VARIABLE MULTISTEP METHODS

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1. Introduction

In the last few years we could read terms like “variable stepsize method”,
“variable formula method”, “variable order method”, “cyclic method”,
and “method with variable coefficients”, “adaptive integration method”,
“generalized multistep method”, “parametric method” in many papers
concerned with numerical methods for ordinary initial value problems.

In all these cases the situation is such that in fact the integration
method may change from one step to the next. This interchange can be
produced by

- step-changing,
- order-changing and formula-switching,
- parameter-alteration.

We realize step-changing either by using a formula constructed for
variable stepsize directly — in this case the integration formula may
change stepwise by itself — or we combine a formula constructed for
constant stepsize with a step-changing technique, i.e., with a technique
of computing the required additional values for the next application of
the constant-step formula. The combinations of constant-step formulas
with step-changing techniques can also be called stepwise varying multi-
step methods.

Obviously, all methods in which formulas of the same type but
of different order are used (for instance, in the well-known subroutine
DIFSUB of C.W.Gear in [5]) are step-by-step variable methods. A simi-
lar situation arises when formulas of the same order but of different
type are combined, as was done by Z. Zlatev ([20]).

All methods depending on one or more parameters which can be
chosen in advance or automatically in an adaptive way per integration

[407]
step naturally form step-by-step variable methods. E. F. Sarkany and W. Liniger ([15]), R. März ([10]) and others have investigated such parametric methods.

Consequently almost all of our methods in their implemented form change stepwise. But there are only a few papers dealing with this variability, whereas numerous books and papers discuss the “constant” methods, their consistency, stability, convergence and asymptotic behaviour on great integration intervals. Therefore, if we want to bring the theory closer to the way in which the methods are really applied, we must admit and investigate stepwise variable methods.

2. Stability and consistency of variable multistep methods over grid-classes

Consider the initial value problem

\[
\begin{align*}
\dot{x}(t) &= f(x(t), t), & t &\in [t_0, T], \\
x(t_0) &= x_0
\end{align*}
\]

in which the continuous vector-valued function \( f: \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n \) is assumed to satisfy the usual Lipschitz condition with a constant \( L_f \). Denote by \( x_*(\cdot) \) the exact solution of this initial value problem.

Let the grid \( t_0 < t_1 < \ldots < t_N = T \) on \([t_0, T]\) be given. Denote by \( h_l = t_l - t_{l-1}, \ l = 1, \ldots, N, \ h_{\text{max}} = \max_{l=1,\ldots,N} h_l, \ h_{\text{min}} = \min_{l=1,\ldots,N} h_l \) the \( l \)th stepsize, the maximal and minimal stepsizes of the chosen grid, respectively. As we have agreed, we try to approximate the values \( x_*(t_l) \) according to the grid points by values \( x_l \), which are the solutions of the nonlinear equations

\[
\begin{align*}
\sum_{j=0}^{k} a_{l,j}^s x_{l-j}^s &= h_l \varphi_l^s(x_l, \ldots, x_{l-k}), \\
a_{k,0}^s &= 1, \quad s = 1, \ldots, m, \quad l = k_1, \ldots, N.
\end{align*}
\]

We assume the starting values \( x_0, \ldots, x_{k-1} \) to be given. In (1) different formulas can be used in each component and each step. Especially the functions \( \varphi_l^s: \mathbb{R}^{m(k+1)} \to \mathbb{R}^1, \ s = 1, \ldots, m, \ l = k_1, \ldots, N, \) are usually different from each other. All the methods mentioned above have form (1) or can easily be transformed into (1).

As a simple example we want to mention the parametric modification of the two-step variable stepsize Adams–Bashforth method ([10], p. 11)

\[
x_l^s - x_{l-1}^s = h_l \left( f_{l-1}^s + \frac{h_{l-1}}{h_l} b \left( a_l^s h_{l-1}, \frac{h_l}{h_{l-1}} \right) (f_{l-1}^s - f_{l-2}^s) \right),
\]
with parameters \( a_i^s \in \mathbb{R}^1, \ s = 1, \ldots, m, \ l = 2, \ldots, N, \) and the coefficient-function

\[
b(\beta, k) = \begin{cases} 
\frac{(1+k)^2-1}{2} & \text{if } \beta = 0, \\
\frac{k + e^{-\beta} \beta^{-1}(e^{-k\beta} - 1)}{1 - e^{-\beta}} & \text{otherwise.}
\end{cases}
\]

Let \( \mathcal{G}_{\text{equ}} \) be the class of all equidistant grids on \([t_0, T]\). Consider a grid-class \( \mathcal{G} \) covering at least all equidistant grids, i.e., \( \mathcal{G}_{\text{equ}} \subseteq \mathcal{G} \).

**Definition.** A variable \( k \)-step method \( M_k \) over the grid-class \( \mathcal{G} \) is said to be the set of the finite sequences of formulas (1) according to the grids from \( \mathcal{G} \).

Among (1) a variable \( k \)-step method may contain some formulas degenerating into \( k_j \)-step formulas, \( k_j < k \). Let us agree to speak of a \( k \)-step method if at least one of the formulas contained in (1) is a "proper" \( k \)-step formula. Denote by \( \mathcal{A}(M_k/\mathcal{G}) = \Pi_k \) the set of all characteristic polynomials \( \sum_{j=0}^{k} a_j^s \lambda^{k-j} \) occurring in the variable \( k \)-step method \( M_k \) over \( \mathcal{G} \), and by \( \Phi(M_k/\mathcal{G}) = [\mathbb{R}^{m(k+1)}, \mathbb{R}^1] \) the set of all functions \( \varphi^s \). The set \( \Phi(M_k/\mathcal{G}) \) may be a finite one or an infinite one. The set \( \mathcal{A}(M_k/\mathcal{G}) \) is finite in the methods published so far.

