COMPUTATIONAL MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 13
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1984

VARIABLE MULTISTEP METHODS

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1. Introduction

In the last few years we could read terms like "variable stepsize method", "variable formula method", "variable order method", "cyclic method", and "method with variable coefficients", "adaptive integration method", "generalized multistep method", "parametric method" in many papers concerned with numerical methods for ordinary initial value problems.

In all these cases the situation is such that in fact the integration method may change from one step to the next. This interchange can be produced by

step-changing,
order-changing and formula-switching,
parameter-alteration.

We realize step-changing either by using a formula constructed for variable stepsize directly — in this case the integration formula may change stepwise by itself — or we combine a formula constructed for constant stepsize with a step-changing technique, i.e., with a technique of computing the required additional values for the next application of the constant-step formula. The combinations of constant-step formulas with step-changing techniques can also be called stepwise varying multistep methods.

Obviously, all methods in which formulas of the same type but of different order are used (for instance, in the well-known subroutine DIFSUB of C.W.Gear in [5]) are step-by-step variable methods. A similar situation arises when formulas of the same order but of different type are combined, as was done by Z. Zlatev ([20]).

All methods depending on one or more parameters which can be chosen in advance or automatically in an adaptive way per integration step naturally form step-by-step variable methods. E. F. Sarkany and W. Liniger ([15]), R. März ([10]) and others have investigated such parametric methods.

Consequently almost all of our methods in their implemented form change stepwise. But there are only a few papers dealing with this variability, whereas numerous books and papers discuss the "constant" methods, their consistency, stability, convergence and asymptotic behaviour on great integration intervals. Therefore, if we want to bring the theory closer to the way in which the methods are really applied, we must admit and investigate stepwise variable methods.

2. Stability and consistency of variable multistep methods over grid-classes

Consider the initial value problem

$$\begin{cases} \dot{x}(t) = f\big(x(t), t\big), & t \in [t_0, T], \\ x(t_0) = x_0 \end{cases}$$

in which the continuous vector-valued function $f\colon \boldsymbol{R}^m \times [t_0, T] \to \boldsymbol{R}^m$ is assumed to satisfy the usual Lipschitz condition with a constant L_f. Denote by $x_*(\cdot)$ the exact solution of this initial value problem.

Let the grid $t_0 < t_1 < \ldots < t_N = T$ on $[t_0, T]$ be given. Denote by $h_l = t_l - t_{l-1}$, $l = 1, \ldots, N$, $h_{\max} = \max\limits_{l=1,\ldots,N} h_l$, $h_{\min} = \min\limits_{l=1,\ldots,N} h_l$ the lth stepsize, the maximal and minimal stepsizes of the chosen grid, respectively. As we have agreed, we try to approximate the values $x_*(t_l)$ according to the grid points by values x_l, which are the solutions of the nonlinear equations

$$\sum_{j=0}^{k} a_{l,j}^s x_{l-j}^s = h_l \varphi_l^s(x_l, \ldots, x_{l-k}), \tag{1}$$
$$a_{k,0}^s = 1, \quad s = 1, \ldots, m, \quad l = k, \ldots, N.$$

We assume the starting values $x_0, \ldots, x_{k-1}$ to be given. In (1) different formulas can be used in each component and each step. Especially the functions $\varphi_l^s\colon \boldsymbol{R}^{m(k+1)} \to \boldsymbol{R}^1$, $s = 1, \ldots, m$, $l = k, \ldots, N$, are usually different from each other. All the methods mentioned above have form (1) or can easily be transformed into (1).

As a simple example we want to mention the parametric modification of the two-step variable stepsize Adams–Bashforth method ([10], p. 11)

$$x_l^s - x_{l-1}^s = h_l \left(f_{l-2}^s + \frac{h_{l-1}}{h_l} b \left(a_l^s h_{l-1}, \frac{h_l}{h_{l-1}} \right) (f_{l-1}^s - f_{l-2}^s) \right),$$

with parameters $a_l^s \in \boldsymbol{R}^1$, $s = 1, \ldots, m$, $l = 2, \ldots, N$, and the coefficient-function

$$b(\beta, k) = \begin{cases} \dfrac{(1+k)^2-1}{2} & \text{if } \beta = 0, \\ \dfrac{k+e^{-\beta}\beta^{-1}(e^{-k\beta}-1)}{1-e^{-\beta}} & \text{otherwise.} \end{cases}$$

Let G_{equ} be the class of all equidistant grids on $[t_0, T]$. Consider a grid-class G covering at least all equidistant grids, i.e., $G_{\mathrm{equ}} \subset G$.

DEFINITION. A *variable k-step method M_k over the grid-class G* is said to be the set of the finite sequences of formulas (1) according to the grids from G.

Among (1) a variable k-step method may contain some formulas degenerating into k_l-step formulas, $k_l < k$. Let us agree to speak of a k-step method if at least one of the formulas contained in (1) is a "proper" k-step formula. Denote by $A(M_k/G) \subset \Pi_k$ the set of all characteristic polynomials $\sum_{j=0}^{k} a_{l,j}^s \lambda^{k-j}$ occurring in the variable k-step method M_k over G, and by $\Phi(M_k/G) \subset [\boldsymbol{R}^{m(k+1)}, \boldsymbol{R}^1)$ the set of all functions φ_l^s. The set $\Phi(M_k/G)$ may be a finite one or an infinite one. The set $A(M_k/G)$ is finite in the methods published so far.

In practice, in implementations we have to select step-by-step the next stepsize in such a way that the resulting grid remains in the given class G.

