{I})$. According to Theorem 19 we may conclude $f(\underline{\gamma}) \sim g(\underline{\gamma})$ in $\varphi(X_0 \times \tilde{I})$ (and hence in Y) provided we show that for each $\eta \in \tilde{R}_+$ there exists $i_0 \in \tilde{N}$ such that $i \geqslant i_0$ implies $f(\gamma_i) \underset{2\eta}{\sim} g(\gamma_i)$ in $\varphi(X_0 \times \tilde{I})$. Let $\eta \in \tilde{R}_+$ be given. The compactness of $X_0 \times \tilde{I}$ implies that φ is uniformly continuous. Thus there exists $\varepsilon \in \tilde{R}_+$ such that if $(x, t), (x', t') \in X_0 \times \tilde{I}$ and $\hat{\varrho}((x, t), (x', t')) < \varepsilon$, then $\varrho'(\varphi(x, t), \varphi(x', t')) < \eta$. In particular if $x, x' \in X_0$ and $\varrho(x, x') < \varepsilon$, then for each $t \in \tilde{I}$, $\varrho'(\varphi(x, t), \varphi(x', t)) < \eta$. Now let $t_0, t_1, \ldots, t_m \in \tilde{I}$ be chosen so that $0 = t_0 < t_1 < \ldots < t_m = 1$ and $|t_j - t_{j+1}| < \varepsilon$ for $j = 0, 1, \ldots, m-1$. Then for each $x \in X_0$, $\varrho'(\varphi(x, t_j), \varphi(x, t_{j+1})) < \eta$ for $j = 0, 1, \ldots, m-1$. For each such value of j we now define a function

$i_j\colon X_0 \to X_0 \times \{t_j\}$ by the formula $i_j(x) = (x, t_j)$, and we denote the composite function $\varphi \cdot i_j$ by the symbol f_j. Thus for $j = 0, 1, \ldots, m-1$ we have a function $f_j\colon X_0 \to Y$ satisfying the condition $f_j(x) = \varphi(x, t_j)$ for each $x \in X_0$. We note that $f_0 = f|X_0$ and $f_m = g|X_0$. For $j = 0, 1, \ldots, m-1$ we shall apply Theorem 30 to the functions f_j and f_{j+1} from X_0 into Y. We must show that conditions (1) and (2) of the hypothesis of Theorem 30 are satisfied. Condition (1) requires that for each $x \in X_0$, $\varrho'(f_j(x), f_{j+1}(x)) < \eta$. Since $f_j(x) = \varphi(x, t_j)$ and $f_{j+1}(x) = \varphi(x, t_{j+1})$ it follows that $\varrho'(f_j(x), f_{j+1}(x)) = \varrho'(\varphi(x,t_j), \varphi(x, t_{j+1}))$ and we have observed above that $\varrho'(\varphi(x,t_j), \varphi(x,t_{j+1})) < \eta$. Condition (2) requires that if $x, x' \in X_0$ and $\varrho(x, x') < \varepsilon$, then $\sigma'(f_j(x), f_j(x')) < \eta$ and $\varrho'(f_{j+1}(x), f_{j+1}(x')) < \eta$. Now we have observed above that under these assumptions $\varrho'(\varphi(x, t), \varphi(x', t)) < \eta$ for each $t \in \tilde{I}$. The desired conclusion follow since

$$\varrho'(f_j(x), f_j(x')) = \varrho'(\varphi(x, t_j), \varphi(x', t_j))$$

and

$$\varrho'(f_{j+1}(x), f_{j+1}(x')) = \varrho'(\varphi(x, t_{j+1}), \varphi(x', t_{j+1})).$$

Thus conditions (1) and (2) are satisfied. We conclude from Theorem 30 that if $\gamma \in Z_n^\varepsilon(X_0, D)$, then $f_j(\gamma), f_{j+1}(\gamma) \in Z_n^\eta(Y, D)$ and there is a chain $\mu \in C_{n+1}^{2\eta}(Y, D)$ such that $\partial\mu = f_{j+1}(\gamma) - f_j(\gamma)$ and $V(\mu) \subseteq V(f_j(\gamma)) \cup V(f_{j+1}(\gamma))$. The latter condition implies that $V(\mu)$ is contained in the compact set $\varphi(X_0 \times \tilde{I})$ so that $\mu \in C_{n+1}^{2\eta}(\varphi(X_0 \times \tilde{I}), D)$. Thus for $j = 0, 1, \ldots, m-1$, $f_j(\gamma) \underset{2\eta}{\sim} f_{j+1}(\gamma)$ in $\varphi(X_0 \times \tilde{I})$ whenever $\gamma \in Z_n^\varepsilon(X_0, D)$. By the transitivity of this homology relation we conclude that $f(\gamma) = f_0(\gamma) \underset{2\eta}{\sim} f_m(\gamma) = g(\gamma)$ in $\varphi(X_0 \times \tilde{I})$ provided $\gamma \in Z_n^\varepsilon(X_0, D)$. Now for sufficiently large values of i, it will be true that $\gamma_i \in Z_n^\varepsilon(X_0, D)$ so that the condition $f(\gamma_i) \underset{2\eta}{\sim} g(\gamma_i)$ in $\varphi(X_0 \times \tilde{I})$ will be satisfied. Therefore as remarked at the beginning of this proof, Theorem 19 enables us to conclude that $f(\underline{\gamma}) \sim g(\underline{\gamma})$ in $\varphi(X_0 \times \tilde{I})$ and hence in Y.

The following corollary to Theorem 31 is easily proved.

THEOREM 32. *Let (X, ϱ) be a metric space and D an abelian group. If the topological space X is contractible, then $Z_n^\infty(X, D) = B_n^\infty(X, D)$.*

4.8. Null translations. Let (X, ϱ) be a metric space and D an abelian group. If $\underline{f} = \langle f_i \rangle$ is a sequence of functions from X into X, we shall write $\underline{f}\colon X \to X$ or $\langle f_i \rangle\colon X \to X$ provided no confusion is likely to result. Suppose that $\underline{f}\colon X \to X$ and there exists $\underline{\eta} = \langle \eta_i \rangle \in \tilde{R}_+^{\tilde{N}}$ with $\underline{\eta} \to 0$ such that $\varrho(x, f_i(x)) < \eta_i$ for each $x \in X$, $i \in \tilde{N}$. Then $\underline{f}$ is called a *null translation of X with majorant $\underline{\eta}$*. It is clear from the triangle inequality that for each $\varepsilon \in \tilde{R}_+$, if $\varrho(a, b) < \varepsilon$, then $\varrho(f_i(a), f_i(b)) < \varepsilon + 2\eta_i$ for $i \in \tilde{N}$, since

$$\varrho(f_i(a), f_i(b)) \leqslant \varrho(f_i(a), a) + \varrho(a, b) + \varrho(b, f_i(b)).$$

Thus f_i induces a homomorphism f_i: $C_n^\varepsilon(X, D) \to C_n^{\varepsilon+2\eta_i}(X, D)$ as described in Theorem 22, and, moreover, $f_i \partial = \partial f_i$: $C_n^\varepsilon(X, D) \to C_{n-1}^{\varepsilon+2\eta_i}(X, D)$. Suppose now that $\underline{\varkappa} = \langle \varkappa_i \rangle \in C_n^\infty(X, D)$ with majorant $\underline{\varepsilon}$, carrier X_0 so that for each $i \in \tilde{N}$, $\varkappa_i \in C_n^{\varepsilon_i}(X, D)$. Then for each $i \in \tilde{N}$, $f_i(\varkappa_i) \in C_n^{\varepsilon_i+2\eta_i}(X, D)$. We denote the sequence $\langle f_i(\varkappa_i) \rangle$ by $\underline{f}(\underline{\varkappa})$ or $\langle f_i \rangle (\underline{\varkappa})$. The following lemma will enable us to prove that $\underline{f}(\underline{\varkappa})$ is itself an infinite chain.

