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QUASI-ISOMORPHIC PERVERSITIES AND OBSTRUCTION THEORY FOR PSEUDOMANIFOLDS

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For the investigation of the singular Poincaré duality homomorphism [KpFi$_1$], of the theorems of Lefschetz type in intersection homology [FiKp$_2$], and of the Vanishing Theorem for Stein spaces [FiKp$_1$] we need the homomorphism

$$\mu = \mu_{pq}^{\cdot}\colon\ H_{\Phi}^{\cdot}(X, P_p^{\cdot}) \to H_{\Phi}^{\cdot}(X, P_q^{\cdot})$$

for appropriate perversities p and q. Obviously μ is an isomorphism if the complexes $P_p^{\cdot}$ and $P_q^{\cdot}$ are quasi-isomorphic, thus, in order to investigate the obstructions against μ being an isomorphism, it is natural to identify such perversities p and q; we call them *quasi-isomorphic perversities* on X and analyze the corresponding equivalence relation.

An equivalence class turns out to be an "interval" $[lp, up]$, where, with respect to the partial ordering $p \leqslant q$, the perversity lp is the lower bound and up the upper bound of the perversities quasi-isomorphic to p. The homomorphism $\mu_{pq}^{\cdot}$ exists in a natural manner if there exist perversities $p' \in [lp, up]$ and $q' \in [lq, uq]$ such that $p' \leqslant q'$; we extend this to a partial ordering $p \subset q$ such that the operations $p \mapsto lp$ and $p \mapsto up$, which are not monotonous with respect to $\leqslant$ (see Section 4), are so with respect to $\subset$. For so called "dualizing" perversities the investigation runs much along the same lines as in the case of coefficients in a field, while in the general situation new difficulties arise.

Our main concern is the intersection homology of complex spaces. But the formal part of the theory does not depend on the existence of a complex structure, thus we consider in this article n-dimensional pseudomanifolds X with a topological stratification X (specific properties of complex spaces are

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discussed in [FiKp₃]). If X is not a p-homology manifold, then there exists a nonzero j such that $H^j P_p^{\cdot}$ does not vanish. In order to measure how much X fails to be a p-homology manifold, we introduce the *obstruction trapezoid* $\mathrm{OT}_p(X)$. Its generalization to a pair of perversities $p \subset q$ provides the essential information for the properties of the homomorphism μ_{pq}; the defining invariants $a(p, q)$ and $b(p, q)$ provide the improved and more symmetric version of the Main Lemma in [KpFi] depicted below:

μ_{pq}^j is	$\cong \cdots\cdots \cong$	injective	onto	$\cong \cdots\cdots \cong$
for $j =$	0 … $a(p,q)$	$a(p,q)+1$ …	… $n-b(p,q)-1$	$n-b(p,q)$ … n

The definition of the invariant $a(p, q)$ is rather easy: it is the maximal natural number a such that $P_p^{\cdot}$ and $P_q^{\cdot}$ are quasi-isomorphic up to order a; the number $b(p, q)$ can be characterized in a similar way. Hence, it remains to investigate the invariants $a(p, q)$ and $b(p, q)$. Let k_p denote the codimension of the p-singular subset of X and $p^* := t - p$ the complementary perversity. Then

$$a(p, q) \geqslant \max(a(q), up(k_q)),$$

$$b(p, q) \geqslant \max(b(q), (lq)^*(k_q)),$$

$$a(p, q) \geqslant b(q^*, p^*) \quad \text{if } p \text{ and } q \text{ are dualizing},$$

$$b(p, q) \geqslant a(q^*, p^* + 1) - 1 \quad \text{if } q \text{ is dualizing},$$

where the numbers $a(p) = a(o, p)$ and $b(p) = b(o, p)$ are easily read of from the Deligne complex $P_p^{\cdot}$ itself. Moreover, $a(-, -)$ and $b(-, -)$ are monotonous in both variables with respect to the partial ordering $p \subset q$. If X is a complex space that is a set-theoretic local complete intersection, then singularities, then, by [KpFi, 5.22],

$$b(p) \geqslant \max(p^*(k_p), a(p) - 1).$$

If X is a complex space that is a set-theoretic local complete intersection, then

$$a(p, q) \geqslant d - 2, \qquad b(p, q) \geqslant d - 2,$$

where d is the complex codimension of the singular locus Σ of X. This is a consequence of [FiKp₃]. Section 4 offers some examples that may illustrate classes of quasi-isomorphic perversities and also the problems with forming lower or upper bounds of perversities.

We are particularly indebted to S. Łojasiewicz for the invitation to participate in the Semester on Singularity Theory of the Stefan Banach Center and to the Department of Mathematics at the University of Warsaw where we could discuss and carry out our ideas on intersection homology of complex spaces.

0. Notations and preliminary remarks

0.1. We use the following notations, according to [Bo, V] and [KpFi], which differ a little bit from that in the basic articles [GoMPh_1] and [GoMPh_2]:

R – denotes a principal ideal domain, L a locally constant sheaf of R-modules with finitely generated stalks;

X – denotes an R-orientable pseudomanifold of topological dimension n with a topological stratification

$$X = (\emptyset = X_{n-u-1} \subset X_{n-u} \subset \ldots \subset \Sigma = X_{n-k} = \ldots = X_{n-2} \subset X_n = X)$$

where Σ denotes the singular subset of X, set $S_j := X_j \setminus X_{j-1}$ and $U_j := X \setminus X_{n-j}$, then $U_{j+1} = U_j \cup S_{n-j}$, and $\theta\colon U_j \hookrightarrow U_{j+1}$ denotes the inclusion mapping;

ϕ – denotes a family of supports on X such that $H^i_\phi(X_j, F) = 0$ for every $i \geqslant j+1$ and for every sheaf F that is locally constant on the strata of X_j; moreover, we set $E(\phi) := \bigcup_{K \in \phi} K$;

p – denotes a perversity, i.e. a mapping $p\colon N_{\geqslant 2} \cup \{\infty\} \to N \cup \{\infty\}$ such that $p(2) = 0$, $p(\infty) = \infty$ and $p(i) \leqslant p(i+1) \leqslant p(i)+1$ for every $i \in N_{\geqslant 2}$; in particular there are the zero-perversity o, the top-perversity t, the lower middle perversity m, the upper middle perversity n, and the complementary perversity $p^* = t - p$ of p; for $s \in N$ set

$$(p-s)(j) := \max(0, p(j)-s),$$

$$(p+s)(j) := \min(j-2, p(j)+s);$$

if q is another perversity, the functions $\min(p, q)$ and $\max(p, q)$ are again perversities;

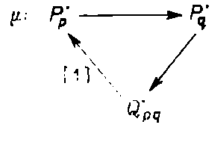
$P^{\cdot}_p := {}_X P^{\cdot}_p L$ – denotes the Deligne complex with coefficients in L;

$I_p H^\phi_j(X, L) := H^{n-j}_\phi(X, P^{\cdot}_p L)$ – is the j-th intersection homology module of X with respect to the perversity p.

0.2. According to the construction of the complex $P^{\cdot}_p$ we frequently have to extend properties from U_i to U_{i+1}. Then, for $x \in S_{n-i}$, denote the link of x with respect to X with $L = L_x$, denote with $U \cong R^{n-i} \times \mathring{c}(L)$ a distinguished neighbourhood of x in X, and set $U' := U \setminus S_{n-i} \cong R^{n-i+1} \times L$.

0.3. *If $\mu\colon P^{\cdot}_p \to P^{\cdot}_q$ is a morphism, then there exists a distinguished triangle (depending on μ)*

$T(p, q)$



and for $x \in S_{n-i}$, *there exist inclusions*

$$(0.3.1) \qquad H^j Q^{\cdot}_{pq,x} \hookrightarrow \begin{cases} H^j(U', Q^{\cdot}_{pq}), & \text{if } j \leqslant p(i), \\ H^j P^{\cdot}_{q,x}, & \text{if } j \geqslant p(i)+1, \\ 0, & \text{if } j \geqslant \max(p, q)(i)+1. \end{cases}$$

Proof. There exists a commutative diagram with $Q^{\cdot} := Q^{\cdot}_{pq}$

$$\begin{array}{ccccccc} H^j P^{\cdot}_{p,x} & \longrightarrow & H^j P^{\cdot}_{q,x} & \longrightarrow & H^j Q^{\cdot}_x & \longrightarrow & H^{j+1} P^{\cdot}_{p,x} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow{\scriptstyle\beta} \\ H^j(U', P^{\cdot}_p) & \longrightarrow & H^j(U', P^{\cdot}_q) & \longrightarrow & H^j(U', Q^{\cdot}) & \longrightarrow & H^{j+1}(U', P^{\cdot}_p) \end{array}$$

By the attachment condition [Bo, V.2.3], the homomorphism β is bijective for $j \leqslant p(i)$, by the vanishing condition β is obviously injective otherwise; an analogous result holds for γ. The Five Lemma implies

$$H^j Q^{\cdot}_x \hookrightarrow H^j(U', Q^{\cdot}) \quad \text{for } j \leqslant p(i),$$

while the other inclusions follow obviously from the upper exact sequence. □

1. Quasi-isomorphic perversities

The . construction of intersection homology modules via the hypercohomology $H^{\cdot}_{\Phi}(X, P^{\cdot}_p L)$ suggests the introduction of the following equivalence relation:

1.1. Definition. Given a pseudomanifold X, we call two perversities p and q *quasi-isomorphic on* X *with respect to* L ($p \cong q$) if there exists a quasi-isomorphism ${}_X P^{\cdot}_p L \cong {}_X P^{\cdot}_q L$.