In practice, in implementations we have to select step-by-step the next stepsize in such a way that the resulting grid remains in the given class \( \mathcal{G} \).

Note that C. W. Gear and D. S. Watanabe ([4]) described a variable formula method in a similar way as "a set of formulas \( \{F_t\} \), step and formula changing techniques and step and formula selection schemes". C. W. Gear suggested the use of the step selection function \( \theta: [t_0, T] \times (0, T-t_0] \rightarrow \mathbb{R}^1 \) as a step selection scheme, i.e., a function with the properties

\[
0 < \Delta \leq \theta(t, h) \leq 1,
\]

\[h_l = h_{\text{max}} \theta(t_{l-1}, h_{\text{max}}), \quad l = 1, \ldots, N.
\]

The step selection function defines the next stepsize \( h_{l+1} \) subject to the grid point \( t_l \) and the admissible maximal stepsize of the grid, but without a memory with respect to the last steps. Because of the possibility of applying implication of the form

"if \( h_l < h_{l-1}, \) then \( h_{l+1} \leq h_l \)"

or similar conditions, the use of grid-classes will be advantageous.
Such kinds of restrictions on the grid are used in almost all implementations. On the other hand, they are also important for the stability of some methods.

Let us now deal with the stability of the variable multistep methods, the basic property for all numerical integration methods.

**Definition.** The variable $k$-step method $M_k$ over $G$ is called stable iff for any grid $t_0 < t_1 < \ldots < t_N = T$ from $G$ and each arbitrary collection of starting values $z_0, \ldots, z_{k-1} \in \mathbb{R}^m$ and perturbations $d_l \in \mathbb{R}^m$, $l = k, \ldots, N$, the equation system

$$
\sum_{j=0}^{l-1} a_{i,j}^s z_{i-j} = h_i q_i^s(z_1, \ldots, z_{l-k}) + h_i d_i^s, \quad s = 1, \ldots, m, \quad l = k, \ldots, N,
$$

is uniquely solvable and if for each two solutions $(z_i)_{i=0}^N$, $(\tilde{z}_i)_{i=0}^N$ corresponding to the two collections of starting values and perturbations $(z_i)_{i=0}^{k-1}$, $(d_i)_{i=0}^N$ and $(\tilde{z}_i)_{i=0}^{k-1}$, $(\tilde{d}_i)_{i=0}^N$ the inequality

$$
\max_{l=0,\ldots,k-1} |z_l - \tilde{z}_l| \leq K \left( \max_{l=0,\ldots,k-1} |z_l - \tilde{z}_l| + \max_{l=k,\ldots,N} |d_l - \tilde{d}_l| \right)
$$

holds with a global constant $K$ for the whole class $G$.

Emphasize once more that the constant $K$ is independent of the chosen grid from $G$. Especially, $K$ is independent of the number $N$.

If $G$ consists of the grid sequence $t_0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_N^{(n)} = T$, $n \in \mathbb{N}$, with $h_{\text{max}}^{(n)} \to 0$, then our stability "over $G" means the usual stability (or so-called zero-stability or inverse stability). Mostly only sequences of equidistant grids are considered here. For implementations we are not interested in infinitely small stepsizes. In practice we like to choose the stepsizes depending on the error control freely within the bounds of the grid-class.

The widest grid-classes we are going to use are classes $G_w = G_w(k_{\text{min}}, k_{\text{max}}, B)$ of all grids with

$$
\begin{align*}
0 < k_{\text{min}} \leq h_l / h_{l-1} \leq k_{\text{max}} < \infty, \\
N \cdot h_{\text{max}} \leq B,
\end{align*}
$$

where $k_{\text{min}}$, $k_{\text{max}}$, $B$ are global constants for the class, i.e., are independent of the special grid.

The "linearly convergent grid sequence" considered by H. J. Stetter ([17]) — here the inequality

$$
\frac{c_1}{N_n} \leq h_l^{(n)} \leq \frac{c_2}{N_n}, \quad n \in \mathbb{N},
$$

(1) By $| \cdot |$ we denote the maximum-norm.
is assumed to be valid — fulfills conditions (3) and forms a subclass of $G_w(c_1/c_2, c_3/c_4, c_5)$. Moreover, conditions (3) are assumed to be fulfilled in all papers dealing with variable stepsize (for instance in [3], [8], [14], [20]).

As usual, the local and global errors produced by $M_k$ with respect to a chosen grid from $G$ we define as

$$
\tau_l^s = \frac{1}{h_l} \sum_{j=0}^{k} a_{i,j}^s x_l^s(t_{l-j}) - \varphi_l^s(x_l(t_l), \ldots, x_l(t_{l-k})),$$

$$s = 1, \ldots, m, \quad l = k, \ldots, N,$$

and

$$e_l = x_l(t_l) - x_l, \quad l = k, \ldots, N.$$ 

Therefore, with a stable variable $k$-step method $M_k$ over $G$ the global error estimation

$$\max_{l=k, \ldots, N} |e_l| \leq K \left( \max_{l=0, \ldots, k-1} |e_l| + \max_{l=k, \ldots, N} |\tau_l| \right)$$

holds. There $e_l = x_l(t_l) - x_l$, $l = 0, \ldots, k-1$ are the errors in the starting values.

If we complete a stable method $M_k$ over $G$ with formulas for computing starting values

$$\sum_{j=0}^{l} a_{i,j}^s x_{l-j} = h_l \varphi_l^s(x_l, \ldots, x_0), \quad s = 1, \ldots, m, \quad l = 1, \ldots, k-1,$$

to a so-called selfstarting method $M_k$ over $G$, we easily obtain

$$\max_{l=1, \ldots, N} |e_l| \leq K \max_{l=1, \ldots, N} |\tau_l|,$$

if the functions $\varphi_l^s: \mathbb{R}^{m(l+1)} \to \mathbb{R}$, $s = 1, \ldots, m$, $l = 1, \ldots, k-1$, are Lipschitz-continuous, and by using implicit formulas the corresponding stepsizes are sufficiently small.