THEOREM 33. *Let (X, ϱ) be a metric space and D an abelian group. Let $\underline{f}$: $X \to X$ be a null translation and let $\underline{\varkappa} \in C_n^\infty(X, D)$ with carrier X_0. Then the set $X_0 \cup \bigcup \{V(f_i(\varkappa_i))$: $i \in \tilde{N}\}$ is compact.*

Proof. Let $\underline{\eta}$ be a majorant for $\underline{f}$. For each $i \in \tilde{N}$ let $N\eta_i(X_0)$ denote the η_i neighbourhood of the set X_0; that is, $N\eta_i(X_0) = \{x \in X$: $\varrho(x, x_0) < \eta_i$ for some $x_0 \in X_0\}$. It is clear that for each $i \in \tilde{N}$, $V(f_i(\varkappa_i)) \subseteq N\eta_i(X_0)$ since $\underline{\eta}$ is a majorant for $\underline{f}$ and since $V(\varkappa_i) \subseteq X_0$. Suppose that $\{U\alpha$: $\alpha \in A\}$ is an open covering of the set $X_0 \cup \bigcup \{V(f_i(\varkappa_i))$: $i \in \tilde{N}\}$. Some finite subcollection, say $U\alpha_1, \ldots, U\alpha_k$, covers X_0 since X_0 is compact. The set $W = \bigcup \{U\alpha_j$: $j = 1, \ldots, k\}$ is an open set containing X_0 and hence W contains each of the sets $N\eta_i(X_0)$ for sufficiently large i, since $\underline{\eta} \to 0$. It follows that for sufficiently large i, say $i \geqslant i_0$, W contains each of the sets $V(f_i(\varkappa_i))$. Since there are only finitely many vertices in the set $\bigcup \{V(f_i(\varkappa_i))$: $i = 1, \ldots, i_0 - 1\}$ we may choose a finite subcollection of $\{U\alpha$: $\alpha \in A\}$, say $U\alpha_{k+1}, \ldots, U\alpha_q$, containing these vertices. The sets $U\alpha_1, \ldots, U\alpha_k, U\alpha_{k+1}, \ldots, U\alpha_q$ form a finite subcollection of $\{U\alpha$: $\alpha \in A\}$ whose union $\bigcup \{U\alpha_j$: $j = 1, \ldots, q\}$ contains $X_0 \cup \bigcup \{V(f_i(\varkappa_i))$: $i \in \tilde{N}\}$.

The following theorem is now easily proved.

THEOREM 34. *Let (X, ϱ) be a metric space and D an abelian group. Suppose that $\underline{f}$: $X \to X$ is a null translation with majorant $\underline{\eta}$. If $\underline{\varkappa} \in C_n^\infty(X, D)$ with majorant $\underline{\varepsilon}$, carrier X_0, then $\underline{f}(\underline{\varkappa}) \in C_n^\infty(X, D)$ with majorant $\underline{\varepsilon} + 2\underline{\eta}$, carrier $X_0 \cup \bigcup \{V(f_i(\varkappa_i))$: $i \in \tilde{N}\}$.*

THEOREM 35. *Let (X, ϱ) be a metric space, D an abelian group, and let $\underline{f}$: $X \to X$ be a null translation. Then $\underline{f}$ induces a homomorphism $\underline{f}$: $C_n^\infty(X, D) \to C_n^\infty(X, D)$ such that the following conditions are satisfied:*

(1) *$\underline{f}$ commutes with the boundary operator; that is, $\underline{f}\partial = \partial \underline{f}$: $C_n^\infty(X, D) \to$ $\to C_{n-1}^\infty(X, D)$.*

(2) *$\underline{f}$ carries cycles into cycles; that is, $\underline{f}(Z_n^\infty(X, D)) \subseteq Z_n^\infty(X, D)$.*

(3) *$\underline{f}$ carries boundaries into boundaries; that is, $\underline{f}(B_n^\infty(X, D)) \subseteq B_n^\infty(X, D)$.*

Proof. Theorem 34 implies that $\underline{f}$ is a function from $C_n^\infty(X, D)$ into $C_n^\infty(X, D)$. To show that $\underline{f}$ is a homomorphism we must prove that $\underline{f}(\underline{\varkappa} + \underline{\lambda}) = \underline{f}(\underline{\varkappa}) + \underline{f}(\underline{\lambda})$ for all $\underline{\varkappa}, \underline{\lambda} \in C_n^\infty(X, D)$. Now $\underline{f}(\underline{\varkappa} + \underline{\lambda}) = \langle f_i(\varkappa_i + \lambda_i) \rangle$ and $\underline{f}(\underline{\varkappa}) + \underline{f}(\underline{\lambda}) = \langle f_i(\varkappa_i) + f_i(\lambda_i) \rangle$. The required equality follows from Theorem 20 according to which f_i: $C_n^\#(X, D) \to C_n^\#(X, D)$ is a homomorphism for each $i \in \tilde{N}$. To show that condition (1) holds we must prove that $\underline{f}(\partial \underline{\varkappa}) = \partial(\underline{f}(\underline{\varkappa}))$ for each $\underline{\varkappa} \in C_n^{\prime}(X, D)$. Now $\underline{f}(\partial \underline{\varkappa}) = \langle f_i(\partial \varkappa_i) \rangle$ and $\partial(\underline{f}(\underline{\varkappa})) = \langle \partial(f_i(\varkappa_i)) \rangle$. The desired result again follows from Theorem 20 which implies that for each $i \in \tilde{N}$ the homomorphism f_i: $C_n^{\#}(X, D) \to C_n^{\#}(X, D)$ commutes with the boundary operator. Finally it is an immediate consequence of condition (1) that conditions (2) and (3) are also satisfied.

It is easy to see that a null translation which carries a subset into itself may be regarded as a null translation of the subset. Thus we have the following theorem:

THEOREM 36. *Let (X, ϱ) be a metric space and let $\underline{f}$: $X \to X$ be a null translation with majorant $\underline{\eta}$. Suppose that A is a subset of X and that for each $i \in \tilde{N}$, $f_i(A) \subseteq A$. Then $\underline{f}$ defines a null translation $\underline{f}$: $A \to A$ with majorant $\underline{\eta}$. More precisely, if for each $i \in \tilde{N}$, a function g_i: $A \to A$ is defined by the rule $g_i(x) = f_i(x)$ for each $x \in A$, then $\underline{g}$: $A \to A$ is a null translation with majorant $\underline{\eta}$.*

THEOREM 37. *The composition of null translations is a null translation. More precisely, let (X, ϱ) be a metric space and suppose that $\underline{f}$: $X \to X$ is a null translation with majorant $\underline{\eta}$ and $\underline{g}$: $X \to X$ is a null translation with majorant $\underline{\zeta}$. Then the composition $\underline{g} \cdot \underline{f}$ of $\underline{f}$ and $\underline{g}$ which is defined to be the sequence $\langle g_i \cdot f_i \rangle$: $X \to X$ is a null translation with majorant $\underline{\eta}+\underline{\zeta}$.*

Proof. Evidently for each $i \in \tilde{N}$, $g_i \cdot f_i$ is a function from X into X. Thus we need only show that for each $i \in \tilde{N}$ and each $x \in X$, $\varrho(x, g_i \cdot f_i(x)) < \eta_i + \zeta_i$. This follows from the triangle inequality; that is, $\varrho(x, g_i \cdot f_i(x)) \leqslant \varrho(x, f_i(x)) + \varrho(f_i(x), g_i(f_i(x)))$ where $\varrho(x, f_i(x)) < \eta_i$ and $\varrho(f_i(x), g_i(f_i(x))) < \zeta_i$ since $\underline{\eta}$ and $\underline{\zeta}$ are majorants for $\underline{f}$ and $\underline{g}$ respectively.