Note that the relation "$\cong$" is of a local nature: if, for every point $x \in X$, there exists an open neighbourhood U_x of x such that $p \cong q$ with respect to $L|_{U_x}$, then $p \cong q$ on X with respect to L. This follows immediately from the fact that the axioms characterizing $P^{\cdot}_p L$ up to quasi-isomorphism (cf. [Bo, V.4.7]) concern essentially only the local behaviour of $P^{\cdot}_p L$.

In Lemma 1.3 we shall see that the corresponding equivalence class of a perversity p is a "closed interval" $[lp, up]$ with respect to the partial ordering $p \leqslant q$; for explicit examples see Section 4.

1.2. Lemma. *If* p *and* q *are perversities such that* $p \cong q$, *then*

$$\min(p, q) \cong p \cong q \cong \max(p, q).$$

Proof. We induct on i in order to show

$$P^{\cdot}_{\min(p,q)}|_{U_i} \cong P^{\cdot}_q|_{U_i} \cong P^{\cdot}_q|_{U_i} \cong P^{\cdot}_{\max(p,q)}|_{U_i}.$$

The case "$i = 2$" is obvious. For the step "$i \Rightarrow i+1$" we use the inclusion mapping

$$\theta\colon U_i \hookrightarrow U_{i+1}.$$

We may assume that $p(i) \leqslant q(i)$; then the construction of the Deligne complex yields on U_{i+1}:

$$\begin{aligned} P^{\cdot}_{\min(p,q)} &\cong \tau_{\leqslant p(i)} R\theta_*(P^{\cdot}_{\min(p,q)}|_{U_i}) \cong \tau_{\leqslant p(i)} R\theta_*(P^{\cdot}_p|_{U_i}) \cong P^{\cdot}_p \cong P^{\cdot}_q \\ &\cong \tau_{\leqslant q(i)} R\theta_*(P^{\cdot}_q|_{U_i}) \cong \tau_{\leqslant q(i)} R\theta_*(P^{\cdot}_{\max(p,q)}|_{U_i}) \cong P^{\cdot}_{\max(p,q)}. \end{aligned}$$ □

The preceding result allows us to associate to a given perversity p two new perversities $lp \cong p$ and $up \cong p$: we call

$$lp := \min\{q;\, q \cong p\}$$

the *lower bound*, and

$$up := \max\{q;\, q \cong p\}$$

the *upper bound* for p.

1.3. LEMMA. *For two perversities p and q the following statements are equivalent*:

(i) $p \cong q$;

(ii) $lp \leqslant q \leqslant up$;

(iii) *for every $i \geqslant 2$ we have*

$$H^j P^{\cdot}_p|_{S_{n-i}} = 0 \quad \textit{for every } j \geqslant q(i)+1,$$

$$H^j P^{\cdot}_q|_{S_{n-i}} = 0 \quad \textit{for every } j \geqslant p(i)+1.$$

Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are immediate.

(iii) $\Rightarrow$ (i) We prove by induction on i that

$$P^{\cdot}_p|_{U_i} \cong P^{\cdot}_q|_{U_i}.$$

The case "$i = 2$" is evident. For the step "$i \Rightarrow i+1$" we may assume that $p(i) \leqslant q(i)$, since (i) and (iii) are symmetric with respect to p and q. Then we obtain on U_{i+1}:

$$P^{\cdot}_p \cong \tau_{\leqslant p(i)} R\theta_*(P^{\cdot}_p|_{U_i}) \cong \tau_{\leqslant p(i)} R\theta_*(P^{\cdot}_q|_{U_i}) \cong \tau_{\leqslant p(i)}(P^{\cdot}_q),$$

by induction hypothesis. The second part of (iii) implies that

$$H^j P^{\cdot}_q|_{U_{i+1}} = 0 \quad \text{for } j \geqslant p(i)+1;$$

hence

$$\tau_{\leqslant p(i)}(P^{\cdot}_q) \cong P^{\cdot}_q \quad \text{on } U_{i+1}.$$

(ii) $\Rightarrow$ (i) Since $p \cong lp \leqslant q$ and $q \leqslant up \cong p$, we obtain $p \subset q$ as well as $q \subset p$; consequently $p \cong q$ (cf. Def. 2.3 and the discussion following it). □

1.4. *Remark.* In particular, we obtain the following characterization of lp and up:

1. lp is the minimal perversity $q \leqslant p$ such that $H^j P_p^{\cdot}|_{S_{n-i}} = 0$ for every i and every $j \geqslant q(i)+1$,

2. up is the maximal perversity $q \geqslant p$ such that $H^j P_q^{\cdot}|_{S_{n-i}} = 0$ for every i and every $j \geqslant p(i)+1$.

For applications of the duality theory it is important to get information about the complementary perversity p^*. For that reason, we consider a particular class of perversities:

1.5. DEFINITION. A perversity p is called *dualizing on X with respect to R* if the natural morphism

$$\sigma_p\colon\ P_{q^*}^{\cdot} R \to DP_p^{\cdot} R[-n]$$

of [KpFi, §5] is a quasi-isomorphism.

We give some examples and counterexamples:

1.6. *Remark.* Let p be a perversity. Then

(α) p is dualizing in each of the following cases:

(i) $p = o$ or $p = t$;

(ii) $p \cong p+1$, in particular for $p \cong t$,

(iii) R is a field,

(iv) $X = (X, \Sigma)$ and $H^{p(k)+1} P_t^{\cdot}$ is a torsionfree sheaf;

(β) p is dualizing iff

(v) p^* is dualizing,

(vi) for every finitely generated torsion module T, the complexes $P_p^{\cdot} T$ and $(P_p^{\cdot} R) \overset{L}{\otimes} T$ are quasi-isomorphic;

(γ) p need not be dualizing even if $p \cong o$; in particular, $p \cong q$ does not imply that $p^* \cong q^*$.

Proof. (i), (iii) and (iv) follow from [KpFi], (ii) will be proved in 1.9, (vi) is a consequence of [GoSi] (in that paper a pseudomanifold X is called *locally p-torsionfree* if p is dualizing on X with respect to $\mathbf{Z}$). (v) follows immediately from biduality [Bo, V.8]. For (γ) we use this example: for $m \geqslant 2$ set

$${}_mX_2^2 := V(\mathbf{P}_{m+1}; z_0^2 + z_1^2 + z_2^2),$$

and $R = \mathbf{Z}$; by [FiKp, 2.2] there are only two different classes of perversities, represented by $o \cong m \cong n$ and $m+1 \cong t$, thus $o^* = t \not\cong n = m^*$. □

For the next result, note that for $X = {}_mX^2_2$ and $R = Z$ this holds:

$$(uo)^* \not\lesseqgtr lo^*, \quad (lm)^* \not\lesseqgtr um^*, \quad (lt)^* \neq ut^*.$$

1.7. Lemma. *If p is a dualizing perversity, then*

$$(up)^* \leqslant lp^* \lesseqqgtr p^* \lesseqqgtr (lp)^* \lesseqqgtr up^*.$$

If R is a field or L is torsion, then

$$lp^* = (up)^* \quad \textit{and} \quad up^* = (lp)^*.$$

Proof. In order to prove the inequalities let us first assume that $p^* \cong (lp)^*$. Then $(lp)^* \leqslant up^*$ by Lemma 1.3, and it follows that

$$lp^* \leqslant p^* \leqslant (lp)^* \leqslant up^*$$

is a chain of quasi-isomorphic perversities. Since p^* is dualizing as well, we obtain a similar chain with the perversity p^* instead of p. An application of the *-operator yields in particular that

$$(up)^* \leqslant lp^*.$$

We now have to verify that $P^{\cdot}_{p*}L \cong P^{\cdot}_{(lp)*}L$. Since this is a local problem and since L has finitely generated stalks, it is enough to consider the cases that $L = R$ and that L is a constant torsion module T. By [KpFi, 5.10] there exists a commutative diagram

$$\begin{array}{ccc}
P^{\cdot}_{p*}T & \longrightarrow & P^{\cdot}_{(lp)*}T \\
\cong\downarrow & & \downarrow\cong \\
DP^{\cdot}_{p}T[1-n] & \xrightarrow[\cong]{D\mu_{lp,p}[1-n]} & DP^{\cdot}_{lp}T[1-n]
\end{array}$$

Hence, the natural morphism $P^{\cdot}_{p*}T \to P^{\cdot}_{(lp)*}T$ is a quasi-isomorphism. For the case $L = R$ we use the natural morphism (see [KpFi, § 5])

$$\sigma_p\colon P^{\cdot}_{p*}R \to DP^{\cdot}_pR[-n].$$

Then there exists an analogous diagram

$$\begin{array}{ccc}
P^{\cdot}_{p*}R & \longrightarrow & P^{\cdot}_{(lp)*}R \\
\cong\downarrow\sigma_p & & \downarrow\sigma_{lp} \\
DP^{\cdot}_{p}R[-n] & \xrightarrow[\cong]{D\mu_{lp,p}[-n]} & DP^{\cdot}_{lp}R[-n]
\end{array}$$

where σ_{lp} is a quasi-isomorphism by Remark 1.8 below. Thus

$$P^{\cdot}_{p*}R \cong P^{\cdot}_{(lp)*}R.$$

Now assume that up and up^* are dualizing (then, by Remark 1.8, all perversities $q \cong p$ respectively $q \cong p^*$ are dualizing as well, e.g., if R is a field) or if L is torsion. Then the first part of Lemma 1.7 is also true with up respectively up^* instead of p and thus

$$(up)^* \cong (lup)^* = (lp)^* \cong p^*,$$

i.e., $(up)^* \geqslant lp^*$; since also $(up)^* \leqslant lp^*$, we obtain $(up)^* = lp^*$. If we dualize this equation and apply it to p^* instead of p, then the second equality follows. □

1.8. *Remark.* If p is a dualizing perversity, then so is every perversity q with $lp \leqslant q \leqslant p$. If moreover $lp^* \leqslant (up)^*$, then every perversity $q \cong p$ is dualizing.