Note that C. W. Gear and D. S. Watanabe ([4]) described a variable formula method in a similar way as "a set of formulas $\{F_i\}$, step and formula changing techniques and step and formula selection schemes". C. W. Gear suggested the use of the step selection function θ: $[t_0, T] \times \times (0, T-t_0] \to \boldsymbol{R}^1$ as a step selection scheme, i.e., a function with the properties

$$0 < \Delta \leqslant \theta(t, h) \leqslant 1,$$

$$h_l = h_{\max}\theta(t_{l-1}, h_{\max}), \quad l = 1, \ldots, N.$$

The step selection function defines the next stepsize h_{l+1} subject to the grid point t_l and the admissible maximal stepsize of the grid, but without a memory with respect to the last steps. Because of the possibility of applying implication of the form

$$\text{"if } h_l < h_{l-1}, \text{ then } h_{l+1} \leqslant h_l\text{"}$$

or similar conditions, the use of grid-classes will be advantageous.

Such kinds of restrictions on the grid are used in almost all implementations. On the other hand, they are also important for the stability of some methods.

Let us now deal with the stability of the variable multistep methods, the basic property for all numerical integration methods.

DEFINITION. The variable k-step method M_k over G is called *stable* iff for any grid $t_0 < t_1 < \ldots < t_N = T$ from G and each arbitrary collection of starting values $z_0, \ldots, z_{k-1} \in \boldsymbol{R}^m$ and perturbations $d_l \in \boldsymbol{R}^m$, $l = k, \ldots, N$, the equation system

$$\sum_{j=0}^{k} a_{l,j}^s z_{l-j}^s = h_l \varphi_l^s(z_l, \ldots, z_{l-k}) + h_l d_l^s, \qquad s = 1, \ldots, m,\ l = k, \ldots, N,$$

is uniquely solvable and if for each two solutions $\{z_l\}_{l=k}^N$, $\{\bar{z}_l\}_{l=k}^N$ corresponding to the two collections of starting values and perturbations $\{z_l\}_{l=0}^{k-1}$, $\{d_l\}_{l=k}^N$ and $\{\bar{z}_l\}_{l=0}^{k-1}$, $\{\bar{d}_l\}_{l=k}^N$ the inequality (1)

$$(2)\qquad \max_{l=k,\ldots,N} |z_l - \bar{z}| \leqslant K\big(\max_{l=0,\ldots,k-1} |z_l - \bar{z}_l| + \max_{l=k,\ldots,N} |d_l - \bar{d}_l|\big)$$

holds with a global constant K for the whole class G.

Emphasize once more that the constant K is independent of the chosen grid from G. Especially, K is independent of the number N.

If G consists of the grid sequence $t_0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{N_n}^{(n)} = T$, $n \in nz$, with $h_{\max}^{(n)} \xrightarrow[n\to\infty]{} 0$, then our stability "over G" means the usual stability (or so-called zero-stability or inverse stability). Mostly only sequences of equidistant grids are considered here. For implementations we are not interested in infinitely small stepsizes. In practice we like to choose the stepsizes depending on the error control freely within the bounds of the grid-class.

The widest grid-classes we are going to use are classes $G_w = G_w(k_{\min}, k_{\max}, B)$ of all grids with

$$(3)\qquad \begin{cases} 0 < k_{\min} \leqslant h_l/h_{l-1} \leqslant k_{\max} < \infty, \\ N \cdot h_{\max} \leqslant B, \end{cases}$$

where $k_{\min}, k_{\max}, B$ are global constants for the class, i.e., are independent of the special grid.

The "linearly convergent grid sequence" considered by H. J. Stetter ([17]) — here the inequality

$$\frac{c_1}{N_n} \leqslant h_l^{(n)} \leqslant \frac{c_2}{N_n}, \qquad n \in nz,$$

(1) By $|\cdot|$ we denote the maximum-norm.

is assumed to be valid — fulfils conditions (3) and forms a subclass of $G_w(c_1/c_2, c_2/c_1, c_2)$. Moreover, conditions (3) are assumed to be fulfilled in all papers dealing with variable stepsize (for instance in [3], [8], [14], [20]).

As usual, the local and global errors produced by M_k with respect to a chosen grid from G we define as

$$\tau_l^s = \frac{1}{h_l}\sum_{j=0}^{k} a_{l,j}^s x_*^s(t_{l-j}) - \varphi_l^s\big(x_*(t_l), \ldots, x_*(t_{l-k})\big),$$
$$s = 1, \ldots, m, \quad l = k, \ldots, N,$$

and

$$\varepsilon_l = x_*(t_l) - x_l, \quad l = k, \ldots, N.$$

Therefore, with a stable variable k-step method M_k over G the global error estimation

$$\max_{l=k,\ldots,N} |\varepsilon_l| \leqslant K\big(\max_{l=0,\ldots,k-1} |\varepsilon_l| + \max_{l=k,\ldots,N} |\tau_l|\big) \tag{4}$$

holds. There $\varepsilon_l = x_*(t_l) - x_l$, $l = 0, \ldots, k-1$ are the errors in the starting values.

If we complete a stable method M_k over G with formulas for computing starting values

$$\sum_{j=0}^{l} a_{l,j}^s x_{l-j}^s = h_l \varphi_l^s(x_l, \ldots, x_0), \quad s = 1, \ldots, m, \ l = 1, \ldots, k-1,$$

to a so-called selfstarting method M_k over G, we easily obtain

$$\max_{l=1,\ldots,N} |\varepsilon_l| \leqslant \tilde{K} \max_{l=1,\ldots,N} |\tau_l|,$$

if the functions $\varphi_l^s\colon \mathbf{R}^{m(l+1)} \to \mathbf{R}^1$, $s = 1, \ldots, m$, $l = 1, \ldots, k-1$, are Lipschitz-continuous, and by using implicit formulas the corresponding stepsizes are sufficiently small.