The following theorem suggests the usefulness of the concept of null translation. Its proof employs the generalized Borsuk lemma, Theorem 30.

THEOREM 38. *A null translation carries each infinite cycle into an infinite cycle homologous to it. More precisely, let (X, ϱ) be a metric space, D an abelian group, and suppose that $\underline{f}$: $X \to X$ is a null translation. If $\underline{\gamma} \in Z_n^{\infty}(X, D)$ (and hence $\underline{f}(\underline{\gamma}) \in Z_n^{\infty}(X, D)$), then $\underline{f}(\underline{\gamma}) \sim$ in X.*

Proof. Suppose that $\underline{\gamma}$ has majorant $\underline{\varepsilon}$, carrier X_0 and that $\underline{f}$ has majorant $\underline{\eta}$. Let id denote the identity function from X into X. For each $i \in \tilde{N}$ we shall apply Theorem 30 to the functions f_i and id from X into X, replacing ε by ε_i and η by $\varepsilon_i + 2\eta_i$. Condition (1) of the hypothesis of Theorem 30 is satisfied since for each $x \in X$, $\varrho(f_i(x), \text{id}(x)) = \varrho(x, f_i(x)) < \eta_i < \varepsilon_i + 2\eta_i$. Condition (2) is also satisfied since if $a, b \in X$ and $\varrho(a, b) < \varepsilon_i$, then $\varrho(f_i(a), f_i(b)) < \varepsilon_i + 2\eta_i$ as noted previously, and $\varrho(\text{id}(a), \text{id}(b)) = \varrho(a, b) < \varepsilon_i < \varepsilon_i + 2\eta_i$. Therefore we may conclude from Theorem 30 that if $\gamma_i \in Z_n^{\varepsilon_i}(X, D)$, then $f_i(\gamma_i)$ and $\text{id}(\gamma_i) = \gamma_i$ are elements of $Z_n^{\varepsilon_i + 2\eta_i}(X, D)$ and $f_i(\gamma_i) \underset{2\varepsilon_i + 4\eta_i}{\sim} \gamma_i$ in X. Moreover, there exists a chain $\mu_i \in C_{n+1}^{2\varepsilon_i + 4\eta_i}(X, D)$ such that $\partial\mu_i = \gamma_i - f_i(\gamma_i)$ and such that $V(\mu_i) \subseteq V(\gamma_i) \cup V(f_i(\gamma_i)) \subseteq X_0 \cup V(f_i(\gamma_i))$. Now letting $\underline{\mu} = \langle \mu_i \rangle$ we see that the sequence $\langle 2\varepsilon_i + 4\eta_i \rangle$ is a majorant for $\underline{\mu}$. Also $V(\underline{\mu})$ is contained in the set $X_0 \cup \bigcup \{V(f_i(\gamma_i)) \colon i \in \tilde{N}\}$ and hence this set is a carrier for $\underline{\mu}$ since it is compact according to Theorem 33. Thus $\underline{\mu} \in C^{\alpha}_{n+1}(X, D)$. Moreover, $\partial\underline{\mu} = \langle \partial\mu_i \rangle = \langle \gamma_i - f_i(\gamma_i) \rangle = \underline{\gamma} - \underline{f}(\underline{\gamma})$, and consequently $\underline{f}(\underline{\gamma}) \sim \underline{\gamma}$ in X.

Chapter V

The Phragmen–Brouwer theorem

5.1. Introduction. The first theorem of this chapter is the classical Phragmen–Brouwer theorem for compact metric spaces [2]. The proof given here is essentially similar to that outlined by Borsuk [6]. It is included for the sake of completeness. References to this proof occur in the proof of the second theorem, which is an extension of the Phragmen–Brouwer theorem to arbitrary metric spaces.

THEOREM 39. *Let X and Y be compact subsets of a metric space (M, ϱ) and let D be an abelian group. Suppose there is a cycle $\underline{\gamma} \in Z^{\infty}_n(X \cap Y, D)$ such that $\underline{\gamma} \notin B^{\alpha}_n(X \cap Y, D)$ but $\underline{\gamma} \in B^{\alpha}_n(X, D)$ and $\underline{\gamma} \in B^{\infty}_n(Y, D)$. Then there is a cycle $\underline{\delta} \in Z^{\alpha}_{n+1}(X \cup Y, D)$ such that $\underline{\delta} \notin B^{\infty}_{n+1}(X \cup Y, D)$.*

Proof. Suppose that $\underline{\gamma} \in Z^{\alpha}_n(X \cap Y, D) \backslash B^{\alpha}_n(X \cap Y, D)$ and $\underline{\gamma} \in B^{\alpha}_n(X, D) \cap B^{\infty}_n(Y, D)$. Since $\underline{\gamma} \in B^{\infty}_n(X, D)$ there exists $\underline{\lambda} \in C^{\infty}_{n+1}(X, D)$ such that $\partial\underline{\lambda} = \underline{\gamma}$, and since $\underline{\gamma} \in B^{\infty}_n(Y, D)$ there exists $\underline{\mu} \in C^{\infty}_{n+1}(Y, D)$ such that $\partial\underline{\mu} = \underline{\gamma}$. Let $\underline{\delta} = \underline{\lambda} - \underline{\mu}$. Then $\underline{\delta} \in C^{\alpha}_{n+1}(X \cup Y, D)$ and in fact $\underline{\delta} \in Z^{\infty}_{n+1}(X \cup Y, D)$ since $\partial(\underline{\delta}) = \partial(\underline{\lambda} - \underline{\mu}) = \underline{\gamma} - \underline{\gamma} = \underline{0}$. We shall prove that $\underline{\delta} \notin B^{\alpha}_{n+1}(X \cup Y, D)$. Let us assume the contrary, that is, that $\underline{\delta} \in B^{\alpha}_{n+1}(X \cup Y, D)$. Then there exists a chain $\underline{\varkappa} \in C^{\alpha}_{n+2}(X \cup Y, D)$ such that $\partial\underline{\varkappa} = \underline{\delta}$. We first construct a null translation $\underline{f} \colon X \cup Y \to X \cup Y$ which carries X into X and Y into Y, and which has the further property that each simplex of $\underline{f}(\underline{\varkappa})$ lies entirely within X or entirely within Y. Now X and Y are compact sets and $X \cap Y \neq \emptyset$ since $\underline{\gamma} \in Z^{\alpha}_n(X \cap Y, D) \backslash B^{\alpha}_n(X \cap Y, D)$. Also $X \cap Y \neq X$ since $\underline{\gamma} \in B^{\alpha}_n(X, D) \backslash B^{\alpha}_n(X \cap Y, D)$, and $X \cap Y \neq Y$ since $\underline{\gamma} \in B^{\alpha}_n(Y, D) \backslash B^{\alpha}_n(X \cap Y, D)$. For each $j \in \tilde{N}$ let U_j be the $1/j$ neighbourhood of $X \cap Y$ in $X \cup Y$, that is, $U_j = \{a \in X \cup Y \colon \varrho(a, b) < 1/j \text{ for some } b \in X \cap Y\}$. Since $X \backslash U_j$ and $Y \backslash U_j$ are disjoint compact sets, there exists $\varepsilon_j \in \tilde{R}_+$ such that $\varrho(x, y) > \varepsilon_j$ whenever $x \in X \backslash U_j$ and $y \in Y \backslash U_j$. And there exists $i_j \in \tilde{N}$ such that $\varkappa_i \in C^{\varepsilon_j}_{n+2}(X \cup Y, D)$ whenever $i > i_j$. We may assume that the sequence $\underline{i} = \langle i_j \rangle$ is strictly increasing. Because $X \cap Y$ is non-empty and compact, there is for each