Proof. We only have to show that the complex $DP^{\cdot}_q R[-n]$ satisfies the axioms for the complex $P^{\cdot}_{q*}$ (see the discussion of σ_p above Lemma 5.4 in [KpFi]). The only (possibly) missing property is that

$$H^{q^*(i)+1}\, DP^{\cdot}_q[-n] \,|\, S_{n-i} = 0 \quad \text{for } 2 \leqslant i \leqslant u.$$

Since $p \cong q$ by Lemma 1.3, there exists a quasi-isomorphism

$$DP^{\cdot}_q R[-n] \cong DP^{\cdot}_p R[-n] \cong P^{\cdot}_{p*}.$$

By assumption, $p^* \leqslant q^*$; consequently,

$$H^j P^{\cdot}_{p*} \,|\, S_{n-i} = 0 \quad \text{for } j \geqslant q^*(i)+1$$

implies the missing vanishing condition. Now it suffices to consider complementary perversities in order to obtain the second statement. □

We know already that the analogue of Remark 1.8 for lp does not hold. The following result tells us that a perversity p is quasi-isomorphic to a dualizing perversity provided that

$$lp(i) < up(i) \quad \text{unless } p(i) = i-2:$$

1.9. PROPOSITION. *If $p \cong p+1$, then p is a dualizing perversity.*

Instead of giving a direct proof, we refer to Proposition 3.10, which obviously implies the result.

In Remark 1.6 (γ) we have seen an example such that $o \cong p$ does not imply $o^* \cong p^*$. However, we can show this:

1.10. COROLLARY. *If $o \cong p+1$, then $o^* \cong p^*$.*

Proof. From Lemma 1.3 we obtain that $lp = 0$. Hence, $p \cong p+1$, and p is dualizing by Proposition 1.9. Now Lemma 1.7 applies:

$$p^* \cong (lp)^* = o^*.$$ □

2. The obstruction trapezoid

A main topic in [KpFi] was to compare the intersection homology modules for two perversities $p \leqslant q$ by means of the canonical morphism

$$\mu_{pq}\colon P_p^{\cdot} \to P_q^{\cdot}.$$

Obviously, such a morphism also exists if we replace p or q with quasi-isomorphic perversities. Hence, it is natural to try to carry over the partial ordering "$\leqslant$" to a partial ordering "$\subset$" in the set of equivalence classes of quasi-isomorphic perversities. The naive attempt "$[lp, up] \subset [lq, uq]$ if there exists perversities $p' \cong p$ and $q' \cong q$ such that $p' \leqslant q$" is not satisfactory, since that does not define a transitive ordering (see Section 4). For that reason we adopt a more axiomatic point of view (see 2.3). We start with some formal considerations:

2.1. Definition. Let $S^{\cdot}$ be a complex of sheaves on X, assume that $S^{\cdot}$ is X-cc. For an open subset $W \subset X$ set

$$a_W(S^{\cdot}) := \sup\{a;\ H^j S^{\cdot}|_W = 0, j \leqslant a\},$$

$$b_W(S^{\cdot}) := \sup\{b;\ H^j S^{\cdot}|_{W \cap S_{n-i}} = 0, j \geqslant i-1-b,\ 0 \leqslant i \leqslant u\}.$$

For a family of supports ϕ on X set

$$a_{\phi}(S^{\cdot}) := \max\{a_W(S^{\cdot});\ E(\phi) \subset W \subset X\}$$

$$b_{\phi}(S^{\cdot}) := \max\{b_W(S^{\cdot});\ E(\phi) \subset W \subset X\}.$$

Note that $a_W(S^{\cdot}) = \infty = b_W(S^{\cdot})$ iff $S^{\cdot}|_W \cong 0$.

The following simple result is needed for induction proofs:

2.2. Lemma. *$H^j_{\phi}(X, S^{\cdot}) = 0$ for $j \leqslant a_{\phi}(S^{\cdot})$ and for $j \geqslant n - b_{\phi}(S^{\cdot}) - 1$.*

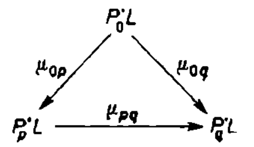
Proof. Since $H^{\cdot}_{\phi}(X, S^{\cdot}) \cong H^{\cdot}_{\phi}(W, S^{\cdot})$ for $E(\phi) \subset W \subset X$, we may assume that $W = X$. There exists a spectral sequence

$$E_2^{s,t} = H^s_{\phi}(X, H^t S^{\cdot}) \;\Rightarrow\; H^{s+t}_{\phi}(X, S^{\cdot}).$$

Thus the result is obvious for $j \leqslant a_{\phi}(S^{\cdot})$; since $\dim \operatorname{supp} H^t S^{\cdot} \leqslant n-2-b_{\phi}(S^{\cdot}) - t$, we also obtain the vanishing for $j \geqslant n - b_{\phi}(S^{\cdot}) - 1$. □

We are interested in the following situation:

2.3. Definition. For a pseudomanifold X, a sheaf L as in 0.1 and two perversities p and q, set $p \subset q$ if there exists a commutative diagram of morphisms



where μ_{op} and μ_{oq} are the canonical morphisms (cf. [KpFi, 0.2]). A chain $p \subset q \subset r$ is always understood to be "commutative", i.e., $\mu_{pr} = \mu_{qr}\mu_{pq}$ for the corresponding morphisms.

If $p \leqslant q$, then $\mathrm{Mor}(P_p^{\cdot}, P_q^{\cdot}) \cong \mathrm{Mor}(P_p^{\cdot}|_{U_2}, P_q^{\cdot}|_{U_2})$, by [Bo, V.9.2]; hence, $p \subset q$ where μ_{pq} is the canonical morphism as considered in [KpFi]. The notion "$\subset$" extends "$\leqslant$" to a partial ordering of the set of equivalence classes of perversities for fixed X and L: while the transitivity is obviously true, the anti-symmetry can be seen in the following way: if $p \subset q$ and $q \subset p$, then there exists a diagram

$$\begin{array}{ccc} P_p^{\cdot} & \xrightarrow{\mu_{pp}} & P_p^{\cdot} \\ & {\scriptstyle \mu_{pq}}\searrow \quad \nearrow{\scriptstyle \mu_{qp}} & \\ & P_q^{\cdot} & \end{array}$$

which commutes on U_2 and thus everywhere, and in which μ_{pp} is quasi-isomorphic to the identity. Interchanging p and q we see that these two perversities are quasi-isomorphic. An interesting application of quasi-isomorphic perversities is in [GoMPh$_3$]. If Σ itself is a stratum, then $p \subset q$ iff $lp \leqslant uq$.

In general, it is not clear whether μ_{pq} is uniquely determined by p and q; nevertheless, we shall show the following in Lemma 2.5: For the distinguished triangle

$$T(p, q) \qquad \begin{array}{ccc} P_p^{\cdot}L & \longrightarrow & P_q^{\cdot}L \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & Q_{pq}^{\cdot}L =: Q_{pq}^{\cdot} & \end{array}$$

the numbers $a_\phi(Q_{pq}^{\cdot})$ and $b_\phi(Q_{pq}^{\cdot})$ depend on p and q, but not on μ_{pq}, which is obvious for $p \leqslant q$ since then μ_{pq} is uniquely determined. Thus we may define

$$a_\phi^L(p, q) := a_\phi(Q_{pq}^{\cdot}) \quad \text{and} \quad b_\phi^L(p, q) := b_\phi(Q_{pq}^{\cdot})$$

(for simplicity we usually omit ϕ and L in the notations). Furthermore, it follows that $a(p, q)$ and $b(p, q)$ do not change if p and q are replaced with quasi-isomorphic perversities. Besides the obvious fact that $Q_{pq}^{\cdot}|_{U_2} \cong 0$, for induction proofs we shall use this consequence of Lemma 2.2:

2.4. Corollary. *Let p and q be perversities such that $p \subset q$, and let x be a point in S_{n-i}.*

(i) *If $H^j Q_{pq}^{\cdot}|_{U_i} = 0$ for $j \leqslant a$, then*

$$H^j Q_{pq,x}^{\cdot} = 0 \quad \text{for } j \leqslant \min(a, p(i)).$$

(ii) *If* $H^j Q^{\cdot}_{pq}|_{S_{n-1}} = 0$ *for* $j \geqslant l-b-1$ *if* $l \leqslant i-1$, *then*

$$H^j Q^{\cdot}_{pq,x} = 0 \quad \text{for } i-b-2 \leqslant j \leqslant p(i).$$

Proof. By (0.3.1), it suffices to show that $H^j(U', Q^{\cdot}_{pq})$ vanishes for the j's under consideration. We apply Lemma 2.2. The case (i) is immediate with U' instead of X; in the case (ii) we use the link L of x in X, which is of dimension $i-1$:

$$H^j(U', Q^{\cdot}_{pq}) \cong H^j(L, Q^{\cdot}_{pq}|_L) = 0$$

for $j \geqslant \dim(L)-b-1 = i-b-2$. □

We now intend to prove that $a(\mu_{pq}) := a(Q^{\cdot}_{pq})$ and $b(\mu_{pq}) := b(Q^{\cdot}_{pq})$ depend only on p and q. For that purpose we need a "truncation" of perversities: for $c \in N$ set

$$\tau_{\leqslant c}\, p := \min(p, o+c), \qquad \tau^{\geqslant c} p := \max(p, t-c).$$

2.5. Lemma. *If* $p \subset q$ *is realized by a morphism* $\mu_{pq}\colon P^{\cdot}_p \to P^{\cdot}_q$, *then*

$$a(\mu_{pq}) = \max\{c;\, \tau_{\leqslant c}\, p \cong \tau_{\leqslant c}\, q\},$$

$$b(\mu_{pq}) = \max\{c;\, \tau^{\geqslant c} p \cong \tau^{\geqslant c} q\}.$$

Proof. Let us first show this auxiliary result

(2.5.1) $$a(\tau_{\leqslant c}\, p, p) \geqslant c, \qquad b(p, \tau^{\geqslant c} p) \geqslant c.$$

Set $r := \tau_{\leqslant c}\, p$; for the first inequality it suffices to prove by induction on i

$$H^j Q^{\cdot}_{rp}|_{U_i} = 0 \qquad \text{for } j \leqslant c.$$

This is obvious for $i = 2$. For the step "$i \Rightarrow i+1$" we may assume that

$$r(i) < p(i) \qquad \text{(and thus } c = r(i)),$$

since, otherwise $r = p$ on $[2, \ldots, i]$ and thus $P^{\cdot}_r|_{U_{i+1}} \cong P^{\cdot}_p|_{U_{i+1}}$. For a point $x \in S_{n-i}$, Corollary 2.4(i) implies $H^j Q^{\cdot}_{rp,x} = 0$ for $j \leqslant c = r(i)$ by induction hypothesis.

Now set $s := \tau^{\geqslant c} p$; for the second inequality it suffices to prove by induction on i

$$H^j Q^{\cdot}_{ps}|_{S_{n-l}} = 0 \qquad \text{for } j \geqslant l-c-1 \text{ if } l \leqslant i-1.$$

For the step "$i \Rightarrow i+1$" we may assume in the same way that $s(i) > p(i)$ and thus $s(i) = i-c-2$. For a point $x \in S_{n-i}$, then $H^j Q^{\cdot}_{ps,x} = 0$ for $j \geqslant s(i)+1 = i-c-1$ by (0.3.1).

Now we can prove (2.5): set $Q^{\cdot} := Q^{\cdot}_{pq}$; in order to show that

$$a := \max\{c;\, \tau_{\leqslant c}\, p \cong \tau_{\leqslant c}\, q\} \leqslant a(\mu_{pq}),$$

$$b := \max\{c;\, \tau^{\geqslant c} p \cong \tau^{\geqslant c} q\} \leqslant b(\mu_{pq})$$

it suffices to verify by induction on i that

$$H^j Q^\cdot|_{U_i} = 0 \quad \text{for } j \leqslant a,$$

$$H^j Q^\cdot|_{S_{n-l}} = 0 \quad \text{for } j \geqslant l-b-1 \text{ if } l < i.$$

For the step "$i \Rightarrow i+1$" we may use Corollary 2.4 in the case that $j \leqslant p(i)$. For $j \geqslant p(i)+1$ there is an inclusion $H^j Q^\cdot_x \hookrightarrow H^j P^\cdot_{q,x}$ for $x \in S_{n-i}$ (cf. 0.3); hence, it suffices to verify

$$H^j P^\cdot_{q,x} = 0 \quad \text{for } p(i)+1 \leqslant j \leqslant q(i), \quad j \leqslant a \text{ or } j \geqslant i-b-1.$$

If $j \leqslant a$, then (2.5.1) implies

$$H^j P^\cdot_{q,x} \cong H^j P^\cdot_{\tau_{\leqslant a} q,x} \cong H^j P^\cdot_{\tau_{\leqslant a} p,x} \cong H^j P^\cdot_{p,x} = 0.$$

If $j \geqslant i-b-1$, then we obtain in the notation of 0.2 for the link L of x, which is of dimension $i-1$,

$$0 = H^j P^\cdot_{p,x} \twoheadrightarrow H^j P^\cdot_{\tau^{\geqslant b} p,x} \cong H^j P^\cdot_{\tau^{\geqslant b} q,x} \underset{j \leqslant q(i)}{\cong} H^j(L, P^\cdot_{\tau^{\geqslant b} q})$$

$$\underset{j \geqslant i-1-b}{\cong} H^j(L, P^\cdot_q) \cong H^j(U', P^\cdot_q) \underset{j \leqslant q(i)}{\cong} H^j P^\cdot_{q,x}.$$

Now we have to show

$$\max\{c; \tau_{\leqslant c} p \cong \tau_{\leqslant c} q\} \geqslant a(\mu_{pq}) =: a,$$

$$\max\{c; \tau^{\geqslant c} p \cong \tau^{\geqslant c} q\} \geqslant b(\mu_{pq}) =: b.$$

That is equivalent to

$$P^\cdot_{\tau_{\leqslant a} p} \cong P^\cdot_{\tau_{\leqslant a} q} \quad \text{and} \quad P^\cdot_{\tau^{\geqslant b} p} \cong P^\cdot_{\tau^{\geqslant b} q}.$$

The first statement is easy to see, since

$$P^\cdot_{\tau_{\leqslant a} p} \cong \tau_{\leqslant a} P^\cdot_p \cong \tau_{\leqslant a} P^\cdot_q \cong P^\cdot_{\tau_{\leqslant a} q}$$

by the definition of a. For the second statement, according to Lemma 1.3, we have to verify for $r := \tau^{\geqslant b} p$, $s := \tau^{\geqslant b} q$, and every $i \geqslant 2$:

(α) $H^j P^\cdot_r|_{S_{n-i}} = 0$ for $j \geqslant s(i)+1$;

(β) $H^j P^\cdot_s|_{S_{n-i}} = 0$ for $j \geqslant r(i)+1$.

Statement (α) is obvious for $r(i) \leqslant s(i)$. Hence, we may assume that $s(i) < r(i)$, which implies $q(i) < p(i)$ and also $p = r$ on $[2, \ldots, i]$, since, otherwise, $p(i) < (t-b)(i)$ yields $s(i) = r(i)$. There is a commutative diagram for $s(i) < j \leqslant p(i) = r(i)$ and $x \in S_{n-i}$

$$\begin{array}{ccccc} H^j P^\cdot_{r,x} & \cong & H^j P^\cdot_{p,x} & \cong & H^j(L, P^\cdot_p) \\ & & \downarrow & & \downarrow \gamma \\ 0 & = & H^j P^\cdot_{q,x} & \longrightarrow & H^j(L, P^\cdot_q) \end{array}$$

where γ is bijective for $j \geqslant i-1-b \leqslant s(i)+1$, by Lemma 2.2.

For the statement β) we may assume that

$$r(i) = \max(p(i),\, i-b-2) < j \leqslant s(i).$$

For $x \in S_{n-i}$ the epimorphism

$$0 = H^j P^{\cdot}_{p,x} \twoheadrightarrow H^j P^{\cdot}_{q,x} \underset{(2.5.1)}{\twoheadrightarrow} H^j P^{\cdot}_{s,x}$$

yields the result. □

One might use Lemma 2.5 in order to define $a(p, q)$ and $b(p, q)$ for arbitrary perversities p and q. This will be of interest in the vanishing theorems for Stein spaces.

An essential step in the comparison of the "obstruction trapezoids" in Theorem 2.9 is provided by the following result:

2.6. Monotony Lemma. *If p, q, r are perversities such that $p \subset q \subset r$, then*

$$a(p, r) = \min(a(p, q), a(q, r)) \quad \textit{and} \quad b(p, r) = \min(b(p, q), b(q, r));$$

furthermore, if $p \subset q \subset r \subset s$, then

$$a(p, s) \leqslant a(q, r) \quad \textit{and} \quad b(p, s) \leqslant b(q, r).$$

We prove the statement in three steps:

(i) $a(p, r) \leqslant a(q, r)$ and $b(p, r) \leqslant b(q, r)$;

(ii) $a(p, r) \leqslant a(p, q)$ and $b(p, r) \leqslant b(p, q)$;

(iii) $a(p, r) = \min(a(p, q), a(q, r))$ and $b(p, r) = \min(b(p, q), b(q, r))$.