**Definition.** The variable $k$-step method $M_k$ over $G$ is said to be consistent with the above initial value problem with the consistency order $p > 0$ iff

$$|\tau_l| \leq C(x_*(\cdot)) \left( \max_{j=0, \ldots, k-1} h_{l-j} \right)^p, \quad l = k, \ldots, N,$$

holds for all grids from $G$ with a class-global constant $C(x_*(\cdot))$.

The proof of consistency for row-wise different formulas is not trivial in general. E. Griepentrog ([6]) investigated the consistency of row-wise different one-step methods. In the case of linear multistep methods
\[ \sum_{j=0}^{k} a_{l,j} x_{l-j}^s = h_l \sum_{j=0}^{k} \left( b_{l,j} f_{l-j}^s + c_{l,j} x_{l-j}^s \right), \quad l = k, \ldots, N, \quad s = 1, \ldots, m, \]

the consistency follows from the consistency of all single methods used in the rows.

With a stable and consistent method \( M_k \) over \( G \), for each tolerancy \( \delta > 0 \) we can select a grid from \( G \) for which the inequality

\[ \max_{i=k, \ldots, N} |\tau_i| \leq \delta \]

is fulfilled. The existence of such a grid is secured by the consistency and because of \( G \supseteq G_{\text{equ}} \). We might choose an equidistant grid with a sufficiently small stepsize. In practice the grid is selected step-by-step by using local error estimations. With (4) for our grid the inequality

\[ \max_{i=k, \ldots, N} |e_i| \leq K \delta \]

holds.

Note that in general the constant \( K \) grows as a consequence of the widening of the grid-class \( G \). But the restriction to the smallest grid-class \( G_{\text{equ}} \) is not ingenious, as the computations have shown for a long time. On the other hand, with grid-classes being too wide, very large constants may arise and there is really no difference to the instability (comp. [3]). Therefore it is an important task to discover useful grid-classes for the methods.

### 3. Some results concerning stability

In the following we denote by \( G^* \) a subclass of \( G \) containing grids with sufficiently small stepsizes.

P. Piotrowski ([14]) proved

**Proposition 1.** The variable multistep method \( M_k \) consisting of Adams–Moulton formulas of fixed order which are directly constructed for variable stepsize over \( G^* \) is stable.

**Proposition 2.** The variable \( k \)-step methods consisting either of variable-step Adams–Moulton formulas with variable orders \( 2 \leq p_i \leq k + 1 \), or of variable-step variable-order Adams–Bashforth formulas with orders \( 1 \leq p_i \leq k \) or of \( k_i \)-step Adams–Bashforth–Moulton predictor-corrector formulas directly designed for variable stepsize with \( k_i \leq k \) are stable over \( G^*_w \) resp. \( G_w \).

The proof may be found in [4], [10].

**Proposition 3.** The variable \( k \)-step method consisting of formulas of
the kind

\[ x_t = a(s_t)x_{t-1} + (1-a(s_t))x_{t-2} + h_t \sum_{j=0}^{s_t} b_{t,j} f_{t-j}, \]

where the coefficients \( b_{t,j}, j = 0, \ldots, s_t \), are computed in such a way that (5) in the \( l \)-th step has order \( s_t \) in case \( b_{t,0} = 0 \) and order \( s_t + 1 \) otherwise, is stable over \( G_w \) or \( G_w^* \) if implicit formulas are used.

The variable \( k \)-step methods consisting in predictor-corrector formulas with correctors of type (5) are stable over \( G_w \).

For the proof see [20].

Note that in (5) with \( a(s_t) = 1 \) the Adams-formulas occur. With \( a(s_t) = 0 \), (5) gives the Nyström or the Milne–Simpson formula. Notice that the case \( a(s_t) = 0 \) is not admitted in Proposition 3.

In the three propositions formulated above the sets \( A(M_k|G_w) \) (or \( A(M_k|G_w^*) \)) are finite ones. In Propositions 1, 2 they contain the single polynomial \( \lambda^k - \lambda^{k-1} \) only. In Proposition 3 the set \( A(M_k|G_w) \) of the method \( M_k \) according to a given function \( a: \{1, 2, \ldots, k\} \to (0, 2) \) with \( a(1) = 1 \) covers the polynomials

\[ \lambda^k - a(n) \lambda^{k-1} - (1-a(n)) \lambda^{k-2}, \quad n = 1, 2, \ldots, k. \]

Furthermore, in all variable \( k \)-step methods \( M_k \) that can be found in Propositions 1–3 the functions \( \varphi_i \) from \( \Phi(M_k|G_w) \) are collectively Lipschitz-continuous, i.e., there exists a constant \( L_\varphi \) with

\[ |\varphi(z_0, \ldots, z_k) - \varphi(\tilde{z}_0, \ldots, \tilde{z}_k)| \leq L_\varphi \sum_{j=0}^{k} |z_j - \tilde{z}_j| \]

for all \( z_j, \tilde{z}_j \in \mathbb{R}^m, \ j = 0, \ldots, k \), and each \( \varphi \in \Phi(M_k|G_w) \). Of course the Lipschitz-constant \( L_\varphi \) depends on the function \( f \) of the original problem.

For the purely linear interpolation methods written as

\[ \sum_{j=0}^{k} a_{t,j} x_{t-j} = h_t \sum_{j=0}^{k} b_{t,j} f_{t-j} \]

the Lipschitz-continuity of the corresponding \( \varphi_i \) follows from the uniform boundedness of the coefficients \( b_{t,j} \) resulting from properties (3) of the grid \( G_w \) and from the Lipschitz-continuity of the function \( f \). With predictor-corrector methods we obtain the Lipschitz-continuity by successive estimations.