$a \in X \cup Y$ at least one point of $X \cap Y$ nearest to a. Thus we may define a function g_0: $X \cup Y \to X \cap Y$ by selecting for each $a \in X \cup Y$ some $g_0(a) \in X \cap Y$ for which the distance $\varrho(a, g_0(a))$ is minimum. Now for each $j \in \tilde{N}$ let g_j: $X \cup Y \to X \cup Y$ be the function defined by the rule $g_j(a) = a$ if $a \in (X \cup Y) \setminus U_j$ and $g_j(a) = g_0(a)$ if $a \in U_j$. It is clear that $\varrho(a, g_j(a)) < 1/j$ for each $a \in X \cup Y$ and that $g_j(X \cup U_j) \subseteq X$ and $g_j(Y \cup U_j) \subseteq Y$. Suppose that $i > i_j$ so that $\varkappa_i \in C^{\varepsilon_j}_{n+2}(X \cup Y, D)$, and let $\sigma \in \Sigma(\varkappa_i)$. Then $V(\sigma) \subseteq X \cup Y$ and $\Delta\varrho(V(\sigma)) < \varepsilon_j$. It follows that $V(\sigma) \subseteq X \cup U_j$ or $V(\sigma) \subseteq Y \cup U_j$ and consequently $g_j(V(\sigma)) \subseteq X$ or $g_j(V(\sigma)) \subseteq Y$. We are now ready to define the desired null translation $\underline{f}$: $X \cup Y \to X \cup Y$. For each $i \in \tilde{N}$ and each $a \in X \cup Y$, let $f_i(a) = g_0(a)$ if $i \leqslant i_1$ and let $f_i(a) = g_j(a)$ if $i_j < i \leqslant i_{j+1}$. Then f_i is a function from $X \cup Y$ into $X \cup Y$ and for each $a \in X \cup Y$, $\varrho(a, f_i(a)) < 1 + \Delta\varrho(X \cup Y)$ if $i \leqslant i_1$ and $\varrho(a, f_i(a)) < 1/j$ if $i_j < i \leqslant i_{j+1}$. Therefore $\underline{f} = \langle f_i \rangle$: $X \cup Y \to X \cup Y$ is a null translation with majorant $\underline{\eta} = \langle \eta_i \rangle$ defined by the rule $\eta_i = 1 + \Delta\varrho(X \cup Y)$ if $i \leqslant i_1$ and $\eta_i = 1/j$ if $i_j < i \leqslant i_{j+1}$. For each $i \in \tilde{N}$ it is clear that $f_i(X) \subseteq X$ and $f_i(Y) \subseteq Y$, and that if $\sigma \in \Sigma(\varkappa_i)$, then $V(f_i(\sigma)) \subseteq X$ or $V(f_i(\sigma)) \subseteq Y$. According to Theorem 35, for each $q \in \tilde{Z}$, $\underline{f}$ induces a homomorphism $\underline{f}$: $C^{\infty}_q(X \cup Y, D) \to$ $\to C^{\infty}_q(X \cup Y, D)$ which commutes with the boundary operator. Moreover, according to Theorem 36, since for each $i \in \tilde{N}$, $f_i(X) \subseteq X$ and $f_i(Y) \subseteq Y$, $\underline{f}$ defines null translations $\underline{f}$: $X \to X$, $\underline{f}$: $Y \to Y$, and $\underline{f}$: $X \cap Y \to X \cap Y$. These null translations again by Theorem 35 induce homomorphisms $\underline{f}$: $C^{\infty}_q(X, D) \to C^{\infty}_q(X, D)$, $\underline{f}$: $C^{\infty}_q(Y, D) \to C^{\infty}_q(Y, D)$, and $\underline{f}$: $C^{\infty}_q(X \cap Y, D) \to$ $\to C^{\infty}_q(X \cap Y, D)$, respectively, each of which commutes with the boundary operator. In fact each of these homomorphisms is defined by the first-mentioned homomorphism $\underline{f}$: $C^{\infty}_q(X \cup Y, D) \to C^{\infty}_q(X \cup Y, D)$. Let the images of $\underline{\varkappa}$, $\underline{\delta}$, $\underline{\lambda}$, $\underline{\mu}$, $\underline{\gamma}$ under the homomorphisms $\underline{f}$ be denoted by $\hat{\underline{\varkappa}}$, $\hat{\underline{\delta}}$, $\hat{\underline{\lambda}}$, $\hat{\underline{\mu}}$, $\hat{\underline{\gamma}}$ respectively. Then the above remarks imply that $\underline{\varkappa}, \hat{\underline{\varkappa}} \in C^{\infty}_{n+2}(X \cup Y, D)$, $\underline{\delta}, \hat{\underline{\delta}} \in Z^{\infty}_{n+1}(X \cup Y, D)$, $\underline{\lambda}, \hat{\underline{\lambda}} \in C^{\infty}_{n+1}(X, D)$, $\underline{\mu}, \hat{\underline{\mu}} \in C^{\infty}_{n+1}(Y, D)$, and $\underline{\gamma}, \hat{\underline{\gamma}} \in Z^{\infty}_n \times$ $\times (X \cap Y, D)$. Moreover, $\partial\underline{\varkappa} = \underline{\delta}$ implies $\partial\hat{\underline{\varkappa}} = \hat{\underline{\delta}}$ since $\partial(\underline{f}(\underline{\varkappa})) = \underline{f}(\partial\underline{\varkappa})$, and $\underline{\delta} = \underline{\lambda} - \underline{\mu}$ implies $\hat{\underline{\delta}} = \hat{\underline{\lambda}} - \hat{\underline{\mu}}$ (since $\underline{f}$ is a homomorphism), and hence $\partial\hat{\underline{\varkappa}} = \hat{\underline{\lambda}} - \hat{\underline{\mu}}$. Also $\underline{\gamma} = \partial\underline{\lambda} = \partial\underline{\mu}$ implies $\hat{\underline{\gamma}} = \partial\hat{\underline{\lambda}} = \partial\hat{\underline{\mu}}$ since $\underline{f}$ commutes with the boundary operator. By applying Theorem 38 to the null translation $\underline{f}$: $X \cap Y \to X \cap Y$ and to the cycle $\underline{\gamma} \in Z^{\infty}_n(X \cap Y, D)$ we find that $\hat{\underline{\gamma}} \sim \underline{\gamma}$ in $X \cap Y$. It follows immediately that $\hat{\underline{\gamma}} \notin B^{\infty}_n(X \cap Y, D)$. The final portion of the proof of the theorem will consist of deducing that $\hat{\underline{\gamma}} \in B^{\infty}_n(X \cap Y, D)$, thus providing the required contradiction. Since for each $i \in \tilde{N}$, $\sigma \in \Sigma(\varkappa_i)$ implies that $V(f_i(\sigma)) \subseteq X$ or $V(f_i(\sigma)) \subseteq Y$, it follows that $\hat{\sigma} \in \Sigma(\hat{\varkappa}_i)$ implies that $V(\hat{\sigma}) \subseteq X$ or $V(\hat{\sigma}) \subseteq Y$. Thus we may represent $\hat{\varkappa}_i$ uniquely in the form $\hat{\varkappa}_i = \hat{\varkappa}^X_i - \hat{\varkappa}^Y_i$, where $\Sigma(\hat{\varkappa}^X_i) \cup \Sigma(\hat{\varkappa}^Y_i) = \Sigma(\hat{\varkappa}_i)$, $V(\hat{\varkappa}^X_i) \subseteq X$, and $V(\hat{\varkappa}^Y_i) \subseteq Y \setminus X$. In this way we obtain two infinite chains $\hat{\underline{\varkappa}}^X = \langle \hat{\varkappa}^X_i \rangle \in C^{\infty}_{n+2}(X, D)$ and $\hat{\underline{\varkappa}}^Y = \langle \hat{\varkappa}^Y_i \rangle \in C^{\infty}_{n+2}(Y, D)$ such that $\hat{\underline{\varkappa}} = \hat{\underline{\varkappa}}^X - \hat{\underline{\varkappa}}^Y$ and consequently $\partial\hat{\underline{\varkappa}} = \partial\hat{\underline{\varkappa}}^X -$ $-\partial\hat{\underline{\varkappa}}^Y$. Then $\hat{\underline{\lambda}} - \hat{\underline{\mu}} = \partial\hat{\underline{\varkappa}}^X - \partial\hat{\underline{\varkappa}}^Y$ since as we have noted previously, $\hat{\underline{\lambda}} - \hat{\underline{\mu}} = \hat{\underline{\delta}}$