DEFINITION. The variable k-step method M_k over G is said to be *consistent with the above initial value problem with the consistency order* $p > 0$ iff

$$|\tau_l| \leqslant C\big(x_*(\cdot)\big)\big(\max_{j=0,\ldots,k-1} h_{l-j}\big)^p, \quad l = k, \ldots, N,$$

holds for all grids from G with a class-global constant $C\big(x_*(\cdot)\big)$.

The proof of consistency for row-wise different formulas is not trivial in general. E. Griepentrog ([6]) investigated the consistency of row-wise different one-step methods. In the case of linear multistep methods

$$\sum_{j=0}^{k} a_{l,j}^{s} x_{l-j}^{s} = h_l \sum_{j=0}^{k} (b_{l,j}^{s} f_{l-j}^{s} + c_{l,j}^{s} x_{l-j}^{s}), \quad l = k, \ldots, N, \quad s = 1, \ldots, m,$$

the consistency follows from the consistency of all single methods used in the rows.

With a stable and consistent method M_k over G, for each tolerancy $\delta > 0$ we can select a grid from G for which the inequality

$$\max_{l=k,\ldots,N} |\tau_l| \leqslant \delta$$

is fulfilled. The existence of such a grid is secured by the consistency and because of $G \supset G_{\text{equ}}$. We might choose an equidistant grid with a sufficiently small stepsize. In practice the grid is selected step-by-step by using local error estimations. With (4) for our grid the inequality

$$\max_{l=k,\ldots,N} |\varepsilon_l| \leqslant K\delta$$

holds.

Note that in general the constant K grows as a consequence of the widening of the grid-class G. But the restriction to the smallest grid-class G_{equ} is not ingenious, as the computations have shown for a long time. On the other hand, with grid-classes being too wide, very large constants may arise and there is really no difference to the instability (comp. [3]). Therefore it is an important task to discover useful grid-classes for the methods.

3. Some results concerning stability

In the following we denote by G^* a subclass of G containing grids with sufficiently small stepsizes.

P. Piotrowski ([14]) proved

PROPOSITION 1. *The variable multistep method M_k consisting of Adams–Moulton formulas of fixed order which are directly constructed for variable stepsize over G_w^* is stable.*

PROPOSITION 2. *The variable k-step methods consisting either of variable-step Adams–Moulton formulas with variable orders $2 \leqslant p_l \leqslant k+1$, or of variable-step variable-order Adams–Bashforth formulas with orders $1 \leqslant p_l \leqslant k$ or of k_l-step Adams–Bashforth–Moulton predictor-corrector formulas directly designed for variable stepsize with $k_l \leqslant k$ are stable over G_w^* resp. G_w.*

The proof may be found in [4], [10].

PROPOSITION 3. *The variable k-step method consisting of formulas of* *the kind*

$$x_l = a(s_l)x_{l-1} + \big(1-a(s_l)\big)x_{l-2} + h_l \sum_{j=0}^{s_l} b_{l,j} f_{l-j}, \tag{5}$$

$$s_l \leqslant k, \quad a(1) = 1, \quad 0 < a(n) < 2 \quad \textit{for } n = 1, \ldots, k,$$

where the coefficients $b_{l,j}$, $j = 0, \ldots, s_l$, *are computed in such a way that* (5) *in the l-th step has order* s_l *in case* $b_{l,0} = 0$ *and order* s_l+1 *otherwise, is stable over* G_w *or* G_w^* *if implicit formulas are used.*

The variable k-step methods consisting in predictor-corrector formulas with correctors of type (5) *are stable over* G_w.

For the proof see [20].

Note that in (5) with $a(s_l) = 1$ the Adams-formulas occur. With $a(s_l) = 0$, (5) gives the Nyström or the Milne–Simpson formula. Notice that the case $a(s_l) = 0$ is not admitted in Proposition 3.

In the three propositions formulated above the sets $A(M_k/G_w)$ (or $A(M_k/G_w^*)$) are finite ones. In Propositions 1, 2 they contain the single polynomial $\lambda^k - \lambda^{k-1}$ only. In Proposition 3 the set $A(M_k/G_w)$ of the method M_k according to a given function a: $\{1, 2, \ldots, k\} \to (0, 2)$ with $a(1) = 1$ covers the polynomials

$$\lambda^k - a(n)\lambda^{k-1} - \big(1-a(n)\big)\lambda^{k-2}, \qquad n = 1, 2, \ldots, k.$$

Furthermore, in all variable k-step methods M_k that can be found in Propositions 1–3 the functions φ_l^s from $\Phi(M_k/G_w)$ are collectively Lipschitz-continuous, i.e., there exists a constant L_Φ with

$$|\varphi(z_0, \ldots, z_k) - \varphi(\bar{z}_0, \ldots, \bar{z}_k)| \leqslant L_\Phi \sum_{j=0}^{k} |z_j - \bar{z}_j|$$

for all $z_j, \bar{z}_j \in \boldsymbol{R}^m$, $j = 0, \ldots, k$, and each $\varphi \in \Phi(M_k/G_w)$. Of course the Lipschitz-constant L_Φ depends on the function f of the original problem.

For the purely linear interpolation methods written as

$$\sum_{j=0}^{k} a_{l,j} x_{l-j} = h_l \sum_{j=0}^{k} b_{l,j} f_{l-j}$$

the Lipschitz-continuity of the corresponding φ_l^s follows from the uniform boundedness of the coefficients $b_{l,j}$ resulting from properties (3) of the grid G_w and from the Lipschitz-continuity of the function f. With predictor-corrector methods we obtain the Lipschitz-continuity by successive estimations.