Before we can formulate the next proposition we are still occupied with the set \( A(M_k|G_k) \). For each polynomial

\[ p(\lambda) = \sum_{j=0}^{k} a_{t,j} \lambda^{k-j} \]
we define the $k \times k$ Frobenius matrix
\[
\begin{bmatrix}
-a_{i,1} & \ldots & -a_{i,k} \\
1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & 0
\end{bmatrix},
\]
whose eigenvalues are the roots of the corresponding polynomial $p(\lambda)$. If this polynomial satisfies the root-condition formulated by G. Dahlquist, then there are such $R^k$-norms $\| \cdot \|_{i,s}$ that for the corresponding induced matrix-norm the inequality
\[
\|A_{i,s}\|_{i,s} \leq 1
\]
holds.

Because of the equivalence of all $R^k$-norms we also find norm transition coefficients
\[
e_{i,l,s} > 0, \quad i = 1, 2, \quad \gamma_{l,s} > 0
\]
with
\[
e_{1,l,s}|w|_{i,s} \leq |w| \leq e_{2,l,s}|w|_{i,s},
\]
\[
|w|_{l,s} \leq \gamma_{l,s} |w|_{l-1,s}, \quad w \in R^k.
\]
Now, if we allow finitely many characteristic polynomials different from each other in $M_k$ over $G$ only, then $A(M_k/G)$ is a finite set and we find $\varepsilon_1, \varepsilon_2$ with
\[
0 < \varepsilon_1 = \min_{s=1, \ldots, m} \varepsilon_{1,s}, \quad \infty > \varepsilon_2 = \max_{s=1, \ldots, m} \varepsilon_{2,s}
\]
and
\[
\varepsilon_1 |w|_{l,s} \leq |w| \leq \varepsilon_2 |w|_{l,s}, \quad w \in R^k,
\]
holds for $s = 1, \ldots, m$, $l = k, \ldots, N$, and all grids from the given grid class $G$.

**Proposition 4.** Let a variable $k$-step method $M_k$ over $G_w$ (or $G_w^*$ respectively) be given. Let $A(M_k/G_w)$ be finite. Let the root-condition of Dahlquist be fulfilled for each polynomial $p \in A(M_k/G_w)$. The functions $\varphi \in \Phi(M_k/G_w)$ are assumed to be collectively Lipschitz-continuous.

Then the following stability-like inequality holds:
\[
\max_{l=k, \ldots, N} |z_i - \tilde{z}_i| \leq K \Gamma_p \left( \max_{l=0, \ldots, k-1} |z_i - \tilde{z}_l| + \max_{l=k, \ldots, N} |d_i - \tilde{d}_l| \right)
\]
where $K_1$ is a class-global constant and

$$
\Gamma_g = \prod_{j=k}^{N} \max \{1, \max_{s=1, \ldots, m} \gamma_{j,s}^g\}
$$

depends on the chosen grid $g$ from $G_w$.

If the constants $\Gamma_g$ are bounded on the class $G \subset G_w$, $\Gamma_g \leq \Gamma$, $g \in G$, then $M_k$ over $G$ is stable with the stability-constant $K = K_1 \Gamma$.

The proof is explained extensively in [10]. It is obtained by collecting the interesting equations component-wise, introducing the $k$-tuples

$$u_i^s = \begin{bmatrix} z_i^s - \bar{z}_i^s \\ \vdots \\ z_{i-k+1}^s - \bar{z}_{i-k+1}^s \end{bmatrix}$$

and taking some trivial equations to

$$u_i^s = A_{l,s} u_{l-1}^s + \bar{h}_l \begin{bmatrix} \varphi_i^s(z_i, \ldots, z_{i-k}) - \varphi_i^s(\bar{z}_i, \ldots, \bar{z}_{i-k}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + h_l \begin{bmatrix} d_i^s - \bar{d}_i^s \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$s = 1, \ldots, m$, $l = k, \ldots, N$.

From this the asserted inequality is derived by successive estimations.

Note that Proposition 4 is valid for grid-classes in which the condition

$$0 < k_{\text{min}} \leq \frac{\bar{h}_l}{h_{l-1}} \leq k_{\text{max}} < \infty$$

is dropped. But, in practice this extension is useless. From Proposition 4 we get sufficient stability conditions for a great many variable multistep methods. First of all Propositions 1, 2 and Proposition 3 with functions $\alpha: \{1, 2, \ldots, k\} \rightarrow [0, 1]$ are their special cases. In fact, in these cases only matrices $A_{l,s}$ of the type

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & \ldots & 1 & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} \alpha(n) & 1 - \alpha(n) & 0 & \ldots & 0 \\ 1 & 0 & \ldots & \cdot & \cdot \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & \ldots & \cdot & 0 & 1 \end{bmatrix}, \quad \alpha(n) \in [0, 1], \ n = 1, \ldots, k,$$
arise. Here we can obviously choose $|\cdot|_{l,s} = |\cdot|$ because the row-sum
norm of these matrices is not greater than one. Therefore we have
$e_1 = e_2 = 1, \gamma_{l,s} = 1, \Gamma_0 = \Gamma = 1$ for all grids $g \in G_w$.

The above consideration is valid for all so-called interpolation for-
mulas.

Further we find trivial stability conditions bounding the number
of admissible formula-switchings with respect to the whole grid-class.