$= \partial\hat{\underline{\varkappa}}$. This last result may be rewritten in the form $\hat{\underline{\lambda}}-\partial\hat{\underline{\varkappa}}^X = \hat{\underline{\mu}}-\partial\hat{\underline{\varkappa}}^Y$, where $\hat{\underline{\lambda}}-\partial\hat{\underline{\varkappa}}^X \in C^{\infty}_{n+1}(X, D)$ and $\hat{\underline{\mu}}-\partial\hat{\underline{\varkappa}}^Y \in C^{\infty}_{n+1}(Y, D)$. It follows that $\hat{\underline{\lambda}}-\partial\hat{\underline{\varkappa}}^X \in C^{\infty}_{n+1}(X \cap Y, D)$. Moreover, $\partial(\hat{\underline{\lambda}}-\partial\hat{\underline{\varkappa}}^X) = \partial\hat{\underline{\lambda}}-\partial\partial\hat{\underline{\varkappa}}^X = \partial\hat{\underline{\lambda}} = \hat{\underline{\gamma}}$. Thus we obtain the contradiction that $\hat{\underline{\gamma}} \in B^{\infty}_n(X \cap Y, D)$.

5.2. The Phragmen–Brouwer theorem for non-compact spaces. The following theorem is an extension of Theorem 39. The restriction that the subsets X and Y of the metric space (M, ϱ) be compact is replaced by the weaker requirement that X and Y be closed subsets of (M, ϱ). The proof is based on the statement and proof of Theorem 39.

THEOREM 40. *Let X and Y be closed subsets of a metric space (M, ϱ) and let D be an abelian group. Suppose there is a cycle $\underline{\gamma} \in Z^{\infty}_n(X \cap Y, D)$ such that $\underline{\gamma} \notin B^{\infty}_n(X \cap Y, D)$ but $\underline{\gamma} \in B^{\infty}_n(X, D)$ and $\underline{\gamma} \in B^{\infty}_n(Y, D)$. Then there is a cycle $\underline{\delta} \in Z^{\infty}_{n+1}(X \cup Y, D)$ such that $\underline{\delta} \notin B^{\infty}_{n+1}(X \cup Y, D)$.*

Proof. Suppose that $\underline{\gamma} \in Z^{\infty}_n(X \cap Y, D)\backslash B^{\infty}_n(X \cap Y, D)$ and $\underline{\gamma} \in B^{\infty}_n(X, D) \cap \cap B^{\infty}_n(Y, D)$. Since $\underline{\gamma} \in B^{\infty}_n(X, D)$ there is a chain $\underline{\lambda}^X \in C^{\infty}_{n+1}(X, D)$ such that $\underline{\gamma} = \partial\underline{\lambda}^X$. Let X_0 be a carrier of $\underline{\lambda}^X$ in X. Then $\underline{\lambda}^X \in C^{\infty}_{n+1}(X_0, D)$ and $\underline{\gamma} \in B^{\infty}_n(X_0, D) \subseteq Z^{\infty}_n(X_0, D)$. Similarly since $\underline{\gamma} \in B^{\infty}_n(Y, D)$ there is a chain $\underline{\lambda}^Y \in C^{\infty}_{n+1}(Y, D)$ such that $\underline{\gamma} = \partial\underline{\lambda}^Y$. Let Y_0 be a carrier of $\underline{\lambda}^Y$ in Y. Then $\underline{\lambda}^Y \in C^{\infty}_{n+1}(Y_0, D)$ and $\underline{\gamma} \in B^{\infty}_n(Y_0, D) \subseteq Z^{\infty}_n(Y_0, D)$. We note that X_0 and Y_0 are compact sets and that $X_0 \cap Y_0 \subseteq X \cap Y$. And $\underline{\gamma} \in Z^{\infty}_n(X_0 \cap Y_0, D)$ since $\underline{\gamma} \in Z^{\infty}_n(X_0, D) \cap Z^{\infty}_n(Y_0, D)$. Moreover, $\underline{\gamma} \notin B^{\infty}_n(X_0 \cap Y_0, D)$ for otherwise we would have $\underline{\gamma} \in B^{\infty}_n(X \cap Y, D)$. Thus the hypothesis of Theorem 39 is satisfied for the case of the compact sets X_0 and Y_0 and the cycle $\underline{\gamma}$. It follows that there exists a cycle $\underline{\delta} \in Z^{\infty}_{n+1}(X_0 \cup Y_0, D)$ such that $\underline{\delta} \notin B^{\infty}_{n+1}(X_0 \cup Y_0, D)$. In fact we see from the proof of Theorem 39 that we may take $\underline{\delta}$ to be the cycle $\underline{\lambda}^X - \underline{\lambda}^Y$. Clearly $\underline{\delta} \in Z^{\infty}_{n+1}(X \cup Y, D)$. Therefore in order to complete the proof of the theorem it is sufficient to show that $\underline{\delta} \notin B^{\infty}_{n+1}(X \cup Y, D)$. Let us assume that contrary, that is, that $\underline{\delta} \in B^{\infty}_{n+1}(X \cup Y, D)$. Then $\underline{\delta} = \partial\underline{\mu}$ for some $\underline{\mu} \in C^{\infty}_{n+2}(X \cup Y, D)$. Let Z_0 be a carrier of $\underline{\mu}$ in $X \cup Y$ so that $\underline{\mu} \in C^{\infty}_{n+2}(Z_0, D)$ and $\underline{\delta} \in B^{\infty}_{n+1}(Z_0, D)$. The set $X \cap Z_0$ is compact since X being closed in M implies that $X \cap Z_0$ is closed in the compact set Z_0. Hence the set $X_1 = X_0 \cup (X \cap Z_0)$ is compact. Similarly the set $Y_1 = Y_0 \cup \cup (Y \cap Z_0)$ is compact since Y is closed in M. Thus X_1 and Y_1 are compact sets such that $X_0 \subseteq X_1 \subseteq X$ and $Y_0 \subseteq Y_1 \subseteq Y$. The following inclusion relations are also satisfied:

$$X_0 \cap Y_0 \subseteq X_1 \cap Y_1 \subseteq X \cap Y. \tag{1}$$

$$X_0 \cup Y_0 \subseteq X_1 \cup Y_1 \subseteq X \cup Y. \tag{2}$$

$$Z_0 \subseteq X_1 \cup Y_1 \subseteq X \cup Y. \tag{3}$$

We now apply Theorem 39 again, using the same cycle $\underline{\gamma}$ but this time using the compact sets X_1 and Y_1. Clearly $\underline{\gamma} \in Z^{\infty}_n(X_1 \cap Y_1, D)$

since $\underline{\gamma} \in Z_n^{\infty}(X_0 \cap Y_0, D)$, and $\underline{\gamma} \notin B_n^{\infty}(X_1 \cap Y_1, D)$ or else we would have $\underline{\gamma} \in B_n^{\infty}(X \cap Y, D)$. And $\underline{\gamma} \in B_n^{\infty}(X_1, D)$ since $\underline{\gamma} = \partial \underline{\lambda}^X$, where $\underline{\lambda}^X \in C_{n+1}^{\infty}(X_0, D) \subseteq C_{n+1}^{\infty}(X_1, D)$. Similarly $\underline{\gamma} \in B_n^{\infty}(Y_1, D)$ since $\underline{\gamma} = \partial \underline{\lambda}^Y$, where $\underline{\lambda}^Y \in C_{n+1}^{\infty}(Y_0, D) \subseteq C_{n+1}^{\infty}(Y_1, D)$. Thus the hypothesis of Theorem 39 is satisfied and we may conclude that there exists a cycle $\underline{\delta}^* \in Z_{n+1}^{\infty}(X_1 \cup Y_1, D)$ such that $\underline{\delta}^* \notin B_{n+1}^{\infty}(X_1 \cup Y_1, D)$. In fact we see from the proof of Theorem 39 that we may take $\underline{\delta}^*$ to be the cycle $\underline{\lambda}^X - \underline{\lambda}^Y$. But $\underline{\lambda}^X - \underline{\lambda}^Y = \underline{\delta}$. That is, $\underline{\delta} \notin B_{n+1}^{\infty}(X_1 \cup Y_1, D)$. However, we have seen previously that $\underline{\delta} \in B_{n+1}^{\infty}(Z_0, D)$ and that $Z_0 \subseteq X_1 \cup Y_1$. This contradiction completes the proof of the theorem.

In Theorem 40 the sets X and Y are required to be closed. The following example shows that this requirement is essential. Let X and Y be the subsets of the plane $\tilde{R}^2$ defined as follows: $X = X_1 \cup X_2 \cup X_3$, where $X_1 = \{(a, b)\colon 0 \leqslant a \leqslant 2 \text{ and } 0 \leqslant b \leqslant 6\}$, $X_2 = \{(a, b)\colon 2 \leqslant a \leqslant 4 \text{ and } 0 \leqslant b \leqslant 2\}$, $X_3 = \{(a, b)\colon 2 \leqslant a \leqslant 4 \text{ and } 4 \leqslant b \leqslant 6\}$; $Y = \{(a, b)\colon 2 < a \leqslant 6 \text{ and } 0 \leqslant b \leqslant 6\}$. Then $X \cup Y$ is homeomorphic to the ball $\tilde{B}^2$ and hence is contractible. Let $p = (3, 1)$ and $q = (3, 5)$. Let $\underline{\gamma}$ be the sequence whose ith term is the 0-cycle $\gamma_i = q - p$. Then $\underline{\gamma} \in Z_0^{\infty}(X \cap Y, \tilde{Z})$ and $\underline{\gamma} \notin B_0^{\infty}(X \cap Y, \tilde{Z})$ (where $\tilde{Z}$ denotes the group of integers). Let $r = (1, 1)$, $s = (1, 3)$, $t = (1, 5)$, $r' = (5, 1)$, $s' = (5, 3)$, $t' = (5, 5)$. The usual subdivision process applied to the 1-dimensional chain $[pr]+[rs]+[st]+[tq]$ yields an infinite chain $\underline{\lambda}^X \in C_1^{\infty}(X, \tilde{Z})$ such that $\partial \underline{\lambda}^X = \underline{\gamma}$. Thus $\underline{\gamma} \in B_0^{\infty}(X, \tilde{Z})$. Similarly by considering the chain $[pr']+[r's']+[s't']+[t'q]$ we find that there is an infinite chain $\underline{\lambda}^Y \in C_1^{\infty}(Y, \tilde{Z})$ such that $\partial \underline{\lambda}^Y = \underline{\gamma}$. Hence $\underline{\gamma} \in B_0^{\infty}(Y, \tilde{Z})$ also. Thus, except for the requirement that X and Y be closed subsets, the hypothesis of Theorem 40 is satisfied. If the conclusion of Theorem 40 held, there would be an infinite cycle $\underline{\delta} \in Z_1^{\infty}(X \cup Y, \tilde{Z})$ such that $\underline{\delta} \notin B_1^{\infty}(X \cup Y, \tilde{Z})$. But this is clearly impossible since the fact that $X \cup Y$ is contractible implies by Theorem 32 that $Z_1^{\infty}(X \cup Y, \tilde{Z}) = B_1^{\infty}(X \cup Y, \tilde{Z})$. Therefore the conclusion of Theorem 40 does not hold in this case.

Chapter VI

The Alexandroff dimension theorem

6.1. Introduction. In this chapter we shall require various results from the topological theory of dimension. For convenience we adopt the definitions and statements of theorems which appear in the book of Nagata [21]. If (X, ϱ) is a metric space we shall mean by the *dimension* of X, denoted $\dim X$, the covering dimension of X. Equivalently $\dim X$ may be regarded as the strong inductive dimension of X. We shall write $\dim X < \infty$ to indicate that X is finite dimensional.

We recall from Section 3.4 that if (X, ϱ) is a metric space and D an abelian group, then an infinite cycle $\underline{\gamma} \in Z_n^{\infty}(X, D)$ is called *essential* if $\underline{\gamma}$ has a carrier X_0 such that $\gamma \notin B_n^{\infty}(X_0, D)$. Throughout this chapter we let D denote the additive group of real numbers modulo 1. The classical Alexandroff dimension theorem [2] may now be stated in the following way.

THEOREM 41. *Let* (X, ϱ) *be a compact metric space with* $\dim X < \infty$. *Then* $\dim X > k$ *if and only if there is an essential cycle* $\underline{\gamma} \in Z_k^{\infty}(X, D)$ *such that* $\gamma \sim \underline{0}$ *in* X.

In this chapter we shall prove that the conclusion of Theorem 41 remains valid when the hypothesis of compactness is replaced by the weaker assumption that the space is a locally countable union of locally compact subspaces. For this purpose we introduce some new terminology.

6.2. Compactly dimensioned spaces. We shall say that a metric space (X, ϱ) satisfies the *Alexandroff equivalence* if the following statement holds:

(1) $\dim X > k$ if and only if there is an essential cycle $\underline{\gamma} \in Z_k^{\infty}(X, D)$ such that $\underline{\gamma} \sim \underline{0}$ in X.