Before we can formulate the next proposition we are still occupied with the set $A(M_k/G_k)$. For each polynomial

$$p(\lambda) \equiv \sum_{j=0}^{k} a_{l,j}^s \lambda^{k-j}$$

we define the $k\times k$ Frobenius matrix

$$\begin{bmatrix} -a^s_{l,1} & \dots & & -a^s_{l,k} \\ 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix},$$

whose eigenvalues are the roots of the corresponding polynomial $p(\lambda)$. If this polynomial satisfies the root-condition formulated by G. Dahlquist, then there are such $\boldsymbol{R}^k$-norms $|\cdot|_{l,s}$ that for the corresponding induced matrix-norm the inequality

$$\|A_{l,s}\|_{l,s}\leqslant 1$$

holds.

Because of the equivalence of all $\boldsymbol{R}^k$-norms we also find norm transition coefficients

$$\varrho_{i,l,s}>0,\quad i=1,2,\qquad \gamma_{l,s}>0$$

with

$$\varrho_{1,l,s}|w|_{l,s}\leqslant |w|\leqslant \varrho_{2,l,s}|w|_{l,s},$$

$$|w|_{l,s}\leqslant \gamma_{l,s}|w|_{l-1,s},\qquad w\in\boldsymbol{R}^k.$$

Now, if we allow finitely many characteristic polynomials different from each other in M_k over G only, then $A(M_k/G)$ is a finite set and we find ϱ_1, ϱ_2 with

$$0<\varrho_1=\min_{\substack{s=1,\dots,m\\ l=k,\dots,N\\ \text{grid from } G}}\varrho_{1,l,s},\qquad \infty>\varrho_2=\max_{\substack{s=1,\dots,m\\ l=k,\dots,N\\ \text{grid from } G}}\varrho_{2,l,s}$$

and

$$\varrho_1|w|_{l,s}\leqslant |w|\leqslant \varrho_2|w|_{l,s},\qquad w\in\boldsymbol{R}^k,$$

holds for $s=1,\dots,m$, $l=k,\dots,N$, and all grids from the given grid-class G.

PROPOSITION 4. *Let a variable k-step method M_k over G_w (or G_w^* respectively) be given. Let $A(M_k/G_w)$ be finite. Let the root-condition of Dahlquist be fulfilled for each polynomial $p\in A(M_k/G_w)$. The functions $\varphi\in\Phi(M_k/G_w)$ are assumed to be collectively Lipschitz-continuous.*

Then the following stability-like inequality holds:

$$\max_{l=k,\dots,N}|z_l-\bar z_l|\leqslant K_1\Gamma_g\Big(\max_{l=0,\dots,k-1}|z_l-\bar z_l|+\max_{l=k,\dots,N}|d_l-\bar d_l|\Big)$$

where K_1 is a class-global constant and

$$\Gamma_g = \prod_{j=k}^{N} \max\{1, \max_{s=1,\dots,m} \gamma_{j,s}\}$$

depends on the chosen grid g from G_w.

If the constants Γ_g are bounded on the class $G \subset G_w$, $\Gamma_g \leqslant \Gamma$, $g \in G$, then M_k over G is stable with the stability-constant $K = K_1 \Gamma$.

The proof is explained extensively in [10]. It is obtained by collecting the interesting equations component-wise, introducing the k-tuples

$$u_l^s = \begin{bmatrix} z_l^s - \bar{z}_l^s \\ \vdots \\ z_{l-k+1}^s - \bar{z}_{l-k+1}^s \end{bmatrix}$$

and taking some trivial equations to

$$u_l^s = A_{l,s} u_{l-1}^s + h_l \begin{bmatrix} \varphi_l^s(z_l, \dots, z_{l-k}) - \varphi_l^s(\bar{z}_l, \dots, \bar{z}_{l-k}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + h_l \begin{bmatrix} d_l^s - \bar{d}_l^s \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$s = 1, \dots, m$, $l = k, \dots, N$.

From this the asserted inequality is derived by successive estimations.

Note that Proposition 4 is valid for grid-classes in which the condition

$$0 < k_{\min} \leqslant \frac{h_l}{h_{l-1}} \leqslant k_{\max} < \infty$$

is dropped. But, in practice this extension is useless. From Proposition 4 we get sufficient stability conditions for a great many variable multistep methods. First of all Propositions 1, 2 and Proposition 3 with functions α: $\{1, 2, \dots, k\} \to [0, 1]$ are their special cases. In fact, in these cases only matrices $A_{l,s}$ of the type

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & \dots 1 \dots & 0 \\ 1 & 0 & \dots \\ \dots & \dots & \dots \\ 0 & \dots & 1 \; 0 \end{bmatrix}, \quad \text{or}$$

$$\begin{bmatrix} \alpha(n) & 1-\alpha(n) & 0 & \dots & 0 \\ 1 & 0 & & & \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & & 1 & 0 \end{bmatrix}, \quad \alpha(n) \in [0, 1], \; n = 1, \dots, k,$$

arise. Here we can obviously choose $|\cdot|_{l,s} = |\cdot|$ because the row-sum norm of these matrices is not greater than one. Therefore we have $\varrho_1 = \varrho_2 = 1$, $\gamma_{l,s} = 1$, $\Gamma_g = \Gamma = 1$ for all grids $g \in G_w$.

The above consideration is valid for all so-called interpolation formulas.

Further we find trivial stability conditions bounding the number of admissible formula-switchings with respect to the whole grid-class.