Moreover, from Proposition 4 we get the stability of the parametric
multistep methods

\begin{equation}
\sum_{j=0}^{k} \hat{a}_{l,j} a_{l-j} = h \sum_{j=0}^{k} \left( \hat{b}_j(a_l h)f_{l-j} + \hat{c}_j(a_l h) a_l a_{l-j} \right)
\end{equation}

over $G_{\text{eq}}$. In (6) the coefficients $\hat{b}_j(a_l h), \hat{c}_j(a_l h)$ are $m \times m$ matrices.
$a_l$ is assumed to be a matrix-parameter. In many papers one tries to
compute $a_l$ as an approximation of the Jacobian of $f$. In [10], [11] and
[15] $a_l$ are diagonal matrices whose elements are computed in an adaptive
way.) The functions $\hat{c}_j(\cdot), \hat{b}_j(\cdot)$ are assumed to be smooth and the domains
in which the parameters $a_l$ may alter are assumed to be bounded. Finally,
let the matrices $\hat{a}_{l,j}$ have diagonal form, $a_{l,j} = \text{diag}(a_{l,1}^j, \ldots, a_{l,m}^j)$. More
precisely, either we admit there stepwise constant coefficients, only,

$$
\hat{a}_{l,j} = \hat{a}_j = \text{diag}(a_{l,1}^j, \ldots, a_{l,m}^j) \quad \text{where} \quad \sum_{j=1}^{k} a_{l,j}^s x^{k-j}, \quad s = 1, \ldots, m,
$$

fulfill the root-condition, or we demand in addition to $\hat{a}_{l,0} = \mathcal{E}$
that for each $s$ and each $l$ among the coefficients $a_{l,1}^s, \ldots, a_{l,k}^s$ a single one is
equal to $-1$ and the others vanish. In the second case we are concerned
with different parametric modifications of interpolation methods (e.g.
[2], [8], [10], [13], [15], [16], [18]). Mostly parametric modifications of
Adams methods are considered.

Note that the parametric modifications of classical multistep methods
are usually designed with constant stepsize. There the variable-step
forms are too expensive.

For the parametric modification of the two-step variable stepsize
Adams–Bashforth method mentioned in Section 2, from Proposition 4
we get the stability over $G_w$ if the domain for parameter-alteration is
assumed to be bounded (comp. [11]). Obviously, for the parametric
modification of the trapezoidal rule used with variable stepsize a similar
result is valid.

4. Stability of combinations of constant-step formulas
with step-changing techniques

Now, let us deal with the combinations of constant-step formulas with
step-changing techniques. First we describe some step-changing tech-
iques.
Let the values $x_{t_1}, \ldots, x_{t_{-k}}, \ldots, x_{t_{-1}}$ approximating the $x_*(t_1), \ldots, x_*(t_{-1})$ be computed. To compute the next value $x_t = \bar{x}_t$ by means of the constant-step formula

\begin{equation}
\sum_{j=0}^{\bar{k}} \bar{a}_{i,j}^s \bar{x}_{t-j}^s = h_i \varphi_i^s(\bar{x}_{t_{-1}}, \ldots, \bar{x}_{t_{-\bar{k}}}), \quad \bar{a}_{i,0}^s = 1, \ s = 1, \ldots, m,
\end{equation}

we need the values $\bar{x}_{t-1} = x_{t-1}, \bar{x}_{t-j}, \ j = 2, \ldots, \bar{k}$, approximating the exact solution $x_*(\cdot)$ at the points $t_{-j} = t_i - jh_i, \ j = 1, \ldots, \bar{k}$.

Step-changing technique 1 (SCHT1): If $t_{-j}$ coincide with the grid-point $t_{l_{-j}}$, we put $\bar{x}_{t_{-j}} = x_{t_{-j}}$ and $\bar{f}_{t_{-j}} = f_{t_{-j}}$ as needed.

For each "assistant" point $t_{-j}$ not being a grid-point at the same time we seek the biggest smaller grid-point ("reference" point) $t_{l_{-j}}$. Then we compute

$$\bar{x}_{t_{-j}}^s = x_{t_{-j}} + (t_{-j} - t_{l_{-j}}) \hat{\varphi}_{t_{-j}}^s(x_{t_{-j}}), \quad s = 1, \ldots, m,$$

with a suitable explicit one-step method characterized by $\hat{\varphi}_{t_{-j}}$. If the value $\bar{f}_{t_{-j}}$ is also needed, we compute $\bar{f}_{t_{-j}} = f(\bar{x}_{t_{-j}}, \bar{f}_{t_{-j}})$ or put $\bar{f}_{t_{-j}} = \hat{\varphi}_{t_{-j}}(x_{t_{-j}})$.

Obviously, in this step-changing variant it may be necessary to keep accumulated very many of the saved values $x_{t_{-j}}, f_{t_{-j}}$.

Step-changing technique 2 (SCHT2): We interpolate $\bar{x}_{t_{-j}} = P_i^*(t_{-j})$ where $P_i^*(\cdot)$ is an interpolation polynomial through the retained values $x_{-k}^s, \ldots, x_{s-1}^s$ and derivatives $f_{s-\bar{k}}^s, \ldots, f_{s-1}^s$ or through some of them.

In general the implementation of SCHT2 is just as complicated as that of multistep formulas constructed for variable stepsize directly. For instance, if for the construction of $P_i^*(\cdot)$ the interpolation conditions

$$P_i^*(t_{-1}) = x_{t_{-1}}, \quad \dot{P}_i^*(t_{-j}) = f_{t_{-j}}, \quad j = 1, \ldots, \bar{k},$$

are used, then the number of operations for the computation of

\begin{equation}
\bar{x}_{t_{-j}} = x_{t_{-1}} + \int_{t_{-1}}^{t_{-j}} \int_{t_{-1}}^{t_{-j}} \frac{s - t_{-1}}{t_{-\mu} - t_{-\nu}} \ dsdf_{t_{-j}}
\end{equation}

is comparable with the number of operations of the $k$-step variable-stepsize Adams–Bashforth method.