Proof of (i). We show by induction on i that for $Q^{\cdot} := Q^{\cdot}_{qr}$

$$H^j Q^{\cdot}|_{U_i} = 0 \quad \text{for } j \leqslant a(p, r),$$

$$H^j Q^{\cdot}|_{S_{n-l}} = 0 \quad \text{for } j \geqslant l - b(p, r) - 1, \text{ if } l < i.$$

For the step "$i \Rightarrow i+1$" and a point $x \in S_{n-i}$, we may use Corollary 2.4 in the case $j \leqslant q(i)$. For $j \geqslant q(i)+1$ there exists a commutative diagram

$$(*) \qquad \begin{array}{ccccccc} & & H^j P^{\cdot}_{p,x} & & & & \\ & & \downarrow{\scriptstyle \mu^j_{pq}} & \searrow{\scriptstyle \mu^j_{pr}} & & & \\ 0 & = & H^j P^{\cdot}_{q,x} & \xrightarrow{\mu^j_{qr}} & H^j P^{\cdot}_{r,x} & \xrightarrow[\cong]{} & H^j Q^{\cdot}_x \end{array}$$

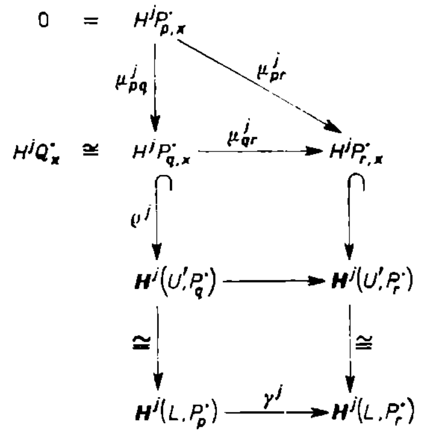
Now μ^j_{pr} is surjective for $j \leqslant a(p, r)$ and for $j \geqslant i - b(p, r) - 1$; hence, $H^j P^{\cdot}_{r,x}$ vanishes.

Proof of (ii). Set $Q^{\cdot} := Q^{\cdot}_{pq}$; we prove by induction on i that

$$H^j Q^{\cdot}|_{U_i} = 0 \quad \text{for } j \leqslant a(p, r),$$

$$H^j Q^{\cdot}|_{S_{n-l}} = 0 \quad \text{for } j \geqslant l - b(p, r) - 1, \text{ if } l < i.$$

For the step "$i \Rightarrow i+1$" we fix a point $x \in S_{n-i}$. In the case that $j \leqslant p(i)$ we may apply Corollary 2.4. For $j \geqslant p(i)+1$ we consider an extended version of a diagram analogous to (*):



Hence, we have to show that $H^j P^{\cdot}_{q,x} = 0$ for the j's under consideration. For $j \leqslant a(p, r)$ and for $j \geqslant i - b(p, r) - 1$ the homomorphism μ^j_{pr} is surjective and thus $H^j P^{\cdot}_{r,x} = 0$. Consequently, it suffices to show that μ^j_{qr} is injective. This is evident for $j \leqslant a(p, r)$, since $a(p, r) \leqslant a(q, r)$ by (i). For $j \geqslant i - b(p, r) - 1$, it is sufficient to verify that γ^j is injective. But that is a consequence of Lemma 2.2 and the induction hypothesis, since $b(q, r) \geqslant b(p, r)$, by (i).

Proof of (iii). As a consequence of (i) and (ii) we obtain

$$a(p, r) \leqslant \min(a(p, q), a(q, r)).$$

The octahedral axiom [Ha, p. 21] provides an exact sequence

$$\ldots \to H^j Q^{\cdot}_{pq} \to H^j Q^{\cdot}_{pr} \to H^j Q^{\cdot}_{qr} \to \ldots .$$

Hence

$$a(p, r) \geqslant \min(a(p, q), a(q, r)).$$

The same argument works for $b(-, -)$. □

For the introduction of the obstruction trapezoids we need a final invariant, which generalizes the codimension of the p-singular set in [KpFi]:

2.7. Definition. For $p \subset q$ and an open subset $W \subset X$ the number $k^L_W(\mu_{pq})$ denotes the minimal codimension k such that $H^{\cdot} Q^{\cdot}_{pq}|_{W \cap S_{n-k}} \neq 0$. If ϕ is a family of supports on X, then set

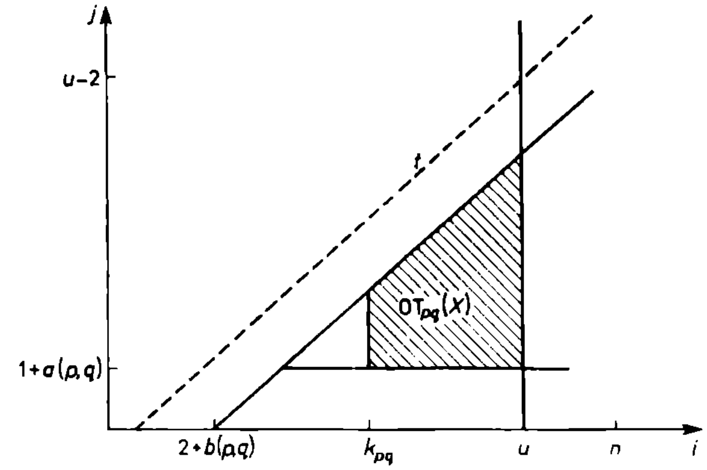
$$k_{pq} := k^{\phi, L}_X(\mu_{pq}) := \max\{k^L_W(\mu_{pq});\ E(\phi) \subset W \subset X\}.$$

For $Q^{\cdot}_{pq} \cong 0$ it is convenient to read this definition as $k_{pq} = \infty$. By Lemma 2.5, the number k_{pq} depends only on p and q, but not on the choice of μ_{pq}.

In the notation of [KpFi, 3.3] we have $k_{oq} = k_q$. A geometric measure to what extent a complex $Q^{\cdot}_{pq}$ is different from the zero-complex, is provided by the *obstruction trapezoid* $\mathrm{OT}_{pq}(X)$:

2.8. DEFINITION. *For $p \subset q$ set*

$$\mathrm{OT}_{pq}(X) = \mathrm{OT}^{\phi,L}_{pq}(X)$$
$$= \{(i, j) \in N^2;\ k_{pq} \leqslant i \leqslant u,\ a(p, q)+1 \leqslant j \leqslant i-2-b(p, q)\}.$$



2.9. THEOREM. *If $p \subset q \subset r \subset s$ is a chain of perversities, then*

$$\mathrm{OT}_{qr}(X) \subset \mathrm{OT}_{ps}(X).$$

Proof. By the Monotony Lemma, we know that $a(q, r) \geqslant a(p, s)$ and $b(q, r) \geqslant b(p, s)$. Hence, the theorem follows from this supplement to the Monotony Lemma:

2.10. LEMMA. *If $p \subset q \subset r$ is a chain of perversities, then*

$$k_{pr} = \min(k_{pq}, k_{qr});$$

furthermore, if $p \subset q \subset r \subset s$, then $k_{ps} \leqslant k_{qr}$.

Proof. By the octahedral axiom, there exists an exact sequence

$$\ldots \to H^j Q^{\cdot}_{pq} \to H^j Q^{\cdot}_{pr} \to H^j Q^{\cdot}_{qr} \to \ldots,$$

and thus we obtain

$$k_{pr} \geqslant \min(k_{pq}, k_{qr}) =: m.$$

Assume that $k_{pr} > m$. Then we may assume that $X = U_{m+1}$, i.e., $k_{pr} = \infty$, while k_{pq} or k_{qr} is finite. Consequently, $a(p, r)$ is infinite, but $a(p, q)$ or $a(q, r)$ is finite, a contradiction to Monotony Lemma. □

3. Application to the Main Lemma

We now come to a generalization of [KpFi, 3.5]. Since we use the invariants $a(p, q)$ and $b(p, q)$, we obtain a more symmetric version. Instead of overloading the Main Lemma with the different properties of $a(p, q)$ and $b(p, q)$, we prefer to interpret these results separately.

3.1. Main Lemma. *Let $p \subset q$ be two perversities. Then the associated homomorphism*

$$\mu^j_{pq}\colon H^j_\Phi(X, P^\cdot_p L) \to H^j_\Phi(X, P^\cdot_q L)$$

is bijective for $j \leqslant a^L_\Phi(p, q)$ and $j \geqslant n - b^L_\Phi(p, q)$, injective for $j = a^L_\Phi(p, q) + 1$, and surjective for $j = n - b^L_\Phi(p, q) - 1$.

Proof. The distinguished triangle $T(p, q)$ induces a long exact sequence in hypercohomology

$$H^{j-1}_\Phi(X, Q^\cdot_{pq}) \to H^j_\Phi(X, P^\cdot_p L) \to H^j_\Phi(X, P^\cdot_q L) \to H^j_\Phi(X, Q^\cdot_{pq}).$$

Hence, Lemma 2.2 yields the result. □

It is convenient to use these abbreviations [KpFi, § 3]

$$a(p) := a(o, p), \quad b(p) := b(o, p), \quad k_p := k_{op}.$$

The complex $Q^\cdot_{op}$ is nothing but the complex $\tau^{\geqslant 1} P^\cdot_p$. For that reason, the invariants $a(p)$, $b(p)$, and k_p can be calculated directly from the complex $P^\cdot_p$.