Moreover, from Proposition 4 we get the stability of the parametric multistep methods

$$\sum_{j=0}^{k} \hat{a}_{l,j} x_{l-j} = h \sum_{j=0}^{k} \left(\hat{b}_j(a_l h) f_{l-j} + \hat{c}_j(a_l h)\, a_l x_{l-j}\right) \tag{6}$$

over G_{equ}. In (6) the coefficients $\hat{b}_j(a_l h)$, $\hat{c}_j(a_l h)$ are $m \times m$ matrices. a_l is assumed to be a matrix-parameter. (In many papers one tries to compute a_l as an approximation of the Jacobian of f. In [10], [11] and [15] a_l are diagonal matrices whose elements are computed in an adaptive way.) The functions $\hat{c}_j(\cdot)$, $\hat{b}_j(\cdot)$ are assumed to be smooth and the domains in which the parameters a_l may alter are assumed to be bounded. Finally, let the matrices $\hat{a}_{l,j}$ have diagonal form, $a_{l,j} = \operatorname{diag}(a^1_{l,j}, \ldots, a^m_{l,j})$. More precisely, either we admit there stepwise constant coefficients, only,

$$\hat{a}_{l,j} \equiv \hat{a}_j \equiv \operatorname{diag}(a^1_j, \ldots, a^m_j) \quad \text{where} \quad \sum_{j=0}^{k} a^s_j \lambda^{k-j}, \quad s = 1, \ldots, m,$$

fulfil the root-condition, or we demand in addition to $\hat{a}_{l,0} = E$ that for each s and each l among the coefficients $a^s_{l,1}, \ldots, a^s_{l,k}$ a single one is equal to -1 and the others vanish. In the second case we are concerned with different parametric modifications of interpolation methods (e.g. [2], [8], [10], [13], [15], [16], [18]). Mostly parametric modifications of Adams methods are considered.

Note that the parametric modifications of classical multistep methods are usually designed with constant stepsize. There the variable-step forms are too expensive.

For the parametric modification of the two-step variable stepsize Adams–Bashforth method mentioned in Section 2, from Proposition 4 we get the stability over G_w if the domain for parameter-alteration is assumed to be bounded (comp. [11]). Obviously, for the parametric modification of the trapezoidal rule used with variable stepsize a similar result is valid.

4. Stability of combinations of constant-step formulas with step-changing techniques

Now, let us deal with the combinations of constant-step formulas with step-changing techniques. First we describe some step-changing techniques.

Let the values $x_1, \ldots, x_{l-\bar{k}}, \ldots, x_{l-1}$ approximating the $x_*(t_1), \ldots$ $\ldots, x_*(t_{l-1})$ be computed. To compute the next value $x_l = \bar{x}_l$ by means of the constant-step formula

$$(7) \qquad \sum_{j=0}^{\bar{k}} \bar{a}_{l,j}^s \bar{x}_{l-j}^s = h_l \bar{\varphi}_l^s(\bar{x}_l, \ldots, \bar{x}_{l-\bar{k}}), \quad \bar{a}_{l,0}^s = 1, \ s = 1, \ldots, m,$$

we need the values $\bar{x}_{l-1} = x_{l-1}$, $\bar{x}_{l-j}$, $j = 2, \ldots, \bar{k}$, approximating the exact solution $x_*(\cdot)$ at the points $\bar{t}_{l-j} = t_l - jh_l$, $j = 1, \ldots, \bar{k}$.

Step-changing technique 1 (SCHT1): If $\bar{t}_{l-j}$ coincide with the grid-point t_{l-i_j}, we put $\bar{x}_{l-j} = x_{l-i_j}$ and $\bar{f}_{l-j} = f_{l-i_j}$ as needed.

For each "assistant" point $\bar{t}_{l-j}$ not being a grid-point at the same time we seek the biggest smaller grid-point ("reference" point) t_{l-i_j}. Then we compute

$$\bar{x}_{l-j}^s = x_{l-i_j}^s + (\bar{t}_{l-j} - t_{l-i_j}) \hat{\varphi}_{l,i_j}^s(x_{l-i_j}), \quad s = 1, \ldots, m,$$

with a suitable explicit one-step method characterized by $\hat{\varphi}_{l,i_j}$. If the value $\bar{f}_{l-j}$ is also needed, we compute $\bar{f}_{l-j} = f(\bar{x}_{l-j}, \bar{t}_{l-j})$ or put $\bar{f}_{l-j} = \hat{\varphi}_{l,i_j}(x_{l-i_j})$.

Obviously, in this step-changing variant it may be necessary to keep accumulated very many of the saved values x_{l-j}, f_{l-j}.

Step-changing technique 2 (SCHT2): We interpolate $\bar{x}_{l-j}^s = P_l^s(\bar{t}_{l-j})$ where $P_l^s(\cdot)$ is an interpolation polynomial through the retained values $x_{l-\bar{k}}^s, \ldots, x_{l-1}^s$ and derivatives $f_{l-\bar{k}}^s, \ldots, f_{l-1}^s$ or through some of them.

In general the implementation of SCHT2 is just as complicated as that of multistep formulas constructed for variable stepsize directly. For instance, if for the construction of $P_l^s(\cdot)$ the interpolation conditions

$$P_l^s(t_{l-1}) = x_{l-1}, \quad \dot{P}_l^s(t_{l-j}) = f_{l-j}, \quad j = 1, \ldots, \bar{k},$$

are used, then the number of operations for the computation of

$$(8) \qquad \bar{x}_{l-j} = x_{l-1} + \sum_{\mu=1}^{\bar{k}} \int_{t_{l-1}}^{\bar{t}_{l-j}} \prod_{\substack{i=1 \\ i \neq \mu}}^{\bar{k}} \frac{s - t_{l-i}}{t_{l-\mu} - t_{l-i}} \, ds f_{l-j}$$

is comparable with the number of operations of the $\bar{k}$-step variable-stepsize Adams–Bashforth method.