The situation will be simpler if in (7) no $\bar{x}_{t_{-j}}$ except $\bar{x}_t = x_t, \bar{x}_{t_{-1}} = x_{t_{-1}}$ are really contained. This situation arises, e.g., if (7) has the form

\begin{equation}
\bar{x}_t^s - \bar{x}_{t_{-1}}^s = h_t \sum_{j=0}^{\bar{k}} \bar{a}_{i,j}^s \bar{f}_{t_{-j}}^s, \quad s = 1, \ldots, m.
\end{equation}
(9) may represent variable order Adams formulas as constructed in [5].
For such methods the following modified technique is useful:

Step-changing technique 2' (SCHT2'): For the use of (9) we put
\[ x_{l-1}, \; \bar{f}_{l-j} = \bar{P}_l(t_{l-j}), \; j = 0, \ldots, \bar{k}, \]
where \( \bar{P}_l(\cdot) \) is the interpolation polynomial passed through \( f_{l-\bar{k}}, \ldots, f_{l-1}, f_l \).

Step-changing technique 3 (SCHT3): Assume \( h_{l-\bar{k}} = h_{l-\bar{k}+1} = \ldots = h_{l-1}, \; h_l \neq h_{l-1}, \) (at least for \( l = \bar{k} - 1 \)). We interpolate as in SCHT2 and obtain the value \( x_l = \bar{x}_l \) by the constant-step formula (7). But for
the next step we save \( \bar{x}_{l-\bar{k}+1}, \ldots, \bar{x}_l, \; \bar{f}_{l-\bar{k}+1}, \ldots, \bar{f}_l \) instead of \( x_{l-\bar{k}+1}, \ldots, x_l, \; f_{l-\bar{k}+1}, \ldots, f_l \), and so on.

In this way we accumulate values for equidistant points in each step and we always change over to new equidistant points. A. Nordsieck
and C.W. Gear noted very practicable formulas for the realization of
this technique.

Step-changing technique 3' (SCHT3'): For the multistep formula
(9) a special modification of SCHT3 is useful. As in SCHT2', the
interpolation polynomial links the \( f_{l-\bar{k}}, \ldots, f_{l-1}, \) only. For the next step
the values \( \bar{f}_{l-\bar{k}+1}, \ldots, \bar{f}_l \) must be retained.

Sometimes it is advantageous to reduce the grid-class and allow
only halvings and doublings of the stepsizes. With this we keep the number
of values which have to be computed additionally (e.g., in SCHT1) small.
After a halving the stepsizes is not doubled immediately, but kept for
at least one further step. Define a grid-class \( G_{HD} \subset G_u \) according to these
restrictions. Let \( G_{HD} \) be the class consisting of all grids which have prop-
erties (3) which satisfy the following conditions:

\[ h_{\text{max}} \leq 2^p h_{\text{min}}, \]
either \( h_i = 2h_{i-1} \) or \( h_i = h_{i-1} \) or \( h_i = 2^{-i}h_{i-1} \) with \( i \leq p, \)
if \( h_{i-1} \leq h_{i-2}/2, \) then \( h_i \leq h_{i-1}. \)

Let the natural number \( p \) be global for the grid-class.

Let us now define a suitable variable \( k \)-step method over \( G_{HD} \) which
combines the constant-step \( \bar{k} \)-step formula (7) with SCHT1.

First we put

\[ k = 2^p \bar{k} - \sum_{i=0}^{p} (2^{p-i} - 1) = 2^p (\bar{k} - 2) + p + 2. \]
The number \( k \) is calculated in such a way that \( t_{l-k} \leq t_l - \bar{k}h_l \) holds.

Now we select those grid-points from our grid which coincide with
assistant points. Certainly \( t_i = \bar{t}_i, \; t_{l-1} = \bar{t}_{l-1} \) holds. Let \( \bar{t}_{l-j} = t_{l-i}, \)
\( u = 0, \ldots, n_i, \; n_j \geq 1, \; j_0^i = i_0^i = 0, \; j_1^i = i_1^i = 1, \) be all these grid-points.
For any assistant point $t_{i-1}^u$, $u = n_t + 1, \ldots, k$, not being a grid-point at the same time we choose the biggest smaller grid-point $t_{i-1}^u$ as a reference point. Note that while the grid-points or the assistant points are different from one another, the reference points may coincide. For example, the case $n_t = 1$, $f_u^t = u$, $i_u^t = 2$, $u = 2, \ldots, \kbar$, occurs as soon as $h_{t-1} > (\kbar - 1) h_t$.

Yet, in any case $2 \leq i_u^t \leq \kbar$ holds for $u = n_t + 1, \ldots, \kbar$.

Now, we compose a $k$-step method (1) resulting from the combination of (7) and SCHT1 over $\mathcal{G}$. With

$$
\bar{x}_{i-1}^s = x_{i-1}^s, \quad \quad u = 0, 1, \ldots, n_t,
$$

$$
\bar{x}_{i-1}^s = x_{i-1}^s + (t_{i-1}^u - t_{i-1}^u) \hat{\varphi}_{i-1}^s (x_{i-1}^u), \quad \quad u = n_t + 1, \ldots, \kbar,
$$

we find

$$
\sum_{u=0}^{\kbar} \bar{a}_{i,j}^s x_{i-1}^u = h_t \left\{ f_t^s (\bar{x}_i, \ldots, \bar{x}_{i-k}) - \sum_{u=n_t+1}^{\kbar} \bar{a}_{i,j}^s \frac{1}{h_t} (t_{i-1}^u - t_{i-1}^u) \times \right.
$$

$$
\times \hat{\varphi}_{i,j}^s (x_{i-1}^u) \right\}, \quad s = 1, \ldots, m.
$$

The expression in the braces on the right-hand side we sum up to the function $f_t^s (x_1, \ldots, x_{i-k})$ in each row. Obviously, the resulting functions $f_t^s (\cdot)$ are collectively Lipschitz-continuous if both the functions $\varphi_t^s$ and the functions $\hat{\varphi}_{i,j}^s$ are collectively Lipschitz-continuous.