3.2. Proposition. *If $p \subset q$, then*

$$a(p, q) \geqslant \max(a(q), up(k_{pq})) \quad \textit{and} \quad b(p, q) \geqslant \max(b(q), (lq)^*(k_{pq})).$$

Proof. The inequalities $a(p, q) \geqslant a(o, q)$ and $b(p, q) \geqslant b(o, q)$ are particular cases of the Monotony Lemma. Since $a(p, q)$, $b(p, q)$, and k_{pq} do not change when p is replaced with up and q with lq, we may assume that $p = up$ and $q = lq$. For the inequality

$$a(p, q) \geqslant p(k_{pq})$$

we prove by induction on i for $Q^\cdot = Q^\cdot_{pq}$ and $k := k_{pq}$

$$H^j Q^\cdot|_{U_i} = 0 \quad \text{for} \quad j \leqslant p(k).$$

This is obvious for $i \leqslant k$, since then $Q^\cdot|_{U_i} \cong 0$. For the step "$i \Rightarrow i+1$" we may assume that $i \geqslant k$. For a point $x \in S_{n-i}$ there exists an inclusion

$$H^j Q^\cdot_x \hookrightarrow H^j(U', Q^\cdot)$$

for $j \leqslant p(i)$ by 0.3; the induction hypothesis and Lemma 2.2 then imply that $H^j(U', Q^\cdot)$ vanishes for $j \leqslant p(k)$.

Finally, we have to verify that

$$b(p, q) \geqslant q^*(k).$$

To that end we induct on i in order to show that

$$H^j Q^{\cdot}|_{S_{n-l}} = 0 \quad \text{for} \quad j \geqslant l - q^*(k) - 1 \text{ if } l \leqslant i-1.$$

That is obviously true for $i \leqslant k$. For the step "$i \Rightarrow i+1$" with $i \geqslant k$ fix a point $x \in S_{n-i}$. For $j \leqslant p(i)$ we may apply Corollary 2.4; for $j \geqslant p(i)+1$ and $j \geqslant i - q^*(k) - 1 = q(k)+(i-k)+1 \geqslant q(i)+1$, we may use (0.3.1). □

3.3. Proposition. *If $p \subset q$, then*

$$a(p, q)+b(p, q)+3 \leqslant k_{pq} \leqslant b(p, q)+2+\max(lp, lq)(k_{pq}).$$

In particular

$$a(p, q)+1 \leqslant \max(lp, lq)(k_{pq}).$$

Proof. Set $k := k_{pq}$ and $\max(i) := \max(lp, lq)(i)$. If $p \not\equiv q$, there exists a pair (k, j) in $\mathrm{OT}_{pq}(X)$ and thus

$$a(p, q)+1 \leqslant j \leqslant k - b(p, q) - 2.$$

On the other side

$$H^j Q^{\cdot}_{pq}|_{S_{n-i}} = 0 \quad \text{for} \quad j \geqslant \max(i)+1 = i-1-(i-2-\max(i)),$$

thus we obtain

$$b(p, q) \geqslant \min\{i-2-\max(i);\, i \geqslant k\} = k-2-\max(k).$$ □

3.4. Remark. For small perversities $p \subset q$ there is a stronger estimate. Set $s := k_{pq}/2 - \max(lp, lq)(k_{pq})$. Then $2(a(p, q)+1+s) \leqslant k_{pq}$. Equality holds iff $a(p, q)+1 = \max(lp, lq)(k_{pq})$; in this case $b(p, q) = k_{pq} - 2 - \max(lp, lq)(k_{pq})$. We come back to this equality in Lemma 3.11.

The application of the universal coefficient formula [KpFi, 5.1] provides additional information if the first nonvanishing sheaf $H^j S^{\cdot}$ is even torsionfree. For that reason we introduce the following invariant

$$a_0(S^{\cdot}) := \begin{cases} a(S^{\cdot}), & \text{if } H^{a+1} S^{\cdot} \text{ is torsionfree for } a := a(S^{\cdot}), \\ a(S^{\cdot})-1 & \text{otherwise.} \end{cases}$$

When we consider "quotient" complexes $Q^{\cdot}_{pq} = Q^{\cdot}_{pq} L$ we have to distinguish between torsion arising from the torsion subsheaf $T \subset L$ and torsion generated by the "free part" L/T of L; hence, we set

$$a_0^L(p, q) := \min\{a(Q^{\cdot}_{pq} T),\, a_0(Q^{\cdot}_{pq}(L/T))\}.$$

Note that $a_0(Q^{\cdot}_{pq}(L/T)) = a_0(Q^{\cdot}_{pq} R)$ if L/T does not vanish on any connected component of X. Obviously

$$a_0^T(p, q) = a^T(p, q) \quad \text{and} \quad a^R(p, q) - 1 \leqslant a_0^R(p, q) \leqslant a^R(p, q).$$

If $p \subset q$ are dualizing perversities then

$$a_0^R(p, q) \leqslant a^L(p, q).$$

Proof. Since we have to show this inequality only locally, we may assume that L is of the form $L = R^m \oplus T$ with some $m \geqslant 0$ and a torsion module T. Then we obviously have $Q^{\cdot}_{pq} L \cong Q^{\cdot}_{pq} R^m \oplus Q^{\cdot}_{pq} T$; moreover $Q^{\cdot}_{pq} T \cong Q^{\cdot}_{pq} R \overset{L}{\otimes} T$, by the Five Lemma, since the analogous formula holds for $P^{\cdot}_p$ and $P^{\cdot}_q$, by Remark 1.5 (vi). Thus, the sheaf version of the covariant universal coefficient formula [KpFi, (5.13.1)] yields the result. □

We now extend two further results of [KpFi, § 6]. For that purpose we have to recall the notion of a pseudomanifold of (exceptional) type E: We say that X is of *type E with respect to L* if $k := k_t$ is even and

$$H^j P^{\cdot}_t(L/\text{Tors } L)|_{S_{n-k}} = \begin{cases} 0, & \text{if } j \geqslant 1, j \neq k/2, \\ \text{nonzero torsion}, & \text{if } j = k/2. \end{cases}$$

For such a space $a_{U_{k+1}}(n) = \infty$. In the following set $c := n(k)+1$ if X is of type E and $c := n(k)$ otherwise.

3.5. Proposition. *If q is a perversity such that $o+c \subset q$, then*

$$a(q) = a(t) \leqslant c-1, \quad a_0(q) = a_0(t), \quad k_q = k_t.$$

Proof. Set $p := o+c$. By Monotony Lemma 2.6, Remark 3.6, and Lemma 2.10 we know that

$$a(p) \geqslant a(q) \geqslant a(t), \quad a_0(p) \geqslant a_0(q) \geqslant a_0(t),$$

and $k_p \geqslant k_q \geqslant k_t$. Hence, we may assume $q = p$. Then [KpFi, 6.1] implies that $k_p = k_t$ for $L = R$; if $L = T$ is a constant torsion sheaf and $k_t < k_p$, then also $k_t < k_{p^*}$, since $p^*(k_t) \leqslant p(k_t)$. We obtain on U_{k+1}: $p^* \cong o \cong p$, and thus $p \cong o^* = t$, since L is torsion, i.e. $t \cong o$ on U_{k+1}, a contradiction. But L is locally of the type $R^m \oplus T$; hence, the equality $k_p = k_t$ holds in general. Since by Proposition 3.2 $a(p, t) \geqslant p(k_{pt}) \geqslant p(k_t)$, the Main Lemma implies

$$H^j P^{\cdot}_p \cong H^j P^{\cdot}_t \quad \text{for } j \leqslant p(k_t).$$

Since, moreover, $a(p)+1 \leqslant p(k_p) = p(k_t)$, it is easy to see that $a_0(p) = a_0(t)$ and $a(p) = a(t)$. □

3.6. *Remark.* If $p \subset q \subset r$, then $a_0(p, q) \geqslant a_0(p, r)$. In particular, $a_0(p) \geqslant a_0(q)$. If even $p \leqslant q$, then $a_0(p, r) \leqslant a_0(q, r)$.

Proof. It is easy to see that we only have to prove for $L = R$ and $j = a(p, r)+1$: if $H^j Q^{\cdot}_{pr}$ is torsionfree, then so is $H^j Q^{\cdot}_{pq}$. To that end, we use the exact sequence

$$H^{j-1} Q^{\cdot}_{qr} \to H^j Q^{\cdot}_{pq} \to H^j Q^{\cdot}_{pr}$$

as in the proof of 2.6. By the Monotony Lemma, the module $H^{j-1} Q^{\cdot}_{qr}$ vanishes.

Now assume that $p \leqslant q$. Then we prove by induction on i that $H^{a+1} Q^{\cdot}_{qr}|_{U_i}$ is torsionfree for $a := a_0(p, r)$. If $a+1 \leqslant q(i)$, then, for $x \in S_{n-i}$,

$$H^{a+1} Q^{\cdot}_{qr,x} \hookrightarrow H^{a+1}(U', Q^{\cdot}_{qr}) \cong H^0(U', H^{a+1} Q^{\cdot}_{qr})$$

is torsionfree by induction hypothesis. If $a+1 > q(i)$, then

$$H^{a+1} Q^{\cdot}_{qr,x} \cong H^{a+1} P^{\cdot}_{p,x} \cong H^{a+1} Q^{\cdot}_{pr,x}$$

is torsionfree. □

Note that the following result implies that $a_0(p, r) = \min\{a_0(p, q), a_0(q, r)\}$ if $p \subset q \subset r$ are dualizing perversities such that $r^* \subset q^* \subset p^*$.