The situation will be simpler if in (7) no $\bar{x}_{l-j}$ except $\bar{x}_l = x_l$, $\bar{x}_{l-1} = x_{l-1}$ are really contained. This situation arises, e.g., if (7) has the form

$$(9) \qquad \bar{x}_l^s - \bar{x}_{l-1}^s = h_l \sum_{j=0}^{\bar{k}} \bar{b}_{l,j}^s \bar{f}_{l-j}^s, \quad s = 1, \ldots, m.$$

(9) may represent variable order Adams formulas as constructed in [5]. For such methods the following modified technique is useful:

Step-changing technique 2' (SCHT2'): For the use of (9) we put $\bar{x}_{l-1} = x_{l-1}$, $\bar{f}_{l-j} = \tilde{P}_l(\bar{t}_{l-j})$, $j = 0, \ldots, \bar{k}$, where $\tilde{P}_l(\cdot)$ is the interpolation polynomial passed through $f_{l-\bar{k}}, \ldots, f_{l-1}, (f_l)$.

Step-changing technique 3 (SCHT3): Assume $h_{l-\bar{k}} = h_{l-\bar{k}+1} = \ldots \ldots = h_{l-1}$, $h_l \neq h_{l-1}$ (at least for $l = \bar{k}-1$). We interpolate as in SCHT2 and obtain the value $x_l = \bar{x}_l$ by the constant-step formula (7). But for the next step we save $\bar{x}_{l-\bar{k}+1}, \ldots, \bar{x}_l, \bar{f}_{l-\bar{k}+1}, \ldots, \bar{f}_l$ instead of $x_{l-\bar{k}+1}, \ldots \ldots, x_l, f_{l-\bar{k}+1}, \ldots, f_l$, and so on.

In this way we accumulate values for equidistant points in each step and we always change over to new equidistant points. A. Nordsieck and C. W. Gear noted very practicable formulas for the realization of this technique.

Step-changing technique 3' (SCHT3'): For the multistep formula (9) a special modification of SCHT3 is useful. As in SCHT2', the interpolation polynomial links the $f_{l-\bar{k}}, \ldots, f_{l-1}$, only. For the next step the values $\bar{f}_{l-\bar{k}+1}, \ldots, \bar{f}_l$ must be retained.

Sometimes it is advantageous to reduce the grid-class and allow only halvings and doublings of the stepsize. With this we keep the number of values which have to be computed additionally (e.g., in SCHT1) small. After a halving the stepsize is not doubled immediately, but kept for at least one further step. Define a grid-class $G_{HD} \subset G_w$ according to these restrictions. Let G_{HD} be the class consisting of all grids which have properties (3) which satisfy the following conditions:

$$h_{\max} \leqslant 2^p h_{\min},$$

$$\text{either } h_l = 2h_{l-1} \text{ or } h_l = h_{l-1} \text{ or } h_l = 2^{-i}h_{l-1} \text{ with } i \leqslant p,$$

$$\text{if } h_{l-1} \leqslant h_{l-2}/2, \text{ then } h_l \leqslant h_{l-1}.$$

Let the natural number p be global for the grid-class.

Let us now define a suitable variable k-step method over G_{HD} which combines the constant-step $\bar{k}$-step formula (7) with SCHT1.

First we put

$$k = 2^p\bar{k} - \sum_{i=0}^{p} (2^{p-i}-1) = 2^p(\bar{k}-2)+p+2.$$

The number k is calculated in such a way that $t_{l-k} \leqslant t_l - \bar{k}h_l$ holds.

Now we select those grid-points from our grid which coincide with assistant points. Certainly $t_l = \bar{t}_l$, $t_{l-1} = \bar{t}_{l-1}$ holds. Let $\bar{t}_{l-j_u^l} = t_{l-i_u^l}$, $u = 0, \ldots, n_l$, $n_l \geqslant 1$, $j_0^l = i_0^l = 0$, $j_1^l = i_1^l = 1$, be all these grid-points.

For any assistant point $\bar{t}_{l-j_u^l}$, $u = n_l+1, \ldots, \bar{k}$, not being a grid-point at the same time we choose the biggest smaller grid-point $t_{l-i_u^l}$ as a reference point. Note that while the grid-points or the assistant points are different from one another, the reference points may coincide. For example, the case $n_l = 1$, $j_u^l = u$, $i_u^l = 2$, $u = 2, \ldots, \bar{k}$, occurs as soon as $h_{l-1} > (\bar{k}-1)h_l$.

Yet, in any case $2 \leqslant i_u^l \leqslant k$ holds for $u = n_l+1, \ldots, \bar{k}$.

Now, we compose a k-step method (1) resulting from the combination of (7) and SCHT1 over G_{HD}. With

$$\bar{x}^s_{l-j_u^l} = x^s_{l-i_u^l}, \qquad u = 0, 1, \ldots, n_l,$$

$$\bar{x}^s_{l-j_u^l} = x^s_{l-i_u^l} + (\bar{t}_{l-j_u^l} - t_{l-i_u^l})\hat{\varphi}^s_{l,j_u^l}(x_{l-i_u^l}), \qquad u = n_l+1, \ldots, \bar{k},$$

we find

$$\sum_{u=0}^{\bar{k}} \bar{a}^s_{l,j_u^l} x^s_{l-i_u^l} = h_l \Big\{ \bar{\varphi}^s_l(\bar{x}_l, \ldots, \bar{x}_{l-k}) - \sum_{u=n_l+1}^{\bar{k}} \bar{a}^s_{l,j_u^l} \frac{1}{h_l} (\bar{t}_{l-j_u^l} - t_{l-i_u^l}) \times \\ \times \hat{\varphi}^s_{l,j_u^l}(x_{l-i_u^l}) \Big\}, \qquad s = 1, \ldots, m.$$

The expression in the braces on the right-hand side we sum up to the function $\varphi^s_l(x_l, \ldots, x_{l-k})$ in each row. Obviously, the resulting functions $\varphi^s_l(\cdot)$ are collectively Lipschitz-continuous if both the functions $\bar{\varphi}^s_l$ and the functions $\hat{\varphi}^s_{l,j}$ are collectively Lipschitz-continuous.