Finally we put

$$
a_{i,0}^s = \bar{a}_{i,0}^s = 1, \quad a_{i,1}^s = \bar{a}_{i,1}^s,
$$

$$
a_{i,j}^s = \sum_{u=0}^{\kbar} \delta_{j,u} \bar{a}_{i,j}^u, \quad j = 2, \ldots, k, \quad s = 1, \ldots, m,
$$

to obtain

$$
\sum_{j=0}^{k} a_{i,j}^s x_{i-1}^j = \sum_{u=0}^{\kbar} \bar{a}_{i,j}^u x_{i-1}^u.
$$

The following situations may arise:

If $t_{i-j}$ acts for $j \in \{2, \ldots, k\}$ neither as an assistant point nor as a reference point, then $a_{i,j}^s = 0$, $s = 1, \ldots, m$.

If $t_{i-j} = t_{i-j}$, $j = 0, \ldots, \kbar$, then the set of reference points is empty and we have $n_t = \kbar$, $i_u^t = j_u^t = u$, $u = 0, \ldots, \kbar$,

$$
a_{i,j}^s = \bar{a}_{i,j}^s, \quad j = 0, \ldots, \kbar,
$$

$$
a_{i,j}^s = 0, \quad j = \kbar + 1, \ldots, k.$$

If \( h_{l-1} > (\overline{k} - 1)h_1 \), then none of the grid-points is also an assistant point, except for \( t_l, t_{l-1} \). Here

\[
a_{i_0}^{s} = \sum_{u=2}^{\overline{k}} a_{i,u}^{s}, \quad a_{i,j}^{s} = 0, \quad j = 3, \ldots, k,
\]
is valid.

If \( h_l = 2^i h_{l-1}, \quad j = 1, \ldots, p, \quad h_{l-j} = h_{l-p}, \quad j = p+1, \ldots, k-1 \), then all assistant points are grid-points at the same time, namely \( \bar{t}_{l-2} = t_{l-(p+2)}, \quad \bar{t}_{l-i} = t_{l-2p(u-u+2)}, \quad i = 3, \ldots, \overline{k} \). Hence we have \( j_{l,u} = u, \quad u = 0, \ldots, \overline{k}, \quad i_{l,u}^{s} = 2^p(u-2) + p + 2, \quad u = 2, \ldots, \overline{k} \), and especially \( \bar{t}_{l,\overline{k}} = \bar{k} \).

Now, if we investigate such formulas as constant-step formulas which possess characteristic polynomials of the form

\[
\lambda^k - \lambda^{k-t_{l,s}}
\]
(e.g. the Adams formulas), then we obtain

\[
a_{i_0}^{s} = 1 \quad \text{and} \quad a_{i_{l,s}}^{s} = -1 \quad \text{for only one } j_{l,s} \in \{1, \ldots, k\},
\]

\[
a_{i,j}^{s} = 0 \quad \text{for } j \neq j_{l,s}, \quad j = 1, \ldots, k.
\]

In the case of Adams-type formulas \( (a_{i,0}^{s} = 1, \quad a_{i,1}^{s} = -1) \) we get

\[
a_{l,0}^{s} = 1, \quad a_{l,1}^{s} = -1, \quad a_{l,j}^{s} = 0, \quad j = 2, \ldots, k,
\]
i.e., the resulting variable \( k \)-step method is also of Adams-type.

**Proposition 5.** The combination of the constant-step \( \overline{k} \)-step formula

\[
\overline{x}_l^s - \overline{x}_{l-t_{l,s}}^s = h_l \overline{\varphi}_l^s(\overline{x}_l, \ldots, \overline{x}_{l-\overline{k}}), \quad 1 \leq t_{l,s} \leq \overline{k}, \quad s = 1, \ldots, m,
\]

with the step-changing technique SCHT1 over the grid-class \( G_{HD} \) represents a variable \( k \)-step method over \( G_{HD}, \quad k = 2^{p}(\overline{k} - 2) + p + 2 \).

This variable \( k \)-step method over \( G_{HD} \) is stable if the functions \( \overline{\varphi}_s^s \) and \( \varphi_l^s \) are both collectively Lipschitz-continuous.

In order to diminish the number of values which must be kept in the computer-memory, we can choose a number \( \overline{k} \),

\[
\overline{k} \leq \bar{k} \leq k = 2^{p}(\overline{k} - 2) + p + 2,
\]
and consider the above combination of (10) with SCHT1 over an additionally restricted grid-class.

By \( \bar{G}_{HD} \) we denote the subclass of \( G_{HD} \) the grids of which satisfy the condition

\[
t_l - t_{l-\overline{k}} = h_l + \ldots + h_{l-\overline{k}+1} \geq \overline{k}h_1.
\]
The combination of (10) and SCHT1 over $\tilde{G}_{HD}$ represents a variable, $\tilde{k}$-step method over $\tilde{G}_{HD}$ which is stable under the assumption of Proposition 5.

Especially, Proposition 5 is valid for variable-coefficient constant-step linear methods of the form

$$\bar{x}_{l-s} - \bar{x}_{l-\mu} = \lambda_1 \sum_{j=0}^{\tilde{k}} \left( \tilde{v}_{l,j}^s \tilde{f}_{l-j}^s + \tilde{v}_{l,j}^\mu \tilde{x}_{l-j}^\mu \right), \quad 1 \leq l-s \leq \tilde{k}, \quad s = 1, \ldots, m,$$

i.e. for all parametric modifications of interpolation methods from [1] [2], [7], [9], [10], [12], [13], [15], [16], [18], [19].