3.7. THEOREM. *Let p, q be perversities such that $p \subset q$ and $q^* \subset p^*$. If L is torsion or if p and q are dualizing, then*

$$a_0(p, q) = b(q^*, p^*).$$

Proof. Since the problem is local, we may assume that L is a constant sheaf.

(a) Assume that $L = {}_X T$ is torsion. Dualizing the distinguished triangle $T(p, q)$ we obtain a new distinguished triangle

$$\begin{array}{ccc} P^{\cdot}_{p^*} \cong DP^{\cdot}_p[1-n] & \longleftarrow & DP^{\cdot}_q[1-n] \cong P^{\cdot}_{q^*} \\ {\scriptstyle [1]} \searrow & & \nearrow \\ & DQ^{\cdot}_{pq}[1-n] \cong Q^{\cdot}_{q^*p^*}[-1] & \end{array}$$

see [KpFi, 5.10]; the last quasi-isomorphism results from turning the canonical triangle $T(q^*, p^*)$. Thus, by Lemma 3.8, we obtain

$$\begin{aligned} a_0(p, q) = a_0(Q^{\cdot}_{pq})+1 = b(DQ^{\cdot}_{pq}[1-n])+1 &= b(DQ^{\cdot}_{pq}[2-n]) \\ &= b(Q^{\cdot}_{q^*p^*}) = b(q^*, p^*). \end{aligned}$$

(b) The argument for $L = {}_X R$ and dualizing perversities is rather similar; the distinguished triangle is of the form

$$\begin{array}{ccc} P^{\cdot}_{p^*} \cong DP^{\cdot}_p[-n] & \longleftarrow & DP^{\cdot}_q[-n] \cong P^{\cdot}_{q^*} \\ {\scriptstyle [1]} \searrow & & \nearrow \\ & DQ^{\cdot}_{pq}[-n] \cong Q^{\cdot}_{q^*p^*}[-1] & \end{array}$$

and we find

$$a_0(p, q) = a_0(Q^{\cdot}_{pq}) = b(DQ^{\cdot}_{pq}[1-n]) = b(Q^{\cdot}_{q^*p^*}) = b(q^*, p^*).$$

(c) For $L = R^m \oplus T$ with $m > 0$ note that

$$a_0^L(p, q) = \min\{a_0^R(p, q), a_0^T(p, q)\}$$

as well as

$$b^L(p, q) = \min\{b^R(p, q), b^T(p, q)\}. \qquad \square$$

3.8. LEMMA. *If a complex of sheaves $S^{\cdot}$ is X-cc, then,*

$$b(DS^{\cdot}[1-n]) = b(DS^{\cdot}[-n]) + 1 = a_0(S^{\cdot}).$$

Proof. We only have to show that, for every integer a, the following statements are equivalent:

(i) $H^j S^{\cdot} = 0$ for every $j \leqslant a$, and $H^{a+1} S^{\cdot}$ is torsionfree;

(ii) $H^j DS^{\cdot}[-n]|_{S_{n-i}} = 0$ for every i and every $j \geqslant i - a$.

The implication (i) $\Rightarrow$ (ii) has been shown in [KpFi, Lemma 4.1] with $v = 1$.

(ii) $\Rightarrow$ (i). Let U be a distinguished neighbourhood of a point $x \in X$. By assumption on $S^{\cdot}$, the R-modules $H^j S^{\cdot}_x \cong H^j(U, S^{\cdot})$ are finitely generated. Since $S^{\cdot} \cong D(DS^{\cdot}[-n])[-n]$, we may use the universal coefficient formula [KpFi, 5.1] in the following form

$$0 \to \operatorname{Ext}(H_c^{n-j+1}(U, DS^{\cdot}[-n]), R) \to H^j(U, S^{\cdot})$$
$$\to \operatorname{Hom}(H_c^{n-j}(U, DS^{\cdot}[-n]), R) \to 0.$$

By Lemma 2.2, condition (ii) implies $H_c^j(U, DS^{\cdot}[-n]) = 0$ for $j \geqslant n - a$. Consequently, $H^j S^{\cdot}_x$ vanishes for $j \leqslant a$ and is torsionfree for $j = a + 1$. $\square$

Theorem 3.7 need not hold if one of the perversities is not dualizing:

3.9. EXAMPLE. If X is a complex space such that $o \cong m \not\cong t$, then $a(o, m) = \infty$ though $b(m, t)$ is finite. In particular, the projective algebraic variety ${}_mX_2^2$ of Section 1 is of this type for $R = Z$.

If only the perversity p in Theorem 3.7 is dualizing, then there are still estimates which in particular show that $b(q, t) \geqslant a_0(q^* + 1)$ for an arbitrary perversity q. That result shows that part (β) of the Main Lemma in [KpFi] is in fact a consequence of the new version 3.1 of the Main Lemma.

3.10. PROPOSITION. *If p and q are perversities such that p is dualizing and $p \leqslant q$, then $a_0(q^*, p^*) \geqslant b(p, q+1)$ and $b(q^*, p^*) \geqslant a_0(p, q+1)$.*

Proof. Since this is a local problem and the estimates hold for a torsion sheaf $L = T$ by Theorem 3.7, we may assume that $L = R$. For $x \in S_{n-i}$ consider the commutative diagram

(*)
$$\begin{array}{ccc} P^{\cdot}_{p,x} & \xrightarrow{\mu_{pq}} & P^{\cdot}_{q,x} \\ \cong \big\downarrow \sigma_{p^{\cdot}} & & \big\downarrow \sigma_{q^{\cdot}} \\ DP^{\cdot}_{p^{\cdot}}[-n]_x & \xrightarrow{D\mu_{q^{\cdot}p^{\cdot}}[-n]} & DP^{\cdot}_{q^{\cdot}}[-n]_x \end{array}$$

Since, by Monotony Lemma 2.6, $b := b(p, q+1) = \min\{b(p, q), b(q, q+1)\}$, we conclude that $D\mu_{q*p*}[-n]^j$ is bijective for $j \geq i-b$ and surjective for $j = i-b-1$, cf. Lemma 3.11. Thus, for the "quotient" complex in the distinguished triangle

$$\begin{array}{ccc} DP^{\cdot}_{p^{\cdot}}[-n] & \longrightarrow & DP^{\cdot}_{q^{\cdot}}[-n] \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & DQ^{\cdot}_{q^{\cdot}p^{\cdot}}[1-n] & \end{array}$$

we obtain

$$a_0(q^*, p^*) = b(DQ^{\cdot}_{q^*p^*}[1-n]) \geq b.$$

In the same manner we see that

$$a(DQ^{\cdot}_{q*p*}[1-n]) \geq \min\{a_0(p, q), a_0(q, q+1)\} \geq a_0(p, q+1) =: a,$$

by Remark 3.6. It remains to prove that $H^{a+1}DQ^{\cdot}_{q*p*}[1-n]$ is torsionfree and thus

$$b(q^*, p^*) = a_0(DQ^{\cdot}_{q*p*}[1-n]) \geq a.$$

To that end we consider the induced morphism of the distinguished triangle associated to μ_{pq} and $D\mu^{\cdot}_{q*p*}[-n]$. Then Lemma 3.11 and the Five Lemma implies that the corresponding homomorphism of quotient complexes

$$\alpha^j\colon H^jQ^{\cdot}_{qp} \to H^jDQ^{\cdot}_{q*p*}[1-n]$$

is bijective even for $j = a+1 \underset{3.6}{\leq} a_0(q, q+1)+1$. □

3.11. Lemma. *If $T^{\cdot}_q$ is the complex determined by the distinguished triangle*

$$\begin{array}{ccc} P^{\cdot}_q R & \xrightarrow{\sigma_{q^{\cdot}}} & DP^{\cdot}_{q^{\cdot}}R[-n] \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & T^{\cdot}_q & \end{array}$$

then $a(T^{\cdot}_q) \geq a_0(q, q+1)+1$ and $b(T^{\cdot}_q) \geq b(q, q+1)$.

Proof. Set $a := a_0(q, q+1)+1$ and $b := b(q, q+1)$; we show by induction on i

$$H^j T_q^{\cdot}|_{S_{n-l}} = 0 \quad \text{for } j \geqslant l-b-1 \text{ and } l < i,$$
$$H^j T_q^{\cdot}|_{U_i} = 0 \quad \text{for } j \leqslant a.$$

For the step "$i \Rightarrow i+1$" fix a point $x \in S_{n-i}$. In the notation of 0.2 we obtain for $j \leqslant q(i)$ that

$$H^j T_{q,x}^{\cdot} \hookrightarrow H^j(U', T_q^{\cdot}):$$

as in (0.3.1) this follows from the Five Lemma using the fact that $H^j DP_{q*}^{\cdot}[-n]_x$ is isomorphic to $H^j(U', DP_{q*}^{\cdot}[-n])$ for $j \leqslant q(i)$ (see the remarks preceding [KpFi, 5.4]). By induction hypothesis, Lemma 2.2 yields the result.