Finally we put

$$a^s_{l,0} = \bar{a}^s_{l,0} = 1, \qquad a^s_{l,1} = \bar{a}^s_{l,1},$$

$$a^s_{l,j} = \sum_{u=0}^{\bar{k}} \delta_{j,i_u^l} \bar{a}^s_{l,j_u^l}, \qquad j = 2, \ldots, k, \quad s = 1, \ldots, m,$$

to obtain

$$\sum_{j=0}^{k} a^s_{l,j} x^s_{l-j} = \sum_{u=0}^{\bar{k}} \bar{a}^s_{l,j_u^l} x^s_{l-i_u^l}.$$

The following situations may arise:

If t_{l-j} acts for $j \in \{2, \ldots, k\}$ neither as an assistant point nor as a reference point, then $a^s_{l,j} = 0$, $s = 1, \ldots, m$.

If $t_{l-j} = \bar{t}_{l-j}$, $j = 0, \ldots, \bar{k}$, then the set of reference points is empty and we have $n_l = \bar{k}$, $i_u^l = j_u^l = u$, $u = 0, \ldots, \bar{k}$,

$$a^s_{l,j} = \bar{a}^s_{l,j}, \qquad j = 0, \ldots, \bar{k},$$

$$a^s_{l,j} = 0, \qquad j = \bar{k}+1, \ldots, k.$$

If $h_{l-1} > (\bar{k}-1)h_l$, then none of the grid-points is also an assistant point, except for t_l, t_{l-1}. Here

$$a^s_{l,2} = \sum_{u=2}^{\bar{k}} \bar{a}^s_{l,u}, \quad a^s_{l,j} = 0, \quad j = 3, \ldots, k,$$

is valid.

If $h_l = 2^j h_{l-j}$, $j = 1, \ldots, p$, $h_{l-j} = h_{l-p}$, $j = p+1, \ldots, k-1$, then all assistant points are grid-points at the same time, namely $\bar{t}_{l-2} = t_{l-(p+2)}$, $\bar{t}_{l-i} = t_{l-(2^p(i-2)+p+2)}$, $i = 3, \ldots, \bar{k}$. Hence we have $j^l_u = u$, $u = 0, \ldots, \bar{k}$, $i^l_u = 2^p(u-2)+p+2$, $u = 2, \ldots, \bar{k}$, and especially $i^l_{\bar{k}} = k$.

Now, if we investigate such formulas as constant-step formulas which possess characteristic polynomials of the form

$$\lambda^{\bar{k}} - \lambda^{\bar{k}-i_{l,s}}$$

(e.g. the Adams formulas), then we obtain

$$a^s_{l,0} = 1 \quad \text{and} \quad a^s_{l,j_{l,s}} = -1 \quad \text{for only one } j_{l,s} \in \{1, \ldots, k\},$$

$$a^s_{l,j} = 0 \quad \text{for} \quad j \neq j_{l,s}, \quad j = 1, \ldots, k.$$

In the case of Adams-type formulas ($\bar{a}^s_{l,0} = 1$, $\bar{a}^s_{l,1} = -1$) we get

$$a^s_{l,0} = 1, \quad a^s_{l,1} = -1, \quad a^s_{l,j} = 0, \quad j = 2, \ldots, k,$$

i.e., the resulting variable k-step method is also of Adams-type.

PROPOSITION 5. *The combination of the constant-step $\bar{k}$-step formula*

$$\text{(10)} \qquad \bar{x}^s_l - \bar{x}^s_{l-i_{l,s}} = h_l \bar{\varphi}^s_l(\bar{x}_l, \ldots, \bar{x}_{l-\bar{k}}), \quad 1 \leqslant i_{l,s} \leqslant \bar{k}, \quad s = 1, \ldots, m,$$

with the step-changing technique SCHT1 *over the grid-class G_{HD} represents a variable k-step method over G_{HD}, $k = 2^p(\bar{k}-2)+p+2$.*

This variable k-step method over G_{HD} is stable if the functions $\hat{\varphi}^s_{l,i}$ and $\bar{\varphi}^s_l$ are both collectively Lipschitz-continuous.

In order to diminish the number of values which must be kept in the computer-memory, we can choose a number $\tilde{k}$,

$$\bar{k} \leqslant \tilde{k} \leqslant k = 2^p(\bar{k}-2)+p+2,$$

and consider the above combination of (10) with SCHT1 over an additionally restricted grid-class.

By $\tilde{G}_{HD}$ we denote the subclass of G_{HD} the grids of which satisfy the condition

$$t_l - t_{l-\tilde{k}} = h_l + \ldots + h_{l-\tilde{k}+1} \geqslant \bar{k} h_l.$$

The combination of (10) and SCHT1 over $\tilde{G}_{HD}$ represents a variable, $\tilde{k}$-step method over $\tilde{G}_{HD}$ which is stable under the assumption of Proposition 5.

Especially, Proposition 5 is valid for variable-coefficient constant-step linear methods of the form

$$(11) \qquad \bar{x}_l^s - \bar{x}_{l-i_{l,s}}^s = h_l \sum_{j=0}^{\bar{k}} \{\bar{b}_{l,j}^s \bar{f}_{l-j}^s + \bar{c}_{l,j}^s \bar{x}_{l-j}^s\}, \quad 1 \leqslant i_{l,s} \leqslant \bar{k}, \quad s = 1, \ldots, m,$$

i.e. for all parametric modifications of interpolation methods from [1] [2], [7], [9], [10], [12], [13], [15], [16], [18], [19].

Now let us deal with the second step-changing technique. Combining the constant-step formula (11) with the special case of SCHT2 given by (8), we get a variable-stepsize $\bar{k}$-step formula over G_w which has the same type as that described by (11). As we mentioned above, SCHT2 is rather complicated in general. Compared to that the implementation of the combination of (9) and SCHT2′ is easy.