We may now restate Theorem 41 in the following way.

THEOREM 42. *Let* (X, ϱ) *be a compact metric space with* $\dim X < \infty$. *Then* X *satisfies the Alexandroff equivalence.*

The following apparently stronger form of Theorem 42 suggests the conclusion desired for a generalization of the classical Alexandroff theorem.

THEOREM 43. *Let* (X, ϱ) *be a compact metric space with* $\dim X < \infty$. *Then each closed subspace of* X *satisfies the Alexandroff equivalence.*

Proof. If A is a closed subspace of X, then A is itself a compact metric space with $\dim A < \infty$. Thus A satisfies the Alexandroff equivalence by Theorem 42.

The next theorem gives a necessary and sufficient condition that a finite dimensional (not necessarily compact) metric space satisfy the Alexandroff equivalence. We first state a definition.

A finite dimensional metric space (X, ϱ) is said to be *compactly dimensioned* if there is a compact subspace X_0 of X such that $\dim X_0 = \dim X$.

THEOREM 44. *Let* (X, ϱ) *be a metric space with* $\dim X < \infty$. *Then* X *satisfies the Alexandroff equivalence if and only if* X *is compactly dimensioned.*

Proof. We first assume that X is compactly dimensioned and show that the Alexandroff equivalence is satisfied. If $\dim X > k$, then $\dim X_0 > k$ for some compact subspace X_0 of X, since X is compactly dimensioned. Then X_0 is a finite dimensional compact metric space and $\dim X_0 > k$

so that by Theorem 42 there is an essential cycle $\underline{\gamma} \in Z_k^{\infty}(X_0, D)$ such that $\underline{\gamma} \sim \underline{0}$ in X_0. It follows from Theorem 3 that $\underline{\gamma} \in Z_k^{\infty}(X, D)$ and that $\underline{\gamma} \sim \underline{0}$ in X. Moreover, Theorem 5 implies that $\underline{\gamma}$ is essential in X. On the other hand, if there is an essential cycle $\underline{\gamma} \in Z_k^{\infty}(X, D)$ such that $\underline{\gamma} \sim \underline{0}$ in X, then there is an infinite chain $\underline{\varkappa} \in C_{k+1}^{\infty}(X, D)$ such that $\partial\underline{\varkappa} = \underline{\gamma}$. Let X_0 be a carrier of $\underline{\varkappa}$. Then X_0 is compact and is also a carrier of $\underline{\gamma}$. Theorem 4 implies that $\underline{\varkappa} \in C_{k+1}^{\infty}(X_0, D)$, $\underline{\gamma} \in Z_k^{\infty}(X_0, D)$, and consequently that $\underline{\gamma} \sim \underline{0}$ in X_0. Another application of Theorem 5 yields the result that $\underline{\gamma}$ is essential in X_0. Therefore X_0 is a finite dimensional compact metric space such that there is an essential cycle $\underline{\gamma} \in Z_k^{\infty}(X_0, D)$ with $\underline{\gamma} \sim \underline{0}$ in X_0. It follows from Theorem 42 that $\dim X_0 > k$. Hence $\dim X > k$ by the monotonicity of the dimension function. Thus we have shown that if X is compactly dimensioned, then the Alexandroff equivalence must be satisfied. Now suppose that X satisfies the Alexandroff equivalence and that $\dim X = n$. Since $\dim X > n-1$ there is an essential cycle $\underline{\gamma} \in Z_{n-1}^{\infty}(X, D)$ such that $\underline{\gamma} \sim \underline{0}$ in X. Then there is an infinite chain $\underline{\varkappa} \in C_n^{\infty}(X, D)$ with $\partial\underline{\varkappa} = \underline{\gamma}$. Letting X_0 be a carrier of $\underline{\varkappa}$ we see in the first part of the proof that X_0 is a finite dimensional compact metric space, that $\underline{\gamma} \in Z_{n-1}^{\infty}(X_0, D)$ with $\underline{\gamma} \sim \underline{0}$ in X_0, and that $\underline{\gamma}$ is essential in X_0. Theorem 42 then implies that $\dim X_0 > n-1$. Since monotonicity of the dimension function implies that $\dim X_0 \leqslant n$, it follows that $\dim X_0 = n$. Thus X is compactly dimensioned.

The two examples which follow indicate the nature of the difficulty of extending the classical Alexandroff theorem to a more general class of metric spaces.

EXAMPLE 1. Not every finite dimensional metric space is compactly dimensioned. In fact there is a separable metric space X with $\dim X = 1$ such that X is not compactly dimensioned. To obtain such a space we appeal to a well-known example due to Knaster and Kuratowski [16]. A simple description of the example appears in the book of Hurewicz and Wallman [14]. In the plane $\tilde{R}^2$ let C denote the Cantor set constructed on the unit interval $\{(x, y)\colon 0 \leqslant x \leqslant 1, y = 0\}$ by successive deletion of open middle thirds. Let $a = (\frac{1}{2}, \frac{1}{2})$ and let $a \cdot C$ denote the join of the point a with the set C; that is, $a \cdot C = \bigcup \{\overline{ap}\colon p \in C\}$, where $\overline{ap}$ denotes the line segment joining a and p. Now if $p \in C$ and p is an endpoint of one of the deleted intervals, let A_p denote the collection of all points (x, y) of $\overline{ap}$ for which y is rational. If $p \in C$ and p is not such an endpoint, let A_p denote the collection of all points (x, y) of $\overline{ap}$ for which y is irrational. Then the set $A = \bigcup \{A_p\colon p \in C\}$ is a connected subset of $a \cdot C$ and the set $X = A \backslash \{a\}$ is totally disconnected. Thus X is a totally disconnected separable metric space and, moreover, $\dim X = 1$ [14]. We show now that X is not compactly dimensioned. Suppose that K is a compact non-empty subset of X.

Then K is a compact totally disconnected metric space. It follows that K is homeomorphic to a subset of the Cantor set C [13], p. 100. But $\dim C = 0$ and hence $\dim K \leqslant 0$ since dimension is a topological invariant and is monotone. Therefore $\dim K \neq \dim X$ and we conclude that X is not compactly dimensioned.

EXAMPLE 2. A finite dimensional metric space may be compactly dimensioned and yet have closed subspaces which are not compactly dimensioned. Let X be the 1-dimensional separable metric space described in Example 1. Let Y denote the plane subset $\{(x, y)\colon 2 \leqslant x \leqslant 3, y = 0\}$. Then $Z = X \cup Y$ is a subset of the plane such that $\dim Z = 1$. Now Y is a compact subset of Z and $\dim Y = \dim Z = 1$ so that Z is compactly dimensioned. However, X is a closed subset of Z and as we saw in Example 1, X is not compactly dimensioned.

6.3. The generalized Alexandroff theorem. The next five theorems are lemmas to be used in proving the above mentioned generalization of Theorem 41.