Now assume $j \geqslant q(i)+1$, then

$$H^j T_{q,x}^{\cdot} \cong H^j DP_{q*}^{\cdot}[-n]_x.$$

By [KpFi, 5.4], that module vanishes for $j \geqslant q(i)+2$, thus we are left with the case $j = q(i)+1$. In the proof of [KpFi, 5.9] it has been shown that then $H^j T_{q,x}^{\cdot} \cong \mathrm{Tors}(H^j(U', DP_{q*}^{\cdot}[-n]))$, which in turn is isomorphic to $\mathrm{Tors}(H^j(U', P_q^{\cdot}))$ and for $j \leqslant a$ and $j \geqslant i-b-1$ by induction hypothesis: for $j \leqslant a$ this is immediate with Lemma 2.2, while for $j \geqslant i-b-1$ we use as usual the restriction to the link L of x. In the same way we derive from the Main Lemma that $H^j(U', P_q^{\cdot})$ is a submodule of $H^j(U', P_{q+1}^{\cdot})$.

Finally we obtain

$$H^j(U', P_{q+1}) \cong \begin{cases} H^{i-1}(L), & \text{if } q(i) = i-2, \\ H^j P_{q+1,x} \cong H^j Q_{q,q+1,x}^{\cdot}, & \text{if } q(i) < i-2 \end{cases}$$

which are torsionfree modules. □

We now come back to a geometric interpretation of some of the invariants.

3.12. Proposition. *For every perversity $p \not\cong o$ we have*

$$b(p) = k_p - 2 - lp(k_p) = (lp)^*(k_p).$$

Proof. By Proposition 3.2, we know that $b(p) \geqslant (lp)^*(k_p)$. If $b(p) > lp^*(k_p)$, then we obtain $H^j P_p^{\cdot}|_{S_{n-i}} = 0$ for $i \geqslant k_p$ and $j \geqslant i-2-(lp)^*(k_p) \geqslant lp(i)$. Hence, Lemma 1.3 implies $lp-1 \cong lp$, which is only possible for $p \cong o$. □

The last result has in particular the following geometric meaning: the graph of a perversity p either lies completely above the obstruction trapezoid $\mathrm{OT}_p(X) := \mathrm{OT}_{op}(X)$ or it enters $\mathrm{OT}_p(X)$ in a point that lies on the upper side of the trapezoid.

Furthermore, we may visualize part of Lemma 1.3 in the following way:

GOING-DOWN-PRINCIPLE. *Up to quasi-isomorphy a perversity p remains unchanged when it is lowered step by step as long as the intersection of the graph of p with* $\mathrm{OT}_p(X)$ *does not change.*

In an analogous way, *going up* is allowed for p with $\mathrm{OT}_t(X)$ instead of $\mathrm{OT}_p(X)$. In particular, we obtain

$$\tau_{\geqslant a(t)}\, p \cong p \cong \tau^{\leqslant b(p)}\, p,$$

where $\tau_{\geqslant c}\, p := \max(o+c,\, p)$ and $\tau^{\leqslant c}\, p := \min(t-c,\, p)$.

4. Examples

In this section we discuss three equivalence classes of perversities for the open real cone $Y := \mathring{c}(X)$, where

$$X := {}_mX_d^g := \{z \in \boldsymbol{P}_{m+1};\ \sum_{j=0}^{d} z_i^g = 0\}$$

for $g \geqslant 3$, $d \geqslant 2$ even, and $m \geqslant d+2$. The intersection homology of X has been calculated in [FiKp$_3$]. Moreover, let $L = \boldsymbol{Z}_r = \boldsymbol{Z}/r\boldsymbol{Z}$ for some divisor $r \neq 1$ of g.

We consider the perversities p, q and r determined up to quasi-isomorphism by the assignment

	p	q	r
$2d$	$d-2$	$d-1$	d
$2m+1$	$d+2$	$d+2$	$d+1$

It is easy to compute $H^{\cdot}_Y P^{\cdot}_s$ for a perversity s: Let v denote the vertex of Y, then

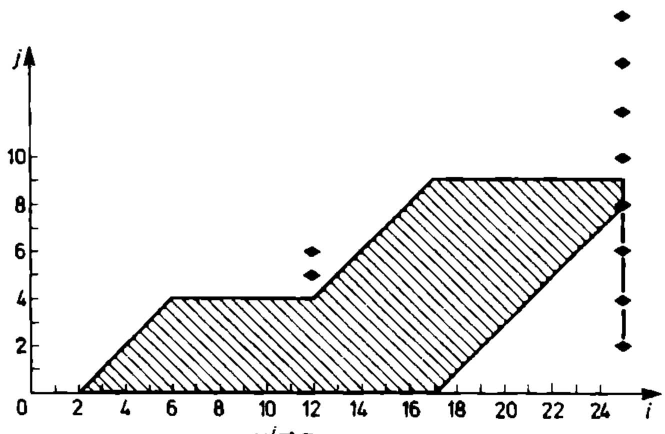
$$\boldsymbol{H}^j(\mathring{c}(X)\setminus\{v\},\, P^{\cdot}_s) \cong \boldsymbol{H}^j(X \times]0,1[,\, P^{\cdot}_s) \cong \boldsymbol{H}^j(X,\, P^{\cdot}_s).$$

Thus we obtain

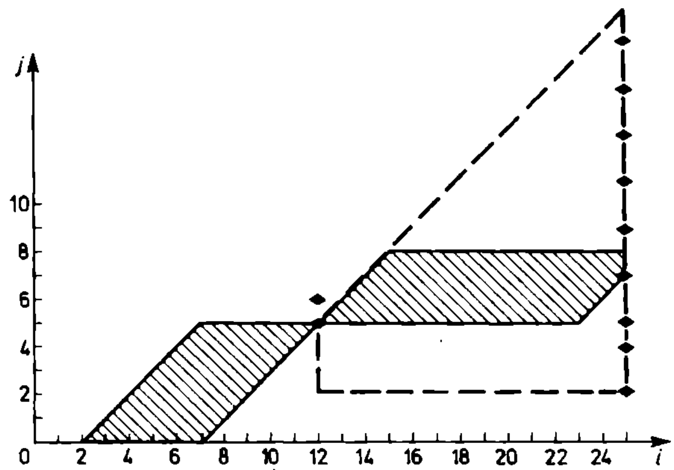
$$H^j P^{\cdot}_{s,v} \cong \begin{cases} \boldsymbol{H}^j(X,\, P^{\cdot}_s), & \text{if } j \leqslant s(2m+1),\\ 0 & \text{otherwise.}\end{cases}$$

[FiKp, 2.1] thus determines explicitly the modules $H^j P^{\cdot}_{s,v}$ for $s = p, q, r$, since, on X, p is quasi-isomorphic to o, q to m, and r to t. On the stratum $S = \Sigma \times]0,1[$ of codimension $2d$ in Y there exists a canonical isomorphism $H^{\cdot}_Y P^{\cdot}_{s,(x,\lambda)} \cong H^{\cdot}_X P^{\cdot}_{s,x}$.

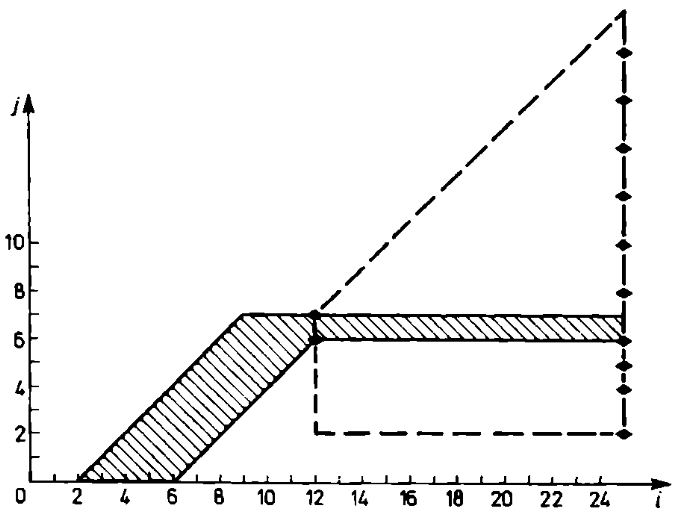
We depict the situation for $d = 6$ and $m = 12$. The obstruction trapezoids $OT_{os}(Y)$ are indicated by broken lines, the "intervals" $[ls, us]$ are shaded, diamonds indicate obstructions against going up or down in the equivalence class of the perversity, see Section 2 (those are the points (i, j) where $j \neq 0$ and $H^j(L, P_s^{\cdot}) \neq 0$ for the link L of a point $y \in S_{25-i}$).



$H^j P_p^{\cdot} Z_{r|S_{25-i}}$



$H^j P_q^{\cdot} Z_{r|S_{25-i}}$



$H^j P_r^{\cdot} Z_{r|S_{25-i}}$

From the pictures we read off the following relations:

$$lp \leqslant uq, \quad lp \nleqslant ur,$$
$$lq \nleqslant lr, \quad lq \leqslant ur,$$
$$uq \nleqslant ur.$$

The construction of the upper bound us and that of the lower bound ls of a perversity s is not monotonous: consider the relation $s \leqslant s'$ for $s := lq$ and $s' := ur$. Finally, it follows that $p \subset r$ though $lp \nleqslant ur$.

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