PROPOSITION 6. *The k-step method over G_w resulting from the combination of* (9) *and* SCHT2′ *is stable if the coefficients $\bar{b}_{l,j}^s$ are uniformly bounded.*

Next we consider the technique SCHT3′ for the use in combination with (9).

Let $h_{l-\bar{k}+1} = \ldots = h_{l-1} = h_l$, the $x_{l-\bar{k}}, \ldots, x_{l-1}, f_{l-\bar{k}}, \ldots, f_{l-1}$ be retained. The value x_l will be computed directly by (9).

Now, let $h_{l+1} \neq h_l$. To compute x_{l+1} by means of (9) we need the values

$$\bar{f}_{l+1-j}^{[1]} = \sum_{\eta=1}^{\bar{k}} \prod_{\mu=1}^{\bar{k}} \frac{t_{l+1} - jh_{l+1} - t_{l+1-\mu}}{t_{l+1-\eta} - t_{l+1-\mu}} f_{l+1-\eta}, \qquad j = 1, \ldots, k.$$

For the next step we retain

$$\bar{f}_{l+2-\bar{k}}^{[1]}, \ldots, \bar{f}_l^{[1]} = f_l, \quad \bar{f}_{l+1}^{[1]} = f_{l+1}, \qquad \bar{x}_{l+1} = x_{l+1}.$$

To compute $x_{l+2} = \bar{x}_{l+2}$ we need

$$\bar{f}_{l+2-j}^{[2]} = \sum_{\eta=1}^{\bar{k}} \prod_{\mu=1}^{\bar{k}} \frac{t_{l+2} - jh_{l+2} - \bar{t}_{l+2-\mu}^{[1]}}{\bar{t}_{l+2-\eta}^{[1]} - \bar{t}_{l+2-\mu}^{[1]}} \bar{f}_{l+2-\eta}^{[1]}, \qquad j = 1, \ldots, \bar{k}.$$

Having computed x_{l+2}, f_{l+2}, we retain

$$\bar{f}_{l+3-\bar{k}}^{[2]}, \ldots, \bar{f}_l^{[2]}, \ \bar{f}_{l+1}^{[2]} = \bar{f}_{l+1}^{[1]} = f_{l+1}, \quad \bar{f}_{l+2}^{[2]} = f_{l+2}, \quad \bar{x}_{l+2} = x_{l+2},$$

and so on.

In this way, as the computation formula for x_{l+q}, $q \geqslant 1$, in the true sense we get a formula containing all values

$$f_{l+1-\bar{k}}, \dots, f_{l+q-1}$$

on the right-hand side.

C. W. Gear and K. W. Tu ([3]) showed that instabilities may arise if the stepsize in the grid (from G_w) changes too often.

Using the idea of [3], we demand

$$(*) \qquad h_{l+1} = h_{l+2} = \dots = h_{l+\bar{k}-1}$$

and calculate

$$\begin{aligned} &\bar{f}^{[\bar{k}-1]}_{l+\bar{k}-1} = f_{l+\bar{k}-1}, \\ &\bar{f}^{[\bar{k}-1]}_{l+\bar{k}-2} = \bar{f}^{[\bar{k}-2]}_{l+\bar{k}-2} = f_{l+\bar{k}-2}, \\ &\dots\dots\dots\dots\dots\dots \\ &\bar{f}^{[\bar{k}-1]}_{l} = \bar{f}^{[\bar{k}-2]}_{l} = \dots = \bar{f}^{[1]}_{l} = f_l. \end{aligned}$$

This means that the restriction $(*)$ guarantees that the "original" values $f_l, \dots, f_{l+\bar{k}-1}, x_{l+\bar{k}-1}$ again occur as retained values for the computation of $x_{l+\bar{k}}$.

Note that we use $\bar{f}^{[\bar{k}-2]}_{l-1}, f_l, \dots, f_{l+\bar{k}-2}, x_{l+\bar{k}-2}$ to compute $x_{l+\bar{k}-1}$. In fact, we use $f_{l-\bar{k}+1}, \dots, f_{l-1}, f_l, \dots, f_{l+\bar{k}-2}, x_{l+\bar{k}-2}$, i.e., the resulting variable multistep method is a $(2\bar{k}-1)$-step method over the subclass of G_w according to our additional assumptions.

PROPOSITION 7. *Let $G_{GT} \subset G_w$ be such a subclass that for each grid of G_{GT} the following condition holds:*

$$\text{if } h_{l+1} \neq h_l, \text{ then } h_{l+1} = h_{l+2} = \dots = h_{l+\bar{k}-1}$$

(after each stepsize-change, $\bar{k}-1$ equal steps must be performed).

The variable $(2\bar{k}-1)$-step method over G_{GT} resulting from the combination of (9) *and* SCHT3′ *over G_{GT} is stable.*

If in SCHT3′ one takes an interpolation polynomial based on $\bar{k}+1$ nodes (instead of the $\bar{k}$ nodes used above), a similar proposition holds with $\bar{k}$ equal steps instead of $\bar{k}-1$ ones in Proposition 7 (comp. Theorem 5 in [3]).

For similar grid-classes in whose grids some consecutive stepsize-changes are admissible, we can formulate further propositions. For instance, if we denote by G_{GT_2} the set of grids from G_w fulfilling the condition "if $h_{l+1} \neq h_l$, then either $h_{l+1} = h_{l+2} = \ldots = h_{l+\bar{k}-1}$ or $h_{l+1} \neq h_{l+2}$ and $h_{l+2} = h_{l+3} = \ldots = h_{l+\bar{k}}$," then the combination of (9) and SCHT3' is really a stable variable $(2\bar{k})$-step method over G_{QT2}.

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Presented to the Semester
Computational Mathematics
February 20 – May 30, 1980