THEOREM 45. *Let (X, ϱ) be a metric space. If X locally compact, then X has a locally finite covering by compact sets.*

Proof. Since X is a metric space, each open covering of X has an open locally finite refinement; that is, X is paracompact. It follows that each open covering of X has a closed locally finite refinement [15], p. 156. Now because X is locally compact there is for each $x \in X$ an open neighbourhood U_x of x whose closure $\bar{U}_x$ is compact. The collection $\mathscr{U} = \{U_x\colon x \in X\}$ is an open covering of X. Let $\mathscr{F} = \{F_\gamma\colon \gamma \in \Gamma\}$ be a closed locally finite refinement of $\mathscr{U}$. Then $\mathscr{F}$ is a locally finite covering of X by the sets F_γ. Moreover, for each $\gamma \in \Gamma$, F_γ is compact since F_γ must lie in some set U_x (because $\mathscr{F}$ refines $\mathscr{U}$) and hence F_γ is a closed subset of the compact set $\bar{U}_x$.

THEOREM 46. *Let (X, ϱ) be a metric space with $\dim X < \infty$. If X is locally compact, then X is compactly dimensioned.*

Proof. Suppose that $\dim X = n$. Theorem 45 implies that X has a locally finite covering $\mathscr{F} = \{F_\gamma\colon \gamma \in \Gamma\}$ such that each F_γ is compact. Then $\mathscr{F}$ is a locally countable covering by closed sets and hence the Sum Theorem of dimension theory [21] may be applied. According to this theorem, if $\dim F_\gamma \leqslant n-1$ for each $\gamma \in \Gamma$, then $\dim X \leqslant n-1$, contrary to the fact that $\dim X = n$. Therefore there exists $\hat{\gamma} \in \Gamma$ such that $\dim F_{\hat{\gamma}} = n$. Since $F_{\hat{\gamma}}$ is a compact set with $\dim F_{\hat{\gamma}} = \dim X$ we conclude that X is compactly dimensioned.

THEOREM 47. *Let (X, ϱ) be a metric space. If L is a locally compact subset of X, then L is a countable union of closed locally compact sets.*

Proof. Since L is locally compact there exists sets V, open in X, and F, closed in X, such that $L = V \cap F$, [9], p. 239. The set V, being an open subset of a metric space, is an F_σ, and hence may be written $V = \bigcup \{C_j\colon j \in \tilde{N}\}$, where each C_j is a closed subset of X. Thus $L = \bigcup \{C_j\colon j \in \tilde{N}\} \cap F = \bigcup \{C_j \cap F\colon j \in \tilde{N}\}$ so that L is a countable union of the closed sets $C_j \cap F$. Moreover, each set $C_j \cap F$ is a locally compact subspace of L since it is a closed subset of L and L is locally compact.

THEOREM 48. *Let (X, ϱ) be a metric space. If X is a locally countable union of locally compact subspaces, then X is a locally countable union of closed locally compact subspaces.*

Proof. Suppose that $X = \bigcup \{L_\gamma\colon \gamma \in \Gamma\}$, where $\{L_\gamma\colon \gamma \in \Gamma\}$ is a locally countable collection of locally compact subspaces. Theorem 47 implies that each L_γ can be written $L_\gamma = \bigcup \{C_\gamma^i\colon i \in \tilde{N}\}$, where each C_γ^i is a closed locally compact subspace of X. Therefore we may write

$$X = \bigcup \{\bigcup \{C_\gamma^i\colon i \in \tilde{N}\}\colon \gamma \in \Gamma\} = \bigcup \{C_\gamma^i\colon \gamma \in \Gamma, i \in \tilde{N}\}.$$

Then X is a union of closed locally compact subspaces C_γ^i and it remains only to show that the collection $\{C_\gamma^i\colon \gamma \in \Gamma, i \in \tilde{N}\}$ is locally countable. Let $p \in X$. Since the collection $\{L_\gamma\colon \gamma \in \Gamma\}$ is locally countable there is an open neighbourhood U of p such that U meets at most countably many of the sets L_γ. Say U meets $L_{\gamma_1}, L_{\gamma_2}, \ldots$; that is, $U \cap L_\gamma = \emptyset$ for $\gamma \neq \gamma_1, \gamma_2, \ldots$ Then $U \cap C_\gamma^i = \emptyset$ for $\gamma \neq \gamma_1, \gamma_2, \ldots$ and for each $i \in \tilde{N}$, since $C_\gamma^i \subseteq L_\gamma$ for each $i \in \tilde{N}$. Therefore if $U \cap C_\gamma^i \neq \emptyset$ it follows that $\gamma \in \{\gamma_1, \gamma_2, \ldots\}$ and $i \in \tilde{N}$. But the collection $\{C_\gamma^i\colon \gamma \in \{\gamma_1, \gamma_2, \ldots\}, i \in \tilde{N}\}$ is countable and hence we have shown that U meets at most countably many of the sets C_γ^i.

THEOREM 49. *Let (X, ϱ) be a metric space with* $\dim X < \infty$. *If X is a locally countable union of locally compact subspaces, then X is compactly dimensioned.*

Proof. Theorem 48 implies that X is a locally countable union of closed locally compact subspaces, say $X = \bigcup \{F_\gamma\colon \gamma \in \Gamma\}$, where $\{F_\gamma\colon \gamma \in \Gamma\}$ is a locally countable collection and each F_γ is a closed locally compact subspace of X. Suppose that $\dim X = n$. Since $\{F_\gamma\colon \gamma \in \Gamma\}$ is a locally countable covering of X by closed sets, the Sum Theorem of dimension theory may again be applied. As in the earlier application we obtain the result that there exists $\hat{\gamma} \in \Gamma$ such that $\dim F_{\hat{\gamma}} = n$. Since $F_{\hat{\gamma}}$ is locally compact, Theorem 46 implies that $F_{\hat{\gamma}}$ is compactly dimensioned. Hence there is a compact set $F_0 \subseteq F_{\hat{\gamma}}$ such that $\dim F_0 = \dim F_{\hat{\gamma}} = n$. Then F_0 is a compact subset of X such that $\dim F_0 = \dim X$ and consequently X is compactly dimensioned.

Finally we are able to prove the generalization of the classical Alexandroff dimension theorem which was mentioned in Section 6.1.

THEOREM 50. *Let (X, ϱ) be a metric space with* $\dim X < \infty$. *Assume* *that is a locally countable union of locally compact subspaces. Then each closed subspaces of X satisfies the Alexandroff equivalence.*

Proof. Let $\{F_\gamma\colon \gamma \in \Gamma\}$ be a locally countable collection of locally compact subspaces of X such that $X = \bigcup\{F_\gamma\colon \gamma \in \Gamma\}$. Suppose that A is a closed subspace of X. Then A is a metric space and $\dim A < \infty$. We show that A is a locally countable union of locally compact subspaces. Since $A = A \cap X$ we may write $A = A \cap \bigcup\{F_\gamma\colon \gamma \in \Gamma\} = \bigcup\{A \cap F_\gamma\colon \gamma \in \Gamma\}$. Each of the sets $A \cap F_\gamma$ is locally compact since A closed in X implies that $A \cap F_\gamma$ is closed in the locally compact space F_γ. Thus A is a union of the locally compact subspaces $A \cap F_\gamma$. If $a \in A$, then $a \in X$ and since $\{F_\gamma\colon \gamma \in \Gamma\}$ is a locally countable collection of subspaces of X there is an X-open set U_a such that $a \in U_a$ and U_a meets at most countably many of the sets F_γ. But then $A \cap U_a$ is an A-open set such that $a \in A \cap U_a$ and evidently $A \cap U_a$ meets at most countably many of the sets $A \cap F_\gamma$. Hence $\{A \cap F_\gamma\colon \gamma \in \Gamma\}$ is a locally countable collection of subspaces of A. It now follows from Theorem 49 that A is compactly dimensioned. Therefore A satisfies the Alexandroff equivalence according to Theorem 44.

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