Now let us deal with the second step-changing technique. Combining the constant-step formula (11) with the special case of SCHT2 given by (8), we get a variable-stepsize $\tilde{k}$-step formula over $G_{\bar{u}}$ which has the same type as that described by (11). As we mentioned above, SCHT2 is rather complicated in general. Compared to that the implementation of the combination of (9) and SCHT2' is easy.

**Proposition 6.** The $k$-step method over $G_{\bar{u}}$ resulting from the combination of (9) and SCHT2' is stable if the coefficients $\tilde{v}_{l,j}^s$ are uniformly bounded.

Next we consider the technique SCHT3' for the use in combination with (9).

Let $h_{l-\tilde{k}+1} = \ldots = h_{l-1} = h_l$, the $x_{l-\tilde{k}}, \ldots, x_{l-1}, f_{l-\tilde{k}}, \ldots, f_{l-1}$ be retained. The value $x_l$ will be computed directly by (9).

Now, let $h_{l+1} \neq h_l$. To compute $x_{l+1}$ by means of (9) we need the values

$$\tilde{f}_{l+1-j} = \sum_{\eta=1}^{\tilde{k}} \tilde{f}_{l+1-\mu} \frac{t_{l+1-j} - \tilde{h}_{l+1-j} - t_{l+1-\mu}}{t_{l+1-\eta} - t_{l+1-\mu}} \tilde{f}_{l+1-\eta}, \quad j = 1, \ldots, \tilde{k}.$$

For the next step we retain

$$\tilde{f}_{l+2-j} = \tilde{f}_{l+1-j}, \quad \tilde{f}_{l+2} = \tilde{f}_{l+1}, \quad \bar{x}_{l+1} = x_{l+1}.$$

To compute $x_{l+2} = \bar{x}_{l+2}$ we need

$$\tilde{f}_{l+2-j} = \sum_{\eta=1}^{\tilde{k}} \tilde{f}_{l+2-\mu} \frac{t_{l+2-j} - \tilde{h}_{l+2-j} - \tilde{h}_{l+2-\mu}}{\tilde{f}_{l+2-\mu} - \tilde{f}_{l+2-\eta}}, \quad j = 1, \ldots, \tilde{k}.$$

Having computed $x_{l+2}, \tilde{f}_{l+2}$, we retain

$$\tilde{f}_{l+3-j} = \tilde{f}_{l+2-j}, \quad \tilde{f}_{l+3} = \tilde{f}_{l+2}, \quad \bar{x}_{l+2} = x_{l+2},$$

and so on.
In this way, as the computation formula for \( a_{t+q} \), \( q \geq 1 \), in the true sense we get a formula containing all values

\[
f_{t+1}, \ldots, f_{t+q-1}
\]
on the right-hand side.

C. W. Gear and K. W. Tu ([3]) showed that instabilities may arise if the stepsize in the grid (from \( G_w \)) changes too often.

Using the idea of [3], we demand

\[
h_{t+1} = h_{t+2} = \ldots = h_{t+k-1}
\]
and calculate

\[
\begin{align*}
\tilde{f}_{t+k-1}^{[k-1]} &= f_{t+k-1}, \\
\tilde{f}_{t+k-2}^{[k-1]} &= \tilde{f}_{t+k-2}^{[k-2]} = f_{t+k-2}, \\
&\ldots &\ldots &\ldots &\ldots &\ldots \\
\tilde{f}_{t}^{[k-1]} &= \tilde{f}_{t}^{[k-2]} = \ldots = \tilde{f}_{t}^{[1]} = f_t.
\end{align*}
\]

This means that the restriction (*) guarantees that the "original" values \( f_t, \ldots, f_{t+k-1}, a_{t+k-1} \) again occur as retained values for the computation of \( a_{t+k} \).

Note that we use \( \tilde{f}_{t+k-2}^{[k-2]}, f_t, \ldots, f_{t+k-2}, a_{t+k-2} \) to compute \( a_{t+k-1} \). In fact, we use \( f_{t+k+1}, \ldots, f_{t-1}, f_t, \ldots, f_{t+k-2}, a_{t+k-2} \), i.e., the resulting variable multistep method is a \( (2k-1) \)-step method over the subclass of \( G_w \) according to our additional assumptions.

**Proposition 7.** Let \( G_{GT} \subset G_w \) be such a subclass that for each grid of \( G_{GT} \) the following condition holds:

\[
\text{if } h_{t+1} \neq h_t, \text{ then } h_{t+1} = h_{t+2} = \ldots = h_{t+k-1}
\]
(after each stepsize-change, \( k-1 \) equal steps must be performed).

The variable \( (2k-1) \)-step method over \( G_{GT} \) resulting from the combination of (9) and SCHT3' over \( G_{GT} \) is stable.

If in SCHT3' one takes an interpolation polynomial based on \( k+1 \) nodes (instead of the \( k \) nodes used above), a similar proposition holds with \( k \) equal steps instead of \( k-1 \) ones in Proposition 7 (comp. Theorem 5 in [3]).

For similar grid-classes in whose grids some consecutive stepsize-changes are admissible, we can formulate further propositions. For instance, if we denote by \( G_{GT2} \) the set of grids from \( G_w \) fulfilling the condition
"if \( h_{t+1} \neq h_t \), then either \( h_{t+1} = h_{t+2} = \ldots = h_{t+k-1} \) or \( h_{t+1} \neq h_{t+2} \)
and \( h_{t+2} = h_{t+3} = \ldots = h_{t+k} \)," then the combination of (9) and SCHT3' is really a stable variable \((2k)\)-step method over \( G_{GT2} \).

References


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